SISO Pole Placement Algorithm: A Linear Transformation Approach

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Abstract: In this paper, an algorithm for SISO Pole Placement based on linear algebra concepts it's developed. This algorithm uses the knowledge of the degrees of certain polynomials associated to the *Internal Model Principle* and *Stable Zero-Pole cancellations* involved in the equation of the closed loop and it's coefficients, generating a linear system of equations for the desired closed loop poles in a systematic way.

Keywords: pole placement, SISO systems, linear algebra, diophantine equations

1. INTRODUCTION

The central problem in control is to find a way to act on a given process such that it behaves close to a desired behavior. Furthermore, this approximate behavior should be achieved in presence of uncertainty of the process and of uncontrollable external disturbances acting on the process. That means, given the closed loop of one degree of freedom shown in Figure 1, where the nominal model of the process to be controlled is $G_0(s)$, find a controller $K(s)$ that ensure that the nominal loop is stable and, if it's possible, to reach a desire behavior previously defined.

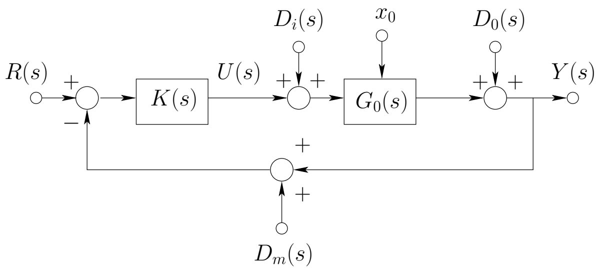

Fig. 1. Closed loop of one degree of freedom

In the loop shown in Figure 1 we use transfer functions and Laplace transforms to describe the relationships between signals in the loop, where $R(s)$ is the reference input, $U(s)$ is the control signal, $Y(s)$ is the output of the loop, $D_i(s)$ is the input disturbance, $D_0(s)$ is the output disturbance and $D_m(s)$ is the measurement noise. We also use x_0 to denote the initial conditions of the model. For linear time-invariant (LTI) systems, the nominal model and the controller can be written as

$$G_0(s) = \frac{B(s)}{A(s)} \qquad K(s) = \frac{P(s)}{L(s)}$$

The poles of the four sensitivity functions governing the closed loop belong to the same set, namely the roots of the characteristic equation $A(s)L(s) + B(s)P(s) = 0$. The poles have a deep impact on the dynamics of a transfer function; they define the stability of the loop. In this way, there exists a technique which deals with the choice of the roots of the characteristic equation, that is, given polynomials $A(s)$, $B(s)$ (defining the model) and given a polynomial $A_{cl}(s)$ (defining the desired location of closed loop poles), it is possible to find polynomials $P(s)$ and $L(s)$ such that

$$A(s)L(s) + B(s)P(s) = A_{cl}(s) \tag{1}$$

The Equation (1) is known as a Diophantine equation and the controller synthesis by solving it is known as pole placement. Polynomial Diophantine equations play a crucial role in the polynomial theory of control systems synthesis. Systems are described by input-output relations, similarly to the classical control techniques, however, the transfer functions are not regarded as functions of complex variable but as algebraic objects. Applications include closed loop pole placement (Kučera, 1993), minimum variance control (Hunt, 1993), LQ and LQG optimal compensators (Kučera, 1991) or adaptive and predictive control (Hunt, 1993). It is well known that, if the controller is biproper, the solution of the equation exists if

$$\deg\{P(s)\} = \deg\{L(s)\} \geq n - 1$$

with $n = \deg\{A(s)\}$. In this context, the minimum order controller is then of degree $n - 1$ and the condition on coprimeness between $A(s)$ y $B(s)$ is necessary to guarantee the existence and uniqueness of the solution (Sylvester theorem) (Goodwin et al., 2001). Solving this equation basically implies solving a linear system of equations, which involves a Sylvester matrix. A suitable and fast algorithm for invert this type of matrices was developed in (Li, 2011).

Many times control objective for the closed loop is to track a specific reference or reject a disturbance of a known frequency. In order to accomplish this we present a systematic way to solve the system equation obtained from using Internal Model Principle (IMP) defined for the first time in (Francis and Wonham, 1975), which establish that the *reference or disturbance generating polynomial* (o simply *generating polynomial*) must be in the denominator of $K(s)$ (Goodwin et al., 2001). This can also be achieved by solving the Diophantine equation (1). Sometimes it is desirable to force the controller to cancel a subset of stable poles and/or zeros of the plant model, this is also taking into account by this systematization, which arises to an algorithm to solve this problem in an automatic way. This approach can be used for design adaptive controllers, or simply synthesize PID controllers.

2. LINEAR TRANSFORMATION APPROACH

2.1 Notation

Let $X(s) = x_n\, s^n + x_{n-1}\, s^{n-1} + \cdots + x_1\, s + x_0$ be a polynomial with real coefficients.

