Evaluating Pauli Errors on Cluster States by Weighted Distances

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Choong Pak Shen, *Davide Girolami**

Dr. Choong Pak Shen
UCSI College, Lot 39890, Jalan Cerdas Taman Connaught Cheras, 56000 Kuala Lumpur, Malaysia
Institute for Mathematical Research, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia
Prof. Davide Girolami
Politecnico di Torino, Corso Duca Degli Abruzzi, 24, 10129 Torino, Italy
Email Address: davegirolami@gmail.com

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We address the problem of evaluating the differences between quantum states before and after being affected by errors encoded in unitary transformations. Standard distance functions, e.g., the Bures length, are not fully adequate for such a task. Weighted distances are instead the appropriate information measures to quantify the distinguishability of multipartite states. Here, we employ the previously introduced weighted Bures length and the newly defined weighted Hilbert-Schmidt distance to quantify how much single-qubit Pauli errors alter cluster states. For both types of weighted distances, we find that different errors of the same dimension change cluster states in a different way, i.e., their detectability is in general different. Indeed, they transform an ideal cluster state into a state whose weighted distances from the input depends on the specific chosen Pauli rotation, as well as the position of the affected qubit in the graph related to the cluster state. As these features are undetected by using standard distances, the study proves the usefulness of weighted distances to monitor key but elusive properties of many-body quantum systems.

Introduction

Quantifying the differences between quantum states is a critical task in both theoretical and experimental quantum information processing. Indeed, several quantifiers of state difference have been invented. [1] An important class of them is represented by geometric distance functions, which are built to compare the physical properties of different configurations in terms of their distances in the abstract space of all possible quantum states of a system. [2,3]

Despite their wide applicability and obvious importance, such "standard" distances are not fully adequate to investigate quantum states of many particles, as they do not take into account their dimension. For such a reason, the concept of *weighted* distances has been introduced. [4] These quantities satisfy a set of desirable mathematical properties, just as standard distances, while factoring in the dimension of the system under scrutiny and, at the same time, the size of the apparatuses that implement appropriate discriminating measurements.

In particular, the weighted Bures length has a compelling operational interpretation: It quantifies the difficulty in distinguishing two quantum states by the most informative measurements. Subsequent studies have found that the weighted Bures length between the input and output states of a computation quantifies the complexity of the process, [5] and signals chaotic behaviour in quantum dynamics. [6,7]

In this paper, we expand the use of weighted distances by applying them to evaluate the difficulty of detecting errors, i.e., uncontrollable, unwanted changes of a state. Specifically, we calculate the weighted Bures length and a newly introduced weighted Hilbert-Schmidt distance between cluster states of three, four, and five qubits, and the same states after they have undergone unplanned single-qubit unitary rotations ("Pauli errors"). Cluster states are an important class of states that are entangled even when information about some system components is lost. That is, they are "robust", as they display genuine multipartite entanglement. [8–10] Also, they are the resource to perform one-way quantum computations. [11–13]

Our results show that there exists a hierarchy between unitary errors in terms of their effects on an initial, ideal cluster state. Specifically, input cluster states are sent by single-qubit Pauli errors into output states whose weighted distance from the input depends on the chosen Pauli error. That is, our ability to detect when a cluster state has been corrupted depends on the specific error. Moreover, we observe another surprising feature. As the number of qubits increases, phase flips (Pauli-Z errors) that affect "internal" qubits of one-dimensional cluster states send the system into a state that is closer to the input state in comparison with phase flips affecting qubits placed at

the boundary. Internally located errors are therefore more difficult to detect, somehow hidden in the structure of the system.
Remarkably, both results are independent of the employed weighted distance, as they describe how errors change subsystems of different size within systems prepared in cluster states. Also, neither phenomena can be observed by using any standard distances.

