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On the long-time behaviour of solutions to unforced evolution Navier–Stokes equations under Navier boundary conditions

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ABSTRACT

We study the asymptotic behaviour of the solutions to Navier–Stokes unforced equations under Navier boundary conditions in a wide class of merely Lipschitz domains of physical interest. The paper draws its main motivation from celebrated results by Foias and Saut (1984) under Dirichlet conditions; here the choice of the boundary conditions requires carefully considering the geometry of the domain Ω, due to the possible lack of the Poincaré inequality in presence of symmetries. In non-axially symmetric domains we show the validity of the Foias–Saut result about the limit at infinity of the Dirichlet quotient, in axially symmetric domains we provide two invariants of the flow which completely characterize the motion and we prove that the Foias–Saut result holds for initial data belonging to one of the invariants.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, where by this we mean that Ω is open, nonempty and connected. For $T > 0$, we put $Q_T := \Omega\times(0,T)$ and we consider the evolution 3D Navier–Stokes problem in Q_T with zero-source and homogeneous Navier boundary conditions:

$$\begin{cases} u_t - \mu\Delta u + (u\cdot\nabla)u + \nabla p = 0 & \text{in } Q_T \\ \nabla\cdot u = 0 & \text{in } Q_T \\ u\cdot\nu = (\mathbf{D}u\cdot\nu)\cdot\tau = 0 & \text{on } \partial\Omega\times(0,T) \\ u(x,y,z,0) = u_0(x,y,z) & \text{in } \Omega\,. \end{cases} \tag{1.1}$$

In the above, $\mu > 0$ denotes the kinematic viscosity, $\mathbf{D}u = (\nabla u + \nabla^T u)/2$ is the strain tensor, ν is the outward normal vector to $\partial\Omega$ while τ is tangential. The pressure p is defined up to an additive constant so that one can fix its mean value as follows

$$\int_\Omega p(t) = 0 \qquad \forall\, t\in(0,T)\,. \tag{1.2}$$

Navier boundary conditions were introduced by Navier [1] in 1827 in the following formulation:

$$u\cdot\nu = \beta u\cdot\tau + (\mathbf{D}u\cdot\nu)\cdot\tau = 0\,, \qquad \text{on } \partial\Omega\times(0,T)\,, \tag{1.3}$$

where $\beta \geq 0$ is a parameter describing friction. In contrast to Dirichlet boundary conditions (also known as no-slip boundary conditions), the hypothesis underlying (1.3) is the presence of a stagnant layer of fluid close to the wall allowing the fluid to

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slip. This type of boundary conditions turn out to be relevant in many physical applications, see e.g., [2, Section 3]; we mention for instance their aptness when taking care of a turbulent boundary layer in the context of LES (Large Eddy Simulations), see [3].

Although the literature is not as extended as in the case of Dirichlet boundary conditions, after the seminal paper of V.A. Solonnikov and V.E. Scadilov [4] in 1973, many papers have been devoted to the study of the Navier–Stokes equations under Navier boundary conditions. A rich overview of results can be found in the Introduction of [5]. Concerning the full initial-value problem, we mention the work [6], where the author proves that when $\beta > 0$ the Stokes operator associated to problem $(1.1)_1$–$(1.1)_2$–(1.3) generates an analytical and compact semigroup of contractions in L^2-space/energy space, which implies the existence of a unique strong solution for small time. Existence, uniqueness and regularity of solutions in a general L^p-setting were instead given in [7] for $\beta = 0$ and brought to a high level of complexity in [5,8] by assuming β non constant and satisfying minimal regularity assumptions. We also recall the paper [9] where the regularity up to the boundary of weak solutions to (1.1) with the so-called "stress-free" boundary conditions (that coincide with $(1.1)_3$ in the portions of the domain with flat boundaries) was investigated.

In general, due to the presence of the derivatives in (1.3), most of known results require that Ω is at least of class $C^{2,1}$. However, in [10], by means of a suitable reflection principle, the authors managed to prove the well-posedness of (1.1) (namely, when $\beta = 0$) in a special class of merely Lipschitz domains, called *sectors*, see Definition 2.3 below. This class is sufficiently wide to contain most of the domains needed in physics and engineering, see [10] for more details and applications. The main purpose of this paper is to study the long-time behaviour of the solution $u(t)$ to (1.1) when Ω is precisely a *sector*. In contrast to the Dirichlet case, the boundary conditions in (1.1) require carefully considering the geometry of the domain Ω, due to the possible lack of the Poincaré inequality in presence of symmetries. This behaviour, which is peculiar of the case $\beta = 0$, occurs in the case called "special" in [11], namely when Ω is generated by revolution around a given axis, i.e. it is *axially symmetric*, see also [2,7]. We remark that, while the Poincaré inequality creates no obstruction when $\beta > 0$, the reflection principle, on which our analysis on sectors relies, fails, see Remark 4.1 for more details. For this reason, since our interest is to study the long time behaviour of solutions on sectors, possibly relating it to the symmetries of the domain, we focus the analysis of the present paper on the case $\beta = 0$.

More precisely, when the domain is non-axially symmetric, we extend to problem (1.1) a celebrated result by Foias and Saut [12] about the limit of the Dirichlet quotient of solutions at infinity, showing that it tends to an eigenvalue Λ of the Stokes operator as $t \to \infty$, see Theorem 3.1 in Section 3.1. By this result it follows that the energy of the solution $u(t)$ of (1.1) concentrates on the modes corresponding to Λ which, in some sense, catch the energy of the system. Furthermore, the decay in time of $u(t)$ is exactly of exponential type, namely there holds

$$\lim_{t\to\infty} e^{\mu\Lambda t}u(t) = e_\Lambda \quad \text{ in } H^1(\Omega),$$

where e_Λ is an eigenfunction corresponding to the eigenvalue Λ, see Proposition 3.5 in Section 3.1.

When Ω is axially symmetric, according to the choice of the initial data, the solution may not decay at 0 converging to a steady state of the flow, see also [13]. In this case we show the existence of a decomposition of the set of initial data into two invariants of the flow, which completely characterizes the motion, see Theorem 3.2 and Remark 3.3 below. In particular, we prove that a Foias–Saut type result holds when the initial datum belongs to one of the invariants.

A natural issue is to establish a priori which modes catch the energy of $u(t)$ and, in turn, to predict its asymptotic behaviour. As one could expect, this depends on the initial datum u_0 but, in general, this dependence is far from being explicit, see Remark 3.6. We complement our analysis by providing two examples in the 2D case where more explicit information can be given. Even if in the lower dimensional case, the examples work as prototypes of the long-time behaviour of solutions under Navier boundary conditions since they exhibit the same dichotomy registered in 3D domains between the asymptotic behaviour in non-axially symmetric and axially symmetric domains.

The paper is organized as follows. In Section 2 we present the notations and some preliminary results needed to prove and state our main results given in Section 3. In particular, in Section 2.1 we recall the functional framework needed to approach the problem, in Section 2.2 we illustrate the notion of *sectors*-type domain, while in Section 2.3 we give the definition of Stokes operator and the characterization of its kernel. Here we also illustrate a decomposition of the functional space based on the geometry of the domain that will be crucial to distinguish the behaviour at infinity of the solution to (1.1) according to the symmetries of Ω. Finally, in Section 2.4 we recall the main properties of the solution to (1.1) needed in the proof of the asymptotic behaviour, such as the analyticity in time proved in Theorem 2.7.

Section 3 contains the main results of the paper, namely Theorems 3.1 and 3.2 stated in Section 3.1, which provide the asymptotic behaviour of the solution, respectively, in the non-axially symmetric and axially symmetric case. Section 3.2 contains a complementary discussion of our results in the framework of our 2D prototypes problems.

Section 4 is devoted to the proofs of the results. In particular, the reflection principle, which is essential to prove the results in the framework of sectors, is illustrated in Section 4.1. Finally, in the Appendix we provide a system of eigenfunctions of the Stokes operator when Ω is the 2D ball.

2. Notations and preliminary results

2.1. Notations

Let Ω be a Lipschitz and bounded domain in $\mathbb{R}^3$. We first recall the definition of the usual spaces in the treatment of the Navier–Stokes equations, which we will denote by H and V. The set H is obtained by taking the closure of $\mathcal{V} := \{u \in C_c^\infty(\Omega) : \nabla\cdot u = 0 \text{ in } \Omega\}$ in $L^2(\Omega)$; by [14, Theorem III.2.3] we have

$$H := \{u \in L^2(\Omega) : \nabla \cdot u = 0 \text{ in } \Omega, u \cdot \nu = 0 \text{ on } \partial\Omega\},$$

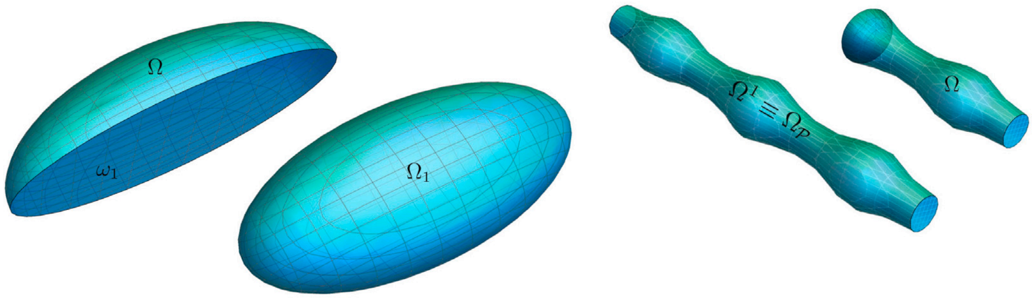

Fig. 1. On the left a sector of type (A) on the right a sector of type (B), Ω_P denotes the cell of periodicity, see Definition 4.2.

in which the solenoidality condition is intended in the sense of weak derivatives and, with an abuse of notation, we denote by $u \cdot \nu$ the generalized normal trace of u (see for instance [14, Theorem III.2.2]). The set V is defined as

$$V := H \cap H^1(\Omega),$$

being $H^1(\Omega)$ the usual Sobolev space. By construction, H is a closed subspace of $L^2(\Omega)$; therefore, V is a closed subspace of $H^1(\Omega)$. When the domain is a generic D, different from Ω, we specify $H(D)$, $V(D)$. In $H(D)$ and $V(D)$, respectively, we consider the bilinear forms

$$(u, w)_D := \int_D u \cdot v \qquad \text{and} \qquad (\mathbf{D}u, \mathbf{D}v)_D := \int_D \mathbf{D}u : \mathbf{D}v.$$

Here ":" indicates the scalar product between matrices, namely $A : B = \sum_{i,j=1}^3 a_{ij} b_{ij}$ if A and B are 3×3 matrices of components a_{ij}, b_{ij}.

For $1 \le p < \infty$, we denote by $\|u\|_{p,D} := \left(\sum_{i=1}^3 \int_D |u_i|^p\right)^{1/p}$ the standard $L^p(D)$-norm. In the following, by standard norms on H and V we mean, respectively, $\|u\|_{2,\Omega}$ and $\|u\|_{H^1(\Omega)} := \|u\|_{2,\Omega} + \|\mathbf{D}u\|_{2,\Omega}$ where $\|\mathbf{D}u\|_{2,\Omega} := \left[(\mathbf{D}u, \mathbf{D}u)_\Omega\right]^{1/2}$. We notice that the fact that $\|\mathbf{D}u\|_{2,\Omega}$ is an equivalent norm on V depends on the geometry of Ω through the spectral properties of the *Stokes operator* under Navier boundary conditions, see Remark 2.5.

2.2. Sectors definition

Our analysis is performed on special Lipschitz domains called sectors. In this section, we recall the definition of *sectors* originally given in [10] to which we refer for further details and examples.

Definition 2.1. We call *face* any bounded open planar domain $\omega \subset \mathbb{R}^2$, and we denote by P_ω the plane containing ω. Let P be a plane and let $\Omega \subset \mathbb{R}^3$ be a bounded domain such that

$$\Omega \cap P = \emptyset \quad \text{and} \quad \bar{\Omega} \cap P \quad \text{is the union of a finite number } h \ge 1 \text{ of (closed) faces;} \tag{2.1}$$

we denote by Ω_P the interior of the closure of the union between Ω and its reflection about P.

Let $P_1, ..P_m$ be m planes ($m \ge 1$). If $\Omega \subset \mathbb{R}^3$ is such that (2.1) holds for the m couples

$$\Omega \text{ and } P_1\,, \quad \Omega_{P_1} \text{ and } P_2\,, \ldots\,, \quad \left(\left(\Omega_{P_1}\right)_{P_2} ...\right)_{P_{m-1}} \text{ and } P_m,$$

then we can define the domain given by iterative reflections $\Omega_{P_1,\ldots,P_m} := \left(\left(\left(\Omega_{P_1}\right)_{P_2} ...\right)_{P_{m-1}}\right)_{P_m}$.

Definition 2.2. A bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ is said *smoothly periodically extendable* if it admits a periodic extension with $C^{2,1}$ boundary and if $\partial\Omega$ has a finite number of faces ω_i, $i = 1, \ldots, k$ with $k \ge 2$, all lying on at most six planes $p_1, \ldots, p_6$ such that:

$$p_s \cap \Omega = \emptyset \;\forall\; s = 1, \ldots, 6 \quad \text{and} \quad p_1 \parallel p_4\,, p_2 \parallel p_5\,, p_3 \parallel p_6\,, \; p_1 \perp p_2\,, p_1 \perp p_3\,, p_2 \perp p_3\,. \tag{2.2}$$

Definition 2.3. A bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ is said a *sector* if one of the two following facts occurs:

(A) there exists a bounded $C^{2,1}$-domain Ω_m having at least $m \ge 0$ planes of symmetry $P_1, \ldots, P_m$, and such that $\Omega_m = \Omega_{P_1,\ldots,P_m}$ when $m \ge 1$; if $m = 0$, then Ω has $C^{2,1}$-boundary (see Fig. 1 left);

(B) there exists a smoothly periodically extendable domain Ω^m having at least $m \ge 0$ planes of symmetry $P_1, \ldots, P_m$, and such that $\Omega^m = \Omega_{P_1,\ldots,P_m}$ when $m \ge 1$; if $m = 0$, then Ω is smoothly periodically extendable domain (see Fig. 1 right).

2.3. The Stokes operator and its kernel

The *Stokes operator* under the boundary conditions in (1.1) is the linear unbounded self-adjoint operator $A : V \to V'$ defined by

$$\langle Au, v\rangle_{V',V} = 2(\mathbf{D}u, \mathbf{D}v)_\Omega \qquad \forall\, u, v \in V.$$

Notice that A can be considered restricted (as an unbounded operator) to H with domain $D(A) := \{u \in V : Au \in H\}$. In particular, if $\Omega \in C^{2,1}$ it is well-known that $D(A)$ coincides with the set

$$W(\Omega) := \{u \in V \cap H^2(\Omega) : (\mathbf{D}u \cdot \nu) \cdot \tau = 0 \text{ on } \partial\Omega\}, \tag{2.3}$$

see Lemma 4.4 for a proof and Remark 4.6 in Section 4.2 for a deeper discussion in the case of sectors. We refer instead to [5, Section 1.3] for a complete analysis including the case of Navier boundary conditions (1.3) with possibly non constant parameter β, in a general L^p setting. By recalling the Green formula (see e.g., [7, formula (2.8)]):

$$-\int_\Omega \Delta u \cdot v + 2\int_{\partial\Omega} [(\mathbf{D}u \cdot \nu) \cdot \tau](v \cdot \tau) = 2\int_\Omega \mathbf{D}u : \mathbf{D}v \qquad \forall\, u \in V \cap H^2(\Omega) \text{ and } \forall\, v \in V, \tag{2.4}$$

it follows that $Au = P(\Delta u)$ for all $u \in D(A)$ where P is the projection operator $L^2(\Omega) \to H$, see [2, Section 5]. Finally, to A we associate the eigenvalue problem

$$(u, \lambda) \in V \times \mathbb{R} : \qquad 2(\mathbf{D}u, \mathbf{D}w)_\Omega = \lambda(u, v)_\Omega \qquad \forall\, v \in V,$$

which in strong form reads

$$\begin{cases} -\Delta u + \nabla p = \lambda u & \text{in } \Omega \\ \nabla \cdot u = 0 & \text{in } \Omega \\ u \cdot \nu = (\mathbf{D}u \cdot \nu) \cdot \tau = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Since the Stokes operator A is linear, compact, self-adjoint and positive, it admits a non-decreasing sequence of eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}_+}$, where the eigenvalues are repeated according to their multiplicity. The set of eigenfunctions $\{e_k\}_{k\in\mathbb{N}_+}$, up to normalization, is a complete orthonormal system in H orthogonal with respect the product $(\mathbf{D}\cdot, \mathbf{D}\cdot)_\Omega$, see [7, Theorem 6.2]. According to the geometry of Ω, problem (2.5) may admit a trivial least eigenvalue, therefore we introduce the kernel of the linear map $v \mapsto \mathbf{D}v$:

$$\mathcal{K}_\Omega := \{u \in V : \mathbf{D}u \equiv 0 \text{ in } \Omega\}, \tag{2.6}$$

so that, when $\mathcal{K}_\Omega$ is non-trivial, the following decomposition holds:

$$\forall\, u \in V, \qquad u = \overline{u} + u_{\mathcal{K}} \quad \text{with} \quad u_{\mathcal{K}} \in \mathcal{K}_\Omega \quad \text{and} \quad \overline{u} \in \mathcal{K}_\Omega^\perp, \tag{2.7}$$

where $\mathcal{K}_\Omega^\perp$ is the orthogonal complement of $\mathcal{K}_\Omega$ in $L^2(\Omega)$. In Section 4.2 we prove the following characterization of $\mathcal{K}_\Omega$ in the context of *sectors*:

