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(Article begins on next page)

Reconciling Kozlov's vakonomic method with the traditional non-holonomic method: solution of two benchmark problems

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Abstract

We study the dynamics of a class of mechanical systems subjected to non-holonomic constraints by employing a method termed "modified vakonomic method" (MVM), and developed by Llibre, Ramírez and Sadovskaia in 2014. In particular, we test the MVM for the "rolling coin" problem, and a variant of the "non-holonomic skate" problem. For our purposes, we divide our work in two parts. For the first one, our point of departure is a paper published in this journal by Lemos in 2022, in which, for the "rolling coin" problem, Kozlov's vakonomic method is shown to lead to inconsistencies with the so-called Lagrange-D'Alembert traditional non-holonomic method (TNHM). In this case, we prove that, if the MVM is used, the equivalence with the TNHM can be restored, and the two methods can be reconciled. In the second part, we formulate a *thought experiment* consisting of an *electrically charged* "non-holonomic skate" interacting with a magnetic field, and we examine its dynamics by means of the MVM. In this case, we point out the differences with the predictions of the TNHM, and we propose a reformulation of the MVM capable of retrieving the results obtained with the TNHM. Moreover, we give some insight into the main computational aspects related to the MVM for non-holonomic constraints linear in the generalized velocities.

Keywords: Modified Vakonomic Method, Non-holonomic constraints, Hamilton-Suslov variational Principle, Rolling coin, Charged skate.

1 Introduction

The motivation for studying mechanical systems subjected to non-holonomic constraints is determined not only by their mathematical beauty, but also by their relevance in physics [18, 36], e.g. for modeling relativistic fluids; in biomechanics, for describing, e.g., tissue growth and remodeling [32, 33, 53], and animal or cellular motion [4]; and in engineering, especially in robotics [29–31, 40]. The ever-green interest for these systems is the search for a *paradigmatic framework* formalizing non-holonomic mechanics, whose applicability should be as universal as possible. This applies, in particular, to problems exhibiting non-holonomic constraints that are either nonlinear in the generalized velocities of a given system [32–35] or involving the accelerations [16, 17].

A peculiar feature of non-holonomic systems is their intrinsic tendency to involve several aspects of mathematics, such as analysis, differential geometry and numerics, with the aim of providing physically consistent formulations of their evolution. This has led to descriptions of non-holonomic systems different from those of standard analytical mechanics, which are based on the Lagrange-D'Alembert principle, typically "augmented" by the technique of Lagrange multipliers, i.e., the so-called "traditional non-holonomic method" (TNHM) [13, 14, 16, 17]. For example, in the early 80's, Kozlov proposed a method for modeling non-holonomic mechanics, known as "vakonomic method" (VM) [7–10], and standing for mechanics of the *"variational axiomatic kind"* [1], which applies Hamilton's principle of stationary action [13] also to systems subjected to non-holonomic constraints. One of the main features of this method is the capability of describing non-holonomic systems through a fully variational procedure, based on the definition of an appropriate *constrained* Lagrangian function [7–10]. We remark that also other types of variational approaches have been introduced to study systems subjected to non-holonomic constraints (see, e.g., [21, 27, 37, 38, 51, 52]).

Clearly, the introduction of new methods that are, in a sense, groundbreaking, since they question the pillars of the standard variational procedures, requires them to be tested against the more traditional ones. This, indeed, has been done quite intensively by having recourse to widely studied and easy-to-reproduce "benchmark" problems, all encoded in the classical literature of Analytical Mechanics (see e.g. [2, 3, 5, 6, 22, 26, 28, 43–47]). Typical examples are a ball rolling over a moving or fixed surface [2, 3, 5], a skate moving on an inclined or horizontal plane [3, 6, 12], a disk rolling on an inclined plane [1, 14, 22, 43–45], and a two-wheeled carriage moving on a plane [2, 22, 43, 45, 47]. In spite of the fact that these case studies were formulated many years ago, and can be found in textbooks, they remain up-to-date (see e.g. [6, 11, 16–18, 22, 36], and, more recently, [1, 12]), since they are archetypal and, thus, serve as a reference even for much more complex physical situations in which the constraints can be reconducted to those characterizing the original case studies.

The vakonomic method [7–10], used *as is*, can lead, in same cases, to motions that differ from the ones obtained with the TNHM, which is known to supply solutions observable experimentally for the majority of mechanical problems [1–3, 5, 6, 22]. In particular, as recently shown by Lemos [1], one of such cases is the problem of the "rolling coin", which describes a rigid coin rolling without slipping on an inclined plane [1, 14], for which the VM is said to lead to unphysical results. Similar problems have also been investigated in [6, 12]. On the other hand, there are problems, e.g. in geometric control theory [4, 28, 48] and in field theory [18, 36], in which it is claimed that the TNHM does not provide physically consistent dynamics, while the VM provides *"interesting results"* [18].

Many authors have compared the VM and the TNHM on the basis of the *variations* introduced to define the associated variational principle [2, 3, 5, 18, 22, 36, 41, 43–45, 48], and in terms of the *geometrical meaning* of such variations [28, 37, 41, 44–46, 48]. Often, one of the purposes of these investigations is to provide conditions that allow to conclude that a solution to a mechanical problem obtained within the TNHM is also a solution of the VM, whereas the converse is, in general, not true (see [3] for a critique on Kozlov's approach). In fact, Lemos' work [1] places itself in this research line, thereby highlighting the discrepancies between the VM and the TNHM for a certain class of problems. Still, in our opinion, Kozlov's idea is worth of being investigated because of its conceptual potential, and, to the best of our understanding, it could be "saved" by performing suitable *modifications* to it (Llibre et al. [12] speak of *"modified vakonomic method"*). Indeed, if our interpretation is correct, it is with this attitude in mind that the works of Ramírez et al. [11] and Llibre et al. [12] have been conceived. From an operative point of view, the modifications to the VM should be able to provide conditions on the variations of the Lagrangian parameters of a given system, and on the associated generalized velocities, in such a way that, under certain working hypotheses, the TNHM solutions can be recovered (see Sect. 2.5). To this end, the variations should satisfy Lagrange-Chetaev's conditions in addition to other conditions specified by the method employed.

To our knowledge, in fact, a possible reconciliation between the vakonomic method and the traditional methodologies is given in a paper by Llibre et al. [12], whose germinal idea was already present in [11], and in which the authors elaborate a modification to the original vakonomic method based on the Hamilton-Suslov variational principle [12, 19, 20] that accounts for the presence of non-vanishing *transpositional relations* [11, 12, 25–27]. Such relations express that, in the presence of a non-holonomic constraint, the variation of a generalized velocity involved in the constraint is generally not equal to the time derivative of the variation of the associated Lagrangian parameter. With this premise, Llibre et al. [12] prove that the equations of motion obtained with their so-called modified vakonomic method (MVM) [12] are equivalent to the ones obtained with the traditional non-holonomic method, up to a *"zero Lebesgue-measure set"* [12]. For their purposes, Llibre et al. [12] propose two *paths*: one is based on the introduction of auxiliary functions that augment the Lagrangian function of the considered problem in a suitable way (see Theorem

2 of [12] and Section 2.2 below), while the other one imposes *a priori* conditions that compel the equivalence of their approach with the TNHM (see Theorem 3 of [12] and our Corollary 1).

Within the context explained above, the scope of this work is:

(i) To add a methodological note to the paper recently published by Lemos [1] by proving that the MVM and the TNHM are in agreement with each other for the "Rolling coin" problem [1, 14].
(ii) To apply the procedure developed in [11, 12] to a variant of the "Non-holonomic skate" presented in [12]. Specifically, we consider a non-holonomic and *electrically charged* skate interacting with an imposed magnetic field, and we show that, for this problem, if the MVM is formulated according to the first *path*, it is not equivalent to the TNHM. The second *path* is equivalent to the TNHM, but, *for our problem*, it is inconsistent with the first *path*. For our purposes, we generalize a benchmark taken from [27].
(iii) To present a reformulation of the MVM that, on the one hand, retrieves the equivalence with the TNHM, and, on the other hand, merges the two *paths* of [12] in a physics-driven way. More in detail, we propose a way to indicate, on physical grounds (e.g. compliance with evident conservation laws), which *path* should be followed.

It should be noticed that, in the following, our efforts are concentrated on the reconciliation of the VM with the TNHM for a class of mechanical problems known to be reliably described by the TNHM, i.e., for which the TNHM is known to produce solutions consistent with the experiments. In fact, our interest for such reconciliation does not require that the TNHM should always be used as a reference, but it resides in the possibility of describing the dynamics of a given mechanical system through a variational approach that supplies the same solutions as the TNHM, and, thus, the experimentally observable motions. In this respect, our work can be seen as a "gym" for testing the MVM [11, 12], and for understanding whether fundamental results of Analytical Mechanics holding in the non-holonomic setting [26] can be extended to study the symmetries of mechanical systems (see Noether's theorem and its extension to non-holonomic problems [49]), and whether, and with which modifications, the MVM can be imported to field theories (e.g. inelastic processes in continuum systems [32, 33]). Hence, in this work we are not interested in comparing the VM and the TNHM, neither on theoretical nor on experimental bases. Rather, we would just like to present our reformulation of the MVM of Llibre et al. [12], which enjoys the efficacy and elegance of the Lagrangian formalism, while providing solutions that are in harmony with the TNHM.

As for the structure of this work, Sect. 2 is dedicated to give the reader an *overview* on the mathematical foundations of the MVM [12]. In particular, in Sect. 2.1 we establish the geometrical setting used for describing the kinematics of a non-holonomically constrained system and the need for introducing non-vanishing *"transpositional relations"* [12, 25–27]; in Sect. 2.2, we have explicit recourse to the Hamilton-Suslov variational principle [12, 19, 20] that leads us to the dynamic equations for the system under investigation, which are then closed under the assumption of suitable *solvability conditions* discussed in Sect. 2.3; in Sect. 2.4, we further elaborate on some computational aspects regarding the MVM, which help us draw the differences and similarities between the MVM and the TNHM, as expanded in Sects. 2.5 and 2.6. Finally, the "rolling coin" [1, 14, 28] and the "charged skate" [12, 27] benchmark problems are examined by means of both the MVM and the TNHM, as seen in Sect. 3 and Sect. 4, respectively. Some geometric remarks related to the analyzed constraints are summarized in "Appendix A".

2 Modified vakonomic method (MVM)

In this section, we summarize the fundamental aspects of the modified vakonomic method developed by Llibre et al. [12] that are relevant for our work. Hereafter, for the sake of brevity, we shall refer to the "modified vakonomic method" [12] as MVM, in order to distinguish it from the original "vakonomic method", referred to as VM, introduced by Kozlov [7–10].

2.1 Variations and non-holonomic constraints

Following [12], we start our discussion with the presentation of the *Hamilton-Suslov variational principle* [12, 19, 20] for mechanical systems featuring non-holonomic constraints that fulfill the *Lagrange-Chetaev conditions* [12].

Let $\mathfrak{M}$ be a mechanical system whose evolution is characterized by $n \geq 1$, $n \in \mathbb{N}$, Lagrangian parameters, i.e., n functions of time, collected in the array $q := (q^1, \ldots, q^n) : [t_{\text{in}}, t_{\text{fin}}] \subseteq \mathscr{T} \to \mathscr{C}$, where $[t_{\text{in}}, t_{\text{fin}}]$ is a closed interval of the *time line* $\mathscr{T}$ [23], and $\mathscr{C}$ is referred to as the *configuration space* of $\mathfrak{M}$ [13]. The latter is regarded here as an n-dimensional manifold representing the set of all possible configurations for $\mathfrak{M}$ and, as required in [12], it is assumed to be smooth in the sequel. In this sense, $q : [t_{\text{in}}, t_{\text{fin}}] \to \mathscr{C}$ is a one-parameter family of configurations of $\mathscr{C}$, thereby parameterizing a curve on this manifold, such that, for each $t \in [t_{\text{in}}, t_{\text{fin}}]$, ϑ is the unique element of $\mathscr{C}$ satisfying $\vartheta = q(t) \in \mathscr{C}$.

For each configuration $\vartheta \in \mathscr{C}$, $T_\vartheta\mathscr{C}$ denotes the tangent space of $\mathscr{C}$ at ϑ, and $T\mathscr{C} = \cup_{\vartheta\in\mathscr{C}}\big(\{\vartheta\} \times T_\vartheta\mathscr{C}\big)$ is the tangent bundle of $\mathscr{C}$, so that, for each pair $(\vartheta, \boldsymbol{\nu}) \in T\mathscr{C}$, $\boldsymbol{\nu} \in T_\vartheta\mathscr{C}$ is a realization of the generalized velocity vector attached at ϑ. Moreover, under the hypothesis that q is differentiable, we denote by $(q, \dot{q}) :]t_{\text{in}}, t_{\text{fin}}[\to T\mathscr{C}$ the map such that, for each time $t \in]t_{\text{in}}, t_{\text{fin}}[$, $\dot{q}(t) \in T_{q(t)}\mathscr{C}$ is the tangent vector to the curve parameterized by q at $q(t)$, and is equal to the vector of generalized velocities attached at $q(t) \in \mathscr{C}$, i.e., there exists $\boldsymbol{\nu} \in T_{q(t)}\mathscr{C}$ such that $\dot{q}(t) = \boldsymbol{\nu}$.

We assume that the system $\mathfrak{M}$ is subjected to $m \in \mathbb{N}$, with $m \leq n$, linearly independent *non-holonomic constraints*, represented by the functions $\mathcal{V}^\alpha : T\mathscr{C} \times \mathscr{T} \to \mathbb{R}$, for $\alpha = 1, \ldots, m$. The presence of such constraints restricts the admissible generalized velocities to the following sub-manifold of the tangent bundle $T\mathscr{C}$ (see e.g. [2, 39])

$$T\mathscr{C}_{\mathrm{c}} := \left\{(\vartheta, \boldsymbol{\nu}) \in T\mathscr{C} \;:\; \mathcal{V}^{\alpha}(\vartheta, \boldsymbol{\nu}, t) = 0, \;\; \alpha = 1, \dots, m, \;\; \forall\, t \in [t_{\mathrm{in}}, t_{\mathrm{fin}}]\right\}. \quad (1)$$

Moreover, as done in [11, 12], we focus on non-holonomic constraints *linear* in the generalized velocities. To this end, for each $\alpha = 1, \dots, m$, and for all $\vartheta \in \mathscr{C}$ and $t \in \mathscr{T}$, we introduce a linear map $\mathcal{G}^{\alpha}(\vartheta, t)[\,\cdot\,] : T_{\vartheta}\mathscr{C} \to \mathbb{R}$, i.e., an element of the cotangent space $T^{*}_{\vartheta}\mathscr{C}$, represented by

$$\mathcal{G}^{\alpha}(\vartheta, t)[\,\cdot\,] := \textstyle\sum_{k=1}^{n} a^{\alpha}{}_{k}(\vartheta, t)\boldsymbol{e}^{k}(\vartheta)[\,\cdot\,], \quad (2)$$

with $\boldsymbol{e}^{k}(\vartheta)[\,\cdot\,]$ denoting the kth basis co-vector of $T^{*}_{\vartheta}\mathscr{C}$, and defined such that

$$\mathcal{G}^{\alpha}(\vartheta, t)[\boldsymbol{\nu}] := \textstyle\sum_{k=1}^{n} a^{\alpha}{}_{k}(\vartheta, t)\nu^{k} \equiv \mathcal{V}^{\alpha}(\vartheta, \boldsymbol{\nu}, t), \quad (3)$$

where the identity $\boldsymbol{e}^{k}(\vartheta)[\boldsymbol{\nu}] \equiv \nu^{k}$ is exploited for each $k = 1, \dots, n$, and, for varying $\alpha = 1, \dots, m$ and $k = 1, \dots, n$, $a^{\alpha}{}_{k} : \mathscr{C} \times \mathscr{T} \to \mathbb{R}$ determines a collection of given scalar-valued functions.

When $\mathcal{V}^{\alpha}$ is evaluated for $\vartheta = q(t)$ and $\boldsymbol{\nu} = \dot{q}(t)$, we use the notation

$$\mathcal{V}^{\alpha}(q(t), \dot{q}(t), t) = [\mathcal{V}^{\alpha} \circ (q, \dot{q}, \tau)](t) =: \hat{\mathcal{V}}^{\alpha}(t), \quad (4)$$

where we have employed the composition of maps to define the scalar-valued function $\hat{\mathcal{V}}^{\alpha} : \mathscr{T} \to \mathbb{R}$, with $\tau : \mathscr{T} \to \mathscr{T}$ being the *time-identity* function [23] defined as $\tau(t) := t$. Finally, for all the pairs $(q, \dot{q})$ that are admissible in the sense specified by Eq. (1), i.e., $(q, \dot{q}) :]t_{\mathrm{in}}, t_{\mathrm{fin}}[\to T\mathscr{C}_{\mathrm{c}}$, we can write

$$\hat{\mathcal{V}}^{\alpha} \equiv \mathcal{V}^{\alpha} \circ (q, \dot{q}, \tau) = [a^{\alpha}{}_{k} \circ (q, \tau)]\dot{q}^{k} = 0, \qquad \alpha = 1, \dots, m. \quad (5)$$

In Eq. (5), we have used Einstein's notation to imply a summation over the repeated index k, and, in the following, we will make large use of such notation, unless otherwise specified. Note that the *linear independence* of the constraints is fulfilled by requiring the functions $[a^{\alpha}{}_{k}]_{k=1,\dots,n}^{\alpha=1,\dots,m}$ to define a (rectangular) matrix of rank m.

Now that the kinematics of the system $\mathfrak{M}$ is described, we can introduce the variations of q and $\dot{q}$ as

$$\tilde{Q} : [t_{\mathrm{in}}, t_{\mathrm{fin}}] \times]-\varepsilon_0, +\varepsilon_0[\to \mathscr{C}, \quad (6a)$$
$$\tilde{\boldsymbol{V}} : [t_{\mathrm{in}}, t_{\mathrm{fin}}] \times]-\varepsilon_0, +\varepsilon_0[\to T_{\tilde{Q}}\mathscr{C}, \quad (6b)$$

with $\varepsilon_0 > 0$ being a smallness parameter, and we require them to be such that

$$\tilde{Q}(t, \varepsilon) \in \mathscr{C}, \qquad \tilde{Q}(t, 0) = q(t), \qquad \forall\, t \in [t_{\mathrm{in}}, t_{\mathrm{fin}}], \quad (7a)$$
$$\tilde{\boldsymbol{V}}(t, \varepsilon) \in T_{\tilde{Q}(t,\varepsilon)}\mathscr{C}, \qquad \tilde{\boldsymbol{V}}(t, 0) = \dot{q}(t), \qquad \forall\, t \in [t_{\mathrm{in}}, t_{\mathrm{fin}}], \quad (7b)$$

where, in Eq. (7b), $\dot{q}$ is prolonged by continuity to $t_{\rm in}$ and $t_{\rm fin}$.

By varying $t \in [t_{\rm in}, t_{\rm fin}]$ and $\varepsilon \in]-\varepsilon_0, +\varepsilon_0[$, one can determine the two-parameter family of tangent spaces $T_{\tilde{Q}}\mathscr{C}$, such that $T_{\tilde{Q}(t,\varepsilon)}\mathscr{C}$ is the element of this family obtained for fixed $(t, \varepsilon) \in [t_{\rm in}, t_{\rm fin}] \times]-\varepsilon_0, +\varepsilon_0[$, and $T_q\mathscr{C} \equiv T_{\tilde{Q}(\,\cdot\,,0)}\mathscr{C}$ is the one-parameter family of tangent spaces identified by $\varepsilon = 0$ and t varying in $[t_{\rm in}, t_{\rm fin}]$.

To avoid technicalities, we assume that both $\tilde{Q}$ and $\tilde{\boldsymbol{V}}$ are C^2 maps over their domain, i.e., $[t_{\rm in}, t_{\rm fin}] \times]-\varepsilon_0, +\varepsilon_0[$, and we set

$$\boldsymbol{\eta}(t) := \frac{\partial \tilde{Q}}{\partial \varepsilon}(t, 0), \qquad \forall t \in [t_{\rm in}, t_{\rm fin}], \tag{8a}$$

$$\boldsymbol{\zeta}(t) := \frac{\partial \tilde{\boldsymbol{V}}}{\partial \varepsilon}(t, 0), \qquad \forall t \in [t_{\rm in}, t_{\rm fin}], \tag{8b}$$

where $\boldsymbol{\eta} : [t_{\rm in}, t_{\rm fin}] \to T_q\mathscr{C}$ and $\boldsymbol{\zeta} : [t_{\rm in}, t_{\rm fin}] \to T_q\mathscr{C}$ are the *first-order variations* of q and $\dot{q}$, respectively, and $\boldsymbol{\eta}$ is such that $\boldsymbol{\eta}(t_{\rm in}) = \boldsymbol{\eta}(t_{\rm fin}) = 0$ [12, 13, 23]. Before proceeding, to simplify the notation, in analogy with the maps $\tilde{Q}$ and $\tilde{\boldsymbol{V}}$, we introduce the auxiliary map $\tilde{\mathfrak{T}} : [t_{\rm in}, t_{\rm fin}] \times]-\varepsilon_0, +\varepsilon_0[\to [t_{\rm in}, t_{\rm fin}]$ defined as $\tilde{\mathfrak{T}}(t, \varepsilon) = t$ so that $\partial_\varepsilon \tilde{\mathfrak{T}}(t, \varepsilon) = 0$ applies everywhere in its domain.