Notation 1. The set of all coefficients of $X(s)$ (in decreasing power order) is denoted by $C_X = \{x_n, x_{n-1}, \cdots, x_1, x_0\}$

Notation 2. The degree of $X(s)$ is denoted by $\deg\{X(s)\}$

Let $\mathbb{V}, \mathbb{W}$ be finite-dimensional vector spaces over a field K and choose bases $V = \{v_1, \ldots, v_m\}$ for $\mathbb{V}$ and $W = \{w_1, \ldots, w_n\}$ for $\mathbb{W}$.

Notation 3. The dimension of $\mathbb{V}$ is denoted by $\dim\{\mathbb{V}\}$.

Notation 4. Let $v^* \in \mathbb{V}$. The coordinates of v^* in the basis V are denoted by $(v^*)_V \in \mathbb{R}^m$.

Notation 5. Let $T : \mathbb{V} \longrightarrow \mathbb{W}$ be a linear transformation from $\mathbb{V}$ to $\mathbb{W}$. The matrix associated to T choosing bases V and W is denoted by T_{VW}.

Definition 1. The external direct sum of $\mathbb{V}$ and $\mathbb{W}$, denoted by $\mathbb{V}\tilde{\oplus}\mathbb{W}$ is defined as the set of all ordered pairs (v, w) with $v \in \mathbb{V}$ and $w \in \mathbb{W}$. Scalar multiplication is defined by $c(v, w) = (cv, cw)$ with $c \in K$, and addition is defined by $(v, w) + (v', w') = (v + v', w + w')$. One checks the other classical axioms for a vector space.

Note that the external direct sum of $\mathbb{V}$ and $\mathbb{W}$ can be expressed as the internal direct sum of $(\mathbb{V}, 0)$ and $(0, \mathbb{W})$. A basis for $\mathbb{V}\tilde{\oplus}\mathbb{W}$ is given by

$$\left\{\{(v_i, 0)\}\bigcup\{(0, w_j)\}\right\}$$

2.2 Pole Placement

Given the control loop of one degree of freedom as in Figure 1. Let $G_0(s)$ be the process nominal model and $K(s)$ the biproper controller defined as

$$G_0(s) = \frac{B(s)}{A(s)} \equiv \frac{B}{A} \qquad K(s) = \frac{P(s)}{L(s)} \equiv \frac{P}{L}$$

where

$$\begin{aligned} A &= a_n\, s^n + a_{n-1} s^{n-1} + \cdots + a_1\, s + a_0 \\ B &= b_m\, s^m + b_{m-1} s^{m-1} + \cdots + b_1\, s + b_0 \\ P &= p_{n-1}\, s^{n-1} + p_{n-2} s^{n-2} + \cdots + p_1\, s + p_0 \\ L &= l_{n-1}\, s^{n-1} + l_{n-2} s^{n-2} + \cdots + l_1\, s + l_0 \end{aligned}$$

The degrees of the polynomials are as it follows

$$\begin{aligned} \deg\{A\} &= n \\ \deg\{B\} &= m \qquad m \le n \\ \deg\{P\} &= n - 1 \\ \deg\{L\} &= n - 1 \end{aligned}$$

The closed loop polynomial $A_{cl}(s) \equiv A_{cl}$ is given by the following Diophantine equation

$$A\, L + B\, P = A_{cl}$$

where

$$\deg\{A_{cl}\} = \deg\{A\} + \deg\{L\} = 2n - 1$$

and so

$$A_{cl} = c_{2n-1}\, s^{2n-1} + c_{2n-2}\, s^{2n-2} + \cdots + c_1\, s + c_0$$

Let $\mathbb{V}_l$, $\mathbb{V}_p$, $\mathbb{W}$ be finite-dimensional vector spaces over $\mathbb{R}$ such as

$$\begin{aligned} \mathbb{V}_l &= \text{span}\{s^{n-1}, s^{n-2}, \ldots, s, 1\} \\ \mathbb{V}_p &= \text{span}\{s^{n-1}, s^{n-2}, \ldots, s, 1\} \\ \mathbb{W} &= \text{span}\{s^{2n-1}, s^{2n-2}, \ldots, s, 1\} \end{aligned}$$