It is important to note that state transformation and state discrimination are "quite different tasks". [14] Indeed, every single unitary error can be corrected by reapplying the very same unitary. More generally, quantifiers of the complexity of an input/output transformation (how difficult it is to transform quantum states) and input/output distinguishability (how difficult it is to distinguish quantum states) are in general different kinds of information measures. Yet, the weighted Bures length, which reliably evaluates our ability to discriminate between two states via measurements, is also a lower bound to the experimental cost, in terms of physical resources, to transform the states into each other. [4]

The paper is organized as follows. In Section 1, we review the notion of weighted distances and their advantages with respect to standard geometric quantifiers of state difference. First, we recall the definition of weighted Bures length introduced in Ref. [4]. Then, we introduce the computationally friendly weighted Hilbert-Schmidt distance and prove some properties that inherits from the related standard distance. In Section 2, we present a case study: We evaluate the difference between cluster states plagued by unitary errors and their textbook form in terms of their weighted distances. In the Conclusion, we draw our final comments and suggest further lines of research.

1 From standard distances to weighted distances

1.1 Standard distances between quantum states

The difference between quantum states, e.g., between a target configuration and what one actually implements in an experiment, is customarily quantified by means of the state fidelity. The fidelity between two pure states is indeed equivalent to the state overlap:

$$F(|\psi\rangle\,,\,|\phi\rangle) = |\langle\psi|\phi\rangle|^2\,. \tag{1}$$

Remarkably, the overlap has a geometric interpretation, being related to the Fubini-Study distance, which reads $\cos^{-1}\sqrt{F(|\psi\rangle\,,\,|\phi\rangle)}$. [1,15–17] Generalizing the fidelity to mixed states, [18–21] one has

$$F(\rho,\,\sigma) = \left(\mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)^2\,. \tag{2}$$

Among many distance functions for mixed states, the Bures length between two density matrices ρ and σ is the natural generalization of the Fubini-Study distance:

$$B(\rho,\,\sigma) = \cos^{-1}\sqrt{F(\rho,\,\sigma)}. \tag{3}$$

State fidelity and Bures length are ubiquitous in quantum information theory. They are well-motivated quantities, as they provably quantify the difficulty in discriminating quantum states by means of a single measurement on the global system. Further, they are the most popular quantifiers of how close an engineered quantum state, say ρ_N, is to the desired target σ_N.
Yet, they are not fully satisfactory metrics when one wants to compare states of multipartite quantum systems. Consider for example the following quantum states:

$$|0\rangle^{\otimes N}\,,\;\; |0\rangle^{\otimes N-k}\,|1\rangle^{\otimes k}\,. \tag{4}$$

If we evaluate their differences by means of their Bures length, they are maximally far for any value of k. Yet, the larger is k, the easier it is to experimentally distinguish them, as there are more (local) measurements that can discriminate between these two states. In general, overlap-based distances reach maximal value when two global states are orthogonal, no matter how close the marginal density matrices may be.

Another relevant example is the comparison between a state of N qubits that displays maximal entanglement, say a GHZ state, with a classically correlated mixed state:

$$\frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes N}+|1\rangle^{\otimes N}\right),\ \frac{1}{2}\left(|0\rangle\langle 0|^{\otimes N}+|1\rangle\langle 1|^{\otimes N}\right). \tag{5}$$

The difficulty to experimentally discriminate these two states increases with N, as only global measurements over all qubits would output different data, while the Bures length between them is independent of the system size.
In order to overcome these issues, while retaining the explanatory power of distance functions, the concept of weighted distances was introduced. [4] They are generalizations of standard distances that take into account the size of both the system and the possible measurements one can perform on it.

