Fourier type operators on Orlicz spaces and the role of Orlicz Lebesgue exponents

FOURIER TYPE OPERATORS ON ORLICZ SPACES AND THE ROLE OF ORLICZ LEBESGUE EXPONENTS

MATTEO BONINO, SANDRO CORIASCO, ALBIN PETERSSON, AND JOACHIM TOFT

Abstract. We deduce continuity properties of classes of Fourier multipliers, pseudo-differential and Fourier integral operators when acting on Orlicz spaces. Especially we show classical results like Hörmander's improvement of Mihlin's Fourier multiplier theorem are extendable to the framework of Orlicz spaces. We also show how some properties of the Young functions Φ of the Orlicz spaces are linked to properties of certain Lebesgue exponents p_Φ and q_Φ emerged from Φ.

0. Introduction

Orlicz spaces, introduced by W. Orlicz in 1932 [12], are Banach spaces which generalize the normal L^p spaces (see Section 1 for notations). Orlicz spaces are denoted by L^Φ where Φ is a Young function, and we obtain the usual L^p spaces, $1 \leqslant p < \infty$, by choosing $\Phi(t) = t^p$. For more facts on Orlicz spaces, see [14].

An advantage of Orlicz spaces is that they are suitable when solving certain problems where L^p spaces are insufficient. As an example, consider the entropy of a probability density function f given by

$$E(f) = -\int f(\xi) \log f(\xi)\, d\xi.$$

In this case, it may be more suitable to work with an Orlicz norm estimate, for instance with $\Phi(t) = t \log(1 + t)$, as opposed to L^1 norm estimates.

The literature on Orlicz spaces is rich, see e.g. [1,4,8,9,11,13] and the references therein. Recent investigations also put pseudo-differential operators in the framework of Orlicz modulation spaces (cf [19], see also [15, 20] for further properties on Orlicz modulation spaces). In this paper, we deal with pseudo-differential operators as well as Fourier multipliers in Orlicz spaces.

Results pertaining to continuity properties on L^p-spaces are well-established. Our approach is to utilize a Marcinkiewicz interpolation-type theorem by Liu and Wang in [7] to extend such continuity properties to also hold on Orlicz spaces. As an initial example, the methods described in the subsequent sections allow us to obtain the following extension of Mihlin's Fourier multiplier theorem (see [10] for the original theorem).

2010 *Mathematics Subject Classification.* primary: 35S05, 46E30, 46A16, 42B35 secondary: 46F10.

Key words and phrases. Orlicz, quasi-Banach, quasi-Young functionals.

Theorem 0.1 (Mihlin). *Let Φ be a strict Young function and $a \in L^\infty(\mathbf{R}^d \setminus \{0\})$ be such that*

$$\sup_{\xi \neq 0} \Big(|\xi|^{|\alpha|} \, |\partial^\alpha a(\xi)| \Big)$$

is finite for every $\alpha \in \mathbf{N}^d$ with $|\alpha| \leqslant [\frac{d}{2}] + 1$. Then $a(D)$ is continuous on $L^\Phi(\mathbf{R}^d)$.

In fact, we also obtain Hörmander's improvement of Mihlin's Fourier multiplier theorem (cf [5]) in the context of Orlicz spaces. This result can be found in Section 3 (Theorem 3.4). In a similar manner, we obtain continuity results for pseudo-differential operators of order 0 in Orlicz spaces as well, see Theorem 3.3. Finally, we show a continuity result for a broad class of Fourier integral operators, under a condition on the order of the amplitude (that is, a loss of derivatives and decay), see Theorem 3.5.

Section 1 also include investigations of Lebesgue exponents p_Φ and q_Φ constructed from the Young function Φ, which are important for the interpolation theorem. These parameters were described in [7], where it was claimed that

$$p_\Phi < \infty \iff \Phi \text{ fulfills the } \Delta_2 \text{ condition} \tag{0.1}$$

and

$$q_\Phi > 1 \iff \Phi \text{ is strictly convex.} \tag{0.2}$$

In Section 1, we confirm that (0.1) is correct, but that neither logical implication of (0.2) is correct. Instead, other conditions on Φ are found which characterize $q_\Phi > 1$ (see Proposition 2.1). At the same time, we deduce a weaker form of the equivalence (0.2) and show that if $q_\Phi > 1$, then there is an equivalent Young function to Φ which is strictly convex. (see Proposition 2.4).

1. Preliminaries

In this section we recall some facts on Orlicz spaces and pseudo-differential operators. Especially we recall Lebesgue exponents given in e. g. [7] and explain some of their features.

1.1. **Orlicz Spaces.** In this subsection we provide an overview of some basic definitions and state some technical results that will be needed. First, we recall the definition of weak L^p spaces.

Definition 1.1. Let $p \in (0, \infty]$. The *weak L^p space* $wL^p(\mathbf{R}^d)$ consists of all Lebesgue measurable functions $f : \mathbf{R}^d \to \mathbf{C}$ for which

$$\|f\|_{wL^p} \equiv \sup_{t>0} t \, (\mu_f(t))^{\frac{1}{p}} \tag{1.1}$$

is finite. Here $\mu_f(t)$ is the Lebesgue measure of the set $\{\, x \in \mathbf{R}^d \, ; \, |f(x) > t| \,\}$.

Remark 1.2. Notice that the wL^p-norm is not a true norm, since the triangular inequality fails. Nevertheless, one has that $\|f\|_{wL^p} \leqslant \|f\|_{L^p}$. In particular, $L^p(\mathbf{R}^d)$ is continuously embedded in $wL^p(\mathbf{R}^d)$.

Next, we recall some facts concerning Young functions and Orlicz spaces. (See [4, 14].)

Definition 1.3. A function $\Phi : \mathbf{R} \to \mathbf{R} \cup \{\infty\}$ is called *convex* if

$$\Phi(s_1t_1 + s_2t_2) \leqslant s_1\Phi(t_1) + s_2\Phi(t_2)$$

when $s_j, t_j \in \mathbf{R}$ satisfy $s_j \geqslant 0$ and $s_1 + s_2 = 1,\ j = 1, 2$.

