Supplementary material for online publication

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Abstract

This is the online supplementary materials for the paper, "Degree evolution in a general growing network". Appendix 1 is the simulation study that was conducted to numerically demonstrate the asymptotic behavior described in the main results, Proposition 2.1 and Proposition 2.2. Appendix 2 contains the proofs of the preliminary results, Lemma 3.2 and Lemma 3.3, which are used in the proofs of Propositions 2.1 and 2.2.

Appendix 1. Simulation study for the main results

Three independent graph processes were simulated under three different deletion probabilities. The number of time steps for each process is set as $t = 10^4$. The probabilities of vertex addition and edge addition at each time step were set as $p_1 = 1/2$ and $p_2 = 1 - p_1 - p_3$, respectively, for all three processes. Note that p_3 is the parameter we tune in the simulation below. More precisely, p_3 is the probability of an edge being deleted and each process was generated under a different deletion probability p_3. One of the three processes was generated using the critical value of $p_3 = 1/3$. The other two processes were generated under $p_3 = 1/5$ and $p_3 = 1/10$, respectively.

Figure 1 shows the vertices degree of the three processes that were simulated under the critical case $p_3 = 1/3$ (red lines), $p_3 = 1/5$ (blue lines) and $p_3 = 1/10$ (green lines), respectively. Vertices that were approximately born at the same time s are plotted in the same figure. In each figure, the solid lines represent the asymptotic rescaled degree given by Propositions 2.1 and 2.2, whilst the dotted lines represent the simulated degree of the vertices.

It is noted that the convergence for the critical case, $p_3 = 1/3$, is not apparent in the figures in comparison to that of $p_3 < 1/3$. This is due to the relatively small range of values for the asymptotic degree in the critical case of $p_3 = 1/3$ given by Proposition 2.1. For example, the asymptotic degrees of the vertices at time step $t = 10^4$ in Figure 1 (a) with $s = 1$ under the cases of $p_3 = 1/3, p_3 = 1/5$ and $p_3 = 1/10$ are $\approx 3, \approx 22$ and ≈ 56, respectively. However, the asymptotic degrees of the vertices in Figure 1 (c) with $s \approx 70$ under the cases of $p_3 = 1/3, p_3 = 1/5$ and $p_3 = 1/10$ are $\approx 2, \approx 5$ and ≈ 8, respecitvely. This uneven scale of the $y-$axis results in the lack of visible growth of the red curves in comparison to the cases of $p_3 < 1/3$.

Appendix 2. Proof of supporting lemmas for the main results

Since Hoeffding's inequality is used for the proof of Lemma 3.2 below, it is provided here for the convenience of the readers.

Lemma (Hoeffding's inequality)**.** Let $X_1, \dots, X_n$ be independent random variables. Assume that, for every $i \in \{1, \dots, n\}$, one can find two constants a_i and b_i with $a_i < b_i$ such that $a_i \le X_i \le b_i$ almost surely for every $i \in \{1, \dots, n\}$. Define $S_n := \sum_{i=1}^n (X_i - \mathbb{E}(X_i))$. Then, for every $x > 0$, setting $D_n := \sum_{i=1}^n (b_i - a_i)^2$ we have $\mathbb{P}(S_n \ge x) \le \exp\{-2x^2/D_n\}$ and $\mathbb{P}(S_n \le -x) \le \exp\{-2x^2/D_n\}$.