We evaluate now the constraints for the varied configuration and varied velocity, defined in Eqs. (6a) and (6b), and we set [12]

$$\tilde{\mathcal{V}}^\alpha := \mathcal{V}^\alpha \circ (\tilde{Q}, \tilde{\boldsymbol{V}}, \tilde{\mathfrak{T}}) = [a^\alpha{}_k \circ (\tilde{Q}, \tilde{\mathfrak{T}})]\tilde{V}^k. \tag{9}$$

Clearly, evaluating $\tilde{\mathcal{V}}^\alpha$ at $\varepsilon = 0$ for each $\alpha = 1, \ldots, m$ yields the original set of constraints, i.e.,

$$\tilde{\mathcal{V}}^\alpha(\,\cdot\,, 0) = \mathcal{V}^\alpha \circ (q, \dot{q}, \tau) = [a^\alpha{}_k \circ (q, \tau)]\dot{q}^k = 0. \tag{10}$$

However, following [12], we also require that each $\tilde{\mathcal{V}}^\alpha$, for $\alpha = 1, \ldots, m$, satisfies the corresponding constraint at the first order in ε, i.e., up to orders $o(\varepsilon)$ in the limit $\varepsilon \to 0$. This requirement, in turn, is expressed by enforcing the set of conditions:

$$\frac{\partial \tilde{\mathcal{V}}^\alpha}{\partial \varepsilon}(\,\cdot\,, 0) = \Big[\frac{\partial a^\alpha{}_i}{\partial q^k} \circ (q, \tau)\Big]\eta^k \dot{q}^i + [a^\alpha{}_k \circ (q, \tau)]\zeta^k = 0, \quad \alpha = 1, \ldots, m. \tag{11}$$

Furthermore, by adhering to the framework developed in [11, 12], we *hypothesize* that, in the jargon of [12], all the constraints considered in this work are *ideal*, in the sense that they satisfy the so-called *Lagrange-Chetaev conditions* [12, 18, 22], which are expressed in the form

$$\sum_{k=1}^{n} \Big[\frac{\partial \mathcal{V}^\alpha}{\partial \dot{q}^k} \circ (q, \dot{q}, \tau)\Big]\eta^k = \sum_{k=1}^{n} [a^\alpha{}_k \circ (q, \tau)]\eta^k = 0, \qquad \alpha = 1, \ldots, m. \tag{12}$$

Note that, since, for each $\alpha = 1, \dots, m$, $\mathcal{V}^\alpha$ is linear in the velocities, Eq. (12) returns the constraints in the form $\mathcal{V}^\alpha \circ (q, \boldsymbol{\eta}, \tau) = 0$, i.e., with $\boldsymbol{\eta}$ replacing $\dot{q}$, thereby prescribing the conditions that must be fulfilled by the first-order variations $\eta^1, \dots, \eta^n$ (see [34, 35]). Thus, by computing the derivative of Eq. (12) with respect to time, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\sum_{k=1}^{n}\Big[\frac{\partial \mathcal{V}^\alpha}{\partial \dot{q}^k}\circ(q,\dot{q},\tau)\Big]\eta^k\Big\} = \frac{\mathrm{d}}{\mathrm{d}t}\Big\{\sum_{k=1}^{n}[a^\alpha{}_k\circ(q,\tau)]\eta^k\Big\}\\
&= \Big[\frac{\partial a^\alpha{}_k}{\partial q^i}\circ(q,\tau)\Big]\eta^k\dot{q}^i + \Big[\frac{\partial a^\alpha{}_k}{\partial \tau}\circ(q,\tau)\Big]\eta^k + [a^\alpha{}_k\circ(q,\tau)]\dot{\eta}^k = 0. \qquad (13)
\end{aligned}$$

Moreover, by computing the difference between Eq. (11) and (13), and by changing the indexes appropriately, we obtain, for each $\alpha = 1, \dots, m$, the relation [12, 16, 17]

$$\begin{aligned}
0 =&\frac{\partial \tilde{\mathcal{V}}^\alpha}{\partial \varepsilon}(\,\cdot\,,0) - \frac{\mathrm{d}}{\mathrm{d}t}\Big\{\sum_{k=1}^{n}\Big[\frac{\partial \mathcal{V}^\alpha}{\partial \dot{q}^k}\circ(q,\dot{q},\tau)\Big]\eta^k\Big\}\\
=&\Big[\Big(\frac{\partial a^\alpha{}_i}{\partial q^k} - \frac{\partial a^\alpha{}_k}{\partial q^i}\Big)\circ(q,\tau)\Big]\eta^k\dot{q}^i + \Big[-\frac{\partial a^\alpha{}_k}{\partial \tau}\circ(q,\tau)\Big]\eta^k\\
&+ [a^\alpha{}_h\circ(q,\tau)](\zeta^h - \dot{\eta}^h). \qquad (14)
\end{aligned}$$

Equation (14) produces a set of m linear relations between the components of $\boldsymbol{\zeta}-\dot{\boldsymbol{\eta}}$ and the components of $\boldsymbol{\eta}$. Thus, by following the framework established in [11, 12], we *hypothesize* (see Remark 1) the existence of a linear transformation between $\boldsymbol{\zeta} - \dot{\boldsymbol{\eta}}$ and $\boldsymbol{\eta}$ represented by a matrix $\mathsf{W} := [W^h{}_k]_{k=1,\dots,n}^{h=1,\dots,n}$ such that [11, 12]

$$\zeta^h - \dot{\eta}^h = \textstyle\sum_{k=1}^{n} W^h{}_k\eta^k, \qquad h = 1, \dots, n. \qquad (15)$$

In the terminology of [11, 12, 25–27, 39], the relations reported in Eq. (15) are referred to as *"transpositional relations"*, since they express, at the first order in ε, the fact that, for each $h = 1, \dots, n$, the variation of the time derivative of the Lagrangian parameter q^h, i.e., ζ^h, is not equal, in general, to the time derivative of the variation of q^h, i.e., $\dot{\eta}^h$. The statement just given here contains a deep geometric meaning, and can be formalized quite rigorously by employing the language of Differential Geometry developed, e.g., in [41]. In this respect, one should introduce the concept of *second tangent bundle* of $\mathscr{C}$, denoted by $TT\mathscr{C}$, and declare $\boldsymbol{\zeta}$ and $\dot{\boldsymbol{\eta}}$ as elements of $TT\mathscr{C}$, but belonging to different fibers of $TT\mathscr{C}$. A study of constrained mechanical systems conducted by having recourse to such concepts of Differential Geometry can be found in [41], which, in turn, employes the framework established in [42].

To the best of our understanding, the variations taken in the MVM proposed by Llibre et al. [12] live in the intersection between two sets of variations introduced in a work by Jóźwikowski&Respondek [22] (based on the paper by

Gràcia et al. [48]), i.e., the set of *"nonholonomic admissible variations"* [22], defined according to *"Chetaev's principle"* [22], and the set of *"vakonomic admissible variations"* [22]. The latter variations are also in agreement with those presented in the vakonomic approach provided by Lemos [1]. The reason for requiring that the variations of the MVM belong to the intersection mentioned above is that we need the consistency with Lagrange-Chetaev's conditions in order to look for coherence between the TNHM (based on the fulfillment of such conditions) and the MVM. More details are reported in "Appendix A".

Remark 1 (Transpositional relations [11, 12, 26, 27, 39, 50])
Granted the approach specified above [11, 12], which, as remarked in [12], places itself in the school of thought of *"Suslov, Voronets, Levi-Civita, Amaldi,..."* [12], the presence of non-holonomic constraints renders the operations of "variation" and "time differentiation" non-commutative in general [16, 17]. More in detail, as discussed in [11, 12, 26, 39, 50], if a generalized velocity featuring in a given constraint is taken as *dependent*, i.e., as a function of the other *independent* velocities, then its corresponding transpositional relation does not vanish. On the contrary, *indipendent* velocities produce null transpositional relations [11, 12, 26, 39, 50].

To contextualize the introduction of the *square* matrix $\mathsf{W} = [W^h{}_k]_{k=1,\ldots,n}^{h=1,\ldots,n}$ in Eq. (15), we briefly review the point of view of Jarzębowska [50], who summarized the discussion of Neimark and Fufaev [26] on the transpositional relations and the analysis of motion in terms of *quasi-velocities* and *quasi-coordinates*. Hence, by slightly modifying Jarzębowska's notation [50], we denote here by $\omega^1, \ldots, \omega^n$ the n quasi-velocities of a given mechanical system (they must be as many as the "true" velocities). Each quasi-velocity ω^h, with $h = 1, \ldots, n$, is, in general, a function defined as $\omega^h := \hat{\omega}^h \circ (q, \dot{q}, \tau)$, which supplies a convenient reformulation of the system's kinematics. For systems featuring $m \leq n$ non-holonomic constraints ($m \geq 1$), it is useful to identify the first m quasi-velocities with the functions expressing the constraints themselves, i.e., $\omega^\alpha := \hat{\omega}^\alpha \circ (q, \dot{q}, \tau) \equiv \mathcal{V}^\alpha \circ (q, \dot{q}, \tau) = 0$, with $\alpha = 1, \ldots, m$ [26, 50] (indeed, this way, the constraints are satisfied as $\omega^1 = 0, \ldots, \omega^m = 0$), while, as remarked in [26], the remaining $n - m$ quasi-velocities can be assigned *arbitrarily* through generic functions $\mathcal{F}^\beta$, i.e., $\omega^\beta := \hat{\omega}^\beta \circ (q, \dot{q}, \tau) \equiv \mathcal{F}^\beta \circ (q, \dot{q}, \tau)$, with $\beta = m + 1, \ldots, n$, provided that they represent a change of variables in the "space" of the velocities. Thus, by denoting by $H^h{}_k := \partial_{\dot{q}^k} \hat{\omega}^h \circ (q, \dot{q}, \tau)$ the coefficients of the non-singular Jacobian matrix H associated with the quasi-velocities, and adapting Jarzębowska's procedure [50] to ours, we rewrite Equation (16) of [50] as

$$\zeta^h - \dot{\eta}^h = -[\mathsf{H}^{-1}]^h{}_i \left\{ \frac{\partial \hat{\omega}^i}{\partial q^k} \circ (q, \dot{q}, \tau) - \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\partial \hat{\omega}^i}{\partial \dot{q}^k} \circ (q, \dot{q}, \tau) \right] \right\} \eta^k, \tag{16}$$

which is obtained by identifying our ζ^h, $\dot{\eta}^h$, and η^k with $\delta \dot{q}_\lambda$, $(\delta q_\lambda)^{\cdot}$, and δq_λ of [50], respectively, and assuming that the difference $\delta\omega_r - (\delta\pi_r)^{\cdot}$ is null for each $r = 1, \ldots, n$ (here $\delta\pi_r$ is the rth form defined by "$\delta\pi_r := [\partial_{\dot{q}_\sigma} \omega_r] \delta q_\sigma$" in [50]). We emphasize that the latter assumption is not made in [50], although Neimark and Fufaev [26] explain that it can be done always. At this stage, if in Eq. (16) we set

$$W^h{}_k := -[\mathsf{H}^{-1}]^h{}_i \left\{ \frac{\partial \hat{\omega}^i}{\partial q^k} \circ (q, \dot{q}, \tau) - \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\partial \hat{\omega}^i}{\partial \dot{q}^k} \circ (q, \dot{q}, \tau) \right] \right\}, \tag{17}$$

we retrieve Eq. (15). Note that the matrix W obtained this way is related to the one defined in [50] through $\mathsf{W}_{\mathrm{J}} = \mathsf{H}\mathsf{W}\mathsf{H}^{-1}$, where "J" stands for "Jarzębowska".

If m is the number of nonholonomic constraints linear in the velocities, i.e., if $\omega^\alpha = \hat{\omega}^\alpha \circ (q, \dot{q}, \tau) = [a^\alpha{}_k \circ (q, \tau)]\dot{q}^k$, as is the case addressed in this work, the terms between braces in Eq. (17), evaluated for $i = \alpha = 1, \ldots, m$, become

$$\frac{\partial \hat{\omega}^\alpha}{\partial q^k} \circ (q, \dot{q}, \tau) - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \hat{\omega}^\alpha}{\partial \dot{q}^k} \circ (q, \dot{q}, \tau)\right] = \left[\left(\frac{\partial a^\alpha{}_i}{\partial q^k} - \frac{\partial a^\alpha{}_k}{\partial q^i}\right) \circ (q, \tau)\right]\dot{q}^i - \frac{\partial a^\alpha{}_k}{\partial \tau} \circ (q, \tau), \tag{18}$$

which, after multiplication by η^k, is the sum of the first two terms on the right-hand side of Eq. (14). In conclusion, in deriving Eq. (15), we have hypothesized the existence of the linear transformation relating $\boldsymbol{\zeta} - \dot{\boldsymbol{\eta}}$ with $\boldsymbol{\eta}$ because in Eq. (14) the matrix corresponding to $[a^\alpha{}_h \circ (q, \tau)]$ is rectangular, and, thus, cannot be inverted to obtain Eq. (16) at once.

Remark 2 (Transpositional relations and the *"Canonical flip"* [22, 41])
A relevant result of the work by Grabowska&Grabowski [41] is their Eq. (2.5), which reproposes the tranpositional relations discussed by Neimark&Fufaev [26], and establishes a peculiar relationship between the quantities that, in our context, are denoted by $\boldsymbol{\zeta}$ and $\dot{\boldsymbol{\eta}}$. In our notation, which follows the one adopted in [26], the transpositional relations could be written as [22, 26, 41]

$$\zeta^h = \dot{\eta}^h + \textstyle\sum_{\ell,k=1}^n \mathcal{C}^h{}_{\ell k}\dot{q}^\ell \eta^k, \tag{19}$$

where $\mathcal{C}^h{}_{\ell k} = -\mathcal{C}^h{}_{k\ell}$ is a collection of functions that is skew-symmetric in the lower indices, and represents, in local coordinates, the fourth slot of the *"Canonical flip"* between $TT\mathscr{C}$ and itself [22, 41] (see Appendix A). However, adapted to our framework, in which the constraints are linear in the velocities (cf. Eq. (5)), and are allowed to depend explicitly on time, Eq. (19) is reformulated as

$$\zeta^h = \dot{\eta}^h + \textstyle\sum_{k=1}^n \mathcal{C}^h{}_{0k}\eta^k + \sum_{\ell,k=1}^n \mathcal{C}^h{}_{\ell k}\dot{q}^\ell \eta^k. \tag{20}$$

This result, in fact, is consistent with the skew-symmetry of the first two terms on the far right-hand side of Eq. (14)[1].

Although obtained in a different context, Eq. (15) (see [11, 12]) can be understood as an equivalent form of the relationship (20) adapted from Grabowska&Grabowski [41], provided the following identification is made:

$$\begin{aligned}\zeta^h - \dot{\eta}^h &= \textstyle\sum_{k=1}^n W^h{}_k \eta^k \equiv \sum_{k=1}^n \mathcal{C}^h{}_{0k}\eta^k + \sum_{\ell,k=1}^n \mathcal{C}^h{}_{\ell k}\, \dot{q}^\ell \eta^k \\ &= \textstyle\sum_{k=1}^n \big\{\mathcal{C}^h{}_{0k} + \sum_{\ell=1}^n \mathcal{C}^h{}_{\ell k}\, \dot{q}^\ell\big\}\eta^k.\end{aligned} \tag{21}$$

In the remainder of our work, and, in particular, in the presentation of the benchmark problems analyzed below, we will show how relationships similar to Eq. (21) are obtained. Moreover, we will emphasize that the skew-symmetry of the functions $\mathcal{C}^h{}_{\ell k}$ in the lower indices is respected as a consequence of the structure of the coefficients $W^h{}_k$. More specifically, we will present two situations in which Eq. (21) is studied with special care: One case (see Sect. 4.2.2) requires redefining the coefficients $\mathcal{C}^h{}_{\ell k}$ as functions of q, $\dot{q}$, and τ (this will involve, in fact, the dependence of these coefficients

[1] Whereas the skew-symmetry of the first term is obvious, that of the second one becomes evident if, for each $\alpha = 1, \ldots, m$, we introduce an identically null function $a^\alpha{}_0 \circ (q, \tau)$, whose only role is to yield the expression $\big[\partial_k a^\alpha{}_0 - \partial_\tau a^\alpha{}_k\big] \circ (q, \tau)$. Note, however, that this is true because, throughout our work, we *are not* considering constraints affine in the velocities. Indeed, should these constraints be considered, the function $a^\alpha{}_0 \circ (q, \tau)$ would not be identically zero, and Eqs. (19) and (20) would change accordingly (see [51] for a discussion on constraints affine in the velocities).

on the generalized momenta of the problem); the second case (see Sect. 4.2.3), instead, requires redefining the generalized velocities $\dot{q}$ by suitably normalizing the generalized momenta mentioned above.

Before closing this section, we remark that one crucial difference between the MVM and the VM is that, while the MVM is formulated by accounting explicitly for the *transpositional relations*, the VM is developed without even introducing such relations [7–12].

2.2 Hamilton-Suslov variational principle

Given the mechanical system $\mathfrak{M}$ introduced above, and assuming $\mathfrak{M}$ to be subjected to m non-holonomic constraints of the type specified in Eq. (5), we denote by $\mathcal{L} : T\mathscr{C} \times [t_{\text{in}}, t_{\text{fin}}] \to \mathbb{R}$ the Lagrangian function that would characterize $\mathfrak{M}$ if none of the m imposed constraints were present. Moreover, we adopt the composition $\hat{\mathcal{L}} := \mathcal{L} \circ (q, \dot{q}, \tau) : [t_{\text{in}}, t_{\text{fin}}] \to \mathbb{R}$ to rephrase $\mathcal{L}$ as a function of time. Then, by following [7–10], we introduce also the *constrained Lagrangian function* of $\mathfrak{M}$ as

$$\hat{\mathcal{L}}_{\text{c}} \equiv \mathcal{L}_{\text{c}} \circ (q, \dot{q}, \tau, \lambda) := \mathcal{L} \circ (q, \dot{q}, \tau) - \textstyle\sum_{\alpha=1}^{m} \lambda_\alpha [\mathcal{V}^\alpha \circ (q, \dot{q}, \tau)], \tag{22}$$

where $\lambda_1, \dots, \lambda_m$ are m Lagrange multipliers associated with the m non-holonomic constraints. For the sake of a compact notation, such Lagrange multipliers can be collected in the array $\lambda := (\lambda_1, \dots, \lambda_m)$. Next, we define the action functional associated with $\mathcal{L}_{\text{c}}$ over the time interval $[t_{\text{in}}, t_{\text{fin}}]$ as [7–10]

$$\begin{aligned} \mathcal{A}_{\text{c}}(q, \lambda) &:= \int_{t_{\text{in}}}^{t_{\text{fin}}} \mathcal{L}_{\text{c}}(q(t), \dot{q}(t), \tau(t), \lambda(t))\, \mathrm{d}t = \int_{t_{\text{in}}}^{t_{\text{fin}}} \hat{\mathcal{L}}_{\text{c}}(t)\, \mathrm{d}t \\ &= \int_{t_{\text{in}}}^{t_{\text{fin}}} \Big\{ \hat{\mathcal{L}}(t) - \textstyle\sum_{\alpha=1}^{m} \lambda_\alpha(t) \hat{\mathcal{V}}^\alpha(t) \Big\} \mathrm{d}t. \end{aligned} \tag{23}$$

To perform the variation of the action functional $\mathcal{A}_{\text{c}}$, and in analogy with the notation given in Eqs. (6a), (6b), and (9), we write $\tilde{\Lambda}(t, \varepsilon)$ to indicate the collection of varied Lagrange multipliers, and [7–10, 12]

$$\tilde{\mathcal{L}}(t, \varepsilon) := \mathcal{L}(\tilde{Q}(t, \varepsilon), \tilde{\boldsymbol{V}}(t, \varepsilon), \tilde{\mathfrak{T}}(t, \varepsilon)) \tag{24}$$

to indicate the varied Lagrangian function. Note that, for each $\alpha = 1, \dots, n$, we set $\tilde{\Lambda}_\alpha(t, 0) = \lambda_\alpha(t)$, and we call $\gamma_\alpha(t) := \partial_\varepsilon \tilde{\Lambda}_\alpha(t, 0)$ the first-order variation [12]. Accordingly, the varied action functional can be written as a function of ε, and reads [7–10]

$$\tilde{\mathcal{A}}_{\text{c}}(\varepsilon) := \int_{t_{\text{in}}}^{t_{\text{fin}}} \mathcal{L}_{\text{c}}(\tilde{Q}(t, \varepsilon), \tilde{\boldsymbol{V}}(t, \varepsilon), \tilde{\mathfrak{T}}(t, \varepsilon), \tilde{\Lambda}(t, \varepsilon))\, \mathrm{d}t$$

$$= \int_{t_{\rm in}}^{t_{\rm fin}} \Big\{ \tilde{\mathcal{L}}(t,\varepsilon) - \textstyle\sum_\alpha \tilde{\Lambda}_\alpha(t,\varepsilon)\tilde{\mathcal{V}}^\alpha(t,\varepsilon) \Big\} \mathrm{d}t, \tag{25}$$

where we have simplified the notation by using the convention according to which, in the summations, the index α runs from 1 to m. A similar notation will be employed in the sequel also for summations over indices running from 1 to n. We remark that, in Eqs. (23) and (25), the structure of the Lagrangian function $\mathcal{L}_{\rm c}$ is taken from [7–10], while the way in which the varied velocity $\tilde{\boldsymbol{V}}(t,\varepsilon)$ is defined rephrases the definition given in [12]. Finally, for the sake of brevity and for future use, from here on we denote by

$$\sharp^{(1)} := (q, \dot{q}, \tau), \qquad\qquad \sharp_{\rm c}^{(1)} := (q, \dot{q}, \tau, \lambda) \tag{26}$$

the lists of arguments of $\mathcal{L}$ and $\mathcal{L}_{\rm c}$, respectively, and by

$$\sharp^{(2)} := (q, \dot{q}, \ddot{q}, \tau), \qquad\qquad \sharp_{\rm c}^{(2)} := (q, \dot{q}, \ddot{q}, \tau, \lambda, \dot{\lambda}) \tag{27}$$

the lists of arguments of the corresponding Euler-Lagrange operators, i.e.,

$$\mathscr{E}_k \mathcal{L} \circ \sharp^{(2)} := \frac{\partial \mathcal{L}}{\partial q^k} \circ \sharp^{(1)} - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}^k} \circ \sharp^{(1)}\right], \qquad k = 1, \ldots, n, \tag{28a}$$

$$\mathscr{E}_k \mathcal{L}_{\rm c} \circ \sharp_{\rm c}^{(2)} := \frac{\partial \mathcal{L}_{\rm c}}{\partial q^k} \circ \sharp_{\rm c}^{(1)} - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}_{\rm c}}{\partial \dot{q}^k} \circ \sharp_{\rm c}^{(1)}\right], \qquad k = 1, \ldots, n. \tag{28b}$$

With all the premises given above, the condition of stationary action at $\varepsilon = 0$ yields [12, 13]

$$\begin{aligned} \tilde{\mathcal{A}}_{\rm c}'(0) = &\int_{t_{\rm in}}^{t_{\rm fin}} \Big\{ \textstyle\sum_k [\mathscr{E}_k \mathcal{L}_{\rm c} \circ \sharp_{\rm c}^{(2)}]\eta^k + \textstyle\sum_\alpha \left[\dfrac{\partial \mathcal{L}_{\rm c}}{\partial \lambda_\alpha} \circ \sharp_{\rm c}^{(1)}\right]\gamma_\alpha \Big\}(t)\,\mathrm{d}t \\ &+ \int_{t_{\rm in}}^{t_{\rm fin}} \Big\{ \textstyle\sum_h \left[\dfrac{\partial \mathcal{L}_{\rm c}}{\partial \dot{q}^h} \circ \sharp_{\rm c}^{(1)}\right](\zeta^h - \dot{\eta}^h) \Big\}(t)\,\mathrm{d}t = 0. \end{aligned} \tag{29}$$

Equation (29) can be further worked out by employing Eq. (15) to rewrite the differences $\zeta^h - \dot{\eta}^h$ in terms of $\boldsymbol{\eta}$, by introducing the operators [11, 12]

$$\mathscr{D}_k \mathcal{L}_{\rm c} \circ \sharp_{\rm c}^{(2)} := \mathscr{E}_k \mathcal{L}_{\rm c} \circ \sharp_{\rm c}^{(2)} + \textstyle\sum_h \left[\dfrac{\partial \mathcal{L}_{\rm c}}{\partial \dot{q}^h} \circ \sharp_{\rm c}^{(1)}\right] {W^h}_k, \qquad k = 1, \ldots, n, \tag{30}$$

and by noticing that the derivative of $\mathcal{L}_{\rm c}$ with respect to λ_α returns, up to the sign, the αth constraint. These considerations lead to the following compact expression of the condition of stationary action:

$$\tilde{\mathcal{A}}_{\rm c}'(0) = \int_{t_{\rm in}}^{t_{\rm fin}} \Big\{ \textstyle\sum_k [\mathscr{D}_k \mathcal{L}_{\rm c} \circ \sharp_{\rm c}^{(2)}]\eta^k + \textstyle\sum_\alpha [-\mathcal{V}^\alpha \circ \sharp^{(1)}]\gamma_\alpha \Big\}(t)\,\mathrm{d}t = 0. \tag{31}$$

Since Eq. (31) must hold for any possible choice of $t_{\rm in}$ and $t_{\rm fin} > t_{\rm in}$, we require the integrand to be zero, and, since the latter condition must be fulfilled for arbitrary variations $\eta^1, \ldots, \eta^n$ and $\gamma_1, \ldots, \gamma_m$, we obtain [12]

$$\mathscr{D}_k \mathcal{L}_{\rm c} \circ \sharp_{\rm c}^{(2)} = 0, \qquad k = 1, \ldots, n, \tag{32a}$$

$$-\mathcal{V}^\alpha \circ \sharp^{(1)} = 0, \qquad \alpha = 1, \ldots, m. \tag{32b}$$

This is a set of $n + m$ equations in the $n + m$ unknowns given by the n Lagrangian parameters $q^1, \ldots, q^n$ and by the m Lagrange multipliers $\lambda_1, \ldots, \lambda_m$. We notice that Eq. (32a) are the dynamic equations of the problem, while Eq. (32b) return the constraints.