1.2 Weighted Bures length

We review the construction of the weighted Bures length between two preparations ρ_N and σ_N of an N-particle system (see Ref. [4] for full details), while the argument applies to any distance function. Suppose cooperating agents independently measure on different subsystems of k_α particles, with $k_\alpha \leq N$, of two copies of the system, which are prepared in the states ρ_N and σ_N, respectively. Each agent evaluates the difference between two states of the assigned subsystem by the Bures length $B(\rho_{k_\alpha}, \sigma_{k_\alpha})$. In particular, they perform the optimal measurement $\mathcal{M}^{k_\alpha} = \rho_{k_\alpha}^{-1/2}\sqrt{\rho_{k_\alpha}^{1/2}\sigma_{k_\alpha}\rho_{k_\alpha}^{1/2}}\rho_{k_\alpha}^{-1/2}$ to discriminate the marginal states of k_α particles. The setup defines a measurement partition $P_{k_\alpha} := \left\{\mathcal{M}^{k_\alpha},\ \sum_a k_\alpha = N\right\}$.
Then, one builds a weighted sum of each agent contribution $B(\rho_{k_\alpha}, \sigma_{k_\alpha})$, assigning a weight $1/k_\alpha$ to each of them, i.e., evaluating their importance as inversely proportional to the size of the related subsystem. The reason is the following: One can quantify the difficulty in realizing a measurement with the size of the measured subsystem k_α. Since a distance must quantify how easy it is to distinguish the two states, i.e., how easy it is to experimentally carry out the related measurement, a sound choice of weight is $1/k_\alpha$. Finally, one defines the weighted Bures length by optimizing the measurement strategy, which implies to maximize the weighted sum of local distances over all possible partitions:

$$D_B(\rho_N, \sigma_N) := \max_{P_{k_\alpha}} \sum_\alpha \frac{1}{k_\alpha} B\left(\rho_{k_\alpha}, \sigma_{k_\alpha}\right). \tag{6}$$

The weighted Bures length meets a set of desirable properties:

- $D_B(\rho_N, \sigma_N) \geq 0$ (non-negativity)
- $D_B(\rho_N, \sigma_N) = 0 \iff \rho_N = \sigma_N$ (faithfulness)
- $D_B(\rho_N, \sigma_N) \geq D(\Lambda_1(\rho_N), \Lambda_1(\sigma_N)),\ \forall\, \Lambda_1$ (contractivity under single qubit operations Λ_1)
- $D_B(\rho_N, \sigma_N) \leq D_B(\rho_N, \tau_N) + D_B(\tau_N, \sigma_N)$ (triangle inequality)

The weighted Bures length also has a second operational meaning that is not shared by the standard Bures length. It is shown to be the lower bound of experimental cost of the state transformation $\rho_N = U\sigma_N U^\dagger, U = e^{-iHt}$:

$$N\,E\,t \geq D_B(\rho_N, \sigma_N), \tag{7}$$

in which $E = (\lambda_{max} - \lambda_{min})/2$, and $\lambda_{max(min)}$ is the largest (smallest) eigenvalue of the Hamiltonian H. Hence, the weighted Bures length lower bounds the cost of transforming ρ_N into σ_N (and vice versa), as quantified by the product of the size of the system N, the energy parameter E, and the available time t. [4]

1.3 Weighted Hilbert-Schmidt distance

While the weighted Bures length enjoys desirable properties, one can build an alternative weighted distance from another standard distance. This effort is useful for two reasons. First, calculating the weighted Bures length is theoretically and experimentally expensive, as the full states ρ_N, σ_N that one wants to compare need to be known. Yet, quantifying a weighted distance built from another standard distance may require only knowledge of some (polynomial) functions of the two states under study, e.g., their purities $\mathrm{Tr}\left(\rho_N^2\right), \mathrm{Tr}\left(\sigma_N^2\right)$, which are both easier to compute on paper and to experimentally reconstruct than the full density matrices. Second, we would like to understand if some interesting phenomenon that is discovered by employing the weighted Bures length is independent on the chosen metric. In other words, we would like to verify if the features of many-body quantum systems found by using the weighted Bures length are truly universal, and they can be therefore explored by using another weighted distance, rather than emerging only because of the technical properties of the Bures length.

