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Article Modeling Excitable Cells with Memristors

Maheshwar Sah^{1,2*}, Alon Ascoli³, Ronald Tetzlaff^{4*}, Vetriveeran Rajamani⁵ and Ram Kaji Budhathoki⁶

- ¹ Department of Electronics and Communication Engineering, Nepal Engineering College, Changunarayan, Bhaktapur, , Nepal, e-mail: sahmaheshwar@gmail.com
- TJX Evansville, Indiana, USA
- ³ Department of Electronics and Telecommunications, Politecnico di Torino, 10129 Turin, Italy; e-mail: alon.ascoli@polito.it
- ⁴ Faculty of Electrical and Computer Engineering, Institute of Circuits and Systems, Technische Universität Dresden, Dresden, Germany; e-email: ronald.tetzlaff@tu-dresden.de
- ⁵ School of Electronics Engineering (SENSE), Vellore Institute of Technology, Vellore, Tamilnadu, India; e-mail: vetriece86@gmail.com
- ⁶ Department of Electrical and Electronics Engineering, Kathmandu University, Dhulikhel, Kavre, Nepal; e-mail: ramkaji@gmail.com
- * Correspondence: tetzlaff@tu-dresden.de; sahmaheshwar@gmail.com

Abstract: This paper presents an in-depth analysis of an excitable membrane of a biological system 18 by proposing a novel approach that the cells of excitable membrane can be modeled as the networks 19 of memristors. We provide compelling evidence from the Chay neuron model that the state-inde-20 pendent mixed ion-channel behaves as a nonlinear resistor, while the state-dependent voltage-sensitive 21 potassium ion-channel and calcium-sensitive potassium ion-channel function as generic memristors from 22 the perspective of electrical circuit theory. The mechanism to give the rise to the periodic oscillation, 23 aperiodic (chaotic) oscillation, spikes and bursting in an excitable cell are also analyzed via small-24 signal model, pole-zero diagram of admittance functions, local activity, edge of chaos and Hopf 25 bifurcation theorem. It is also proved that the zeros of the admittance functions are equivalent to 26 the eigen values of the Jacobian matrix and the presence of the positive real parts of the eigen values 27 between the two bifurcations points lead to the generation of complicated electrical signals in an 28 excitable membrane. The innovative concepts outlined in this paper pave the way for a deeper un-29 derstanding of the dynamic behavior of excitable cells, offering potent tools for simulating and ex-30 ploring the fundamental characteristics of biological neurons. 31

Keywords: Memristor; excitable cell; oscillation; chaos; spikes; bursting; Chay model; small-signal model; pole-zero diagram; local activity; edge of chaos; Hopf bifurcation

1. Introduction

The electrical activities of neurons are characterized by a diverse array of dynamic 35 phenomena, such as action potential, oscillation, spike, chaos and bursting. Understand-36 ing these qualitative features are essential for unraveling the principles underlying neu-37 ronal excitability. The popular Hodgkin-Huxley (HH) model developed in 1952 consists 38 of membrane voltage, potassium conductance, sodium conductance and leakage conduct-39 ance, provide a framework for understanding the propagation of action potential based 40on the squid giant axon experiments [1]. Recent analysis revealed that the potassium ion-41 channel and sodium ion-channel in the HH model, initially interpreted as time-varying 42 conductances are in fact generic memristors from the perspective of electrical circuit the-43 ory [2-5]. The HH model has spurred significant interests to design electrical circuit mod-44 els and observe the experimental results in the wide varieties of complex system of the 45 membrane potential, nervous system, barnacle giant muscle fiber, Purkinje fibers, solitary 46 hair cells, auditory periphery, mini review of neuromorphic architectures and implemen-47 tation, organic synapses for neuromorphic electronics, and photochromic compounds [6-48

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Copyright: © 2024 by the authors. Submitted for possible open access publication under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/license s/by/4.0/). 16]. Similarly, extensive researches have been conducted to observe the varieties of oscil-49 lations in pancreatic β -cells inspired by the HH model [17-26]. The mathematical model 50 of an excitable membrane in pancreatic β -cells consist of voltage-sensitive channels that 51 allow the Na⁺ and Ca²⁺ to enter the cell and, voltage-sensitive K⁺ channels and voltage-52 insensitive K+ channel which allow to leave K⁺ ion and activate intracellular calcium ion 53 respectively [27-29]. Therefore, the outward current carried by K^+ ions pass through the 54 voltage and calcium-sensitive channels, and inward current carried by Na⁺ and Ca²⁺ ions 55 pass through the voltage-sensitive Na⁺ and Ca²⁺ channels. However, the above models 56 consist of complicated nonlinear differential equations associated with membrane volt-57 age. Later a modified model was presented by Chay [30], assuming the β -cells of the volt-58 age-sensitive Na⁺ conductance is almost inactive, and the inward current is almost exclu-59 sively carried by Ca^{2+} ions through the voltage-sensitive Ca^{2+} channel. Therefore, the as-60 sumption of a mixed effective conductance was formulated without affecting the results 61 by expressing the total inward current in terms of a single mixed conductance g_I, and re-62 versal potential E1 of the two functionally independent Na⁺ and Ca²⁺ channels. The model 63 consists of three nonlinear differential equations in contrast to the other complicated mod-64 els of an excitable membrane of pancreatic β -cell. Our studies in this paper typically focus 65 on the simplified Chay neuron model of an excitable cell [30]. 66

The scientific novelty of this study is to model the excitable cells using memristive 67 theory. By characterizing the state-independent voltage-sensitive mixed ion channel gi as 68 a nonlinear resistor, and the state-dependent voltage-sensitive potassium ion-channel gK,V 69 and calcium-sensitive potassium ion-channel $g_{K,Ca}$ as time-invariant memristors in the 70 Chay neuron model, this research introduces a novel approach to study the behavior of 71 ion channels in excitable cells. This unique modeling framework extends memristive the-72 ory to the realm of neuroscience, opening up new avenues for investigating the complex 73 dynamics of excitable cells and their role in neural information processing. Moreover, the 74 study employs comprehensive analytical tools such as small signal equivalent circuit 75 model, pole-zero diagrams, the local activity principle, edge of chaos theory, and Hopf 76 bifurcation theorem with the goal of gaining deeper insights in to the dynamic behavior 77 the excitable cells. By integrating these analytical tools, the study provides a comprehen-78 sive perspective on the dynamic behavior of excitable cells in the framework of memris-79 tive theory, potentially uncovering new insights and relationships that were previously 80 unexplored. The contributions of the study have the potential to advance our understand-81 ing of excitable cell dynamics and their implications for neural function. 82

The paper is organized as follows. We introduce the Chay neuron model and its comparison analyses with HH, FitzHugh-Nagumo and Morris-Lecar(ML) models in section 2. Section 3 describes the pinched hysteresis fingerprints of ion-channel memristors. Section 4 presents Direct Current (DC) analysis of Memristive Chay neuron model. Section 5 provides the small-signal analysis. Section 6 explores the application of the local activity principle, edge of chaos theorem, and Hopf bifurcations in memristive Chay neuron. Finally, section 7 concludes the paper.

2. Chay Neuron Model of an Excitable Cell

Excitable cells are specialized cells in the body and neurons that are capable of 91 generating electrical signals in response to certain stimuli. These cells are crucial for the 92 functioning of various physiological processes, including nerve signaling, muscle 93 contraction, and hormone release. Excitability in these cells is primarily due to the 94 presence of specialized proteins called ion-channels in their cell membranes. These ion 95

channels control the movement of ions such as sodium (Na⁺), potassium (K⁺), calcium 96 (Ca^{2+}) , and chloride (Cl^{-}) across the membrane, leading to changes in the cell's membrane 97 potential and the generation of electrical signals. The study of excitable cells encompasses 98 a wide array of topics, and our primary aim is to present a unified model for both neuronal 99 and secretory excitable membranes based on the Chay neuron model. The Chay neuron 100 model, which focuses on a simplified representation of neuronal and secretory excitable 101 membranes, aims to provide a unified framework for understanding the complex 102 electrical activity observed in excitable cells. This model typically involves just three 103 ordinary differential equations(ODEs) to capture the essential features of an excitable cell 104 membrane. The model consists of (a) mixed ion-channel g_l (b) the state-dependent voltage-105 sensitive potassium ion-channel $g_{K,V}$ (c) calcium-sensitive potassium ion-channel $g_{K,Ca}$ 106 and (d) leakage channel are described by the following differential equation: 107

$$\frac{dV}{dt} = \frac{I - g_I m_{\infty}^3 h_{\infty} (V - E_I) - g_{K,V} n^4 (V - E_K) - g_{K,Ca} \frac{Ca}{1 + Ca} (V - E_K) - g_L (V - E_L)}{C_m}$$
(1a) 108

$$\frac{dn}{dt} = \frac{n_{\infty} - n}{\tau_n} \tag{1b} \quad 109$$

$$\frac{dCa}{dt} = -\rho \left[m_{\infty}^3 h_{\infty} \left(V - E_{Ca} \right) + k_{Ca} Ca \right]$$
(1c) 110

where

$$n_{\infty} = \frac{\alpha_n}{\alpha_n + \beta_n}, \ \alpha_n = \frac{0.01(V + 20)}{1 - e^{-0.1(V + 20)}}, \ \beta_n = 0.125e^{\left(\frac{-(V + 30)}{80}\right)}$$
(1d) 112

$$m_{\infty} = \frac{\alpha_m}{\alpha_m + \beta_m}, \ \alpha_m = \frac{0.1(V + 25)}{1 - e^{-0.1(V + 25)}}, \ \beta_m = 4e^{\left(\frac{-(V + 50)}{18}\right)}$$
(1e) 113

$$h_{\infty} = \frac{\alpha_{h}}{\alpha_{h} + \beta_{h}}, \ \alpha_{h} = 0.07e^{\left(\frac{-(V+50)}{20}\right)}, \ \beta_{h} = \frac{1}{1 + e^{-0.1(V+20)}}, \ \tau_{n} = \frac{1}{\lambda_{n}(\alpha_{n} + \beta_{n})}$$
(1f) 114

Fig. 1(a) shows the typical circuit of Chay model with external current stimulus, 115 denoted as I^1 . It consists of membrane potential V of capacitance C_m , potentials E_I , E_K and 116 E_L for mixed Na^+-Ca^{2+} ions, K^+ and leakage ions respectively. The conductances g_I , $g_{K,V}$, 117 $g_{K,Ca}$, and g_L , represent the voltage-sensitive mixed ion-channel, voltage-sensitive potassium 118 ion-channel, calcium-sensitive potassium ion-channel and leakage channel respectively. In the 119

¹Electrical model is not given in the Chay paper [30]. We have designed a typical electrical circuit model following the differential equation of the membrane potential. The symbolic representation of the conductances and potentials are assumed in different notations compared to the original representation. Fig. 1(a) shows an electrical circuit model following the conventional assumption of HH model.

upcoming session, we will provide a rigorous proof that the state independent mixed ion-120 channel functions as a nonlinear resistor. However, the commonly held belief regarding 121 state-dependent ion-channels exhibiting time-varying conductances is found to be 122 conceptually incorrect from the perspective of electrical circuit theory. Contrary to this 123 conventional assumption, these ion-channels do not adhere to time-varying conductance 124 principles. Instead, they align more accurately with the characteristics of time-invariant 125 generic memristors from a circuit theoretic standpoint. A rigorous proof will be 126 demonstrated in the subsequent section. The parameters for this model are summarized 127 in Table 1² and list of abbreviations of the model parameters are illustrated in Appendix. 128 The comparison analyses of the HH model[1], FitzHugh-Nagumo model[31], ML 129 model[8], and the Chay model[30] are sumarized in Table 2 along with their respective 130 strengths and limitations. It is notable that each model possesses distinct advantages and 131 drawbacks making them suitable for different research contexts and questions. The choice 132 of model depends on the level of detail required, computational resources available, and 133 specific phenomena under investigation. This study predominantly centers on the Chay 134 neuron model of excitable cells. 135



Figure 1. Typical Chay neuron model of an excitable cell [30]. (a) Electrical circuit model, following137conventional assumption as time varying conductances [1]. (b) Equivalent memristive Chay model138based on Chua's memristive theory [2-4]. The potential E_{Ca} for Ca^{2+} ion given in the rate of the139calcium concentration in (1c) is not an external battery source and not shown in external Fig. 1(a)140and Fig. 1(b), respectively.141

² The unit of conductances of mixed ion-channel, voltage sensitive potassium ion-channel, calcium sensitive potassium ion-channel and leakage ion-channel in Chay model [30] are assumed as

 $g^* = \frac{conductance}{membrane capacitance} = \frac{mS / cm^2}{mF / cm^2} = \frac{1}{second(s)} = s^{-1}$. As, we are assuming the value of membrane

capacitance $(C_m)=1mF/cm^2$, we use the unit of all the conductances of the ion-channels $g=mS/cm^2$ throughout this study, which is also the equivalent unit g^* of original Chay model. Due to the periodic and dynamic nature of the conductance $g^*(g)$, it can also be considered as the "conductance periodic factor".