It must be emphasized, however, that the system of equations just obtained differs from the one of standard Analytical Mechanics for the following facts:

(i) The dynamic equations (32a) search for functions $q^1, \ldots, q^n$ that yield, for each $k = 1, \ldots, n$, the vanishing of the quantity $\mathscr{D}_k \mathcal{L}_{\rm c}$ introduced by Llibre et al. [12], instead of the *standard* Euler-Lagrange operator applied to $\mathcal{L}_{\rm c}$, i.e., $\mathscr{E}_k \mathcal{L}_{\rm c}$. In this respect, it should be noticed that, if one followed Kozlov's vakonomic method [7–10], the dynamic equations would still be formulated in terms of the standard Euler-Lagrange operators applied to $\mathcal{L}_{\rm c}$, since the transpositional relations (15) would not be explicitly considered. Exactly at this point the "modification" to the vakonomic dynamics proposed in [11, 12] comes into play. Indeed, Llibre et al. [12] resolve explicitly the transpositional relations in Eq. (15) through the introduction of the matrix W, which, in turn, leads to the definition of the quantity $\mathscr{D}_k \mathcal{L}_{\rm c}$. From here on, with a slight abuse of notation, we shall refer to this quantity and to other similar ones as "operator". To conclude, we mention that, in the case of holonomic constraints, Eq. (14) complies with the equalities $\zeta^h = \dot{\eta}^h$ for each $h = 1, \ldots, n$, thereby requiring W to be the null matrix.

(ii) Even though the number of equations (given by the sum of the number of the dynamic equations and of the constraints) equals the number of unknowns, Eqs. (32a) and (32b) are *not* closed, because, for each $k = 1, \ldots, n$, the operator $\mathscr{D}_k \mathcal{L}_{\rm c}$ features the n^2 unknown coefficients of W. To overcome this problem, n^2 auxiliary conditions need to be imposed (see the difference with the approach sketched in Remark 1).

(iii) In this theory, the equations for the Lagrange multipliers are first order differential equations, whereas they are algebraic in the TNHM.

2.3 Solvability conditions

Equations (32a) and (32b) can be rewritten as

$$\mathscr{D}_k \mathcal{L} \circ \sharp^{(2)} + \sum\nolimits_\alpha \dot{\lambda}_\alpha \left[\frac{\partial \mathcal{V}^\alpha}{\partial \dot{q}^k} \circ \sharp^{(1)} \right] - \sum\nolimits_\alpha \lambda_\alpha [\mathscr{D}_k \mathcal{V}^\alpha \circ \sharp^{(2)}] = 0, \; k = 1, \ldots, n, \tag{33a}$$

$$-\mathcal{V}^\alpha \circ \sharp^{(1)} = 0, \; \alpha = 1, \ldots, m. \tag{33b}$$

As anticipated in Sect. 2.2, we need to assign n^2 additional conditions, denoted hereafter as the *solvability conditions* of the MVM, in order to close the system (33a) and (33b). In particular, due to the properties of the constraints highlighted in Eqs. (14) and (15), mn conditions can be found by substituting Eq. (15) into Eq. (14), which amounts to setting

$$\mathscr{D}_k \mathcal{V}^\alpha \circ \sharp^{(2)} = 0, \qquad k = 1, \dots, n, \qquad \alpha = 1, \dots, m. \tag{34}$$

The last $n(n-m)$ conditions are assigned via the prescription of *Ansätze*, in which either $n-m$ auxiliary functions satisfying certain conditions [12] are introduced or the fulfillment of some physics-based conditions is assumed.

2.3.1 Ansatz 1: approach based on the auxiliary functions

To solve Eqs. (33a) and (33b), Llibre et al. [12] determine univocally the coefficients of W by introducing $n-m$ auxiliary functions $\mathcal{F}^\beta \circ (q, \dot{q}, \tau)$, with $\beta = m+1, \dots, n$, which yield the following $n(n-m)$ conditions

$$\mathscr{D}_k \mathcal{F}^\beta \circ \sharp^{(2)} = 0, \qquad k = 1, \dots, n, \qquad \beta = m+1, \dots, n. \tag{35}$$

As anticipated in Remark 1, the rationale behind the introduction of the set of functions $\{\mathcal{F}^\beta\}_{\beta=m+1}^n$ relies on the concept of quasi-velocities (see e.g. [26, 27, 50]), i.e., a set of functions $\{\omega^h\}_{h=1}^n$ constituting a reparameterization of the velocities $\{\dot{q}^k\}_{k=1}^n$ such that $\omega^h(t) = \hat{\omega}^h(q(t), \dot{q}(t), t)$, for all $h = 1, \dots, n$, and the Jacobian $H^h{}_k(t) := \partial_{\dot{q}^k} \hat{\omega}^h(q(t), \dot{q}(t), t)$ is non-singular. In fact, for a mechanical system subjected to m constraints on the velocities, it is possible to choose the first m quasi-velocities coincident with the constraints themselves (see Remark 1), and to identify the remaining $n-m$ quasi-velocities with the "arbitrary" functions $\omega^{m+1} \equiv \mathcal{F}^{m+1}, \dots, \omega^n \equiv \mathcal{F}^n$. However, in spite of this arbitrariness, some selection rules are necessary (see [11, 12]). To this end, the identification of the functions $\{\mathcal{F}^\beta\}_{\beta=m+1}^n$ with the corresponding quasi-velocities $\{\omega^\beta\}_{\beta=m+1}^n$ notwithstanding, it is instructive to critically review the method outlined in [12] because it involves conditions on the auxiliary functions that are called for by the variational procedure.

As a starting point, the functions $\mathcal{F}^{m+1}, \dots, \mathcal{F}^n$ must be such that the matrix $\mathsf{H} = [H^i{}_k]_{k=1,\dots,n}^{i=1,\dots,n}$, with components

$$H^i{}_k = \begin{cases} [\partial_{\dot{q}^k} \mathcal{V}^\alpha] \circ (q, \dot{q}, \tau), & i = \alpha = 1, \dots, m, \qquad k = 1, \dots, n, \\ [\partial_{\dot{q}^k} \mathcal{F}^\beta] \circ (q, \dot{q}, \tau), & i = \beta = m+1, \dots, n, \quad k = 1, \dots, n, \end{cases} \tag{36}$$

is *non-singular*, i.e., $\det \mathsf{H} \neq 0$, almost everywhere in $T\mathscr{C} \times [t_{\text{in}}, t_{\text{fin}}]$. However, on this point, we would like to emphasize that:

(i) The conditions in Eq. (35) may be regarded as *gauge conditions* [24] for the problem under study. In fact, through the introduction of $\mathcal{F}^{m+1}, \dots, \mathcal{F}^n$, we can transform *a posteriori* the Lagrangian $\mathcal{L}_{\mathrm{c}}$ in Eq. (22) as [12]

$$\mathcal{L}_{\mathrm{c}} \circ \sharp_{\mathrm{c}}^{(1)} \mapsto \mathcal{L}_{\mathrm{c}} \circ \sharp_{\mathrm{c}}^{(1)} - \textstyle\sum_{\beta} \kappa_{\beta} [\mathcal{F}^{\beta} \circ \sharp^{(1)}] =: \mathcal{L}_{\mathrm{L}} \circ (\sharp_{\mathrm{c}}^{(1)}, \kappa), \tag{37}$$

where the summation over β is done from $m+1$ to n; $\kappa_{m+1}, \ldots, \kappa_n$ are constant-valued parameters, i.e., $\kappa_\beta(t) = \kappa_{\beta 0} \in \mathbb{R}$, for $\beta = m+1, \ldots, n$, and for all $t \in [t_{\mathrm{in}}, t_{\mathrm{fin}}]$; and $\kappa := (\kappa_{m+1}, \ldots, \kappa_n)$ collects all such parameters. Therefore, by applying the variational procedure described in Sect. 2.2 to the case in which the Lagrangian is transformed as in Eq. (37), it is necessary to impose Eq. (35) in order for the dynamic equations associated with $\mathcal{L}_{\mathrm{L}}$ to be *invariant* with respect to those associated with $\mathcal{L}_{\mathrm{c}}$, reported in Eq. (33a). However, this invariance of the dynamic equations is only sufficient for the equivalence of the descriptions provided by $\mathcal{L}_{\mathrm{L}}$ and $\mathcal{L}_{\mathrm{c}}$. Indeed, also the next remark has to be considered.

(ii) The transformation in Eq. (37) has to be consistent with the Principle of Stationary Action [13], in the sense that the Action associated with the transformed Lagrangian needs to have the same stationary points as the non-transformed Action. In other words, upon setting

$$\mathcal{A}_{\mathrm{L}}(q, \lambda, \kappa) := \int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}} \mathcal{L}_{\mathrm{L}}(q(t), \dot{q}(t), t, \lambda(t), \kappa(t)) \mathrm{d}t, \tag{38}$$

it must hold true that $\mathcal{A}_{\mathrm{L}}(q, \lambda, \kappa) = \mathcal{A}_{\mathrm{c}}(q, \lambda) + C$, where C is a real constant, i.e., the Actions $\mathcal{A}_{\mathrm{L}}(q, \lambda, \kappa)$ and $\mathcal{A}_{\mathrm{c}}(q, \lambda)$ only differ additively by a constant term [13]. To this end, the generic function $\mathcal{F}^\beta$ will be chosen either as a total derivative, e.g. a generalized velocity, or as a function of the generalized velocities (quasi-velocity) that carries some physical interpretation for the problem at hand. Note that, in the second case, the associated coefficient κ_β must be zero in order to satisfy the previous hypothesis on $\mathcal{A}_{\mathrm{L}}$ [12]. To see this, assume that there exists $\bar{\beta} \in \{m+1, \ldots, n\}$ such that $\mathcal{F}^{\bar{\beta}}$ *is not* the total derivative of a function of the type $f \circ (q, \tau)$, while all other functions $\mathcal{F}^\beta$ are so, with $\beta \in \{m+1, \ldots, n\} \backslash \{\bar{\beta}\}$. Then, the Action $\mathcal{A}_{\mathrm{L}}(q, \lambda, \kappa)$ becomes

$$\begin{aligned}
\mathcal{A}_{\mathrm{L}}(q, \lambda, \kappa) := & \mathcal{A}_{\mathrm{c}}(q, \lambda) - \sum_{\beta \neq \bar{\beta}} \int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}} \kappa_\beta(t) \mathcal{F}^\beta(\sharp^{(1)}(t)) \mathrm{d}t - \int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}} \kappa_{\bar{\beta}}(t) \mathcal{F}^{\bar{\beta}}(\sharp^{(1)}(t)) \mathrm{d}t \\
= & \mathcal{A}_{\mathrm{c}}(q, \lambda) \underbrace{- \sum_{\beta \neq \bar{\beta}} \kappa_{\beta 0} \int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}} \mathcal{F}^\beta(\sharp^{(1)}(t)) \mathrm{d}t}_{=:C} - \kappa_{\bar{\beta} 0} \int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}} \mathcal{F}^{\bar{\beta}}(\sharp^{(1)}(t)) \mathrm{d}t \\
= & \mathcal{A}_{\mathrm{c}}(q, \lambda) + C - \kappa_{\bar{\beta} 0} \int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}} \mathcal{F}^{\bar{\beta}}(\sharp^{(1)}(t)) \mathrm{d}t,
\end{aligned} \tag{39}$$

which means that $\kappa_{\bar{\beta} 0}$ must be zero.

(iii) Last but not least, the auxiliary functions must be chosen in such a way that the coefficients of W computed through Eq. (35) must yield *zero transpositional relations* for those velocities that do not feature in the constraints or that, if featuring in them, can be varied independently of the other ones (see Remark 1, and [51]). Operatively, if β^* corresponds to a velocity that can be chosen as independent or absent in the constraints, one can choose $\mathcal{F}^{\beta^*} \circ \sharp^{(1)} = \dot{q}^{\beta^*}$. A function of this type complies trivially with the requirements (i), (ii), and with the present one, and implies that the β^*th row of W is identically null.

Given these considerations on the choice of the auxiliary functions, Eqs. (34) and (35) can be rewritten in explicit form as [12]

$$\textstyle\sum_h \left[\frac{\partial \mathcal{V}^\alpha}{\partial \dot{q}^h} \circ \sharp^{(1)} \right] W^h{}_k = -\mathscr{E}_k \mathcal{V}^\alpha \circ \sharp^{(2)},\ k = 1, \ldots, n,\ \alpha = 1, \ldots, m, \tag{40a}$$

$$\textstyle\sum_h \left[\frac{\partial \mathcal{F}^\beta}{\partial \dot{q}^h} \circ \sharp^{(1)} \right] W^h{}_k = -\mathscr{E}_k \mathcal{F}^\beta \circ \sharp^{(2)},\ k = 1, \ldots, n,\ \beta = m+1, \ldots, n, \tag{40b}$$

so that the coefficients $W^h{}_k$ are obtained by solving (40a) and (40b). Note that, Eqs. (40a) and (40b) together are equivalent to Eq. (17).

In spite of the selection rules mentioned above, we remark that, in the "charged skate" benchmark studied in Sect. 4, we found that an auxiliary function that works well in the uncharged case, yields to results that, in our opinion, are unphysical when the skate is charged and subjected to an interaction with a magnetic field. According to our calculations, indeed, there occur inconsistencies with the fulfillment of certain conservation laws, that can be related to the way in which the chosen auxiliary function depends on the velocities that *are* involved in the constraint. To amend these shortcomings, we suggest to switch to a different formulation, that we report in the following *Ansatz 2*.

2.3.2 Ansatz 2: physics-based conditions

In order to close Eqs. (33a) and (33b), we can require the system to respect the symmetries that may be naturally present in the Lagrangian function $\mathcal{L}$ and in the constraints. For instance, if we assume the existence of a Lagrangian parameter $q^{\bar{k}}$ *"ignorable"* [13] in $\mathcal{L}$, i.e., not explicitly featuring among its arguments, and such that $\partial_{\dot{q}^{\bar{k}}} \mathcal{V}^\alpha \circ \sharp^{(1)} = 0$ for all $\alpha = 1, \ldots, m$, then, by consistency with classical Analytical Mechanics, expressed by the Lagrange-D'Alembert method, we may impose the condition

$$\sum_h p_h W^h{}_{\bar{k}} = 0, \qquad\qquad p_h := \frac{\partial \mathcal{L}}{\partial \dot{q}^h} \circ \sharp^{(1)}, \tag{41}$$

to restore the conservation of the generalized momentum $p_{\bar{k}}$ [13].

Another scenario could be the one in which all the m constraints, say $f^1 \circ (q, \tau) = 0, \ldots, f^m \circ (q, \tau) = 0$, are holonomic. In this case, it is known that the equations of motion are $\mathscr{E}_k\mathcal{L} + \sum_\alpha \mu_\alpha[\partial_{q^k} f^\alpha \circ (q,\tau)] = 0$, for $k = 1,\ldots,n$ (see, e.g., [13]), and one can put the constraints in "non-holonomic" form by setting $\mathcal{V}^\alpha \circ \sharp^{(1)} \equiv \mathrm{d}_t[f^\alpha \circ (q,\tau)]$, with $\alpha = 1,\ldots,m$ and d_t indicating the total time derivative. In this case, however, it automatically applies $\mathscr{E}_k\mathcal{V}^\alpha \circ \sharp^{(2)} = 0$, for all $k = 1,\ldots,n$, and for all $\alpha = 1,\ldots,m$. Thus, if we take $\mathcal{F}^\beta \circ \sharp^{(1)} = \mathrm{d}_t[g^\beta \circ (q,\tau)]$, with $\beta = m+1,\ldots,n$, and $g^\beta \circ (q,\tau)$ being arbitrary scalar functions such that the matrix H in Eq. (36) is non-singular, then W turns out to be the null matrix, as prescribed by Eqs. (40a) and (40b).

Note that, the conditions in *Ansatz 2* are, in general, not enough to close the problem, so that a combination of the conditions featuring in *Ansatz 1* and *Ansatz 2* is required. From an operative point of view, our *Ansatz 2* suggests to introduce, as shown in *Ansatz 1*, a number ℓ^* of auxiliary functions $\{\mathcal{F}^\beta\}_{\beta=m+1}^{m+\ell^*}$, if this is the number of velocities that either do not feature in the constraints or can be chosen as independent. Then, to determine the remaining $n-(m+\ell^*)$ conditions, *Ansatz 2* indicates to apply a criterion that will be formalized in Theorem 1. Indeed, this criterion provides automatically the "physics-based conditions" mentioned at the beginning of this section, also for those cases in which it can be hard to impose them from the outset.

In conclusion, when matrix W is found, either by *Ansatz 1* or *Ansatz 2*, then the dynamic equations of the "modified vakonomic method" read [11, 12]

$$\mathscr{E}_k\mathcal{L} \circ \sharp^{(2)} + \textstyle\sum_h p_h {W^h}_k + \sum_\alpha \dot{\lambda}_\alpha \left[\frac{\partial \mathcal{V}^\alpha}{\partial \dot{q}^k} \circ \sharp^{(1)}\right] = 0, \quad k = 1,\ldots,n, \tag{42a}$$

$$-\mathcal{V}^\alpha \circ \sharp^{(1)} = 0, \quad \alpha = 1,\ldots,m. \tag{42b}$$

In the presence of generalized forces $\mathcal{Q}_1,\ldots,\mathcal{Q}_k$ that cannot be obtained from a scalar potential (Lanczos [13] refers to such forces as *"polygenic"*), Eq. (42a) acquires, up to the sign, the right-hand side $\mathcal{Q}_k$, so that the set of Eqs. (42a) and (42b) becomes

$$\mathscr{E}_k\mathcal{L} \circ \sharp^{(2)} + \textstyle\sum_h p_h {W^h}_k + \sum_\alpha \dot{\lambda}_\alpha \left[\frac{\partial \mathcal{V}^\alpha}{\partial \dot{q}^k} \circ \sharp^{(1)}\right] = -\mathcal{Q}_k, \quad k = 1,\ldots,n, \tag{43a}$$

$$-\mathcal{V}^\alpha \circ \sharp^{(1)} = 0, \qquad \alpha = 1,\ldots,m. \tag{43b}$$

2.4 Some computational aspects regarding the MVM

In this section, we discuss some computational aspects regarding the solution of the set of Eqs. (42a) and (42b) (or (43a) and (43b)). We remark that having a general methodology for solving numerically the MVM dynamic equations is a key aspect of the theory, since not all problems allow for an analytical solution.

To include problems that involve electromagnetic interactions [24] as well as time-dependent holonomic constraints [13], we assume $\mathcal{L}$ to be of the type

$$\mathcal{L} \circ \sharp^{(1)} = \tfrac{1}{2}\textstyle\sum_{h,k=1}^{n} [G_{hk} \circ (q,\tau)]\dot{q}^h\dot{q}^k + \sum_{h=1}^{n} [Z_h \circ (q,\tau)]\dot{q}^h + \mathcal{U} \circ (q,\tau), \tag{44}$$

where G_{hk} is the generic component of the "metric tensor" associated with the kinetic energy [13]; Z_h is the hth component of a co-vector field accounting for Maxwell's (co-)vector potential [24], and for the possible explicit time-dependence of holonomic constraints; $\mathfrak{U}$ is a potential function that collects both electric and mechanical interactions as well as the contribution to the kinetic energy stemming from the possible presence of explicitly time-dependent holonomic constraints. Moreover, for such type of Lagrangian functions, their associated kth Euler-Lagrange operator reads

$$\begin{aligned}\mathscr{E}_k\mathcal{L}\circ\sharp^{(2)} =& -[G_{kh}\circ(q,\tau)]\left\{\ddot{q}^h+[\Gamma^h{}_{pl}\circ(q,\tau)]\dot{q}^p\dot{q}^l\right\}-\left[\frac{\partial G_{kh}}{\partial\tau}\circ(q,\tau)\right]\dot{q}^h\\ &+\left[\left(\frac{\partial Z_h}{\partial q^k}-\frac{\partial Z_k}{\partial q^h}\right)\circ(q,\tau)\right]\dot{q}^h+\left(\frac{\partial\mathfrak{U}}{\partial q^k}-\frac{\partial Z_k}{\partial\tau}\right)\circ(q,\tau),\end{aligned}\tag{45}$$

where $\Gamma^h{}_{pl}\circ(q,\tau)$ is the generic Christoffel symbol induced by the metric tensor, i.e. (see, e.g., [6, 15]),

$$\Gamma^h{}_{pl} := \frac{1}{2}\sum_{r=1}^{n} G^{hr}\left(\frac{\partial G_{rp}}{\partial q^l}+\frac{\partial G_{rl}}{\partial q^p}-\frac{\partial G_{pl}}{\partial q^r}\right),\tag{46}$$

and G^{hr} are the components of the inverse of the metric tensor.

Before going further, we notice that the substitution of Eq. (45) into (42a) renders the latter one of second order in the Lagrangian parameters $q^1,\ldots,q^n$, whereas Eq. (42b) is, by definition, of the first order in these variables. Following [6], it is convenient to "promote" the constraints to second-order ordinary differential equations, as to store the constraints in the first non-singular matrix associated with the highest derivatives of q, i.e., the mass matrix. This can be accomplished by differentiating Eq. (42b) with respect to time, thereby obtaining

$$\overline{\mathcal{V}^\alpha\circ\sharp^{(1)}}^{\,\boldsymbol{\cdot}}=[a^\alpha{}_k\circ(q,\tau)]\ddot{q}^k+\left[\frac{\partial a^\alpha{}_k}{\partial q^h}\circ(q,\tau)\right]\dot{q}^k\dot{q}^h+\left[\frac{\partial a^\alpha{}_k}{\partial\tau}\circ(q,\tau)\right]\dot{q}^k=0,\tag{47}$$

and by solving the resulting equations together with Eq. (45). For this purpose, it is convenient to introduce the following notation:

$$M_{kh} := G_{kh}\circ(q,\tau),\tag{48a}$$

$$C^h{}_{pl} := \Gamma^h{}_{pl}\circ(q,\tau),\tag{48b}$$

$$\Omega_{kh} := \left[\frac{\partial Z_h}{\partial q^k}-\frac{\partial Z_k}{\partial q^h}\right]\circ(q,\tau)-\frac{\partial G_{kh}}{\partial\tau}\circ(q,\tau),\tag{48c}$$

$$F_k := \left[\frac{\partial\mathfrak{U}}{\partial q^k}-\frac{\partial Z_k}{\partial\tau}\right]\circ(q,\tau),\tag{48d}$$

$$A^\alpha{}_k := a^\alpha{}_k\circ(q,\tau),\tag{48e}$$

$$\Lambda^\alpha{}_{kh} := \frac{\partial a^\alpha{}_k}{\partial q^h} \circ (q, \tau), \tag{48f}$$

$$\Theta^\alpha{}_k := \frac{\partial a^\alpha{}_k}{\partial \tau} \circ (q, \tau). \tag{48g}$$

Hence, by substituting Eq. (45) in (42a), and changing sign to the resulting expression, Eqs. (42a) and (42b) take on the form

$$M_{kh}\ddot{q}^h + M_{kh}C^h{}_{pl}\dot{q}^p\dot{q}^l - \Omega_{kh}\dot{q}^h - F_k - p_h W^h{}_k - \dot{\lambda}_\alpha A^\alpha{}_k = 0, \tag{49a}$$

$$-A^\alpha{}_h\ddot{q}^h - \Lambda^\alpha{}_{pl}\dot{q}^p\dot{q}^l - \Theta^\alpha{}_h\dot{q}^h = 0. \tag{49b}$$

Moreover, Eqs. (49a) and (49b) can be recast in matrix form as

$$\begin{bmatrix} \mathsf{M} & -\mathsf{A}^{\mathrm{T}} \\ -\mathsf{A} & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\mathsf{q}} \\ \dot{\lambda} \end{Bmatrix} = \begin{Bmatrix} -\mathsf{M}[\mathsf{C} : \dot{\mathsf{q}} \otimes \dot{\mathsf{q}}] + \Omega\dot{\mathsf{q}} + \mathsf{F} + \mathsf{W}^{\mathrm{T}}\mathsf{p} \\ \Lambda : \dot{\mathsf{q}} \otimes \dot{\mathsf{q}} + \Theta\dot{\mathsf{q}} \end{Bmatrix} \equiv \begin{Bmatrix} \mathsf{b}_{\mathsf{q}} \\ \mathsf{b}_{\mathsf{c}} \end{Bmatrix} + \begin{Bmatrix} \mathsf{W}^{\mathrm{T}}\mathsf{p} \\ 0 \end{Bmatrix}, \tag{50}$$

where the symbols in Sans Serif font represent the matrices and vectors (in the sense of arrays) associated with the quantities in Eqs. (49a) and (49b).