Notice that $L \in \mathbb{V}_l$, $P \in \mathbb{V}_p$ and $A_{cl} \in \mathbb{W}$. Although in this case $\mathbb{V}_l$ is exactly the same space as $\mathbb{V}_p$, we keep the subscripts for the sake of clarity. Let $\mathbb{V}_l\tilde{\oplus}\mathbb{V}_p$ be the external direct sum of $\mathbb{V}_l$ and $\mathbb{V}_p$. Let V and W be a basis for $\mathbb{V}_l\tilde{\oplus}\mathbb{V}_p$ and $\mathbb{W}$ respectively, such as

$$\begin{aligned} V = &\Big\{\{(s^{n-1}, 0), (s^{n-2}, 0), \ldots, (s, 0), (1, 0)\}\bigcup \\ &\{(0, s^{n-1}), (0, s^{n-2}), \ldots, (0, s), (0, 1)\}\Big\} \end{aligned} \tag{2}$$

$$W = \{s^{2n-1}, s^{2n-2}, \ldots, s, 1\} \tag{3}$$

Define the linear transformation Φ as it follows

$$\begin{aligned} &\Phi : \mathbb{V}_l\tilde{\oplus}\mathbb{V}_p \longrightarrow \mathbb{W} \\ &\Phi\{(l, p)\} \mapsto A\, l + B\, p \end{aligned}$$

The construction of the matrix associated to the linear transformation Φ in the bases V and W starts by computing the transformation of every vector of V

$$\begin{aligned} \Phi\{(s^{n-1}, 0)\} &\mapsto A\, s^{n-1} = a_n\, s^{2n-1} + \cdots + a_0\, s^{n-1} \\ \Phi\{(s^{n-2}, 0)\} &\mapsto A\, s^{n-2} = a_n\, s^{2n-2} + \cdots + a_0\, s^{n-2} \\ &\vdots \\ \Phi\{(s, 0)\} &\mapsto A\, s = a_n\, s^{n+1} + \cdots + a_0\, s \\ \Phi\{(1, 0)\} &\mapsto A = a_n\, s^n + \cdots + a_0 \\ \Phi\{(0, s^{n-1})\} &\mapsto B\, s^{n-1} = b_m\, s^{m+n-1} + \cdots + b_0\, s^{n-1} \\ \Phi\{(0, s^{n-2})\} &\mapsto B\, s^{n-2} = b_m\, s^{m+n-2} + \cdots + b_0\, s^{n-2} \\ &\vdots \\ \Phi\{(0, s)\} &\mapsto B\, s = b_m\, s^{m+1} + \cdots + b_0\, s \\ \Phi\{(0, 1)\} &\mapsto B = b_m\, s^m + \cdots + b_0 \end{aligned} \tag{4}$$

Getting coordinates in basis W yields

$$\begin{aligned} (A\, s^{n-1})_W &= (C_A, 0, 0, \ldots\ldots\ldots\ldots, 0, 0, 0) \\ (A\, s^{n-2})_W &= (0, C_A, 0, \ldots\ldots\ldots\ldots, 0, 0, 0) \\ &\vdots \\ (A\, s)_W &= (0, 0, 0, \ldots\ldots\ldots\ldots, 0, C_A, 0) \\ (A)_W &= (0, 0, 0, \ldots\ldots\ldots\ldots, 0, 0, C_A) \\ (B\, s^{n-1})_W &= (\overbrace{0, \ldots, 0}^{n-m}, C_B, 0, 0, \ldots, 0, 0, 0) \\ (B\, s^{n-2})_W &= (0, \ldots, 0, 0, C_B, 0, \ldots, 0, 0, 0) \\ &\vdots \\ (B\, s)_W &= (0, \ldots, 0, 0, 0, 0, \ldots, 0, C_B, 0) \\ (B)_W &= (0, \ldots, 0, 0, 0, 0, \ldots, 0, 0, C_B) \end{aligned} \tag{5}$$

Notice that every vector in $\mathbb{R}^{2n}$ defined above it's a shift of the coefficients of A and B polynomials respectively. Define the following submatrices $\xi_A \in \mathbb{R}^{2n \times n}$ and $\xi_B \in \mathbb{R}^{2n \times n}$ (in columns)

$$\xi_A = \underbrace{\left[\, (A\, s^{n-1})_W^T \; \cdots \; (A)_W^T \,\right]}_{\dim\{\mathbb{V}_l\}=n}$$

$$\xi_B = \underbrace{\left[\, (B\, s^{n-1})_W^T \; \cdots \; (B)_W^T \,\right]}_{\dim\{\mathbb{V}_p\}=n}$$

Then, the matrix associated to the transformation in the bases V and W (in columns)

$$\Phi_{VW} = [\, \xi_A \,|\, \xi_B \,]$$

Where $\Phi_{VW} \in \mathbb{R}^{2n \times 2n}$ is a *Sylvester Matrix* associated to the polynomials A and B, and $|\Phi_{VW}| \neq 0$ because A and B are coprime. Moreover, because of the shifting property of the columns of ξ_A and ξ_B (and knowing the dimensions of $\mathbb{V}_l$, $\mathbb{V}_p$ and $\mathbb{W}$) constructing the matrix it's straightforward. With Φ_{VW} computed (which is systematic) knowing the coefficients of L and P it's reduced to solve the following linear system of equations:

$$[\, \xi_A \,|\, \xi_B \,] \begin{bmatrix} C_L \\ C_P \end{bmatrix} = C_{A_{cl}} \longrightarrow \begin{bmatrix} C_L \\ C_P \end{bmatrix} = [\, \xi_A \,|\, \xi_B \,]^{-1} \, C_{A_{cl}}$$