Hence, we define an alternative weighted distance function, which is simpler to compute at both theoretical and experimental level. We here introduce the weighted Hilbert-Schmidt distance:

$$D_{HS}(\rho_N, \sigma_N) := \max_{P_{k_\alpha}} \sum_a \frac{1}{k_\alpha} d_{HS}(\rho_{k_\alpha}, \sigma_{k_\alpha}), \tag{8}$$

where d_{HS} is the (square root of the) standard Hilbert-Schmidt distance, which is defined as

$$d_{HS}(\rho_N, \sigma_N) := \sqrt{\mathrm{Tr}\left(\rho_N^2 + \sigma_N^2 - 2\,\rho_N \sigma_N\right)}. \tag{9}$$

Note that the square root here is crucial. We will run a numerical comparison with the weighted Bures length, so both quantities need to be functions of the state eigenvalues elevated to the same powers.
The weighted Hilbert-Schmidt distance inherits the following properties:

- $D_{HS}(\rho_N, \sigma_N) \geq 0$ (non-negativity)
  Proof: A weighted distance is the sum of standard distances with positive weights, so the property is satisfied.
- $D_{HS}(\rho_N, \sigma_N) = 0 \iff \rho_N = \sigma_N$ (faithfulness)
  Proof: Since all the weights are positive, the weighted sum for an arbitrary partition P_{k_α} is non-zero if and only if there is at least a non-zero term $d_{HS}(\rho_{k_\alpha}, \sigma_{k_\alpha})$. Hence, the claim is proven.
- $D_{HS}(\rho_N, \sigma_N) \leq D_{HS}(\rho_N, \tau_N) + D_{HS}(\tau_N, \sigma_N)$ (triangle inequality)
  Proof: For an arbitrary term of the weighted sum related to an arbitrary partition, one has $\frac{1}{k_\alpha} d_{HS}\left(\rho_{k_\alpha}, \sigma_{k_\alpha}\right) \leq \frac{1}{k_\alpha}\left\{d_{HS}\left(\rho_{k_\alpha}, \tau_{k_\alpha}\right) + d_{HS}\left(\tau_{k_\alpha}, \sigma_{k_\alpha}\right)\right\}$. Calling $\hat{k}_\alpha$ the weights related to the partition that maximizes the weighted sum of the terms $d_{HS}\left(\rho_{k_\alpha}, \sigma_{k_\alpha}\right)$, one has

$$\begin{aligned} D_{HS}(\rho_N, \sigma_N) &= \sum_\alpha \frac{1}{\hat{k}_\alpha} d_{HS}\left(\rho_{\hat{k}_\alpha}, \sigma_{\hat{k}_\alpha}\right) && (10)\\ &\leq \sum_\alpha \frac{d_{HS}\left(\rho_{\hat{k}_\alpha}, \tau_{\hat{k}_\alpha}\right) + d_{HS}\left(\tau_{\hat{k}_\alpha}, \sigma_{\hat{k}_\alpha}\right)}{\hat{k}_\alpha} && (11)\\ &\leq D_{HS}(\rho_N, \tau_N) + D_{HS}(\tau_N, \sigma_N). && (12) \end{aligned}$$

Note that the weighted Hilbert-Schmidt distance is not monotonically decreasing under local operations, as the related standard distance can increase under noisy channels. [2,22] Yet, since the Hilbert-Schmidt distance is a function of state purities $\mathrm{Tr}(\rho_N^2), \mathrm{Tr}(\sigma_N^2)$ and state overlaps $\mathrm{Tr}\left(\rho_N \sigma_N\right)$, it is manifestly easier to compute than the Bures length, both theoretically and experimentally. Indeed, a plethora of measurement schemes has been designed to evaluate state purities and overlaps without full state tomography and full spectrum reconstruction. [23–29]

In the next section, we will run a comparison of the weighted Bures length and the weighted Hilbert-Schmidt distance. We will employ them to evaluate the difference between ideal cluster states and their corrupted version, when subject to unitary errors. We will observe that we can rank such errors based on how difficult it is their detection, and such a classifications is independent of the employed weighted distance.