Proposition 2.4. *Let $\Omega \subset \mathbb{R}^3$ be a sector, then $\mathcal{K}_\Omega \neq \emptyset$ if and only if Ω is axially symmetric. In particular, if Ω is axially symmetric two cases may occur:*

(i) Ω is a ball and $\mathcal{K}_\Omega = \{\vec{a} + \vec{\ell} \wedge (x, y, z) : \vec{a}, \vec{\ell}, (x, y, z) \in \mathbb{R}^3\}$;
(ii) Ω is monoaxially symmetric with axis parallel to some unit vector $\vec{\ell}$ and

$$\mathcal{K}_\Omega = \{\vec{a} + c_0\, \vec{\ell} \wedge (x, y, z) : c_0 \in \mathbb{R} \text{ and } \vec{a}, (x, y, z) \in \mathbb{R}^3\}.$$

Remark 2.5. By Proposition 2.4, the first eigenvalue of (2.5) is positive and, in turn, the Poincaré inequality holds, if and only if Ω is non-axially symmetric. When Ω is axially symmetric, by the same reasoning, a Poincaré type inequality holds only in $\mathcal{K}_\Omega^\perp$. Therefore, recalling (2.7), we conclude that there exists $C_\Omega > 0$ such that:

$$\|u\|_{2,\Omega} \le C_\Omega \begin{cases} \|\mathbf{D}u\|_{2,\Omega} & \text{if } \Omega \text{ is non-axially symmetric} \\ \|u_{\mathcal{K}}\|_{2,\Omega} + \|\mathbf{D}\overline{u}\|_{2,\Omega} & \text{if } \Omega \text{ is axially symmetric} \end{cases} \qquad \forall\, u \in V. \tag{2.8}$$

Since, by Korn's inequality, $\|u\|_{H^1(\Omega)} \le c(\|u\|_{2,\Omega} + \|\mathbf{D}u\|_{2,\Omega})$ for all $u \in V$ and for some $c = c(\Omega) > 0$, in view of (2.8), in the sequel we will often take as norms on V (equivalent to the H^1-norm): $\|\mathbf{D}u\|_{2,\Omega}$ if Ω is non-axially symmetric and $\|u_{\mathcal{K}}\|_{2,\Omega} + \|\mathbf{D}\overline{u}\|_{2,\Omega}$ if Ω is axially symmetric.

2.4. Existence and regularity properties of solutions

In this section we state some useful properties of solutions that we will exploit in the proof of our main results. First we make precise our definition of solution.

We say that $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ is a *weak solution* to (1.1) if

$$\int_0^T (u(t),v)_\Omega\, \phi'(t)\,dt + \phi(0)(u_0,v)_\Omega = \int_0^T \{2\mu(\mathbf{D}u(t),\mathbf{D}v)_\Omega + \int_\Omega (u(t)\cdot\nabla)u(t)\cdot v\}\phi(t)\,dt \tag{2.9}$$

for all $v \in V$ and for all $\phi \in C_c^\infty[0,T)$. For sufficiently small initial data, it was proved in [10] that a weak solution to (1.1) is strong and, in turn, unique. More precisely, the following statement holds:

Proposition 2.6 (*[10, Theorem 2]*). *Let $\Omega \subset \mathbb{R}^3$ be a sector and let $u_0 \in V$. There exists $C = C(\Omega,\mu) > 0$ such that if*

$$\|u_0\|_{2,\Omega}^2(\|\mathbf{D}u_0\|_{2,\Omega}^2 + 1) < C\,, \tag{2.10}$$

then problem (1.1)–(1.2) admits a weak solution u which satisfies

$$u \in L^\infty(\mathbb{R}^+;V) \qquad \text{and} \qquad u_t, Au, \nabla p \in L^2(\mathbb{R}^+;L^2(\Omega))\,, \tag{2.11}$$

so that it is strong, unique and global in time.

Let W be the set defined in (2.3). If $u_0 \in V$ and (2.10) holds, condition (2.11) yields $u \in L^2(0,T;W)$ and $u_t \in L^2(0,T;W')$ for every $T > 0$, whence $u \in C^0([0,\infty);V)$, see Remark 4.6 in Section 4.3. In Section 4.4 we prove that $u(t)$ is also analytic as stated in the following:

Theorem 2.7. *If $u_0 \in V$ satisfies (2.10), the solution $u(t)$ given by Proposition 2.6 is analytic on $(0,\infty)$ as a W-valued function.*

An immediate consequence of the uniqueness given in Proposition 2.6 and the analyticity given in Theorem 2.7 is the fundamental property of backward uniqueness for strong solutions:

Corollary 2.8. *Let $u_{0,1}, u_{0,2} \in V$ satisfy (2.10) for some constant $C = C(\Omega,\mu)$, and let u_1, u_2 be the corresponding strong solutions fulfilling (2.11). If for some $\bar{t} \in (0,\infty)$, $u_1(\bar{t}) = u_2(\bar{t})$, then $u_1(t) = u_2(t)$ for all $t \in [0,\infty)$. In particular, if $u_{0,1} \neq u_{0,2}$ then u_1 and u_2 do not intersect.*

From Corollary 2.8 it follows that the solution given by Proposition 2.6 with $u_0 \not\equiv 0$ satisfies $u(t) \neq 0$ for $t \geq 0$; hence, the Dirichlet quotient:

$$\lambda(t) := \frac{2\|\mathbf{D}u(t)\|_{2,\Omega}^2}{\|u(t)\|_{2,\Omega}^2} \tag{2.12}$$

is well-defined. The study of the asymptotic behaviour of $\lambda(t)$ will be one of the main object of the next section.

3. Main results

3.1. Asymptotic behaviour

In this section we will always denote by $u(t)$ the unique strong solution of (1.1)–(1.2) with $u_0 \not\equiv 0$, given by Proposition 2.6 under the assumption (2.10) and we study its asymptotic behaviour from several points of view. The geometry of the domain requires considering two cases. When Ω is non-axially symmetric, we extend [12, Theorem 1] to Navier boundary conditions, namely we prove that the quotient (2.12) tends to an eigenvalue of the Stokes operator as $t \to \infty$, and in turn, we obtain that both $\|u(t)\|_{2,\Omega}$ and $\|\mathbf{D}u(t)\|_{2,\Omega}$ decay at infinity with the same exponential rate, see Theorem 3.1 and Proposition 3.5 below.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^3$ be a non-axially symmetric sector. Then, there exists an eigenvalue Λ of problem (2.5) such that*

$$\lim_{t\to+\infty} \lambda(t) = \Lambda \tag{3.1}$$

and

$$\lim_{t\to+\infty} \frac{\log\|\mathbf{D}u(t)\|_{2,\Omega}}{t} = \lim_{t\to+\infty} \frac{\log\|u(t)\|_{2,\Omega}}{t} = -\mu\Lambda. \tag{3.2}$$

Instead, when Ω is characterized by some axial symmetry, three different kind of decay may be observed and the energy of the solution may not vanish at infinity.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^3$ be an axially symmetric sector and let $\mathcal{K}_\Omega$ be as defined in (2.6). Furthermore, in view of (2.7), write $u_0 = \bar{u}_0 + u_{0,\mathcal{K}}$ with $\bar{u}_0 \in \mathcal{K}_\Omega^\perp$ and $u_{0,\mathcal{K}} \in \mathcal{K}_\Omega$. Then one of the following holds:*

(i) if $\bar{u}_0 = 0$, then

$$u(t) = u_{0,\mathcal{K}} \in \mathcal{K}_\Omega \qquad \text{for all } t > 0$$

namely $u_{0,\mathcal{K}}$ is a stationary solution of (1.1);

(ii) *if* $u_{0,\mathcal{K}} = 0$*, then*

$$u(t) \in \mathcal{K}_\Omega^\perp \qquad \textit{for all } t > 0$$

and satisfies (3.1) *and* (3.2)*, being* Λ *a nontrivial eigenvalue of problem* (2.5)*;*

(iii) *if* $u_{0,\mathcal{K}} \neq 0$ *and* $\bar{u}_0 \neq 0$*, then*

$$u(t) - u_{0,\mathcal{K}} \in \mathcal{K}_\Omega^\perp \qquad \textit{for all } t > 0 \tag{3.3}$$

and $\|u(t) - u_{0,\mathcal{K}}\|_{2,\Omega}^2 = O(e^{-2\mu\lambda_1 t})$ *as* $t \to +\infty$*, with* λ_1 *being the first non trivial eigenvalue of* (2.5)*. Hence,*

$$\lim_{t\to+\infty} \|u(t)\|_{2,\Omega}^2 = \|u_{0,\mathcal{K}}\|_{2,\Omega}^2.$$

Remark 3.3. The existence of steady flows in axially symmetric domains was already known, as well as the fact that a nonstationary flow converges to a steady flow, see e.g., [11] and [13, Section 6]. However, these results were not settled on sectors and, at our best knowledge, the fact that $\mathcal{K}_\Omega$ and $\mathcal{K}_\Omega^\perp$ are invariants of the flow was not previously highlighted. In particular, Theorem 3.2 points out that, whenever Ω admits an axial symmetry, if the initial datum has non trivial components in the kernel $\mathcal{K}_\Omega$ (case (i) or (iii)), then the fluid velocity will never extinguish, although being characterized by a zero-acceleration. In other words, if the initial datum driving the motion of the fluid complies with the symmetry of the domain and the fluid is allowed to slip on the boundary with no friction, then the fluid will fall into a perpetual motion. On the other hand, if the initial datum is chosen in the complement of the kernel $\mathcal{K}_\Omega^\perp$ (case (ii)), it remains in $\mathcal{K}_\Omega^\perp$ and all results given in Theorem 3.1 in the non-axially symmetric case hold.

In order to further characterize the asymptotic behaviour of $u(t)$, we give:

Definition 3.4. We say that an (L^2-normalized) eigenfunction e_k of (2.5) is an **active mode** for $u(t)$ if

$$c_k(t) := \int_\Omega u(t) \cdot e_k \not\equiv 0 \qquad \text{in } (0,\infty)\,.$$

According to Theorem 3.2, the eigenfunctions corresponding to the null eigenvalue are the only active modes in case (i), while they are not active in case (ii). By [7, Theorem 6.2] we know that the sequence of eigenfunctions $\{e_k\}_{k\in\mathbb{N}_+}$ of problem (2.5) is an Hilbert basis for H, thus we can expand the solution in a Fourier series, that is,

$$u(t) = \sum_{k=1}^{\infty} c_k(t) e_k, \tag{3.4}$$

where the sum is extended only to active modes and the indices k are ordered in such a way that the sequence of eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}_+}$ of A is non-decreasing.

Under the assumptions of Theorem 3.1 or Theorem 3.2-(ii), (3.1) holds and we derive from it further information on the asymptotics. To this aim, we introduce the set of indices:

$$J_\Lambda = \{k \in \mathbb{N}_+ \quad \text{s.t.} \quad \lambda_k = \Lambda\}.$$

The next statement shows that, as $t \to \infty$, the energy concentrates on the modes corresponding to Λ, thus identified by the indices belonging to J_Λ.

Proposition 3.5. *Let the hypotheses of Theorem 3.1 or Theorem 3.2-(ii) hold, and let Λ be as given there. Furthermore, let $u(t)$ be expanded as in (3.4). Then, one has*

$$\sum_{k\in J_\Lambda} c_k^2(t) \sim \|u(t)\|_{2,\Omega}^2 \qquad \textit{and} \qquad \sum_{k\notin J_\Lambda} c_k^2(t) = o(\|u(t)\|_{2,\Omega}^2) \qquad \textit{as} \quad t \to \infty\,. \tag{3.5}$$

Moreover, there holds

$$\lim_{t\to\infty} e^{\mu\Lambda t} u(t) = e_{\lambda_{k_0}} \qquad \textit{in } H \textit{ and in } V, \tag{3.6}$$

where $e_{\lambda_{k_0}}$ is an eigenfunction of A (not necessarily L^2 normalized) corresponding to the eigenvalue λ_{k_0} for some $k_0 \in J_\Lambda$. Hence,

$$c_{k_0}(t) = O(e^{-\mu\Lambda t}) \qquad \textit{and} \qquad c_k(t) = o(e^{-\mu\Lambda t}) \quad \forall\, k \neq k_0 \qquad \textit{as} \quad t \to \infty.$$

The proof of Proposition 3.5 relies on the validity of the limit (3.1) and it is along the lines of the conclusions drawn in [12] for the Dirichlet problem, therefore we omit it. For more details, we refer to [12, Proposition 1] for the proof of (3.5) and to [12, Proposition 3] for the proof of (3.6).

Remark 3.6. A natural issue is to establish a priori which index/indices belong to the set J_Λ and, in turn, to predict the asymptotic behaviour of the solution. If the hypotheses of Theorem 3.1 or those of Theorem 3.2-(ii) hold, one can reproduce in our context the proof of [12, Theorem 3] showing that which indices belong to J_Λ depend on the initial data. In particular, it can be proved that there exists in the set $\mathcal{R}$ of initial data given by (2.10) a flag of smooth analytic manifolds $\mathcal{R} = M_0 \supset M_1 \supset M_2 \supset \ldots$ such that $\Lambda = \lambda_j$ if and only if $u_0 \in M_{j-1} \setminus M_j$ for some $j \in \mathbb{N}_+$ (see also [15]). Each M_j is constructed as the kernel of a nonlinear analytic map $\Phi_j : \mathcal{R} \to R_{\lambda_j} V$, where R_{λ_j} denotes the projection of V on the eigenspace of λ_j. Moreover, $S(t)M_j \subset M_j$ for all $j \in \mathbb{N}_+$. Hence, M_j are invariants of the flow and, recalling Theorems 3.1 and 3.2, we conclude that:

(i) if Ω is non-axially symmetric, the flow (1.1) admits the invariants M_j;
(ii) if Ω is axially symmetric, $\mathcal{K}_\Omega$ and $\mathcal{K}_\Omega^\perp$ are invariants of the flow (1.1). Moreover, the set $\mathcal{K}_\Omega^\perp$ can be further decomposed in the spectral manifolds M_j which are also invariants.

It is worth noticing that, due to their *nonlinear* nature, the definition of the manifolds M_j in terms of the initial data is possible from a theoretical point of view, but it remains quite implicit. In Section 3.2 we provide two examples of 2D domains, one of them axially symmetric (the disk), for which more explicit information can be found.

Remark 3.7. The existence of one mode catching the energy of the system, highlighted in Proposition 3.5, is very similar to that observed in some mathematical models for bridges, when studying the Tacoma Narrows Bridge oscillations, and for which the definition of *prevailing mode* was formally given in [16]. This behaviour depends on the amount of energy inserted into the system by means of the initial data and it is related to the *nonlinear* nature of the models. See also [17–20].