Moreover, we can exploit the saddle-point nature of the system (50) by employing the Schur complement technique [6], and, by doing so, we can recast the system (50) as

$$\ddot{\mathsf{q}} = \left[\mathsf{M}^{-1} - (\mathsf{M}^{-1}\mathsf{A}^{\mathrm{T}})\mathsf{S}^{-1}(\mathsf{A}\mathsf{M}^{-1})\right](\mathsf{b}_{\mathsf{q}} + \mathsf{W}^{\mathrm{T}}\mathsf{p}) - (\mathsf{M}^{-1}\mathsf{A}^{\mathrm{T}})\mathsf{S}^{-1}\mathsf{b}_{\mathsf{c}}, \tag{51a}$$

$$\dot{\lambda} = -\mathsf{S}^{-1}\left[(\mathsf{A}\mathsf{M}^{-1})(\mathsf{b}_{\mathsf{q}} + \mathsf{W}^{\mathrm{T}}\mathsf{p}) + \mathsf{b}_{\mathsf{c}}\right], \tag{51b}$$

with $\mathsf{S} := \mathsf{A}\mathsf{M}^{-1}\mathsf{A}^{\mathrm{T}}$ being the Schur complement of the block-wise system. Note that the formalism used in Eq. (51a) is similar to the one adopted in [49].

2.5 Differences and similarities with the TNHM

In this section, we will acknowledge the differences and the similarities between the MVM and the "traditional non-holonomic method", in short TNHM, which is based on a generalization of the Lagrange-D'Alembert Principle (see, e.g., [16, 17]). In particular, let us briefly recall the main results related to dynamic equations characterizing the TNHM [1, 16–18].

If we consider a mechanical system described by n Lagrangian parameters and constrained by m non-holonomic constraints $\mathcal{V}^\alpha \circ \sharp^{(1)} = 0$, with $\alpha = 1, \dots, m$, then the TNHM is characterized by the $n + m$ equations [1, 3, 6]

$$\mathscr{E}_k\mathcal{L} \circ \sharp^{(2)} + \sum_{\alpha=1}^{m} \mu_\alpha \left[\frac{\partial \mathcal{V}^\alpha}{\partial \dot{q}^k} \circ \sharp^{(1)}\right] = 0, \qquad k = 1, \dots, n, \tag{52a}$$

$$-\mathcal{V}^\alpha \circ \sharp^{(1)} = 0, \qquad \alpha = 1, \dots, m, \tag{52b}$$

in which μ_α is the αth Lagrange multiplier associated with the αth constraint.

By comparing Eq. (42a) with Eq. (52a) for the case of non-holonomic constraints linear in the generalized velocities, we notice that the quantities $\mathscr{E}_k\mathcal{L}\circ\sharp^{(2)}$ are the same for both equations, and that the derivatives $\partial_{\dot{q}^k}\mathcal{V}^\alpha\circ\sharp^{(1)}$ return the coefficients $a^\alpha{}_k\circ(q,\tau)$ of Eq. (5), for all $\alpha=1,\ldots,m$ and for all $k=1,\ldots,n$. Therefore, the only visual differences between these equations are due to the presence of the quantities $\sum_{h=1}^n p_h W^h{}_k$ and of $\dot{\lambda}_\alpha$ in lieu of μ_α in Eq. (42a). Moreover, for each $\alpha=1,\ldots,m$, the physical dimensions of the Lagrange multiplier λ_α in Eq. (42a) are equal to the physical dimensions of the corresponding multiplier μ_α multiplied by a characteristic time [32, 33].

As previously done for the dynamic equations of the MVM in Eq. (50), we can specialize Eqs. (52a) and (52b) for the case of the Lagrangian function in Eq. (44). In particular, by referring to the same notation introduced in Eqs. (48a)–(48g), and by defining the array of Lagrange multipliers of the TNHM, i.e., $\mathsf{\mu}:=\{\mu_1,\ldots,\mu_m\}^{\mathrm{T}}$, Eqs. (52a) and (52b) can be written in matrix form as [6]

$$\begin{bmatrix}\mathsf{M} & -\mathsf{A}^{\mathrm{T}}\\ -\mathsf{A} & 0\end{bmatrix}\begin{Bmatrix}\ddot{\mathsf{q}}\\ \mathsf{\mu}\end{Bmatrix}=\begin{Bmatrix}-\mathsf{M}[\mathsf{C}:\dot{\mathsf{q}}\otimes\dot{\mathsf{q}}]+\Omega\dot{\mathsf{q}}+\mathsf{F}\\ \Lambda:\dot{\mathsf{q}}\otimes\dot{\mathsf{q}}+\Theta\dot{\mathsf{q}}\end{Bmatrix}\equiv\begin{Bmatrix}\mathsf{b}_\mathsf{q}\\ \mathsf{b}_\mathrm{c}\end{Bmatrix}. \tag{53}$$

Moreover, by defining $\mathsf{S}:=\mathsf{A}\mathsf{M}^{-1}\mathsf{A}^{\mathrm{T}}$, the Schur complement technique allows us to invert Eq. (53), which reads as follows:

$$\ddot{\mathsf{q}}=[\mathsf{M}^{-1}-(\mathsf{M}^{-1}\mathsf{A}^{\mathrm{T}})\mathsf{S}^{-1}(\mathsf{A}\mathsf{M}^{-1})]\mathsf{b}_\mathsf{q}-(\mathsf{M}^{-1}\mathsf{A}^{\mathrm{T}})\mathsf{S}^{-1}\mathsf{b}_\mathrm{c}, \tag{54a}$$

$$\mathsf{\mu}=-\mathsf{S}^{-1}[(\mathsf{A}\mathsf{M}^{-1})\mathsf{b}_\mathsf{q}+\mathsf{b}_\mathrm{c}]. \tag{54b}$$

Note that, both for the MVM and for the TNHM, we obtain the same block-wise matrix on the left-hand side of Eqs. (50) and (53). However, the right-hand side of these equations is different since $\mathsf{W}^{\mathrm{T}}\mathsf{p}$ features in Eq. (50), only, as a result of the followed procedure.

At this stage, we find it convenient to formalize the results obtained so far by providing a definition of *equivalence* between the "modified vakonomic method" (MVM) and the "traditional non-holonomic method" (TNHM).

Definition 1 (Equivalence between MVM and TNHM)
Given the same set of initial conditions on the generalized coordinates and velocities, we say that the MVM, i.e., Eqs. (42a) and (42b), and the TNHM, i.e., Eqs. (52a) and (52b), are *equivalent*, if they return the same solution q.

Theorem 1 (Characterization of the equivalence between MVM and TNHM)
Let us consider a mechanical system described by a Lagrangian function of the type given in Eq. (44) and by the set of non-holonomic constraints featuring in Eq. (5), which are linear in the generalized velocities. Then, the MVM, i.e., Eqs. (51a) and (51b), and the TNHM, i.e., Eqs. (54a) and (54b), are equivalent if, and only if, the following conditions are met

$$\left[\mathsf{M}^{-1}-(\mathsf{M}^{-1}\mathsf{A}^{\mathrm{T}})\mathsf{S}^{-1}(\mathsf{A}\mathsf{M}^{-1})\right]\mathsf{W}^{\mathrm{T}}\mathsf{p}=0, \tag{55a}$$

$$\dot{\lambda} = \mu - (\mathsf{S}^{-1}\mathsf{A}\mathsf{M}^{-1})\mathsf{W}^{\mathrm{T}}\mathsf{p}. \tag{55b}$$

Proof If the MVM and the TNHM predict the same motion, which implies that the collection $q = (q^1, \dots, q^n)$, represented by the array q, satisfies Eqs. (51a) and (51b) as well as Eqs. (54a) and (54b) at the same time, then subtracting Eq. (54a) from Eq. (51a), and (54b) from Eq. (51b) yields Eqs. (55a) and (55b).

Conversely, if Eqs. (55a) and (55b) hold true, then Eq. (51a) becomes identical to Eq. (54a), and it is possible to establish a univocally determined relationship between the multipliers of the two methods, i.e., $\dot{\lambda}$ and μ, thereby predicting the same solution. □

As stated in Theorem 1, the fulfillment of Eqs. (55a) and (55b) provides the equivalence between the MVM and the TNHM. Moreover, it contributes to the understanding of whether a problem formulated with the vakonomic method yields, in a general context, the same solutions as the TNHM. In this sense, Theorem 1 seems to give an affirmative answer, since it prescribes the conditions under which the vakonomic method, modified as indicated by the MVM of [11, 12], returns the same results obtained within the TNHM.

We emphasize that, granted the equivalence between the MVM and the TNHM, the identification between the Lagrange multipliers in Eq. (55b) is similar to the ones originally presented in [12]. Note also that, even though it is possible to *formally* find relations as in Eq. (55b), they alone are not sufficient, in general, to guarantee the equivalence between the TNHM and the MVM.

Corollary 1 (Sufficient condition for the equivalence between MVM and TNHM) *Given a mechanical system of the type addressed in Theorem 1, the MVM is* equivalent *to the TNHM, if it holds true that*

$$\mathsf{W}^{\mathrm{T}}\mathsf{p} = 0, \tag{56a}$$

$$\dot{\lambda} = \mu. \tag{56b}$$

Proof If the condition $\mathsf{W}^{\mathrm{T}}\mathsf{p} = 0$ applies, then Eq. (55a) is trivially satisfied, and, thus, the MVM is equivalent to the TNHM. In addition, Eq. (56b) follows directly from Eq. (55b). □

We emphasize that our Theorem 1 is a reinterpretation of Theorem 2 in [12], while our Corollary 1, and, by extension, also our *Ansatz 2*, aim to reinterpret Theorem 3 of Llibre et al. [12] and Section 3.2 of Ramírez et al. [11], as a sufficient condition for Theorem 1.

Remark 3 (On the role of the Lagrangian, its generalized momenta p, and W) The condition (56a) requires that the array of the generalized momenta p belongs to the kernel of the matrix W^{T}, and it amounts to requiring that the second term on the left-hand side of Eq. (42a) vanishes identically, i.e., $\sum_{h=1}^{n} p_h {W^h}_k = 0$, for $k = 1, \dots, n$. For a given Lagrangian function $\mathcal{L}$, which identifies the components $p_h = \partial_{\dot{q}^h}\mathcal{L} \circ \sharp^{(1)}$ of p, whether or not p belongs to the kernel of W^{T} depends both on

W itself, and, thus, in general, on the conditions imposed to determine the coefficients of W, and on the form of p. The conditions on W, in fact, can be obtained by following the *Ansatz 1* and/or the *Ansatz 2*. On the other hand, for a given matrix W, determined e.g. through the assignment of suitable functions $\mathcal{F}^{m+1}, \dots, \mathcal{F}^n$ (we recall that the constraints provide indications on the adequacy of such functions, which are none other than additional quasi-velocities), the array of momenta stemming from a given $\mathcal{L}$ does not necessarily belong to the kernel of W^{T}, depending on the interactions accounted for by $\mathcal{L}$. Indeed, the Lagrangian function addressed in our work, specified in Eq. (44), is more general than the ones typically considered in the context of vakonomic mechanics, since it accounts for interactions that do not allow to write it as the sum of a kinetic energy plus a potential depending solely on the Lagrangian parameters and possibly on time. This will become clearer below, when we shall analyze a modification of the classical benchmark of the non-holonomic skate, by including magnetic interactions that yield the presence in the Lagrangian function of a term linear in the generalized velocities. In fact, this generalization leads to momenta that are affine in the velocities and that, because of their nature, cannot belong to the kernel of W^{T}, and do not trivially satisfy Theorem 1. This issue, however, will be discussed in Sect. 2.6, and specialized to the case of the "charged skate" in Sect. 4.2.1.

Before closing this section, we deem it worthwhile to emphasize that, to the best of our understanding, the MVM proposed by Llibre et al. [12] could be viewed as a *variational version* of the method based on the quasi-velocities (see e.g. [26, 50]), in which the variational form of Newton's second law of dynamics is obtained by employing D'Alembert principle, but admitting that the time derivative of the variations are different from the variations of the velocities. In this respect, the sufficient conditions for the equivalence between the MVM and the TNHM, i.e., Eqs. (55a) and (55b), are a consequence of the method of the quasi-velocities.

2.6 The case of momenta linear in the velocities

In Eqs. (43a) and (43b) of Sect. 2.4, we have shown the equations of the MVM in the presence of *"polygenic forces"* [13]. In fact, it is possible to obtain the same form also in the case in which the interaction that leads to the term $\sum_{h=1}^{n}[Z_h \circ (q,\tau)]\dot{q}^h$ in the Lagrangian function of Eq. (44) is formally treated as a force of this type (although this force is not polygenic *per se*). Indeed, by making the identification

$$-\mathcal{Q}_k \equiv -\left[\left(\frac{\partial Z_h}{\partial q^k} - \frac{\partial Z_k}{\partial q^h}\right)\circ(q,\tau)\right]\dot{q}^h + \frac{\partial Z_k}{\partial \tau}\circ(q,\tau), \tag{57}$$

and, thus, consistently omitting the term $\sum_{h=1}^{n}[Z_h \circ (q,\tau)]\dot{q}^h$ in the Lagrangian function of Eq. (44), i.e., redefining $\mathcal{L}\circ\sharp^{(1)}$ as

$$\mathcal{L}_0 \circ \sharp^{(1)} = \tfrac{1}{2}\textstyle\sum_{h,k=1}^{n}[G_{hk}\circ(q,\tau)]\dot{q}^h\dot{q}^k + \mathcal{U}\circ(q,\tau), \tag{58}$$

Equations (42a) and (42b) are recast in the form

$$\mathscr{E}_k \mathcal{L}_0 \circ \sharp^{(2)} + \textstyle\sum_h p_{0h} W^h{}_k + \sum_\alpha \dot{\lambda}_\alpha \Big[\dfrac{\partial \mathcal{V}^\alpha}{\partial \dot{q}^k} \circ \sharp^{(1)} \Big] = -\mathcal{Q}_k, \quad k = 1, \ldots, n, \tag{59a}$$

$$-\mathcal{V}^\alpha \circ \sharp^{(1)} = 0, \qquad \alpha = 1, \ldots, m, \tag{59b}$$

with $p_{0h} := \partial_{\dot{q}^h} \mathcal{L}_0 \circ \sharp^{(1)} = \sum_{\ell=1}^n [G_{h\ell} \circ (q, \tau)] \dot{q}^\ell$. This last result, which stipulates that the momenta are linear functions of the velocities, has deep repercussions on the fulfillment of conditions (55a) and (55b), with p_0 substituting p. More details on this issue will be discussed in Remark 8 of Sect. 4.2.1 for the specific case of the "charged skate".

3 The "rolling coin" benchmark

In this section, we compute analytically the MVM equations of motion for the benchmark problem of the "rolling coin", recently presented in [1, 14], and adopted as a *benchmark* for comparing the VM and the TNHM in [22, 43–45].

The mechanical system that we are considering is composed by a coin, hereafter denoted by $\mathfrak{C}$, idealized as a bi-dimensional rigid disk of radius R and mass m, that can roll without slipping over a plane inclined of an angle $\alpha \in]0, \pi/2[$ with respect to a horizontal plane.

As done in [1, 14], we specify a coordinate system $\{O, (x, y, z)\}$ in which the y-axis is parallel to the inclined plane and is aligned along the direction of steepest descent; the z-axis "enters" the inclined plane and is orthogonal to it; the x-axis is such that the x-, y-, and z-axis form a right-handed triad, whose origin O is a point having the same z coordinate as the center of mass of $\mathfrak{C}$, hereafter denoted by G. Moreover, similarly to [14], we introduce the coordinates of G, i.e., $(x_G, y_G, 0)$; the angle of rolling, ϕ, taken clockwise from the z-axis; and the angle θ between the y-axis and the axis along which rolling occurs. Note that x_G, y_G, ϕ, and θ are the Lagrangian parameters of the mechanical system under study, and are thus intended as functions of time, defined over the interval of observation $[t_{\text{in}}, t_{\text{fin}}]$.

From here on, the identification $q \equiv (q^1, q^2, q^3, q^4) = (x_G, y_G, \phi, \theta)$ is made, which implies that the system satisfies automatically two holonomic constraints: one requires G to experience only motions parallel to the inclined plane; the other one prescribes that the coin $\mathfrak{C}$ remains orthogonal to the inclined plane during its entire motion.

If gravity is the only interaction accounted for, the Lagrangian function of the mechanical system under study reads [1, 14, 26]

$$\mathcal{L} \circ (q, \dot{q}) = \underbrace{\tfrac{1}{2} m \left[(\dot{q}^1)^2 + (\dot{q}^2)^2 + \tfrac{1}{2} R^2 (\dot{q}^3)^2 + \tfrac{1}{4} R^2 (\dot{q}^4)^2 \right]}_{\mathcal{K} \circ \dot{q}} + \underbrace{(mg \sin\alpha) q^2}_{\mathcal{U} \circ q}, \tag{60}$$

where $\mathcal{K}$ is the kinetic energy; $\mathcal{U}$ is the gravitational potential; and g denotes the magnitude of the gravitational acceleration.

The assumption that the coin *"rolls without slipping"* [1, 14] determines $m = 2$ non-holonomic constraints given by [22, 26, 43–45, 47]

$$\mathcal{V}^1 \circ (q, \dot{q}) := \dot{q}^1 - R\dot{q}^3 \sin q^4 = 0, \tag{61a}$$

$$\mathcal{V}^2 \circ (q, \dot{q}) := \dot{q}^2 - R\dot{q}^3 \cos q^4 = 0. \tag{61b}$$

Note that, by indicating with $\boldsymbol{\tau} = \sin\theta\, \boldsymbol{e}_x + \cos\theta\, \boldsymbol{e}_y$ the unit vector defining the direction of rolling, Eqs. (61a) and (61b) express, in terms of the Lagrangian parameters of the model and their derivatives, the fact that the velocity of the coin's center of mass, $\boldsymbol{v}_G$, satisfies the condition $\boldsymbol{v}_G = R\dot{\phi}\boldsymbol{\tau}$, given in the physical space.

3.1 The traditional non-holonomic approach

In the benchmark that we are analyzing, the dynamic equations of the TNHM, namely Eqs. (52a) and (52b), admit the expressions [1, 14, 22, 43–45, 47]

$$-m\ddot{q}^1 + \mu_1 = 0, \tag{62a}$$

$$mg \sin\alpha - m\ddot{q}^2 + \mu_2 = 0, \tag{62b}$$

$$-\tfrac{1}{2}mR^2\ddot{q}^3 - \mu_1 R \sin q^4 - \mu_2 R \cos q^4 = 0, \tag{62c}$$

$$-\tfrac{1}{4}mR^2\ddot{q}^4 = 0. \tag{62d}$$

As shown in [1], Eqs. (62a)–(62d) can be solved analytically by exploiting the direct integrability of Eq. (62d), so we will not further investigate this solution.

3.2 The "modified vakonomic" approach

In this section, we study the benchmark problem introduced above by means of the MVM [12], thereby determining the motion from the dynamic equations (42a) and (42b) in compliance with the *solvability conditions* (40a) and (40b).

By applying the Euler-Lagrange operators defined in Eq. (28a) to the Lagrangian function in Eq. (60), we obtain

$$\mathscr{E}_1\mathcal{L} \circ \sharp^{(2)} = -m\ddot{q}^1, \tag{63a}$$

$$\mathscr{E}_2\mathcal{L} \circ \sharp^{(2)} = -m\ddot{q}^2 + mg \sin\alpha, \tag{63b}$$

$$\mathscr{E}_3\mathcal{L} \circ \sharp^{(2)} = -\tfrac{1}{2}mR^2\ddot{q}^3, \tag{63c}$$

$$\mathscr{E}_4\mathcal{L} \circ \sharp^{(2)} = -\tfrac{1}{4}mR^2\ddot{q}^4. \tag{63d}$$

Furthermore, we write explicitly the differential operators in Eq. (34) for the expressions of the two constraints in (61a) and (61b), respectively. Hence, for

$\alpha = 1$, we have

$$\mathscr{D}_1\mathcal{V}^1 \circ \sharp^{(2)} = W^1{}_1 - (R\sin q^4)W^3{}_1 = 0, \tag{64a}$$
$$\mathscr{D}_2\mathcal{V}^1 \circ \sharp^{(2)} = W^1{}_2 - (R\sin q^4)W^3{}_2 = 0, \tag{64b}$$
$$\mathscr{D}_3\mathcal{V}^1 \circ \sharp^{(2)} = W^1{}_3 - (R\sin q^4)W^3{}_3 + (R\cos q^4)\dot{q}^4 = 0, \tag{64c}$$
$$\mathscr{D}_4\mathcal{V}^1 \circ \sharp^{(2)} = W^1{}_4 - (R\sin q^4)W^3{}_4 - (R\cos q^4)\dot{q}^3 = 0, \tag{64d}$$

while, for $\alpha = 2$, we obtain

$$\mathscr{D}_1\mathcal{V}^2 \circ \sharp^{(2)} = W^2{}_1 - (R\cos q^4)W^3{}_1 = 0, \tag{65a}$$
$$\mathscr{D}_2\mathcal{V}^2 \circ \sharp^{(2)} = W^2{}_2 - (R\cos q^4)W^3{}_2 = 0, \tag{65b}$$
$$\mathscr{D}_3\mathcal{V}^2 \circ \sharp^{(2)} = W^2{}_3 - (R\cos q^4)W^3{}_3 - (R\sin q^4)\dot{q}^4 = 0, \tag{65c}$$
$$\mathscr{D}_4\mathcal{V}^2 \circ \sharp^{(2)} = W^2{}_4 - (R\cos q^4)W^3{}_4 + (R\sin q^4)\dot{q}^3 = 0. \tag{65d}$$

By choosing the arbitrary *auxiliary functions* as [11, 12]

$$\mathcal{F}^3 \circ \sharp^{(1)} := \dot{q}^3, \tag{66a}$$
$$\mathcal{F}^4 \circ \sharp^{(1)} := \dot{q}^4, \tag{66b}$$

Equation (35), for $\beta = 3$ and $\beta = 4$, implies that

$$\mathscr{D}_k\mathcal{F}^3 \circ \sharp^{(2)} = W^3{}_k = 0, \qquad k = 1, \ldots, 4, \tag{67a}$$
$$\mathscr{D}_k\mathcal{F}^4 \circ \sharp^{(2)} = W^4{}_k = 0, \qquad k = 1, \ldots, 4. \tag{67b}$$

We can represent the coefficients $W^h{}_k$ in a more compact way by assembling the 4×4 matrix W that, by having recourse to Eqs. (64a)–(64d), (65a)–(65d), and (67a) and (67b), is given by

$$\mathsf{W} = \begin{bmatrix} 0 & 0 & -(R\cos q^4)\dot{q}^4 & (R\cos q^4)\dot{q}^3 \\ 0 & 0 & (R\sin q^4)\dot{q}^4 & -(R\sin q^4)\dot{q}^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{68}$$

Our choice of the auxiliary functions in Eqs. (66a) and (66b) is done in compliance with Remark 1, since, by inspection of the constraints (61a) and (61b), the generalized velocities $\dot{q}^3$ and $\dot{q}^4$ are *independent*, thereby leading to the vanishing of the corresponding transpositional relations $\zeta^3 - \dot{\eta}^3$ and $\zeta^4 - \dot{\eta}^4$. In fact, this choice is a *strong* way of guaranteeing this result because the conditions (67a) and (67b), implied by the selected auxiliary functions, ensure that the third and fourth row of W are null.