C_m	1mF/cm ²	$g_{K,V}$	1700 mS/cm ²
E_K	-75 mV	g_I	1800 mS/cm ²
E_I	100 mV	g_L	$7 mS/cm^2$
E_L	-40 mV	8 _{K,Ca}	10 mS/cm ²
E_{Ca}	100mV	k_{Ca}	3.3/18 mV
λ_n	230	ρ	0.27

Table 1. Parameters of the Chay neuron model of an exctiable cell

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	Table 2. Comparision analyses of HH, FitzHugh-Nagumo, ML and Chay 1	nodels
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Models	Memristive Models	Strengths	Limitations
HH [1]	Potassium ion-channel and sodium ion-	It is a framework to understand the emergence	Difficult to generalize to all
	channel in the HH model are represented with	of action potential propagation in neuron based	neurons. Incapable of producing
	generic memristors [3]	on the experimental data of squid giant axon.	bursting
FitzHugh-	It doesn't follow the state dependent Ohm's	Simplified model of neuronal excitation.	Not accurately represent all
Nagumo	law and can not model with memristors.		neuronal behaviors. Incapable of
[31]			producing bursting
ML [8]	Modeled that the state independent	Initially presented a model for the barnacle	Limited in capturing certain
	(dependent) calcium ion channel act as a	muscle fiber and later it is considered as a	neuronal dynamics. Can't
	nonlinear resistor(generic memristor) and	popular and simplified representation of the	produce bursting patterns.
	state dependent potassium ion channels acts as	neuron model.	
	a generic memristor [9-10].		
Chav[30]	We are proposing a framework that the cells of	Novel model of excitable cells to capture	Limited validation in
	excitable membranes can be modeled as the	multiple neuronal states, such as action	experimental contexts and lack
	networks of memristors.	potentials, periodic oscillations, aperiodic	details for some applications.
		oscillations, spikes and bursting patterns.	

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3. Pinched Hysteresis Fingerprints of the Ion-Channel Memristor

A generic memristor driven by a current source or voltage source is a two-terminal 147 electrical circuit element whose instantaneous current or voltage obeys a state-dependent 148 Ohm's law. A generic memristor driven by a current source can be expressed as follows 149 150

in terms of state \dot{x}_n :

$$v = R(x_1, x_2, ..., x_n)i$$
 (2a) 151

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$$\dot{x}_n = f_1(x_1, x_2, ..., x_n; i)$$
 (2b) 152

where $R(\mathbf{x})$ is the memristance of the memristor and depends on "n" (n ≥ 1) states variables, ¹⁵³

Similarly, a voltage-controlled memristor is defined in terms of the memductance G(x) 154 and the state variables $x_1, x_2, ..., x_n$, as follows: 155

$$i = G(x_1, x_2, ..., x_n)v$$
 (3a) 156

$$\dot{x}_n = f_1(x_1, x_2, ..., x_n; v)$$
 (3b) ¹⁵⁷

Eqs. (2) and (3) play significant importances to distinguish the memristive and non-158memristive system [32-33]. They provide evidence that the state independent voltage-159sensitive mixed ion-channel functions as a nonlinear resistor and, state dependent voltage-160sensitive potassium ion-channel and calcium-sensitive potassium ion-channel behaves as time-161invariant generic memristors.162

3.1. Voltage-sensitive mixed ion-channel nonlinear resistor

The time varying *voltage sensitive mixed ion-channel* with input voltage v_1 and current i_1 ¹⁶⁵ in the second element (from left) in Fig. 1(a) is given by, ¹⁶⁶ $V - E_I = v_1$ (4a) ¹⁶⁷

and

$$i_I = G_I(m_{\infty}, h_{\infty}) v_I \tag{4b}$$

and the conductance of the voltage sensitive mixed ion channel is given by

$$G_I(m_{\infty}, h_{\infty}) = g_I m_{\infty}^3 h_{\infty} \tag{4c}$$

where m_{∞} and h_{∞} are computed using (1e) and (1f)

$$m_{\infty} = \frac{0.1(v_{I} + E_{I} + 25)}{0.1(v_{I} + E_{I} + 25) + 4(1 - e^{-0.1(v_{I} + E_{I} + 25)})e^{\left(\frac{-(v_{I} + E_{I} + 50)}{18}\right)}}$$

$$(4d) \quad 173$$

$$h_{\infty} = \frac{0.07 \left(1 + e^{-0.1(v_I + E_I + 20)}\right) e^{\left(\frac{-(v_I + E_I + 50)}{20}\right)} + 1}$$
(4e) 174

Observe (4b)–(4e) are not identical to (2a)-2(b) or (3a)–(3b) in terms of state dependent ¹⁷⁵ Ohm's law. Consequently, the time-varying *voltage-sensitive mixed ion-channel* can be ¹⁷⁶ substituted by a nonlinear resistor³ as depicted in the second element (from the left) in ¹⁷⁷ Fig. 1(b). To verify the *voltage-sensitive mixed ion-channel* is a nonlinear resistor, an ¹⁷⁸ extensive numerical simulation for a sinusoidal input voltage source $v_l = 100sin(2\pi ft) mV$ ¹⁷⁹

³ Mixed ion-channel is a nonlinear voltage controlled resistor with conductance $G_l(m_{\infty}, h_{\infty})$ where m_{∞} , and h_{∞} are functions of the voltage v_l across the two-terminal element.

180 is performed for the three different frequencies namely, f=100 Hz, 200 Hz, and 1 KHz, 181 respectively. Fig. 2 shows the corresponding output nonlinear waveform on currente *i* vs. 182 voltage v_i plane for these frequencies, confirming that the mixed ion-channel exhibits the 183 properties of a nonlinear resistor only.

Figure 2. Output waveform plotted on *i* vs. *v* plane when the input voltage $v_1 = 100 \sin(2\pi ft) mV$ is applied 185 with three different frequencies, namely f = 100 Hz, 200 Hz, 1 KHz to the voltage-sensitive mixed ion-186 channel. The output nonlinear waveform observed in Fig. 2 for different frequencies confirm the 187 mixed ion channel is a nonlinear resistor. 188

3.2. Voltage-sensitive potassium ion-channel memristor

191 Let us define the voltage across the voltage-sensitive potassium ion-channel shown in 192 third (from left) element in Fig. 1(a) is $v_{K,V}$ and current is $i_{K,V}$, then 193 $V - E_K = v_{K,V}$ (5a) 194

and current entering to the channel is

$$i_{K,V} = G_{K,v}(n) v_{K,V}$$
 (5b) ¹⁹⁵

where the memductance is given by

$$G_{K,V}(n) = g_{K,V} n^4$$
 (5c) ¹⁹⁷

198 and the state equation describing the channel in terms of *n* can be simplified from 1(b) as,

$$\frac{dn}{dt} = f(n; v_{K,V}) = \lambda_n \left[\frac{0.01 \left(v_{K,V} + E_K + 20 \right)}{1 - e^{-0.1 \left(v_{K,V} + E_K + 20 \right)}} (1 - n) - 0.125 e^{\left(\frac{-\left(v_{K,V} + E_K + 30 \right)}{80} \right)} n \right]$$
(5d)

200 Note that (5b)-(5d) are identical to the voltage controlled generic memristor defined 201 in (3a)-(3b) with first order differential equation. Hence, the time-varying conductance 202 shown in Fig. 1(a) of voltage-sensitive potassium ion-channel is replaced with voltage-203 sensitive potassium ion-channel memristor as shown in the third element (from left) in Fig. 2041(b).

205 We observed the memristive fingerprint of the voltage-sensitive potassium ion-channel 206 memristor by applying sinusoidal bipolar signal under different frequencies. This



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207 property asserts that beyond some frequency f^* , the pinched hysteresis loops 208 characterized by a memristor shrinks to a single-valued function through the origin as 209 frequency $f > f^*$ tends to infinity. To verify this property, a sinusoidal voltage source 210 $v_{\mathcal{K},\mathcal{V}}(t)=100sin(2\pi ft)mV$ is applied to the voltage-sensitive potassium ion-channel with 211 frequencies f=100 KHz, 500 KHz, and 4 MHz respectively. As shown in Fig. 3, the zero 212 crossing pinched hysteresis loops shrink as the frequencies increase and tend to a straight 213 line at 4 MHz which confirms that the voltage-sensitive potassium ion-channel is a generic 214 memristor. All of these pinched hysteresis loops exhibit the fingerprints of a memristor 215 [33].



Figure 3. Pinched hysteresis loops of *voltage-sensitive potassium ion-channel* memristor at frequencies217f=100 KHz, 500 KHz, and 4 MHz for the input signal $v_{K,V}(t) = 100sin(2\pi ft) mV$.218

3.3. *Calcium-sensitive potassium ion-channel memristor*

Let us consider the input voltage of the *calcium-sensitive potassium ion-channel*, the ²²¹ fourth element (from left) in Fig. 1(a) is $v_{\mathcal{K},\mathcal{C}_d}$ ⁴ and current is $i_{\mathcal{K},\mathcal{C}_d}$ then the current ²²² entering to the channel is given by ²²³

$$i_{K,Ca} = G_{K,Ca}(Ca)v_{K,Ca}$$
(6a) ²²⁴

where

$$V - E_{K} = v_{K,Ca}$$
 (6b) ²²⁶

and the memductance of the calcium-sensitive potassium channel is given by

$$G_{K,Ca}(Ca) = g_{K,Ca} \frac{Ca}{1+Ca}$$
 (6c) ²²⁸

⁴ Since the same potential E_{κ} is shared by the *voltage-sensitive potassium ion-channel* memristor and *calcium- sensitive potassium ion-channel* memristor, the voltage assumed $V-E_{\kappa} = v_{\kappa,V}$ in (5a) and $V-E_{\kappa} = v_{\kappa,Ca}$ in (6b) are identical. The voltages $v_{\kappa,V}$ and $v_{\kappa,Ca}$ are assumed to distinguish the input voltage applied to *voltage-sensitive potassium ion-channel* memristor and *calcium- sensitive potassium ion-channel* memristor.

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229 The state equation in terms of calcium concentration from 6(b) and (1c) is given by,

$$\frac{dCa}{dt} = f\left(Ca; V_{K,Ca}\right) = -\rho \left[m_{\infty}^{3} h_{\infty} \left(v_{K,Ca} + E_{K} - E_{Ca} \right) + k_{Ca} Ca \right]$$
(6d) 230

231 Observe that (6b)-(6d) are examples of a voltage-controlled memristor defined in 232 (3a)–(3b) in terms of the calcium concentration channel Ca. Since only one state equation 233 is defined in terms of Ca, we call this memristor as a first order calcium-sensitive potassium 234 ion-channel generic memristor. Therefore, the time varying calcium-sensitive potassium ion-235 channel is replaced with calcium-sensitive potassium ion-channel memristor as shown in the 236 fourth element (from left) in Fig. 1(b).

237 Let us verify the fingerprint of the frequency-dependent pinched hysteresis loops of 238 the calcium-sensitive potassium ion-channel by applying sinusoidal voltage source $v_{KCa}(t) =$ 239 100sin(2*π*ft) mV with frequencies f=10 Hz, 30 Hz and 200 Hz respectively. Observe from 240 Fig. 4 that, all the zero crossing pinched hysteresis loops shrink as the frequencies of the 241 input signal increase and tend to a straight line for the frequency f=200 Hz. All of the 242 pinched hysteresis fingerprint confirm that the calcium-sensitive potassium ion-channel is a 243 generic memristor.



Figure 4. Pinched hysteresis loops of calcium-sensitive potassium ion-channel memristor at frequencies 245 246 f = 10Hz, 30Hz and 200Hz for the input signal $v_{K,Ca}(t) = 100sin(2\pi ft) mV$.