3.2. Complementary results: prototype problems in the 2D case

As already explained in Remark 3.6, which indices belong to the set J_Λ, namely which mode catches the energy of the system definitively, depends on the initial data through the manifolds M_j, however their geometry far from the origin is unknown, see [21, Remark 2.1]. In this section we focus on two prototype problems in the 2D case where more explicit information can be obtained. More precisely, we consider (1.1) when Ω is the square $Q = (0,\pi)^2$ or the disk $B = \{(x,y) \in \mathbb{R}^2 \,:\, x^2 + y^2 < 1\}$, respectively, our prototypes of 2D non-axially symmetric and 2D axially symmetric domains. The starting point of our analysis is the validity, in both cases, of the following identity:

$$-\int_\Omega (u\cdot\nabla)u\cdot Au = 0 \quad \text{for all } u \in W\,, \tag{3.7}$$

see Lemma 4.7 in Section 4.7 for a proof. Taking advantage of the 2D context and (3.7), it is easy to show that, without smallness assumptions on the initial datum, there holds

$$\begin{gathered} \textit{for all } u_0 \in V, \textit{problem (1.1)–(1.2) with } \Omega = Q \textit{ or } \Omega = B \textit{ admits a weak solution } u(t) \\ \textit{which satisfies (2.11) so that it is strong, unique and global in time,} \end{gathered}$$

see Remark 4.8 in Section 4.7. For what concerns the asymptotic behaviour of $u(t)$, Theorems 3.1 and 3.2 hold with no changes in the proofs except some simplifications coming from the validity of (3.7), see the proof of Theorem 3.8 below. In this respect we notice that B is the only axially symmetric set in the plane and the kernel $\mathcal{K}_\Omega$ of the linear map $v \mapsto \mathbf{D}v$ satisfies $\mathcal{K}_\Omega \neq \emptyset$ if and only if $\Omega = B$. According to the proof of Proposition 2.4 it is possible to characterize $\mathcal{K}_B$ as follows

$$\mathcal{K}_B = \{\vec{a} + c_0(y,-x) \,:\, c_0 \in \mathbb{R} \text{ and } \vec{a}, (x,y) \in \mathbb{R}^2\}.$$

By exploiting identity (3.7) we have also established that the quotient $\lambda(t)$ in (2.12) is non-increasing in time; this provides an upper bound for Λ in terms of u_0 and, in turn, allows to establish a priori a majorant for the set J_Λ. More precisely, in Section 4.7 we prove:

Theorem 3.8. *Let $\Omega = Q$ or $\Omega = B$. Furthermore, in the latter case assume that $u_0 \in \mathcal{K}_B^\perp$. Then, $u(t)$ satisfies (3.1) and (3.2), being Λ a nontrivial eigenvalue of problem (2.5). Moreover the map $[0,+\infty) \ni t \mapsto \lambda(t)$ is nonincreasing and satisfies*

$$\Lambda \le \lambda(t) \le \frac{2\|\mathbf{D}u_0\|_{2,\Omega}^2}{\|u_0\|_{2,\Omega}^2} \qquad \text{for all } t > 0\,. \tag{3.8}$$

In particular, by denoting $\{\lambda_k\}_{k\in\mathbb{N}_+}$ the nontrivial eigenvalues of (2.5) with (L^2-normalized) eigenfunctions e_k, we have

(i) if $0 \not\equiv u_0 = c_{0,1}e_1$ for some $c_{0,1} \in \mathbb{R}$, then $\Lambda = \lambda_1$ and $u(t) = c_{0,1}e^{-\mu\lambda_1 t}e_1$;
(ii) if $0 \not\equiv u_0 = c_{0,N}e_N$ and $\Lambda = \lambda_N$ for some $c_{0,N} \in \mathbb{R}, N \in \mathbb{N}_+$, then $u(t) = c_{0,N}e^{-\mu\lambda_N t}e_N$;
(iii) if $0 \not\equiv u_0 = \sum_{k=1}^N c_{0,k}e_k$ for some $c_{0,k} \in \mathbb{R}$ and $N \in \mathbb{N}_+$, then $\Lambda \le \lambda_N$ and at least one among the first N modes of u is active, namely if $u(t)$ is expanded as in (3.4) then there exists $1 \le k_0 \le N$ such that $c_{k_0}(t) \not\equiv 0$.

Remark 3.9. According to Theorem 3.8-(i), $span(e_1)$ is an invariant of the flow. When $\Omega = Q$ a direct inspection reveals that the sets $span(e_k)$ are all invariants of the flow, see Proposition 4.10 in Section 4.7. The proof relies on the explicit form of the eigenfunctions and on their simple analytical expression. We conjecture a similar behaviour when $\Omega = B$, but, unfortunately, the same computations seem difficult to be reproduced since the eigenfunctions have more involved expressions, see Appendix.

4. Proofs

Notation. Since in the following we will deal with several multiplying (positive) constants whose exact value is irrelevant to our purposes, we will frequently use the general symbol C. The actual value may therefore change from line to line, without explicit reference. However, when it is significant to specify the dependence of C on suitable parameters, we will write it explicitly; we will use different symbols only when needed to avoid ambiguity.

4.1. Reflection principle

The general strategy to deal with *sectors* is to handle the irregularity of the domain by introducing suitable *auxiliary problems*, obtained by applying a reflection principle. This strategy was introduced in [10] and adopted to prove Proposition 2.6, however for the sake of completeness we recall the main ideas in this section. When Ω is a sector of type (A) the auxiliary problem is set on Ω_m, see Definition 2.3, while in the case of a sector (B) we use the cell of periodicity $\Omega_{\mathcal{P}}$, see Definition 4.2 below. More in details, we distinguish between the following cases:

- **Sectors of type** (A) **with** $m = 1$. We introduce the domain $\Omega_1 = \Omega_{P_{\omega_1}}$, where ω_1 is the one face satisfying $\partial\Omega = \overline{\omega_1 \cup \Gamma}$, for some Γ having $C^{2,1}$-regularity, see e.g., Fig. 1 on the left. Then, we make two key observations. The first is that Ω_1 is of class $C^{2,1}$. The second is that if a vector field $v \in H^2(\Omega_1)$ is symmetric with respect to P_{ω_1} (in a suitable sense, see (4.1) below) and satisfies the boundary conditions (1.1) in Ω_1, then it satisfies the boundary conditions (1.1) in Ω as well, see Remark 4.1. Thus, the *auxiliary problem* is obtained by setting (1.1) on Ω_1, with the further constraint that the solution is symmetric with respect to P_{ω_1}. This requires working in $H^{\mathcal{E}}$ and $V^{\mathcal{E}}$, which are closed subspaces of H and V containing $\mathcal{E}$-symmetric vector fields with respect to P_{ω_1}. In order to clarify what we mean by $\mathcal{E}$-symmetric functions, let us assume, up to translations and rotations, that ω_1 lies on the plane $z = 0$. Then, a vector field $\Psi = (\Psi_1, \Psi_2, \Psi_3) : \Omega_1 \times (0,T) \to \mathbb{R}^3$, and a scalar function $q : \Omega_1 \times (0,T) \to \mathbb{R}$ are said to be $\mathcal{E}$-symmetric if, for all $(x, y, z, t) \in \Omega_1 \times (0,T)$,

$$\begin{aligned} &\Psi_i(x,y,z,t) = \Psi_i(x,y,-z,t) \quad (i = 1,2)\,, \\ &\Psi_3(x,y,z,t) = -\Psi_3(x,y,-z,t)\,, \\ &q(x,y,z,t) = q(x,y,-z,t)\,. \end{aligned} \tag{4.1}$$

  The space $H^{\mathcal{E}}(\Omega_1)$ (respectively, $V^{\mathcal{E}}(\Omega_1)$) contain vector fields belonging to $H(\Omega_1)$ (respectively, $V(\Omega_1)$) satisfying $(4.1)_1$–$(4.1)_2$. Let (u, p) be the unique strong solution to (1.1) given by Proposition 2.6, under assumption (2.10). Define $\hat{u} = (\hat{u}_1, \hat{u}_2, \hat{u}_3) : \Omega_1 \times \mathbb{R}^+ \to \mathbb{R}^3$ and $\hat{p} : \Omega_1 \times \mathbb{R}^+ \to \mathbb{R}$ to be the symmetrized versions of u and p according to (4.1). The auxiliary problem then reads as follows:

$$\begin{cases} \hat{u}_t - \mu\Delta\hat{u} + (\hat{u}\cdot\nabla)\hat{u} + \nabla\hat{p} = 0 & \text{in } \Omega_1 \times (0,T)\,, \\ \nabla\cdot\hat{u} = 0 & \text{in } \Omega_1 \times (0,T)\,, \\ \hat{u}\cdot\nu = (\mathbf{D}\hat{u}\cdot\nu)\cdot\tau = 0 & \text{on } \partial\Omega_1 \times (0,T)\,, \end{cases} \tag{4.2}$$

  with the initial condition

$$\hat{u}(x,y,z,0) = \hat{u}_0(x,y,z) \qquad \text{in} \quad \Omega_1\,. \tag{4.3}$$

  In the following, results will be proven for problem (4.2)–(4.3) in the spaces $H^{\mathcal{E}}(\Omega_1)$ and $V^{\mathcal{E}}(\Omega_1)$, and then coming back to the original problem by restricting $\hat{u}$ to Ω.

Remark 4.1. The key idea of the above symmetrization argument is the following. Let $\hat{u} \in C^1(\Omega_1)$ and $\mathcal{E}$−symmetric with respect to the plane containing $\omega_1 : z = 0$, characterized by the unit normal $\nu = (0,0,1)$ and tangential vector $\tau = (\tau_1, \tau_2, 0)$; then $\hat{u}$ satisfies Navier boundary conditions on ω_1. Indeed, in view of (4.1), we have

$$\hat{u}\cdot\nu|_{\omega_1} = \hat{u}_3(x,y,0,t) = 0 \qquad \Rightarrow \qquad \nabla(\hat{u}\cdot\nu)\cdot\tau|_{\omega_1} = 0.$$

On the other hand, again from (4.1), it follows that $(\hat{u}_i)_z(x,y,0,t) = 0$ for $i = 1,2$, by which

$$\nabla(\hat{u}\cdot\tau)\cdot\nu|_{\omega_1} = (\hat{u}_1)_z(x,y,0,t)\tau_1 + (\hat{u}_2)_z(x,y,0,t)\tau_2 = 0$$

implying

$$(\mathbf{D}\hat{u}\cdot\nu)\cdot\tau|_{\omega_1} = \frac{1}{2}\nabla(\hat{u}\cdot\nu)\cdot\tau|_{\omega_1} + \frac{1}{2}\nabla(\hat{u}\cdot\tau)\cdot\nu|_{\omega_1} = 0.$$

The above identities can be extended in terms of weak derivatives.

We notice that, this argument fails if we consider the boundary conditions (1.3) with $\beta \neq 0$, due to the presence of the tangential velocity in the boundary conditions.

- **Sectors of type**(A) **with** $m \geq 2$. In this case, we have that $\partial\Omega = \overline{\bigcup_{i=1}^m \Gamma \cup \omega_i}$ for some Γ having $C^{2,1}$-regularity, and ω_i, $i = 1,\dots,m$ faces as in Definition 2.1. The procedure used to handle this type of sectors is inductive: for $m \geq 2$, we exploit the result obtained in the case $m-1$. For instance, in the case $m = 2$, we define $\Omega_2 = \left(\Omega_{P_{\omega_1}}\right)_{P_{\omega_2}}$; then we consider the problem in $\Omega_{P_{\omega_1}}$, where we have only the face ω_2, thus we can apply the results for $m = 1$. Observe that in this case the spaces $H^{\mathcal{E}}(\Omega_m)$ and $V^{\mathcal{E}}(\Omega_m)$ contain vector fields with $m \geq 2$ symmetries, which are obtained iteratively as $\mathcal{E}$-symmetric extensions of the corresponding vector fields at step $m-1$.

- **Sectors of type** (B), **smoothly periodically extendable in one direction.** We first introduce the notion of cell of periodicity $\Omega_{\mathcal{P}}$ (which should not be confused with the reflected domains Ω_P).

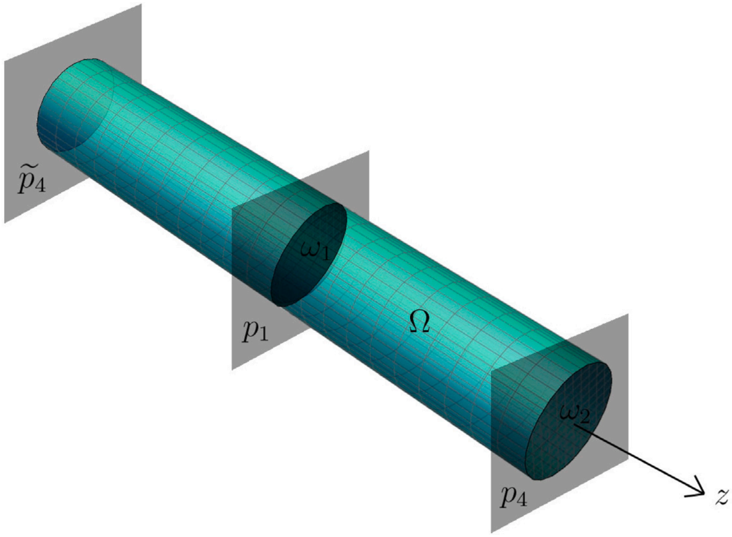

Fig. 2. The cylinder Ω is a sector of type (B) its cell of periodicity is the double cylinder.

Definition 4.2. Let Ω^m be as given in Definition 2.3. The *cell of periodicity* $\Omega_{\mathcal{P}}$ is the set obtained by reflecting Ω^m in each of the directions of periodicity, except for those that have already been used to obtain Ω^m.

In Figs. 1 (left) and 2 we provide two examples where the cell of periodicity $\Omega_{\mathcal{P}}$ is obtained from reflections due to different reasons. Indeed, the *bottle* Ω is reflected one time for topological reasons, in order to get the smoothly periodically extendable domain Ω^1 which coincides with $\Omega_{\mathcal{P}}$. On the other hand, the cylinder Ω in Fig. 2 is already smoothly periodically extendable but, in order to get $\Omega_{\mathcal{P}}$, it is reflected one time (since it has one direction of periodicity) for analytical reasons that we explain in a while. Notice that $\Omega_{\mathcal{P}}$ is only Lipschitzian, while its periodic extension is $C^{2,1}$. In order to overcome this difficulty, the strategy developed in [10] is to introduce an auxiliary problem obtained by setting (1.1) on $\Omega_{\mathcal{P}}$ with two constraints: the solution must be symmetric with respect the planes of reflections used to obtain $\Omega_{\mathcal{P}}$ and it must be periodic with respect to the directions of periodicity of $\Omega_{\mathcal{P}}$. Both conditions are included in the functional setting.

As an explicit example, let $p_1 \parallel p_4$ be the unique couple of planes in (2.2). Let us assume that $p_1 : z = 0$ and $p_4 : z = z_0$ for some $z_0 \in \mathbb{R}^+$, while $\widetilde{p}_4 : z = -z_0$ is the symmetric of p_4 with respect to p_1, see e.g., Fig. 2. Instead of working with H and V, one works with $H^{\mathcal{E}}(\Omega_{\mathcal{P}}) \cap L^2_{\mathcal{P}}$ and $V^{\mathcal{E}}(\Omega_{\mathcal{P}}) \cap H^1_{\mathcal{P}}$, where $H^{\mathcal{E}}(\Omega_{\mathcal{P}})$ (respectively, $V^{\mathcal{E}}(\Omega_{\mathcal{P}})$) are the subsets of H (respectively, V) of vector fields symmetric with respect to the planes of reflection used to obtain $\Omega_{\mathcal{P}}$ while

$$H^s_{\mathcal{P}} := \left\{ u \in H^s(\Omega_{\mathcal{P}}) \quad : \quad u = \sum_{k\in\mathbb{Z}} c_k e^{\frac{i\pi k}{z_0} z} \quad \text{and} \quad \sum_{k\in\mathbb{Z}} k^{2s} |c_k|^2 < \infty \right\} \qquad (s = 0, 1, 2),$$

with the convention that $H^0_{\mathcal{P}} = L^2(\Omega_{\mathcal{P}})$. In general, $u|_{p_1} \neq u|_{p_4}$, that is why, for instance, in the case of the cylinder Ω one has to perform one reflection with respect to p_1 so as to obtain functions which are periodic in the z direction.

Still referring to the above explicit example, let (u, p) be the unique strong solution to (1.1) given by Proposition 2.6 under assumption (2.10), we define u^m, p^m, u^m_0 to be the symmetrized corresponding vector fields, coherent with the symmetries of Ω^m and defined iteratively as in the case of type (A) sectors. If $\Omega^m \equiv \Omega_{\mathcal{P}}$, we put $u^{\mathcal{P}} = u^m, p^{\mathcal{P}} = p^m, u^{\mathcal{P}}_0 = u^m_0$. Otherwise, we define $u^{\mathcal{P}}, p^{\mathcal{P}}, u^{\mathcal{P}}_0$ to be the $\mathcal{E}$-symmetric extensions of u^m, p^m, u^m_0 on $\Omega_{\mathcal{P}}$. The proof of Proposition 2.6 in [10] is achieved by dealing with the ***auxiliary problem*** obtained by setting (1.1) on $\Omega_{\mathcal{P}}$ and replacing $(1.1)_3$ with

$$\begin{aligned} &u^{\mathcal{P}}(x, y, -z_0, t) = u^{\mathcal{P}}(x, y, z_0, t) \quad \text{on} \quad \Gamma_1 \times (0, T), \\ &u^{\mathcal{P}} \cdot \nu = (\mathbf{D}u^{\mathcal{P}} \cdot \nu) \cdot \tau = 0 \quad \text{on} \quad \Gamma_N \times (0, T), \end{aligned} \tag{4.4}$$

where $\Gamma_1 := \partial\Omega_{\mathcal{P}} \cap (p_4 \cup \widetilde{p}_4)$ and $\Gamma_N := \partial\Omega_{\mathcal{P}} \setminus \Gamma_1$. We notice that the first in (4.4), combined with the second in (4.1), yields $u^{\mathcal{P}} \cdot \nu|_{p_4} = 0$, hence, under the $\mathcal{E}$-symmetry assumption, one finally recovers the boundary conditions in (1.1) for $\Omega_{\mathcal{P}}$. In particular, in [10] it was first proved that $u^{\mathcal{P}} \in L^\infty(\mathbb{R}^+; V^{\mathcal{E}}(\Omega_{\mathcal{P}}) \cap H^1_{\mathcal{P}}(\Omega_{\mathcal{P}}))$ and $u^{\mathcal{P}}_t, Au^{\mathcal{P}}, \nabla p^{\mathcal{P}} \in L^2(\mathbb{R}^+; L^2(\Omega_{\mathcal{P}}))$ and then the statement for (u, p) was gained by taking the restriction of $u^{\mathcal{P}}$ to Ω.

