Quasi-Banach Schatten-von Neumann properties in Weyl-Hörmander calculus

QUASI-BANACH SCHATTEN-VON NEUMANN PROPERTIES IN WEYL-HÖRMANDER CALCULUS

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Abstract. We study structural properties of $\mathit{WL}^{q,p}_{g,\theta}$, which are Wiener-Lebesgue spaces with respect to a slowly varying metric g and with parameters $p,q \in (0,\infty]$, $\theta \in \mathbb{R}$. For $p \in (0,1]$, we deduce Schatten-p properties for pseudo-differential operators whose symbols, together with their derivatives, obey suitable $\mathit{WL}^{q,p}_{g,\theta}$-boundedness conditions. Especially, we perform such investigations for the Weyl-Hörmander calculus. Finally, we apply our results to global-type SG and Shubin pseudo-differential operators.

0. Introduction

The theory of pseudo-differential operators naturally arises in e. g. partial differential equations, statistics, quantum mechanics, and signal processing. A pseudo-differential calculus is a rule which associates a suitable function $a(x,\xi)$, defined on the phase space $W = V \times V' \asymp \mathbf{R}^{2d}$, to a linear operator $\operatorname{Op}(a)$. (See [11] or Section 1 for notations.) The function $a(x,\xi)$ is called the symbol of $\operatorname{Op}(a)$. The partial differential operators are obtained by choosing the symbols to be polynomials in the momentum variable $\xi \in V'$. Hence, pseudo-differential operators are a generalization of the concept of differential operators.

The Weyl quantization $a \mapsto \operatorname{Op}^w(a)$ is unique because it is the only pseudo-differential calculus which is invariant under affine symplectic transformations. This property is fundamental in quantum mechanics, making the Weyl quantization of special interest in several fields. This symplectic structure also facilitates calculations which are otherwise more cumbersome. Therefore, the Weyl calculus naturally lends itself to deeper analysis.

An important question in the theory pseudo-differential operators is to find suitable conditions on the symbol classes in order to guarantee L^2-continuity and compactness properties of the corresponding operators. More detailed studies on compactness are then possible in the framework of Schatten-von Neumann classes, a family $\{\mathscr{I}_p\}_{p\in(0,\infty]}$ of operator spaces characterized by the decay properties of their singular values.

In the paper, we find sufficient conditions on symbols in the Hörmander class $S(m,g)$ in order for corresponding pseudo-differential operators to be Schatten operators of degree $0 < p \leq 1$ on L^2.

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In the case that $1 \leq p \leq \infty$, investigations related to ours can be found in [5,6,16]. It is then assumed that the weight function m fulfills different types of L^p boundedness conditions. More precisely, suppose that g is strongly feasible on W, $p \in [1,\infty]$ and m is g-continuous and (σ,g)-temperate. In [16] it is then proved that

$$m \in L^p \quad \iff \quad \operatorname{Op}^w(a) \in \mathscr{I}_p, \quad \text{when} \quad a \in S(m,g), \tag{0.1}$$

and in [6], (0.1) it is proved that

$$a \in L^p \quad \iff \quad \operatorname{Op}^w(a) \in \mathscr{I}_p, \quad \text{when} \quad h_g^{k/2} m \in L^p,\ a \in S(m,g). \tag{0.2}$$

We observe that (0.1) deals with Schatten-von Neumann properties for the whole symbol class $S(m,g)$, while (0.2) is focused on more individual symbols. In the case $p \in (0,1]$, the right implication

$$m \in L^p \quad \Longrightarrow \quad \operatorname{Op}^w(a) \in \mathscr{I}_p, \quad \text{when} \quad a \in S(m,g), \tag{0.3}$$

in (0.1) was proved in [19]. We also remark that the right implication

$$a \in L^p \quad \Longrightarrow \quad \operatorname{Op}^w(a) \in \mathscr{I}_p, \quad \text{when} \quad h_g^{k/2} m \in L^p,\ a \in S(m,g). \tag{0.4}$$

in (0.2) was deduced already in [10] in the case $p = 1$, and in [16] for general $p \in [1,\infty]$. For $p \leq 2$, it suffices to assume that g should be feasible instead of strongly feasible, in order for (0.3) and (0.4) to hold.

In the paper, we improve (0.3) and obtain a version of (0.4) in the case $p \in (0,1]$, by introducing Wiener-Lebesgue spaces $\mathit{WL}_g^{q,p}$ with respect to a slowly varying metric g. By replacing L^p with $\mathit{WL}_g^{1,p}$ in (0.3) and (0.4), we obtain stronger results than in previous investigations, because we neither need to assume that m is g-continuous nor (σ,g)-temperate. At first glance, it might seem that we are more restrictive since $\mathit{WL}_g^{1,p}$ is contained in L^p when $p \in (0,1]$. However, if in addition m is g-continuous, which is the case in [19], then $m \in L^p$, if and only if $m \in \mathit{WL}_g^{1,p}$. (See Lemma 3.9.) Since there are no prior investigations of $\mathit{WL}_g^{q,p}$-spaces, a significant part of the paper is devoted to their study.

The paper is organized as follows. In Section 1, we recall definitions and some facts on symplectic vector spaces, pseudo-differential operators, the symbol class $S(m,g)$, and Schatten-von Neumannn classes. Here, we also introduce the Wiener-Lebesgue spaces $\mathit{WL}_{g,\theta}^{q,p}$.

In Section 2, we examine the structure of the $\mathit{WL}_g^{q,p}$-spaces, or even more general $\mathit{WL}_{g,\theta}^{q,p}$-spaces. We deduce some invariance properties. We also show that $\mathit{WL}_g^{q,p}$ is essentially increasing with respect to the slowly varying metric g.

In Section 3, we employ the results from Section 2 to draw conclusions about Schatten-p properties of pseudo-differential operators on L^2. Section 3.1 is devoted to the standard Hörmander-Weyl calculus and in Section 3.2 we restrict ourselves to split metrics g in order to find analogous results for more general pseudo-differential calculi.

Lastly, in Section 4 we apply our results to pseudo-differential operators with SG symbols or Shubin symbols.

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1. Preliminaries

In this section we recall some facts on symplectic vector spaces and the symplectic Fourier transform. Thereafter we focus on the Hörmander symbol classes $S(m,g)$, pseudo-differential operators and Schatten-von Neumann operators, and recall some basic facts for them. In the last part of the section we introduce Wiener-Lebesgue spaces $W^{q,p}_{g,\theta}(W)$, and discuss some basic properties.

1.1. Integrations on real vector spaces.

Let V be a real vector space of dimension d, with basis $e_1,\dots,e_d$, and let V' be its dual, with dual basis $\varepsilon_1,\dots,\varepsilon_d$. In particular,

$$\langle e_j,\varepsilon_k\rangle = \delta_{jk},$$

where $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{V,V'}$ is the dual form between V and V'. For any $f \in L^1(V)$, we put

$$\int_V f\,dx \equiv \int\cdots\int_{\mathbf{R}^d} f(x_1e_1+\cdots+x_de_d)\,dx_1\cdots dx_d.$$

For any $f\in L^1(V)$, we define the Fourier transform by

$$(\mathscr{F}f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-\frac{d}{2}}\int_V f(x)e^{-i\langle x,\xi\rangle}\,dx, \qquad \xi\in V'.$$

It follows that $\mathscr{F}$ restricts to a homeomorphism from $\mathscr{S}(V)$ to $\mathscr{S}(V')$, which in turn is uniquely extendable to a homeomorphism from $\mathscr{S}'(V)$ to $\mathscr{S}'(V')$, and to a unitary map from $L^2(V)$ to $L^2(V')$.

1.2. Symplectic vector spaces.

The real vector space W of dimension $2d<\infty$ is called *symplectic* with symplectic form σ, if σ is a non-degenerate anti-symmetric bilinear form on W, i.e. $\sigma(X,Y) = -\sigma(Y,X)$ for every $X,Y\in W$, and if $\sigma(X,Y)=0$ for every $Y\in W$, then $X=0$. The coordinates $X=(x,\xi)$ are called symplectic if the corresponding basis $e_1,\dots,e_d,\varepsilon_1,\dots,\varepsilon_d$ is symplectic, i.e. it satisfies

$$\sigma(e_j,e_k)=\sigma(\varepsilon_j,\varepsilon_k)=0,\quad \sigma(e_j,\varepsilon_k)=-\delta_{jk},\quad j,k=1,\dots,d.$$

It follows that W in a canonical way may be identified with $\mathbf{R}^d\oplus\mathbf{R}^d=\mathbf{R}^{2d}$, and that σ is given by

$$\sigma(X,Y) = \langle y,\xi\rangle - \langle x,\eta\rangle,\quad X=(x,\xi)\in W,\quad Y=(y,\eta)\in W. \tag{1.1}$$

Here $\langle\cdot,\cdot\rangle$ is the usual scalar product on $\mathbf{R}^d$. Moreover, let π_1 and π_2 be the projections $\pi_1(x,\xi)=x$ and $\pi_2(x,\xi)=\xi$ respectively, and set $V=\pi_1W$ and $V'=\pi_2W$, which are identified with $\{\,(x,0)\in W\,;\,x\in V\,\}$ and $\{\,(0,\xi)\in W\,;\,\xi\in V'\,\}$ respectively. Then the dual space of V may be identified with

V' through the symplectic form σ, and W agrees with the cotangent bundle (or phase space) $T^*V = V \oplus V'$.