2.3 Internal Model Principle

Adding the Internal Model Principle to the loop, given by the *generating polynomial* $\Gamma(s) \equiv \Gamma$ where

$$\deg\{\Gamma\} = q$$

the pole placement problem can be reformulated: the *generating polynomial* must appear as part of the denominator of the controller. To accomplish that goal, one chooses

$$L = \Gamma \bar{L}$$

and the closed loop equation can be rewritten as

$$\bar{A}\, \bar{L} + B\, P = A_{cl} \quad \text{where} \quad \bar{A} = \Gamma\, A$$

including Γ inside the term that represents the denominator of the plant, creating an equivalent model of degree $\bar{n} = n + q$. Now, using the same criterion of design a biproper controller with one degree less than the plant:

$$\begin{aligned} \deg\{P\} &= \bar{n} - 1 = n + q - 1 \\ \deg\{L\} &= \bar{n} - 1 = n + q - 1 \\ \deg\{A_{cl}\} &= 2n + q - 1 \end{aligned}$$

and

$$\deg\{\bar{L}\} = \deg\{L\} - \deg\{\Gamma\} = n - 1$$

Let ${}^{\mathrm{I}}\mathbb{V}_{\bar{l}}$, ${}^{\mathrm{I}}\mathbb{V}_p$, ${}^{\mathrm{I}}\mathbb{W}$ be vector spaces over $\mathbb{R}$ such as

$$\begin{aligned} {}^{\mathrm{I}}\mathbb{V}_{\bar{l}} &= \mathrm{span}\{s^{n-1}, s^{n-2}, \ldots, s, 1\} \\ {}^{\mathrm{I}}\mathbb{V}_p &= \mathrm{span}\{s^{n+q-1}, s^{n+q-2}, \ldots, s, 1\} \\ {}^{\mathrm{I}}\mathbb{W} &= \mathrm{span}\{s^{2n+q-1}, s^{2n+q-2}, \ldots, s, 1\} \end{aligned}$$

So that $\bar{L} \in {}^{\mathrm{I}}\mathbb{V}_{\bar{l}}$, $P \in {}^{\mathrm{I}}\mathbb{V}_p$ and $A_{cl} \in {}^{\mathrm{I}}\mathbb{W}$. Let ${}^{\mathrm{I}}\mathbb{V}_{\bar{l}} \tilde{\oplus} {}^{\mathrm{I}}\mathbb{V}_p$ be the external direct sum of ${}^{\mathrm{I}}\mathbb{V}_{\bar{l}}$ and ${}^{\mathrm{I}}\mathbb{V}_p$. Construct bases ${}^{\mathrm{I}}V$ and ${}^{\mathrm{I}}W$ in the same way as in (2) and (3). Define the linear transformation ${}^{\mathrm{I}}\Phi$ as it follows

$$\begin{aligned} {}^{\mathrm{I}}\Phi : {}^{\mathrm{I}}\mathbb{V}_{\bar{l}} \tilde{\oplus} {}^{\mathrm{I}}\mathbb{V}_p &\longrightarrow \mathbb{W} \\ {}^{\mathrm{I}}\Phi\{(\bar{l}, p)\} &\mapsto \bar{A}\, \bar{l} + B\, p \end{aligned} \tag{6}$$

Computing the corresponding maps to every vector in ${}^{\mathrm{I}}V$ in the same way as in (4) and getting it's coordinates in the basis ${}^{\mathrm{I}}W$ as in (5), construct the submatrices ${}^{\mathrm{I}}\xi_{\bar{A}} \in \mathbb{R}^{2n+q \times n}$ and ${}^{\mathrm{I}}\xi_B \in \mathbb{R}^{2n+q \times n+q}$ as it follows (in columns)

$${}^{\mathrm{I}}\xi_{\bar{A}} = \underbrace{\left[\, (\bar{A}\, s^{n-1})_{{}^{\mathrm{I}}W}^T \; \cdots \; (\bar{A})_{{}^{\mathrm{I}}W}^T \,\right]}_{\dim\{{}^{\mathrm{I}}\mathbb{V}_{\bar{l}}\}=n}$$

$${}^{\mathrm{I}}\xi_B = \underbrace{\left[\, (B\, s^{n+q-1})_{{}^{\mathrm{I}}W}^T \; \cdots \; (B)_{{}^{\mathrm{I}}W}^T \,\right]}_{\dim\{{}^{\mathrm{I}}\mathbb{V}_p\}=n+q}$$