4. DC analysis of Memristive Chay Model of an Excitable Cell

249 The primary objectives to analyze the DC behavior of the memristive Chay model is 250 to identify its equilibrium points of the nonlinear equations. These equilibrium points rep-251 resent the steady-state solutions obtained by equating the rate of change of equilibrium 252 voltage V_{m_r} gate activation n of the voltage-sensitive potassium ion-channel memristor and 253 concentration of calcium-sensitive Ca of the calcium sensitive potassium ion-channel memris-254 tor to zero from (1a), (1b) and (1c) respectively. By determining these equilibrium points,

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insights can be gained into the behavior of the excitable cell under different conditions, ²⁵⁵ such as varying input stimuli or parameter values, and can be expressed as a function of ²⁵⁶ current *I as*: ²⁵⁷

$$n = n_{\infty}(V_m) \triangleq \hat{n}(V_m) \tag{7a} 258$$

$$Ca = Ca_{\infty}(V_m) \triangleq C\hat{a}(V_m)$$
(7b) 259

$$I = g_I m_{\infty}^3 h_{\infty} \left(V_m - E_I \right) + g_{K,V} \hat{n}^4 \left(V_m - E_K \right) + g_{K,Ca} \frac{C\hat{a}}{1 + C\hat{a}} \left(V_m - E_K \right) + g_L \left(V_m - E_L \right)$$
(7c) 260



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Figure 5. (a) Memristive DC Chay model at equilibrium voltage V_m . (b) DC V-I curve of mixed ion-262channel nonlinear resistor at equilibrium voltage $V_{I=}V_m$ - E_I . (c) DC V-I curve of voltage sensitive263potassium ion-channel memristor at equilibrium voltage $V_{K,V}=V_m$ - E_K . (d) DC V-I curve of calcium264sensitive potassium ion-channel memristor at equilibrium voltage $V_{K,Ca}=V_m$ - E_K . (e) DC V-I curve of265leakage channel at equilibrium voltage $V_L=V_m$ -EL.(f) Plot of DC V-I curve of memristive Chay model266at membrane voltage V_m .267

The external current I expressed as the function of membrane voltage V_m in (7c) gives269the *explicit* formula of the DC V-I curve of the memristive Chay model. We have plotted270

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the individual DC V-I curve of voltage sensitive mixed ion channel non-linear resistor, 271 voltage sensitive potassium ion-channel memristor, calcium sensitive potassium ion-272 channel memristor and leakage channel at equilibrium voltage VI, VK,V, VK,Ca and VL as 273 shown in the Fig. 5(b), Fig. 5(c), Fig. 5(d) and Fig. 5(e) respectively. Fig. 5(f) shows DC V-I 274 curve of Fig. 5(a) over the range of DC voltage -50 mV <Vm <-24 mV. For any DC value of 275 V_m , we calculated the corresponding value of I as the vertical axis. Our extensive calcula-276 tions show that, the two Hopf bifurcation points occur at V_m =-48.763 mV (resp., I=-66.671 277 μA) and V_m=-27.984 mV (resp., I=433.594 μA) respectively. Details of these two bifurcation 278 points will be discussed in upcoming section 279

5. Small-Signal Circuit Model

The small-signal equivalent circuit is the linearized method to predict the response of281electronic circuits when a small input signal is applied to an equilibrium point *Q*. The282objective of this section is to analyze the small-signal response of voltage-sensitive mixed283ion-channel nonlinear resistor, voltage-sensitive potassium ion-channel memristor and cal-284cium-sensitive potassium ion-channel memristor using Taylor series expansion and Laplace285transformation.286

5.1. Small-signal circuit model of the mixed ion-channel nonlinear resistor

The small signal equivalent circuit of the *mixed ion-channel* nonlinear resistor at an equilibrium point Q_{I} ⁵ on the *DC* V_{I} - I_{I} curve is derived as follows ²⁹⁰

$$v_I = V_I(Q_I) + \partial v_I \tag{6a}$$

$$i_I = I_I(Q_I) + \partial i_I \tag{8b} ^{292}$$

Applying Taylor series expansion to the *voltage-sensitive mixed ion-channel* nonlinear resistor defined in (8a)-(8b) at the DC operating point *Q*_{*l*}, we get 294

$$i_{I} = f(V_{I} + \delta v_{I}) = a_{00}(Q_{I}) + a_{12}(Q_{I})\delta v_{I} + h.o.t.$$

$$= I_{I}(Q_{I}) + \delta i_{I}$$
(8c) 295

Where *h.o.t* denotes higher order terms and coefficients can be computed as,

$$a_{00}(Q_I) = G_I(Q_I)V_I(Q_I) = I_I(Q_I)$$
(8d) 297

$$a_{12}(Q_I) = \frac{\partial f(v_I)}{\partial v_I}$$
(8e) 298

Linearize (8c) by neglecting the h.o.t. then,

$$\delta i_I = a_{12}(Q_I) \delta v_I \tag{8f} \quad \text{300}$$

Taking the Laplace transform of (8f), we obtain

⁵ The equilibrium point Q_l at $v_i = V_l$ is obtained by solving 4(b).

$$\hat{i}_{I}(s) = a_{12}(Q_{I})\hat{v}_{I}(s)$$
 (8g) 302

The admittance $Y_l(s; Q_l)$ of the small-signal equivalent circuit of *the voltage sensitive* 303 *mixed ion-channel* nonlinear resistor at the DC operating point Q_l is given by, 304

$$Y_{I}(s; Q_{I}) = \frac{\hat{i}_{I}(s)}{\hat{v}_{I}(s)} = a_{12}(Q_{I}) = \frac{1}{\frac{1}{a_{12}(Q_{I})}} = \frac{1}{R_{1I}}$$
(8h) 305

where

$$R_{1,I} = 1/a_{12}(Q_I) \tag{8i} \quad 307$$



Figure 6. (a) Small-signal circuit model of the *voltage-sensitive mixed ion-channel* nonlinear resistor 310 about the DC equilibrium point Q_l (V_l , I_l). (b) Plot of the coefficient a_{12} and resistance $R_{1,l}$ as a 311 function of the DC equilibrium voltage $V_l = V_m - E_l$ where $E_l = 100 \text{ mV}$. $R_{1,l} < 0$ over the range of local 312 activity, edge of chaos 1 and edge of chaos 2 of the mixed ion channel nonlinear resistor is identified 313 with respect to V_l of the entire Chay circuit in Fig. 1(b) and Fig. 13 . 314

315

$a_{12}(Q_{I}) = g_{I}\left[m(V_{I})^{3}h(V_{I}) + m(V_{I})^{3}V_{I}\frac{d_{I}h(V_{I})}{dV_{I}} + h(V_{I})V_{I}\frac{d_{I}m(V_{I})^{3}}{dV_{I}}\right] \qquad \qquad$
$h(V_{I}) = \frac{\left(1 + e^{-0.1(V_{I} + E_{I} + 20)}\right)0.07e^{\left(\frac{-(V_{I} + E_{I} + 50)}{20}\right)}}{\left(1 + e^{-0.1(V_{I} + E_{I} + 20)}\right)0.07e^{\left(\frac{-(V_{I} + E_{I} + 50)}{20}\right)} + 1} \qquad \boxed{R_{1,I} = 1/a_{12}(Q_{I})}$ Small-signal Equivalent Circuit
$h_{d}(V_{I}) = \frac{dh(V_{I})}{dV_{I}} = -h(V_{I})^{2} \left(\frac{\frac{0.07}{20} \left(1 + e^{-0.1(V_{I} + E_{I} + 20)}\right) e^{\left(\frac{-(V_{I} + E_{I} + 50)}{20}\right)} + 0.007 e^{\left(\frac{-(V_{I} + E_{I} + 50)}{20}\right)} e^{-0.1(V_{I} + E_{I} + 20)}}{\left(\left(1 + e^{-0.1(V_{I} + E_{I} + 20)}\right)0.07 e^{\left(\frac{-(V_{I} + E_{I} + 50)}{20}\right)}\right)^{2}}\right)$
$\overline{m_d(V_I) = \frac{dm(V_I)^3}{dV_I} = 3m(V_I)^2 \frac{dm(V_I)}{dV_I} = 3m(V_I)^2 m_{dd}(V_I)} = \frac{0.1(V_I + E_I + 25)}{0.1(V_I + E_I + 25) + 4(1 - e^{-0.1(V_I + E_I + 25)})e^{\left(\frac{-(V_I + E_I + 50)}{18}\right)}}$
$m_{dd}(V_{I}) = \frac{-40m_{\infty}^{2}}{(V_{I} + E_{I} + 25)^{2}} \left((V_{I} + E_{I} + 25)m_{ddd} - (1 - e^{-0.1(V_{I} + E_{I} + 25)})e^{\left(\frac{-(V_{I} + E_{I} + 50)}{18}\right)} \right)$
$m_{ddd}(V_I) = \frac{-1}{18} \left(1 - e^{-0.1(V_I + E_I + 25)}\right) e^{\left(\frac{-(V_I + E_I + 50)}{18}\right)} + 0.1e^{\left(\frac{-(V_I + E_I + 50)}{18}\right)} e^{-0.1(V_I + E_I + 25)}$

Table 3. Explicit formulas for computing the coefficients $a_{12}(Q_l)$ of the voltage-sensitive mixed ion317*channel* nonlinear resistor318

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327 328

From (8h), it is followed that the small-signal admittance function of the *mixed ionchannel* nonlinear resistor is equivalent to a linear resistor. The corresponding small-signal equivalent circuit and a plot of the coefficient $a_{12}(Q_l)$ and resistance $R_{1,l}$ as a function of the DC equilibrium voltage $V_l = V_m - E_l$ where $E_l = 100 \text{ mV}$ are shown in Fig. 6(a) and Fig. 6(b), respectively. The explicit formulas for computing coefficient $a_{12}(Q_l)$ are given in Table 3 for readers' convenience. 326

5.2. Small-signal circuit model of the voltage-sensitive potassium ion-channel memristor

The small-signal circuit model of the *voltage sensitive potassium ion-channel* memristor 329 at an equilibrium point $Q_{\mathcal{K},V^6}$ on the DC $V_{\mathcal{K},V}$ - $I_{\mathcal{K},V}$ curve is derived by defining 330

$$n = n_{Q_{KV}} + \delta n \tag{9a} \quad 331$$

$$v_{K,V} = V_{K,V}(Q_{K,V}) + \delta v_{K,V}$$
 (9b) 332

$$i_{K,V} = I_{K,V}(Q_{K,V}) + \delta i_{K,V}$$
 (9c) 333

⁶ The equilibrium point $Q_{K,V}$ at $v_{K,V} = V_{K,V}$ is obtained from (5d) by solving $f(n; V_{K,V}) = 0$ for $n = n_{K,V}$. The explicit formula for $n(V_{K,V})$ is given in Table 4. Expanding $i_{K,V} = G_{K,V}(n)v_{K,V}$ from (5b) in a Taylor series about the equilibrium 334

point
$$(N(Q_{K,V}), V_{K,V}(Q_{K,V}))$$
, we obtain, 335

$$i_{K,V} = a_{00}(Q_{K,V}) + a_{11}(Q_{K,V})\delta n + a_{12}(Q_{K,V})\delta v_{K,V} + h.o.t.$$
(9d) 336

$$=I_{K,V}(Q_{K,V})+\delta i_{K,V}$$

where

$$\delta n = n - n_{Q_{K,V}}, \quad \delta v_{K,V} = v_{K,V} - V_{K,V}(Q_{K,V})$$
338

$$\delta i_{K,V} = i_{K,V} - I_{K,V}(Q_{K,V})$$
(9e) 339

and

$$a_{00}(Q_{K,V}) = G_{K,V}(Q_{K,V}) V_{K,V}(Q_{K,V}) = I_{K,V}(Q_{K,V})$$
(9f) 341

$$a_{11}(Q_{K,V}) = V_{K,V}(Q_{K,V})G'_{K,V}\left(n_{Q_{K,V}}\right)$$
(9g) 342

$$a_{12}(Q_{K,V}) = G_{K,V}(n_{Q_{K,V}})$$
(9h) 343

and h.o.t denotes the higher-order terms. Let us linearize the nonlinear equation by 344 neglecting the h.o.t. in (9d), then: 345

$$\delta i_{K} = a_{11}(Q_{K,V})\delta n + a_{12}(Q_{K,V})\delta v_{K,V}$$
(9i) 346