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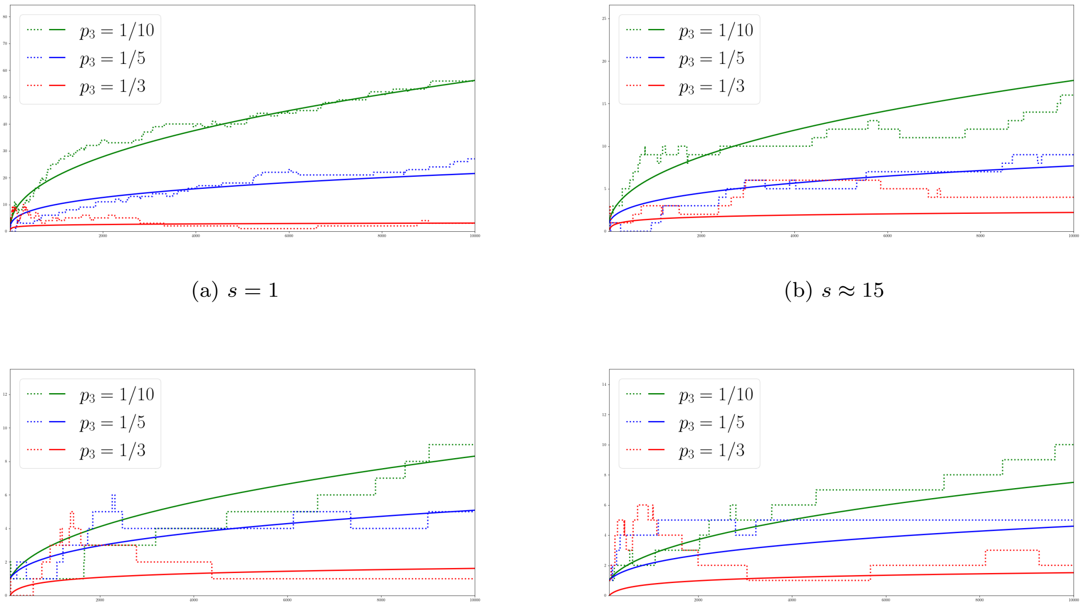

(a) $s = 1$ (b) $s \approx 15$

(c) $s \approx 70$ (d) $s \approx 105$

Figure 1: Vertices degree against time $t = 10^4$.

Proof of Lemma 3.2

Lemma 3.2 was used in the proof of Proposition 2.1 and is provided here for the convenience of the readers.

Lemma 3.2. Let $p_3 \in (0, 1/2)$, and let $\lambda = \lambda_t$ be such that $C \leq \lambda = o(t)$ as $t \to \infty$, for some finite constant $C > 1$. Then there exists a $t_0 \in \mathbb{N}$ such that, for all $t \geq t_0$, we have

$$\mathbb{P}(|E_t - t(1-2p_3)| > \lambda) \leq 3e^{-\frac{\lambda^2(1-1/C)^2}{8t}}. \tag{1}$$

Proof of Lemma 3.2. Let $(U_t)_{t\geq 1}$ be a sequence of independent and identically distributed random variables taking values in $\{-1, 1\}$ with $\mathbb{P}(U_1 = 1) = p_3 = 1 - \mathbb{P}(U_1 = -1)$. Define $\widehat{E}_0 := 1$ and, iteratively,

$$\widehat{E}_t = \widehat{E}_{t-1} + 1 - 2\mathbb{1}_{\{U_t=1\}}\mathbb{1}_{\{\widehat{E}_{t-1}\geq 1\}} \tag{2}$$

for $t \geq 1$. Then $\widehat{E}_t \overset{d}{=} E_t$ for every $t \geq 0$ (this can be seen e.g. using induction on t). Note that, defining $X_i := \mathbb{1}_{\{U_i=1\}}\mathbb{1}_{\{\widehat{E}_{i-1}\geq 1\}}$ for $i \geq 1$, from (2) we see that $\widehat{E}_t = 1 + \sum_{i=1}^{t}(1 - 2X_i)$. Then, since $X_i \leq \mathbb{1}_{\{U_i=1\}}$, we see that $\widehat{E}_t \geq \sum_{i=1}^{t}(1 - 2\mathbb{1}_{\{U_i=1\}})$ for all t and by Hoeffding's inequality we obtain

$$\begin{aligned}\mathbb{P}\left(E_t \leq t(1-2p_3) - \lambda\right) &\leq \mathbb{P}\left(\sum_{i=1}^{t}(1 - 2\mathbb{1}_{\{U_i=1\}}) \leq t(1-2p_3) - \lambda\right) \\ &= \mathbb{P}\left(\sum_{i=1}^{t}(\mathbb{1}_{\{U_i=1\}} - p_3) \geq \lambda/2\right) \leq e^{-\lambda^2/2t}.\end{aligned} \tag{3}$$