Then, the matrix associated to the transformation in the bases ${}^{\mathrm{I}}V$ and ${}^{\mathrm{I}}W$ (in columns)

$${}^{\mathrm{I}}\Phi_{{}^{\mathrm{I}}V{}^{\mathrm{I}}W} = \left[\, {}^{\mathrm{I}}\xi_{\bar{A}} \,|\, {}^{\mathrm{I}}\xi_B \,\right]$$

Where ${}^{\mathrm{I}}\Phi_{{}^{\mathrm{I}}V{}^{\mathrm{I}}W} \in \mathbb{R}^{2n+q \times 2n+q}$ is a *Sylvester Matrix* associated to the polynomials $\bar{A}$ and B; and $|{}^{\mathrm{I}}\Phi_{{}^{\mathrm{I}}V{}^{\mathrm{I}}W}| \neq 0$ because $\bar{A}$ and B are coprime.

Example 1. Given

$$G_0(s) = \frac{s+1}{s^2 + 4\, s + 4}$$

we aim to design a biproper controller applying the Internal Model Principle with the *generating polynomial* $\Gamma = s(s^2+1)$.

The degrees of the polynomials

$$\begin{aligned} \deg\{A\} &= n = 2 \\ \deg\{B\} &= m = 1 \\ \deg\{\Gamma\} &= q = 3 \\ \deg\{L\} = \deg\{P\} &= n + q - 1 = 4 \\ \deg\{\bar{L}\} &= n - 1 = 1 \\ \deg\{A_{cl}\} &= 2n + q - 1 = 6 \end{aligned}$$

The corresponding dimensions

$$\dim\{{}^{\mathrm{I}}\mathbb{V}_{\bar{l}}\} = 2, \quad \dim\{{}^{\mathrm{I}}\mathbb{V}_p\} = 5, \quad \dim\{{}^{\mathrm{I}}\mathbb{W}\} = 7 \tag{7}$$

Computing $\bar{A}$ yields

$$\begin{aligned} \bar{A} &= \Gamma\, A = s^5 + 4\, s^4 + 5\, s^3 + 4\, s^2 + 4\, s \\ C_{\bar{A}} &= \{1, 4, 5, 4, 4, 0\} \end{aligned}$$

After defining the transformation ${}^{\mathrm{I}}\Phi$ as in (6) the corresponding ${}^{\mathrm{I}}\xi_{\bar{A}} \in \mathbb{R}^{7\times 2}$ and ${}^{\mathrm{I}}\xi_B \in \mathbb{R}^{7\times 5}$ (which are shifts of the coefficients of the polynomials $\bar{A}$ and B according to the dimensions stated in (7))

$${}^{\mathrm{I}}\xi_{\bar{A}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 4 & 1 \\ 5 & 4 \\ 4 & 5 \\ 4 & 4 \\ 0 & 4 \\ 0 & 0 \end{bmatrix}}_{\dim\{{}^{\mathrm{I}}\mathbb{V}_{\bar{l}}\}} \qquad {}^{\mathrm{I}}\xi_B = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\dim\{{}^{\mathrm{I}}\mathbb{V}_p\}}$$

Computing the coefficients of $\bar{L}$ and P involves the following linear system

$$\begin{bmatrix} \bar{l}_1 \\ \bar{l}_0 \\ p_4 \\ p_3 \\ p_2 \\ p_1 \\ p_0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 1 & 0 & 0 & 0 & 0 \\ 5 & 4 & 1 & 1 & 0 & 0 & 0 \\ 4 & 5 & 0 & 1 & 1 & 0 & 0 \\ 4 & 4 & 0 & 0 & 1 & 1 & 0 \\ 0 & 4 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1}}_{{}^{\mathrm{I}}\Phi^{-1}_{{}^{\mathrm{I}}V{}^{\mathrm{I}}W}} \begin{bmatrix} c_6 \\ c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix}$$

Where $C_{A_{cl}} = \{c_6, c_5, c_4, c_3, c_2, c_1, c_0\}$ are the coefficients of the desired closed loop polynomial A_{cl}. If we choose $A_{cl} = (s+3)^6$, then the controller transfer function is given by

$$K(s) = \frac{45s^4 + 209s^3 + 482s^2 + 853s + 729}{s(s^2+1)(s-31)} \quad (8)$$

The steady state response of the closed loop of Figure 1 is indeed the one chosen, but the transient response is affected by the dynamics of the zeros of the controller and the nominal model. The behavior of this zeros is undesirable if they are located at the right side of the poles in the left semi-plane of the complex plane (Seron et al., 1997). This can be sometimes avoided if we propose *Stable Zero-Pole cancellations* between the controller and the plant, as we explain in the following section.