We observe that Φ might not be continuous, because we permit ∞ as function value. For example,

$$\Phi(t) = \begin{cases} c, & \text{when } t \leqslant a \\ \infty, & \text{when } t > a \end{cases}$$

is convex but discontinuous at $t = a$.

Definition 1.4. Let Φ be a function from $[0, \infty)$ to $[0, \infty]$. Then Φ is called a *Young function* if

(1) Φ is convex,

(2) $\Phi(0) = 0$,

(3) $\lim\limits_{t\to\infty} \Phi(t) = +\infty$.

It is clear that Φ in Definition 1.4 is non-decreasing, because if $0 \leqslant t_1 \leqslant t_2$ and $s \in [0, 1]$ is chosen such that $t_1 = st_2$, then

$$\Phi(t_1) = \Phi(st_2 + (1 - s)0) \leqslant s\Phi(t_2) + (1 - s)\Phi(0) \leqslant \Phi(t_2),$$

since $\Phi(0) = 0$ and $s \in [0, 1]$.

The Young functions Φ_1 and Φ_2 are called *equivalent*, if there is a constant $C \geqslant 1$ such that

$$C^{-1}\Phi_2(t) \leqslant \Phi_1(t) \leqslant C\Phi_2(t), \qquad t \in [0, \infty].$$

We recall that a Young function is said to fulfill the Δ_2-*condition* if there is a constant $C \geqslant 1$ such that

$$\Phi(2t) \leqslant C\Phi(t), \qquad t \in [0, \infty].$$

We also introduce the following condition. A Young function is said to fulfill the Λ-*condition* if there is a $p > 1$ such that

$$\Phi(ct) \leqslant c^p\Phi(t), \qquad t \in [0, \infty],\ c \in (0, 1]. \tag{1.2}$$

The following characterization of Young functions fulfilling the Δ_2-condition follows from the fact that any Young function is increasing. The verifications are left for the reader.

Proposition 1.5. *Let Φ be a Young function. Then the following conditions are equivalent:*

(1) *Φ satisfies the Δ_2-condition;*

(2) *for every constant $c > 0$, the Young function $t \mapsto \Phi(ct)$ is equivalent to Φ;*

(3) *for some constant $c > 0$ with $c \neq 1$, the Young function $t \mapsto \Phi(ct)$ is equivalent to Φ.*

For any Young function Φ, t The *upper* and *lower Lebesgue exponents* for a Young function Φ are defined by

$$p_\Phi \equiv \sup_{t>0} \left(\frac{t\Phi'_+(t)}{\Phi(t)} \right) = \sup_{t>0} \left(\frac{t\Phi'_-(t)}{\Phi(t)} \right) \tag{1.3}$$

and

$$q_\Phi \equiv \inf_{t>0} \left(\frac{t\Phi'_+(t)}{\Phi(t)} \right) = \inf_{t>0} \left(\frac{t\Phi'_-(t)}{\Phi(t)} \right), \tag{1.4}$$

respectively. We recall that these exponents are essential in the analysis in [7]. We observe that for any $r_1, r_2 > 0$,

$$t^{p_\Phi} \lesssim \Phi(t) \lesssim t^{q_\Phi} \quad \text{when} \quad t \leqslant r_1 \tag{1.5}$$

and

$$t^{q_\Phi} \lesssim \Phi(t) \lesssim t^{p_\Phi} \quad \text{when} \quad t \geqslant r_2. \tag{1.6}$$

In order to shed some light on this as well as demonstrate arguments used in the next section we here show these relations.

By (1.3) we obtain

$$\frac{t\Phi'_+(t)}{\Phi(t)} - p_\Phi \leqslant 0 \quad \Leftrightarrow \quad \left(\frac{\Phi(t)}{t^{p_\Phi}} \right)'_+ \leqslant 0.$$

Hence $\Phi(t) = t^{p_\Phi} h(t)$ for some decreasing function $h(t) > 0$. This gives

$$\Phi(t) = t^{p_\Phi} h(t) \geqslant t^{p_\Phi} h(r_1) \gtrsim t^{p_\Phi}$$

for $t \leqslant r_1$ and

$$\Phi(t) = t^{p_\Phi} h(t) \leqslant t^{p_\Phi} h(r_2) \lesssim t^{p_\Phi}$$

for $t \geqslant r_2$. This shows the relations between t^{p_Φ} and $\Phi(t)$ in (1.5) and (1.6). The remaining relations follow in similar ways.

In our investigations we need to assume that our Young functions are *strict* in the following sense.

Definition 1.6. The Young function Φ from $[0, \infty)$ to $[0, \infty]$ is called *strict* or a *strict Young function*, if

(1) $\Phi(t) < \infty$ for every $t \in [0, \infty)$,
(2) Φ satisfies the Δ_2-condition,
(3) Φ satisfies the Λ-condition.

In Section 2 we give various kinds of characterizations of the conditions (2) and (3) in Definition 1.6. In particular we show that (2) and (3) in Definition 1.6 are equivalent to $p_\Phi < \infty$ and $q_\Phi > 1$, respectively. (See Proposition 2.3.)

It will also be useful to rely on regular Young functions, which is possible due to the following proposition.

Proposition 1.7. *Let Φ be a Young function which satisfies the Δ_2 condition. Then there is a Young function Ψ such that the following is true:*

(1) *Ψ is equivalent to Φ and $\Psi \leqslant \Phi$;*
(2) *Ψ is smooth on $\mathbf{R}_+$;*
(3) *$\Psi'_+(0) = \Phi'_+(0)$.*

Proof. Let $\phi \in C_0^\infty[0,1]$ be such that $\phi \geqslant 0$ and $\int_0^1 \phi(s)\,ds = 1$. Put

$$\Psi(t) = \int_0^1 \Phi(t - \tfrac{1}{2}st)\phi(s)\,ds.$$

Then using this formula and

$$\Psi(t) = \int_{t/2}^t \Phi(s)\phi(s - 2s/t)\frac{t}{s}\,ds,$$

we reach the result. $\square$

It follows that Ψ in Proposition 1.7 fulfills the Δ_2 condition, because Φ satisfy that condition and Ψ is equivalent to Φ.