Finally, if we substitute Eqs. (63a)-(63d) and the coefficients collected in Eq. (68) into the Euler-Lagrange equations of the MVM written in (42a), we can recast Eqs. (42a) and (42b) in the form

$$m\ddot{q}^1 - \dot{\lambda}_1 = 0, \tag{69a}$$
$$m\ddot{q}^2 - \dot{\lambda}_2 = mg\sin\alpha, \tag{69b}$$
$$\tfrac{1}{2}mR^2\ddot{q}^3 + \dot{\lambda}_1 R\sin q^4 + \dot{\lambda}_2 R\cos q^4 = -mR(\dot{q}^1\cos q^4 - \dot{q}^2\sin q^4)\dot{q}^4, \tag{69c}$$
$$\tfrac{1}{4}mR^2\ddot{q}^4 = mR(\dot{q}^1\cos q^4 - \dot{q}^2\sin q^4)\dot{q}^3, \tag{69d}$$
$$\dot{q}^1 = R\dot{q}^3\sin q^4, \tag{69e}$$
$$\dot{q}^2 = R\dot{q}^3\cos q^4. \tag{69f}$$

Before going further, we notice that the quantity $\dot{q}^1\cos q^4 - \dot{q}^2\sin q^4$, which features both in Eqs. (69c) and (69d), is none other than the component along the axis $\boldsymbol{e}_z$ of the cross product $\boldsymbol{v}_G \times \boldsymbol{\tau}$, which is null because the velocity of the center of mass of the coin has to be parallel to the unit vector $\boldsymbol{\tau}$. In particular, it should be noticed that this condition is naturally satisfied by the constraint of rolling without slipping, but it is, in fact, more general than the latter, since it expresses a merely geometric fact. In conclusion, it applies that [26]

$$(\boldsymbol{v}_G \times \boldsymbol{\tau}) \cdot \boldsymbol{e}_z = \dot{q}^1\cos q^4 - \dot{q}^2\sin q^4 = 0. \tag{70}$$

By virtue of this result, Eqs. (69a)–(69f) become

$$m\ddot{q}^1 - \dot{\lambda}_1 = 0, \tag{71a}$$
$$m\ddot{q}^2 - \dot{\lambda}_2 = mg\sin\alpha, \tag{71b}$$
$$\tfrac{1}{2}mR^2\ddot{q}^3 + \dot{\lambda}_1 R\sin q^4 + \dot{\lambda}_2 R\cos q^4 = 0, \tag{71c}$$
$$\tfrac{1}{4}mR^2\ddot{q}^4 = 0. \tag{71d}$$
$$\dot{q}^1 = R\dot{q}^3\sin q^4, \tag{71e}$$
$$\dot{q}^2 = R\dot{q}^3\cos q^4. \tag{71f}$$

By making the identifications $\mu_\alpha \equiv \dot{\lambda}_\alpha$, for $\alpha = 1, \ldots, m$, the set of equations (71a)–(71d) is equal to the set of dynamic equations obtained within the TNHM, i.e. Eqs. (62a)–(62d). This shows that, in agreement with [12], the MVM is indeed *equivalent* with the TNHM for the "rolling coin" problem.

Remark 4 (The MVM and the TNHM for the "rolling coin" problem)
It should be noticed that the equivalence between the MVM and the TNHM, and the identification between their respective Lagrange multipliers, could have been proven in advance since, in the case considered above, the matrix W in Eq. (68) and the array of momenta p satisfy the hypothesis of Corollary 1, namely $\mathsf{W}^{\mathrm{T}}\mathsf{p} = \mathsf{0}$.

Remark 5 (*"Canonical flip"* [22, 41] for the "rolling coin" problem)
Looking at the structure of the matrix W featuring in Eq. (68), and computing the quantities $\zeta^h - \dot{\eta}^h$, with $h = 1, \dots, n$, once as $\zeta^h - \dot{\eta}^h = \sum_{k=1}^{n} W^h{}_k \eta^k$ and once as $\zeta^h - \dot{\eta}^h = \sum_{\ell,k=1}^{n} \mathcal{C}^h{}_{\ell k} \dot{q}^\ell \eta^k$ (cf. Eq. (21)), we can choose

$$\mathcal{C}^1{}_{43}\dot{q}^4 = W^1{}_3 = -(R\cos q^4)\dot{q}^4 \qquad \Rightarrow \qquad \mathcal{C}^1{}_{43} \equiv \hat{\mathcal{C}}^1{}_{43} \circ q = -R\cos q^4, \tag{72a}$$
$$\mathcal{C}^1{}_{34}\dot{q}^3 = W^1{}_4 = +(R\cos q^4)\dot{q}^3 \qquad \Rightarrow \qquad \mathcal{C}^1{}_{34} \equiv \hat{\mathcal{C}}^1{}_{34} \circ q = +R\cos q^4, \tag{72b}$$
$$\mathcal{C}^2{}_{43}\dot{q}^4 = W^2{}_3 = +(R\sin q^4)\dot{q}^4 \qquad \Rightarrow \qquad \mathcal{C}^2{}_{43} \equiv \hat{\mathcal{C}}^2{}_{43} \circ q = +R\sin q^4, \tag{72c}$$
$$\mathcal{C}^2{}_{34}\dot{q}^3 = W^2{}_4 = -(R\sin q^4)\dot{q}^3 \qquad \Rightarrow \qquad \mathcal{C}^2{}_{34} \equiv \hat{\mathcal{C}}^2{}_{34} \circ q = -R\sin q^4, \tag{72d}$$

while all the other entries of $\mathcal{C}^h{}_{\ell k}$ can be set equal to zero. We notice that, in the case studied in this remark, the nonzero entries of $\mathcal{C}^h{}_{\ell k}$, reported in Eqs. (72a)–(72d), depend on q, and, in particular, on q^4, only.

3.3 Analytical solution

In this section, we are interested in finding an analytical solution to the dynamical equations previously obtained within the MVM, i.e., Eqs. (71a)-(71d). To this end, we introduce the following set of initial conditions, i.e., at $t = t_{\text{in}}$, for the generalized coordinates and velocities:

$$q^1(t_{\text{in}}) = q^1_{\text{in}}, \qquad q^2(t_{\text{in}}) = q^2_{\text{in}}, \qquad q^3(t_{\text{in}}) = q^3_{\text{in}}, \qquad q^4(t_{\text{in}}) = q^4_{\text{in}}, \tag{73a}$$
$$\dot{q}^1(t_{\text{in}}) = \nu^1_{\text{in}}, \qquad \dot{q}^2(t_{\text{in}}) = \nu^2_{\text{in}}, \qquad \dot{q}^3(t_{\text{in}}) = \nu^3_{\text{in}}, \qquad \dot{q}^4(t_{\text{in}}) = \nu^4_{\text{in}}, \tag{73b}$$

where their specified values q^k_{in} and ν^k_{in}, for $k = 1, \dots, 4$, should satisfy the constraints in Eqs. (69e) and (69f), both for physical and for numerical consistency [6]. Furthermore, we introduce the initial values for the multipliers

$$\lambda_1(t_{\text{in}}) = \lambda_{1\,\text{in}}, \qquad\qquad \lambda_2(t_{\text{in}}) = \lambda_{2\,\text{in}}. \tag{74}$$

By taking inspiration from the solution strategy employed by Lemos [1], from the direct integration of Eq. (71d) we obtain that q^4 is *affine* in time, that is

$$q^4(t) = \Omega[t - t_{\text{in}}] + q^4_{\text{in}} = \Omega t + \theta_0, \qquad\qquad \forall\, t \in [t_{\text{in}}, t_{\text{fin}}], \tag{75}$$

where $\Omega := \dot{q}^4(t) \equiv \dot{q}^4(t_{\text{in}}) = \nu^4_{\text{in}}$ for all $t \in [t_{\text{in}}, t_{\text{fin}}]$, and, for a better readability, we write $\theta_0 := q^4_{\text{in}} - \Omega t_{\text{in}}$. Note that the angular velocity $\dot{q}^4(t) = \Omega$ is constant in time.

We can exploit the *saddle-point nature* of the system of equations under study by decoupling the dynamic equations (71a)–(71d) from the constraints, i.e., (71e) and (71f), so that the latter ones can be computed *a posteriori*. Hence, by substituting the constraints in (71e) and (71f) into Eqs. (71a) and

(71b), we obtain

$$\dot{\lambda}_1 = m\ddot{q}^3 R\,\mathrm{sin}q^4 + m\dot{q}^3\dot{q}^4 R\,\mathrm{cos}q^4, \tag{76a}$$
$$\dot{\lambda}_2 = m\ddot{q}^3 R\,\mathrm{cos}q^4 - m\dot{q}^3\dot{q}^4 R\,\mathrm{sin}q^4 - mg\,\mathrm{sin}\alpha. \tag{76b}$$

With the expressions obtained in Eqs. (76a) and (76b) for the two time derivatives of the Lagrange multipliers, and with the expression of q^4 in Eq. (75), we can re-frame Eq. (71c) as follows

$$\ddot{q}^3(t) = \frac{2g\,\mathrm{sin}\alpha}{3R}\cos(\Omega t + \theta_0). \tag{77}$$

In the following, we introduce, for each $k = 1, \ldots, 4$ and for each $\alpha = 1, 2$, the notations $q^k(t) \equiv \hat{q}^k(t;\,\Omega)$ and $\lambda_\alpha(t) \equiv \hat{\lambda}_\alpha(t;\,\Omega)$ in order to emphasize the dependence of the solution of the system under study on the parameter Ω. In particular, depending on the value of Ω, we can distinguish two cases.

Case 1 $\Omega \neq 0$

By assuming $\Omega \neq 0$, the right-hand side of Eq. (77) represents an oscillatory forcing term with angular frequency Ω and initial phase θ_0. By integrating Eq. (77), we obtain

$$q^3(t) \equiv \hat{q}^3(t;\,\Omega) = \hat{q}^3_{\mathrm{osc}}(t;\,\Omega) + \nu^3_{\mathrm{in}}[t - t_{\mathrm{in}}] + q^3_{\mathrm{in}}, \tag{78}$$

where we have introduced the auxiliary notation

$$\begin{aligned}\hat{q}^3_{\mathrm{osc}}(t;\,\Omega) :=\ & \Phi_{\mathrm{s}}(\Omega)\big\{\sin(\Omega t) - \sin(\Omega t_{\mathrm{in}})\big\} - \Phi_{\mathrm{c}}(\Omega)\big\{\cos(\Omega t) - \cos(\Omega t_{\mathrm{in}})\big\}\\ & - \Omega[t - t_{\mathrm{in}}]\big\{\Phi_{\mathrm{c}}(\Omega)\sin(\Omega t_{\mathrm{in}}) + \Phi_{\mathrm{s}}(\Omega)\cos(\Omega t_{\mathrm{in}})\big\},\end{aligned} \tag{79a}$$
$$\Phi(\Omega) := \frac{2g\,\mathrm{sin}\alpha}{3R\Omega^2}, \qquad \Phi_{\mathrm{c}}(\Omega) := \Phi(\Omega)\,\mathrm{cos}\theta_0, \qquad \Phi_{\mathrm{s}}(\Omega) := \Phi(\Omega)\,\mathrm{sin}\theta_0. \tag{79b}$$

Note that, by virtue of Eq. (78), the Lagrangian parameter $q^3(t)$ features an oscillatory contribution, given by Eq. (79a), which is characterized by the amplitude $\Phi(\Omega)$ in Equation (79b), and a contribution that is affine in time, i.e., $\nu^3_{\mathrm{in}}[t - t_{\mathrm{in}}] + q^3_{\mathrm{in}}$. Moreover, by differentiating in time Eq. (79a), one obtains

$$\begin{aligned}\frac{\partial \hat{q}^3_{\mathrm{osc}}}{\partial t}(t;\,\Omega) &= \Omega\big\{\Phi_{\mathrm{s}}(\Omega)\big[\cos(\Omega t) - \cos(\Omega t_{\mathrm{in}})\big] + \Phi_{\mathrm{c}}(\Omega)\big[\sin(\Omega t) - \sin(\Omega t_{\mathrm{in}})\big]\big\}\\ &= \Omega\Phi(\Omega)\big[\sin(\Omega t + \theta_0) - \sin(\Omega t_{\mathrm{in}} + \theta_0)\big].\end{aligned} \tag{80}$$

Thus, by integrating Eqs. (71e) and (71f), $q^1(t)$ and $q^2(t)$ read as

$$\begin{aligned}q^1(t) \equiv \hat{q}^1(t;\,\Omega) =\ & -\tfrac{1}{4}R\Phi(\Omega)\big[\sin(2\Omega t + 2\theta_0) - \sin(2\Omega t_{\mathrm{in}} + 2\theta_0)\big]\\ & - R\big[\tfrac{\nu^3_{\mathrm{in}}}{\Omega} - \Phi(\Omega)\sin(\Omega t_{\mathrm{in}} + \theta_0)\big]\big\{\cos(\Omega t + \theta_0) - \cos(\Omega t_{\mathrm{in}} + \theta_0)\big\}\end{aligned}$$

$$
\begin{aligned}
&+\tfrac{1}{2}R\Omega\Phi(\Omega)[t-t_{\text{in}}]+q^1_{\text{in}}, && \text{(81a)}\\
q^2(t)\equiv \hat{q}^2(t;\Omega) &= -\tfrac{1}{4}R\Phi(\Omega)\big[\cos(2\Omega t+2\theta_0)-\cos(2\Omega t_{\text{in}}+2\theta_0)\big]\\
&+R\big[\tfrac{\nu^3_{\text{in}}}{\Omega}-\Phi(\Omega)\sin(\Omega t_{\text{in}}+\theta_0)\big]\big\{\sin(\Omega t+\theta_0)-\sin(\Omega t_{\text{in}}+\theta_0)\big\}\\
&+q^2_{\text{in}}. && \text{(81b)}
\end{aligned}
$$

In light of the calculations above, we conclude that, for $\Omega \neq 0$, $q^1(t)$ and $q^2(t)$ exhibit two oscillatory contributions: one with angular frequency Ω and the other one with angular frequency 2Ω. In particular, $q^1(t)$ is unbounded in the limit $t \to +\infty$, whereas $q^2(t)$ is bounded. Therefore, the center of mass of the coin moves indefinitely along the x-axis and, in addition, will never reach the end of the inclined plane along the y-axis.

Finally, we compute the Lagrange multipliers *a posteriori* by integrating Eqs. (76a) and (76b) in light of the relations obtained for $q^3(t)$ and $q^4(t)$, i.e., Eqs. (75) and (80). By doing this, we obtain

$$
\begin{aligned}
\lambda_1(t)\equiv\hat{\lambda}_1(t;\Omega) &= -\tfrac{1}{2}mR\Omega\Phi(\Omega)\big[\cos(2\Omega t+2\theta_0)-\cos(2\Omega t_{\text{in}}+2\theta_0)\big]\\
&+mR\big[\nu^3_{\text{in}}-\Omega\Phi(\Omega)\sin(\Omega t_{\text{in}}+\theta_0)\big]\{\sin(\Omega t+\theta_0)-\sin(\Omega t_{\text{in}}+\theta_0)\}\\
&+\lambda_{1\,\text{in}}, && \text{(82a)}\\
\lambda_2(t)\equiv\hat{\lambda}_2(t;\Omega) &= \quad\tfrac{1}{2}mR\Omega\Phi(\Omega)\big[\sin(2\Omega t+2\theta_0)-\sin(2\Omega t_{\text{in}}+2\theta_0)\big]\\
&+mR\big[\nu^3_{\text{in}}-\Omega\Phi(\Omega)\sin(\Omega t_{\text{in}}+\theta_0)\big]\{\cos(\Omega t+\theta_0)-\cos(\Omega t_{\text{in}}+\theta_0)\}\\
&-(mg\sin\alpha)[t-t_{\text{in}}]+\lambda_{2\,\text{in}}. && \text{(82b)}
\end{aligned}
$$

Case 2 $\Omega = 0$

In the case $\Omega = 0$, we find that $q^4(t) = \theta_0 \equiv q^4_{\text{in}}$ for all $t \in [t_{\text{in}}, t_{\text{fin}}]$, and that the right-hand side of Eq. (77) is constant in time, thereby returning a uniformly accelerated angular motion for q^3. Thus, by integrating Eq. (77) two times, we obtain

$$
q^3(t)\equiv\hat{q}^3(t;0)=\tfrac{1}{2}\xi\,[t-t_{\text{in}}]^2+\nu^3_{\text{in}}[t-t_{\text{in}}]+q^3_{\text{in}},\quad \xi:=\frac{2g\sin\alpha\cos\theta_0}{3R}, \qquad (83)
$$

where ξ represent the angular acceleration associated with q^3.

Since the evolution in time of q^3 is known, we can deduce the evolution of q^1 and q^2 by substituting Eq. (83) in the constraints, i.e., Eqs. (71e) and (71f). By integrating the resulting expressions, we get

$$
q^1(t)=\tfrac{1}{2}\xi(R\sin\theta_0)[t-t_{\text{in}}]^2+\underbrace{\nu^3_{\text{in}}(R\sin\theta_0)}_{\equiv\nu^1_{\text{in}}}[t-t_{\text{in}}]+q^1_{\text{in}}, \qquad (84a)
$$

$$
q^2(t)=\tfrac{1}{2}\xi(R\cos\theta_0)[t-t_{\text{in}}]^2+\underbrace{\nu^3_{\text{in}}(R\cos\theta_0)}_{\equiv\nu^2_{\text{in}}}[t-t_{\text{in}}]+q^2_{\text{in}}. \qquad (84b)
$$

Finally, we can compute λ_1 and λ_2 *a posteriori* by integrating Eqs. (76a) and (76b) as follows:

$$\lambda_1(t) = m\xi(R\sin\theta_0)[t - t_{\text{in}}] + \lambda_{1\,\text{in}}, \tag{85a}$$

$$\lambda_2(t) = \{m\xi(R\cos\theta_0) - mg\sin\alpha\}[t - t_{\text{in}}] + \lambda_{2\,\text{in}}. \tag{85b}$$

We conclude noting that, for $\Omega = 0$, the Lagrange multipliers $\lambda_1(t)$ and $\lambda_2(t)$ are *affine* in time, whereas their time derivatives are constants. This means that, since $\dot{\lambda}_1(t)$ and $\dot{\lambda}_2(t)$ are equal to the Lagrange multipliers $\mu_1(t)$ and $\mu_2(t)$, respectively, of the TNHM (see Corollary 1 and Remark 4), and since the latter ones measure the magnitude of the reaction forces associated with the imposed constraints, we can say that the reaction forces are constants.

Remark 6 (Continuity of the third Lagrangian parameter with respect to Ω) From Case 1 and Case 2, it follows that the solution of Eq. (77) is defined as

$$q^3(t) \equiv \hat{q}^3(t;\Omega) = \begin{cases} \hat{q}^3_{\text{osc}}(t;\Omega) \quad + \nu^3_{\text{in}}[t - t_{\text{in}}] + q^3_{\text{in}}, & \text{if } \Omega \neq 0, \\ \frac{1}{2}\xi\,[t - t_{\text{in}}]^2 + \nu^3_{\text{in}}[t - t_{\text{in}}] + q^3_{\text{in}}, & \text{if } \Omega = 0, \end{cases} \tag{86}$$

where $\hat{q}^3_{\text{osc}}(t;\Omega)$ is reported in Eq. (79a). In particular, we emphasize that $\hat{q}^3(t,\,\cdot\,)$ is continuous with respect to Ω uniformly in $[t_{\text{in}}, t_{\text{fin}}]$, since it holds that

$$\lim_{\Omega\to 0} \hat{q}^3(t;\Omega) = \hat{q}^3(t;0), \qquad t \in [t_{\text{in}}, t_{\text{fin}}]. \tag{87}$$

3.4 Numerical simulations

In this section, we present the graphical results of the "Rolling coin" problem with the purpose of visualizing that the MVM is equivalent, for the considered problem, to the TNHM.

Hence, we numerically solve the system of Eqs. (71a)–(71f), put in a more general form as in Eq. (50), over the interval $[t_{\text{in}}, t_{\text{fin}}] \equiv [0, T]$. To this end, we introduce the set of *normalized* Lagrangian parameters

$$\mathsf{q}^1 := q^1/R, \qquad \mathsf{q}^2 := q^2/R, \qquad \mathsf{q}^3 := q^3, \qquad \mathsf{q}^4 := q^4, \tag{88}$$

and the set of *normalized* Lagrange multipliers as

$$\mathsf{\lambda}_1 := \lambda_1/(mR), \qquad \mathsf{\lambda}_2 := \lambda_2/(mR). \tag{89}$$

Note that the normalization of a given Lagrangian parameter is necessary only when it represents a translational kinematic descriptor, and not an angular one, while both the normalized Lagrange multipliers "absorb" the coin's mass m and radius R. As a consequence of (88) and (89), Eqs. (71a)–(71f) are thus normalized as follows:

$$\ddot{\mathsf{q}}^1 - \dot{\mathsf{\lambda}}_1 = 0, \tag{90a}$$

$$\ddot{\mathfrak{q}}^2 - \dot{\lambda}_2 = \frac{g \sin\alpha}{R}, \tag{90b}$$

$$\frac{1}{2}\ddot{\mathfrak{q}}^3 + \dot{\lambda}_1 \sin\mathfrak{q}^4 + \dot{\lambda}_2 \cos\mathfrak{q}^4 = 0, \tag{90c}$$

$$\frac{1}{4}\ddot{\mathfrak{q}}^4 = 0, \tag{90d}$$

$$\dot{\mathfrak{q}}^1 = \dot{\mathfrak{q}}^3 \sin\mathfrak{q}^4, \tag{90e}$$

$$\dot{\mathfrak{q}}^2 = \dot{\mathfrak{q}}^3 \cos\mathfrak{q}^4. \tag{90f}$$

Finally, Eqs. (90a)–(90f) are solved numerically using the "trapezium rule" in the case in which the physical and numerical parameters are the ones reported in Table 1.

Parameter	Value	Units	Description
R	11.625	mm	Radius of the coin
m	7.5	g	Mass of the coin
g	9.81	m/s^2	Gravitational acceleration
α	$\pi/6$	rad	Angle of the inclined plane
T	2	s	Total time of simulation
Δt	10^{-4}	s	Time-step

Table 1: Physical and numerical parameters chosen for the simulation of a standard "1 Euro" coin rolling down an infinitely long inclined plane

Parametric sweep of Ω

The first scenario we examine is the case of a coin, initially tilted towards the direction of steepest descent of the plane, i.e., $q^4_{\text{in}} = 0$, that rolls without slipping from the top of the inclined plane. Moreover, we consider the initial angular velocity $\nu^4_{\text{in}} \equiv \Omega$, and we study its effect on the evolution of the system. Then, we initialize the remaining generalized coordinates as $q^1_{\text{in}} = 0$, $q^2_{\text{in}} = 0$ and $q^3_{\text{in}} = 0$, and the Lagrange multipliers as $\lambda_{1\,\text{in}} = 0$ and $\lambda_{2\,\text{in}} = 0$. Further, we assume that $\nu^3_{\text{in}} = 0$, which implies that the initial velocities ν^1_{in} and ν^2_{in} are zero too, since they have to satisfy the constraints in Eqs. (61a) and (61b) at time $t = t_{\text{in}} = 0$.

In Fig. 1, we show the trajectories of the center of mass of the coin for different values of Ω, e.g. $\Omega \in \{\pi/2, \pi, 2\pi\}$ rad/s. We observe that gyroscopic effects arise when non-vanishing values of Ω are considered, as predicted by the analytical solution of the problem (see e.g. Section 3.3) [1, 14]. These effects, which are produced by the action of the constraints, are due to an oscillatory contribution characterized by angular frequency 2Ω, since, if we specialize the analytical solution to the case under examination, the coordinates of the center of mass of the coin read

$$q^1(t) \equiv \hat{q}^1(t;\Omega) = -\frac{1}{4}R\Phi(\Omega)\sin(2\Omega t) + \frac{1}{2}R\Omega\Phi(\Omega)t, \tag{91a}$$

$$q^2(t) \equiv \hat{q}^2(t;\Omega) = -\frac{1}{4}R\Phi(\Omega)\big[\cos(2\Omega t) - 1\big]. \tag{91b}$$

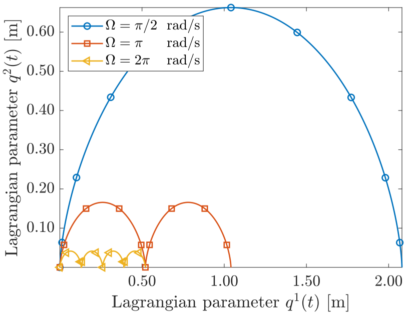

Figure 1: Trajectories of the center of mass of the coin for different values of $\Omega \in \{\pi/2, \pi, 2\pi\}$ rad/s simulated from $t_{\text{in}} = 0\,\text{s}$ to $t_{\text{fin}} = 2\,\text{s}$.