On the other hand, if V is a vector space of dimension $d < \infty$ with dual space V' and duality $\langle \cdot, \cdot \rangle$, then $W = V \oplus V'$ is a symplectic vector space with symplectic form given by (1.1).

A linear map T on W is called symplectic if $\sigma(TX, TY) = \sigma(X, Y)$ for every $X, Y \in W$. For each pairs of symplectic bases $e_1, \dots, e_d, \varepsilon_1, \dots, \varepsilon_d$ and $\tilde{e}_1, \dots, \tilde{e}_d, \tilde{\varepsilon}_1, \dots, \tilde{\varepsilon}_d$, there is a unique linear symplectic map T such that $Te_j = \tilde{e}_j$ and $T\varepsilon_j = \tilde{\varepsilon}_j$ for every $j = 1, \dots, d$. On the other hand, if T is linear and symplectic and $e_1, \dots, e_d, \varepsilon_1, \dots, \varepsilon_d$ is a symplectic basis, then $Te_1, \dots, T\varepsilon_d$ is also a symplectic basis. Consequently, there is a one-to-one relation between linear symplectic mappings, and representations of W as cotangent boundles T^*V. We refer to [11] for more facts about symplectic vector spaces.

The symplectic volume form is defined by $dX = \sigma^n/d!$, and if $U \subseteq W$ is measurable, then $|U|$ denotes the measure of U with respect to dX. This implies that

$$\int_W a(X)\, dX = \int \cdots \int_{\mathbf{R}^d \oplus \mathbf{R}^d} a(x_1 e_1 + \cdots + \xi_d \varepsilon_d)\, dx_1 \cdots d\xi_d$$

is independent of the choice of the symplectic coordinates $X = (x, \xi)$ when $f \in L^1(W)$. Consequently, $\mathscr{D}'(W)$ and its usual subspaces only depend on σ and are independent of the choice of symplectic coordinates.

The symplectic Fourier transform $\mathscr{F}_\sigma$ on $\mathscr{S}(W)$ is defined by the formula

$$\mathscr{F}_\sigma a(X) = \widehat{a}(X) \equiv \pi^{-n} \int_W a(Y) e^{2i\sigma(X,Y)}\, dY,$$

when $a \in \mathscr{S}(W)$. Then $\mathscr{F}_\sigma$ is a homeomorphism on $\mathscr{S}(W)$ which extends to a homeomorphism on $\mathscr{S}'(W)$, and to a unitary operator on $L^2(W)$. Moreover, $\mathscr{F}_\sigma^2$ is the identity operator. Note also that $\mathscr{F}_\sigma$ is defined without any reference to symplectic coordinates. By straight-forward computations it follows that

$$\mathscr{F}_\sigma(a * b)(X) = \pi^d \widehat{a}(X)\widehat{b}(X), \quad \mathscr{F}_\sigma(ab)(X) = \pi^{-d}(\widehat{a} * \widehat{b})(X),$$

when $a \in \mathscr{S}'(W)$, $b \in \mathscr{S}(W)$, and $*$ denotes the usual convolution. We refer to [7, 15] for more facts about the symplectic Fourier transform.

1.3. **Symbol classes and feasible metrics.** Next we recall the definition of the symbol classes. (See [9–11].) Let $N \geq 0$ be an integer, V be a finite-dimensional vector space, a belongs to $\mathscr{C}^N(V)$, the set of continuously differentiable functions of order N, g be an arbitrary Riemannian metric on V, and let $0 < m \in L^\infty_{loc}(V)$. For each $k = 0, \dots, N$, let

$$|a|_k^g(x) \equiv \sup |a^{(k)}(x; y_1, \dots, y_k)|, \tag{1.2}$$

where the supremum is taken over all $y_1, \dots, y_k \in V$ such that $g_x(y_j) \leq 1$ for every $j = 1, \dots, k$. Also set

$$\|a\|_{N,m}^g \equiv \sum_{k=0}^N \sup_{x \in V} \big(|a|_k^g(x)/m(x)\big). \tag{1.3}$$

We let $S_N(m,g)$ be the set of all $a \in \mathscr{C}^N(V)$ such that $\|a\|^g_{N,m}$ is finite. Also set

$$S(m,g) = S_\infty(m,g) \equiv \bigcap_{N\ge 0} S_N(m,g).$$

It follows that $S_N(m,g)$ is a Banach space and $S(m,g)$ is a Fréchet space.

In our applications, V here above agrees with the symplectic vector space W, and $S_N(m,g)$ when $0 \le N \le \infty$ are the symbol classes for the Weyl operators.

Next we recall some properties for the weight function m and the metric g on W. It follows from Section 18.6 in [11] that for each fixed $X \in W$, there are symplectic coordinates $Z = (z,\zeta)$ which diagonalize g_X, i. e. g_X takes the form

$$g_X(Z) = \sum_{j=1}^d \lambda_j(X)(z_j^2 + \zeta_j^2), \quad Z = (z,\zeta) \in W, \tag{1.4}$$

where

$$\lambda_1(X) \ge \lambda_2(X) \ge \cdots \ge \lambda_d(X) > 0 \tag{1.5}$$

only depend on g_X and are independent of the choice of symplectic coordinates which diagonalize g_X.

The *dual metric* g^σ and *Planck's function* h_g with respect to g and the symplectic form σ are defined by

$$g^\sigma_X(Z) \equiv \sup_{Y\neq 0} \Big(\frac{\sigma(Y,Z)^2}{g_X(Y)} \Big) \quad \text{and} \quad h_g(X) = \sup_{Z\neq 0} \Big(\frac{g_X(Z)}{g^\sigma_X(Z)} \Big)^{1/2}$$

respectively. It follows that if (1.4) and (1.5) are fulfilled, then $h_g(X) = \lambda_1(X)$ and

$$g^\sigma_X(Z) = \sum_{j=1}^d \lambda_j(X)^{-1}(z_j^2 + \zeta_j^2), \quad Z = (z,\zeta) \in W. \tag*{$(1.4)'$}$$

We usually assume that

$$h_g(X) \le 1 \quad \Longleftrightarrow \quad g_X \le g^\sigma_X, \qquad X \in W, \tag{1.6}$$

i. e. the *uncertainly principle* holds.

The metric g is called *symplectic* if $g_X = g^\sigma_X$ for every $X \in W$. It follows that g is symplectic if and only if $\lambda_1(X) = \cdots = \lambda_d(X) = 1$ in (1.4). If g_X is given by (1.4), then the corresponding *symplectic metric* is given by

$$g^0_X(Z) = \sum_{j=1}^d (z_j^2 + \zeta_j^2).$$

We observe that g^0 is defined in a symplectically invariant way (cf. [16]).

Let $X \in W$ be fixed, and let $g = g_X$ be as above. Then the operator Δ_g is defined by $\mathscr{F}_\sigma(\Delta_g f) = -4g^\sigma \cdot \widehat{f}$ when $f \in \mathscr{S}'(W)$. The operator Δ_g is related to the Laplace-Beltrami operator for g, and is obviously symplectically invariantly defined, since similar facts hold for $\mathscr{F}_\sigma$ and g^σ. If

$Z = (z, \zeta)$ are symplectic coordinates such that (1.4) holds, then it follows by straight-forward computation that

$$\Delta_{g_X} = \sum_{j=1}^{d} \lambda_j(X)^{-1}(\partial_{z_j}^2 + \partial_{\zeta_j}^2).$$

The Riemannian metric g on W is called *slowly varying* if there are positive constants c and C such that

$$\begin{gathered} g_X(Y-X) \le c \quad \Rightarrow \\ C^{-1} g_Y(Z) \le g_X(Z) \le C g_Y(Z) \quad \text{for every} \quad Z \in W. \end{gathered} \tag{1.7}$$

If g and G are Riemannian metrics, then G is called *g-continuous*, if there are positive constants c and C such that

$$\begin{gathered} g_X(Y-X) \le c \quad \Rightarrow \\ C^{-1} G_Y(Z) \le G_X(Z) \le C G_Y(Z) \quad \text{for every} \quad Z \in W. \end{gathered} \tag{1.8}$$

Lastly, a positive function m is called *g-continuous* if there are positive constants c and C such that

$$\begin{gathered} g_X(Y-X) \le c \quad \Rightarrow \\ C^{-1} m(Y) \le m(X) \le C m(Y). \end{gathered} \tag{1.9}$$

The metric g is called *σ-temperate*, if there are positive constants c, C, and N such that

$$g_Y(Z) \le g_X(Z)(1 + g_Y^\sigma(X-Y))^N, \qquad X, Y, Z \in W.$$

As in [6,16], g is called *feasible* if it is slowly varying and satisfies (1.6), and *strongly feasible* if it is feasible and σ-temperate.