Definition 1.8. Let Φ be a Young function. The *Orlicz space* $L^\Phi(\mathbf{R}^d)$ consists of all Lebesgue measurable functions $f : \mathbf{R}^d \to \mathbf{C}$ such that

$$\|f\|_{L^\Phi} \equiv \inf \left\{ \lambda > 0\,;\, \int_{\mathbf{R}^d} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leqslant 1 \right\}$$

is finite.

Definition 1.9. Let Φ be a Young function. The *weak Orlicz space* $wL^\Phi(\mathbf{R}^d)$ consists of all Lebesgue measurable functions $f : \mathbf{R}^d \to \mathbf{C}$ such that

$$\|f\|_{wL^\Phi} \equiv \inf \left\{ \lambda > 0\,;\, \sup_{t>0} \left(\Phi\left(\frac{t}{\lambda}\right) \mu_f(t) \right) \leqslant 1 \right\}$$

is finite. Here $\mu_f(t)$ is the Lebesgue measure of the set $\{\, x \in \mathbf{R}^d\,;\, |f(x) > t| \,\}$.

In accordance with the usual Lebesgue spaces, $f, g \in wL^\Phi(\mathbf{R}^d)$ are equivalent whenever $f = g$ a. e.

1.2. **Pseudo-differential operators.** Let $\mathbf{M}(d,\Omega)$ be the set of all $d \times d$-matrices with entries in the set Ω, and let $a \in \mathscr{S}(\mathbf{R}^{2d})$ and $A \in \mathbf{M}(d,\mathbf{R})$ be fixed. Then the pseudo-differential operator $\operatorname{Op}_A(a)$ is the linear and continuous operator on $\mathscr{S}(\mathbf{R}^d)$, given by

$$(\operatorname{Op}_A(a)f)(x) = (2\pi)^{-d} \iint a(x - A(x-y),\xi) f(y) e^{i\langle x-y,\xi\rangle}\,dyd\xi, \tag{1.7}$$

when $f \in \mathscr{S}(\mathbf{R}^d)$. For general $a \in \mathscr{S}'(\mathbf{R}^{2d})$, the pseudo-differential operator $\operatorname{Op}_A(a)$ is defined as the linear and continuous operator from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ with distribution kernel given by

$$K_{a,A}(x,y) = (2\pi)^{-d/2}(\mathscr{F}_2^{-1}a)(x - A(x-y), x-y). \tag{1.8}$$

Here $\mathscr{F}_2 F$ is the partial Fourier transform of $F(x,y) \in \mathscr{S}'(\mathbf{R}^{2d})$ with respect to the y variable. This definition makes sense, since the mappings

$$\mathscr{F}_2 \quad \text{and} \quad F(x,y) \mapsto F(x - A(x-y), x-y) \tag{1.9}$$

are homeomorphisms on $\mathscr{S}'(\mathbf{R}^{2d})$. In particular, the map $a \mapsto K_{a,A}$ is a homeomorphism on $\mathscr{S}'(\mathbf{R}^{2d})$.

An important special case appears when $A = t \cdot I$, with $t \in \mathbf{R}$. Here and in what follows, $I \in \mathbf{M}(d,\mathbf{R})$ denotes the $d \times d$ identity matrix. In this case we set

$$\operatorname{Op}_t(a) = \operatorname{Op}_{t\cdot I}(a).$$

The normal or Kohn-Nirenberg representation, $a(x,D)$, is obtained when $t=0$, and the Weyl quantization, $\operatorname{Op}^w(a)$, is obtained when $t=\frac{1}{2}$. That is,

$$a(x,D)=\operatorname{Op}_0(a) \quad \text{and} \quad \operatorname{Op}^w(a)=\operatorname{Op}_{1/2}(a).$$

For any $K\in \mathscr{S}'(\mathbf{R}^{d_1+d_2})$, we let T_K be the linear and continuous mapping from $\mathscr{S}(\mathbf{R}^{d_1})$ to $\mathscr{S}'(\mathbf{R}^{d_2})$, defined by the formula

$$(T_Kf,g)_{L^2(\mathbf{R}^{d_2})}=(K,g\otimes \overline{f})_{L^2(\mathbf{R}^{d_1+d_2})}. \tag{1.10}$$

It is well-known that if $A\in \mathbf{M}(d,\mathbf{R})$, then it follows from Schwartz kernel theorem that $K\mapsto T_K$ and $a\mapsto \operatorname{Op}_A(a)$ are bijective mappings from $\mathscr{S}'(\mathbf{R}^{2d})$ to the set of linear and continuous mappings from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ (cf. e. g. [6]).

In particular, for every $a_1\in \mathscr{S}'(\mathbf{R}^{2d})$ and $A_1,A_2\in \mathbf{M}(d,\mathbf{R})$, there is a unique $a_2\in \mathscr{S}'(\mathbf{R}^{2d})$ such that $\operatorname{Op}_{A_1}(a_1)=\operatorname{Op}_{A_2}(a_2)$. The following result explains the relations between a_1 and a_2.

Proposition 1.10. *Let $a_1,a_2\in \mathscr{S}'(\mathbf{R}^{2d})$ and $A_1,A_2\in \mathbf{M}(d,\mathbf{R})$. Then*

$$\operatorname{Op}_{A_1}(a_1)=\operatorname{Op}_{A_2}(a_2) \quad \Leftrightarrow \quad e^{i\langle A_2D_\xi,D_x\rangle}a_2(x,\xi)=e^{i\langle A_1D_\xi,D_x\rangle}a_1(x,\xi).$$

In [18], a proof of the previous proposition is given, which is similar to the proof of the case $A=t\cdot I$ in [6,17,21].