Note that, by expanding Eqs. (91a) and (91b) in a neighborhood of $\Omega = 0$, and for a fixed time $t \in [t_{\text{in}}, t_{\text{fin}}]$, we obtain, in the limit $\Omega \to 0$, that

$$q^1(t) \equiv \hat{q}^1(t;\,\Omega) = \tfrac{1}{3} R\Phi(\Omega)\Omega^3 t^3 + o(\Omega) = \tfrac{2}{9} g \sin\alpha\, \Omega t^3 + o(\Omega), \tag{92a}$$

$$q^2(t) \equiv \hat{q}^2(t;\,\Omega) = \tfrac{1}{2} R\Phi(\Omega)\Omega^2 t^2 + o(1) = \tfrac{1}{3} g \sin\alpha\, t^2 + o(1). \tag{92b}$$

Therefore, we find that $\hat{q}^1(t;\,0) = \lim_{\Omega\to 0} \hat{q}^1(t;\,\Omega)$, and that the leading term of $\hat{q}^2(t;\ \Omega)$ does not depend on Ω and grows quadratically in time in the considered neighborhood.

In conclusion, if $\Omega = 0$, the coin rolls downwards indefinitely and reaches the end of the inclined plane, when the latter one is finite. On the other hand, when $\Omega \neq 0$, the center of mass of the coin exhibits a behavior similar to the one of a "yo-yo", since it oscillates back and forth in the direction of the y-axis, while moving indefinitely along the x-axis, and the amplitude of the oscillations depends on Ω. Moreover, even for inclined planes whose slope is finite, yet sufficiently longer than the amplitude of the y-oscillations, there exist values of Ω such that the coin does not reach the end of the slope.

In Fig. 2, we have reported four plots describing the qualitative and quantitative behavior of the Lagrange multipliers λ_1 and λ_2 in the case in which Ω is either $\pi\,\text{rad/s}$ or $2\pi\,\text{rad/s}$. In particular, in Fig. 2a–2c, we introduce ρ_1 and ρ_2 as the "reactive forces" associated to q^1 and q^2, respectively, and ρ_3 the "reactive moment" associated to q^3, so that the corresponding pairs of "Lagrangian parameter – reactive force" could be compared. Note that, the expressions of ρ_1, ρ_2 and ρ_3 follow from Eqs. (71a)–(71c) and read

$$\rho_1 := \dot{\lambda}_1, \qquad \rho_2 := \dot{\lambda}_2, \qquad \rho_3 := -\dot{\lambda}_1 R \sin q^4 - \dot{\lambda}_2 R \cos q^4, \tag{93}$$

while, by virtue of Eq. (71d), the "reactive moment" associated to q^4, i.e., ρ_4, is identically zero.

In Fig. 2a we observe that in the instants of time in which the reactive force ρ_1 vanishes, the Lagrangian parameter q^1 exhibits, in the same instants, inflection points. This phenomenon is a direct consequence of Eq. (71e). Moreover, as one can see in Fig. 2b,c, for the instants of time in which q^2 or q^3 admit a local maximum, their corresponding reactive force and moment, i.e., ρ_2 and ρ_3, feature a local minimum, and vice versa. Indeed, this is in compliance with the analytical expressions of the Lagrange multipliers $\lambda_1(t)$ and $\lambda_2(t)$ featuring in Eqs. (82a) and (82b), and reported in Figure 2d, since, in the case we are considering, they read

$$\lambda_1(t) \equiv \hat{\lambda}_1(t;\Omega) = -\tfrac{1}{2} mR\Omega\Phi(\Omega)[\cos(2\Omega t) - 1], \tag{94a}$$

$$\lambda_2(t) \equiv \hat{\lambda}_2(t;\Omega) = \quad \tfrac{1}{2} mR\Omega\Phi(\Omega)\sin(2\Omega t) - (mg\sin\alpha)t, \tag{94b}$$

which, in turn, produce the following reactions

$$\rho_1(t) = \tfrac{2}{3} mg\sin\alpha\,\sin(2\Omega t), \tag{95a}$$

$$\rho_2(t) = \tfrac{2}{3} mg\sin\alpha\,\cos(2\Omega t) - mg\sin\alpha, \tag{95b}$$

$$\rho_3(t) = \tfrac{1}{3} mgR\sin\alpha\,\cos(\Omega t). \tag{95c}$$

Therefore, by comparing the expression in Eqs. (95a)–(95c) with Eqs. (91a), (91b) and (78), we conclude that:

(i) The constraint $\mathcal{V}^1$ has the effect to change the curvature of the trajectory of the center of mass of the coin, i.e., the second time derivative of $q^1(t)$.
(ii) The constraint $\mathcal{V}^2$ produces a reaction force on q^2 opposing to the motion that, in turn, determines the oscillatory behavior of the center of mass of the coin along the y-axis.

Effect of a non-zero θ_0

The second scenario we examine for the "rolling coin" problem concerns how considering non-zero values of $\theta_0 \equiv q^4_{\text{in}}$, for the fixed value of $\Omega = 2\pi\,\text{rad/s}$, affects the overall motion of the system. In particular, we will compare the solutions when the chosen value of θ_0 belongs to the set $\{0, \pi/2\}$ (see Fig. 3). Note that, the two cases above describe the situations in which the initial direction of rolling, indicated by the unit vector $\boldsymbol{\tau}$, is either aligned with the direction of steepest descent ($\theta_0 = 0$) or is orthogonal to it ($\theta_0 = \pi/2$). As in the study above, we initialize the remaining generalized coordinates as $q^1_{\text{in}} = 0$, $q^2_{\text{in}} = 0$ and $q^3_{\text{in}} = 0$, and the Lagrange multipliers as $\lambda_{1\,\text{in}} = 0$ and $\lambda_{2\,\text{in}} = 0$.

In Fig. 3, we observe that in the case in which $\theta_0 = 0\,\text{rad}$, the trajectory of the center of mass of the coin features an oscillatory motion that makes the center of mass move in the positive direction of the x-axis without ever going backward in the same direction. Instead, for $\theta_0 = \pi/2\,\text{rad}$, the center of mass of the coin experiences an oscillatory motion that moves the coin initially in the negative direction of the x-axis. Since this behavior is periodic in the motion, a knot is produced in the trajectory of the center of mass when $t \simeq 1\,\text{s}$.

Finally, from Fig. 4, we have compared, as done previously in Fig. 2, the pairs of "Lagrangian parameters – Reactive forces" that are obtained when Ω is

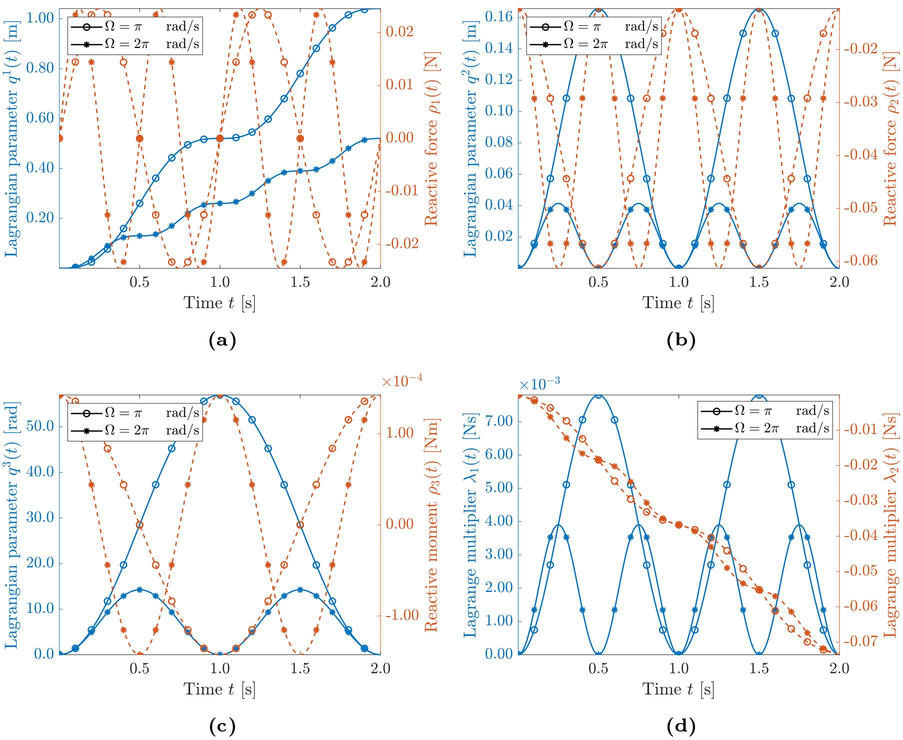

Figure 2: In 2a–2c the pairs "Lagrangian parameter – Reactive force" are represented by continuous and dotted lines, respectively. In Figure 2d, $\lambda_1(t)$ and $\lambda_2(t)$ are represented with a continuous and a dotted line, respectively.

fixed at 2π rad/s and $\theta_0 \in \{0, \pi/2\}$ rad. In this case, by looking at Fig. 4a–4c, we can draw similar conclusions to what we draw for the parametric study on Ω. Nevertheless, we acknowledge the fact that, when $\theta_0 \neq 0$, then the reactive forces ρ_1 and ρ_2 are determined by an apposition of oscillatory motion with different angular frequencies (see Fig. 4a–4b). On the other hand, as shown in Fig. 4c, ρ_3 is defined by only one cosinusoidal contribution that is shifted by the presence of a non-zero θ_0. Indeed, these phenomena are explained by the analytical expressions for the three generalized reactive forces, which, in the case of $t_{\text{in}} = 0$ and $\theta_0 \neq 0$, read as follows:

$$\rho_1(t) = \tfrac{2}{3} mg\,\sin\alpha \left[\sin(2\Omega t + 2\theta_0) - \sin\theta_0\,\cos(\Omega t + \theta_0)\right], \tag{96a}$$

$$\rho_2(t) = \tfrac{2}{3} mg\,\sin\alpha \left[\cos(2\Omega t + 2\theta_0) + \sin\theta_0\,\sin(\Omega t + \theta_0)\right] - mg\,\sin\alpha, \tag{96b}$$

$$\rho_3(t) = \tfrac{1}{3} mg\,\sin\alpha R\,\cos(\Omega t). \tag{96c}$$

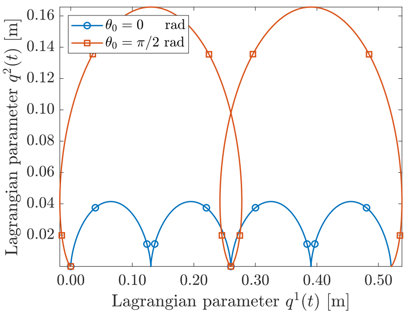

Figure 3: Trajectories of the center of mass of the coin for $\theta_0 \in \{0, \pi/2\}$ rad which are simulated from $t_{\text{in}} = 0$ s to $t_{\text{fin}} = 2$ s.

Note that the presence of non-zero θ_0 provides the same "reactive moment" $\rho_3(t)$, i.e., Eq. (96c), that we had in the case with $\theta_0 = 0$, i.e., Eq. (95c), thereby manifesting the independence of $\rho_3(t)$ with respect to θ_0.

4 The "charged skate" benchmark

In this section, we study the dynamic equations for the benchmark problem of the "non-holonomic skate" (see e.g. [2, 5, 6, 12, 26, 28, 43, 44, 47]), which we have modified by introducing an electromagnetic interaction (see also [27]). Also in this case, we employ the "modified vakonomic method" (MVM) with the objective of showing how an interaction of this type can "break" the equivalence between the MVM and the TNHM, thereby violating the hypothesis of Theorem 1. In particular, the role of the *Ansätze*, needed for the closure of Eqs. (42a) and (42b), is investigated in the sequel.

We consider a three-dimensional rigid skate $\mathfrak{S}$, shaped as a rectangular parallelepiped having mass m, length ℓ, and cross section of area σ^2, that slides over an inclined plane of an angle α with respect to a horizontal plane. We assume that the skate is electrically charged, with volumetric charge density e distributed uniformly in $\mathfrak{S}$, so that the total electric charge of the skate is $Q := e\ell\sigma^2$. Moreover, we let the skate interact with a field of magnetic induction associated with the (co-)vector potential $\boldsymbol{A}$ [24].

A coordinate system $\{O, (x, y, z)\}$ is prescribed such that the x-axis is parallel to the inclined plane and is aligned with the direction of steepest descent; the z-axis "exits" the inclined plane and is orthogonal to it; the y-axis is such that the x-, y- and z-axis form a right-handed triad; the origin O is fixed at the same z-coordinate as the center of mass of the skate, hereafter denoted by G. Furthermore, we denote by $\{\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z\}$ and $\{\boldsymbol{e}^x, \boldsymbol{e}^y, \boldsymbol{e}^z\}$ the basis unit vectors and co-vectors, respectively, associated with the coordinate system $\{O, (x, y, z)\}$.

By introducing $(x_G, y_G, 0)$ as the coordinates of G, and θ as the angle between the x-axis and the axis of the skate, we choose x_G, y_G and θ as the

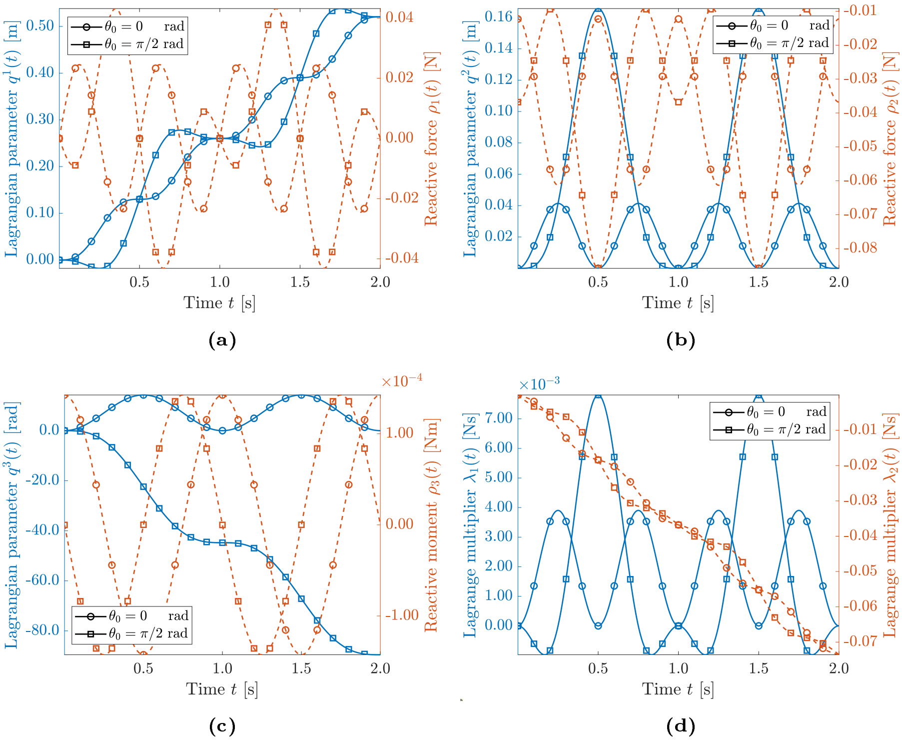

Figure 4: In 4a, 4b and 4c the pairs "Lagrangian parameter – Reactive force" are represented by continuous and dotted lines, respectively. In Fig. 4d, $\lambda_1(t)$ and $\lambda_2(t)$ are represented with a continuous and a dotted line, respectively.

$n = 3$ Lagrangian parameters of the mechanical system under study, thereby leading to the identification $q \equiv (q^1, q^2, q^3) \equiv (x_G, y_G, \theta)$ [12].

In the following, we will only consider the case in which the magnetic induction field $\boldsymbol{B} = \mathrm{curl}\boldsymbol{A}$ is homogeneous in space and orthogonal to the inclined plane, i.e., $\boldsymbol{B}(x, y, z, t) = B_0(t)\boldsymbol{e}_z$, so that the resulting Lorentz force, acting on the skate, is parallel to the inclined plane itself. To this end, we design the (co-)vector potential to be $\boldsymbol{A}(x, y, z, t) = [-\frac{1}{2}B_0(t)y]\boldsymbol{e}^x + [\frac{1}{2}B_0(t)x]\boldsymbol{e}^y$.

Given the premises above, and if, in addiction to the considered magnetic interactions, we also account for the gravitational interaction, then the Lagrangian of the skate reads [5, 6, 12, 43]

$$\begin{aligned} \mathcal{L} \circ (q, \dot{q}, \tau) = & \underbrace{\tfrac{1}{2}m\left[(\dot{q}^1)^2 + (\dot{q}^2)^2 + \tfrac{\ell^2+\sigma^2}{12}(\dot{q}^3)^2\right]}_{\mathcal{K}\circ\dot{q}} + \underbrace{(mg\,\sin\alpha)q^1}_{\mathcal{U}_\mathrm{g}\circ q} \\ & + \underbrace{\left[-\tfrac{QB_0}{2}q^2\right]\dot{q}^1 + \left[\tfrac{QB_0}{2}q^1\right]\dot{q}^2 + \left[\tfrac{QB_0(\ell^2+\sigma^2)}{24}\right]\dot{q}^3}_{\mathcal{U}_\mathrm{m}\circ(q,\dot{q},\tau)}, \end{aligned} \tag{97}$$

where $\mathcal{K}$ is the kinetic energy; $\mathcal{U}_{\mathrm{g}}$ and $\mathcal{U}_{\mathrm{m}}$ are the gravitational and magnetic potential functions, respectively; g is the magnitude of the gravity acceleration vector, and $B_0 \equiv B_0 \circ \tau$ is the magnitude of the magnetic induction field.

By denoting by $\boldsymbol{n} = \cos\theta \boldsymbol{e}_x + \sin\theta \boldsymbol{e}_y$ the unit vector aligned with the skate's axis, we assume the velocity of the center of mass of the skate, $\boldsymbol{v}_G$, to remain always parallel to $\boldsymbol{n}$. This condition, written in the physical space as $\boldsymbol{v}_G \times \boldsymbol{n} = \boldsymbol{0}$, prescribes the non-holonomic constraint to be [5, 6, 12, 43–45]

$$\mathcal{V}^1 \circ (q, \dot{q}) := (\sin q^3)\dot{q}^1 - (\cos q^3)\dot{q}^2 = 0. \tag{98}$$

Note that the absence of $\dot{q}^3$ in the constraint (98) is sufficient to conclude that the transpositional relation $\zeta^3 - \dot{\eta}^3$ must be zero, i.e., $\zeta^3 - \dot{\eta}^3 = \sum_{h=1}^{3} W^3{}_h \eta^h = 0$. In turn, this means that the coefficients $W^3{}_1$, $W^3{}_2$, and $W^3{}_3$ must be either identically zero or such that their combination with η^1, η^2, and η^3 is zero because of Lagrange-Chetaev's conditions (12).

4.1 The traditional non-holonomic approach

In this section, we compute the dynamic equations produced by the TNHM for the "charged skate" problem. Hence, we specify the Euler-Lagrange operators in Eq. (28a) for the Lagrangian in Eq. (97), i.e.,

$$\mathscr{E}_1\mathcal{L} \circ \sharp^{(2)} = -m\ddot{q}^1 + QB_0\dot{q}^2 + \tfrac{1}{2}Q\dot{B}_0 q^2 + mg\sin\alpha, \tag{99a}$$
$$\mathscr{E}_2\mathcal{L} \circ \sharp^{(2)} = -m\ddot{q}^2 - QB_0\dot{q}^1 - \tfrac{1}{2}Q\dot{B}_0 q^1, \tag{99b}$$
$$\mathscr{E}_3\mathcal{L} \circ \sharp^{(2)} = -\tfrac{1}{12}m(\ell^2 + \sigma^2)\ddot{q}^3 - \tfrac{1}{24}Q\dot{B}_0(\ell^2 + \sigma^2), \tag{99c}$$

and, by substituting Eqs. (99a)–(99c) as well as the constraint (98) into Eqs. (52a) and (52b), we find the TNHM dynamic equations for the "charged skate" problem to be as follows:

$$m\ddot{q}^1 - \mu_1 \sin q^3 = mg\sin\alpha + QB_0\dot{q}^2 + \tfrac{1}{2}Q\dot{B}_0 q^2, \tag{100a}$$
$$m\ddot{q}^2 + \mu_1 \cos q^3 = -QB_0\dot{q}^1 - \tfrac{1}{2}Q\dot{B}_0 q^1, \tag{100b}$$
$$\tfrac{1}{12}m(\ell^2 + \sigma^2)\ddot{q}^3 = -\tfrac{1}{24}Q(\ell^2 + \sigma^2)\dot{B}_0, \tag{100c}$$
$$(\sin q^3)\dot{q}^1 - (\cos q^3)\dot{q}^2 = 0. \tag{100d}$$

We notice that Eqs. (100a)–(100c) can be recast in a more suggestive form by rewriting them in terms of the canonical momenta:

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\left[m\dot{q}^1 - \tfrac{1}{2}QB_0 q^2\right]}_{=:p_1} = \mu_1 \sin q^3 + mg\sin\alpha + \tfrac{1}{2}QB_0\dot{q}^2, \tag{101a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\left[m\dot{q}^2 + \tfrac{1}{2}QB_0q^1\right]}_{=:p_2} = -\mu_1\,\mathrm{cos}q^3 - \tfrac{1}{2}QB_0\dot{q}^1, \tag{101b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\left[\tfrac{1}{12}m[\ell^2+\sigma^2]\dot{q}^3 + \tfrac{1}{24}Q[\ell^2+\sigma^2]B_0\right]}_{=:p_3} = 0. \tag{101c}$$

Note also that, since the Lagrangian parameter q^3 is *"ignorable"* [13], Eq. (101c) is equivalent to state the conservation of the generalized momentum conjugated with q^3, which, thus, turns out to be an integral of the motion, i.e.,

$$p_3(t) = \frac{\partial\mathcal{L}}{\partial\dot{q}^3}(\sharp^{(1)}(t)) = \tfrac{1}{12}m[\ell^2+\sigma^2]\dot{q}^3(t) + \tfrac{1}{24}[\ell^2+\sigma^2]QB_0(t) = C, \tag{102}$$

with C being an integration constant.

4.2 The "modified vakonomic" approach

Now, we apply the procedure reported in Sect. 2 to the problem under study. Hence, we solve Eqs. (42a) and (42b) in compliance with the *solvability conditions* (40a) and (40b), for the case in which the Lagrangian function and the constraint are of the type specified in Eqs. (97) and (98), respectively.