The weight function m is called *(σ, g)-temperate*, if there are positive constants c, C, and N such that

$$m(Y) \le m(X)(1 + g_Y^\sigma(X-Y))^N, \qquad X, Y \in W.$$

1.4. **An extended family of pseudo-differential calculi.** Next we discuss some issues in pseudo-differential calculus. Let V be a real vector space of dimension d and $a \in \mathscr{S}(V \times V')$ be fixed. Suppose also that A belongs to $\mathcal{L}(V)$, the set of all linear mappings on V. Then the pseudo-differential operator $\operatorname{Op}_A(a)$ is the linear and continuous operator on $\mathscr{S}(V)$, given by

$$(\operatorname{Op}_A(a)f)(x) = (2\pi)^{-d} \iint_{V \times V'} a(x - A(x-y), \xi) f(y) e^{i\langle x-y, \xi\rangle}\, dy d\xi, \tag{1.10}$$

when $f \in \mathscr{S}(V)$. For general $a \in \mathscr{S}'(V \times V')$, the pseudo-differential operator $\operatorname{Op}_A(a)$ is defined as the linear and continuous operator from $\mathscr{S}(V)$ to $\mathscr{S}'(V)$ with distribution kernel given by

$$K_{a,A}(x,y) = (2\pi)^{-\frac{d}{2}} (\mathscr{F}_2^{-1} a)(x - A(x-y), x-y). \tag{1.11}$$

Here $\mathscr{F}_2 F$ is the partial Fourier transform of $F(x,y) \in \mathscr{S}'(V \times V)$ with respect to the y variable. This definition makes sense, since the mappings

$$\mathscr{F}_2 \quad \text{and} \quad F(x,y) \mapsto F(x - A(x-y), x-y) \tag{1.12}$$

are homeomorphisms on $\mathscr{S}'(V \times V')$ and on $\mathscr{S}'(V \times V)$, respectively. In particular, the map $a \mapsto K_{a,A}$ is a homeomorphism from $\mathscr{S}'(V \times V')$ to $\mathscr{S}'(V \times V)$.

An important special case appears when $A = t \cdot I$, with $t \in \mathbf{R}$. Here and in what follows, $I = I_V$ is the identity map on V. In this case we set

$$\operatorname{Op}_t(a) = \operatorname{Op}_{t \cdot I}(a).$$

The *normal* or *Kohn-Nirenberg representation*, $a(x, D)$, is obtained when $t = 0$, and the *Weyl quantization*, $\operatorname{Op}^w(a)$, is obtained when $t = \frac{1}{2}$. That is,

$$a(x, D) = \operatorname{Op}_0(a) \quad \text{and} \quad \operatorname{Op}^w(a) = \operatorname{Op}_{1/2}(a).$$

We recall that if $A \in \mathcal{L}(V)$, then it follows from the kernel theorem of Schwartz and Fourier's inversion formula that the map $a \mapsto \operatorname{Op}_A(a)$ is bijective from $\mathscr{S}'(V \times V')$ to the set of linear and continuous mappings from $\mathscr{S}(V)$ to $\mathscr{S}'(V')$ (cf. e.g. [9,18]). We refer to [11,18] for the proof of the following result, concerning transitions between different pseudo-differential calculi.

Proposition 1.1. *Let $a_1, a_2 \in \mathscr{S}'(V \times V')$ and $A_1, A_2 \in \mathcal{L}(V)$. Then*

$$\operatorname{Op}_{A_1}(a_1) = \operatorname{Op}_{A_2}(a_2) \qquad \Longleftrightarrow \qquad e^{i\langle A_2 D_\xi, D_x\rangle} a_2(x, \xi) = e^{i\langle A_1 D_\xi, D_x\rangle} a_1(x, \xi). \tag{1.13}$$

Note here that the latter equality in (1.13) makes sense since it is equivalent to

$$e^{i\langle A_2 x, \xi\rangle} \widehat{a}_2(\xi, x) = e^{i\langle A_1 x, \xi\rangle} \widehat{a}_1(\xi, x),$$

and that the map $a \mapsto e^{i\langle Ax, \xi\rangle} a$ is continuous on $\mathscr{S}'(V \times V')$ (cf. e.g. [18]).

For any $A \in \mathcal{L}(V)$, the A-product, $a \#_A b$ between $a \in \mathscr{S}'(V \times V')$ and $b \in \mathscr{S}'(V \times V')$ is defined by the formula

$$\operatorname{Op}_A(a \#_A b) = \operatorname{Op}_A(a) \circ \operatorname{Op}_A(b), \tag{1.14}$$

provided the right-hand side makes sense as a continuous operator from $\mathscr{S}(V)$ to $\mathscr{S}'(V)$. Since the Weyl case is especially important, we write $\#$ instead of $\#_A$ when $A = \frac{1}{2} I_V$.

We shall mainly consider pseudo-differential operators with symbols in $S(m, g)$. This family of operators possesses several convenient properties. For example, suppose that g is strongly feasible, m_k is g-continuous and (σ, g)-temperate, and that $a_k \in S(m_k, g)$, $k = 1, 2$. Then there is a unique $a \in S(m_1 m_2, g)$ such that

$$\operatorname{Op}^w(a_1) \circ \operatorname{Op}^w(a_2) = \operatorname{Op}^w(a).$$

That is,

$$S(m_1, g) \# S(m_2, g) \subseteq S(m_1 m_2, g). \tag{1.15}$$

1.5. **Schatten-von Neumann classes.** In order to discuss full range of Schatten-von Neumann classes, we recall the definition of quasi-Banach spaces.

Definition 1.2. A *quasi-norm* $\| \cdot \|_{\mathcal{B}}$ of order $p \in (0, 1]$, or a *p-norm*, to the vector space $\mathcal{B}$, is a functional on $\mathcal{B}$ such that the following is true:

(i) $\|f\|_{\mathcal{B}} \ge 0$, when $f \in \mathcal{B}$, with equality only for $f = 0$;

(ii) $\|\alpha f\|_{\mathcal{B}} = |\alpha| \, \|f\|_{\mathcal{B}}$, when $f \in \mathcal{B}$ and $\alpha \in \mathbf{C}$;

(iii) $\|f + g\|_{\mathcal{B}}^p \le \|f\|_{\mathcal{B}}^p + \|g\|_{\mathcal{B}}^p$, when $f, g \in \mathcal{B}$.

We equip $\mathcal{B}$ with the topology induced by $\|\cdot\|_{\mathcal{B}}$. The space $\mathcal{B}$ is called a *quasi-Banach space of order* p, or a p*-Banach space*, if $\mathcal{B}$ is complete under this topology.

Evidently, a topological vector space is a Banach space, if and only if it is a quasi-Banach space of order 1.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, and let T be a linear map from $\mathcal{H}_1$ to $\mathcal{H}_2$. For every integer $j \ge 1$, the *singular number* of T of order j is given by

$$\sigma_j(T) = \sigma_j(\mathcal{H}_1, \mathcal{H}_2, T) \equiv \inf \|T - T_0\|_{\mathcal{H}_1 \to \mathcal{H}_2},$$

where the infimum is taken over all linear operators T_0 from $\mathcal{H}_1$ to $\mathcal{H}_2$ with rank at most $j-1$. Therefore, $\sigma_1(T)$ equals $\|T\|_{\mathcal{H}_1 \to \mathcal{H}_2}$, while $\sigma_j(T)$ is non-negative and decreases with j.

For any $p \in (0,\infty]$ we set

$$\|T\|_{\mathscr{I}_p} = \|T\|_{\mathscr{I}_p(\mathcal{H}_1,\mathcal{H}_2)} \equiv \|\{\sigma_j(\mathcal{H}_1,\mathcal{H}_2,T)\}_{j=1}^\infty\|_{\ell^p}$$

(which might attain $+\infty$). The operator T is called a *Schatten-von Neumann operator* of order p from $\mathcal{H}_1$ to $\mathcal{H}_2$, if $\|T\|_{\mathscr{I}_p}$ is finite, i. e. $\{\sigma_j(\mathcal{H}_1,\mathcal{H}_2,T)\}_{j=1}^\infty$ should belong to ℓ^p. The set of all Schatten-von Neumann operators of order p from $\mathcal{H}_1$ to $\mathcal{H}_2$ is denoted by $\mathscr{I}_p = \mathscr{I}_p(\mathcal{H}_1,\mathcal{H}_2)$. We note that $\mathscr{I}_\infty(\mathcal{H}_1,\mathcal{H}_2)$ agrees with $\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$ (also in norms), the set of linear and bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. If $p<\infty$, then $\mathscr{I}_p(\mathcal{H}_1,\mathcal{H}_2)$ is contained in $\mathcal{K}(\mathcal{H}_1,\mathcal{H}_2)$, the set of linear and compact operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. The spaces $\mathscr{I}_p(\mathcal{H}_1,\mathcal{H}_2)$ for $p \in (0,\infty]$ and $\mathcal{K}(\mathcal{H}_1,\mathcal{H}_2)$ are quasi-Banach spaces which are Banach spaces when $p \ge 1$. Furthermore, $\mathscr{I}_2(\mathcal{H}_1,\mathcal{H}_2)$ is a Hilbert space and agrees with the set of Hilbert-Schmidt operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ (also in norms). We set $\mathscr{I}_p(\mathcal{H}) = \mathscr{I}_p(\mathcal{H},\mathcal{H})$.

The set $\mathscr{I}_1(\mathcal{H}_1,\mathcal{H}_2)$ is the set of trace-class operators from $\mathcal{H}_1$ to $\mathcal{H}_2$, and $\|\cdot\|_{\mathscr{I}_1(\mathcal{H}_1,\mathcal{H}_2)}$ coincides with the trace-norm. If in addition $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, then the trace

$$\operatorname{Tr}_{\mathcal{H}}(T) \equiv \sum_\alpha (Tf_\alpha, f_\alpha)_{\mathcal{H}}$$

is well-defined and independent of the orthonormal basis $\{f_\alpha\}_\alpha$ in $\mathcal{H}$.