Let $r,\rho,\delta\in \mathbf{R}$ be such that $0\leqslant \delta\leqslant \rho\leqslant 1$ and $\delta<1$. Then we recall that the Hörmander class $S^r_{\rho,\delta}(\mathbf{R}^{2d})$ consists of all $a\in C^\infty(\mathbf{R}^{2d})$ such that

$$\sum_{|\alpha|,|\beta|\leqslant N}\sup_{x,\xi\in\mathbf{R}^d}\Big(\langle \xi\rangle^{-r+\rho|\alpha|-\delta|\beta|}|D_\xi^\alpha D_x^\beta a(x,\xi)|\Big)$$

is finite for every integer $N\geqslant 0$.

We recall the following continuity property for pseudo-differential operators acting on L^p-spaces (see e. g. [22]).

Proposition 1.11. *Let $p\in(1,\infty)$, $A\in\mathbf{M}(\mathbf{R},d)$ and $a\in S^0_{1,0}(\mathbf{R}^{2d})$. Then $\operatorname{Op}_A(a)$ is continuous on $L^p(\mathbf{R}^d)$.*

In the next proposition we essentially recall Hörmander's improvement of Mihlin's Fourier multiplier theorem.

Proposition 1.12. *Let $p\in(1,\infty)$ and $a\in L^\infty(\mathbf{R}^d\setminus 0)$ be such that*

$$\sup_{R>0}\left(R^{-d+2|\alpha|}\int_{A_R}|\partial^\alpha a(\xi)|^2\,d\xi\right) \tag{1.11}$$

is finite for every $\alpha\in\mathbf{N}^d$ with $|\alpha|\leqslant[\frac{d}{2}]+1$, where A_R is the annulus $\{\,\xi\in\mathbf{R}^d\,;\,R<|\xi|<2R\,\}$. Then $a(D)$ is continuous on $L^p(\mathbf{R}^d)$.

1.3. **Fourier integral operators of SG type.** We recall that the so-called SG-symbol class $S^{m,\mu}(\mathbf{R}^{2d})$, $m,\mu\in\mathbf{R}$, consists of all $a\in C^\infty(\mathbf{R}^{2d})$ such that

$$\sum_{|\alpha|,|\beta|\leqslant N}\sup_{x,\xi\in\mathbf{R}^d}\Big(\langle x\rangle^{-m+|\alpha|}\langle\xi\rangle^{-\mu+|\beta|}|D_x^\alpha D_\xi^\beta a(x,\xi)|\Big)$$

is finite for every integer $N\geqslant 0$. Following [3], we say that $\varphi\in C^\infty(\mathbf{R}^d\times(\mathbf{R}^d\setminus 0))$ is a phase-function if it is real-valued, positively 1-homogeneous with respect to ξ, that is, $\varphi(x,\tau\xi)=\tau\varphi(x,\xi)$ for all $\tau>0$, $x,\xi\in\mathbf{R}^d$, $\xi\neq 0$, and satisfies, for all $x,\xi\in\mathbf{R}^d$, $\xi\neq 0$,

$$\begin{aligned}|\det\partial_x\partial_\xi\varphi(x,\xi)|\geq C>0,\quad \partial_x^\alpha\varphi(x,\xi)\prec\langle x\rangle^{1-|\alpha|}|\xi| \text{ for all } \alpha\in\mathbf{N}^d,\\ \langle\varphi'_\xi(x,\xi)\rangle\sim\langle x\rangle,\ \langle\varphi'_x(x,\xi)\rangle\sim\langle\xi\rangle.\end{aligned}\tag{1.12}$$

In the sequel, we will denote the set of all such phase-functions by $\mathfrak{P}_r^{\mathrm{hom}}$.

For any $a\in S^{m,\mu}(\mathbf{R}^{2d})$ and $\varphi\in\mathfrak{P}_r^{\mathrm{hom}}$, the Fourier integral operator $\mathrm{Op}_\varphi(a)$ is the linear and continuous operator from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$, given by

$$(\mathrm{Op}_\varphi(a)f)(x)=\int_{\mathbf{R}^d}e^{i\varphi(x,\xi)}a(x,\xi)\widehat{f}(\xi)\,d\xi,\quad f\in\mathscr{S}(\mathbf{R}^d).\tag{1.13}$$

We recall the following (global on $\mathbf{R}^d$) L^p-boundedness result, proved in [3].

Theorem 1.13. *Let $p\in(1,\infty)$, $m,\mu\in\mathbf{R}$ be such that*

$$m\leq-(d-1)\left|\frac{1}{p}-\frac{1}{2}\right|\ \text{ and }\ \mu\leq-(d-1)\left|\frac{1}{p}-\frac{1}{2}\right|,\tag{1.14}$$

and suppose that $a\in S^{m,\mu}(\mathbf{R}^{2d})$ is such that $|\xi|\geq\varepsilon$, for some $\varepsilon>0$, on the support of a. Then $\mathrm{Op}_\varphi(a)$ from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ extends uniquely to a continuous operator on $L^p(\mathbf{R}^d)$.

Remark 1.14. As it is well-known, in view of the presence of a phase function $\varphi\in\mathfrak{P}_r^{\mathrm{hom}}$, assumed different from $\varphi(x,\xi)=x\cdot\xi$ (for which (1.13) actually becomes a pseudo-differential operator), the uniform boundedness of the amplitude a is, in general, not enough to guarantee that $\mathrm{Op}_\varphi(a)$ continuously maps L^p into itself, even if the support of f is compact (see the celebrated paper [16]), except when $p=2$. This is, of course, in strong contrast with Proposition 1.11. Notice, in (1.14), the *loss of decay* (that is, the condition on the x-order m of the amplitude), together with the well-known *loss of smoothness* (that is, the condition on the ξ-order μ of the amplitude). Notice also that no condition of compactness of the support of f is needed in Theorem 1.13 (see [3] and the references quoted therein for more details).

2. The role of upper and lower Lebesgue exponents for Young functions

In this section we investigate the Orlicz Lebesgue exponents p_Φ and q_Φ and link conditions on these exponents to various properties on their Young functions Φ. Especially we show that both implications in (0.2) involving q_Φ are wrong (see Proposition 2.4). Instead we deduce other conditions Φ which characterize $q_\Phi>1$ (see Propositions 2.1 and 2.3).

In the following proposition we list some basic properties of relations between Young functions and their upper and lower Lebesgue exponents.