- **Sectors of type (B), smoothly periodically extendable in two or three directions.** The previous case is extended by properly modifying the auxiliary problem. This requires introducing the cell of periodicity $\Omega_{\mathcal{P}}$, changing the functional spaces so as to include periodicity and symmetries in two or three directions, and defining the suitable $\mathcal{E}$-symmetric extension of (u, p), see [10, Appendix 1] for more details and figures.

4.2. Proof of Proposition 2.4

Parts of the proof were already known but not yet stated in the case of sectors, see e.g., [7, Lemma 3.3], [11, Appendix 1], [22, Theorem 1] and [13]. We start by noticing that

$$\mathbf{D}z = 0 \qquad \Longleftrightarrow \qquad z_{\vec{\ell}} = \vec{a} + \vec{\ell} \wedge (x, y, z) \qquad \forall (x, y, z) \in \Omega \quad \text{for some } \vec{a}, \vec{\ell} \in \mathbb{R}^3.$$

If Ω is a sector of type (A) with $m = 0$, it is readily seen that $z_{\vec{\ell}} \cdot \nu = 0$ on $\partial\Omega$ if and only if Ω is axially symmetric with respect to an axis parallel to $\vec{\ell}$. In particular, if two functions $z_{\vec{\ell}_1}, z_{\vec{\ell}_2}$ satisfy $z_{\vec{\ell}_1} \cdot \nu = z_{\vec{\ell}_2} \cdot \nu = 0$ on $\partial\Omega$ for $\vec{\ell}_1 \neq \vec{\ell}_2$, then, in view of [22, Lemma 1], Ω is a ball.

In all other cases, we observe that the boundary $\partial\Omega$ of a sector Ω may be written as

$$\partial\Omega = \overline{\bigcup_{i=1}^{k} \omega_i \cup \Gamma},$$

for some Γ having $C^{2,1}$ regularity, and $\omega_i, i = 1, \dots, k$ faces according to Definition 2.1. In particular each of the faces of a sector "sticks orthogonally" to the smooth part Γ. If $k = 1$, then, $z_{\vec{\ell}} \cdot \nu = 0$ on ω_1 if and only if $\vec{\ell} = c_0\nu$ for some $c_0 \in \mathbb{R}$, hence Ω is axially symmetric with respect to ν. If $k = 2$, by the same argument we find that $\mathcal{K}_\Omega \neq \emptyset$ if and only if ω_1 and ω_2 are parallel, and Ω is axially symmetric with respect to ν, being ν the unit vector normal to ω_1 and ω_2. The case $k \geq 3$ gives a contradiction.

Remark 4.3. By the reflection principle outlined in Section 4.1 we can properly relate the eigenvalue problem (2.5) posed on a sector Ω with those posed on the auxiliary domains Ω_m or $\Omega_{\mathcal{P}}$. In particular, with reference to the notations of Section 4.1, if Ω is a sector of type (A), the system of eigenfunctions $\{e_k\}_{k\in\mathbb{N}_+}$ of A in $H(\Omega)$ can be obtained by restricting to Ω the set of eigenfunctions $\{\hat{e}_k\}_{k\in\mathbb{N}_+}$ of A in $H^{\mathcal{E}}(\Omega_m)$. Moreover, as for V, in $V^{\mathcal{E}}(\Omega_m)$, the usual decomposition holds:

$$\forall\, v \in V^{\mathcal{E}}(\Omega_m), \qquad v = \bar{v} + v_{\mathcal{K}} \qquad \text{with} \quad v_{\mathcal{K}} \in \mathcal{K}_{\Omega_m} \cap V^{\mathcal{E}}(\Omega_m) \quad \text{and} \quad \bar{v} \in \mathcal{K}^{\perp}_{\Omega_m} \cap V^{\mathcal{E}}(\Omega_m).$$

We observe that, if the sector Ω is non-axially symmetric, then $\mathcal{K}_{\Omega_m} \cap V^{\mathcal{E}}(\Omega_m) = \emptyset$ even if Ω_m is axially symmetric, otherwise, restricting to Ω, we obtain $\mathcal{K}_\Omega \neq \emptyset$ contradicting Proposition 2.4. The case of sectors (B) works similarly.

4.3. Technical lemmas

In this section we prove two technical lemmas needed in the proofs of Theorems 2.7, 3.1 and 3.2. The statements are given for smooth domains while at the end of the section we explain how they can be exploited in the framework of sectors by means of the reflection principle in Section 4.1. With reference to the notations of Section 2.3, we prove

Lemma 4.4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,1}$. Then $D(A) = W(\Omega)$ and*

(i) if Ω is non-axially symmetric, $\|Au\|_{2,\Omega}$ is a norm on $D(A)$, equivalent to the norm induced by $H^2(\Omega)$;
(ii) if Ω is axially symmetric, $\|Au\|_{2,\Omega} + \|u_{\mathcal{K}}\|_{2,\Omega}$ is a norm on $D(A)$ equivalent to the norm induced by $H^2(\Omega)$.

Proof. Parts of the proof were already known, see e.g., [11, Theorem 1.2] or [5, Theorem 1.3.1], however, for the sake of completeness, we provide a short but full proof here. We start by considering the case where Ω is *non-axially symmetric*. Let $u \in D(A)$, by definition $Au \in L^2(\Omega)$, namely there exists $f \in L^2(\Omega)$ such that

$$2\int_\Omega \mathbf{D}u : \mathbf{D}v = \int_\Omega f \cdot v\,, \qquad \forall v \in V\,. \tag{4.5}$$

By interpreting (4.5) as the weak formulation of a stationary Stokes problem, we apply [7, Theorem 4.1] and we deduce that $u \in H^2(\Omega)$ and

$$\|u\|_{H^2(\Omega)} \leq C\|f\|_{2,\Omega} = C\|Au\|_{2,\Omega},$$

for some $C = C(\Omega) > 0$. Since the inverse inequality is trivial, it follows that $\|Au\|_{2,\Omega}$ is a norm on $V \cap H^2(\Omega)$. Then, by exploiting formula (2.4), we also deduce that $(\mathbf{D}u \cdot \nu) \cdot \tau = 0$ on $\partial\Omega$, hence $u \in W(\Omega)$. Viceversa, if $u \in W(\Omega)$, by exploiting again (2.4), it is readily seen that (4.5) holds with $f = -P(\Delta u) \in L^2(\Omega)$, therefore $u \in D(A)$.

If Ω is *axially symmetric*, by Proposition 2.4, $\mathcal{K}_\Omega$ is non-trivial. Thus, for $u \in D(A)$ we have $u = \bar{u} + u_{\mathcal{K}}$, where $u_{\mathcal{K}} \in K_\Omega$, and $\bar{u} \in \mathcal{K}^{\perp}_{\Omega}$ and, as above, $Au \in L^2(\Omega)$ is equivalent to

$$2\int_\Omega \mathbf{D}\bar{u} : \mathbf{D}v = \int_\Omega f \cdot v\,, \qquad \forall v \in V,$$

for some $f \in L^2(\Omega)$. Then, by [7, Theorem 4.1] (which holds in $\mathcal{K}^{\perp}_{\Omega}$), $\|\bar{u}\|_{H^2(\Omega)} \leq C\|Au\|_{2,\Omega}$ for some $C = C(\Omega) > 0$. Therefore, since $\|u\|^2_{H^2(\Omega)} = \|\bar{u}\|^2_{H^2(\Omega)} + \|u_{\mathcal{K}}\|^2_{2,\Omega}$, (ii) follows by recalling Remark 2.5. Once this established, the fact that $D(A) = W(\Omega)$ follows by exploiting (2.4) as explained in the non-axially symmetric case. □

We are now ready to provide some estimates for the convection term. To this aim, we first recall the identities below which follows by integrating by parts and that will be exploited throughout the paper:

$$\int_\Omega (u\cdot\nabla)v\cdot w = -\int_\Omega (u\cdot\nabla)w\cdot v \quad\text{and}\quad \int_\Omega (u\cdot\nabla)v\cdot v = 0 \qquad \forall u\in V\,, \forall v,w\in H^1(\Omega)\,. \tag{4.6}$$

Lemma 4.5. *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain of class $C^{2,1}$. For all $u,w\in D(A)$ the following inequalities hold:*

(i) if Ω is non-axially symmetric:

$$|((u\cdot\nabla)u, Aw)_\Omega| \le \begin{cases} C_1\|\mathbf{D}u\|_{2,\Omega}^{3/2}\|Au\|_{2,\Omega}^{1/2}\|Aw\|_{2,\Omega} \\ C_2\|u\|_{2,\Omega}^{1/2}\|\mathbf{D}u\|_{2,\Omega}^{1/2}\|Au\|_{2,\Omega}\|Aw\|_{2,\Omega}; \end{cases} \tag{4.7}$$

(ii) if Ω is axially symmetric:

$$|((u\cdot\nabla)u, Aw)_\Omega| \le \begin{cases} C_3\left(\|u_\mathcal{K}\|_{2,\Omega}+\|\mathbf{D}\overline{u}\|_{2,\Omega}\right)^{3/2}\left(\|u_\mathcal{K}\|_{2,\Omega}+\|Au\|_{2,\Omega}\right)^{1/2}\|Aw\|_{2,\Omega} \\ C_4\|u\|_{2,\Omega}^{1/2}\left(\|u_\mathcal{K}\|_{2,\Omega}+\|\mathbf{D}\overline{u}\|_{2,\Omega}\right)^{1/2}\left(\|u_\mathcal{K}\|_{2,\Omega}+\|Au\|_{2,\Omega}\right)\|Aw\|_{2,\Omega}\,, \end{cases}$$

for some $C_i = C_i(\Omega)>0$ with $i=1,\dots,4$.

Proof. Let Ω be non-axially symmetric. Then, by Proposition 2.4 we know that the norms $\|\nabla\cdot\|_{2,\Omega}$ and $\|\mathbf{D}\cdot\|_{2,\Omega}$ are equivalent. Thus, by suitable Sobolev embeddings and Lemma 4.4, for all $v\in V$ and $w\in D(A)$ we have

$$\|v\|_{6,\Omega}\le C\|\mathbf{D}v\|_{2,\Omega} \tag{4.8a}$$
$$\|\nabla w\|_{6,\Omega}\le C\|Aw\|_{2,\Omega} \tag{4.8b}$$
$$\|\nabla w\|_{3,\Omega}\le \|\nabla w\|_{2,\Omega}^{1/2}\|\nabla w\|_{6,\Omega}^{1/2}\le C\|\mathbf{D}w\|_{2,\Omega}^{1/2}\|Aw\|_{2,\Omega}^{1/2}\,. \tag{4.8c}$$

Then, inequalities (4.7) can then be obtained as an immediate consequence, by using $\|(v\cdot\nabla)u\|_{2,\Omega}\le\|v\|_{6,\Omega}\|\nabla u\|_{3,\Omega}$ in $(4.7)_1$ and $\|(v\cdot\nabla)u\|_{2,\Omega}\le\|v\|_{3,\Omega}\|\nabla u\|_{6,\Omega}$ in $(4.7)_2$ for all $v,u\in H^2(\Omega)\cap V$.

If Ω is non-axially symmetric, let $v=\overline{v}+v_\mathcal{K}\in V$ and $w=\overline{w}+w_\mathcal{K}\in D(A)$, where $v_\mathcal{K},w_\mathcal{K}\in K_\Omega$ and $\overline{v},\overline{w}\in\mathcal{K}_\Omega^\perp$. Inequalities (4.8a)–(4.8b)–(4.8c) change as

$$\|v\|_{6,\Omega}\le C\left(\|v_\mathcal{K}\|_{2,\Omega}+\|\mathbf{D}v\|_{2,\Omega}\right)$$
$$\|\nabla w\|_{6,\Omega}\le C\left(\|w_\mathcal{K}\|_{2,\Omega}+\|Aw\|_{2,\Omega}\right)$$
$$\|\nabla w\|_{3,\Omega}\le\|\nabla w\|_{2,\Omega}^{1/2}\|\nabla w\|_{6,\Omega}^{1/2}\le C\left(\|w_\mathcal{K}\|_{2,\Omega}+\|\mathbf{D}\overline{w}\|_{2,\Omega}\right)^{1/2}\left(\|w_\mathcal{K}\|_{2,\Omega}+\|Aw\|_{2,\Omega}\right)^{1/2}\,,$$

for all $v\in V$ and $w\in D(A)$. □

Remark 4.6. In the framework of sectors, recalling the notations in Section 4.1, the conclusions of Lemmas 4.4 and 4.5 are needed only for the auxiliary problems which are set on Ω_m or $\Omega_\mathcal{P}$, respectively for type (A) or type (B) sectors. The sets Ω_m are $C^{2,1}$ therefore Lemmas 4.4 and 4.5 apply. Concerning with $\Omega_\mathcal{P}$, the statement hold by modifying the domain of A as follows

$$D_\mathcal{P}(A) := \{u\in V^\mathcal{E}(\Omega_\mathcal{P})\cap H^1_\mathcal{P}\ :\ Au\in L^2_\mathcal{P}\}.$$

With this position, we may repeat the proof of Section 4.1 with problem (4.5) set on $\Omega=\Omega_\mathcal{P}$ with $f\in L^2_\mathcal{P}$. Here, the lack of smoothness of $\Omega_\mathcal{P}$ in the directions of periodicity can be overcome by viewing this problem as the restriction to $\Omega_\mathcal{P}$ of (4.5) settled on a suitable periodic (smooth) extension of $\Omega_\mathcal{P}$. Making reference to Fig. 2, one can define the extended domain by adding a cylinder (namely a copy of $\Omega_\mathcal{P}$) on both sides of $\Omega_\mathcal{P}$. Finally, we recover the regularity on $\Omega_\mathcal{P}$ by *local* regularity results for (4.5) settled on the extended domain. In particular, we establish that

$$D_\mathcal{P}(A) = W_\mathcal{P}(\Omega_\mathcal{P}) := \{u\in V^\mathcal{E}(\Omega_\mathcal{P})\cap H^2_\mathcal{P}\ :\ (\mathbf{D}u\cdot\nu)\cdot\tau = 0\,\text{on}\,\Gamma_N\},$$

Similarly, from local a priori estimates we recover the desired equivalence of the norms $\|Au\|_{2,\Omega_\mathcal{P}}$ or $\|Au\|_{2,\Omega_\mathcal{P}}+\|u_\mathcal{K}\|_{2,\Omega_\mathcal{P}}$ on $W_\mathcal{P}(\Omega_\mathcal{P})$, respectively in the non-axially symmetric or axially symmetric case.

We conclude the discussion by noticing that, if u is the solution of (1.1) given by Proposition 2.6, by construction (as explained in Section 4.1), there holds:

(i) if Ω is a type (A) sector then the $\mathcal{E}$-symmetric extension $\hat{u}$ of u to Ω_m satisfies $A\hat{u}\in L^2(\Omega_m)$;
(ii) Ω is a type (B) sector then the $\mathcal{E}$-symmetric and periodic extension $u_\mathcal{P}$ of u to $\Omega_\mathcal{P}$ satisfies $Au_\mathcal{P}\in L^2(\Omega_\mathcal{P})$.

In particular, for what remarked above, for type (A) sectors we deduce that $\hat{u}\in W(\Omega_m)$ and, in turn, arguing as in Remark 4.1, that $u\in W(\Omega)$. The same conclusion holds for type (B) sectors.

4.4. Proof of Theorem 2.7

The proof goes along the lines of that of [23, Lemma 3.1] in the Dirichlet case but with nontrivial changes in the estimates needed to conclude (see also [24, Chapter 12] and [25, Chapter 2, Section 8]). We give the proof when Ω has a $C^{2,1}$-boundary; the result can be extended to sectors of type (A) (resp., (B)) using the auxiliary domain Ω_m (resp., the cell of periodicity $\Omega_{\mathcal{P}}$) by applying the reflection principle outlined in Section 4.1. We recall that if Ω is a non-axially symmetric sector, then for the corresponding Ω_m (resp., $\Omega_{\mathcal{P}}$) one has $\mathcal{K}_{\Omega_m} \cap V^{\mathcal{E}}(\Omega_m) = \emptyset$ (resp., $\mathcal{K}_{\Omega_{\mathcal{P}}} \cap V^{\mathcal{E}}(\Omega_{\mathcal{P}}) = \emptyset$).