2.4 Stable Zero-Pole cancellations

In addition to the implementation of the Internal Model Principle, it's from interest to obtain a systematic way to perform *Stable Zero-Pole cancellations*. To achieve that goal, the controller denominator(numerator) must include the pole(zero) dynamics to cancel. Suppose that the stable dynamics to cancel are represented by two polynomials $\alpha(s) \equiv \alpha$ (poles) and $\beta(s) \equiv \beta$ (zeros) such that

$$A = \alpha\,\tilde{A}$$
$$B = \beta\,\tilde{B}$$

where

$$\deg\{\alpha\} = w$$
$$\deg\{\beta\} = z$$

The Diophantine equation associated to the closed loop

$$A\,L + B\,P = A_{cl} \longrightarrow \alpha\,\tilde{A}\,L + \beta\,\tilde{B}\,P = A_{cl} \quad (9)$$

Choosing $L = \beta\,\tilde{L}$ and $P = \alpha\,\tilde{P}$ the equation (9) can be expressed as

$$\tilde{A}\,\tilde{L} + \tilde{B}\tilde{P} = \tilde{A}_{cl}$$

with $A_{cl} = \alpha\,\beta\,\tilde{A}_{cl}$ so that the remaining closed loop poles after the cancellations ($\tilde{A}_{cl}$) can be chose arbitrarily. The corresponding degrees using the same design criterion (biproper controller of one degree less than the plant) remains as follows

$$\deg\{\tilde{L}\} = n - z - 1$$
$$\deg\{\tilde{P}\} = n - w - 1$$
$$\deg\{\tilde{A}_{cl}\} = 2n - z - w - 1$$

Let ${}^{Z}\mathbb{V}_{\tilde{l}}$, ${}^{Z}\mathbb{V}_{\tilde{p}}$, ${}^{Z}\mathbb{W}$ be vector spaces over $\mathbb{R}$ such as

$${}^{Z}\mathbb{V}_{\tilde{l}} = \text{span}\{s^{n-z-1}, s^{n-z-2}, \ldots, s, 1\}$$
$${}^{Z}\mathbb{V}_{\tilde{p}} = \text{span}\{s^{n-w-1}, s^{n-w-2}, \ldots, s, 1\}$$
$${}^{Z}\mathbb{W} = \text{span}\{s^{2n-z-w-1}, s^{2n-z-w-2}, \ldots, s, 1\}$$

So that $\tilde{L} \in {}^{Z}\mathbb{V}_{\tilde{l}}$, $\tilde{P} \in {}^{Z}\mathbb{V}_{\tilde{p}}$ and $\tilde{A}_{cl} \in {}^{Z}\mathbb{W}$. Let ${}^{Z}\mathbb{V}_{\tilde{l}}\tilde{\oplus}{}^{Z}\mathbb{V}_{\tilde{p}}$ be the external direct sum of ${}^{Z}\mathbb{V}_{\tilde{l}}$ and ${}^{Z}\mathbb{V}_{\tilde{p}}$. Construct bases ${}^{Z}V$ and ${}^{Z}W$ in the same way as in (2) and (3). Define the linear transformation ${}^{Z}\Phi$ as it follows

$${}^{Z}\Phi : {}^{Z}\mathbb{V}_{\tilde{l}}\tilde{\oplus}{}^{Z}\mathbb{V}_{\tilde{p}} \longrightarrow {}^{Z}\mathbb{W}$$
$${}^{Z}\Phi\{(\tilde{l}, \tilde{p})\} \mapsto \tilde{A}\,\tilde{l} + \tilde{B}\,\tilde{p}$$

Using the same criterion as in (4) and (5) with bases ${}^{Z}V$ and ${}^{Z}W$ respectively, the submatrices ${}^{Z}\xi_{\tilde{A}} \in \mathbb{R}^{2n-z-w \times n-z}$ and ${}^{Z}\xi_{\tilde{B}} \in \mathbb{R}^{2n-z-w \times n-w}$ (in columns)

$${}^{Z}\xi_{\tilde{A}} = \underbrace{\left[\,(\tilde{A}\,s^{n-z-1})^{T}_{{}^{Z}W} \;\cdots\; (\tilde{A})^{T}_{{}^{Z}W}\,\right]}_{\dim\{{}^{Z}\mathbb{V}_{\tilde{l}}\}=n-z}$$
$${}^{Z}\xi_{\tilde{B}} = \underbrace{\left[\,(\tilde{B}\,s^{n-w-1})^{T}_{{}^{Z}W} \;\cdots\; (\tilde{B})^{T}_{{}^{Z}W}\,\right]}_{\dim\{{}^{Z}\mathbb{V}_{\tilde{p}}\}=n-w}$$

Then, the matrix associated to the transformation in the bases ${}^{Z}V$ and ${}^{Z}W$

$${}^{Z}\Phi_{{}^{Z}V{}^{Z}W} = \left[\,{}^{Z}\xi_{\tilde{A}} | {}^{Z}\xi_{\tilde{B}}\,\right]$$

Where ${}^{Z}\Phi_{{}^{Z}V{}^{Z}W} \in \mathbb{R}^{2n-z-w \times 2n-z-w}$ is a *Sylvester Matrix* associated to the polynomials $\tilde{A}$ and $\tilde{B}$; and $|{}^{Z}\Phi_{{}^{Z}V{}^{Z}W}| \neq 0$ because $\tilde{A}$ and $\tilde{B}$ are coprime.