Similarly, expanding the state equation $f(n_{K,V}, V_{K,V})$ in (5d) using a Taylor series 347 about the equilibrium point $(n(Q_{K,V}), V_{K,V}(Q_{K,V}))$, we obtain 348

$$f(n_{Q_{K,V}} + \delta n, V_{K,V}(Q_{K,V}) + \delta v_{K,V}) = f(n_Q, V_{K,V}(Q_{K,V})) + b_{11}(Q_{K,V}) \delta n + b_{12}(Q_{K,V}) \delta v_{K,V} + h.o.t.$$
(9j) 349

$$\int \langle Q_{K,V}, Y, K,V \rangle \langle \mathcal{L}_{K,V} \rangle \rangle = 0 \prod \langle \mathcal{L}_{K,V} \rangle \rangle \langle \mathcal{L}_{K,V} \rangle \langle \mathcal{L}_{K,V} \rangle \langle$$

where

,

$$b_{11}(Q_{K,V}) = \frac{\partial f_n(n, v_{K,V})}{\partial n}\Big|_{Q_{K,V}}$$
(9k) 351

$$b_{12}(Q_{K,V}) = \frac{\partial f_N(n, v_{K,V})}{\partial v_{K,V}} \bigg|_{Q_{K,V}}$$
(91) 352

Linearizing the nonlinear state equation (9j) by neglecting the h.o.t., we get 353

$$\frac{d(\partial n)}{dt} = b_{11}(Q_{K,V})\,\delta n + b_{12}(Q_{K,V})\,\delta v_{K,V} \tag{9m} 354$$

Taking Laplace transform of (9i) and (9m), we obtain

14 of 41

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$$\hat{i}_{K,V}(s) = a_{11}(Q_{K,V})\hat{n}(s) + a_{12}(Q_{K,V})\hat{v}_{K,V}(s)$$
(9n) 356

$$s \ \hat{n}(s) = b_{11}(Q_{K,V}) \ \hat{n}(s) + b_{12}(Q_{K,V}) \ \hat{v}_{K,V}(s)$$
(90) 357

Solving (90) for $\hat{n}(s)$ and substituting the result into (9n), we obtain the following 358 admittance $Y_{K,V}(s; Q_{K,V})$ for the small-signal equivalent circuit of the *voltage sensitive* 359 *potassium ion-channel* memristor at equilibrium point $Q_{K,V}$: 360

$$Y_{K,V}(s; Q_{K,V}) = \frac{\hat{i}_{K,V}(s)}{\hat{v}_{K,V}(s)} = \left[\frac{1}{\frac{s}{a_{11}(Q_{K,V})b_{12}(Q_{K,V})} - \frac{b_{11}(Q_{K,V})}{a_{11}(Q_{K,V})b_{12}(Q_{K,V})} + \frac{1}{\frac{1}{a_{12}(Q_{K,V})}}\right] (9p) \quad 361$$

362

$$Y_{K,V}(s; Q_{K,V}) = \left(\frac{1}{\left(sL_{K,V} + R_{1K,V}\right)} + \frac{1}{R_{2K,V}}\right)$$
(9q) 363

where
$$L_{K,V} \triangleq \frac{1}{a_{11}(Q_{K,V})b_{12}(Q_{K,V})}$$
 (9r) 364

$$R_{1K,V} \triangleq -\frac{b_{11}(Q_{K,V})}{a_{11}(Q_{K,V})b_{12}(Q_{K,V})}$$
(9s) 365

$$R_{2K,V} \triangleq \frac{1}{a_{12}(Q_{K,V})}$$
 (9t) 366



Figure 7. Small-signal equivalent circuit model of the voltage-sensitive potassium ion-channel368memristor about the DC equilibrium point $Q_{K,V}(V_{K,V,} I_{K,V})$.369

373



Figure 8. Plot of coefficients (a) a_{11} (b) a_{12} (c) b_{11} and (d) b_{12} of the voltage-sensitive potassium ion-371channel memristor as a function of the DC equilibrium voltage $V_{K,V}$.372

It follows from (9r)-(9t) that the small-signal admittance function of the first-order 374 voltage sensitive potassium ion-channel memristor is equivalent to the series connection of 375 an inductor and a resistor in parallel with another resistor as shown in Fig. 7. The 376 corresponding coefficients a_{11} , a_{12} , b_{11} , b_{12} and inductance $L_{K,V}$, resistance $R_{1K,V}$ and resistance 377 $R_{2K,V}$ as a function of the DC equilibrium voltage $V_{K,V} = V_m - E_K$ where $E_K = -75mV$ are shown in 378 Figs. 8 and Figs. 9, respectively. Please note that the local activity, edge of chaos 1 and 379 edge of chaos 2 shown in Figs. 9 are not the local activity and edge of chaos domains of 380 the separate two terminal of the voltage sensitive potassium ion-channel memristor. The 381 small signal positive inductance and resistances (*i.e.* $L_{K,V}>0$, $R_{1K,V}>0$ and, $R_{2K,V}>0$) of the 382 potassium ion-channel memristor observed over the local activity, edge of chaos 1 and 383 edge of chaos 2 regime are just corresponding range of the voltage with respect to $V_{\mathcal{K}\mathcal{V}}$ of 384 the entire connected Chay small-signal equivalent circuit of Fig. 1(b) and Fig. 13. For the 385 readers' convenience, the explicit formulas for computing the coefficients $a_{11}(Q_{K,V})$, 386 $a_{12}(Q_{K,V})$, $b_{11}(Q_{K,V})$, $b_{12}(Q_{K,V})$ and $L_{K,V}$, $R_{1K,V}$, $R_{2K,V}$ are summarized in Table 4. 387



Figure 9. (a) Inductance $L_{K,V}$ (b) resistance $R_{1K,V}$ and (c) resistance $R_{2K,V}$ of the voltage-sensitive391potassium ion-channel memristor as a function of DC equilibrium voltage $V_{K,V} = V_m - E_K$ where $E_K = -75mV$.392 $L_{K,V}>0$, $R_{1KV}>0$ and $R_{2KV}>0$ shown in figures over the local activity, edge of chaos 1 and edge of chaos3932 are just corresponding range of the voltage with respect to $V_{K,V}$ of the entire connected Chay small394signal equivalent circuit of Fig. 1(b) and Fig. 13.395

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Table 4: Explicit formulas for computing the coefficients $a_{11}(Q_{K,V})$, $a_{12}(Q_{K,V})$, $b_{11}(Q_{K,V})$, $b_{12}(Q_{K,V})$ and403 $L_{K,V}$, $R_{1K,V}$, $R_{2K,V}$ of the voltage sensitive potassium ion-channel memristor.404

$\boxed{a_{11}(Q_{K,V}) = 4g_{K,V}n^3(V_{K,V})V_{K,V}}_{a_{12}(Q_{K,V}) = g_{K,V}n^4(V_{K,V})}_{a_{12}(Q_{K,V}) = g_{K,V}n^4(V_{K,V})} = \frac{\alpha_n(V_{K,V})}{(\alpha_n(V_{K,V}) + \beta_n(V_{K,V}))}$	
$\alpha_n(V_{K,V}) = \frac{0.01(V_{K,V} + E_K + 20)}{1 - e^{-0.1(V_{K,V} + E_K + 20)}} \qquad $	
$\beta_{n}(V_{K,V}) = 0.125e^{\left(\frac{-(V_{K,V} + E_{K} + 30)}{80}\right)} \qquad b_{11}(Q_{K,V}) = -\lambda_{n}\left[\alpha_{n}(V_{K,V}) + \beta_{n}(V_{K,V})\right]} \qquad R_{2K,V} = \frac{1}{a_{12}(Q_{K,V})} \qquad $	405
$b_{12}(Q_{K,V}) = \lambda_n \left[\left(\frac{0.01 - 0.1 \alpha_n(V_{K,V}) e^{-0.1(V_{K,V} + E_K + 20)}}{\left(1 - e^{-0.1(V_{K,V} + E_K + 20)}\right)} \right) (1 - n(V_{K,V})) + \frac{\beta_n(V_{K,V})}{80} n(V_{K,V}) \right] \qquad $	

5.3. Small-signal circuit model of the calcium-sensitive potassium ion-channel memristor
 The small-signal circuit model of the calcium-sensitive potassium-channel memristor at
 an equilibrium point Q_{K,Ca⁷} in the DC V_{K,Ca}-I_{K,Ca} curve is derived by defining
 409

$$Ca = Ca_{\mathcal{Q}_{K,Ca}} + \delta Ca \tag{10a} \quad 410$$

$$v_{K,Ca} = V_{K,Ca}(Q_{Ca}) + \delta v_{K,Ca}$$
 (10b) 411

$$i_{K,Ca} = I_{K,Ca}(Q_{K,Ca}) + \delta i_{K,Ca}$$
 (10c) 412

Expanding $i_{K,Ca} = G_{K,Ca}$ (Ca) $v_{K,Ca}$ from (6a) in a Taylor series about the equilibrium 413

point (
$$Ca(Q_{K,Ca}), V_{Ca}(Q_{K,Ca})$$
), we obtain 414

$$i_{K,Ca} = a_{00}(Q_{K,Ca}) + a_{11}(Q_{K,Ca})\delta Ca + a_{12}(Q_{K,Ca})\delta v_{K,Ca} + h.o.t.$$

= $I_{K,Ca}(Q_{Ca}) + \delta i_{K,Ca}$ (10d) 415

where

416

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$$\delta Ca = Ca - Ca_{Q_{K,Ca}}, \quad \delta v_{K,Ca} = v_{K,Ca} - V_{K,Ca}(Q_{K,Ca})$$
⁴¹⁷

$$\delta i_{K,ca} = i_{K,Ca} - I_{K,Ca}(Q_{K,Ca})$$
(10e) 418

and

$$a_{00}(Q_{K,Ca}) = G_{Ca}(Q_{K,Ca}) V_{Ca}(Q_{K,Ca}) = I_{K,Ca}(Q_{K,Ca})$$
(10f) 420

$$a_{11}(Q_{K,Ca}) = V_{K,Ca}(Q_{K,Ca})G'_{K,Ca}(Ca_{Q_{K,Ca}})$$
(10g) 421

⁷ The equilibrium point $Q_{K,Ca}$ at $v_{K,Ca} = V_{K,Ca}$ is obtained from (6d) by solving $f(Ca; V_{K,Ca}) = 0$ for $Ca = Ca_{K,Ca}$. The explicit formula for $Ca(V_{K,Ca})$ is given in Table 5.

$$a_{12}(Q_{K,Ca}) = G_{K,Ca}(Ca_{Q_{K,Ca}})$$
(10h) 422

and h.o.t denotes the higher-order terms. Let us linearize the nonlinear equation by423neglecting the h.o.t. in (10d) then:424

$$\delta i_{K,Ca} = a_{11} (Q_{K,Ca}) \delta Ca + a_{12} (Q_{K,Ca}) \delta v_{K,Ca}$$
(10i) 425

Similarly, expanding the state equation $f(Ca_{K,Ca}, V_{K,Ca})$ of (6d) in a Taylor series 426

about the equilibrium point (Ca(Q,K,Ca), VCa(Q,K,Ca)), we obtain 427

$$f(Ca_{QK,_{Ca}} + \delta Ca, V_{K,Ca}(Q_{K,Ca}) + \delta v_{K,Ca}) = f(Ca_{QK,_{Ca}}, V_{Ca}(Q_{K,Ca})) + b_{11}(Q_{K,Ca}) \delta Ca + b_{12}(Q_{K,Ca}) \delta v_{K,Ca} + h.o.t.$$
(10j) 428

where

$$b_{11}(Q_{K,Ca}) = \frac{\partial f\left(\operatorname{Ca}, v_{K,Ca}\right)}{\partial Ca}\Big|_{Q_{K,Ca}}$$
(10k) 430

$$b_{12}(Q_{K,Ca}) = \frac{\partial f(Ca, v_{K,Ca})}{\partial v_{K,Ca}} \bigg|_{Q_{K,Ca}}$$
(101) 431

Linearizing the nonlinear state equation (10j) by neglecting the h.o.t., we get 432

$$\frac{d(\partial Ca)}{dt} = b_{11}(Q_{K,Ca})\,\delta Ca + b_{12}(Q_{K,Ca})\,\delta v_{K,Ca} \tag{10m}$$

Taking Laplace transform of (10i) and (10m), we obtain

$$\hat{i}_{K,Ca}(s) = a_{11}(Q_{K,Ca})C\hat{a}(s) + a_{12}(Q_{K,Ca})\hat{v}_{Ca}(s)$$
(10n) 435

$$s \ C\hat{a}(s) = b_{11}(Q_{K,Ca})C\hat{a}(s) + b_{12}(Q_{K,Ca})\hat{v}_{K,Ca}(s)$$
(100) 436