The procedure relies on the complexification of H, V and V', which we denote by H_C, V_C and V'_C. The complexified space H_C of H (respectively V_C, V'_C of V, V') is given by

$$H_C = \{u_1 + iu_2 \ : \ u_1, u_2 \in H\}$$

and it can be endowed with the scalar product:

$$(u, v)_{H_C} = (u_1 + iu_2, v_1 + iv_2)_{H_C} := (u_1, v_1)_\Omega + (u_2, v_2)_\Omega + i\left[(u_2, v_1)_\Omega - (u_1, v_2)_\Omega\right].$$

By linearity, the Stokes operator A extends to a self-adjoint operator in H_C. Let $\{e_k\}_{k=1}^\infty \subset V_C$ be the set of eigenfunctions giving an orthogonal basis in V_C, and an orthonormal basis in H_C. Fix $\theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ and take $t = se^{i\theta}$ for $s > 0$. Then we consider the extension to complex t of the nth-order approximation of (1.1), that is

$$\begin{cases} (u_t^n(t), e_k)_{H_C} + \mu\langle Au^n(t), e_k\rangle_{V'_C, V_C} = -((u^n(t)\cdot\nabla)u^n(t), e_k)_{H_C}, \quad k = 1, \dots, n \\ u^n(0) = \sum_{k=1}^n (u_0, e_k)_{H_C} e_k. \end{cases} \tag{4.10}$$

The system has a unique analytic solution $u^n(t)$ for $|t|$ small, of the form $u^n(x,t) := \sum_{k=1}^n c_k^n(t) e_k(x)$. In particular,

$$\frac{d}{ds}\|u^n(se^{i\theta})\|_{H_C}^2 = 2\mathrm{Re}\left[e^{i\theta}\left(\frac{d}{dt}u^n(se^{i\theta}), u^n(se^{i\theta})\right)_{H_C}\right]$$

and

$$\frac{d}{ds}\|\mathbf{D}u^n(se^{i\theta})\|_{H_C}^2 = \frac{1}{2}\frac{d}{ds}\langle Au^n(se^{i\theta}), u^n(se^{i\theta})\rangle_{V'_C, V_C} = \mathrm{Re}\left[e^{i\theta}\langle\frac{d}{dt}u^n(se^{i\theta}), Au^n(se^{i\theta})\rangle_{V'_C, V_C}\right].$$

We multiply $(4.10)_1$ by $e^{i\theta}$, then by $\overline{c_k^n(t)}$, we sum over k and we take the real part to obtain

$$\frac{1}{2}\frac{d}{ds}\|u^n(se^{i\theta})\|_{H_C}^2 + 2\mu\cos\theta\|\mathbf{D}u^n(se^{i\theta})\|_{H_C}^2 = 0. \tag{4.11}$$

On the other hand, by multiplying $(4.10)_1$ for $e^{i\theta}$, then by $\overline{\lambda_k c_k^n(t)} = \lambda_k\overline{c_k^n(t)}$, where $\{\lambda_k\}_{k=1}^\infty$ is the set of eigenvalues of (2.5), summing over k and taking the real part, we have

$$\frac{d}{ds}\|\mathbf{D}u^n(se^{i\theta})\|_{H_C}^2 + \mu\cos\theta\|Au^n(se^{i\theta})\|_{H_C}^2 = -\mathrm{Re}\left[e^{i\theta}((u^n(se^{i\theta})\cdot\nabla)u^n(se^{i\theta}), Au^n(se^{i\theta}))_{H_C}\right]. \tag{4.12}$$

Next, we need to manipulate (4.11)–(4.12). This leads to distinguish between two cases. Before proceeding, we emphasize that all estimates obtained in the real case still hold in the complex framework, with the bounding constants being larger.

(ii) *Case Ω non-axially symmetric.*
Inequality $(2.8)_1$ applied to (4.11) gives $\frac{d}{ds}\|u^n(se^{i\theta})\|_{H_C}^2 + \frac{4\mu}{C}\cos\theta\|u^n(se^{i\theta})\|_{H_C}^2 \le 0$ which, for $s > 0$, yields

$$\|u^n(se^{i\theta})\|_{H_C}^2 \le \|u_0\|_{2,\Omega}^2 e^{-\frac{4\mu}{C}(\cos\theta)s}. \tag{4.13}$$

If we integrate (4.11) over $(0, \tau)$, we get

$$\|u^n(\tau e^{i\theta})\|_{H_C}^2 + 4\mu\cos\theta\int_0^\tau \|\mathbf{D}u^n(se^{i\theta})\|_{H_C}^2\,ds = \|u_0\|_{2,\Omega}^2,$$

which, recalling (4.13), gives $\int_0^\infty \|\mathbf{D}u^n(se^{i\theta})\|_{H_C}^2\,ds = \frac{1}{4\mu\cos\theta}\|u_0\|_{2,\Omega}^2$. To estimate the right-hand side of (4.12), we use $(4.7)_1$. By Young's inequality, we deduce

$$|((u^n\cdot\nabla)u^n, Au^n)_{H_C}| \le C\|\mathbf{D}u^n\|_{H_C}^{3/2}\|Au^n\|_{H_C}^{3/2} \le \frac{\mu}{2}\cos\theta\|Au^n\|_{H_C}^2 + \frac{C}{\mu^3\cos^3\theta}\|\mathbf{D}u^n\|_{H_C}^6.$$

Plugging this into (4.12), we obtain

$$\frac{d}{ds}\|\mathbf{D}u^n(se^{i\theta})\|_{H_C}^2 + \frac{\mu}{2}\cos\theta\|Au^n(se^{i\theta})\|_{H_C}^2 \le \frac{C}{\mu^3\cos^3\theta}\|\mathbf{D}u^n(se^{i\theta})\|_{H_C}^6. \tag{4.14}$$

We apply [10, Lemma 2-(i)] to (4.14) and we obtain that, if $\|u_0\|_{2,\Omega}^2\|\mathbf{D}u_0\|_{2,\Omega}^2 < C\mu^4$, then

$$\|\mathbf{D}u^n(se^{i\theta})\|_{H_C}^2 \le K := \frac{4\mu^4}{\mu^4 - C\|u_0\|_{2,\Omega}^2\|\mathbf{D}u_0\|_{2,\Omega}^2} \qquad \forall s > 0, \forall\theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]. \tag{4.15}$$

Since the right-hand side of (4.15) is uniformly bounded with respect to n, we found that $u^n(t)$ can be actually extended to be analytic in the interior $\mathring{T}$ of the set

$$T = \left\{ t = se^{i\theta} \quad \text{s.t} \quad s \geq 0, \quad \theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4} \right] \right\}.$$

From (4.15) and by the compact embedding of V_C into H_C, we deduce that, for all $t \in T$, we can extract a subsequence, which we still denote by $\{u^n(t)\}_{n=1}^\infty$, converging in H_C. Hence, the Vitali's Theorem (see, e.g., [26, Theorem 3]) applies, so that we can extract a subsequence $\{u^n(t)\}_{n=1}^\infty$ uniformly converging in H_C, on every compact set in $\mathring{T}$, to an analytic H_C-valued function $\tilde{u}(t)$. However, by [10], we know that, under assumption (2.10), for real times $t > 0$, the sequence $\{u^n(t)\}_{n=1}^\infty$ converges in H, uniformly on every compact set included in $[0, \infty)$, to the solution $u(t)$ of (1.1). Thus $\tilde{u}(t)$ is the analytic H_C-valued extension of $u(t)$ to $\mathring{T}$.

As next step we prove that $A\tilde{u}(t)$ is also an analytic H_C-valued function. Let P_k denote the orthogonal projection in H_C onto the space spanned by the first k eigenfunctions $\{e_1, e_2, \dots, e_k\}$. Then, for every k fixed and every compact set M included in $\mathring{T}$, we infer that

$$\|AP_k u^n(t) - AP_k \tilde{u}(t)\|_{H_C} \leq C_k \|u_n(t) - \tilde{u}(t)\|_{H_C} \to 0 \qquad \text{as} \quad n \to \infty, \forall t \in M \subset \mathring{T}.$$

Thus, by Lebesgue's theorem, integrating (4.14) and using (4.15), we have

$$\begin{aligned} \int_M \|AP_k\tilde{u}(t)\|_{H_C}^2\,dt = \lim_{n\to\infty}\int_M \|AP_k u^n(t)\|_{H_C}^2\,dt &\leq \liminf_{n\to\infty}\int_{-\pi/4}^{\pi/4}\int_0^{\bar{s}} \|Au^n(se^{i\theta})\|_{H_C}^2\,s\,ds\,d\theta \\ &\leq C\|\mathbf{D}u_0\|_{2,\Omega}^2 + \frac{CK^3}{\mu^3}\frac{4\pi}{(\sqrt{2})^3}\bar{s} =: K_M, \end{aligned}$$

where $\bar{s}$ is chosen sufficiently large so that $M \subset \{se^{i\theta} : 0 \leq s \leq \bar{s}, \quad \theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]\}$, by which we deduce that

$$\sup_{k=1,2,..}\int_M \|AP_k\tilde{u}(t)\|_{H_C}^2\,dt < K_M < \infty.$$

Since $\{\|AP_k\tilde{u}(t)\|_{H_C}^2\}_{k=1}^\infty$ is increasing, it must be: $\lim_{k\to\infty}\|AP_k\tilde{u}(t)\|_{H_C}^2 < \infty$ a.e. on M.

Hence,

$$0 = \lim_{k\to\infty}\|A\tilde{u}(t) - AP_k\tilde{u}(t)\|_{H_C}^2 = \lim_{k\to\infty}(\|A\tilde{u}(t)\|_{H_C}^2 - \|AP_k\tilde{u}(t)\|_{H_C}^2) \qquad \text{a.e. on } M,$$

by which we infer that $u(t) \in D(A)$ for a.e. $t \in M$. Next, by the Monotone Convergence Theorem, we immediately infer that

$$\begin{aligned} \lim_{k\to\infty}\|A\tilde{u}(t) - AP_k\tilde{u}(t)\|_{L^2(M;H_C)} &= \lim_{k\to\infty}\int_M \|A\tilde{u}(t) - AP_k\tilde{u}(t)\|_{H_C}^2\,dt \\ &= \lim_{k\to\infty}\int_M (\|A\tilde{u}(t)\|_{H_C}^2 - \|AP_k\tilde{u}(t)\|_{H_C}^2)\,dt = 0, \end{aligned} \tag{4.16}$$

for any compact set $M \subset \mathring{T}$. The convergence in (4.16) implies that $\{AP_k\tilde{u}(t)\}_{t=1}^\infty$ converges to $A\tilde{u}(t)$ in H_C uniformly on any compact set included in $\mathring{T}$. Thus, $A\tilde{u}(t)$ is a H_C-valued analytic function on the whole $\mathring{T}$, which implies that $\tilde{u}(t)$ is an analytic $D(A)$-valued function, in $\mathring{T}$, which in turn yields the thesis.

(ii) *Case Ω axially symmetric.*

We rewrite (4.11) as

$$\frac{d}{ds}\left(\|\bar{u}^n(se^{i\theta})\|_{H_C}^2 + \|u_{\mathcal{K}}^n(se^{i\theta})\|_{H_C}^2\right) + 4\mu\cos\theta\|\mathbf{D}u^n(se^{i\theta})\|_{H_C}^2 = 0 \tag{4.17}$$

which yields that $\|u_{\mathcal{K}}^n(se^{i\theta})\|_{H_C}^2 \leq \|u_0\|_{2,\Omega}^2$ and that $\|u^n(se^{i\theta})\|_{H_C}^2$ is non-increasing with respect to s, thus it admits a (finite) limit as $s \to \infty$. By integrating (4.17) on $s \in (0, \infty)$ we get

$$4\mu\cos\theta\int_0^\infty \|\mathbf{D}u^n(se^{i\theta})\|_{H_C}^2\,ds \leq \|u_0\|_{2,\Omega}^2 - \lim_{\tau\to\infty}\|u^n(\tau e^{i\theta})\|_{H_C}^2.$$

Next, we estimate the right-hand side of (4.12). Proceeding as in [10, (35)], we arrive at

$$|((u^n\cdot\nabla)u^n, Au^n)_{H_C}| \leq \frac{\mu}{2}\cos\theta\|Au^n\|_{H_C}^2 + C(1 + \|\mathbf{D}\bar{u}^n\|_{H_C}^6) \tag{4.18}$$

with $C = C(\Omega, \mu, \|u_0\|_{2,\Omega})$. Plugging (4.18) into (4.12), we infer that

$$\frac{d}{ds}\|\mathbf{D}u^n(se^{i\theta})\|_{H_C}^2 + \frac{\mu}{2}\cos\theta\|Au^n(se^{i\theta})\|_{H_C}^2 \leq C(1 + \|\mathbf{D}\bar{u}^n\|_{H_C}^6),$$

and

$$\frac{d}{ds}\|\mathbf{D}u^n(se^{i\theta})\|_{H_C}^2 = \frac{d}{ds}\|\mathbf{D}\bar{u}^n(se^{i\theta})\|_{H_C}^2 \leq C(1 + \|\mathbf{D}\bar{u}^n\|_{H_C}^6). \tag{4.19}$$

Finally, we apply [10, Lemma 2-(ii)] to (4.19) and we deduce that, if $\frac{1}{4\mu}\|u_0\|_{2,\Omega}^2(\|\mathbf{D}\bar{u}_0\|_{2,\Omega}^2 + 1) < \frac{1}{C}$, then there exists $K := K(\Omega, \mu, \|\bar{u}_0\|_{2,\Omega}, \|\mathbf{D}\bar{u}_0\|_{2,\Omega}) > 0$, independent of n, s, θ, such that

$$\|\mathbf{D}\bar{u}^n(se^{i\theta})\|_{H_C}^2 \leq K \quad \forall s > 0, \forall\theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]. \tag{4.20}$$

Once proved (4.20), all subsequent steps follows as in case (i).

4.5. Proof of Theorem 3.1

The proof is achieved expanding on the proof of [12, Theorem 1] for the Dirichlet problem and relies on the validity of the Poincaré inequality which holds if Ω is non-axially symmetric, see e.g., Lemma 4.5-(i). For the sake of completeness, we recall the main steps. We assume that Ω has a $C^{2,1}$-boundary, then, as already explained, the result can be extended to sectors of type (A) (resp., (B)) using the auxiliary domains Ω_m (resp., the cell of periodicity $\Omega_{\mathcal{P}}$) by applying the reflection principle outlined in Section 4.1.

Proof of (3.1). Since Ω is non-axially symmetric the set $\mathcal{K}_\Omega$, defined in (2.6), is empty, thus $u_0 \in \mathcal{K}_\Omega^\perp$. Let $\lambda(t)$ be as given in (3.1), we compute

$$\frac{1}{2}\frac{d}{dt}\lambda(t) = \frac{1}{\|u(t)\|_{2,\Omega}^2}\left[2(\mathbf{D}u, \mathbf{D}u_t)_\Omega - \lambda(u, u_t)_\Omega\right] = \frac{1}{\|u(t)\|_{2,\Omega}^2}\left(Au - \lambda u, u_t\right)_\Omega. \tag{4.21}$$

Denoting by $B(u,u)$ the projection onto H of $(u\cdot\nabla)u$ we write (1.1) projected onto H as

$$u_t + \mu Au + B(u,u) = 0 \quad \text{in } Q_T\,. \tag{4.22}$$

Therefore, (4.21) becomes

$$\frac{1}{2}\frac{d}{dt}\lambda(t) = -\frac{\mu}{\|u(t)\|_{2,\Omega}^2}\|Au - \lambda u\|_{2,\Omega}^2 - \frac{1}{\|u(t)\|_{2,\Omega}^2}\left(B(u,u)\,,\ Au - \lambda u\right)_\Omega. \tag{4.23}$$

In the following, two useful identities will be needed several times. The first is obtained by taking the scalar product of (4.22) with the solution u and exploiting the Green formula (2.4):

$$\frac{d}{dt}\|u(t)\|_{2,\Omega}^2 + 4\mu\|\mathbf{D}u(t)\|_{2,\Omega}^2 = 0\,. \tag{4.24}$$

By taking the scalar product of (4.22) with $Au(t)$, which is possible due to (2.11), we have the second identity:

$$\frac{d}{dt}\|\mathbf{D}u(t)\|_{2,\Omega}^2 + \mu\|Au(t)\|_{2,\Omega}^2 = ((u(t)\cdot\nabla)u(t), Au(t))_\Omega. \tag{4.25}$$

In order to estimate the right-hand side of (4.23) we proceed by steps.