Proposition 2.1. *Let Φ be a Young function which is non-zero outside the origin, and let q_Φ and p_Φ be as in* (1.4) *and* (1.3)*. Then the following is true:*

(1) $1\leqslant q_\Phi\leqslant p_\Phi$;

(2) $p_\Phi=1$, *if and only if Φ is a linear map;*

(3) $p_\Phi < \infty$, *if and only if* Φ *fulfills the* Δ_2*-condition;*

(4) $q_\Phi > 1$, *if and only if there is a* $p > 1$ *such that* $\frac{\Phi(t)}{t^p}$ *increases.*

Remark 2.2. Taking into account that Φ in Proposition 2.1 is a Young function, we find that (4) is equivalent to

(4)$'$ $q_\Phi > 1$, *if and only if there is a* $p > 1$ *such that* $\frac{\Phi(t)}{t^p}$ *increases,*

$$\lim_{t\to 0+} \frac{\Phi(t)}{t^p} = 0 \quad \textit{and} \quad \lim_{t\to\infty} \frac{\Phi(t)}{t^p} = \infty.$$

Most of Proposition 2.1 and Remark 2.2 are well-known. In order to be self-contained we here present a proof.

Proof of Proposition 2.1. Since Φ and its left and right derivatives are increasing, the mean-value theorem gives that for some $c = c_t \in [0,1]$, we have

$$\Phi(t) = \Phi(t) - \Phi(0) \leqslant t\Phi_+(ct) \leqslant t\Phi_+(t).$$

This gives (1).

If Φ is linear, then $\frac{t\Phi'(t)}{\Phi(t)} = 1$, giving that $q_\Phi = p_\Phi = 1$. Suppose instead that $p_\Phi = 1$. Then

$$\frac{t\Phi'(t)}{\Phi(t)} = 1,$$

in view of (1) and its proof. This implies that $\Phi(t) = Ct$ for some constant C, and (2) follows.

In order to prove (3), we first suppose that $p_\Phi < \infty$. Then

$$\frac{t\Phi'_+(t)}{\Phi(t)} \leqslant R \quad \Leftrightarrow \quad t\Phi'_+(t) - R\Phi(t) \leqslant 0,$$

for some $R > 0$. Since $\Phi(0) = 0$, we obtain

$$\Phi(t) = t^R h(t), \quad t > 0,$$

for some positive decreasing function $h(t)$, $t > 0$. This gives

$$\Phi(2t) = (2t)^R h(2t) \leqslant 2^R t^R h(t) = 2^R \Phi(t),$$

and it follows that Φ satisfies the Δ_2-condition when $p_\Phi < \infty$.

Suppose instead that Φ satisfies the Δ_2-condition. By the mean-value theorem and the fact that $\Phi'_+(t)$ is increasing we obtain

$$\Phi'_+(t)t \leqslant \Phi(2t) - \Phi(t) \leqslant \Phi(2t) \leqslant C\Phi(t),$$

for some constant $C > 0$. Here the last inequality follows from the fact that Φ satisfies the Δ_2-condition. This gives

$$\frac{t\Phi'_+(t)}{\Phi(t)} \leqslant C,$$

giving that $p_\Phi \leqslant C < \infty$, and we have proved (3).

Next we prove (4). Suppose that $q_\Phi > 1$. Then there is a $p > 1$ such that

$$\frac{t\Phi'_\pm(t)}{\Phi(t)} > p$$

for all $t > 0$, which gives

$$t\Phi'_\pm(t) - p\Phi(t) > 0.$$

Hence

$$\frac{t^p \Phi'_\pm(t) - pt^{p-1}\Phi(t)}{t^{2p}} > 0,$$

or equivalently

$$\left(\frac{\Phi(t)}{t^p}\right)'_\pm > 0.$$

Hence, the result now holds. If we instead suppose that $\frac{\Phi(t)}{t^p}$ is increasing for some $p > 1$, then applying the arguments above in reverse order now yields $q_\Phi \geqslant p > 1$. $\square$

For the equivalence in (4) of Proposition 2.1 we note further.

Proposition 2.3. *Let Φ be a Young function which is non-zero outside the origin, and let q_Φ be as in* (1.4)*. Then the following conditions are equivalent:*

(1) $q_\Phi > 1$*;*

(2) *there is a $p > 1$ such that $\frac{\Phi(t)}{t^p}$ increases;*

(3) *there are $p, q > 1$ such that $\frac{\Phi(t)}{t^p}$ increases near the origin and $\frac{\Phi(t)}{t^q}$ increases at infinity;*

(4) *there is a $p > 1$ such that for every $t > 0$ and every $c \in (0, 1]$, $\Phi(ct) \leqslant c^p\Phi(t)$.*

Proof. The equivalence of (1) and (2) was established in Proposition 2.1. Trivially, (2) implies (3). Moreover, $\frac{\Phi(t)}{t^p}$ increases if and only if for any $t > 0$ and any $c \in (0, 1]$,

$$\frac{\Phi(ct)}{(ct)^p} \leqslant \frac{\Phi(t)}{t^p}$$

which is equivalent to (4), hence (2) is equivalent to (4). We will now show that (3) implies (1), yielding the result.

Suppose that (3) holds. Then by assumption, there are $R_1, R_2 > 0$ such that $\Phi(t)$ is increasing for $t \in (0, R_1) \cup (R_2, \infty)$,

$$q_1 = \inf_{t\in(0,R_1)} \left(\frac{t\Phi'_+(t)}{\Phi(t)}\right) \geqslant p > 1 \quad \text{and} \quad q_3 = \inf_{t\in(R_2,\infty)} \left(\frac{t\Phi'_+(t)}{\Phi(t)}\right) \geqslant q > 1.$$

Let $q_2 = \inf_{t\in[R_1,R_2]} \frac{t\Phi'_+(t)}{\Phi(t)}$. We want to show that $q_2 > 1$, which will in turn yield $q_\Phi = \inf\{q_1, q_2, q_3\} > 1$, completing the proof.