Solving (10o) for $C\hat{a}(s)$ and substituting the result into (10n), we obtain the following 437 admittance $Y_{K,Ca}(s; Q_{K,Ca})$ of the small-signal equivalent circuit of the *calcium sensitive* 438 *potassium ion-channel* memristor at equilibrium point $Q_{K,Ca}$: 439

$$Y_{K,Ca}(s;Q_{K,Ca}) = \frac{\hat{i}_{K,Ca}(s)}{\hat{v}_{K,Ca}(s)} = \left[\frac{1}{\frac{s}{a_{11}(Q_{K,Ca})b_{12}(Q_{K,Ca})} - \frac{b_{11}(Q_{K,Ca})}{a_{11}(Q_{K,Ca})b_{12}(Q_{K,Ca})} + \frac{1}{\frac{1}{a_{12}(Q_{K,Ca})}}\right]$$
(10p) 440

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$$Y_{Ca}(s; Q_{Ca}) = \left(\frac{1}{(sL_{K,Ca} + R_{1K,Ca})} + \frac{1}{R_{2K,Ca}}\right)$$
(10q) 442

where
$$L_{K,Ca} \triangleq \frac{1}{a_{11}(Q_{K,Ca})b_{12}(Q_{K,Ca})}$$
 (10r) 443

$$R_{1K,Ca} \triangleq -\frac{b_{11}(Q_{K,Ca})}{a_{11}(Q_{K,Ca})b_{12}(Q_{K,Ca})}$$
(10s) 444

$$R_{2K,Ca} \triangleq \frac{1}{a_{12}(Q_{K,Ca})}$$
(10t) 445

It follows from (10r)-(10t) that the small-signal admittance function of the first-order 446 calcium-sensitive potassium ion-channel memristor is equivalent to the series connection of an 447 inductor and a resistor in parallel with another resistor as shown in Fig. 10. The 448 corresponding coefficients a11, a12, b11, b12 and inductance LK,Ca, resistance R1K,Ca, and resistance 449 R2K,Ca as a function of the DC equilibrium voltage VK,Ca are shown in Figs. 11 and Figs. 12, 450 respectively. The small-signal inductance and resistances(i.e. *L*_K,*c*₄>0, *R*_{1K},*c*₄>0 and *R*_{2K},*c*₄>0) 451 over the edge of chaos 1 and edge of chaos 2 with respect to the $V_{K,Ca}$ are shown in Fig. 452 12(a), Fig. 12(b) and Fig. 12(c) respectively. Please note that the local activity, edge of chaos 453 1 and edge of chaos 2 shown in Fig. 12(a), Fig. 12(b) and Fig. 12(c) are not the local activity, 454 edge of chaos 1 and edge of chaos 2 of the individual calcium sensitive potassium ion 455 channel memristor. The local activity, edge of chaos domains are just an information 456 showing the corresponding range of voltage with respect to $V_{K,Ca}$ when measured across 457 the individual calcium sensitive potassium ion channel memristor of the entire connected 458 Chay small-signal equivalent circuit of Fig. 1(b) and Fig. 13. For the readers' convenience, 459 the explicit formulas for computing the coefficients $a_{11}(Q_{K,Ca})$, $a_{12}(Q_{K,Ca})$, $b_{11}(Q_{K,Ca})$, $b_{12}(Q_{K,Ca})$ 460 and $L_{K,Ca}$, $R_{1K,Ca}$, $R_{2K,Ca}$ are summarized in Table 5. 461



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Figure 10. Small-signal equivalent circuit model of the calcium-sensitive potassium ion-channel464memristor about the DC equilibrium point $Q_{K,Ca}$ ($V_{K,Ca}$, $I_{K,Ca}$).465



Figure 11. Plot of coefficients (a) a_{11} (b) a_{12} (c) b_{11} and (d) b_{12} of the calcium-sensitive potassium ion-467channel memristor as a function of the DC equilibrium voltage $V_{K,Ca}$.468



Figure 12. (a) Inductance $L_{K,Ca}$ (b) resistance $R_{1K,Ca}$ and (c) resistance $R_{2K,Ca}$ of the *calcium-sensitive*470*potassium ion-channel memristor* as a function of DC equilibrium voltage $V_{K,Ca}$. $L_{K,Ca}>0$, $R_{1K,Ca}>0$ and471R_{2KCa}>0 over the edge of chaos 1 and edge of chaos 2 with respect to $V_{K,Ca}$ of the entire connected472Chay small-signal equivalent circuit of Fig.1(b) and Fig. 13.473

Table 5: Explicit formulas for computing the coefficients $a_{11}(Q_{K,Ca})$, $a_{12}(Q_{K,Ca})$, $b_{11}(Q_{K,Ca})$, $b_{12}(Q_{K,Ca})$ and474 $L_{K,Ca}$, $R_{1K,Ca}$, $R_{2K,Ca}$ of the calcium-sensitive potassium ion-channel memristor.475



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5.4. Small-signal circuit model of the memristive Chay model

Let us replace the voltage-sensitive mixed ion-channel nonlinear resistor, the voltage-479 sensitive potassium ion-channel memristor, and the calcium-sensitive potassium ion-480 channel memristor in the memristive Chay neuron circuit of Fig. 1(b) with their small-481 signal models about DC operating voltages $V_I = V_m - E_I$, $V_{K,V} = V_m - E_K$, and $V_{K,Ca} = V_m - E_K$, 482 respectively. Short-circuiting all the batteries, the equivalent small-signal circuit model of 483 the third-order neuron circuit from Fig. 1(b) about the operating point $V_m(Q \text{ is found to be})$ 484 composed of one capacitor, two inductors, and six resistors as shown in Fig. 13. The local 485 admittance $Y(s;V_m(Q))$ of this linear circuit seen from the port and formed by the capacitor 486 terminals about Q is given by 487

$$Y(s; V_m(Q)) = sC_m + \frac{1}{sL_{K,V} + R_{1K,V}} + \frac{1}{sL_{K,Ca} + R_{1K,Ca}} + \frac{1}{R_{1,I}} + \frac{1}{R_{2K,V}} + \frac{1}{R_{2K,Ca}} + G_L$$
(11) 488

The corresponding range of local activity, edge of chaos 1 and edge of chaos 2 at 489 equilibrium voltage $V_m(Q)$ (*resp. I*) are also given in Fig. 13 for readers' convenience. We 490 will cover the details of these regimes in the section on locally activity and edge of chaos. 491 The circuit element $R_{1,I}$ is obtained by calculating the small signal model of the voltage-492 sensitive mixed ion-channel nonlinear resistor from Table 3 at equilibrium voltage 493

 $V_m(Q)=V_I+E_I$. Similarly, $L_{K,V}$, $R_{IK,V}$, and $R_{2K,V}$ are calculated from the small-signal 494 equivalent circuit of the *voltage sensitive potassium ion-channel memristor* from Table 4 and 495 $L_{K,Ca}$, $R_{IK,Ca}$, and $R_{2K,Ca}$ are calculated from the small signal equivalent circuit of the 496 *calcium-sensitive potassium ion channel memristor* from Table 5 at equilibrium voltage $V_m(Q)$ 497 respectively. Note that $V_{K,V}+E_K$ and $V_{K,Ca}+E_K$ must be replaced by $V_m(Q)$ in Table 4 and Table 498 5 by the small signal model of the *voltage-sensitive potassium ion-channel memristor* and 499 *calcium-sensitive potassium ion-channel memristor*, respectively. 500



Figure 13. Small-signal equivalent circuit model of the memristive Chay model. The DC equilibrium503voltage V_m is computed at $V_m = V_t + E_t$ for mixed ion channel non-linear resistor, $V_m = V_{K,V} + E_K$ for voltage504sensitive potassium ion-channel memristor and $V_m = V_{K,Ca} + E_K$ for calcium sensitive potassium ion-505channel memristor, respectively.506

502

501

Table 6: Explicit formulas for computing the coefficients of Y(s;Vm(Q)).

$$b_{3} = L_{K,V}L_{K,Ca}R_{1,I}R_{2K,V}R_{2K,Ca}C_{m}$$

$$b_{2} = \left(L_{K,V}R_{1K,ca} + L_{K,Ca}R_{1K,V}\right)R_{1,I}R_{2K,V}R_{2K,Ca}C_{m} + L_{K,V}L_{K,Ca}R_{2K,V}R_{2K,Ca}\right)$$

$$+ L_{K,V}L_{K,Ca}R_{1,I}R_{2K,Ca} + L_{K,V}L_{K,Ca}R_{1,I}R_{2K,V} + L_{K,V}L_{K,Ca}R_{1,I}R_{2K,V}R_{2K,Ca}G_{L}$$

$$b_{1} = R_{1,I}R_{1K,V}R_{1K,ca}R_{2K,V}R_{2K,Ca}C_{m} + \left(L_{K,Ca}R_{1,I}R_{2K,V}R_{2K,Ca}\right) + \left(L_{K,V}R_{1,I}R_{2K,V}R_{2K,Ca}\right)$$

$$+ \left(L_{K,V}R_{1K,ca} + L_{K,Ca}R_{1K,V}\right)R_{2K,V}R_{2K,Ca} + \left(L_{K,V}R_{1K,ca} + L_{K,Ca}R_{1K,V}\right)R_{1,I}R_{2K,V}R_{2K,Ca}\right)$$

$$+ \left(L_{K,V}R_{1K,ca} + L_{K,Ca}R_{1K,V}\right)R_{1,I}R_{2K,V} + \left(L_{K,V}R_{1K,ca} + L_{K,Ca}R_{1K,V}\right)R_{1,I}R_{2K,V}R_{2K,Ca}\right)$$

$$b_{0} = R_{1,I}R_{1K,ca}R_{2K,V}R_{2K,Ca} + R_{1,I}R_{1K,V}R_{2K,V}R_{2K,Ca} + R_{1K,V}R_{1K,ca}R_{2K,V}R_{2K,Ca}G_{L}$$

$$b_{0} = R_{1,I}R_{1K,V}R_{1K,ca}R_{2K,V}R_{2K,Ca} + R_{1,I}R_{1K,V}R_{1K,ca}R_{2K,V} + R_{1,I}R_{1K,V}R_{1K,ca}R_{2K,V}R_{2K,Ca}G_{L}$$

$$a_{1} = \left(L_{K,V}R_{1K,ca} + L_{K,Ca}R_{1K,V}\right)R_{1,I}R_{2K,V}R_{2K,Ca}$$

$$a_{0} = R_{1,I}R_{1K,V}R_{1K,ca}R_{2K,V}R_{2K,Ca}$$

5.4.1. Frequency Response

R R

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 b_1

a

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C

A convenient way to find the total admittance Y(s; Vm(Q)) by recasting (11) into a 512 rational function of the complex frequency variable s, is as follows: 513

$$Y(s; V_m(Q)) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{a_2 s^2 + a_1 s + a_0}$$
(12a) 514

where the explicit formulas for computing the coefficients b₃, b₂, b₁, b₀, a₂, a₁, and a₀ are 515 summarized in Table 6. 516

Substituting $s = i\omega$ in (12a), we obtain the following small-signal admittance 517 function at the equilibrium voltage $V_m(Q)$: 518

$$Y(i\omega; V_m(Q)) = \frac{(b_0 - b_2\omega^2)(a_0 - a_2\omega^2) + a_1\omega^2(b_1 - b_3\omega^2)}{(a_0 - a_2\omega^2)^2 + (a_1\omega)^2} + i\omega \left[\frac{\left[(b_1 - b_3\omega^2)(a_0 - a_2\omega^2) - a_1(b_0 - b_2\omega^2)\right]}{(a_0 - a_2\omega^2)^2 + (a_1\omega)^2}\right]$$
(12b) 519

The corresponding real part Re Y ($i\omega$; $V_m(Q)$) and imaginary part Im Y ($i\omega$; $V_m(Q)$) from 520 (12b) are given by, 521

$$\operatorname{Re} Y(i\omega; V_{m}(Q)) = \left[\frac{(b_{0} - b_{2}\omega^{2})(a_{0} - a_{2}\omega^{2}) + a_{1}\omega^{2}(b_{1} - b_{3}\omega^{2})}{(a_{0} - a_{2}\omega^{2})^{2} + (a_{1}\omega)^{2}}\right]$$

$$\operatorname{Im} Y(i\omega; V_{m}(Q)) = \omega \left[\frac{\left[(b_{1} - b_{3}\omega^{2})(a_{0} - a_{2}\omega^{2}) - a_{1}(b_{0} - b_{2}\omega^{2})\right]}{(a_{0} - a_{2}\omega^{2})^{2} + (a_{1}\omega)^{2}}\right]$$
(12c) 522