– *Step 1.* We estimate $\|u(t)\|_{2,\Omega}$, $\|\mathbf{D}u(t)\|_{2,\Omega}$ and $\|Au(t)\|_{2,\Omega}$. By using $(2.8)_1$ in (4.24) and observing that $C_\Omega^2 = 2/\lambda_1$ (where λ_1 is the first eigenvalue for (2.5)), we get

$$\frac{d}{dt}\|u(t)\|_{2,\Omega}^2 + \frac{4\mu}{C_\Omega^2}\|u(t)\|_{2,\Omega}^2 \le 0 \quad \Rightarrow \quad \|u(t)\|_{2,\Omega}^2 \le \|u_0\|_{2,\Omega}^2 e^{-2\mu\lambda_1 t}; \tag{4.26}$$

while integrating (4.24) over $[t,T]$ we obtain

$$\|u(T)\|_{2,\Omega}^2 + 4\mu\int_t^T \|\mathbf{D}u(\tau)\|_{2,\Omega}^2 d\tau = \|u(t)\|_{2,\Omega}^2 \qquad \forall\, T \ge t \ge 0.$$

In turn, letting $T\to\infty$ we get

$$\int_t^\infty \|\mathbf{D}u(\tau)\|_{2,\Omega}^2 d\tau = \frac{\|u(t)\|_{2,\Omega}^2}{4\mu} \le \frac{1}{4\mu}\|u_0\|_{2,\Omega}^2 e^{-\frac{4\mu}{C_\Omega^2}t} < \infty \tag{4.27}$$

which, for $t=0$, gives $\int_0^\infty \|\mathbf{D}u(t)\|_{2,\Omega}^2 dt = \frac{\|u_0\|_{2,\Omega}^2}{4\mu}$. Thanks to (4.27), there exists $\mathcal{T}>0$, depending on β, sufficiently large, such that

$$\|\mathbf{D}u(t)\|_{2,\Omega} \le \beta \qquad \forall\, t \ge \mathcal{T}. \tag{4.28}$$

Inequalities (4.26) and (4.28) prove that

$$\|u(t)\|_{2,\Omega}\|\mathbf{D}u(t)\|_{2,\Omega} \le \beta\|u_0\|_{2,\Omega}e^{-\mu\lambda_1 t} \qquad\qquad \forall\, t \ge \mathcal{T}\,. \tag{4.29}$$

By $(4.7)_2$ and (4.29), we infer that

$$|((u(t)\cdot\nabla)u(t), Au(t))_\Omega| \le C\sqrt{\beta}\,\|u_0\|_{2,\Omega}^{1/2}\,e^{-(\mu\lambda_1/2)t}\|Au(t)\|_{2,\Omega}^2\,.$$

The above inequality, inserted in (4.25), gives

$$\frac{d}{dt}\|\mathbf{D}u(t)\|_{2,\Omega}^2 + (\mu - C\sqrt{\beta}\|u_0\|_{2,\Omega}^{1/2}\,e^{-(\mu\lambda_1/2)t})\|Au(t)\|_{2,\Omega}^2 \le 0. \tag{4.30}$$

Integrating (4.30) from t to T ($T \ge t \ge \mathcal{T}$), and letting $T\to+\infty$ we get

$$(\mu - C\sqrt{\beta}\|u_0\|_{2,\Omega}^{1/2}\,e^{-(\mu\lambda_1/2)t})\int_t^\infty \|Au(s)\|_{2,\Omega}^2\,ds \le \|\mathbf{D}u(t)\|_{2,\Omega}^2 \qquad \forall\, t\ge\mathcal{T},$$

implying the existence of $C > 0$ such that

$$\int_t^\infty \|Au(\tau)\|_{2,\Omega}^2 d\tau \le C\|\mathbf{D}u(t)\|_{2,\Omega}^2 \qquad \forall\, t \ge \mathcal{T} > \frac{2}{\mu\lambda_1}\log\left(\frac{C\sqrt{\beta}\|u_0\|_{2,\Omega}^{1/2}}{\mu}\right). \tag{4.31}$$

– *Step 2.* We prove that $\lim_{t\to\infty}\lambda(t) < \infty$. By (4.7)$_1$ and Young's inequality, there exists $C > 0$:

$$|\left(B(u,u)\,,\; Au - \lambda u\right)_\Omega| \le \frac{C}{2\mu}\|\mathbf{D}u\|_{2,\Omega}^3\|Au\|_{2,\Omega} + \frac{\mu}{2}\|Au - \lambda u\|_{2,\Omega}^2$$

which, plugged into (4.23), yields

$$\frac{d}{dt}\lambda(t) + \mu\frac{\|Au(t) - \lambda u(t)\|_{2,\Omega}^2}{\|u(t)\|_{2,\Omega}^2} \le \frac{C}{\mu}\frac{\lambda(t)}{2}\|\mathbf{D}u(t)\|_{2,\Omega}\|Au(t)\|_{2,\Omega}. \tag{4.32}$$

Thanks to (4.27) and (4.31), by Holder inequality we have

$$\int_t^\infty \|\mathbf{D}u(\tau)\|_{2,\Omega}\|Au(\tau)\|_{2,\Omega} d\tau \le C\|u(t)\|_{2,\Omega}\|\mathbf{D}u(t)\|_{2,\Omega} \quad \forall\, t \ge \mathcal{T}, \tag{4.33}$$

for some $C > 0$, which combined with (4.32) implies that

$$\lambda_1 \le \limsup_{T\to\infty}\lambda(T) \le \lambda(t)e^{\frac{C}{2\mu}\int_t^\infty \|\mathbf{D}u(\tau)\|_{2,\Omega}\|Au(\tau)\|_{2,\Omega}d\tau} \le \lambda(t)e^{C\|u(t)\|_{2,\Omega}\|\mathbf{D}u(t)\|_{2,\Omega}}$$

for some $C > 0$ and for all $\mathcal{T} \le t \le T$. In particular, from the above formula we deduce that

$$\sup_{t\ge\mathcal{T}}\lambda(t) \le \lambda(\mathcal{T})e^{C\|u(\mathcal{T})\|_{2,\Omega}\|\mathbf{D}u(\mathcal{T})\|_{2,\Omega}} < \infty\,. \tag{4.34}$$

Then, by (4.29), we conclude that $\lambda_1 \le \limsup_{T\to+\infty}\lambda(T) \le \liminf_{t\to+\infty}\lambda(t) < \infty$, proving the existence of $\lim_{t\to+\infty}\lambda(t) = \Lambda \in \mathbb{R}^+$.

– *Step 3.* We prove that Λ is an eigenvalue of (2.5). Set $v(t) = u(t)/\|u(t)\|_{2,\Omega}$ so that $\lambda(t) = 2\|\mathbf{D}v(t)\|_{2,\Omega}^2$. Thanks to (4.32)–(4.33) we get

$$\mu\int_\tau^\infty \|Av(s) - \lambda v(s)\|_{2,\Omega}^2\, ds \le C\sup_{t\ge\tau}\lambda(t)\|u(\tau)\|_{2,\Omega}\|\mathbf{D}u(\tau)\|_{2,\Omega} + \lambda(\tau) \qquad \forall\, \tau \ge \mathcal{T},$$

for some $C > 0$, implying, by (4.29) and (4.34), that $\|Av - \lambda v\|_{2,\Omega} \in L^2(\tau,\infty)$ for $\tau \ge \mathcal{T}$ and the existence of $t_j \to \infty$ such that $[Av(t_j) - \lambda(t_j)v(t_j)] \to 0$ as $t_j \to \infty$. Since $\|\mathbf{D}v(t_j)\|_{2,\Omega}^2 \to \Lambda$, up to a subsequence, we have that $v(t_j) \to \overline{v}$ in H as $t_j \to \infty$; hence,

$$Av(t_j) = (Av(t_j) - \lambda(t_j)v(t_j)) + (\lambda(t_j) - \Lambda)v(t_j) + \Lambda v(t_j) \to \Lambda\overline{v} \qquad \text{in } H \text{ as } t_j \to \infty.$$

Being A a closed operator, $\overline{v}$ belongs to $D(A)$ and we have $A\overline{v} = \Lambda\overline{v}$; since $\|v(t_j)\|_{2,\Omega} = 1$, then $\overline{v} \ne 0$, implying that Λ is an eigenvalue of A. □

Proof of (3.2). We buy the lines of [12, Corollary 1]. Since $\lambda(t) \to \Lambda$ for $t \to \infty$, from (4.24), for all $\varepsilon > 0$ there exists $t_1 > 0$ sufficiently large such that

$$-2\mu(\Lambda + \varepsilon)\|u(t)\|_{2,\Omega}^2 \le \frac{d}{dt}\|u(t)\|_{2,\Omega}^2 \le -2\mu(\Lambda - \varepsilon)\|u(t)\|_{2,\Omega}^2 \qquad \forall\, t \ge t_1.$$

This implies

$$-\varepsilon \le \left(\frac{\log\|u(t)\|_{2,\Omega}}{\mu t} - \frac{\log\|u(t_1)\|_{2,\Omega}}{\mu t} - \Lambda\frac{t_1}{t} + \Lambda\right)\left(1 - \frac{t_1}{t}\right)^{-1} \le \varepsilon \qquad \forall\, t \ge t_1$$

and, in turn, $\lim_{t\to+\infty}\frac{\log\|u(t)\|_{2,\Omega}}{t} = -\mu\Lambda$. On the other hand, by (4.34) there exists $C > 0$: $\left|\log\frac{\|\mathbf{D}u\|_{2,\Omega}}{\|u\|_{2,\Omega}}\right| \le C$ for all $t \ge 0$, implying (3.2). □

4.6. Proof of Theorem 3.2

From Proposition 2.4, we know that either Ω is a ball or it is monoaxially symmetric. The proof of item (i) follows by verifying that

$$2\mu(\mathbf{D}u_{0,\mathcal{K}}, \mathbf{D}v)_\Omega + \int_\Omega (u_{0,\mathcal{K}}\cdot\nabla)u_{0,\mathcal{K}}\cdot v = 0 \qquad \text{for all } v \in V,$$

which yields that $u_{0,\mathcal{K}}$ is a weak and, in turn, a strong solution of (1.1) (for a suitable pressure p). By definition of $\mathcal{K}_\Omega$, we already know that $\mathbf{D}u_{0,\mathcal{K}} = 0$. In order to show that the second term of the above formula also vanishes, we consider the case where Ω is monoaxially symmetric. With no loss of generality, we take $\vec{a} = 0$ and $\vec{\ell} = (0,0,1)$, then $\mathcal{K}_\Omega = \text{span}\{W^z\}$, with $W^z = (y,-x,0)$, and $u_{0,\mathcal{K}}(x,y,0) = c_0(y,-x,0) \in \mathcal{K}_\Omega$ for some $c_0 \in \mathbb{R}$. Finally, we have

$$\int_\Omega (u_{0,\mathcal{K}}\cdot\nabla)u_{0,\mathcal{K}}\cdot v = -\int_\Omega (x,y,0)\cdot v\, dx\,dy\,dz = -\int_\Omega (v\cdot\nabla)u_{0,\mathcal{K}}\cdot u_{0,\mathcal{K}}\, dx\,dy\,dz = 0.$$

When Ω is a ball the proof follows similarly once noticed that $\mathcal{K}_\Omega = \mathrm{span}\{W^x, W^y, W^z\}$, with $W^x = (0, z, -y)$ and $W^y = (-z, 0, x)$.

Next we turn to the proof of (3.3) which yields the first part of the proof of both (ii) (by taking $u_{0,\mathcal{K}} = 0$) and (iii). We divide the proof into two cases.

– *Case (a):* Ω is monoaxially symmetric. With no loss of generality, we assume that $\mathcal{K}_\Omega = \mathrm{span}\{W^z\}$. Then, for any $(x, y, z, t) \in \Omega \times \mathbb{R}^+$, the solution u of (1.1) can be written as

$$u(x, y, z, t) = \bar{u}(x, y, z, t) + u_{\mathcal{K}}(x, y, z, t) \tag{4.35}$$

with

$$\bar{u}(x, y, z, t) = (\bar{u}_1, \bar{u}_2, \bar{u}_3) \in \mathcal{K}_\Omega^\perp \quad \text{and} \quad u_{\mathcal{K}}(x, y, t) = c(t)W^z \in \mathcal{K}_\Omega,$$

where $c \in C^0([0, \infty]; \mathbb{R})$ is such that $c(0) = c_0 = (u_0, W^z)_\Omega$ (namely, $u_{0,\mathcal{K}}(x, y) = c_0 W^z$).
For all $t > 0$, u satisfies $u_t + \mu Au + B(u, u) = 0$ in H and taking the scalar product with W^z we deduce that

$$\begin{aligned} 0 =& \|W^z\|^2_{2,\Omega} c'(t) + \int_\Omega (u \cdot \nabla) u \cdot W^z \, dx\, dy\, dz = \\ & \|W^z\|^2_{2,\Omega} c'(t) + \int_\Omega (\bar{u} \cdot \nabla) \bar{u} \cdot W^z \, dx\, dy\, dz + \int_\Omega (u_{\mathcal{K}} \cdot \nabla) \bar{u} \cdot W^z \, dx\, dy\, dz, \end{aligned} \tag{4.36}$$

where we have used the fact that $\int_\Omega (u \cdot \nabla) u_{\mathcal{K}} \cdot W^z \, dx\, dy\, dz = c(t) \int_\Omega (u \cdot \nabla) W^z \cdot W^z \, dx\, dy\, dz = 0$, and $(4.6)_2$. By $(4.6)_2$, we also deduce that

$$0 = \int_\Omega (\bar{u} \cdot \nabla) W^z \cdot W^z \, dx\, dy\, dz = \int_\Omega (x\bar{u}_1 + y\bar{u}_2)\, dx\, dy\, dz. \tag{4.37}$$

Some computations yield

$$\begin{aligned} \int_\Omega (\bar{u} \cdot \nabla) \bar{u} \cdot W^z \, dx\, dy\, dz &= -\int_\Omega (\bar{u} \cdot \nabla) W^z \cdot \bar{u} \, dx\, dy\, dz \\ &= -\int_\Omega (\bar{u}_2, -\bar{u}_1, 0) \cdot (\bar{u}_1, \bar{u}_2, \bar{u}_3)\, dx\, dy\, dz = 0, \end{aligned} \tag{4.38}$$

and, using (4.37),

$$\begin{aligned} \int_\Omega (u_{\mathcal{K}} \cdot \nabla) \bar{u} \cdot W^z \, dx\, dy\, dz &= -c(t) \int_\Omega (W^z \cdot \nabla) W^z \cdot \bar{u} \, dx\, dy\, dz \\ &= c(t) \int_\Omega \left(x\bar{u}_1 + y\bar{u}_2 \right) \, dx\, dy\, dz = 0. \end{aligned} \tag{4.39}$$

Plugging (4.38) and (4.39) in (4.36), we infer that $c'(t) = 0$ for $t > 0$, thus $c(t) \equiv c_0$ for all $t \geq 0$ and (3.3) follows.

– *Case (b)*: Ω is a ball. Then, $\mathcal{K}_\Omega = \mathrm{span}\{W^x, W^y, W^z\}$ and the solution can be written as in (4.35), where, for any $(x, y, z, t) \in \Omega \times \mathbb{R}^+$, we have

$$u_{\mathcal{K}}(x, y, z, t) = c_1(t)(0, z, -y) + c_2(t)(-z, 0, x) + c_3(t)(y, -x, 0)$$

with $c_1, c_2, c_3 \in C^0([0, \infty); \mathbb{R})$ such that $c_1(0) = c_{1,0} = (u_0, W^x)_\Omega$, $c_2(0) = c_{2,0} = (u_0, W^y)_\Omega$ and $c_3(0) = c_{3,0} = (u_0, W^z)_\Omega$. As in case (a), by taking the scalar product of the equation with W^x (resp., with W^y and W^z), since $\{W^z, W^y, W^z\}$ are orthogonal in H, we obtain

$$\begin{cases} \|W^x\|^2_{2,\Omega} c_1'(t) = & -\int_\Omega (\bar{u} \cdot \nabla) \bar{u} \cdot W^x \, dx\, dy\, dz - \int_\Omega (u_{\mathcal{K}} \cdot \nabla) u \cdot W^x \, dx\, dy\, dz \\ & -\int_\Omega (\bar{u} \cdot \nabla) u_{\mathcal{K}} \cdot W^x \, dx\, dy\, dz \\ \|W^y\|^2_{2,\Omega} c_2'(t) = & -\int_\Omega (\bar{u} \cdot \nabla) \bar{u} \cdot W^y \, dx\, dy\, dz - \int_\Omega (u_{\mathcal{K}} \cdot \nabla) u \cdot W^y \, dx\, dy\, dz \\ & -\int_\Omega (\bar{u} \cdot \nabla) u_{\mathcal{K}} \cdot W^y \, dx\, dy\, dz \\ \|W^z\|^2_{2,\Omega} c_3'(t) = & -\int_\Omega (\bar{u} \cdot \nabla) \bar{u} \cdot W^z \, dx\, dy\, dz - \int_\Omega (u_{\mathcal{K}} \cdot \nabla) u \cdot W^z \, dx\, dy\, dz \\ & -\int_\Omega (\bar{u} \cdot \nabla) u_{\mathcal{K}} \cdot W^z \, dx\, dy\, dz. \end{cases} \tag{4.40}$$

Arguing as for (4.38), the first terms on the right hand side of each line of system (4.40) vanish. To handle the other terms, we observe that

$$\begin{aligned} & -\int_\Omega (u_{\mathcal{K}} \cdot \nabla) u \cdot W^x \, dx\, dy\, dz - \int_\Omega (\bar{u} \cdot \nabla) u_{\mathcal{K}} \cdot W^x \, dx\, dy\, dz \\ &= \int_\Omega (u_{\mathcal{K}} \cdot \nabla) W^x \cdot u \, dx\, dy\, dz + \int_\Omega (\bar{u} \cdot \nabla) W^x \cdot u_{\mathcal{K}} \, dx\, dy\, dz \\ &= \int_\Omega (0, u_{\mathcal{K},3}, -u_{\mathcal{K},2}) \cdot (u_1, u_2, u_3)\, dx\, dy\, dz + \int_\Omega (0, \bar{u}_3, -\bar{u}_2) \cdot (u_{\mathcal{K},1}, u_{\mathcal{K},2}, u_{\mathcal{K},3})\, dx\, dy\, dz \\ &= \int_\Omega \left(u_{\mathcal{K},3}(u_2 - \bar{u}_2) - u_{\mathcal{K},2}(u_3 - \bar{u}_3) \right) \, dx\, dy\, dz = \int_\Omega \left(u_{\mathcal{K},3} u_{\mathcal{K},2} - u_{\mathcal{K},2} u_{\mathcal{K},3} \right) \, dx\, dy\, dz = 0 \end{aligned}$$

and, with similar computations, that

$$\int_\Omega \left(-(u_\mathcal{K}\cdot\nabla)u\cdot W^y-(\bar u\cdot\nabla)u_\mathcal{K}\cdot W^y\right)\,dx\,dy\,dz=\int_\Omega \left(-u_{\mathcal{K},3}u_{\mathcal{K},1}+u_{\mathcal{K},1}u_{\mathcal{K},3}\right)\,dx\,dy\,dz=0,$$

$$\int_\Omega \left(-(u_\mathcal{K}\cdot\nabla)u\cdot W^z-(\bar u\cdot\nabla)u_\mathcal{K}\cdot W^z\right)\,dx\,dy\,dz=\int_\Omega \left(u_{\mathcal{K},2}u_{\mathcal{K},1}-u_{\mathcal{K},1}u_{\mathcal{K},2}\right)\,dx\,dy\,dz=0.$$

Therefore, all terms on the right hand side of each line of system (4.40) vanish and we infer that $c_1'(t)=c_2'(t)=c_3'(t)=0$ for $t>0$, by which $c_1(t)\equiv c_{1,0}, c_2(t)\equiv c_{2,0}$ and $c_3(t)\equiv c_{3,0}$ for all $t\geq 0$ and (3.3) follows also in this case.