Now let $\mathcal{H}_3$ be another Hilbert space and let T_k be a linear and continuous operator from $\mathcal{H}_k$ to $\mathcal{H}_{k+1}$, $k=1,2$. Then we recall the Hölder relation

$$\|T_2 \circ T_1\|_{\mathscr{I}_r(\mathcal{H}_1,\mathcal{H}_3)} \le \|T_1\|_{\mathscr{I}_{p_1}(\mathcal{H}_1,\mathcal{H}_2)} \|T_2\|_{\mathscr{I}_{p_2}(\mathcal{H}_2,\mathcal{H}_3)} \qquad \text{when} \quad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r} \tag{1.16}$$

(cf. e. g. [14,17]).

In particular, the map $(T_1,T_2) \mapsto T_2^* \circ T_1$ is continuous from $\mathscr{I}_p(\mathcal{H}_1,\mathcal{H}_2) \times \mathscr{I}_{p'}(\mathcal{H}_1,\mathcal{H}_2)$ to $\mathscr{I}_1(\mathcal{H}_1)$, giving that

$$(T_1,T_2)_{\mathscr{I}_2(\mathcal{H}_1,\mathcal{H}_2)} \equiv \operatorname{Tr}_{\mathcal{H}_1}(T_2^* \circ T_1) \tag{1.17}$$

is well-defined and continuous from $\mathscr{I}_p(\mathcal{H}_1,\mathcal{H}_2) \times \mathscr{I}_{p'}(\mathcal{H}_1,\mathcal{H}_2)$ to $\mathbf{C}$. If $p=2$, then the product, defined by (1.17) agrees with the scalar product in $\mathscr{I}_2(\mathcal{H}_1,\mathcal{H}_2)$.

The proof of the following result is omitted, since it can be found in e. g. [2,14].

Proposition 1.3. *Let $p \in [1,\infty]$, $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, and let T be a linear and continuous map from $\mathcal{H}_1$ to $\mathcal{H}_2$. Then the following is true:*

(i) *if $q \in [1,p']$, then*

$$\|T\|_{\mathscr{I}_p(\mathcal{H}_1,\mathcal{H}_2)} = \sup |(T,T_0)_{\mathscr{I}_2(\mathcal{H}_1,\mathcal{H}_2)}|,$$

where the supremum is taken over all $T_0 \in \mathscr{I}_q(\mathcal{H}_1,\mathcal{H}_2)$ such that $\|T_0\|_{\mathscr{I}_{p'}(\mathcal{H}_1,\mathcal{H}_2)} \le 1$;

(ii) *if in addition $p<\infty$, then the dual of $\mathscr{I}_p(\mathcal{H}_1,\mathcal{H}_2)$ can be identified through the form* (1.17).

Later on we are especially interested in finding necessary and sufficient conditions on symbols, in order for the corresponding pseudo-differential operators to belong to $\mathscr{I}_p(\mathcal{H}_1,\mathcal{H}_2)$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ satisfy

$$\mathscr{S}(V) \hookrightarrow \mathcal{H}_1, \mathcal{H}_2 \hookrightarrow \mathscr{S}'(V).$$

Therefore, for such Hilbert spaces and $p \in (0,\infty]$, let

$$s_{A,p}(\mathcal{H}_1,\mathcal{H}_2) \equiv \{\, a \in \mathscr{S}'(V \times V')\,;\, \operatorname{Op}_A(a) \in \mathscr{I}_p(\mathcal{H}_1,\mathcal{H}_2)\,\}$$

and

$$\|a\|_{s_{A,p}(\mathcal{H}_1,\mathcal{H}_2)} \equiv \|\operatorname{Op}_A(a)\|_{\mathscr{I}_p(\mathcal{H}_1,\mathcal{H}_2)}. \tag{1.18}$$

Since the map $a \mapsto \operatorname{Op}_A(a)$ is bijective from $\mathscr{S}'(V\times V')$ to the set of all linear and continuous operators from $\mathscr{S}(V)$ to $\mathscr{S}'(V)$, it follows from the definitions that the map $a \mapsto \operatorname{Op}_A(a)$ restricts to a bijective and isometric map from $s_{A,p}(\mathcal{H}_1,\mathcal{H}_2)$ to $\mathscr{I}_p(\mathcal{H}_1,\mathcal{H}_2)$. We put

$$s_{A,p}(W) = s_{A,p}(\mathcal{H}_1,\mathcal{H}_2) \quad \text{when} \quad \mathcal{H}_1 = \mathcal{H}_2 = L^2(V).$$

For convenience we also put $s_p^w = s_{A,p}$ in the Weyl case (i.e. when $A = \frac{1}{2}\cdot I_V$).

1.6. **Wiener Lebesgue spaces with respect to slowly varying metrics.** Before defining the Wiener-Lebesgue spaces, we recall some facts about g-balls, which are given by

$$U_{X,R} = U_{g,X,R} \equiv \{\, Y \in W\,;\, g_X(Y-X) < R^2\,\}, \tag{1.19}$$

when $X \in W$ and $R>0$. The following lemma is a consequence of Lemma 1.4.9 and the proof of Theorem 1.4.10 in [11]. The proof is therefore omitted.

Lemma 1.4. *Let g be slowly varying on W and let c and C be as in* (1.7). *Then there exists a sequence $\{X_j\}_{j=1}^\infty$ such that if*

$$U_j = U_{X_j,R},$$

for some $R>0$ such that $\frac{c}{2} < R^2 < c$, then the following is true:

(i) *$g_{X_j}(X_j - X_k) \ge \frac{c}{2C}$ for every $j,k = 1,2,\dots$ such that $j \ne k$;*

(ii) *$W = \bigcup_{j=1}^\infty U_j$;*

(iii) *if $j \in \mathbf{Z}_+$ is fixed, then $U_j \cap U_k \ne \emptyset$ for at most $(4C^3+1)^{2d}$ numbers of k.*

Definition 1.5. Let g be slowly varying on W, c and C be as in (1.7). Then the family of g-balls $\{U_j\}_{j=1}^{\infty}$ in Lemma 1.4 is called an *admissible g-covering* of W.

Remark 1.6. Let $\{X_j\}_{j=1}^{\infty}$ be as in Lemma 1.4. For future reference, we observe that if $Y \in W$, $r, R_1, R_2 > 0$ satisfy

$$\frac{c}{2} < R_1^2 < R_2^2 < c, \quad r < \frac{R_2 - R_1}{2C},$$

and $U_{Y,r} \cap U_{X_j,R_1} \neq \emptyset$ for some $j \in \mathbf{Z}_+$, then $U_{Y,r} \subseteq U_{X_j,R_2}$.

As a consequence of Lemma 1.4 there are at most $(4C^3+1)^{2d}$ numbers of U_{X_j,R_1} or U_{X_j,R_2} which intersect with $U_{Y,r}$.

In fact, suppose $Z \in U_{Y,r} \cap U_{X_j,R_1}$. Then, for every $X \in U_{Y,r}$ we have that

$$\begin{aligned}\big(g_{X_j}(X - X_j)\big)^{\frac{1}{2}} &= \big(g_{X_j}(X - Z + Z - X_j)\big)^{\frac{1}{2}} \\ &\le \big(g_{X_j}(Z - X_j)\big)^{\frac{1}{2}} + \big(g_{X_j}(X - Z\big)^{\frac{1}{2}}.\end{aligned}$$

By the fact that g is slowly varying, we obtain that $g_{X_j} \le C g_Z \le C^2 g_Y$. Hence, we have

$$\begin{aligned}\big(g_{X_j}(Z - X_j)\big)^{\frac{1}{2}} + \big(g_{X_j}(X - Z)\big)^{\frac{1}{2}} &\le R_1 + C\big(g_Y(Z - X)\big)^{\frac{1}{2}} \\ &\le R_1 + 2Cr < R_2,\end{aligned}$$

which shows that $X \in U_{X_j,R_2}$, and the assertion follows.

Definition 1.7. Let $p, q \in (0,\infty]$, $\theta \in \mathbf{R}$, g be a slowly varying metric on W, $\{U_j\}_{j=1}^{\infty}$ be an admissible g-covering, and let $U \subseteq \mathbf{R}^d$ be an open ball such that $\{j + U\}_{j\in\mathbf{Z}^d}$ covers $\mathbf{R}^d$.