2 Case Study: Evaluation of unitary errors in cluster states by weighted distances

2.1 Cluster states

Cluster states are a subset of graph states which are simultaneous eigenstates of commuting Pauli operators with eigenvalue one. They enable to perform one-way (also called "measurement-based") quantum computations. [11–13] By using the stabilizer formalism, the density matrix of N-qubit cluster states can be written in a compact form: [9,30]

$$\rho = \prod_{i=1}^{N} \frac{I_i + g_i}{2}, \tag{13}$$

where I_i is the N-body identity matrix and $g_i = X_i \bigotimes_{j \in N(i)} Z_j$ are the stabilizing operators. Here, X_i and Z_j are Pauli X and Z matrices, respectively, while $N(i)$ denotes the neighbourhood of i-th qubit (consisting of all the qubits that are adjacent to the i-th qubit in the graphical representation of the state). Each cluster state represents a specific correlation structure of qubits placed at vertices of a graph.

The stabilizer formalism allows us to write the reduced density matrices of cluster states in terms of their stabilizing operators. [8,10] For the sake of clarity, we provide the explicit expressions of the global and reduced density matrices of the cluster states that we selected for this study in the Supporting Information file available online. Figure 1 depicts four different cluster states that we study explicitly in this work.

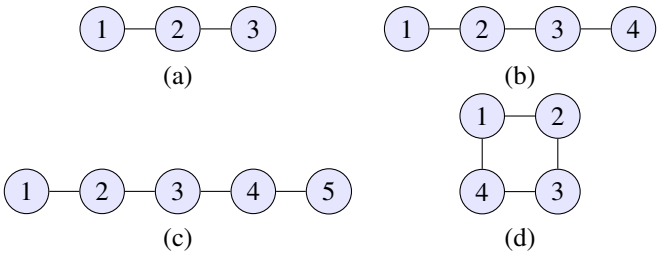

Figure 1: The graphs (a)-(d) depict the four different configurations of cluster states that are here under scrutiny.

2.2 Single-qubit Pauli errors

We focus our study on the detectability of single-qubit Pauli errors, i.e., unwanted implementations of unitary evolutions generated by Pauli matrices. This choice is justified as follows. Given an N-qubit quantum computer, any error can be described in terms of tensor products of Pauli errors. [31,32] Specifically, given a stabilizer state, such as a cluster state, any superposition of Pauli errors can be represented by a statistical mixture of Pauli errors. [30] Consequently, the ability to detect and correct (products of) Pauli errors is sufficient to detect and correct any noise affecting these states. Further, single-qubit (and two-qubit) errors are the most frequent in experimental implementations, and error detection schemes almost always rely on single-qubit measurements. [32] Indeed, the preparation of cluster states only requires controlled-Z gates to act between the nearest-neighbour qubits in the $|+\rangle$ basis. As a result, the most common experimental errors are single-qubit unwanted rotations (bit/phase flips) on a specific system component or the neighbouring qubits. [8–13,30] Hence, it is important to be able to properly quantify the influence of single-qubit Pauli errors on cluster states, as they are arguably the most relevant ones for practical purposes.

Any single-qubit Pauli errors will transform cluster states into orthogonal states. [30] Thus, standard distance functions, when being computed between the ideal cluster state and its corrupted version, will always reach the maximal value, regardless of how many and what kind of single-qubit Pauli errors are present in the cluster states. In other words, overlap-based distances are not fully adequate to discriminate how much different errors affect a quantum computation. We therefore employ weighted distances to evaluate error detectability in this scenario.

Table 1: Weighted Bures length of cluster states before and after they have been affected by single-qubit Pauli errors. In the first column, we list the cluster states under scrutiny. In the second and third columns, we identify the Pauli errors that generate output states whose weighted Bures lengths are $\pi/6$ and $\pi/4$ from the input cluster states, respectively. Note that the standard Bures length reaches the maximal value $\pi/2$ in all these cases.