To this end, we express the conditions in Eq. (34), for $k \in \{1,2,3\}$ and with $\alpha = 1$, in the case in which the considered constraint is the one reported in Eq. (98):

$$\mathscr{D}_1\mathcal{V}^1 \circ \sharp^{(2)} = (\mathrm{sin}q^3)W^1{}_1 - (\mathrm{cos}q^3)W^2{}_1 - (\mathrm{cos}q^3)\dot{q}^3 = 0, \tag{103a}$$

$$\mathscr{D}_2\mathcal{V}^1 \circ \sharp^{(2)} = (\mathrm{sin}q^3)W^1{}_2 - (\mathrm{cos}q^3)W^2{}_2 - (\mathrm{sin}q^3)\dot{q}^3 = 0, \tag{103b}$$

$$\mathscr{D}_3\mathcal{V}^1 \circ \sharp^{(2)} = (\mathrm{sin}q^3)W^1{}_3 - (\mathrm{cos}q^3)W^2{}_3 + (\mathrm{cos}q^3)\dot{q}^1 + (\mathrm{sin}q^3)\dot{q}^2 = 0. \tag{103c}$$

To find all the coefficients $W^h{}_k$, we proceed as follows:

4.2.1 Case A: Llibre et al.'s auxiliary functions [12]

In the sequel, we take the same auxiliary functions as those suggested by Llibre et al. [12] for the case of the "non-holonomic skate" without magnetic field (see also [27] for comparison). However, we flip their order from the one reported in [12], thereby writing

$$\mathcal{F}^2 \circ \sharp^{(1)} := \dot{q}^3, \tag{104a}$$

$$\mathcal{F}^3 \circ \sharp^{(1)} := (\mathrm{cos}q^3)\dot{q}^1 + (\mathrm{sin}q^3)\dot{q}^2, \tag{104b}$$

to emphasize that the choice of these functions must respect the criterion given in *Ansatz 1*. In particular, this means that they must respect the transpositional relation $\zeta^3 - \dot{\eta}^3 = \sum_{h=1}^{3} W^3{}_h\eta^h = 0$, and, indeed, $\mathcal{F}^2 \circ \sharp^{(1)} := \dot{q}^3$ ensures the fulfillment of this condition in *strong* way, as can be seen by the equations

$$\mathscr{D}_1\mathcal{F}^2 \circ \sharp^{(2)} = W^3{}_1 = 0, \tag{105a}$$
$$\mathscr{D}_2\mathcal{F}^2 \circ \sharp^{(2)} = W^3{}_2 = 0, \tag{105b}$$
$$\mathscr{D}_3\mathcal{F}^2 \circ \sharp^{(2)} = W^3{}_3 = 0. \tag{105c}$$

Conversely, the auxiliary function in Eq. (104b) leads to the conditions

$$\mathscr{D}_1\mathcal{F}^3 \circ \sharp^{(2)} = (\cos q^3)W^1{}_1 + (\sin q^3)W^2{}_1 + (\sin q^3)\dot{q}^3 = 0, \tag{106a}$$
$$\mathscr{D}_2\mathcal{F}^3 \circ \sharp^{(2)} = (\cos q^3)W^1{}_2 + (\sin q^3)W^2{}_2 - (\cos q^3)\dot{q}^3 = 0, \tag{106b}$$
$$\mathscr{D}_3\mathcal{F}^3 \circ \sharp^{(2)} = (\cos q^3)W^1{}_3 + (\sin q^3)W^2{}_3 - \underbrace{[(\sin q^3)\dot{q}^1 - (\cos q^3)\dot{q}^2]}_{=0} = 0. \tag{106c}$$

We remark that, as anticipated in the *Ansatz 1* of Sect. 2.3, Eq. (104b) identifies $\mathcal{F}^3$ with the component of the projection of $\boldsymbol{v}_G$ onto the unit vector $\boldsymbol{n}$, namely $\mathcal{F}^3 \circ \sharp^{(1)} \equiv \boldsymbol{v}_G \cdot \boldsymbol{n}$, which is the geometric interpretation of a quasi-velocity.

By solving the linear algebraic system in Eqs. (103a)–(103c), (105a)–(105c), and (106a)–(106c) for the unknown coefficients $W^h{}_k$, we have [12]

$$\mathsf{W} = \begin{bmatrix} 0 & \dot{q}^3 & -\dot{q}^2 \\ -\dot{q}^3 & 0 & \dot{q}^1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{107}$$

Hence, by substituting Eqs. (99a)–(99c) and (107) into Eqs. (42a) and (42b), we obtain that the dynamic equations returned by the MVM for the "charged skate" problem read

$$m\ddot{q}^1 - \dot{\lambda}_1 \sin q^3 = mg\sin\alpha - m\dot{q}^2\dot{q}^3 + QB_0\dot{q}^2 + \tfrac{1}{2}Q\dot{B}_0q^2 - \tfrac{1}{2}QB_0q^1\dot{q}^3, \tag{108a}$$
$$m\ddot{q}^2 + \dot{\lambda}_1 \cos q^3 = m\dot{q}^1\dot{q}^3 - QB_0\dot{q}^1 - \tfrac{1}{2}Q\dot{B}_0q^1 - \tfrac{1}{2}QB_0q^2\dot{q}^3, \tag{108b}$$
$$\tfrac{1}{12}m(\ell^2 + \sigma^2)\ddot{q}^3 = -\tfrac{1}{24}Q(\ell^2 + \sigma^2)\dot{B}_0 + \tfrac{1}{2}QB_0(q^1\dot{q}^1 + q^2\dot{q}^2), \tag{108c}$$
$$(\sin q^3)\dot{q}^1 - (\cos q^3)\dot{q}^2 = 0. \tag{108d}$$

For ease of comparison with the TNHM, we rewrite Eqs. (108a)–(108c) by highlighting the canonical momenta, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\left[m\dot{q}^1 - \tfrac{1}{2}QB_0q^2\right]}_{=:p_1} = \dot{\lambda}_1 \sin q^3 + mg\sin\alpha + \tfrac{1}{2}QB_0\dot{q}^2 - p_2\dot{q}^3, \tag{109a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\left[m\dot{q}^2+\tfrac{1}{2}QB_0q^1\right]}_{=:p_2}=-\dot{\lambda}_1\cos q^3-\tfrac{1}{2}QB_0\dot{q}^1+p_1\dot{q}^3, \tag{109b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\left[\tfrac{1}{12}m[\ell^2+\sigma^2]\dot{q}^3+\tfrac{1}{24}Q[\ell^2+\sigma^2]B_0\right]}_{=:p_3}=\underbrace{\tfrac{1}{2}QB_0[q^1\dot{q}^1+q^2\dot{q}^2]}_{=-p_1\dot{q}^2+p_2\dot{q}^1}. \tag{109c}$$

From Eq. (109c), we notice that the generalized momentum p_3 is not conserved with the functions $\mathcal{F}^2$ and $\mathcal{F}^3$ of Eqs. (104a) and (104b) characterizing *Ansatz 1* for this problem. The criticality with this choice of functions is that they do not guarantee the equivalence between the MVM and the TNHM, i.e., the characterizing condition of Theorem 1 in Eq. (55a) is not satisfied. Indeed, such conditon would require the vanishing of the quantities

$$(\cos q^3)(\mathsf{W}^{\mathrm{T}}\mathsf{p})_1+(\sin q^3)(\mathsf{W}^{\mathrm{T}}\mathsf{p})_2 \qquad \text{and} \qquad (\mathsf{W}^{\mathrm{T}}\mathsf{p})_3, \tag{110}$$

which, in the case considered, i.e., with W as in Eq. (107), are instead

$$(\cos q^3)(\mathsf{W}^{\mathrm{T}}\mathsf{p})_1+(\sin q^3)(\mathsf{W}^{\mathrm{T}}\mathsf{p})_2=-\tfrac{1}{2}QB_0[q^1\cos q^3+q^2\sin q^3]\dot{q}^3\neq 0, \tag{111a}$$

$$(\mathsf{W}^{\mathrm{T}}\mathsf{p})_3=\tfrac{1}{2}QB_0[q^1\dot{q}^1+q^2\dot{q}^2]\neq 0, \tag{111b}$$

where the right-hand sides of Eqs. (111a) and (111b) stem from the terms making the momenta *affine* in the velocities.

Hence, if we take the same auxiliary functions as in [12] for the problem at hand, then, although the transpositional relations are maintained, the MVM is *not equivalent* to the TNHM, since Eq. (55a) of Theorem 1 is not satisfied, and a conservation law, which should exist, is lost.

Remark 7 (*"Canonical flip"* [22, 41] for the Case A of the "charged skate") Similarly to what has been done in Remark 5, the matrix W in Eq. (107) allows to choose the coefficients $\mathcal{C}^h{}_{\ell k}$ of the associated "Canonical flip" as follows

$$\mathcal{C}^1{}_{32}\dot{q}^3=W^1{}_2=+\dot{q}^3 \qquad \Rightarrow \qquad \mathcal{C}^1{}_{32}\equiv\hat{\mathcal{C}}^1{}_{32}\circ q=+1, \tag{112a}$$

$$\mathcal{C}^1{}_{23}\dot{q}^2=W^1{}_3=-\dot{q}^2 \qquad \Rightarrow \qquad \mathcal{C}^1{}_{23}\equiv\hat{\mathcal{C}}^1{}_{23}\circ q=-1, \tag{112b}$$

$$\mathcal{C}^2{}_{31}\dot{q}^3=W^2{}_1=-\dot{q}^3 \qquad \Rightarrow \qquad \mathcal{C}^2{}_{31}\equiv\hat{\mathcal{C}}^2{}_{31}\circ q=-1, \tag{112c}$$

$$\mathcal{C}^2{}_{13}\dot{q}^1=W^2{}_3=+\dot{q}^1 \qquad \Rightarrow \qquad \mathcal{C}^2{}_{13}\equiv\hat{\mathcal{C}}^2{}_{13}\circ q=+1. \tag{112d}$$

As for the "rolling coin", all the other entries of $\mathcal{C}^h{}_{\ell k}$ can be set equal to zero.

Remark 8 (The "charged skate" in the case of momenta linear in the velocities) The fact that, with the matrix W of Eq. (107), the results (111a) and (111b) spoil the fulfillment of Theorem 1, and make, thus, the MVM not equivalent to the TNHM, is a direct consequence of the last three summands of the Lagrangian function (97), which correspond to $\sum_{h=1}^{3}[Z_h\circ(q,\tau)]\dot{q}^h$, and render the momenta affine (rather than linear) in the generalized velocities. However, if we follow the approach presented in Sect. 2.6, so that we deal with Eqs. (59a) and (59b), the dynamic equations (108a)–(108d) for the charged non-holonomic skate with the magnetic interactions regarded as (pseudo-)polygenic [13] become

$$m\ddot{q}^1 - \dot{\lambda}_1 \sin q^3 = \underbrace{-m\dot{q}^2\dot{q}^3}_{=(\mathsf{W}^\mathrm{T}\mathsf{p}_0)_1} + \underbrace{[QB_0\dot{q}^2 + \tfrac{1}{2}Q\dot{B}_0q^2]}_{=\mathcal{Q}_1} + mg\sin\alpha, \tag{113a}$$

$$m\ddot{q}^2 + \dot{\lambda}_1 \cos q^3 = \underbrace{+m\dot{q}^1\dot{q}^3}_{=(\mathsf{W}^\mathrm{T}\mathsf{p}_0)_2} + \underbrace{[-QB_0\dot{q}^1 - \tfrac{1}{2}Q\dot{B}_0q^1]}_{=\mathcal{Q}_2}, \tag{113b}$$

$$\tfrac{1}{12}m(\ell^2+\sigma^2)\ddot{q}^3 = \underbrace{0}_{=(\mathsf{W}^\mathrm{T}\mathsf{p}_0)_3} + \underbrace{[-\tfrac{1}{24}Q(\ell^2+\sigma^2)\dot{B}_0]}_{=\mathcal{Q}_3}, \tag{113c}$$

$$(\sin q^3)\dot{q}^1 - (\cos q^3)\dot{q}^2 = 0. \tag{113d}$$

Accordingly, the condition (55a) of Theorem 1, *instead of* Eqs. (111a) and (111b), produces

$$\begin{aligned}(\cos q^3)(\mathsf{W}^\mathrm{T}\mathsf{p}_0)_1 + (\sin q^3)(\mathsf{W}^\mathrm{T}\mathsf{p}_0)_2 &= -(\cos q^3)m\dot{q}^2\dot{q}^3 + (\sin q^3)m\dot{q}^3\dot{q}^1 \\ &= m\dot{q}^3[(\sin q^3)\dot{q}^1 - (\cos q^3)\dot{q}^2] = 0, \end{aligned}\tag{114a}$$

$$(\mathsf{W}^\mathrm{T}\mathsf{p}_0)_3 = -\dot{q}^2 m\dot{q}^1 + \dot{q}^1 m\dot{q}^2 = 0, \tag{114b}$$

with the right-hand side of Eq. (114a) being null by virtue of the constraint. Therefore, within the present formulation, Eq. (55a) of Theorem 1 is automatically satisfied, and Eq. (55b) follows by working out the terms $\dot{\lambda}_1 \sin q^3 - m\dot{q}^2\dot{q}^3$ and $-\dot{\lambda}_1 \cos q^3 + m\dot{q}^1\dot{q}^3$.

4.2.2 Case B: Direct use of Theorem 1

In addition to what has been done in Remark 8, we may also reason in a different way, which constitutes the core of *Ansatz 2*. Specifically, to compute W, we adopt the conditions supplied in Eqs. (103a)–(103c) and (105a)–(105c), which stem from the constraint and from the use of $\mathcal{F}^2 \circ \sharp^{(1)} = \dot{q}^3$, respectively, and, in lieu of introducing $\mathcal{F}^3$, we invoke directly Theorem 1, thereby requiring the vanishing of the quantities in Eq. (110):

$$\begin{aligned}&(\cos q^3)(\mathsf{W}^\mathrm{T}\mathsf{p})_1 + (\sin q^3)(\mathsf{W}^\mathrm{T}\mathsf{p})_2 \\ &= p_1 \cos q^3 W^1{}_1 + p_1 \sin q^3 W^1{}_2 + p_2 \cos q^3 W^2{}_1 + p_2 \sin q^3 W^2{}_2 = 0, \end{aligned}\tag{115a}$$

$$(\mathsf{W}^\mathrm{T}\mathsf{p})_3 = p_1 W^1{}_3 + p_2 W^2{}_3 = 0. \tag{115b}$$

This guarantees the equivalence between the MVM and the TNHM.

It is important to remark that the conditions (115a) and (115b) amount to requiring that the vector associated with $\mathsf{W}^\mathrm{T}\mathsf{p}$ lies on the plane on which the skate's motion takes place and *is orthogonal* to the skate's axis. Thus, the plane projection of this vector is orthogonal also to the velocity of the skate's center of mass $\boldsymbol{v}_G$, so that the force $\mathsf{W}^\mathrm{T}\mathsf{p}$ produces no power on $\boldsymbol{v}_G$. Moreover, by virtue of Eq. (115b), $\mathsf{W}^{\mathrm{T}}\mathsf{p}$ produces no source/sink of momentum for p_3, so that it does not spoil its conservation.

The eight conditions (103a)–(103c), (105a)–(105c), (115a) and (115b) permit to determine the nine coefficients of the resulting matrix W up to an arbitrary function ϱ, i.e.,

$$W^1{}_1 = \varrho, \tag{116a}$$

$$W^1{}_2 = \frac{p_2\dot{q}^3}{\sin q^3[p_1\cos q^3 + p_2\sin q^3]} - \varrho\cot q^3, \tag{116b}$$

$$W^1{}_3 = -p_2\frac{\dot{q}^1\cos q^3 + \dot{q}^2\sin q^3}{p_1\cos q^3 + p_2\sin q^3}, \tag{116c}$$

$$W^2{}_1 = -\dot{q}^3 + \varrho\tan q^3, \tag{116d}$$

$$W^2{}_2 = -\dot{q}^3\frac{p_1\sin q^3 - p_2\cos q^3}{p_1\cos q^3 + p_2\sin q^3} - \varrho, \tag{116e}$$

$$W^2{}_3 = p_1\frac{\dot{q}^1\cos q^3 + \dot{q}^2\sin q^3}{p_1\cos q^3 + p_2\sin q^3}, \tag{116f}$$

$$W^3{}_1 = W^3{}_2 = W^3{}_3 = 0. \tag{116g}$$

Hence, the equations of motion (42a) and (42b) take on the form

$$\dot{p}_1 = \dot{\lambda}_1\sin q^3 + mg\sin\alpha + \tfrac{1}{2}QB_0\dot{q}^2 - p_2\dot{q}^3 + \varrho\big[p_1 + p_2\tan q^3\big], \tag{117a}$$

$$\dot{p}_2 = -\dot{\lambda}_1\cos q^3 - \tfrac{1}{2}QB_0\dot{q}^1 + p_2\dot{q}^3\cot q^3 - \varrho\big[p_2 + p_1\cot q^3\big], \tag{117b}$$

$$\dot{p}_3 = 0. \tag{117c}$$

Thus, as predicted by Theorem 1, the MVM is equivalent to the TNHM, provided the following identification of the Lagrange multipliers is made:

$$\mu_1 \equiv \dot{\lambda}_1 - p_2\dot{q}^3\csc q^3 + \varrho\big[p_1\csc q^3 + p_2\sec q^3\big]. \tag{118}$$

Note that, because of the equivalence between the two methods, the present reformulation of the MVM conserves the momentum p_3, that is, Eq. (117c) is identical to Eq. (101c). In this respect, it should also be noticed that, even though Eqs. (117a)–(117c) apparently depend on ϱ, which is still unknown, this dependence is not effective. Indeed, if the MVM has to be equivalent to the TNHM, this dependence cancels out by virtue of Eq. (55a) of Theorem 1 when the equations of motion are put in the form (51a). This means that, if the equivalence between the MVM and the TNHM is maintained, the determination of the motion does not require the complete knowledge of W and, thus, of ϱ. However, as predicted by Eq. (51b), the Lagrange multiplier λ_1 does depend on ϱ, thereby yielding a one-parameter family of solutions that are *all* equivalent to those obtained by the TNHM and lead to one, and only one, μ_1. Clearly, as anticipated in *Ansatz 2*, ϱ can be determined through a physics-based condition, or by adopting Corollary 1, which requires $\mathsf{W}^{\mathrm{T}}\mathsf{p} = \mathsf{0}$. In particular, the latter case retrieves the condition pointed out by Ramírez and Sadovskaia [11], and, later, by Llibre et al. [12] (see, in particular, Theorem 3 of [12]).

Before closing this section, we find it convenient to summarize the results that we deem particularly noteworthy in the following Remarks.

Remark 9 (Consequences of the adopted methodology)
Transpositional relations. Given the matrix W, whose coefficients are reported in Eqs. (116a)–(116g), the transpositional relations characterizing the problem at hand are

$$\zeta^1 - \dot{\eta}^1 = \frac{p_2 \dot{q}^3}{\operatorname{sin}q^3 \left[p_1 \operatorname{cos}q^3 + p_2 \operatorname{sin}q^3\right]}\eta^2 - \frac{p_2 \dot{q}^2}{\operatorname{sin}q^3 \left[p_1 \operatorname{cos}q^3 + p_2 \operatorname{sin}q^3\right]}\eta^3, \tag{119a}$$

$$\zeta^2 - \dot{\eta}^2 = -\dot{q}^3\eta^1 + \frac{p_2 \operatorname{cos}q^3 - p_1 \operatorname{sin}q^3}{p_1 \operatorname{cos}q^3 + p_2 \operatorname{sin}q^3}\dot{q}^3\eta^2 + \frac{\dot{q}^1 \operatorname{cos}q^3 + \dot{q}^2 \operatorname{sin}q^3}{p_1 \operatorname{cos}q^3 + p_2 \operatorname{sin}q^3}p_1\eta^3, \tag{119b}$$

$$\zeta^3 - \dot{\eta}^3 = 0. \tag{119c}$$

It is interesting to notice that the unknown parameter ϱ featuring in some coefficients of W, i.e., in Eqs. (116a), (116b), (116d) and (116e), does not enter the transpositional relations in Eqs. (119a)–(119c) because of the *Lagrange-Chetaev condition* in Eq. (12).
Indetermination of the matrix W. Our formulation of the MVM according to *Ansatz 2*, which renounces to one of the auxiliary functions, and invokes directly Theorem 1, does not determine univocally all the 9 entries of W. This is testified by the independent unknown ϱ, and follows from the fact that the conditions delivered by Eq. (55a) are only two (difference between the total number of Lagrangian parameters, i.e., 3, and number of constraints, i.e., 1). Even though this could seem to be a deficiency of our approach with respect to the one developed in [12], one can assign or determine ϱ through other conditions, e.g. physically inspired, as suggested by *Ansatz 2*.

Remark 10 (*"Canonical flip"* [22, 41] for the Case B of the "charged skate")
Although the case analyzed here shares some similarities with Remark 7, some changes arise, which are worth of being investigated. This time, we start looking at Eqs. (119a)–(119c), and we compare them with the relationships (21). To this end, we notice that Eqs. (119a) and (119b) can be rewritten as

$$\zeta^1 - \dot{\eta}^1 = \frac{p_2 \operatorname{csc}q^3}{p_1 \operatorname{cos}q^3 + p_2 \operatorname{sin}q^3}\dot{q}^3\eta^2 - \frac{p_2 \operatorname{csc}q^3}{p_1 \operatorname{cos}q^3 + p_2 \operatorname{sin}q^3}\dot{q}^2\eta^3, \tag{120a}$$

$$\begin{aligned}\zeta^2 - \dot{\eta}^2 = &-\frac{p_1 \operatorname{cos}q^3}{p_1 \operatorname{cos}q^3 + p_2 \operatorname{sin}q^3}\dot{q}^3\eta^1 + \frac{p_1 \operatorname{cos}q^3}{p_1 \operatorname{cos}q^3 + p_2 \operatorname{sin}q^3}\dot{q}^1\eta^3 \\ &- \frac{p_1 \operatorname{sin}q^3}{p_1 \operatorname{cos}q^3 + p_2 \operatorname{sin}q^3}\dot{q}^3\eta^2 + \frac{p_1 \operatorname{sin}q^3}{p_1 \operatorname{cos}q^3 + p_2 \operatorname{sin}q^3}\dot{q}^2\eta^3,\end{aligned} \tag{120b}$$

where, to obtain Eq. (120b), we have made use of the Lagrange-Chetaev condition $(\operatorname{sin}q^3)\eta^1 - (\operatorname{cos}q^3)\eta^2 = 0$. Hence, a direct inspection yields

$$\mathcal{C}^1{}_{32} \equiv \hat{\mathcal{C}}^1{}_{32} \circ (q,p) = \frac{p_2 \operatorname{csc}q^3}{p_1 \operatorname{cos}q^3 + p_2 \operatorname{sin}q^3} = -\mathcal{C}^1{}_{23} \equiv -\hat{\mathcal{C}}^1{}_{23} \circ (q,p), \tag{121a}$$

$$\mathcal{C}^2{}_{31} \equiv \hat{\mathcal{C}}^2{}_{31} \circ (q,p) = -\frac{p_1 \operatorname{cos}q^3}{p_1 \operatorname{cos}q^3 + p_2 \operatorname{sin}q^3} = -\mathcal{C}^2{}_{13} \equiv -\hat{\mathcal{C}}^2{}_{13} \circ (q,p), \tag{121b}$$

$$\mathcal{C}^2{}_{32} \equiv \hat{\mathcal{C}}^2{}_{32} \circ (q, p) = -\frac{p_1 \sin q^3}{p_1 \cos q^3 + p_2 \sin q^3} = -\mathcal{C}^2{}_{23} \equiv -\hat{\mathcal{C}}^2{}_{23} \circ (q, p), \qquad (121c)$$

whereas all the other coefficients of $\mathcal{C}^h{}_{\ell k}$ can be set equal to zero. It should be noticed that, in the present situation, the nonzero coefficients reported in Eqs. (121a)–(121c) are functions of *both* the Lagrangian parameters q *and* of the generalized momenta p. This result is, in fact, a consequence of the Lagrangian function featuring a term linear in the velocities, which, in turn, renders the momenta affine functions of the velocities themselves, and is in harmony with the functional dependence prescribed in [11, 12] for the entries of the matrix W. We emphasize that, in Eqs. (121a)–(121c), the identifications of the functions $\mathcal{C}^h{}_{\ell k}$ is *purely* formal, since we have not determined them through the assignment of a set of quasi-velocities.