(i) The *Wiener-Lebesgue space* $\mathit{WL}^{q,p}(\mathbf{R}^d)$ (with respect to p and q) consists of all measurable functions f such that $\|f\|_{\mathit{WL}^{q,p}}$ is finite, where

$$\|f\|_{\mathit{WL}^{q,p}} \equiv \big\|\{\|f\|_{L^q(j+U)}\}_{j\in\mathbf{Z}^d}\big\|_{\ell^p(\mathbf{Z}^d)}.$$

(ii) The *Wiener-Lebesgue space* $\mathit{WL}^{q,p}_{g,\theta}(W)$ (with respect to p, q, θ and g) consists of all measurable functions a such that $\|a\|_{\mathit{WL}^{q,p}_{g,\theta}}$ is finite, where

$$\|a\|_{\mathit{WL}^{q,p}_{g,\theta}} \equiv \big\|\{\|a\|_{L^q(U_j)} \cdot |U_j|^{\theta}\}_{j\in\mathbf{Z}_+}\big\|_{\ell^p(I)}.$$

We remark that $\mathit{WL}^{q,p}_{g,\theta}(W)$ is a quasi-Banach space of order $\min(1,p,q)$, and independent of the choice of admissible g-covering $\{U_j\}_{j\in\mathbf{Z}_+}$ in Definition 1.7 (cf. Proposition 2.1 below). In particular, it follows that $\mathit{WL}^{q,p}(\mathbf{R}^d)$ is independent of the choice of U in Definition 1.7. (This follows from [8] as well.) If $p, q \ge 1$, then $\mathit{WL}^{q,p}_{g,\theta}(W)$ is a Banach space.

For $p \in (0,1]$ and $q \in (0,\infty]$, the choice of parameter $\theta = \frac{1}{p} - \frac{1}{q}$ in the $\mathit{WL}^{q,p}_{g,\theta}$ spaces is of special interest. For this reason we let

$$\mathit{WL}^{q,p}_{g} = \mathit{WL}^{q,p}_{g,\theta} \quad \text{when} \quad \theta = \frac{1}{p} - \frac{1}{q}.$$

2. Structural properties for Wiener-Lebesgue spaces

In this section we show some basic properties for $WL^{q,p}_{g,\theta}$-spaces. First we show that such spaces are invariantly defined with respect to the choice of admissible g-covering. Then we show that such spaces increase if we replace the metrics with corresponding symplectic metrics.

Proposition 2.1. *Let $p, q \in (0, \infty]$, $\theta \in \mathbf{R}$ and g be slowly varying on W. Then $WL^{q,p}_{g,\theta}(W)$ is independent of the choice of admissible g-covering $\{U_j\}_{j\in\mathbf{Z}_+}$ in Definition 1.7.*

Remark 2.2. Since $WL^{q,p}_{g,\theta}(W)$ is defined through quasi-norm estimates, it follows from Proposition 2.1 that different admissible coverings give rise to equivalent quasi-norms for $WL^{q,p}_{g,\theta}(W)$.

Proof of Proposition 2.1. We only prove the result when $p \le q < \infty$. The other cases follow by similar arguments and are left to the reader. By considering $b(X) = |a(X)|^q$, we reduce ourselves to the case when $q = 1$ and $p \le 1$. We may also replace θ by θ/p.

Let $\mathcal{U} = \{U_j\}_{j\in\mathbf{Z}_+}$ and $\mathcal{V} = \{V_k\}_{k\in\mathbf{Z}_+}$ be admissible g-coverings, let

$$\|a\|^p_{\mathcal{U}} = \sum_{j=0}^{\infty} \left(\int_{U_j} |a(X)|\, dX \right)^p |U_j|^\theta,$$

and let

$$\|a\|^p_{\mathcal{V}} = \sum_{k=0}^{\infty} \left(\int_{V_k} |a(X)|\, dX \right)^p |V_k|^\theta.$$

By [11, Lemma 18.4.4], there is a bounded sequence $\{\varphi_k\}_{k=0}^\infty$ in $S(1, g)$ such that $\varphi_k \ge 0$, $\operatorname{supp} \varphi_k \subseteq V_k$ for every k, and $\sum_{k=0}^\infty \varphi_k = 1$.

We have

$$\begin{aligned}
\|a\|^p_{\mathcal{U}} &= \sum_{j=0}^{\infty} \left(\int_{U_j} |a(X)|\, dX \right)^p |U_j|^\theta \\
&\asymp \sum_{j=0}^{\infty} \left(\int_{U_j} \sum_{k=0}^{\infty} |\varphi_k(x) a(X)|\, dX \right)^p |U_j|^\theta \\
&\le \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left(\int_{U_j} |\varphi_k(x) a(X)|\, dX \right)^p |U_j|^\theta \\
&\asymp \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left(\int_{U_j} |\varphi_k(x) a(X)|\, dX \right)^p |V_k|^\theta,
\end{aligned}$$

where the last relation follows from the fact that $|U_j| \asymp |V_k|$ when $U_j \cap V_k \neq \emptyset$ in combination with the fact that g is slowly varying. Since there is an upper bound of intersections between U_j and V_k in view of Remark 1.6, we obtain

$$
\begin{aligned}
\|a\|_{\mathcal{U}}^p &\lesssim \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} \left(\int_{U_j} |\varphi_k(x) a(X)| \, dX \right)^p |V_k|^\theta \right) \\
&\asymp \sum_{k=0}^{\infty} \left(\int_{W} |\varphi_k(x) a(X)| \, dX \right)^p |V_k|^\theta \\
&\leq \sum_{k=0}^{\infty} \left(\int_{V_k} |a(X)| \, dX \right)^p |V_k|^\theta \\
&= \|a\|_{\mathcal{V}}^p. \qquad \square
\end{aligned}
$$

Next we show that $WL_g^{q,p}(W)$ is contained in $WL_{g^0}^{q,p}(W)$, when g is feasible. For that reason we need the following proposition.

Proposition 2.3. *Let g be a slowly varying metric on W, G be a g-continuous metric such that $g \leq G$, and let $\{U_{X_j,R}\}_{j=1}^{\infty}$ be an admissible g-covering of W. Then there exists an admissible G-covering $\{U_{G,k}\}_{k=1}^{\infty}$ of W given by*

$$
U_{G,k} = U_{G,Y_k,r} = \{\, X \in W \,;\, G_{Y_k}(X - Y_k) < r^2 \,\}, \quad k \in \mathbf{Z}_+
$$

such that

$$
C_1 \frac{|U_{X_j,R}|}{|U_{G,X_j,r}|} \leq N_j \leq C_2 \frac{|U_{X_j,R}|}{|U_{G,X_j,r}|} \tag{2.1}
$$

when N_j is the number of $U_{G,k}$ intersecting $U_{X_j,R}$, and the constants $C_1, C_2 > 0$ are independent of $j \in \mathbf{Z}_+$.

Proof. Let U_{X_j,R_1} and U_{X_j,R_2} be as in Remark 1.6. If $r > 0$ is chosen small enough, then there is an admissible G-covering of W, given by

$$
U_{G,k} = \{\, X \in W \,;\, G_{Y_k}(X - Y_k) < r^2 \,\}, \quad k \in \mathbf{Z}_+
$$

such that $U_{G,k} \subseteq U_{X_j,R_2}$ when $U_{G,k}$ intersects U_{X_j,R_1}. The facts that G is g-continuous and $g \leq G$ guarantees that such r exists. Also, let Ω_j be the set of all $k \in \mathbf{Z}_+$ such that $U_{G,k}$ intersects U_j and let $N_j = |\Omega_j|$.

We have

$$
U_{X_j,R_1} \subseteq \bigcup_{k \in \Omega_j} U_{G,k} \subseteq U_{X_j,R_2}.
$$

Since the balls $\{U_{G,k}\}_{k \in I}$ form an admissible covering of W, there is an upper bound M of overlapping $U_{G,k}$. This gives

$$
\frac{1}{M} \sum_{k \in \Omega_j} |U_{G,k}| \leq |U_{X_j,R_2}| \quad \iff \quad \sum_{k \in \Omega_j} |U_{G,k}| \leq M |U_{X_j,R_2}|.
$$

Since G is g-continuous, we have

$$
C_3 |U_{G,X_j,r}| \leq |U_{G,k}| \leq C_4 |U_{G,X_j,r}|
$$

for some constants $C_3, C_4 > 0$ which are independent of k. A combination of these estimates gives

$$C_3 N_j\, |U_{G,X_j,r}| = C_3|\Omega_j|\, |U_{G,X_j,r}| \le \sum_{k\in\Omega_j} |U_{G,k}| \le M|U_{X_j,R_2}|,$$

which leads to the second inequality in (2.1).

We also have

$$|U_{X_j,R_1}| \le \left| \bigcup_{k\in\Omega_j} U_{G,k} \right| \le \sum_{k\in\Omega_j} |U_{G,k}| \le C_4 N_j |U_{G,X_j,r}|,$$

giving the first inequality in (2.1), giving the result. □

Since all g-balls are of the same size when g is symplectic, the previous proposition takes the following form.

Corollary 2.4. *Let g be a feasible metric on W, and let $\{U_{X_j,R}\}_{j=1}^{\infty}$ be an admissible g-covering of W. Then there exists an admissible g^0-covering $\{U_k^0\}_{k=1}^{\infty}$ of W given by*

$$U_k^0 = \{\, X \in W \,;\, g_{Y_k}(X - Y_k) < r^2 \,\}, \quad k \in \mathbf{Z}_+$$

such that $N_j \le C|U_{X_j,R}|$, where N_j is the number of U_k^0 intersecting $U_{X_j,R}$, and the constant $C > 0$ is independent of $j \in \mathbf{Z}_+$.