Cluster states	$D_B = \pi/6$	$D_B = \pi/4$
Three-qubit one-dimensional	$X_1,\ X_3,\ Z_2$	$Z_1,\ Z_3,\ X_2,\ Y_1,\ Y_2,\ Y_3$
Four-qubit one-dimensional	$X_1,\ X_4,\ Z_2,\ Z_3$	$Z_1,\ Z_4,\ X_2,\ X_3,\ Y_1,\ Y_2,\ Y_3,\ Y_4$
Four-qubit two-dimensional	$X_1,\ X_2,\ X_3,\ X_4$	$Z_1,\ Z_2,\ Z_3,\ Z_4,\ Y_1,\ Y_2,\ Y_3,\ Y_4$
Five-qubit one-dimensional	$X_1,\ X_3,\ X_5,\ Z_2,\ Z_3,\ Z_4,\ Y_3$	$Z_1,\ Z_5,\ X_2,\ X_4,\ Y_1,\ Y_2,\ Y_4,\ Y_5$

2.3 Evaluation of Pauli errors by weighted Bures length

We will now evaluate the detectability of Pauli errors when they are perturbing these cluster states. Unlike standard distance functions, weighted distances take into the consideration the statistical distinguishability of both global and marginal density matrices, in order to assess the differences between two quantum states. The construction is particularly appealing when we want to compare states that are invariant under (possibly all, or at least some) permutation of subsystems. In such a case, the maximization of the weighted sum over all system partitions in Eq. (6) is easily computable.

First, we employ the weighted Bures length to quantify how cluster states are affected by Pauli errors. We consider four different cluster states: Three-, four-, and five-qubit one-dimensional cluster states, and four-qubit two-dimensional cluster states, as shown in Figure 1. We show how different single-qubit rotations change such states, by evaluating the weighted Bures length between the cluster states before and after the error is applied. The results are reported in Table 1. They show that, surprisingly, different single-qubit rotations generate states that are placed at different distances from the original cluster states in the manifold of quantum states. Hence, the detectability of such errors is different.

For example, consider the Z_2 error on three-qubit one-dimensional cluster states. The possible elements of a three-qubit measurement partition P_{k_α} are $\{\{1\}\ ,\ \{2\}\ ,\ \{3\}\ ,\ \{12\}\ ,\ \{13\}\ ,\ \{23\}\ ,\ \{123\}\}$. Since the one-body marginals of three-qubit cluster states are maximally mixed, local unitary operations will not be detectable by one-body measurements: $B(\rho_i, \sigma_i) = 0,\ i = 1, 2, 3$, in which ρ_i, σ_i are the input and output one-body marginals of three-qubit cluster states, respectively. The Z_2 error cannot be detected even by two-body measurements, since it is part of the stabilizers of ρ_{12} and ρ_{23}. Therefore, only global measurements will detect the Z_2 error: the weighted Bures length between a three-qubit one-dimensional cluster state and its altered version is $D_B = \pi/6$. See the Supporting Information file for the full study on this cluster state.

Next, consider the Z_2 error on four-qubit two-dimensional cluster states: it can be detected by two-body measurements on ρ_{24}^{2D} and its corrupted version. On the other hand, the Z_2 error cannot be detected by performing two-site measurements on four-qubit one-dimensional cluster states, so in this case the output state is "closer" to the ideal cluster state. Instead, it changes the three-body marginals like ρ_{123}^{1D}. Therefore, the effect of the Z_2 gate on four-qubit one-dimensional cluster states is different than on four-qubit two-dimensional cluster states, as captured by different values of weighted Bures length. The reason is that the system partition that maximizes the weighted sum in Eq.(6) is different.

Also, we observe that the geometry of cluster states determines the subset of single-qubit Pauli errors with a smaller weighted distance measure. Specifically, unwanted phase flips on the "internal" qubits of one-dimensional cluster states ($\{Z_2\}$ for three-qubit one-dimensional cluster states, $\{Z_2, Z_3\}$ for four-qubit one-dimensional cluster states and $\{Z_2, Z_3, Z_4\}$ for five-qubit one-dimensional cluster states) are more difficult to detect than phase flips affecting qubits at the boundary.