Fig. 14(a) and Fig. 14(b) show $ReY(i\omega; V_m(Q))$ vs. ω , Im $Y(i\omega; V_m(Q))$ vs. ω , and the 523 Nyquist plot Im $Y(i\omega; V_m(Q))$ vs. Re $Y(i\omega; V_m(Q))$ at the DC equilibrium voltage V_m = -48.763 524 mV(resp., I=-66.671 μA), and Vm=-27.984 mV(resp., I=433.594 μA), respectively. Observe 525 from Fig. 14(a) and Fig. 14(b) that $ReY(i\omega; V_m(Q)) < 0$, thereby confirming the memristive 526 Chay model is a *locally active* at each of the two operating points. Our extensive numerical 527 computations show the two DC equilibria coincide with two-Hopf bifurcation points are 528 the origin of generating the oscillation, spikes, chaos and bursting in excitable cells. We 529 will discuss about these two bifurcation points in next section with pole-zeros and eigen 530 values diagram. 531



Figure 14. Small-signal admittance frequency response and Nyquist plot of the memristive Chay534neuron model at (a) V_m = -48.763 mV(resp., I= -66.671 μ A) and (b) V_m =-27.984 mV(resp., I=433.594 μ A).535Observe that $ReY(i\omega;V_m(Q))<0$ at the two Hopf-bifurcation points.536

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538 5.4.2. Pole-zero diagram of the small-signal admittance function Y(s; Vm(Q)) and eigen values of 539 the jacobian matrix

540 The location of the poles and zeros of the small signal admittance function $Y(s; V_m(Q))$ 541 of (12a) is computed by factorizing it's denominator and numerators as

$$Y(s; V_m(Q)) = \frac{k(s-z_1)(s-z_2)(s-z_3)}{(s-p_1)(s-p_2)}$$
(13)⁵⁴²

The poles of the small-signal admittance function $Y(s; V_m(Q))$ as a function of the 543 voltage V_m over -200 mV < Vm < 200 mV is shown in Fig. 15. Observe from Fig. 15(a) and 544 Fig. 15(b) that the two poles $Re(p_1)$, $Re(p_2)$ are negative while $Im(p_1)$, $Im(p_2)$ remain 545 consistently zero for the specified DC input V_m . This observation confirms that the two 546 poles of the admittance function possess no complex frequencies. 547

Fig. 16(a) shows the Nyquist plot, i.e. loci of the imaginary part $Im(z_i)$ versus the real 548 part $Re(z_i)$ of the zeros as a function of the input voltage V_m over the interval $-55 mV \le V_m$ 549 $\leq 25mV$. Observe that the real part of the two zeros z_2 and z_3 are zero at V_m = -48.763 mV(resp., 550 $I=-66.671 \ \mu A)$ and $V_m=-27.984 \ mV(resp., I=433.594 \ \mu A)$, respectively. The corresponding 551 points when $Re(z_i)=0$ are known as Hopf bifurcation points in bifurcation theory. Fig. 16(b) 552 and Fig. 16(c) show the zoomed version of Fig. 16(a) near to the two bifurcation points 553 respectively. It is also observed that the $Re(z_2)$ and $Re(z_3)$ lie in open right half plane(RHP) 554 between the bifurcation points -48.763 mV <V_m<-27.984 mV(resp. -66.671 μ A <I < 433.594 555 μA). Observe from Fig. 17 that the eigenvalues, computed from the Jacobian matrix, 556 associated to the ODE (1a)-(1c) are identical to the zeros of the neuron local admittance 557 $Y(s; V_m(Q))$, as inferable from Fig. 16, and expected from the Chua theory [3]-[4]. 558



Figure 15. Poles diagram of the small-signal admittance function $Y(s; V_m(Q))$ as a function of V_m over 560 -200 mV <V_m<200 mV (a) Top and bottom figures are the plot of the real part of the pole 1 $Re(p_1)$ and 561 Imaginary part of pole 1 $Im(p_1)$ respectively. (b) Top and bottom figures are the plot of the real part 562 of the pole 2 $Re(p_2)$ and Imaginary part of pole 2 $Im(p_2)$ respectively. 563

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Figure 16. Zeros diagram of the small-signal admittance function $Y(s; V_m(Q))$ (a) Nyquist plot of the zeros z_1, z_2, z_3 in $Im(z_i)$ vs. $Re(z_i)$ plane (b) Nyquist plot near the Hopf-bifurcation point 1, V_m =-48.763 567 $mV(resp., I = -66.671 \mu A)$. (c) Nyquist plot near the Hopf-bifurcation point 2, V_m =-27.984 mV(resp., 568 I=433.594 μA). 569



Figure 17. Plot of the loci of the eigen values of the Jacobian Matrix (a) Nyquist plot of the eigen values λ_1 , λ_2 , λ_3 in $Im(\lambda_i)$ vs. $Re(\lambda_i)$ plane Nyquist plot near the Hopf-bifurcation point 1, V_m =-48.763 573 $mV(resp., I = -66.671 \ \mu A)$. (c) Nyquist plot near the Hopf-bifurcation point 2, V_m =-27.984 $mV(resp., 574 \ I=433.594 \ \mu A)$. Our numerical computations confirm the zeros of the admittance functions $Y(s; V_m(Q))$ 575 obtained in Fig. 16 are identical to the eigen values of the Jacobian matrix 576

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6. Local Activity, Edge of Chaos and Hopf-Bifurcation in Memristive Chay Model 577

Local Activity and edge of chaos are the powerful mathematical quantitative theories 578 to predict whether the nonlinear system exhibits complexity or not. Local activity refers 579 to a characteristic of nonlinear systems wherein infinitesimal fluctuations in energy are 580 amplified, leading to the emergence of complex dynamical behavior in the system [34-39]. 581 This section presents an extensive analysis of the memristive Chay model using the principle of local activity, edge of chaos and Hopf-bifurcation theorem to predict the mechanism of generating the complicated electrical signals in an excitable cell. 584

6.1. Locally active regime

The local activity theorem developed by Chua reveals that a nonlinear system must 587 satisfy at least one of the following conditions, concerning its local transfer function about 588 a given operating point in order to support the emergence of complexity[36]. 590 (i) The zero of the admittance function $Y(s; V_m(Q))$ lie in open-right plane where $Re(s_z) > 0$ 590 (ii) $Y(s; V_m(Q))$ has multiple zero on the imaginary axis 591 (iii) $Y(s; V_m(Q))$ has simple zero on the imaginary axis $s = i\omega_z$ on the imaginary axis and 592

 $K_Q(i\omega_z) \triangleq \lim_{s \to i\omega_z} = (s - i\omega_z)Y(s; V_m(Q))$ is either a negative real number, or a complex ⁵⁹³

number.

(iv) $ReY(i\omega; V_m(Q)) < 0$ for some $\omega \in [-\infty, +\infty]$ 595

596 In another words, the emergence of action potentials, oscillations, chaos, burstings or 597 spikes in neurons are impossible unless the cells are locally active. Therefore, restricting 598 the behavior of a nonlinear system to its local activity operating regime reduces the 599 considerable time necessary to identify the complex phenomena, which may emerge 600 across its physical medium as compared to a standard trial-and-error numerical 601 investigation. In order to restrict the above dynamical behavior in memristive Chay model 602 of an excitable cell in local activity regime, we performed comprehensive numerical 603 analyses within the range of the DC equilibrium voltage V_m = -50 mV (resp. I= -74.316 μA) 604 to V_m =-23.5 mV (resp. I= 1.76×10³ μ A). Observe from Fig. 18(a), the real part of the 605 admittance of the frequency response $ReY(i\omega; Vm(Q))>0$ at $V_m = -50 mV$ (resp. I= -74.316 μA), 606 thereby confirming locally passive at this equilibrium point. However, when *V*_m>-50*mV*, 607 our in depth simulation in Fig. 18(b) shows that $ReY(i\omega;Vm(Q))=0$ at $V_m=-49.455 \text{ mV}$ (resp. 608 I= -70.919 μ A) and Fig. 18(c) and Fig. 18(d) show that $ReY(i\omega; Vm(Q)) < 0$ at V_m = -48.1 609 $mV(resp. I = -62.681 \ \mu A)$ and $V_m = -26.5 \ mV$ (resp. I=746.457 μA) respectively for some 610 frequency ω , confirming an excitable cell is locally active at these equilibria. Our 611 simulations in Fig. 18(e) shows, a further increase in the DC equilibrium voltage at V_m = -612 24.685 mV (resp. I=1.291×10³ μ A), the loci is tangential to the ω axis i.e. Re Y (i ω ; Vm(Q))=0. However, when V_m >-24.685 mV, say V_m =-23.5 mV(resp. I= 1.76×10³ µA), it is observed from 613 614 Fig. 18(f) that Re $Y(i\omega; Vm(Q))>0$, and the memrisitve Chay model is no more locally active 615 confirming the cell is locally passive at this equilibrium. Therefore, the local activity

616 regime which started above V_m =- 49.455 mV (resp. I= -70.919 μ A) exists over the following 617 regime

$$\begin{array}{c}
Local \ Activity \ Re \ gime \\
-49.455 \ mV < V_m < -24.685 \ mV \\
-70.919 \ \mu A < I < 1.291 \times 10^3 \ \mu A
\end{array}$$
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Figure 18. Plot of $Re(i\omega; Vm(Q))$ to illustrate the local activity principle at (a) V_m -50 mV (resp. I -622 74.316 µA) (b) Vm= - 49.455 mV (resp. I= -70.919 µA), (c) Vm= -48.1 mV(resp. I= -62.681 µA), (d) Vm= -26.5 623 mV (resp. I=746.457 µA), (e) V_m= -24.685 mV (resp. I=1.291×10³ µA), (f) V_m=-23.5mV(resp. I= 1.76×10³ μA), respectively. 625

6.2. Edge of chaos regime

Edge of chaos is a tiny subset of the locally-active domain where the zeros of the 628 admittance function Y(s; Vm(Q)) (equivalent to the eigen values of Jacobian matrix) lie in 629 the open left-half plane, i.e. $Re(z_p)<0$ (eigen values $\lambda i<0$) as well as $ReY(i\omega; Vm(Q))<0$. Fig. 630 17(a) and Fig. 17(b) show the real part of the *eigen values* vanish at V_m =-48.7631 mV (resp. 631 I=-66.671 μ A) with pair of complex eigen values $\lambda_{2,3}$ = ± 0.557i. It follows from the edge of 632 chaos theorem that the corresponding equilibrium point is no longer asymptotically stable, 633 and becomes unstable thereafter confirming the 1st edge of chaos regime over the following 634 small interval: 635

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Edge of chaos domain 1
$-49.455 \ mV < V_m < -48.763 \ mV$
$-70.919 \ \mu A < I < -66.671 \ \mu A$

Observe from Fig. 17(c) that the real part of the eigen values vanish at $\lambda_{2,3} = \pm 85.606i$ at 637 DC equilibrium voltage V_m =-27.984 *mV*(resp. *I*= 433.594 μ A). It follows that the 638 corresponding equilibrium point $V_m(Q)$ is no longer *asymptotically* stable below this 639 equilibrium point, therefore confirming the existence of a 2*nd edge of chaos* regime over the 640 following interval: 641

Edge of chaos Domain 2

$$-27.984 \text{ mV} < V_m < -24.685 \text{ mV}$$

 $433.594 \ \mu A < I < 1.291 \times 10^3 \ \mu A$

The nonlinear dynamical behavior of the memristive Chay model in this paper is 643 controlled as the function the input stimulus I. The local activity, edge of chaos 1 and edge 644 of chaos 2 regime computed in this paper under the assumption of departing the input 645 parameter *I* from lower stimulus to higher stimulus (resp. low DC equilibrium voltage $V_m(Q)$ to high equilibrium voltage $V_m(Q)$). 647

6.3. Hopf-bifurcation

Hopf-bifurcation namely, super-critical and sub-critical bifurcations are local 650 bifurcation phenomenon in which an equilibrium point changes its stability as the 651 parameter of the nonlinear system changes under certain conditions. When an unstable 652 equilibrium point surrounded by a stable limit cycle results to a super-critical Hopf 653 bifurcation whereas a subcritical Hopf bifurcation refers to a qualitative change in the 654 behavior of a system where a stable equilibrium point transitions to instability, giving rise 655 to sustained oscillations or limit cycles as a parameter is varied. Our careful simulation at 656 Hopf-bifurcation point 1 at $V_m = -48.763 \text{ mV}(\text{resp. }I = -66.671 \mu A)^8$ shows that stimulus 657 current I should be chosen within very small edge of chaos domain 1, where the real part 658 of the eigen values are negative, the result converges to DC equilibrium for any initial 659 conditions. Likewise, I is selected within the bifurcation point 1, where the real part of 660 eigen values are positive, the result converges to a stable limit cycle. Therefore, it follows 661 from the bifurcation theory that bifurcation point 1 is a super-critical Hopf bifurcation. 662 Fig. 19(a) and Fig. 19(b) show the numerical simulations at $I = -68.118 \ \mu A$ and $I = -65.077 \ \mu A$ 663 respectively. Observe, from Fig. 19(a) and Fig. 19(b) that $I = -68.118 \ \mu A$ lying within the 664 tiny subset of edge of chaos domain 1 converges to DC equilibrium and I= -65.077 μ A lying 665 in open right half-plane(RHP) converges to a spikes, respectively, confirming the 666 bifurcation point 1 is a super-critical Hopf bifurcation. 667

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⁸ The super critical Hopf bifurcation point 1 and point 2 observed in this paper are just for the parameters listed in Table 2. The bifurcations phenomenon may vary for different parameters.