Proposition 2.5. *Let g be a slowly varying metric on W and let G be a g-continuous metric such that $g \le G$. Also, suppose that and $0 < p \le q < \infty$. Then*

$$WL_g^{q,p}(W) \subseteq WL_G^{q,p}(W).$$

Proof. Let $p_0 = \frac{p}{q} \in (0, 1]$ and $b(X) = |a(X)|^q$. The inequalities in (2.1) shall be combined with

$$\sum_{k=1}^{N} x_k^{p_0} \le N^{1-p_0} \left(\sum_{k=1}^{N} x_k \right)^{p_0}, \qquad x_1, \dots, x_N \ge 0, \tag{2.2}$$

which follows by concavity of $t \mapsto t^{p_0}$.

We use the same notations as in the proof of Proposition 2.3. Since $\|a\|^p_{WL_G^{q,p}} = \|b\|^{p_0}_{WL_G^{1,p_0}}$, we obtain

$$\begin{aligned}
\|a\|^p_{WL_G^{q,p}} &\asymp \sum_{k=1}^{\infty} \left(\int_{U_{G,k}} |b(X)|\, dX \right)^{p_0} |U_{G,k}|^{1-p_0} \\
&\le \sum_{j=1}^{\infty} \left(\sum_{k\in\Omega_j} \left(\int_{U_{G,k}} |b(X)|\, dX \right)^{p_0} |U_{G,k}|^{1-p_0} \right) \\
&\le \sum_{j=1}^{\infty} \left(|\Omega_j|^{1-p_0} \left(\sum_{k\in\Omega_j} \int_{U_{G,k}} |b(X)|\, dX \right)^{p_0} |U_{G,k}|^{1-p_0} \right),
\end{aligned}$$

where the last inequality follows from (2.2). Since there is a bound M of overlapping $U_{G,k}$,

$$|U_{X_j,R_1}| \asymp |U_{X_j,R_2}|, \quad \text{and} \quad |U_{G,k}| \asymp |U_{G,X_j,r}|,$$

when $U_{G,k}$ intersects with U_{X_j,R_1}, Proposition 2.3 gives

$$
\begin{aligned}
\|a\|^p_{WL^{q,p}_G} &\lesssim \sum_{j=1}^{\infty} \left(\left(\frac{|U_{X_j,R_2}|}{|U_{G,X_j,r}|} \right)^{1-p_0} \left(\sum_{k\in\Omega_j} \int_{U_{G,k}} |b(X)|\, dX \right)^{p_0} |U_{G,X_j,r}|^{1-p_0} \right) \\
&\le \sum_{j=1}^{\infty} \left(\left(M \int_{U_{X_j,R_2}} |b(X)|\, dX \right)^{p_0} |U_{X_j,R_2}|^{1-p_0} \right) \\
&\asymp \|b\|^{p_0}_{WL^{1,p_0}_g} = \|a\|^p_{WL^{q,p}_g} ,
\end{aligned}
$$

and the result follows from these estimates. $\square$

Since g^0 is g-continuous and $g \le g^0$ whenever g is feasible, the following corollary is an immediate consequence of Proposition 2.5.

Corollary 2.6. *Let g be feasible on W and $0 < p \le q < \infty$. Then*

$$WL^{q,p}_g(W) \subseteq WL^{q,p}_{g^0}(W).$$

3. Quasi-Banach Schatten-von Neumann properties in pseudo-differential calculus

In this section we deduce Schatten-von Neumann properties, with respect to $p \in (0,1]$, for pseudo-differential operators with symbols in $S(m,g)$ and with m or a belonging to $WL^{1,p}_g(W)$. In Section 3.1 we deal with Weyl operators, where in the first part the assumptions on m and g are minimal, and the operators are acting on $L^2(V)$. The second part of Section 3.1 is devoted to operators acting between (different) Bony-Chemin Sobolev-type spaces $H(m,g)$. Here, we restrict ourselves and assume that m and g satisfy the usual conditions in the Weyl-Hörmander calculus. In Section 3.2, we consider more general pseudo-differential calculi, but with some additional restrictions on g.

3.1. The case of Hörmander-Weyl calculus.

Theorem 3.1. *Let $p \in (0,1]$, g be feasible on W, and $m \in WL^{1,p}_g(W)$ be a positive function on W. Then $S(m,g) \subseteq s^w_p(W)$.*

For the proof we need the following lemma on embeddings between $s^w_p(W)$ and Sobolev-type spaces of distributions with suitable numbers of derivatives belonging to $WL^{1,p}(W)$.

Lemma 3.2. *Let $p \in (0,1]$. Then there is an integer $N \ge 1$ and a constant $C > 0$ which only depends on p and the dimension of W such that*

$$\|a\|_{s^w_p(W)} \le C\|(1-\Delta)^N a\|_{WL^{1,p}(W)}.$$

Proof. The symbol $b(X) = (1+|X|^2)^{-N}$ belongs to $s^w_p(W)$, provided that $N \ge 1$ is chosen large enough (see e.g. [15, Theorem 2.6]). It follows that $\varphi = \mathscr{F}_\sigma b \in s^w_p(W)$, since $s^w_p(W)$ is invariant under the symplectic Fourier transform. This gives

$$\|a\|_{s_p^w} = \|(1-\Delta)^{-N}((1-\Delta)^N a)\|_{s_p^w}$$
$$\asymp \|\varphi * ((1-\Delta)^N a)\|_{s_p^w} \lesssim \|\varphi\|_{s_p^w}\|(1-\Delta)^N a\|_{WL^{1,p}}.$$

Here the inequality follows from [3, Proposition 5.11]. This gives the result. $\square$

Proof of Theorem 3.1. By $g \le g^0$, Corollary 2.6, and the fact that $S(m,g)$ increases with g, it suffices to prove the result with g^0 in place of g. Hence we may assume that g is symplectic.

Let U_j and φ_k be the same as in the proof of Proposition 2.1, with $V_k = U_k$. Also, let $g_j = g_{X_j}$ and $U_{j,k} = U_j \cap U_k$. By Lemma 3.2 and the fact that $s_p^w(W)$ are invariant under symplectic transformations we obtain

$$\|\varphi_j a\|_{s_p^w} \le C\|(1-\Delta_{g_j})^N(\varphi_j a)\|_{WL^{1,p}_{g_j}}$$

Hence (3.3), $\operatorname{supp}\varphi_j \subseteq U_j$, and the fact that $p \le 1$ give

$$\begin{aligned}
\|a\|_{s_p^w}^p &= \Big\|\sum_{j=1}^\infty (\varphi_j a)\Big\|_{s_p^w}^p \le \sum_{j=1}^\infty \|\varphi_j a\|_{s_p^w}^p \\
&\lesssim \sum_{j=1}^\infty \|(1-\Delta_{g_j})^N(\varphi_j a)\|_{WL^{1,p}_{g_j}}^p \\
&\asymp \sum_{j=1}^\infty\sum_{k=1}^\infty \left(\int_{U_{j,k}} |(1-\Delta_{g_j})^N(\varphi_j(X)a(X))|\,dX\right)^p \\
&\lesssim \sum_{j=1}^\infty\sum_{k=1}^\infty \sum_{|\alpha|\le 2N}\left(\int_{U_{j,k}} |(\partial_{g_j}^\alpha a)(X)|\,dX\right)^p \\
&\lesssim \|a\|^p \sum_{j=1}^\infty\sum_{k=1}^\infty \left(\int_{U_{j,k}} |m(X)|\,dX\right)^p
\end{aligned}$$

Here $\|a\|$ denotes a semi-norm of a in $S(m,g)$. Since there is a bound of overlapping U_j, it follows from these estimates that

$$\|a\|_{s_p^w}^p \lesssim \|a\|^p \sum_{j=1}^\infty \left(\int_{U_j} |m(X)|\,dX\right)^p \asymp \|a\|^p\|m\|_{WL^{1,p}_g}^p,$$

which gives the result. $\square$

The next result improves Theorem 3.1. It also extends [10, Theorem 3.9].

Theorem 3.3. *Let $p \in (0,1]$, g be feasible on W, m be a positive function on W such that $h_g^{k/2}m \in WL^{1,p}_g(W)$ for some $k \ge 0$, and suppose $a \in S(m,g) \cap WL^{1,p}_g(W)$. Then $a \in s_p^w(W)$.*

For the proof we need the following lemmas.