2.4 Evaluation of Pauli errors by weighted Hilbert-Schmidt distance

Now, we repeat the calculations by employing the weighted Hilbert-Schmidt distance introduced in Eq. (8), which is in general simpler to evaluate than the weighted Bures length. Also, we want to verify whether unitaries of the

Table 2: Weighted Hilbert-Schmidt distance between cluster states before and after they have been affected by single-qubit Pauli errors. In the first column, we list the cluster states under scrutiny. In the second and third columns, we list the Pauli errors and the related weighted Hilbert-Schmidt distance between input and output states. Note that the standard Hilbert-Schmidt distance takes the maximal value $\sqrt{2}$ in all these cases.

Cluster states	$\mathbf{D_{HS} = \sqrt{2}/3}$	$\mathbf{D_{HS} = 1/2}$
Three-qubit one-dimensional	$X_1,\ X_3,\ Z_2$	$Z_1,\ Z_3,\ X_2,\ Y_1,\ Y_2,\ Y_3$
Cluster states	$\mathbf{D_{HS} = \sqrt{2}/4}$	$\mathbf{D_{HS} = 1/2}$
Four-qubit one-dimensional	$X_1,\ X_4,\ Z_2,\ Z_3$	$Z_1,\ Z_4,\ X_2,\ X_3,\ Y_1,\ Y_2,\ Y_3,\ Y_4$
Cluster states	$\mathbf{D_{HS} = \sqrt{2}/4}$	$\mathbf{D_{HS} = 1/2}$
Four-qubit two-dimensional	$X_1,\ X_2,\ X_3,\ X_4$	$Z_1,\ Z_2,\ Z_3,\ Z_4,\ Y_1,\ Y_2,\ Y_3,\ Y_4$
Cluster states	$\mathbf{D_{HS} = 1/3}$	$\mathbf{D_{HS} = 1/2}$
Five-qubit one-dimensional	$X_1,\ X_3,\ X_5,\ Z_2,\ Z_3,\ Z_4,\ Y_3$	$Z_1,\ Z_5,\ X_2,\ X_4,\ Y_1,\ Y_2,\ Y_4,\ Y_5$

same dimension still generate states with different distances from the input. That is, we investigate if this surprising fact depends on the employed distance function, rather than being a generic feature of quantum dynamics.

As we said before, single-qubit Pauli errors transform ideal cluster states into orthogonal states. Therefore, the state overlap of input and output states is zero. This means that the weighted Hilbert-Schmidt distance of cluster states to their alterations depends only on their global and marginal purities (as the overlap between orthogonal states is zero). The results are reported in Table 2.
We observe a similarity between the results obtained by employing the weighted Bures length and the weighted Hilbert-Schmidt distance. As a result, the relation between detectability of errors in 1D cluster states and their location seems not a byproduct of choosing the weighted Bures length. That is, there may be a relation between error detectability and their location. We stress again that the Hilbert-Schmidt distance is easier to compute for arbitrary quantum states than the weighted Bures length and even other legit choices (e.g., a "weighted trace distance"), since it is a function of state purity and overlaps. This practical reason justifies the use of the standard Hilbert-Schmidt distance, and therefore its "weighted" version, in quantum information science, even if it is not contractive under noisy channels. [22]. The fact that by using such a metric we can replicate the results obtained by computing the weighted Bures length is a promising first step towards a full-fledged study of error detectability in cluster states of arbitrary size, which may be impossible if full state knowledge is needed.

Specifically, we can establish a hierarchy among Pauli matrices in terms of how much they affect the cluster states, independently of the employed distance measures. For example, for three-qubit one-dimensional cluster states, the very same X_1, X_3, Z_2 errors that have the least effects on the initial state, as quantified by a smaller weighted Bures length, also output the closest state according to the weighted Hilbert-Schmidt distance, which takes the value $\sqrt{2}/3 \approx 0.47$. For the rest of the Pauli errors, the weighted Hilbert-Schmidt distance is $1/2$, since the optimal partition is made of a pair of two-body marginal states. The explicit calculations for the three-qubit one-dimensional cluster states are presented in the Supporting Information file.