3. DEVELOPMENT OF THE ALGORITHM

Combining the criterion developed in Section 2.3 and Section 2.4 one can construct a linear transformation that takes into account the Internal Model Principle and Stable Zero-Pole cancellations at the same time, providing a systematic way to obtain the matrix involved in the determination of the coefficients of the desired closed loop polynomial. Choosing

$$L = \Gamma\,\beta L^*$$
$$P = \alpha\,P^* \quad (10)$$
$$A_{cl} = \alpha\,\beta A^*_{cl}$$

The corresponding Diophantine equation remains as follows

$$A^* L^* + B^* P^* = A^*_{cl}$$

where

$$A^* = \frac{\Gamma}{\alpha} A \quad \text{and} \quad B^* = \frac{1}{\beta} B \quad (11)$$

The corresponding degrees are

$$\deg\{L^*\} = n - z - 1$$
$$\deg\{P^*\} = n + q - w - 1$$
$$\deg\{A^*_{cl}\} = 2n + q - z - w - 1$$

Let ${}^{*}\mathbb{V}_{l^*}$, ${}^{*}\mathbb{V}_{p*}$, ${}^{*}\mathbb{W}$ be vector spaces over $\mathbb{R}$ such as

$${}^{*}\mathbb{V}_{l^*} = \text{span}\{s^{n-z-1}, s^{n-z-2}, \ldots, s, 1\}$$
$${}^{*}\mathbb{V}_{p^*} = \text{span}\{s^{n+q-w-1}, s^{n+q-w-2}, \ldots, s, 1\}$$
$${}^{*}\mathbb{W} = \text{span}\{s^{2n+q-z-w-1}, s^{2n+q-z-w-2}, \ldots, s, 1\}$$

So that $L^* \in {}^{*}\mathbb{V}_{l^*}$, $P^* \in {}^{*}\mathbb{V}_{p^*}$ and $A^*_{cl} \in {}^{*}\mathbb{W}$. Let ${}^{*}\mathbb{V}_{l^*}\tilde{\oplus}{}^{*}\mathbb{V}_{p^*}$ be the external direct sum of ${}^{*}\mathbb{V}_{l^*}$ and ${}^{*}\mathbb{V}_{p^*}$. Construct bases ${}^{*}V$ and ${}^{*}W$ in the same way as in (2) and (3). Define the linear transformation ${}^{*}\Phi$ as it follows

$${}^{*}\Phi : {}^{*}\mathbb{V}_{l^*}\tilde{\oplus}{}^{*}\mathbb{V}_{p^*} \longrightarrow {}^{*}\mathbb{W}$$
$${}^{*}\Phi\{(l^*, p^*)\} \mapsto A^*\,l^* + B^*\,p^*$$

Using the same criterion as in (4) and (5) with bases ${}^{*}V$ and ${}^{*}W$ respectively, the construction of the submatrices ${}^{*}\xi_{A^*} \in \mathbb{R}^{2n+q-z-w \times n-z}$ and ${}^{*}\xi_{B^*} \in \mathbb{R}^{2n+q-z-w \times n+q-w}$ (in columns)

$${}^{*}\xi_{A^*} = \underbrace{\left[\,(A^*\,s^{n-z-1})^{T}_{{}^{*}W} \;\cdots\; (A^*)^{T}_{{}^{*}W}\,\right]}_{\dim\{{}^{*}\mathbb{V}_{l^*}\}=n-z}$$
$${}^{*}\xi_{B^*} = \underbrace{\left[\,(B^*\,s^{n+q-w-1})^{T}_{{}^{*}W} \;\cdots\; (B^*)^{T}_{{}^{*}W}\,\right]}_{\dim\{{}^{*}\mathbb{V}_{p^*}\}=n+q-w} \quad (12)$$

Then, the matrix associated to the transformation in the bases $^*V$ and $^*W$

$$^*\Phi_{^*V^*W} = \left[\, ^*\xi_{A^*} \,\middle|\, ^*\xi_{B^*} \,\right]$$

Where $\Phi^*_{VW} \in \mathbb{R}^{2n+q-z-w \times 2n+q-z-w}$ is a *Sylvester Matrix* associated to the polynomials A^* and B^*.