Next, let $H = H_t \in \mathbb{N}$ with $H \ll t$ to be specified later, and define the event $\mathcal{E}_H = \{\widehat{E}_k \geq 1 \;\forall\; H \leq k \leq t-1\}$. Then, since $E_t \overset{d}{=} \widehat{E}_t$, we can write

$$\mathbb{P}\left(E_t \geq t(1-2p_3) + \lambda\right) \leq \mathbb{P}\left(\{\widehat{E}_t \geq t(1-2p_3) + \lambda\} \cap \mathcal{E}_H\right) + \mathbb{P}\left(\mathcal{E}_H^c\right).$$

Since $X_i \geq 0$, we clearly have $\widehat{E}_H = 1 + \sum_{i=1}^{H}(1-2X_i) \leq H+1$. Because, on the event $\mathcal{E}_H$ we have $1-2X_i = 1-2\mathbb{1}_{\{U_i=1\}}$ for $H+1 \leq i \leq t$, we then obtain

$$\begin{aligned}\mathbb{P}\left(\{\widehat{E}_t \geq t(1-2p_3)+\lambda\} \cap \mathcal{E}_H\right) &= \mathbb{P}\left(\left\{\widehat{E}_H + \sum_{i=H+1}^{t}(1-2X_i) \geq t(1-2p_3)+\lambda\right\} \cap \mathcal{E}_H\right)\\ &\leq \mathbb{P}\left(\sum_{i=H+1}^{t}(1-2\mathbb{1}_{\{U_i=1\}}) \geq t(1-2p_3)+\lambda-H-1\right)\\ &= \mathbb{P}\left(\sum_{i=H+1}^{t}(\mathbb{1}_{\{U_i=1\}}-p_3) \leq Hp_3 - \frac{\lambda}{2}\left(1-\frac{1}{\lambda}\right)\right). \qquad (4)\end{aligned}$$

Taking $H := \lfloor \frac{\lambda}{4p_3}(1-1/\lambda)\rfloor$ we see that the probability in (4) is at most

$$\mathbb{P}\left(\sum_{i=H+1}^{t}(\mathbb{1}_{\{U_i=1\}}-p_3) \leq -\frac{\lambda}{4}(1-1/\lambda)\right) \leq e^{-2\frac{\lambda^2(1-1/\lambda)^2}{16(t-H)}} \leq e^{-\frac{\lambda^2(1-1/C)^2}{8t}},$$

where the first inequality follows again from Hoeffding's inequality. To bound $\mathbb{P}(\mathcal{E}_H^c)$ we note that, by a union bound,

$$\mathbb{P}(\mathcal{E}_H^c) = \mathbb{P}\left(\exists k \in [H, t-1] : \widehat{E}_k = 0\right) \leq \sum_{k=H}^{t-1}\mathbb{P}\left(1+\sum_{i=1}^{k}(1-2X_i)=0\right).$$

Since $1-2X_i \geq 1-2\mathbb{1}_{\{U_i=1\}}$ we have that $\sum_{i=1}^{k}(1-2X_i) \geq \sum_{i=1}^{k}(1-2\mathbb{1}_{\{U_i=1\}})$ and hence we arrive at

$$\sum_{k=H}^{t-1}\mathbb{P}\left(\sum_{i=1}^{k}(1-2\mathbb{1}_{\{U_i=1\}}) \leq 0\right) \leq \sum_{k=H}^{t-1}\mathbb{P}\left(\sum_{i=1}^{k}(\mathbb{1}_{\{U_i=1\}}-p_3) \geq \frac{k}{2}(1-2p_3)\right) \leq \sum_{k=H}^{t-1} e^{-\frac{k(1-2p_3)^2}{2}}, \qquad (5)$$