Let $\varphi_1(t) = k_1t - m_1$ and $\varphi_2(t) = k_2t - m_2$, with $k_j = \Phi'_+(R_j)$ and m_j chosen so that $\varphi_j(R_j) = \Phi(R_j)$, $j = 1, 2$. Given that Φ is a Young function, is convex, and fulfills (3), it is clear that $k_1 \leqslant k_2$, $m_1 \leqslant m_2$ and $m_j > 0$ for $j = 1, 2$.

We now approximate $\Phi(t)$ with linear segments forming polygonal chains for $R_1 \leqslant t \leqslant R_2$. Pick points $R_1 = t_0 < t_1 < \cdots < t_n = R_2$ and define functions $f_j(t) = a_jt - b_j$ such that $f_j(t_j) = \Phi(t_j)$ and $f_j(t_{j+1}) = \Phi(t_{j+1})$. Let $\Phi_n(t)$ be the polygonal chain on $[R_1, R_2]$ formed by connecting the functions f_j, meaning $\Phi_n(t) = f_j(t)$ whenever $t \in [t_j, t_{j+1}]$.

Since Φ is convex and increasing, we have $k_1 \leqslant a_j \leqslant k_2$ and $m_1 \leqslant b_j \leqslant m_2$ for all $j = 1, \dots, n$. Hence, for any $j = 1, \dots, n$,

$$\inf_{t\in[t_j,t_{j+1}]} \left(\frac{t(f_j)'_+(t)}{f_j(t)}\right) = \inf_{t\in[t_j,t_{j+1}]} \left(1 + \frac{b_j}{a_jt_j - b_j}\right) > 1 + \frac{m_1}{\Phi(R_2)},$$

where the last inequality follows from the fact that $b_j > m_1$ and $a_j t_j - b_j = f_j(t_j) \leqslant f_n(t_n) = \Phi(R_2)$. From this, it is clear that

$$q_{\Phi_n} = \inf_{t \in [R_1, R_2]} \left(\frac{t(\Phi_n)'_+(t)}{\Phi_n(t)} \right) > 1 + \frac{m_1}{\Phi(R_2)}$$

independent of the choice of n and the points t_j, $j = 1, \dots n-1$, and therefore

$$q_2 = \lim_{n\to\infty} q_{\Phi_n} \geqslant 1 + \frac{m_1}{\Phi(R_2)} > 1.$$

This gives (1), completing the proof. $\square$

The following proposition shows that the condition $q_\Phi > 1$ cannot be linked to strict convexity for the Young function Φ.

Proposition 2.4. *Let Φ and Ψ be Young functions which are non-zero outside the origin, and let q_Φ be as in* (1.4)*. Then the following is true:*

(1) *if $q_\Phi > 1$, then there is an equivalent Young function to Φ which is strictly convex;*

(2) *Φ can be chosen such that $q_\Phi > 1$ but Φ is not strictly convex;*

(3) *Φ can be chosen such that $q_\Phi = 1$ but Φ is strictly convex.*

Remark 2.5. In [7] it is stated that (1) in Proposition 2.4 can be replaced by

$(1)'$ $q_\Phi > 1$, if and only if Φ is strictly convex.

This is equivalent to that the following conditions should hold:

$(2)'$ if $q_\Phi > 1$, then Φ is strictly convex;

$(3)'$ if Φ is strictly convex, then $q_\Phi > 1$.

(See remark after (1.1) in [7].) Evidently, the assertion in [7] is (strictly) stronger than Proposition 2.4 (1). On the other hand, Proposition 2.4 (2) shows that $(2)'$ can not be true and Proposition 2.4 (3) shows that $(3)'$ can not be true. Consequently, both implications in $(1)'$ are false.

Proof of Proposition 2.4. We begin by proving (1). Therefore assume that $q_\Phi > 1$. Suppose that Φ fails to be strict convex in $(0, \varepsilon)$, for some $\varepsilon > 0$. Then $\Phi''(t) = 0$ when $t \in (0, \varepsilon)$. This implies that $\Phi(t) = ct$ when $t \in (0, \varepsilon)$, for some $c \geqslant 0$, which in turn gives $q_\Phi = 1$, violating the condition $q_\Phi > 1$. Hence Φ must be strict convex in $(0, \varepsilon)$, for some choice of $\varepsilon > 0$.

Let

$$\Psi(t) = \int_0^t \Phi(t-s)e^{-s}\, ds.$$

Then

$$\Psi''(t) = \Phi'(0) + \int_0^t \Phi''(t-s)e^{-s}\, ds \geq \int_{t-\varepsilon}^t \Phi''(t-s)e^{-s}\, ds > 0,$$

since $\Phi''(t-s) > 0$ when $s \in (t-\varepsilon, t)$. This shows that Ψ is a strictly convex Young function.

Since Φ is increasing we also have

$$\Psi(t) \leqslant \Phi(t),$$

because

$$\Psi(t) = \int_0^t \Phi(t-s)e^{-s}\, ds \leq \Phi(t) \int_0^t e^{-s}\, ds \leq \Phi(t) \int_0^\infty e^{-s}\, ds = \Phi(t).$$

This implies that

$$\Phi_1(t) \equiv \Phi(t) + \Psi(t)$$

is equivalent to $\Phi(t)$. Since Ψ is strictly convex, it follows that Φ_1 is strictly convex as well. Consequently, Φ_1 fulfills the required conditions for the searched Young function, and (4) follows.

In order to prove (2), we choose

$$\Phi(t) = \begin{cases} 2t^2, & \text{when } t \leqslant 1 \\ 4t-2, & \text{when } 1 \leqslant t \leqslant 2 \\ t^2+2, & \text{when } t \geqslant 2 \end{cases}$$

which is not strictly convex. Then

$$\begin{aligned} q_\Phi &= \inf_{t>0} \left(\frac{t\Phi'(t)}{\Phi(t)} \right) \\ &= \min \left\{ \inf_{t \leqslant 1} \left(\frac{4t^2}{2t^2} \right), \inf_{1 \leqslant t \leqslant 2} \left(\frac{4t}{4t-2} \right), \inf_{t \geqslant 2} \left(\frac{2t^2}{t^2+2} \right) \right\} = \frac{4}{3} > 1, \end{aligned}$$

which shows that Φ satisfies all the searched properties. This gives (2).