In synthesis, the algorithm can be summarized in the following simple steps

(i) Choose the *generating polynomial* Γ and the stable dynamics to cancel α (poles) and β (zeros).
(ii) Compute A^* and B^* as in (11) and extract their corresponding coefficients.
(iii) Using the information of the degrees of the denominator of the plant A (n), the generating polynomial Γ (q), the desired pole cancellations α (w) and the desired zero cancellations β (z) construct the submatrices $^*\xi_{A^*}$ and $^*\xi_{B^*}$ performing the corresponding shifts to the coefficients of the polynomials A^* and B^* as in (12).
(iv) Choose the desired dynamics for the closed loop polynomial A^*_{cl} of degree $(2n+q-z-1)$.
(v) Solve the corresponding linear equation system involving the matrix associated to the linear transformation $\Phi^* \in \mathbb{R}^{2n+q-z-w \times 2n+q-z-w}$ to find the coefficients of L^* and P^*.
(vi) Compute L, P and A_{cl} as in (10).

Example 2. (Example 1 revisited). We recall the Example 1, but this time we will force the Stable Zero-pole cancellations in addition of the Internal Model Principle using the algorithm stated before. In this case, we cancel all stable factors, that is $z=1$ and $w=2$, and

$$A^* = s(s^2+1) \quad \text{and} \quad B^* = 1.$$

$$\begin{aligned} \deg\{L^*\} &= n - z - 1 = 0 \\ \deg\{P^*\} &= n + q - w - 1 = 2 \\ \deg\{A^*_{cl}\} &= 2n + q - z - w - 1 = 3 \end{aligned}$$

The corresponding dimensions

$$\dim\{^*\mathbb{V}_{l^*}\} = 1, \quad \dim\{^*\mathbb{V}_{p^*}\} = 3, \quad \dim\{^*\mathbb{W}\} = 4$$

Constructing the submatrices $^*\xi_{A^*} \in \mathbb{R}^{4\times 1}$ and $^*\xi_{B^*} \in \mathbb{R}^{4\times 3}$ yields

$$^*\xi_{A^*} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\dim\{^*\mathbb{V}_{l^*}\}} \qquad ^*\xi_{B^*} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\dim\{^*\mathbb{V}_{p^*}\}}$$

Computing the coefficients of L^* and P^* involves the following linear system

$$\begin{bmatrix} l_0^* \\ p_2^* \\ p_1^* \\ p_0^* \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}}_{^*\Phi^{-1}_{^*V^*W}} \begin{bmatrix} 1 \\ 9 \\ 27 \\ 27 \end{bmatrix}$$

Solving the system of equations using the matrix of the linear transformation $^*\Phi_{^*V^*W}$ computed before, the final controller is given by the following transfer function

$$K(s) = \frac{\alpha P^*}{\Gamma \beta L^*} = \frac{9(s+2)^2(s^2+2.889s+3)}{s(s^2+1)(s+1)} \tag{13}$$

In Figure 2 we show the step response from reference input to output using the controller developed in (8) and using the controller with zero-pole cancellations (13). An output disturbance $d_0(t) = sin(t)$ was injected at $t = 5[sec.]$.

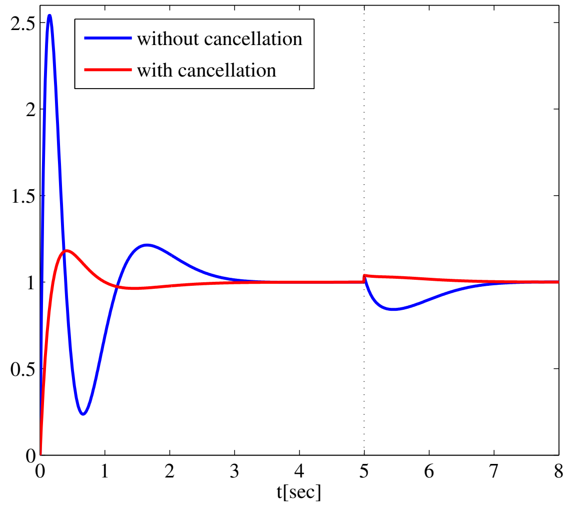

Fig. 2. Step response from reference input to output

4. CONCLUSIONS

In this work, we developed a simple and systematic algorithm to design SISO controllers based on a input-output mathematical model using linear algebra concepts. It considers the *Internal Model Principle* and allows to perform *Stable Zero-Pole cancellations* in the same linear transformation. The extrapolation of this algorithm to the discrete domain it's straightforward, which implies that it can be easily implemented in a microcontroller. In this way, it can be coupled to a model identification system turning the controller into an adaptive one, showing the versatility of the algorithm developed.

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