Lemma 3.4. *Let $p \in (0,\infty]$, $q \in [1,\infty]$, $N \in \mathbf{N}$ and $f \in WL^{q,p}(\mathbf{R}^d) \cap \mathscr{C}^N(\mathbf{R}^d)$. Then there exists a constant $C > 0$ such that*

$$\|\partial^\alpha f\|_{WL^{q,p}}^p \le C \left(\|f\|_{WL^{q,p}}^p + \sum_{|\beta|=N} \|\partial^\beta f\|_{WL^{q,p}}^p \right). \tag{3.1}$$

Lemma 3.5. *Let g be a feasible metric on W, $\alpha \in [0,1]$ and set $G = h_g^{-\alpha} g$. Also, assume that $N \ge 0$ is an integer which is fixed, $m > 0$ is a weight function on W, $a \in \mathscr{C}^N(W)$, and set*

$$m_0 = \sum_{n=0}^{N-1} |a|_n^G + h_g^{\alpha N/2} m.$$

Then the following are true:

(i) *if $p \in (0,1]$, then*

$$\|m_0\|_{WL_g^{1,p}} \le C(\|a\|_{WL_g^{1,p}} + \|h_g^{\alpha N/2} m\|_{WL_g^{1,p}}); \tag{3.2}$$

(ii) *if $a \in WL_g^{1,p}(W)$ and $h_g^{\alpha N/2} m \in WL_g^{1,p}(W)$, then $m_0 \in WL_g^{1,p}(W)$.*

Proof of Lemma 3.4. Let U be as in Definition 1.7. Then there exists a constant $C > 0$ such that, for any $|\alpha| \le N$ and $j \in \mathbf{Z}^d$,

$$\|\partial^\alpha f\|_{L^q(j+U)} \le C \left(\|f\|_{L^q(j+U)} + \sum_{|\beta|=N} \|\partial^\beta f\|_{L^q(j+U)} \right).$$

(See e. g. [1].) Hence for a (possibly new) constant $C > 0$, we obtain

$$\|\partial^\alpha f\|_{L^q(j+U)}^p \le C \left(\|f\|_{L^q(j+U)}^p + \sum_{|\beta|=N} \|\partial^\beta f\|_{L^q(j+U)}^p \right).$$

Summing up with respect to $j \in \mathbf{Z}^d$ we have

$$\begin{aligned} \|\partial^\alpha f\|_{WL^{q,p}}^p &= \sum_{j\in\mathbf{Z}^d} \|\partial^\alpha f\|_{L^q(j+U)}^p \\ &\le C \left(\sum_{j\in\mathbf{Z}^d} \|f\|_{L^q(j+U)}^p + \sum_{j\in\mathbf{Z}^d} \sum_{|\beta|=N} \|\partial^\beta f\|_{L^q(j+U)}^p \right) \\ &= C \left(\|f\|_{WL^{q,p}}^p + \sum_{|\beta|=N} \|\partial^\beta f\|_{WL^{q,p}}^p \right). \qquad \square \end{aligned}$$

Proof of Lemma 3.5. It suffices to prove (i). By [16, Lemma 6.1], it follows that $|a|_k^G \le C m_0$ for some constant $C > 0$. Let $V_j = U_j$, and let φ_j and U_j for $j \in \mathbf{Z}_+$ be as in the proof of Proposition 2.1. Also, let $\{\psi_j\}_{j=1}^\infty$ be a bounded sequence in $S(1,g)$ such that $\psi_j \in C_0^\infty(U_j)$ and $\psi_j = 1$ in the support of φ_j. Lastly, let $g_j = g_{X_j}$ and $G_j = G_{X_j}$. Then

$$|\varphi_j a|_N^{G_j} = h_{g_j}^{\alpha N/2} |\varphi_j a|_N^{g_j} \le C h_{g_j}^{\alpha N/2} \psi_j m,$$

where the constant C is independent of $j \in \mathbf{Z}_+$. For every $j \in \mathbf{Z}_+$, let G_j define the Euclidean structure on W. By Lemma 3.4, and the fact that C in (3.1) is invariant under changes of symplectic structures on W, it follows that

$$\| \, |\varphi_j a|_n^{G_j} \|_{L^1} \le C \big(\|\varphi_j a\|_{L^1} + \|h_{g_j}^{\alpha N/2} \psi_j m\|_{L^1} \big),$$

where the constant C neither depends on $j \in \mathbf{Z}_+$ nor on $n \in \{0, \dots, N\}$.

We have

$$\| \, |a|_n^G \|_{WL_g^{1,p}}^p = \| \, | \sum_{l=1}^\infty \varphi_l a|_n^G \|_{WL_g^{1,p}}^p = \sum_{j=1}^\infty \left(\int_{U_j} | \sum_{l=1}^\infty \varphi_l a|_n^G (X) \, dX \right)^p |U_j|^{1-p}.$$

Since there is a bound of overlapping sets U_j when $j \in \mathbf{Z}_+$, we get

$$\left(\int_{U_j} | \sum_{l=1}^\infty \varphi_l a|_n^G (X) \, dX \right)^p \le C_1 \left(\sum_{k=0}^n \int_{U_j} |a|_k^G (X) \, dX \right)^p ,$$

where the constant C_1 is independent of j. By Lemma 3.4 we obtain

$$\begin{aligned}
\| \, |a|_n^G \|_{WL_g^{1,p}}^p &\le C_1 \sum_{j=1}^\infty \left(\sum_{k=0}^n \int_{U_j} |a|_k^G (X) \, dX \right)^p |U_j|^{1-p} \\
&\le C_2 \sum_{j=1}^\infty \left(\int_{U_j} \big(|a(X)| + |a|_N^G (X) \big) \, dX \right)^p |U_j|^{1-p} \\
&\le C_3 \Bigg(\sum_{j=1}^\infty \left(\int_{U_j} |a(X)| \, dX \right)^p |U_j|^{1-p} \\
&\qquad\qquad + \sum_{j=1}^\infty \left(\int_{U_j} h_{g_j}^{\alpha N/2} (X) m(X) \, dX \right)^p |U_j|^{1-p} \Bigg) \\
&\asymp \|a\|_{WL_g^{1,p}}^p + \|h_g m\|_{WL_g^{1,p}}^p ,
\end{aligned}$$

for some constants C_2 and C_3. This gives (3.2), and the proof is complete. $\square$

Remark 3.6. By the proof of Lemma 3.4, it follows that the constant C in (3.1) only depends on the dimension of W and on N.

In particular, by changing the coordinates in suitable ways, and using that there is a bound of overlapping U_j, it follows that

$$\big\| |a|_k^g \big\|_{WL_{g,\theta}^{q,p}}^p \le C \left(\|a\|_{WL_{g,\theta}^{q,p}}^p + \big\| |a|_N^g \big\|_{WL_{g,\theta}^{q,p}}^p \right), \quad k = 0, 1, \dots, N. \tag{3.3}$$

Proof of Theorem 3.3. Let G and m_0 be as in Lemma 3.5. We observe that if $a \in S(m, g)$, then $a \in S(m_0, G)$, in view of [16, Lemma 6.1]. The result now follows from Theorem 3.1. $\square$

If the involved weight functions are g-continuous, we can replace the conditions on them as in the next two theorems, where the first one agrees with [19, Theorem 4.1] when $p \le 1$.

Theorem 3.7. *Let $p \in (0, 1]$, g be feasible on W, and $m \in L^p(W)$ be a positive g-continuous function on W. Then $S(m, g) \subseteq s_p^w(W)$.*

Theorem 3.8. *Let $p \in (0,1]$, g be feasible on W, m be a positive g-continuous function on W such that $h_g^{k/2} m \in L^p(W)$ for some $k \ge 0$, and suppose $a \in S(m,g) \cap WL_g^{1,p}(W)$. Then $a \in s_p^w(W)$.*

Theorems 3.7 and 3.8 are straight-forward consequences of Theorems 3.1 and 3.3, combined with the following lemma. The details are left for the reader.

Lemma 3.9. *Let $p, q \in (0,\infty]$, g be slowly varying, and m be g-continuous on W. Then*

$$m \in L^p(W) \qquad \Longleftrightarrow \qquad m \in WL_g^{q,p}(W).$$

Proof. Suppose $m \in L^p(W)$, and let $\{U_j\}_{j \in \mathbf{Z}_+}$ be an admissible g-covering of W with centers in $X_j \in W$, $j \in \mathbf{Z}_+$. Since m is g-continuous and g is slowly varying, it follows that

$$\|m\|_{L^p}^p \asymp \sum_{j=1}^{\infty} m(X_j)^p |U_j|.$$

By using the g-continuity again, it also follows that

$$\|m\|_{WL_{g,\theta}^{q,p}}^p \asymp \sum_{j=1}^{\infty} m(X_j)^p |U_j|^{\frac{p}{q}} |U_j|^{\theta p} = \sum_{j=1}^{\infty} m(X_j)^p |U_j|,$$

and the asserted equivalence follows from these relations. □

Remark 3.10. Suppose that, in addition to the assumptions of Theorem 3.7, the metric g and the weight m are σ-temperate and (σ,g)-temperate, respectively. Then there is a natural extension of Theorem 3.7 to Weyl operators acting on Sobolev-type Hilbert spaces, $H(m,g)$, introduced by Bony and Chemin in [4], which is especially suitable for the Weyl-Hörmander calculus. (See also Section 2.6 in [12].)

In fact, suppose that m and m_0 are g-continuous and (σ,g)-temperate, and $a \in S(m,g)$. Then

$$\operatorname{Op}^w(a) : H(m_0,g) \to H(m_0/m,g)$$

is continuous. In [4, 12] it is also shown that there are $a_0 \in S(m,g)$ and $b_0 \in S(1/m,g)$ such that

$$\operatorname{Op}^w(b_0) = \operatorname{Op}^w(a_0)^{-1}, \qquad a_0 \in S(m,g),\ b_0 \in S(1/m,g). \tag{3.4}$$

Especially, it follows that

$$\operatorname{Op}^w(a_0) : H(m_0,g) \to H(m_0/m,g) \quad \text{and} \quad \operatorname{Op}^w(b_0) : H(m_0/m,g) \to H(m_0,g)$$

are continuous bijections, which are inverses to each other. In particular, from these mapping properties it follows that equality is attained in (1.15).