Figure 19. Numerical simulations to confirm the super-critical Hopf bifurcation at bifurcation point 669 1. Plot of membrane potential V_m at (a) $I = -68.118 \ \mu A$ which lies inside the tiny subset of edge of 670 chaos domain 1 and beyond bifurcation point 1 converges to the DC equilibrium, (b) and I = -65.077 671 μA , chosen just to the right of bifurcation point 1, where the real parts of two zeros of the neuron 672 local admittance lie on the open right half plane (RHP) converges to the spikes 673

Similarly, our careful examination predicts a stable DC equilibrium point when 675 current I is chosen within a very small edge of chaos 2, confirming supercritical Hopf 676 bifurcation at bifurcation point V_m = -27.984 mV (resp. I= 433.594 μ A). The possibility of 677 above scenario is illustrated in Figs. 20. Fig. 20(a) shows the membrane potential V_m 678 converges to stable DC equilibrium point when $I=440 \ \mu A$ chosen within the edge of chaos 679 domain 2. Fig. 20(b) shows when $I=430.884 \ \mu A$ chosen very close and inside the bifurcation 680 point 2, where the real part of the eigen value is positive and lie in open right half 681 plane(RHP), the transient waveform converges to stable limit cycle as predicted by Hopf 682 supercritical bifurcation theorem. 683

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Figure 20. Numerical simulations to confirm the super-critical Hopf bifurcation at bifurcation point 686 2. (a) Plot of membrane potential V_m which converges to stable DC equilibrium when $I=440 \ \mu A$ 687 chosen inside the tiny subset of edge of chaos domain 2 and, near and beyond the bifurcation point 688 2. (b) Membrane potential converging to oscillation as predicted by Hopf bifurcation theorem when 689 $I=430.884 \ \mu A$ is chosen inside the bifurcation point (open right-half pane). 690

Table 7 illustrates the computation of the potassium ion-channel activation *n*, calcium 692 concentration *Ca* and eigen values (λ_1 , λ_2 and λ_3) as a function of the DC stimulus current 693 I (resp. membrane potential V_m) at the DC equilibrium point Q. It is observed from Table 694 7 and Fig. 17(a) to Fig. 17(c) that the two Hopf bifurcations points 1 and 2 occur at Vm = -695 48.763 mV(resp. I=-66.671 μA) and V_m= -27.984 mV (resp. I= 433.594 μA) respectively, where 696 the eigen values are purely imaginary at these two equilibria. As I decreases (resp. Vm 697 decreases) from the Hopf bifurcation point 1, the eigen values migrated to the left-hand 698 side confirming the real parts of the eigen values are no longer positive and thereby 699 confirming the first negative real eigen values regime exists over the following interval. 700

Negative real eigen values regime 1:

$$-\infty < V_m < -48.763 \text{ mV}$$

 $-\infty < I < -66.671 \mu A$

Similarly, as *I* increases(resp. V_m *increases*) from the second bifurcation points, the positive real part of the eigen values migrated from open right half to the open left half, there by confirming the second negative real eigen values regime over the following 704

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interval:

Negative real eigen values regime 2

$$-27.984 \text{ mV} < V_m < +\infty$$

 $433.594 \mu A < I < +\infty$

707 Table 7. Computation of the potassium ion-channel activation n, calcium concentration Ca and eigen 708 values (λ_1 , λ_2 and λ_3) as a function of the stimulus current *I* (resp. membrane potential V_m). Rows 5 709 to 7 pertain to the edge of chaos 1, rows 8 to 16 pertain to the unstable local activity domain and 710 rows 17 to 20 pertain to the edge of chaos 2. Rows 7 and 17 pertain to Hopf bifurcation point 1 and 711 Hopf bifurcations points 2, respectively for the memristive Chay neuron model.

S. N	V _m (Vm)	Ι(μΑ)	n	Ca	λ ₁	λ_2	λ ₃
1.	-52.00	-87.02	0.08	0.04	-40.515	-3.842	-0.084
2.	-51.00	-80.63	0.08	0.05	-40.107	-2.871	-0.111
3.	-50.50	-77.46	0.09	0.06	-39.891	-2.289	-0.139
4.	-50.00	-74.32	0.09	0.07	39.666	-1.617	-0.196
5.	-49.455	-70.919	0.94	0.08	-39.408	-0.533-0.174i	-0.533+0.174i
6.	-49.00	-68.12	0.1	0.1	-39.181	-0.19-0.525i	-0.19+0.525i
7.	-48.763	-66.671	0.1	0.1	-39.058	0-0.557i	0+0.557i
8.	-48.50	-65.08	0.1	0.11	-38.917	0.222-0.51i	0.2215+0.5097i
9.	-46.00	-51.02	0.12	0.21	-37.32	0.046	5.736
10.	-45.00	-46.37	0.13	0.27	-36.512	0.027	8.498
11.	-42.00	-39.37	0.16	0.53	-33.218	0.0001	18.604
12.	-40.00	-42.78	0.18	0.79	-29.899	-0.0084	25.99
13.	-38.00	-51.26	0.21	1.13	-24.898	-0.0112	31.992
14.	-32.00	17.59	0.29	2.57	-0.061	11.669-38.01i	11.669+38.01i
15	-30.00	160.68	0.32	3.12	-0.053	8.049-61.778i	8.049+61.778i
16.	-28.00	430.84	0.35	3.65	-0.051	0.08-85.421i	0.08+85.421i
17.	-27.984	433.594	0.35	3.65	-0.051	0-85.606i	0+85.606i
18.	-27.00	628.91	0.36	3.89	-5.556-97.197i	-5.556+97.197i	-0.051
19.	-25.50	1.02×10 ³	0.39	4.22	-15.942-114.607i	-15.942+114.607i	-0.0501
20.	-24.685	1.291×10 ³	0.40	4.37	-22.466-123.858i	-22.466+123.858i	-0.0499
21	-23.00	1.99×10 ³	0.43	4.64	-37.643-142.384i	-37.643+142.384i	-0.0497
22	-22.00	2.5×10^{3}	0.44	4.75	-47.529-152.923i	-47.529+152.923i	-0.0496

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Observe from Table 7 and Fig. 17(a)-Fig. 17(c) that two eigenvalues of the Jacobian 714 matrix associated to the ODE set (1a)-(1c) lie on the open RHP for each operating point Q corresponding to a DC current I value between Hopf bifurcation point 1 and Hopf bifur-716 cation point 2. Therefore, the generation of periodic, bursting, spikes and chaos signals

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predicted by Hopf bifurcation theorem in an excitable cell exists over the following in-718 terval:

Unstable (periodic, bursting, chaos, spikes) regime :

$$-48.763 \text{ mV} < V_m < -27.984 \text{ mV}$$

 $-66.671 \mu A < I < 433.594 \mu A$

721 The convergence of the membrane potential to a stable and unstable DC equilibrium 722 points are verified by numerical simulations at different values of I and are illustrated in 723 Figs. 21. Fig. 21(a) shows the transient waveform of membrane potential V_m converging to 724 a stable DC equilibrium at I= -90 μ A, confirming the Hopf bifurcation theorem no longer 725 holds at this equilibrium. Similarly, when DC simulus currents I= $-50 \ \mu A$ and $-10 \ \mu A$ are 726 chosen inside the two bifurcation points $I=-66.671 \ \mu A$ and $I=433.594 \ \mu A$, we observed 727 different patterns of oscillations as shown in Fig. 21(b) and Fig. 21(c), confirming the 728 bifurcation theorem holds in this regime. Likewise, when DC stimulus currents $I=10 \, \mu A$ 729 and $I=2000\mu A$ are applied within the bifurcation points $I=-66.671 \mu A$ and I=433.594, 730 respectively, oscillation patterns emerge as depicted in Fig. 22(a) and Fig. 22(b). Similarly, 731 Fig. 22(c) illustrates the transient waveform of the membrane potential V_m , indicating its 732 convergence to a stable DC equilibrium at $I=500 \ \mu A$. This observation suggests that the 733 Hopf bifurcation theorem no longer holds at this equilibrium point.

735 Figs. 23 and Figs. 24 show the different patterns of oscillations when the conductance 736 g_{KCa} of calcium sensitive potassium ion channel memristor is varied from 10 mS/cm^2 to 11.5737 mS/cm^2 at stimulus current I=0. Fig. 23(a) shows the excitable membrane cell has a stable 738 limit cycle with period one at $g_{K,Ca}=10 \text{ mS/cm}^2$. As the parameter $g_{K,Ca}$ increases to 10.7 739 mS/cm², 10.75 mS/cm² and 10.77 mS/cm² the cell fires period two, four and eight as shown 740 in Fig. 23(b), Fig. 23(c) and Fig. 24(a) respectively. The change in the period doubling is 741 more apparent in calcium *concentration* (*Ca*) vs. time and, V_m vs. *Ca* as shown in the bottom 742 of Fig. 23(b), Fig. 23(c) and Fig. 24(a) respectively. Fig. 24(b) shows the waveform of the 743 memrisive Chay model confirming the existence of aperiodic oscillation (chaos) at gK,Ca=11 744 mS/cm^2 . The firing of aperiodic oscillations from cell can be clearly seen from the plot of 745 the Ca vs. time and V_m vs. Ca in Fig. 24 (b). A further increase in $g_{K,Ca}$ to 11.5 mS/cm^2 gives 746 rise to the firing of the cell from aperiodic to rhythmic bursting as shown in Fig. 24(c).

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Figure 21. Patterns of oscillations when stimulus current *I* is chosen beyond and inside the bifurcations points. (a) DC pattern observed when *I*= -90 μ *A* chosen beyond bifurcation point 1(*I*=-66.671 μ *A*). Different patterns of oscillations when *I* is chosen between the two bifurcation points *I*=-66.671 μ *A* and *I*= 433.594 μ *A*, at (b) *I*= -50 μ *A*, (c) *I*= -10 μ *A*.