It remains to study the behaviour of u as $t\to+\infty$; the same proof works both in the case where Ω is a ball and it is monoaxially symmetric. In case (ii), from what proved above, we have that $u(t)=\bar u(t)\in\mathcal{K}_\Omega^\perp$ for all $t\geq 0$. In $\mathcal{K}_\Omega^\perp$ the Poincaré inequality holds, therefore the proof of Theorem 3.1 applies and we conclude that $\bar u$ satisfies (3.1) and (3.2). In case (iii), we have instead that

$$u(t)=\bar u(t)+u_{0,\mathcal{K}}\qquad\text{for all } t\geq 0. \tag{4.41}$$

By taking the scalar product of the equation with u and using (2.4), we derive (4.24) which, in view of (4.41), yields

$$\frac{d}{dt}\|\bar u(t)\|_{2,\Omega}^2+4\mu\|\mathbf{D}\bar u(t)\|_{2,\Omega}^2=0.$$

The above equation implies that $\|\bar u(t)\|_{2,\Omega}$ goes monotonically to 0 as $t\to+\infty$ with $\|\bar u(t)\|_{2,\Omega}^2\leq\|\bar u_0\|_{2,\Omega}^2e^{-2\mu\lambda_1 t}$ for all $t\geq 0$, where λ_1 is the first nontrivial eigenvalue. Combined with (4.41) this information completes the proof of (iii). □

4.7. Proof of the results in the 2D case

The key ingredient in our proofs will be identity (3.7), therefore we start by proving it:

Lemma 4.7. *Let $\Omega=Q$ or $\Omega=B$. There holds*

$$-\int_\Omega (u\cdot\nabla)u\cdot Au=0\quad\textit{for all } u\in W. \tag{4.42}$$

Proof. *Proof of* (4.42) *when $\Omega=Q$.* We write $u=(u_1,u_2)$ and $Au=\left((Au)_1,(Au)_2\right)$. Let $L_1=\{0\}\times(0,\pi)$, $L_2=(0,\pi)\times\{\pi\}$, $L_3=\{\pi\}\times(0,\pi)$ and $L_4=(0,\pi)\times\{0\}$. Recalling the definition of W in (2.3), since the boundaries are flat, it is well-known (see e.g., [27]) that u satisfies the mixed Dirichlet-Neumann type boundary conditions:

$$u_1=0 \text{ on } L_1\cup L_3,\quad \partial_y u_1=0 \text{ on } L_2\cup L_4,\quad u_2=0 \text{ on } L_2\cup L_4,\quad \partial_x u_2=0 \text{ on } L_1\cup L_3. \tag{4.43}$$

Developing the l.h.s. of (4.42) in components we obtain

$$-\int_\Omega \left[u_1\partial_x u_1(Au)_1+u_2\partial_y u_1(Au)_1+u_1\partial_x u_2(Au)_2+u_2\partial_y u_2(Au)_2\right]\,dx\,dy.$$

Integrating the first term we get

$$\begin{aligned}-\int_\Omega u_1\partial_x u_1(Au)_1\,dxdy&=-\int_\Omega\nabla(u_1\partial_x u_1)\cdot\nabla u_1\,dx\,dy+\int_{L_1\cup L_2\cup L_3\cup L_4}u_1\partial_x u_1\nabla u_1\cdot\nu\,d\sigma\\&=-\int_\Omega\nabla(u_1\partial_x u_1)\cdot\nabla u_1\,dx\,dy-\int_{L_1}u_1\partial_x u_1\partial_x u_1\,dy+\int_{L_3}u_1\partial_x u_1\partial_x u_1\,dy\\&\quad-\int_{L_2}u_1\partial_x u_1\partial_y u_1\,dx+\int_{L_4}u_1\partial_x u_1\partial_y u_1\,dx=-\int_\Omega\nabla(u_1\partial_x u_1)\cdot\nabla u_1\,dx\,dy,\end{aligned}$$

where the terms on the boundary vanish because of (4.43). As a consequence, denoting $(\partial_1,\partial_2)=(\partial_x,\partial_y)$, we deduce that

$$I_u=-\sum_{i,j,k=1}^2\int_\Omega\partial_k(u_i\partial_i u_j)\partial_k u_j\,dx\,dy=-\sum_{i,j,k=1}^2\int_\Omega\left[\partial_k u_i\partial_i u_j\partial_k u_j+u_i\partial_{ik}u_j\partial_k u_j\right]\,dx\,dy.$$

Next, observe that, expanding the first term, we get

$$\begin{aligned}\sum_{i,j,k=1}^2\int_\Omega\partial_k u_i\partial_i u_j\partial_k u_j\,dx\,dy&=\int_\Omega\partial_x u_1\partial_x u_1\partial_x u_1\,dx\,dy+\int_\Omega\partial_x u_2\partial_y u_1\partial_x u_1\,dx\,dy\\&+\int_\Omega\partial_y u_1\partial_x u_1\partial_y u_1\,dx\,dy+\int_\Omega\partial_y u_2\partial_y u_1\partial_y u_1\,dx\,dy+\int_\Omega\partial_x u_1\partial_x u_2\partial_x u_2\,dx\,dy\\&+\int_\Omega\partial_x u_2\partial_y u_2\partial_x u_2\,dx\,dy+\int_\Omega\partial_y u_1\partial_x u_2\partial_y u_2\,dx\,dy+\int_\Omega\partial_y u_2\partial_y u_2\partial_y u_2\,dx\,dy\\&=\int_\Omega(\partial_x u_1)^3\,dx\,dy+\int_\Omega(\partial_y u_2)^3\,dx\,dy=0,\end{aligned} \tag{4.44}$$

due to the divergence-free condition, which entails that $\partial_x u_1 = -\partial_y u_2$. The second term also vanishes. Indeed, through integration by parts, we have

$$\sum_{i,j,k=1}^{2} \int_\Omega u_i \partial_{ik} u_j \partial_k u_j \, dx\, dy = -\sum_{j,k=1}^{2} \int_\Omega \mathrm{div} u \frac{(\partial_k u_j)^2}{2} \, dx\, dy + \sum_{i,j,k=1}^{2} \int_{\partial\Omega} u_i \frac{(\partial_k u_j)^2}{2} \hat{e}_i \cdot \nu \, d\sigma = 0.$$

This concludes the proof of (4.42) when $\Omega = Q$.

Proof of (4.42) *when* $\Omega = B$. In this framework it is convenient introducing a new reference system with versors $(\mathbf{i}^r, \mathbf{j}^\varphi)$, so that $v \in V$ writes $v = (v_1, v_2) = v_1 \mathbf{i} + v_2 \mathbf{j} = (v_1 \cos\theta + v_2 \sin\theta)\mathbf{i}^r + (v_2 \cos\theta - v_1 \sin\theta)\mathbf{j}^\varphi$ and

$$\begin{cases} v^r := v^r(\rho,\theta) = v_1(\rho,\theta)\cos\theta + v_2(\rho,\theta)\sin\theta \\ v^\varphi := v^\varphi(\rho,\theta) = v_2(\rho,\theta)\cos\theta - v_1(\rho,\theta)\sin\theta\,. \end{cases} \tag{4.45}$$

Then, all $u \in W$ satisfy the boundary conditions

$$u^r = 0 \text{ on } \{1\}\times[0,2\pi) \quad \text{and} \quad u^\varphi_\rho - u^\varphi = 0 \text{ on } \{1\}\times[0,2\pi) \tag{4.46}$$

and the divergence free condition writes $u^r_\rho + \dfrac{u^\varphi_\theta}{\rho} + \dfrac{u^r}{\rho} = 0$ in $(0,1)\times[0,2\pi)$, see (4.49) in Appendix.

Finally, we consider the l.h.s. of (4.42). By integrating by parts, we obtain

$$-\int_\Omega (u\cdot\nabla)u \cdot Au \, dxdy = -\int_\Omega \nabla\big((u\cdot\nabla)u\big) : \nabla u \, dx\, dy + \int_{\partial\Omega} (u\cdot\nabla)u \cdot \nabla u \cdot \nu \, d\sigma$$

where, for $u = (u_1, u_2)$, the boundary term writes

$$\int_{\partial\Omega} (u\cdot\nabla)u \cdot \nabla u \cdot \nu \, d\sigma = \int_{\partial\Omega} (u_1\partial_x u_1 + u_2 \partial_y u_1)\nabla u_1 \cdot \nu \, d\sigma + \int_{\partial\Omega} (u_1 \partial_x u_2 + u_2 \partial_y u_2)\nabla u_2 \cdot \nu \, d\sigma.$$

Passing to polar coordinates, we observe that

$$\nabla u_i \cdot \nu = \left(\partial_\rho u_i \cos\theta - \partial_\theta u_i \frac{\sin\theta}{\rho}\,,\, \partial_\rho u_i \sin\theta + \partial_\theta u_i \frac{\cos\theta}{\rho}\right)\cdot(\cos\theta, \sin\theta) = \partial_\rho u_i \qquad i = 1,2,$$

giving

$$\begin{aligned} &\int_{\partial\Omega} (u\cdot\nabla)u \cdot \nabla u \cdot \nu \, d\sigma = \\ &\int_0^{2\pi} \left[u_1 \partial_\rho u_1 \left(\partial_\rho u_1 \cos\theta - \partial_\theta u_1 \sin\theta\right) + u_2 \partial_\rho u_1 \left(\partial_\rho u_1 \sin\theta + \partial_\theta u_1 \cos\theta\right)\right] d\theta + \\ &\int_0^{2\pi} \left[u_1 \partial_\rho u_2 \left(\partial_\rho u_2 \cos\theta - \partial_\theta u_2 \sin\theta\right) + u_2 \partial_\rho u_2 \left(\partial_\rho u_2 \sin\theta + \partial_\theta u_2 \cos\theta\right)\right] d\theta, \end{aligned}$$

where every function is considered in $\rho = 1$. Recalling (4.45) we may rewrite the equation as

$$\begin{aligned} &\int_{\partial\Omega} (u\cdot\nabla)u \cdot \nabla u \cdot \nu \, d\sigma = \int_0^{2\pi} \left\{\left[(\partial_\rho u_1)^2 + (\partial_\rho u_2)^2\right]u^r + \left[\partial_\rho u_1 \partial_\theta u_1 + \partial_\rho u_2 \partial_\theta u_2\right]u^\varphi\right\} d\theta = \\ &\int_0^{2\pi} \left[\partial_\rho u_1 \partial_\theta u_1 + \partial_\rho u_2 \partial_\theta u_2\right]u^\varphi d\theta, \end{aligned}$$

where in the last equality we used the boundary condition (4.46). In particular, we obtain

$$\int_0^{2\pi} \left[\partial_\rho u_1 \partial_\theta u_1 + \partial_\rho u_2 \partial_\theta u_2\right]u^\varphi d\theta = \int_0^{2\pi} u^\varphi \left[\partial_\rho u^\varphi \partial_\theta u^\varphi - \partial_\rho u^r u^\varphi\right] d\theta\,,$$

where we also exploit the fact that $u^r(1,\theta) = 0$ for all $\theta \in [0,2\pi)$, hence $u^r_\theta(1,\theta) = 0$. Thanks to (4.46) and the divergence condition, we infer

$$\begin{aligned} &\int_0^{2\pi} u^\varphi \left[\partial_\rho u^\varphi \partial_\theta u^\varphi - \partial_\rho u^r u^\varphi\right] d\theta = \int_0^{2\pi} u^\varphi (u^\varphi \partial_\theta u^\varphi + \partial_\theta u^\varphi u^\varphi) d\theta = \\ &2\int_0^{2\pi} (u^\varphi)^2 \partial_\theta u^\varphi \, d\theta = \frac{2}{3}[(u^\varphi(1,2\pi))^3 - (u^\varphi(1,0))^3] = 0. \end{aligned}$$

As a consequence, denoting $(\partial_1, \partial_2) = (\partial_x, \partial_y)$, we deduce that the l.h.s. of (4.42) equals to:

$$-\sum_{i,j,k=1}^{2} \int_\Omega \partial_k (u_i \partial_i u_j) \partial_k u_j \, dx\, dy = -\sum_{i,j,k=1}^{2} \int_\Omega \left[\partial_k u_i \partial_i u_j \partial_k u_j + u_i \partial_{ik} u_j \partial_k u_j\right] \, dx\, dy.$$

Next, we observe that (4.44) holds for all 2D domains, hence we apply it to the first term. The second term also vanishes. Indeed, through integration by parts, we have

$$\sum_{i,j,k=1}^{2} \int_\Omega u_i \partial_{ik} u_j \partial_k u_j \, dx\, dy = -\sum_{j,k=1}^{2} \int_\Omega \mathrm{div} u \frac{(\partial_k u_j)^2}{2} \, dx\, dy + \sum_{i,j,k=1}^{2} \int_{\partial\Omega} u_i \frac{(\partial_k u_j)^2}{2} \hat{e}_i \cdot \nu \, d\sigma = 0$$

since, passing to polar coordinates, we have $\sum_{i=1}^{2} u_i \hat{e}_i \cdot \nu = \sum_{i=1}^{2} u_i \nu_i = u \cdot \nu = 0$ on $\partial\Omega$. This concludes the proof of (4.42) when $\Omega = B$. □

Remark 4.8. As already remarked in Section 3.2, to prove the existence of a unique weak solution to (1.1)–(1.2) when $\Omega = Q$ or $\Omega = B$ and $u_0 \in V$, one can repeat the proof of [10, Theorem 2] (our Proposition 2.6) which is based a standard Galerkin construction (see [28, Chapter 3]) and relies on the proof of suitable a priori bounds for the finite approximate solution. Herein, by exploiting (4.42), we just formally show how to obtain the key bounds in our framework. We point out that the fact that Q is merely Lipschitz can be handled by considering it as a type (B) sector, by means of the reflection principle in Section 4.1. In particular, Q needs to be replaced by the periodicity cell which is here a larger square that, for the sake of simplicity, we will keep denoting by Q. As concerns the axially symmetric set B, the Poincaré inequality does not hold, hence we exploit the decomposition (2.7) as outlined in [10].

The first (formal) energy estimate is obtained when testing (1.1) by u. As for (4.24), we get (both for the square and the disk) the equality

$$\|u(t)\|_{2,\Omega}^2 + 4\mu \int_0^T \|\mathbf{D}u(\tau)\|_{2,\Omega}\, d\tau = \|u_0\|_{2,\Omega}^2 \quad \text{for a.e. } t \in (0,T).$$

This equality, combined with Poincaré inequality (2.8), is sufficient to infer that u is bounded in $L^\infty(0,T;H) \cap L^2(0,T;V)$ and to pass to the limit in the approximate finite-dimensional system to obtain the existence of at least one weak solution to (1.1)–(1.2). Then, we test (1.1) by Au, getting

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{D}u(t)\|_{2,\Omega}^2 + \mu\|Au(t)\|_{2,\Omega}^2 = -\int_\Omega (u(t)\cdot\nabla)u(t)\cdot Au(t) \quad \text{for a.e. } t \in (0,T).$$

In view of (4.42), the second (formal) energy estimate associated to (1.1) (both for the square and the disk) is

$$\|\mathbf{D}u(t)\|_{2,\Omega}^2 \le \|\mathbf{D}u_0\|_{2,\Omega}^2,$$

by which we deduce that the weak solution is indeed strong and unique.