Next we prove (3). Let

$$\Phi(t) = t\ln(1+t), \quad t \geqslant 0.$$

Then Φ is a Young function, and it follows by straight-forward computations that $q_\Phi = 1$. We also have $\Phi''(t) > 0$, giving that Φ is strictly convex. Consequently, Φ satisfies all searched properties, and (3) follows.

This gives the result. □

3. Continuity for pseudo-differential operators, Fourier multipliers, and Fourier integral operators on Orlicz spaces

In this section we extend properties on L^p continuity for various types of Fourier type operators into continuity on Orlicz spaces. Especially we perform such extensions for Hörmander's improvement of Mihlin's Fourier multiplier theorem (see Theorem 3.4). We also deduce Orlicz space continuity for suitable classes of pseudo-differential and Fourier integral operators (see Theorems 3.3 and 3.5). Our investigations are based on a special case of MarcinKiewicz type interpolation theorem for Orlicz spaces, deduced in [7].

We now recall the following interpolation theorem on Orlicz spaces, which is a special case of [7, Theorem 5.1].

Proposition 3.1. *Let Φ be a strict Young function and $p_0, p_1 \in (0, \infty]$ are such that $p_0 < q_\Phi \leqslant p_\Phi < p_1$, where q_Φ and p_Φ are defined in* (1.4) *and* (1.3)*. Also let*

$$T : L^{p_0}(\mathbf{R}^d) + L^{p_1}(\mathbf{R}^d) \to wL^{p_0}(\mathbf{R}^d) + wL^{p_1}(\mathbf{R}^d) \tag{3.1}$$

be a linear and continuous map which restricts to linear and continuous mappings

$$T : L^{p_0}(\mathbf{R}^d) \to wL^{p_0}(\mathbf{R}^d) \qquad \text{and} \qquad T : \ L^{p_1}(\mathbf{R}^d) \to wL^{p_1}(\mathbf{R}^d).$$

Then (3.1) *restricts to linear and continuous mappings*

$$T : L^\Phi(\mathbf{R}^d) \to L^\Phi(\mathbf{R}^d) \qquad \text{and} \qquad T : wL^\Phi(\mathbf{R}^d) \to wL^\Phi(\mathbf{R}^d). \tag{3.2}$$

Remark 3.2. Let Φ and T be the same as in Proposition 3.1. Then the continuity of the mappings in (3.2) means

$$\|Tf\|_{L^\Phi} \lesssim \|f\|_{L^\Phi}, \qquad f \in L^\Phi(\mathbf{R}^d)$$

and

$$\|Tf\|_{wL^\Phi} \lesssim \|f\|_{wL^\Phi}, \quad f \in wL^\Phi(\mathbf{R}^d).$$

A combination of Propositions 1.11 and 3.1 gives the following result on continuity properties for pseudo-differential operators on L^Φ-spaces.

Theorem 3.3. *Let Φ be a strict Young function, $A \in \mathbf{M}(d, \mathbf{R})$ and $a \in S^0_{1,0}(\mathbf{R}^{2d})$. Then*

$$\operatorname{Op}_A(a) : L^\Phi(\mathbf{R}^d) \to L^\Phi(\mathbf{R}^d) \quad \text{and} \quad \operatorname{Op}_A(a) : wL^\Phi(\mathbf{R}^d) \to wL^\Phi(\mathbf{R}^d)$$

are continuous.

Proof. By Proposition 2.1 it follows that $q_\Phi > 1$ and $p_\Phi < \infty$. Choose $p_0, p_1 \in (1, \infty)$ such that $p_0 < q_\Phi$ and $p_1 > p_\Phi$. In view of Remark 1.2 and Proposition 1.11,

$$\|\operatorname{Op}(a)f\|_{wL^{p_j}} \leqslant \|\operatorname{Op}(a)f\|_{L^{p_j}} \leqslant C\|f\|_{L^{p_j}}, \quad f \in L^{p_j}(\mathbf{R}^d),\ j = 0, 1. \tag{3.3}$$

Then it follows that $\operatorname{Op}_A(a)$ extends uniquely to a continuous map from $L^{p_0}(\mathbf{R}^d) + L^{p_1}(\mathbf{R}^d)$ to $wL^{p_0}(\mathbf{R}^d) + wL^{p_1}(\mathbf{R}^d)$ (see e. g. [2]). Hence the conditions of Proposition 3.1 are fulfilled and the result follows. □

By using Proposition 1.12 instead of Proposition 1.11 in the previous proof we obtain the following extension of Hörmander's improvement of Mihlin's Fourier multiplier theorem. The details are left for the reader.

Theorem 3.4. *Let Φ be a strict Young function and $a \in L^\infty(\mathbf{R}^d \setminus 0)$ be such that*

$$\sup_{R>0} \left(R^{-d+2|\alpha|} \int_{A_R} |\partial^\alpha a(\xi)|^2 \, d\xi \right) \tag{3.4}$$

is finite for every $\alpha \in \mathbf{N}^d$ with $|\alpha| \leqslant [\frac{d}{2}] + 1$, where A_R is the annulus $\{\, \xi \in \mathbf{R}^d \,;\, R < |\xi| < 2R \,\}$. Then $a(D)$ is continuous on $L^\Phi(\mathbf{R}^d)$ and on $wL^\Phi(\mathbf{R}^d)$.

Finally, employing Theorem 1.13, we prove the following continuity result for Fourier integral operators on L^Φ-spaces.