In the case of four-qubit one-dimensional cluster states, given the X_1, X_4, Z_2, Z_3 errors, the optimal partition is the full four-body one: Even though three-body measurements can be used to detect those errors, since the purity of mixed states is obviously lower than one, the Hilbert-Schmidt distance computed between the global pure states is larger ($\sqrt{2}/4 \approx 0.35$) than between three-body marginals ($1/3$). Similarly, for X_1, X_2, X_3, X_4 errors, the weighted Hilbert-Schmidt distance between inputs and outputs for four-qubit two-dimensional states is $\sqrt{2}/4$. The least affecting errors for the five-qubit case are instead X_1, X_3, X_5, Z_2, Z_3, Z_4, Y_3: The optimal measurement is a three-body one and the weighted Hilbert-Schmidt distance takes the value $1/3$.

Finally, by employing the weighted Hilbert-Schmidt distance, we observe again how the geometry of cluster states affects the detectability of single-qubit Pauli errors. Internal Z errors are still the most difficult to spot on one-dimensional cluster states via measurements. We conjecture that, considering one-dimensional cluster states of increasing size, Z errors on the inner qubits of the cluster states would still generate states that are closer to

the inputs. Also, more complex two-dimensional states than the one studied here may be prone to hide internally located errors.

Conclusion

We evaluated how local unitary errors alter cluster states by means of the weighted Bures length and the newly defined weighted Hilbert-Schmidt distance. We showed that, while all single-qubit Pauli errors transform ideal cluster states into orthogonal erroneous states, the distinguishability of such corrupted states with respect to the input configuration can be different. Unitary errors of the same dimension (affecting single qubits) produce states that have different weighted distance from the initial state, which physically means that the size of the optimal measurements that detect them depends on the specific error. Note that this feature is independent of the employed weighted distance.

Our results show how weighted distances can be more informative than standard distances in evaluating important properties of many-body quantum systems.
As potential follow-on projects related to error correction, we anticipate that, just like standard distances, weighted distances can be split into genuine "quantum" and "classical" parts. [33] These more refined weighted quantum distances could be employed to evaluate how errors affect quantum resources rather than the whole quantum states. This is much needed to investigate properties of large-dimensional entangled states, [34] and their resistance to various error sources. Other interesting avenues of investigation are extensions of our case study to other classes of multipartite quantum states with complex correlation structure, e.g., other types of graph states. Finally, our findings call for a rigorous proof to our conjecture that the detectability of errors depends on the geometry of the register under scrutiny, regardless of the employed weighted distance. In particular, we do expect that the location of errors in $2D$ and $3D$ cluster states affects their detectability.

Further, the weighted distances may help explore properties of many-body systems that cannot be investigated by means of standard distances. Indeed, important computational resources like entanglement and other types of quantum correlations are not invariant under unitary transformations that involve more than one qubit (e.g., a two-qubit CNOT gate, or a two-qubit error), so the unitarily invariant quantities like the standard distances cannot be the most informative proxies for such properties. [35] Weighted distances are instead only invariant under single-qubit unitaries, [4] being more appropriate measures of distinguishability of multipartite states. In particular, the weighted Hilbert-Schmidt distance is remarkably easy to compute, conversely to other recently introduced metrics that overcome the limitations of standard distances, such as the quantum "Wasserstein" distance introduced in [35] . Of course, since there are infinite classes of multipartite entangled states, [36] it is unlikely to find a one-fits-all method to efficiently detect errors. Hence, we find it remarkable that, whenever the analysis can be limited to cluster states, the weighted distances showed to be reliable indicators of noise affecting quantum states.

Supporting Information
Supporting Information is available from the Wiley Online Library or from the author.

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