Figure 22. Patterns of oscillations when stimulus current *I* is chosen inside and beyond the 755 bifurcations points. Oscillations patterns when *I* is chosen between the two bifurcation points *I*=- 756 66.671 μ A and *I*= 433.594 μ A, at (a) *I*=10 μ A, and (b) *I*=200 μ A. (c) DC pattern when *I*=500 μ A is chosen 757 beyond the bifurcation point 2(*I* = 433.594 μ A). 758



Figure 23. Different patterns of oscillations when gKCa varied from 10 mS/cm² to 10.75 mS/cm² at DC situmulus current I=0. (a) Period-1 oscillation at gK,Ca=10 mS/cm² (b) Period-2 oscillation at gK,Ca=10.7 mS/cm^2 (c) Period-4 oscillation at $g_{K,Ca}=10.75 mS/cm^2$. The simulations were performed at the initial conditions *V*_m(0)=-50mV, n(0)=0.1 and *C*a(0)=0.48. 764



Figure 24. Different patterns of oscillations when g_{KCa} varied from $10.77 \ mS/cm^2$ to $11.5 \ mS/cm^2$ at 767 DC stimulus current *I*=0. (a) Period-8 oscillation at $g_{K,Ca}=10.77 \ mS/cm^2$ (b) Aperiodic (chaotic) 768 oscillation at $g_{K,Ca}=11 \ mS/cm^2$ (c) Bursting at $g_{K,Ca}=11.5 \ mS/cm^2$. The simulations were performed at 769 the initial conditions $V_m(0)=-50mV, n(0)=0.1$ and Ca(0)=0.48. 770

7. Concluding Remarks

This paper has provided a comprehensive and quantitative analysis of a biological 773 excitable cell using the Chay neuron model. Through memristive theory, we have 774 demonstrated that the *voltage-sensitive mixed ion-channel* functions as a nonlinear resistor, 775 while the *voltage-sensitive potassium ion-channel* and *calcium-sensitive potassium ion-channel* 776 in an excitable cell are indeed time-invariant first-order generic memristors. 777

Furthermore, we have conducted in-depth analyses to derive the small signal model, 778 admittance function, pole-zero diagram, frequency response of admittance functions, and 779 Nyquist plot at the DC equilibrium point Q. Our investigations revealed the existence of 780 the local activity regime in the memristive Chay model within the voltage range of 781 -49.455 *mV to* -24.685 *mV*, and identified edge of chaos regime domains 1 and 2 within the 782 voltage ranges of -49.455 *mV to* -48.763 *mV*, and -27.984 *mV to* -24.685 *mV*, respectively. 783

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Moreover, consistent with the predictions of the Hopf bifurcation theorem, we observed 784 the presence of an oscillating regime between two bifurcation points within the voltage 785 range of -48.763 mV to -27.984 mV. Our numerical simulations confirmed the super-critical 786 Hopf bifurcation with complex conjugates of eigenvalues coincide on the purely 787 imaginary axis at ±0.557*i* and ±85.606*i* respectively. It was also observed that a tiny change 788 in external stimulus current I in excitable cells, far from the bifurcation points no longer 789 holds the Hopf bifurcation theorem as it crosses the imaginary axis from right to left 790 confirming that the real part of the eigenvalues becomes negative and converges to a DC 791 equilibrium point. 792

Our comprehensive comparison of the HH, FitzHugh-Nagumo, ML, and the Chay 793 models presented in Table 2 along with their individual strengths and limitations reveals 794 distinct advantages and drawbacks making them suitable for different research contexts 795 and questions. The selection of the particular model depends on the specific objectives. 796 We primarily focused to advance the understanding of excitable cells by modeling with 797 the networks of memristors and predicting their responses with the concept of 798 memristor theory, DC steady state analyses, small signal equivalent circuit, local activity 799 principle, edge of chaos theorem and hopf bifurcations. In Conclusion, the theoretical 800 framework outlined in this paper confirms the significance of memristors in simulating 801 action potentials in excitable cells and also establishes a foundation for their application 802 in neuron modeling, artificial intelligence, and brain-like machine interfaces. Our 803 proposed model offers potential for enhancing adaptive neural networks, 804 neuroprosthetics, neuromorphic computing architectures, and the broader scope of 805 artificial intelligence, thereby aiding in the development of brain-like information 806 processing systems. 807

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Appendix: Abbreviations of the Model Parameters

C _m =Membrane Capacitance
E_{K} =Potential across K ⁺ ion channel memristor
E _I =Potential across mixed ion channel memristor
$E_{\rm L}$ =Potential across leakage channel
E _{Ca} =Potential across Ca ⁺² ion channel memristor
g _{K,V} =Voltage-sensitive K+ ion-channel conductance
g _I =Voltage-sensitive mixed ion channel conductance
g _L =Leakage channel conductance
g _{KCa} =Calcium activated potassium conductance
k _{Ca} =Rate constant for the efflux of the intracellular Ca ⁺² ions
ρ =Proportionality constant
$\lambda_n = Rate constant for k^+ ion-channel opening$
m_{∞} = Probability of activation of the mixed ion channel in steady state α_m = The rate at which the activation of the mixed ion channel closed gates transition to an open state(s ⁻¹) β_m = The rate at which the activation of the mixed ion channel open gates transition to the close state(s ⁻¹) h_{∞} = Probability of inactivation of the mixed ion channel in steady state α_n =The rate at which the inactivation of the mixed ion channel closed gates transition to an open state(s ⁻¹) β_n =The rate at which the inactivation of the mixed ion channel closed gates transition to an open state(s ⁻¹) n=Probability of n opening of the K+ ion channel memristor n_{∞} =Steady state value of n α_n =The rate at which K ⁺ ion channel closed gates transition to an open state(s ⁻¹) θ_n =The rate at which K ⁺ ion channel closed gates transition to an open state(s ⁻¹)

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Ref	erences	815
1.	Hodgkin, A.L.; Huxley, A.F. A quantitative description of membrane current and its application to conduction and excitation in nerve. <i>J. Physiol</i> 1952 , 117, 500-544.	816 817
2. 3.	Chua, L.O.; Kang, S.M. Memristive devices and systems. <i>Proc. the IEEE</i> . 1976 , 64, 209-223. Chua, L.O; Sbitnev, V.I.; Kim, H. Hodgkin-Huxley axon is made of memristors. <i>Int. J. Bifurcation and Chaos</i> . 2012 , 22, 1230011(1) - 1230011.	818 819 820
4.	Chua, L.O.; Sbitnev, V.I.; Kim, H. Neurons are poised near the edge of chaos. <i>Int. J. Bifurcation and Chaos.</i> 2012 , 22, 1250098 (1) - 1250098 (49).	821 822
5. 6. 7.	Chua, L. Hodgkin–Huxley equations implies Edge of chaos kernel. Jpn. J. Applied Physics. 2022 , 61(SM) SM0805(1)-(36). Chua, L. Everything you wish to know about memristor but are afraid to ask. <i>Radioengineering</i> . 2015 , 24, 319-368. Hodgkin, A.L.; Keynes, R.D. Experiments on the injection of substances into squid giant axons by means of microsyringe. <i>J.</i>	823 824 825
8. 9.	Morris, C.; Lecar, H. Voltage oscillations in the Barnacle giant muscle fiber. <i>J. of Biophysical Society</i> . 1981 , 35, 193-213. Sah, M.P.; Kim, H.; Eroglu, A.; Chua, L. Memristive model of the Barnacle giant muscle fibers. <i>Int. J. Bifurcation and Chaos</i> . 2016 , 26, 1630001 (1)-(40).	826 827 828 829
10.	Rajamani, V.; Sah, M.P.; Mannan, Z.I.; Kim, H.; Chua, L. Third-order memristive Morris-Lecar model of barnacle muscle fiber. <i>Int. J. Bifurcation and Chaos.</i> 2017 , <i>27</i> , 1730015 (1)-(58).	830 831
11.	Noble, D. A modification of the Hodgkin-Huxley equations applicable to Purkinje fibre action and peacemaker potentials. <i>J. of Physiology</i> . 1962 , 160, 317-352.	832 833
12.	Hudspeth, A. J.; Lewis, R.S. A model for electrical resonance and frequency tuning in saccular hair cells of the bull-frog Rana catesbeiana. <i>J. Physiology</i> . 1988 , 400, 275-297.	834 835
13.	Giguère, C.; Woodland, P.C. A computational model of the auditory periphery for speech and hearing research. <i>The Journal of the Acoustical Society of America</i> . 1994 , 95, 331-342.	836 837
14.	Nawrocki, R.A; Voyles, R. M.; Shaheen, S. E. A mini review of neuromorphic architectures and implementations. <i>IEE Transac-</i> <i>tions on Electron Devices</i> . 2016 , 63, 3819-3829.	838 839
15.	Lee, Y; Lee, T. W. Organic synapses for neuromorphic electronics: from brain inspired computing to sensorimotor nervetronics. <i>American Chemical Society</i> . 2019, 52, 964–974.	840 841
16.	Gentili, P. L. Photochromic and luminescent materials for the development of chemical artificial intelligence. <i>Dyes and Pigments</i> . 2022 , 205, 1-14.	842 843
17.	Peercy, B. E, Sherman, A. S. How pancreatic beta-cells distinguish long- and short-time scale CAMP signals. <i>Biophys. J.</i> 2010 , 99. 398-406.	844 845
18.	Pedersen. M. G. Contributions of mathematical modeling of Beta-cells to the understanding of beta-cell oscillations and insulin secretion. <i>Diabetes Technology Society</i> , 2009. 3. 12-20.	846 847
19.	Felix-Martinez, G. J.; Godlinez-Fernandez, J. R. <i>Mathematical models of electrical activity of the pancreatic</i> β - <i>cell. Islets.</i> 2014 . 6. e949195(1)- e949195(13).	848 849
20.	Kaestner, K. H.; Thompson, M. C.; Dor, Y.; Gill, R. G.; Glaser, B.; Kim, S. K.; Sander, M.; Stabler, C.; Stewart, A. F.; Powers, A. C. What is a β-cell ? -Chapter I in the Human Islet Research Network (HIRN) review series. <i>Molecular Metabolism</i> . 2021 , 53, 101323(1)- 101323 (6).	850 851 852
21.	Lenzen, S. The pancreatic beta cell: an intricate relation between anatomical structure, the signalling mechanism of glucose-in- duced insulin secretion, the low antioxidative defence, the high vulnerability and sensitivity to diabetic stress. <i>ChemTexts</i> . 2021 , 7, 1-6	853 854 855
22.	Marinelli, I.; Thompson, B. M; Parekh, V. S. Fletcher P. A. L. G. Giorda, A. S. Sherman, L. S. Satin, and R. Bertram, Oscillations in K(ATP) conductance drive slow calcium oscillations in pancreatic β-cells. <i>Biophysical Journal</i> , 2022 , 121, 1449-1464.	856 857
23.	Marinelli, I.; Parekh, V.; Fletcher, P.; Thompson, B.; Ren, J.; Tang, Xi.; Saunders, T. L.; Ha, J.; Sherman, A.; Bertram, R.; Satin, L. S. Slow oscillations persist in pancreatic beta cells lacking phosphofructokinase M, <i>Biophysical Journal</i> . 2022 , 121. 692-704.	858 859
24.	Mukai, E.; Fujimoto, S.; Inagaki, N. Role of Reactive Oxygen Species in Glucose Metabolism Disorder in Diabetic Pancreatic β- Cells. <i>MDPI, Biomolecules</i> . 2022, 12091228(1)-(15).	860 861
25.	Millette, K.; Rodriguez, K.; Sheng, X.; Finley, S. D.; Georgia, S. Exogenous Lactogenic Signaling Stimulates Beta Cell Replication In Vivo and In Vitro. <i>MDPI, Biomolecules</i> . 2022 , 12020215(1)-(10).	862 863
26.	Bertram R.; Marinell, I.; Fletcher, P. A.; Satin, L. S.; Sherman, A.S. Deconstructing the integrated oscillator model for pancreatic β -cells, <i>Mathematical Biosciences</i> . 2023 , 365, 1-14.	864 865
27.	Plant, R.E. Bifurcation and resonance in a model for bursting nerve cells. <i>Journal of mathematical biology</i> . 1981 , 11, 15-32.	866

28.	Chay, T.R. Eyring rate theory in excitable membranes. Application to neural oscillations. <i>Journal of physical chemistry</i> . 1983 , 87, 2935-2940.	867 868
29. 30.	Chay, T.R.; Keizer, J. Minimal model for membrane oscillations in the pancreatic β -cell. <i>J. biophysical society</i> . 1983 , 42, 181-190. Chay, T.R. Chaos in a three-variable model of an excitable cell. <i>Physics D</i> . 1985 , 16, 233-242.	869 870
31.	FitzHugh, R. Impulses and physiological states in theoretical models of nerve membrane. <i>Biophysical Journal</i> . 1961 , 1, 445-466.	871
32.	Chua, L. Five non-volatile memristor enigmas solved. <i>Applied Physics A Materials Science and Processing</i> . 2018 , 124, 563(1)-(43).	872
33.	Chua, L. If it's pinched it's a memristor. Semicond. Sci. Technol. 2014, 29, 104001(1)-104001(42).	873
34.	Chua, L.O.; Desoer, C.A. Kuh E. S. Linear and Nonlinear Circuits . McGraw-Hill book Co. New York. 1987.	874
35.	Chua, L.O. CNN: A Paradigm for Complexity, World Scientific. 1998.	875
36.	Chua, L.O. Local activity is the origin of complexity. Int. J. Bifurcation and Chaos. 2005. 15. 3435-3456.	876
37.	Chua, L. O. Local activity principle: Chua's riddle, "Turing machine, and universal computing rule 137", in The Chua Lectures:	877
	From Memristors and Cellular Nonlinear Networks to the Edge of Chaos. World Scientific Nonlinear Science Series A. 2020.	878
38.	Ascoli, A.; Demirkol, A.S.; Chua, L.O.; Tetzlaff, R. Edge of Chaos Theory Resolves Smale Paradox. <i>IEEE Trans. Circuit and System-</i> <i>L</i> 2022 69 1252-1265	879 880
39.	Sah, M.P.; Mannan, Z.I.; Kim, H.; Chua, L. Oscillator made of only One memristor and one battery. <i>Int. J. Bifurcation and Chaos.</i> 2015 , 25, 1530010(1)-(28).	881 882
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