Proof of Theorem 3.8. We first prove (3.8). We proceed as in the proof of Theorem 3.1. In particular, given $\lambda(t)$ as in (3.1), by exploiting (4.42) in (4.23), we infer that

$$\frac{1}{2}\frac{d}{dt}\lambda(t) = -\frac{\mu}{\|u(t)\|_{2,\Omega}^2}\|Au - \lambda u\|_{2,\Omega}^2 \tag{4.47}$$

which implies that the map $[0,+\infty) \ni t \mapsto \lambda(t)$ is non-increasing and admits limit $\lim_{t\to+\infty}\lambda(t) = \Lambda \in \mathbb{R}^+$. Moreover, integrating (4.47) on $t \in (0,\infty)$, we deduce that

$$\lambda_1 \le \lambda(t) \le \lambda(0) = \frac{2\|\mathbf{D}u_0\|_{2,\Omega}^2}{\|u_0\|_{2,\Omega}^2}. \tag{4.48}$$

We finally prove that Λ is an eigenvalue of (2.5). Setting $v(t) = u(t)/\|u(t)\|_{2,\Omega}$ so that $\lambda(t) = 2\|\mathbf{D}v(t)\|_{2,\Omega}^2$, thanks to (4.47) we get

$$\mu \int_0^\infty \|Av(s) - \lambda v(s)\|_{2,\Omega}^2\, ds \le \lambda(0)/2 \qquad \forall\, \tau \ge 0.$$

Therefore, $\|Av - \lambda v\|_{2,\Omega} \in L^2(0,\infty)$ and we can argue precisely as in the proof of Theorem 3.1-*Step 3*. This concludes the proof of (3.8). Passing to the proof of item (i), we first note that if $u_0 = c_{0,1}e_1$ for some $c_{0,1} \in \mathbb{R}$, then $\lambda(0) = \lambda_1$. By (4.48), this implies that $\lambda(t) \equiv \lambda_1$ and, in turn, from (4.47) we infer that $\|Au - \lambda_1 u\|_{2,\Omega}^2 \equiv 0$. Hence, $u(t) = c(t)e_1$ for some function c. This inserted into (1.1), testing with e_1 and recalling (4.42), yields

$$\begin{cases} \dfrac{d}{dt}\big[c(t)\big] + \mu\lambda_1 c(t) = 0 \quad \text{for a.e. } t \in (0,T) \\ c(0) = c_{0,1}. \end{cases}$$

Hence, $c(t) = c_{0,1}e^{-\mu\lambda_1 t}$ and we have proved item (i). The proof of (ii) follows similarly by noticing that under the given assumptions then $\lambda(t) \equiv \lambda_N$. Concerning with (iii) the upper bound for Λ follows from (4.48) simply by noticing that if $0 \ne u_0 = \sum_{k=1}^N c_{0,k}e_k$ for some $c_{0,k} \in \mathbb{R}$, then

$$\frac{2\|\mathbf{D}u_0\|_{2,\Omega}^2}{\|u_0\|_{2,\Omega}^2} = \frac{\sum_{k=1}^N \lambda_k c_{0,k}^2}{\sum_{k=1}^N c_{0,k}^2} \le \lambda_N.$$

On the other hand, if $u(t)$ is expanded as in (3.4) and $c_k(t) \equiv 0$ for all $1 \le k \le N$, then $\lambda(t) \ge \lambda_M$ where M is the least positive integer such that $c_M(t) \not\equiv 0$. Since $M > N$ this yields a contradiction and concludes the proof. □

When $\Omega = Q$ we enrich the statement of Theorem 3.8 by showing that the sets spanned by all eigenfunctions (not only the first) are invariants of the flow. To this aim we need the explicit form of the eigenfunctions that we recall here below.

Proposition 4.9 *([29, Proposition 1]). Let $\Omega = Q$. For $m, n \in \mathbb{N}_+$ the eigenvalues of (2.5)–(1.2) are $\lambda_{m,n} = m^2 + n^2$ and they correspond to the L^2-normalized eigenfunctions:*

$$v_{m,n}(x, y) = C_{m,n} \begin{pmatrix} n \sin(mx)\cos(ny) \\ -m\cos(mx)\sin(ny) \end{pmatrix}$$

$$p_{m,n}(x, y) = 0,$$

for all $(x, y) \in Q$ with $C_{m,n} := \frac{2}{\sqrt{\pi^3(m^2+n^2)}}$. The least eigenvalue is $\lambda_{1,1} = 2 > 0$ and is simple. Furthermore, the set $\{v^{m,n}\}_{m,n=1}^{\infty}$ is a complete system of V.

We refer to [30, Section 2.2] for the resolution of the 3D analogous, namely the Stokes eigenvalues problem in the cube. Finally, making reference to the notations of Proposition 4.9, we prove:

Proposition 4.10. *Let $u(t)$ be the unique strong solution of problem (1.1)–(1.2) with $u_0 = c_{0,n,m} v_{n,m}$ and $c_{0,n,m} \in \mathbb{R}$. Then, $u(t) = c_{0,n,m} e^{-\mu\lambda_{n,m}t} v_{n,m}$.*

Proof. We only need to check that $u(t) = c_{0,n,m} e^{-\mu\lambda_{n,m}t} v_{n,m}$ solves (2.9). This follows once proved that

$$\int_Q (v_{n,m} \cdot \nabla) v_{n,m} \cdot v = 0 \quad \text{for all } v \in V.$$

Integrating by parts, recalling that $(v_1)_x = -(v_2)_y$ in Q and exploiting the boundary conditions in (4.43), we get

$$\begin{aligned}
&\int_Q (v_{n,m} \cdot \nabla) v_{n,m} \cdot v = C_{m,n} \int_Q \left(n^2\, m \sin(2mx) v_1(x, y) + nm^2 \sin(2ny) v_2(x, y)\right) dx\, dy \\
&= nm\, \frac{C_{m,n}}{2} \left[\int_0^\pi \int_0^\pi \left(\cos(2mx)(v_1)_x(x, y) dx\, dy + \cos(2nx)(v_2)_y(x, y)\right) dy\, dx\right] \\
&= -nm\, \frac{C_{m,n}}{2} \left[\int_0^\pi \cos(2mx) \int_0^\pi (v_2)_y(x, y) dy\, dx + \int_0^\pi \cos(2ny) \int_0^\pi (v_1)_x(x, y) dx\, dy\right] = 0. \qquad \square
\end{aligned}$$

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Appendix

In this section, by separating variables, we provide a system of eigenfunctions to (2.5) with $\Omega = B$. Exploiting the reference system $(\mathbf{i}^r, \mathbf{j}^\varphi)$, with v^r and v^φ as defined in (4.45), (2.5) with $\Omega = B$ also writes

$$\begin{cases}
v^r_{\rho\rho} + \dfrac{v^r_\rho}{\rho} + \dfrac{v^r_{\theta\theta}}{\rho^2} - \dfrac{2v^\varphi_\theta}{\rho^2} - \dfrac{v^r}{\rho^2} + \lambda v^r = p_\rho & \text{in } (0,1) \times [0, 2\pi) \\
v^\varphi_{\rho\rho} + \dfrac{v^\varphi_\rho}{\rho} + \dfrac{v^\varphi_{\theta\theta}}{\rho^2} + \dfrac{2v^r_\theta}{\rho^2} - \dfrac{v^\varphi}{\rho^2} + \lambda v^\varphi = \dfrac{p_\theta}{\rho} & \text{in } (0,1) \times [0, 2\pi) \\
v^r_\rho + \dfrac{v^\varphi_\theta}{\rho} + \dfrac{v^r}{\rho} = 0 & \text{in } (0,1) \times [0, 2\pi) \\
p_{\rho\rho} + \dfrac{p_\rho}{\rho} + \dfrac{p_{\theta\theta}}{\rho^2} = 0 & \text{in } (0,1) \times [0, 2\pi) \\
v^r = 0 & \text{on } \{1\} \times [0, 2\pi) \\
v^\varphi_\rho - v^\varphi = 0 & \text{on } \{1\} \times [0, 2\pi).
\end{cases} \tag{4.49}$$

Denoting by $J_m(\cdot)$ the first kind Bessel functions of order $m \geq 0$, we prove:

Proposition 4.11. *The least eigenvalue of (4.49) is $\lambda_1 = 0$, it is simple and the corresponding eigenfunction is (v^1, p^1) with $v^1 = \left(v^{r,1}, v^{\varphi,1}\right)$, where*

$$v^{r,1} = 0, \qquad v^{\varphi,1}(\rho) = -\rho, \qquad p^1 = 0 \qquad \forall \rho \in (0,1].$$

Furthermore, for any $m \geq 1$ there exists a sequence of eigenvalues $\lambda := \lambda_{m,k} > 0$ satisfying

$$2\sqrt{\lambda}J_{m-1}(\sqrt{\lambda}) + (\lambda - 4\,m)J_m(\sqrt{\lambda}) = 0, \tag{4.50}$$

and corresponding to the eigenfunctions $(v^{m,k}, p^{m,k})$ with $v^{m,k} = \left(v^{r,m,k}, v^{\varphi,m,k}\right)$ and $k \geq 1$, where

$$\begin{aligned}
v^{r,m,k}(\rho,\theta) &= \frac{1}{\rho}\left[J_m(\sqrt{\lambda_{m,k}}\rho) - J_m(\sqrt{\lambda_{m,k}})\rho^m\right]\cos(m\theta)\\
v^{\varphi,m,k}(\rho,\theta) &= \left[\frac{\sqrt{\lambda_{m,k}}}{2m}J_{m+1}(\sqrt{\lambda_{m,k}}\rho) - \frac{\sqrt{\lambda_{m,k}}}{2m}J_{m-1}(\sqrt{\lambda_{m,k}}\rho) + \frac{J_m(\sqrt{\lambda_{m,k}})}{\rho}\rho^m\right]\sin(m\theta)\\
p^{m,k}(\rho,\theta) &= -J_m(\sqrt{\lambda_{m,k}})\frac{\lambda_{m,k}}{m}\rho^m\cos(m\theta)
\end{aligned}$$

for all $(\rho,\theta) \in (0,1] \times [0,2\pi)$. The set $\{v^1, v^{m,k}\}_{m,k=1}^{\infty}$ is an orthogonal system of V.

Proof. The proof follows by adapting to Navier boundary conditions the computations given in [31, Section 5] under Dirichlet boundary conditions. We highlight the main steps. Looking for solutions in the form: $p(\rho,\theta) = e^{im\theta}p_m(\rho)$, $v^r(\rho,\theta) = e^{im\theta}u_{r,m}(\rho)$, $v^\varphi(\rho,\theta) = e^{im\theta}u_{\varphi,m}(\rho)$, for $m \geq 1$ one gets that $p_m(\rho) = c\rho^m$ for some $c \in \mathbb{R}$ while $u_{r,m}$ and $u_{\varphi,m}$ satisfy:

$$\begin{aligned}
&u''_{r,m}(\rho) + \frac{3}{\rho}u'_{r,m}(\rho) + \left(\lambda - \frac{m^2-1}{\rho^2}\right)u_{r,m}(\rho) = cm\rho^{m-1} \qquad \rho \in (0,1)\\
&u''_{\varphi,m}(\rho) + \frac{1}{\rho}u'_{\varphi,m}(\rho) + \left(\lambda - \frac{m^2+1}{\rho^2}\right)u_{\varphi,m}(\rho) = cim\rho^{m-1} - \frac{2im}{\rho^2}u_{r,m}(\rho) \qquad \rho \in (0,1)\\
&u'_{r,m}(\rho) + \frac{1}{\rho}u_{r,m}(\rho) + \frac{im}{\rho}u_{\varphi,m}(\rho) = 0 \qquad \rho \in (0,1)\\
&\lim_{\rho\to 0^+} u_{r,m}(\rho) \in \mathbb{R}\,, \qquad u_{r,m}(1) = 0, \qquad \lim_{\rho\to 0^+} u_{\varphi,m}(\rho) \in \mathbb{C}\,, \qquad u_{\varphi,m}(1) = u'_{\varphi,m}(1)\,.
\end{aligned}$$

The case $m = 0$ is simpler since the first two equations are already decoupled and the statement about the first eigenvalue λ_1 follows at once.

By setting $w_{r,m}(\rho) = \rho u_{r,m}(\rho) - \frac{cm}{\lambda}\rho^m$, the above system yields the following family of problems involving Bessel equations:

$$\begin{aligned}
&w''_{r,m}(\rho) + \frac{1}{\rho}w'_{r,m}(\rho) + \left(\lambda - \frac{m^2}{\rho^2}\right)w_{r,m}(\rho) = 0 \quad \rho \in (0,1)\\
&\qquad\qquad \lim_{\rho\to 0^+} w_{r,m}(\rho) = 0\,, \qquad w_{r,m}(1) = -\frac{cm}{\lambda}\,,
\end{aligned}$$

admitting nontrivial solutions $w_{r,m}(\rho) = aJ_m(\sqrt{\lambda}\rho)$ for all $\lambda > 0$ and for $a \in \mathbb{R}$ only if

$$aJ_m(\sqrt{\lambda}) = -\frac{cm}{\lambda}\,. \tag{4.51}$$

From the free divergence condition, we readily get that $u_{\varphi,m}(\rho) = \frac{cim}{\lambda}\rho^{m-1} + \frac{ai}{m}\frac{d}{d\rho}J_m(\sqrt{\lambda}\rho)$. Then, by setting $w_{\varphi,m}(\rho) = u_{\varphi,m}(\rho) - \frac{cim}{\lambda}\rho^{m-1} - \frac{ai\sqrt{\lambda}}{2m}\left(J_{m-1}(\sqrt{\lambda}\rho) - J_{m+1}(\sqrt{\lambda}\rho)\right) = u_{\varphi,m}(\rho) - \frac{cim}{\lambda}\rho^{m-1} - \frac{ai}{m}\frac{d}{d\rho}J_m(\sqrt{\lambda}\rho)$, the second equation of the system (with the above choice of $w_{r,m}$, instead of $u_{r,m}$) yields the equations:

$$w''_{\varphi,m}(\rho) + \frac{1}{\rho}w'_{\varphi,m}(\rho) + \left(\lambda - \frac{m^2+1}{\rho^2}\right)w_{\varphi,m}(\rho) = 0 \qquad \rho \in (0,1),$$

which are clearly satisfied by $w_{\varphi,m} \equiv 0$. It remains to impose the boundary condition $u_{\varphi,m}(1) = u'_{\varphi,m}(1)$ which gives the further condition

$$a\frac{d^2}{d\rho^2}J_m(\sqrt{\lambda}) - a\frac{d}{d\rho}J_m(\sqrt{\lambda}) + \frac{cm^2(m-2)}{\lambda} = 0\,. \tag{4.52}$$

Summing up, by combining (4.51) and (4.52), $\lambda > 0$ is an eigenvalue if and only if:

$$\lambda J''_m(\sqrt{\lambda}) - \sqrt{\lambda}J'_m(\sqrt{\lambda}) - m(m-2)J_m(\sqrt{\lambda}) = 0,$$

which is equivalent to (4.50) by recalling the equation satisfied by J_m and some of its properties. Finally, the statement of Proposition 4.11 follows by taking $c = -aJ_m(\sqrt{\lambda})\dfrac{\lambda}{m}$, and showing that, for any $m \geq 1$, (4.50) admits simple zeros (this easily implies that all the eigenfunctions provided are orthogonal in V). To this aim, we introduce the function

$$f(x) := 2xJ_{m-1}(x) + (x^2 - 4\,m)J_m(x);$$

let $\overline{x} > 0$ be such that $f(\overline{x}) = 0$, then $J_{m-1}(\overline{x}) = \frac{4m-\overline{x}^2}{2\overline{x}} J_m(\overline{x})$. Being $f \in C^\infty(\mathbb{R})$, to prove that $\overline{x}$ is simple we show that $f'(\overline{x}) \neq 0$. We compute $f'(x) = (x^2 - 2\,m)J_{m-1}(x) + \frac{m}{x}(4m - x^2)J_m(x)$, getting $f'(\overline{x}) = \dfrac{J_m(\overline{x})}{2}(4m - \overline{x}^2)\overline{x}$. Being $\overline{x} > 0$, $f'(\overline{x})$ is not zero since the first positive zero of $J_{m-1}(x)$ is greater than $2\sqrt{m}$, see e.g. [32], and if $J_m(\overline{x}) = 0$ then $J_{m-1}(\overline{x}) \neq 0$ $(m \geq 1)$.

Since also the functions $p(\rho, \theta) = e^{-im\theta} p_m(\rho)$, $v^r(\rho, \theta) = e^{-im\theta} u_{r,m}(\rho)$, $v^\varphi(\rho, \theta) = -e^{-im\theta} u_{\varphi,m}(\rho)$, for $m \geq 1$ are solutions we combine them properly to get the real eigenfunctions in the statement. □

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