where the last inequality follows once more from Hoeffding's inequality. Using the formula for the geometric sum, we obtain

$$\sum_{k=H}^{t-1} e^{-\frac{k(1-2p_3)^2}{2}} = \frac{1-e^{-\frac{t(1-2p_3)^2}{2}}}{1-e^{-\frac{(1-2p_3)^2}{2}}} - \frac{1-e^{-\frac{H(1-2p_3)^2}{2}}}{1-e^{-\frac{(1-2p_3)^2}{2}}} \leq \frac{e^{-\frac{H(1-2p_3)^2}{2}}}{1-e^{-\frac{(1-2p_3)^2}{2}}}.$$

We recall that $H = \lfloor \frac{\lambda}{4p_3}(1-1/\lambda)\rfloor$ and then

$$\frac{H(1-2p_3)^2}{2} \geq \frac{\lambda}{8}\left(1-\frac{1}{C}\right)^2\frac{(1-2p_3)^2}{p_3}.$$

Thus, setting $c_1 := \left(1-e^{-\frac{(1-2p_3)^2}{2}}\right)^{-1}$ and $c_2 := \left(1-\frac{1}{C}\right)^2\frac{(1-2p_3)^2}{8p_3}$, we finally obtain $\mathbb{P}(\mathcal{E}_H^c) \leq c_1 e^{-c_2\lambda}$. Summarizing,

$$\mathbb{P}(E_t \geq t(1-2p_3)+\lambda) \leq e^{-\frac{\lambda^2(1-1/C)^2}{8t}} + c_1 e^{-c_2\lambda}, \qquad (6)$$

and since $\lambda \ll t$, the desired result follows. □

Proof of Lemma 3.3

Lemma 3.3 is an auxiliary estimate which was used to bound L_t, the function that appears in the proof of Proposition 2.2. Lemma 3.3 and its elementary proof is provided here for completeness.

Lemma 3.3. There exists $f : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{R}_+$ with $f(s,t) \to 0$ as $s, t \to \infty$ such that $(1-f(s,t))G_s(t) \leq L_t \leq (1+f(s,t))G_s(t)$.

Proof of Lemma 3.3. Recall that $a, \lambda \ll h$. Setting

$$g(h) = g(h, p_1, p_2, p_3) := \frac{1 - 3p_3 - p_2/(hp_1 - 1 + a)}{2h(1 - 2p_3) - 2\lambda}$$

we see that $|g(h)| < 1$. By a Taylor series expansion we immediately see that

$$L_t = \exp\left\{\sum_{h=s}^{t} \log\left(1 + g(h)\right)\right\} = \exp\left\{\sum_{h=s}^{t} \log\left(g(h) + O\big(g^2(h)\big)\right)\right\}$$

Observe that

$$\begin{aligned}\sum_{h=s}^{t} g(h) &= \frac{1 - 3p_3}{2h(1 - 2p_3)} \sum_{h=s}^{t} \frac{1}{h}\left(1 + O\big(1/h\big)\right) + O\left(\sum_{h=s}^{t} \frac{1}{h^2}\right) \\ &= \frac{1 - 3p_3}{2h(1 - 2p_3)} \sum_{h=s}^{t} \frac{1}{h} + O\left(\sum_{h=s}^{t} h^{-2}\right).\end{aligned}$$

Therefore, since we also have that $g(h) = h^{-2}$, we obtain

$$L_t = \exp\left\{\frac{1 - 3p_3}{2h(1 - 2p_3)} \sum_{h=s}^{t} \frac{1}{h}\right\} \exp\left\{O\left(\sum_{h=s}^{t} h^{-2}\right)\right\} = (t/s)^{\frac{1-3p_3}{2h(1-2p_3)}} \exp\left\{O\left(\sum_{h=s}^{t} h^{-2}\right)\right\},$$

where the last equality follows from the asymptotic $\sum_{i=1}^{k} 1/i \sim \log k$ as $k \to \infty$. Finally, note that if $s, t \to \infty$, then the sum within the exponential converges to zero. $\square$