Theorem 3.5. *Let Φ be a strict Young function, $\varphi \in \mathfrak{P}^{\mathrm{hom}}_r$ a phase function, $a \in S^{m,\mu}(\mathbf{R}^{2d})$ an amplitude function such that*

$$m < \mathfrak{T}_{d,\Phi} \text{ and } \mu < \mathfrak{T}_{d,\Phi}, \tag{3.5}$$

where

$$\mathfrak{T}_{d,\Phi} = -(d-1) \max \left\{ \left| \frac{1}{p_\Phi} - \frac{1}{2} \right|, \left| \frac{1}{q_\Phi} - \frac{1}{2} \right| \right\}.$$

Moreover, assume that $|\xi| \geq \varepsilon$ on the support of a, for some $\varepsilon > 0$. Then,

$$\operatorname{Op}_\varphi(a) \colon L^\Phi(\mathbf{R}^d) \to L^\Phi(\mathbf{R}^d) \quad \text{and} \quad \operatorname{Op}_\varphi(a) : wL^\Phi(\mathbf{R}^d) \to wL^\Phi(\mathbf{R}^d)$$

are continuous.

Remark 3.6. Notice the strict inequality in (3.5), differently from condition (1.14) in Theorem 1.13 for the L^p-boundedness of the Fourier integral operators in (1.13). The sharpness of condition (3.5) will be investigated elsewhere.

Proof. As above, by Proposition 2.1 it follows that $q_\Phi > 1$ and $p_\Phi < \infty$. Choose $p_0, p_1 \in (1, \infty)$ such that $p_0 < q_\Phi$ and $p_1 > p_\Phi$, and, as it is possible, by continuity and the hypothesis (3.5), such that

$$m < -(d-1)\left|\frac{1}{p_j} - \frac{1}{2}\right| \text{ and } \mu < -(d-1)\left|\frac{1}{p_j} - \frac{1}{2}\right|, \quad j = 0, 1.$$

In view of Remark 1.2 and Theorem 1.13,

$$\|\mathrm{Op}_\varphi(a)f\|_{wL^{p_j}} \leqslant \|\mathrm{Op}_\varphi(a)f\|_{L^{p_j}} \leqslant C\|f\|_{L^{p_j}}, \quad f \in L^{p_j}(\mathbf{R}^d), \ j = 0, 1. \tag{3.6}$$

By Proposition 3.1, the claim follows, arguing as in the final step of the proof of Theorem 3.3. □

References

[1] J. Appell, A. Kalitvin, P. Zabreiko *Partial integraloperators in Orlicz spaces with mixed norm* in: Colloquium Mathematicum, **78**, 1998, pp. 293–306.

[2] J. Bergh, J. Löfström *Interpolation Spaces, An Introduction*, Springer-Verlag, Berlin Heidelberg NewYork, 1976.

[3] S. Coriasco, M. Ruzhansky *Global L^p-continuity of Fourier Integral Operators*, Trans. Amer. Math. Soc. **366**, 5 (2014), 2575–2596.

[4] P. Harjulehto, P. Hästö, *Orlicz Spaces and Generalized Orlicz Spaces* Springer, (2019).

[5] L. Hörmander *Estimates for translation invariant operators in L^p spaces*, Acta Math. **104** (1960), 93–140.

[6] L. Hörmander *The Analysis of Linear Partial Differential Operators*, vol I–III, Springer-Verlag, Berlin Heidelberg NewYork Tokyo, 1983, 1985.

[7] PeiDe Liu, MaoFa Wang *Weak Orlicz spaces: some basic properties and their applications to harmonic analysis*, Science China Mathematics, **56**, Springer, 2013, 789–802.

[8] W. A. Majewski, L. E. Labuschagne *On applications of Orlicz spaces to statistical physics*, Ann. Henri Poincaré **15** (2014), 1197–1221.

[9] W. A. Majewski, L. E. Labuschagne *On entropy for general quantum systems*, Adv. Theor. Math. Phys. **24** (2020), 491–526.

[10] S. G. Michlin *Fourier integrals and multiple singular integrals* (Russian) Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **12** (1957), 143–155.

[11] M. Milman *A note on $L(p, q)$ spaces and Orlicz spaces with mixed norms*, Proc. Ame. Math. Soc. **83** 1981, 743–746.

[12] W. Orlicz *Über eine gewisse Klasse von Räumen vom Typus B*, (German) Bull. Int. Acad. Polon. Sci. A 1932, 207–220 (1932).

[13] A. Osançliol, S. Öztop *Weighted Orlicz algebras on locally compact groups*, J. Aust. Math. Soc. **99** (2015), 399–414.

[14] M. M. Rao, Z. D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, New York, 1991.

[15] C. Schnackers, H. Führ *Orlicz Modulation Spaces*, Proceedings of the 10th International Conference on Sampling Theory and Applications.

[16] A. Seeger, C.D. Sogge, E.M. Stein, *Regularity properties of Fourier integral operators*, Ann. of Math. **134** (1991), 231–251.

[17] M. Shubin *Pseudodifferential operators and the spectral theory*, Springer Series in Soviet Mathematics, Springer Verlag, Berlin 1987.

[18] J. Toft *Gabor analysis for a broad class of quasi-Banach modulation spaces* in: S. Pilipović, J. Toft (eds), *Pseudo-differential operators, generalized functions*, Operator Theory: Advances and Applications **245**, Birkhäuser, 2015, 249–278.

[19] J. Toft, R. Üster *Pseudo-differential operators on Orlicz modulation spaces* J. Pseudo-Differ. Oper. Appl. **14** (2023), Paper no. 6.

[20] J. Toft, R. Üster, E. Nabizadeh and S. Öztop, *Continuity and Bargmann mapping properties of quasi-Banach Orlicz modulation spaces* Forum. Math. **34** (2022), 1205–1232.

[21] G. Tranquilli *Global normal forms and global properties in function spaces for second order Shubin type operators* PhD Thesis, 2013.

[22] M. W. Wong *An introduction to pseudo-differential operators*, Ser. Anal. Appl. Comput. **6**, World Scientific, Hackensack, 2014.

Dipartimento di Matematica "G. Peano", Universitá degli Studi di Torino
Email address: `matteo.bonino@unito.it`

Dipartimento di Matematica "G. Peano", Universitá degli Studi di Torino
Email address: `sandro.coriasco@unito.it`

Department of Mathematics, Linnæus University, Sweden
Email address: `albin.petersson@lnu.se`

Department of Mathematics, Linnæus University, Sweden
Email address: `joachim.toft@lnu.se`