Now let $p \in (0,1]$, g be strongly feasible on W, and m, m_1, and m_2 be positive g-continuous and (σ,g)-temperate functions on W such that

$$\frac{m_2 m}{m_1} \in L^p(W).$$

A combination of Theorem 3.7 and (3.4) then gives

$$S(m,g) \subseteq s_{A,p}(\mathcal{H}_1,\mathcal{H}_2), \quad \text{when} \quad \mathcal{H}_1 = H(m_1,g),\ \mathcal{H}_2 = H(m_2,g).$$

(See also [19, Theorem 4.4].) Since $H(1,g) = L^2(V)$, in view of [4, 12], we regain Theorem 3.7 in the case when m is g-continuous and (σ, g)-temperate, by choosing $m_1 = m_2 = 1$.

3.2. **Split metrics and more general pseudo-differential calculi.** In order to state analogous results for more general pseudo-differential calculi, we need to impose further restrictions on the metric g and weight function m.

We recall that a feasible metric g on W is called *split*, if there are global symplectic coordinates $Y = (y, \eta)$ such that

$$g_X(y, -\eta) = g_X(y, \eta),$$

for all $X \in W$.

The next proposition follows from [11, Theorem 18.5.10] and its proof. The details are left for the reader.

Proposition 3.11. *Let $A, B \in \mathcal{L}(V)$, g be strongly feasible and split on $W = T^*V$, and let m be g-continuous and (σ, g)-temperate weight function. Then*

$$\mathrm{Op}_A(S(m, g)) = \mathrm{Op}_B(S(m, g)).$$

A combination of Theorem 3.7, Theorem 3.8, and Proposition 3.11 gives the following. The details are left for the reader.

Theorem 3.12. *Let $A \in \mathcal{L}(V)$, $p \in (0, 1]$, g be strongly feasible and split on W, and $m \in L^p(W)$ be a positive g-continuous and (σ, g)-temperate function on W. Then $S(m, g) \subseteq s_{A,p}(W)$.*

Theorem 3.13. *Let $A \in \mathcal{L}(V)$, $p \in (0, 1]$, g be strongly feasible and split on W, m be a positive g-continuous and (σ, g)-temperate function on W such that $h_g^{k/2} m \in L^p(W)$ for some $k \geq 0$. Also, suppose $a \in S(m, g) \cap WL_g^{1,p}(W)$. Then $a \in s_{A,p}(W)$.*

4. Applications to special families of pseudo-differential operators

In this section we apply the results from previous sections to obtain Schatten-von Neumann properties for pseudo-differential operators with symbols in the well-known Shubin classes and SG classes (see [13]). We first recall their definitions. Here, let

$$\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}, \qquad x \in \mathbf{R}^d.$$

Definition 4.1. Let $r, \rho \in \mathbf{R}$.

(i) The *Shubin class* $\mathrm{Sh}^r(\mathbf{R}^d)$ is the set of all $f \in \mathscr{C}^\infty(\mathbf{R}^d)$ such that

$$|D^\alpha f(x)| \leq C_\alpha \langle x \rangle^{r - |\alpha|}, \qquad x \in \mathbf{R}^d.$$

(ii) The *SG class* $\mathrm{S}^{r,\rho}(\mathbf{R}^{2d})$ is the set of all $a \in \mathscr{C}^\infty(\mathbf{R}^{2d})$ such that

$$|D_x^\alpha D_\xi^\beta a(x, \xi)| \leq C_{\alpha,\beta} \langle x \rangle^{r - |\alpha|} \langle \xi \rangle^{\rho - |\beta|}, \quad x, \xi \in \mathbf{R}^d.$$

Remark 4.2. Let $p \in (0, 1]$. For the symbol classes in Definition 4.1, we observe the following:

(i) if $r \in \mathbf{R}$, then $S(m,g) = \mathrm{Sh}^r(\mathbf{R}^{2d})$ when

$$g_{x,\xi}(y,\eta) = \frac{|y|^2 + |\eta|^2}{\langle (x,\xi)\rangle^2} \quad \text{and} \quad m(x,\xi) = \langle (x,\xi)\rangle^r. \tag{4.1}$$

Furthermore, $h_g(x,\xi) = \langle (x,\xi)\rangle^{-2}$ and

$$h_g^{k/2} m \in L^p(\mathbf{R}^{2d}), \quad \text{when} \quad k > r + \frac{2d}{p};$$

(ii) if $r, \rho \in \mathbf{R}$, then $S(m,g) = \mathrm{S}^{r,\rho}(\mathbf{R}^{2d})$ when

$$g_{x,\xi}(y,\eta) = \frac{|y|^2}{\langle x\rangle^2} + \frac{|\eta|^2}{\langle \xi\rangle^2} \quad \text{and} \quad m(x,\xi) = \langle x\rangle^r \langle \xi\rangle^\rho. \tag{4.2}$$

Furthermore, $h_g(x,\xi) = (\langle x\rangle\langle\xi\rangle)^{-1}$ and

$$h_g^{k/2} m \in L^p(\mathbf{R}^{2d}), \quad \text{when} \quad k > 2\max(r,\rho) + \frac{2d}{p}.$$

In both (i) and (ii), g is strongly feasible and m is g-continuous and (σ, g)-temperate.

In the next result we show how Lemma 3.9 and Theorem 3.13 can be combined with Remark 4.2, in order to obtain quasi-Banach Schatten-von Neumann properties for the Shubin classes and the SG classes.

Proposition 4.3. *Let $p \in (0,1]$, A be a real $d\times d$-matrix, and $r,\rho \in \mathbf{R}$. Then the following is true:*

(i) *if g is given by* (4.1)*, then*

$$\mathrm{Sh}^r(\mathbf{R}^{2d}) \cap \mathit{WL}^{1,p}_g(\mathbf{R}^{2d}) \subseteq s_{A,p}(\mathbf{R}^{2d});$$

(ii) *if g is given by* (4.2)*, then*

$$\mathrm{S}^{r,\rho}(\mathbf{R}^{2d}) \cap \mathit{WL}^{1,p}_g(\mathbf{R}^{2d}) \subseteq s_{A,p}(\mathbf{R}^{2d}).$$

References

[1] C. Bennett and R. Sharpley *Interpolation operators*, Academic Press, Boston SanDiego NewYork Berkley London Sydney Tokyo Toronto, **129**, 1988.

[2] Birman, Solomyak *Estimates for the singular numbers of integral operators (Russian)*, Usbehi Mat. Nauk. **32**, (1977), 17–84.

[3] D. Bhimani, J. Toft *Factorizations in quasi-Banach modules and applications, part I*, preprint, arXiv:2307.01590.

[4] J. M. Bony, J. Y. Chemin *Espaces functionnels associés au calcul de Weyl-Hörmander*, Bull. Soc. math. France **122** (1994), 77–118.

[5] E. Buzano, F. Nicola *Pseudo-differential operators and Schatten-von Neumann classes,* in: P. Boggiatto, R. Ashino, M. W. Wong (eds), Advances in Pseudo-Differential Operators, Proceedings of the Fourth ISAAC Congress, Operator Theory: Advances and Applications, Birkhäuser Verlag, Basel, 2004.

[6] E. Buzano, J. Toft *Schatten-von Neumann properties in the Weyl calculus*, J. Funct. Anal. **259** (2010), 3080–3114.

[7] G. B. Folland *Harmonic analysis in phase space*, Princeton U. P., Princeton, 1989.

[8] K. Gröchenig. *Foundations of time-frequency analysis.* Birkhäuser Boston Inc., Boston, MA, 2001.

[9] L. Hörmander *The Weyl calculus of pseudo-differential operators*, Comm. Pure Appl. Math. **32** (1979), 359–443.

[10] L. Hörmander *On the asymptotic distributions of the eigenvalues of pseudodifferential operators in $\mathbb{R}^n$*, Ark. Mat. **17** (1979), 297–313.

[11] L. Hörmander *The analysis of linear partial differential operators*, I, III, Springer-Verlag, Berlin Heidelberg NewYork Tokyo, 1983, 1985.
[12] N. Lerner *The Wick calculus of pseudo-differential operators and some of its applications*, CUBO, **5** (2003), 213–236.
[13] M. A. Shubin *Pseudodifferential operators and spectral theory*, Springer-Verlag, Berlin, 1987.
[14] B. Simon *Trace ideals and their applications* I, London Math. Soc. Lecture Note Series, Cambridge University Press, Cambridge London New York Melbourne, 1979.
[15] J. Toft *Continuity properties for non-commutative convolution algebras with applications in pseudo-differential calculus*, Bull. Sci. Math. (2) **126** (2002), 115–142.
[16] J. Toft *Schatten-von Neumann properties in the Weyl calculus, and calculus of metrics on symplectic vector spaces*, Ann. Global Anal. Geom. **30** (2006), 169–209.
[17] J. Toft *Multiplication properties in pseudo-differential calculus with small regularity on the symbols*, J. Pseudo-Differ. Oper. Appl. **1** (2010), 101–138.
[18] J. Toft *Matrix parameterized pseudo-differential calculi on modulation spaces* in: M. Oberguggenberger, J. Toft, J. Vindas, P. Wahlberg (eds)*, Generalized functions and Fourier analysis*, Operator Theory: Advances and Applications **260** Birkhäuser, 2017, pp. 215–235.
[19] J. Toft *Continuity and compactness for pseudo-differential operators with symbols in quasi-Banach spaces or Hörmander classes*, Anal. Appl. **15** (2017), 353–389.

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