Remark 11 (Absence of electromagnetic interactions [12])
If the interaction with the magnetic field is switched off (i.e., if $B_0(t) = 0$ is zero at all times), Eqs. (109a)–(109c), as well as Eqs. (117a)–(117c) upon setting $\varrho = 0$, simplify to the case already addressed in [12]. In particular, we obtain the same dynamic equations found by Llibre et al. [12] by employing W as in Eq. (107) for the "non-holonomic skate" problem, i.e., [12]

$$\dot{p}_1 \equiv m\ddot{q}^1 = -m\dot{q}^2\dot{q}^3 + mg\sin\alpha + \dot{\lambda}_1 \sin q^3, \qquad (122a)$$

$$\dot{p}_2 \equiv m\ddot{q}^2 = \quad m\dot{q}^1\dot{q}^3 - \dot{\lambda}_1 \cos q^3, \qquad (122b)$$

$$\dot{p}_3 = 0. \qquad (122c)$$

Moreover, the matrix W with coefficients in Eqs. (116a)–(116g) trivially reduces to the one in Eq. (107). Hence, by removing the magnetic interaction, the equivalence between the MVM and the TNHM is restored again for Case A, while, in Case B, the equivalence was already present even with the magnetic interaction. In both cases, the assumptions $B_0 \equiv 0$ and $\varrho = 0$ induce the same relation between the Lagrange multipliers of the two methods, which reads

$$\mu_1 \equiv \dot{\lambda}_1 - m\dot{q}^2\dot{q}^3 \csc q^3 = \dot{\lambda}_1 - m\big[\dot{q}^1 \cos q^3 + \dot{q}^2 \sin q^3\big]\dot{q}^3, \qquad (123)$$

where the last equality is obtained by working out $\csc q^3$ and employing the constraint. Finally, the absence of the magnetic interaction allows the transpositional relations to simplify to the ones found in [12], i.e.,

$$\zeta^1 - \dot{\eta}^1 = \dot{q}^3\eta^2 - \dot{q}^2\eta^3, \qquad \zeta^2 - \dot{\eta}^2 = \dot{q}^1\eta^3 - \dot{q}^3\eta^1, \qquad \zeta^3 - \dot{\eta}^3 = 0. \qquad (124)$$

4.2.3 Case C: New formulation of the constraint

Looking at the relationship connecting the generalized velocities and the momenta, the constraint (98), the expression of the functions $\mathcal{F}^2$ and $\mathcal{F}^3$ supplied by Llibre et al. [12], and the matrix W of Eq. (107), we notice that there exists a sort of "natural pattern" among all these characteristic features of the considered problem. Indeed, by referring to Case A, both in Eq. (98) and in $\mathcal{F}^3$ there appear the same *constrained velocities*, whereas $\mathcal{F}^2$ involves the "unconstrained velocity" $\dot{q}^3$. Moreover, the structure of W is such that: its first 2×2 block is skew-symmetric in $\dot{q}^3$; its last column features the components of the vector $\boldsymbol{e}_3 \times \boldsymbol{v}_G$, which has zero projection onto the skate's unit vector $\boldsymbol{n}$, so that the mixed product $(\boldsymbol{e}_3 \times \boldsymbol{v}_G) \cdot \boldsymbol{n}$ vanishes, thereby returning the constraint; and its last row is null. These results render the MVM equivalent to the TNHM in the absence of the magnetic field, as highlighted in Remark 11, although they lead to a loss of equivalence in the presence of the magnetic field, as made evident in Eqs. (109a)–(109c) and in the subsequent discussion. It was indeed this broken equivalence, and the need for restoring it, that made us "re-design" W to obtain the matrix in Eqs. (116a)–(116g) under the guidance of Theorem 1. However, this required to renounce to the "natural pattern" mentioned above. Yet, this pattern can be recovered by redefining the constraint in such a way that the *effective velocities* of the skate, i.e., p_1/m and p_2/m (or $(\mathsf{M}^{-1}\mathsf{p})_1$ and $(\mathsf{M}^{-1}\mathsf{p})_2$, in matrix notation), rather than $\dot{q}^1$ and $\dot{q}^2$, are constrained. Clearly, this amounts to modifying the original problem, but in a still physically sound way, so as to account for the velocity shift induced by the magnetic field by passing from a constraint linear in the velocities $\dot{q}^1$ and $\dot{q}^2$ to one *affine* in these velocities (see Eq. (125) below).

By virtue of the discussion above, and setting $\chi := \frac{QB_0}{2m}$, we introduce

$$\begin{aligned}\mathcal{V}^1_{\text{new}} \circ \sharp^{(1)} &= (\sin q^3)\tfrac{p_1}{m} - (\cos q^3)\tfrac{p_2}{m}\\ &= \sin q^3\big[\dot{q}^1 - \chi q^2\big] - \cos q^3\big[\dot{q}^2 + \chi q^1\big]\\ &= (\sin q^3)\dot{q}^1 - (\cos q^3)\dot{q}^2 - \chi\big[(\sin q^3)q^2 + (\cos q^3)q^1\big] = 0. \end{aligned} \tag{125}$$

The new constraint expressed in Eq. (125) suggests that the "natural pattern" discussed above can be recovered by defining also the auxiliary functions in terms of the effective velocities, i.e.,

$$\mathcal{F}^2_{\text{new}} \circ \sharp^{(1)} = \underbrace{\dot{q}^3 + \chi}_{=:\,12p_3/m[\ell^2+\sigma^2]}, \tag{126a}$$

$$\mathcal{F}^3_{\text{new}} \circ \sharp^{(1)} = \cos q^3\,\underbrace{\big[\dot{q}^1 - \chi q^2\big]}_{=:\,p_1/m} + \sin q^3\,\underbrace{\big[\dot{q}^2 + \chi q^1\big]}_{=:\,p_2/m}. \tag{126b}$$

Accordingly, the new solvability conditions read

$$\mathscr{D}_1\mathcal{V}^1_{\text{new}} \circ \sharp^{(2)} = (\sin q^3)W^1{}_1 - (\cos q^3)W^2{}_1 - (\cos q^3)\big[\dot{q}^3 + \chi\big] = 0, \tag{127a}$$

$$\mathscr{D}_2\mathcal{V}^1_{\text{new}} \circ \sharp^{(2)} = (\sin q^3)W^1{}_2 - (\cos q^3)W^2{}_2 - (\sin q^3)\big[\dot{q}^3 + \chi\big] = 0, \tag{127b}$$

$$\mathscr{D}_3\mathcal{V}^1_{\text{new}} \circ \sharp^{(2)} = (\sin q^3)W^1{}_3 - (\cos q^3)W^2{}_3 + (\cos q^3)\tfrac{p_1}{m} + (\sin q^3)\tfrac{p_2}{m} = 0, \tag{127c}$$

$$\mathscr{D}_1\mathcal{F}^2_{\text{new}} \circ \sharp^{(2)} = W^3{}_1 = 0, \tag{127d}$$

$$\mathscr{D}_2\mathcal{F}^2_{\text{new}} \circ \sharp^{(2)} = W^3{}_2 = 0, \tag{127e}$$

$$\mathscr{D}_3\mathcal{F}^2_{\text{new}} \circ \sharp^{(2)} = W^3{}_3 = 0, \tag{127f}$$

$$\mathscr{D}_1\mathcal{F}^3_{\text{new}} \circ \sharp^{(2)} = (\cos q^3)W^1{}_1 + (\sin q^3)W^2{}_1 + (\sin q^3)\big[\dot{q}^3 + \chi\big] = 0, \tag{127g}$$

$$\mathscr{D}_2\mathfrak{F}^3_{\text{new}} \circ \sharp^{(2)} = (\cos q^3)W^1{}_2 + (\sin q^3)W^2{}_2 - (\cos q^3)\left[\dot{q}^3 + \chi\right] = 0, \tag{127h}$$

$$\mathscr{D}_3\mathfrak{F}^3_{\text{new}} \circ \sharp^{(2)} = (\cos q^3)W^1{}_3 + (\sin q^3)W^2{}_3 \underbrace{-(\sin q^3)\tfrac{p_1}{m} + (\cos q^3)\tfrac{p_2}{m}}_{=0} = 0. \tag{127i}$$

Finally, by virtue of these results, the new matrix W_{new}, which solves Eqs. (127a)–(127i), takes on the form

$$\mathsf{W}_{\text{new}} = \begin{bmatrix} 0 & \left(\dot{q}^3 + \chi\right) & -\left(\dot{q}^2 + \chi q^1\right) \\ -\left(\dot{q}^3 + \chi\right) & 0 & \left(\dot{q}^1 - \chi q^2\right) \\ 0 & 0 & 0 \end{bmatrix}, \tag{128}$$

which corresponds to *shifting* the original matrix W_{old} in Eq. (107) by a matrix

$$\mathsf{W}_{\text{mag}} = \chi \begin{bmatrix} 0 & 1 & -q^1 \\ -1 & 0 & -q^2 \\ 0 & 0 & 0 \end{bmatrix}, \tag{129}$$

which accounts for the interaction of the skate with the magnetic field, i.e., $\mathsf{W}_{\text{new}} = \mathsf{W}_{\text{old}} + \mathsf{W}_{\text{mag}}$.

We notice that, since the new constraint $\mathcal{V}^1_{\text{new}}$ in Eq. (125) is the sum of the original constraint in Eq. (98) and of an additional term not involving the generalized velocities, the condition in Eq. (55a), which guarantees the equivalence between the MVM and the TNHM, requires the vanishing of the same terms displayed in Eq. (110). Indeed, by computing the product $\mathsf{W}^{\mathrm{T}}_{\text{new}}\mathsf{p}$, we observe that the equivalence is satisfied, since the following identities hold:

$$(\cos q^3)(\mathsf{W}^{\mathrm{T}}_{\text{new}}\mathsf{p})_1 + (\sin q^3)(\mathsf{W}^{\mathrm{T}}_{\text{new}}\mathsf{p})_2 = \frac{12p_3}{\ell^2 + \sigma^2}\left[\sin q^3 \frac{p_1}{m} - \cos q^3 \frac{p_2}{m}\right] = 0, \tag{130a}$$

$$(\mathsf{W}^{\mathrm{T}}_{\text{new}}\mathsf{p})_3 = -\frac{p_2}{m}p_1 + \frac{p_1}{m}p_2 = 0. \tag{130b}$$

Moreover, the dynamic equations returned by the MVM with the new formulation of both the constraint in Eq. (125) and the auxiliary functions in Eqs. (126a) and (126b) read

$$\dot{p}_1 = \dot{\lambda}_1 \sin q^3 + mg\sin\alpha + \tfrac{1}{2}QB_0\dot{q}^2 - p_2\left(\dot{q}^3 + \chi\right), \tag{131a}$$

$$\dot{p}_2 = -\dot{\lambda}_1 \cos q^3 - \tfrac{1}{2}QB_0\dot{q}^1 + p_1\left(\dot{q}^3 + \chi\right), \tag{131b}$$

$$\dot{p}_3 = 0. \tag{131c}$$

Finally, the relation between the Lagrange multipliers of the MVM and the TNHM is a direct consequence of Eqs. (131a) and (131b), and, following the same procedure as in Eq. (118), it holds that

$$\mu_1 \equiv \dot{\lambda}_1 - \left(p_1 \cos q^3 + p_2 \sin q^3\right)\left(\dot{q}^3 + \chi\right). \tag{132}$$

Remark 12 (*"Canonical flip"* [22, 41] for the Case C of the "charged skate") Looking at Eq. (128), it is immediate to notice that, in spite of the presence of the factor χ, the structure of the "new" matrix $\mathsf{W}_{\mathrm{new}}$ is the same as the one obtained for the Case A (see Eq. (107)), with the sole difference that the entries of $\mathsf{W}_{\mathrm{new}}$ coincide with the components of the momenta of the theory, p, normalized by the corresponding components of the mass matrix. A direct consequence of this fact is that the coefficients $\mathcal{C}^h{}_{\ell k}$ of the "Canonical flip" characterizing this version of the considered problem are exactly those determined in Remark 7, provided that the transpositional relations are written as

$$\zeta^1 - \dot{\eta}^1 = +\,(\dot{q}^3 + \chi)\eta^2 - (\dot{q}^2 + \chi q^1)\eta^3 = +\underbrace{\frac{12\,p_3}{m[\ell^2 + \sigma^2]}}_{=:\pi^3}\eta^2 - \underbrace{\frac{p_2}{m}}_{=:\pi^2}\eta^3, \tag{133a}$$

$$\zeta^2 - \dot{\eta}^2 = -\,(\dot{q}^3 + \chi)\eta^1 + (\dot{q}^1 - \chi q^2)\eta^3 = -\underbrace{\frac{12\,p_3}{m[\ell^2 + \sigma^2]}}_{=:\pi^3}\eta^1 + \underbrace{\frac{p_1}{m}}_{=:\pi^1}\eta^3, \tag{133b}$$

$$\zeta^3 - \dot{\eta}^3 = 0, \tag{133c}$$

where π^1, π^2, and π^3 are the normalized momenta, and can be interpreted as the *effective velocities* of the problem at hand (in the sense that they are the physical quantities, having physical dimensions of velocities, that are effectively constrained).

5 Conclusions

In this work, we have employed the "modified vakonomic method" (MVM), introduced by Llibre et al. [12], to achieve four main results that, in our opinion, may deepen the understanding of vakonomic mechanics:

(i) We have shown that, for the "rolling coin" problem, the TNHM and MVM are equivalent. By doing so, we have proven that the methodology outlined in [12], which, however, was not adopted therein for this problem, allows to reconcile the vakonomic approach with the traditional one, at least for the considered case. This result, in fact, confirms the statement given by Lemos [1], who spoke of *"complete inequivalence between the vakonomic and the non-holonomic method"*, but it contextually indicates how to overcome this inconsistency by using the procedure developed in [12].

(ii) After testing the methodological power of the MVM with the "rolling coin", we have considered another widely studied problem, namely, the "non-holonomic skate", and we have elaborated a variant of it obtained by assuming the skate to be electrically charged and exposed to an imposed, homogeneous magnetic field (we have, in fact, modified the framework outlined in [27]). We emphasize, in this respect, that, whereas Llibre et al. [12] studied the uncharged "non-holonomic skate" by using their MVM, we have studied the variant to this problem described above in two steps. First, we have employed the MVM *as is* (i.e., as formulated in [12]), and we have seen that, according to our results, it *is not* equivalent to the TNHM as long as the auxiliary functions of the MVM are taken like in [12]. However, our second step was to show that, by following the rationale supplied by Theorem 1, the equivalence can be restored through a suitable set of conditions on W entering the transpositional relations.

(iii) Theorem 1, in fact, is the result that states the characterizing conditions by which the MVM can always be made consistently equivalent to the TNHM. This, however, requires to further modify the MVM according to the physical motivations given in *Ansatz 2*. Although the germinal idea of this modification is already present in [11] (cf. the condition $\mathsf{W}^{\mathrm{T}}\mathsf{p} = 0$, in our notation) and in [12] (cf. the unnumbered equation in the Remark 19 of [12] with our Eqs. (55a) and (55b)), we have reinterpreted and expanded it to formulate a new criterion, which we may call "M^{2}VM", where "M^2" stands for "Modified Modified". The reason for this, as remarked in Sect. 4, is that one can formulate problems for which it is not straightforward to find a full set of auxiliary functions guaranteeing the equivalence between the MVM and the TNHM. Therefore, in our opinion, the approach outlined in [12] should be augmented by invoking Theorem 1.

(iv) In Sect. 4.2.3, we have highlighted the existence of a "natural pattern", and we have discussed how this "pattern" is not respected when the charged non-holohomic skate is subjected to the classical constraint (98), and the auxiliary functions supplied in Eqs. (104a) and (104b) are used (cf. [12]). Thus, we have proposed a variant of the constraint in which the constrained velocities are the *effective ones*, i.e., those obtained by dividing the momenta p_1 and p_2 by the mass, and p_3 by the corresponding coefficient of the mass matrix, and we have re-defined the auxiliary functions in terms of such velocities. As reported in Sect. 4.2.3, this restores the pattern and allows to apply the MVM by Llibre et al. [12] *as is*. In addition, this way of looking at a given mechanical problem could lead to a more physical conception of the constraints which may result, for instance, in the passage from a constraint linear in the velocities to one that is affine in them.

We are aware of studies (see, e.g., [18, 28, 36]) suggesting that, for some problems of field theory or geometric control theory, the vakonomic method leads to results that are sometimes believed to be physically remarkable in comparison with those obtained with the traditional non-holonomic method. This belief, in our opinion, promotes further investigations on the vakonomic method, with its generalizations, and on its relationship with the TNHM. In this respect, it is important to emphasize that the aim of our work is not to affirm that the TNHM should always be taken as reference for all types of problems, and that, thus, other methods (e.g. the MVM) should always be made equivalent to it. However, we are saying that, if there are physical reasons —even very subtle ones— hinting that the MVM can be made equivalent to the TNHM, then this equivalence must be accounted for. Indeed, following this philosophy, we highlighted, through Theorem 1, the possibility of applying the variational approach of Llibre et al. [12] to a class of problems that the scientific community regards as correctly described by the TNHM.

It should be noted that, by promoting the further study of the *vakonomic procedure*, we intend the study of *generalized variational procedures* that well-suit the non-holonomic setting and, when expected, should be consistent with the TNHM. In fact, to our knowledge, there is no experimental evidence implying that the VM is better than the TNHM in modeling classical mechanical systems subjected to non-holonomic constraints (see e.g. [3, 5]), but this discussion is out of the scope of our work. In light of the previous sentence, our interest for the MVM relies on the possibility of having a fully-variational procedure specialized to non-holonomic systems, which, while being consistent with the TNHM, allows to exploit the Lagrangian formalism.

As a last remark, we would like to emphasize that our work served, essentially, two main purposes. The first one was to satisfy our need to test the MVM in its original formulation by [12] against both standard (see Sect. 3) and non-standard (see Sect. 4) problems in order to see whether or not the consistency with the TNHM holds. Exactly in this spirit, we view our work as an instructive "exercise" on the MVM. The second purpose, instead, was to propose some *modifications* of the MVM that ensure both the consistency with the TNHM (see Theorem 1) and the fulfillment of the appropriate transpositional relations (see Remark 1). Given the results above, future investigations could make use of the insight gained in this work on the MVM to look for symmetries in complicated systems (see [49] for a review on Noether's theorem in non-holonomic systems), and for extending the MVM to the continuum setting as a tool for approaching the rheonomy and non-holonomy of the inelastic processes [32, 33].

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Statements and Declarations

- **Competing interests**
  The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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Appendix A

The scope of this appendix is to put the variations characterizing the MVM within the context of Differential Geometry. We start combining Jóźwikowski&Respondek's formalism [22], which implements the framework established by Gràcia et al. [48], with ours. In doing this, however, for the sake of simplicity, we consider here only constraints of the type $\mathcal{V}^\alpha \circ (q, \dot{q}) = [a^\alpha{}_k \circ q]\dot{q}^k = 0$, i.e., that do not depend explicitly on time. We denote by $\gamma := (q, \dot{q}) : [t_{\rm in}, t_{\rm fin}] \to T\mathscr{C}$ a *"path"* [22], identified with the *"tangent lift"* [22] of q at t, i.e., $\gamma(t) \equiv \mathrm{T}_t q(t) = (q(t), \dot{q}(t))$. In terms of $\tilde{Q} : [t_{\rm in}, t_{\rm fin}] \times]-\varepsilon_0, +\varepsilon_0[\to \mathscr{C}$ (see Eq. (6a)), we also define $\tilde{\Gamma} := (\tilde{Q}, \partial_t \tilde{Q}) \equiv \mathrm{T}_t \tilde{Q} : [t_{\rm in}, t_{\rm fin}] \times]-\varepsilon_0, +\varepsilon_0[\to T\mathscr{C}$, so that the chain of equalities

$$\gamma(t) = (q(t), \dot{q}(t)) = (\tilde{Q}(t, 0), \partial_t \tilde{Q}(t, 0)) = \mathrm{T}_t \tilde{Q}(t, 0) = \tilde{\Gamma}(t, 0) \tag{A1}$$

applies. Moreover, by differentiating $\tilde{Q}$ with respect to ε, and evaluating the result at $\varepsilon = 0$, we obtain, for each $t \in [t_{\rm in}, t_{\rm fin}]$, the quantity (see [22])

$$\xi(t) := \mathrm{T}_\varepsilon \tilde{Q}(t, 0) = (\tilde{Q}(t, 0), \partial_\varepsilon \tilde{Q}(t, 0)) \equiv (q(t), \boldsymbol{\eta}(t)), \tag{A2}$$

in which the identification $\boldsymbol{\eta}(t) \equiv \partial_\varepsilon \tilde{Q}(t, 0) \in T_{q(t)}\mathscr{C}$ has been made by virtue of Eq. (8a). Looking at the structure of $\xi(t)$, we notice that the second tangent bundle formalism arises naturally by iterating the operation of tangent lift, i.e., by computing the tangent lift of ξ, which yields

$$\mathrm{T}_t\xi(t) = (q(t), \boldsymbol{\eta}(t), \dot{q}(t), \dot{\boldsymbol{\eta}}(t)) \in T T\mathscr{C}. \tag{A3}$$

Finally, following [22], we compute the quantity

$$\begin{aligned}\mathrm{T}_\varepsilon\tilde{\Gamma}(t,0) = \mathrm{T}_\varepsilon\mathrm{T}_t\tilde{Q}(t,0) &= (q(t), \dot{q}(t), \partial_\varepsilon\tilde{Q}(t,0), \partial_\varepsilon\partial_t\tilde{Q}(t,0)) \\ &= (q(t), \dot{q}(t), \boldsymbol{\eta}(t), \dot{\boldsymbol{\eta}}(t)) \in TT\mathscr{C},\end{aligned} \tag{A4}$$

thereby recovering the identity $\mathrm{T}_\varepsilon\tilde{\Gamma}(t,0) = \kappa_{\mathscr{C}}(\mathrm{T}_t\xi(t))$, where the map $\kappa_{\mathscr{C}} : TT\mathscr{C} \to TT\mathscr{C}$ is referred to as *"canonical flip"* in [22, 41], since it changes the position of the entries of the second and third slots of the elements of $TT\mathscr{C}$. In the jargon adopted in [22, 41], the quantity $\mathrm{T}_\varepsilon\tilde{\Gamma}(t,0)$ is referred to as *"natural variation"* and is also denoted by $\delta_\xi\gamma(t) \equiv \mathrm{T}_\varepsilon\tilde{\Gamma}(t,0)$ to highlight that the natural variation is *generated* by $\xi(t)$ through the canonical flip.

In analogy with the reasoning leading to Eq. (A1), and with the purpose of putting the approach of Llibre et al. [12] in the language used in [22], we also define the pair $\tilde{\Sigma} := (\tilde{Q}, \tilde{\boldsymbol{V}}) : [t_{\text{in}}, t_{\text{fin}}] \times]-\varepsilon_0, +\varepsilon_0[\to T\mathscr{C}$, where we require $\tilde{\boldsymbol{V}}$ to satisfy the condition $\tilde{\boldsymbol{V}}(t,0) = \dot{q}(t)$, for $t \in [t_{\text{in}}, t_{\text{fin}}]$ (see Eq. (6b)). Hence, we compute $\mathrm{T}_\varepsilon\tilde{\Sigma}(t,0)$, thereby obtaining

$$\begin{aligned}\mathrm{T}_\varepsilon\tilde{\Sigma}(t,0) &= (\tilde{Q}(t,0), \tilde{\boldsymbol{V}}(t,0), \partial_\varepsilon\tilde{Q}(t,0), \partial_\varepsilon\tilde{\boldsymbol{V}}(t,0)) \\ &= (q(t), \dot{q}(t), \boldsymbol{\eta}(t), \boldsymbol{\zeta}(t)) \in TT\mathscr{C}, \qquad \text{with } \boldsymbol{\zeta}(t) := \partial_\varepsilon\tilde{\boldsymbol{V}}(t,0).\end{aligned} \tag{A5}$$

The quantity $\mathrm{T}_\varepsilon\tilde{\Sigma}(t,0)$ in Eq. (A5) defines a generic variation. To specify such variation for the MVM, we need to impose the following conditions: (a) The path $(q, \dot{q})$ must satisfy the m given constraints $\mathcal{V}^\alpha(q(t), \dot{q}(t)) = 0$, with $\alpha = 1, \ldots, m$, at all times, so that the pair $(q(t), \dot{q}(t))$ belongs to $T\mathscr{C}_{\mathrm{c}}$, defined in Eq. (1). (b) The pair $(q, \boldsymbol{\eta}) \in T\mathscr{C}$ is compelled to fulfill Lagrange-Chetaev's conditions (12), which here becomes $[\partial_{\dot{q}}\mathcal{V}^\alpha(q(t), \dot{q}(t))]\boldsymbol{\eta}(t) \equiv [\partial_{\dot{q}^k}\mathcal{V}^\alpha(q(t), \dot{q}(t))]\eta^k(t) = 0$, for $\alpha = 1, \ldots, m$. (c) As specified in [12], for every $t \in [t_{\text{in}}, t_{\text{fin}}]$, the vector $\boldsymbol{\zeta}(t)$ is required to satisfy the constraints *at the first order in* ε. Thus, upon defining the functions $\tilde{\mathcal{V}}^\alpha : [t_{\text{in}}, t_{\text{fin}}] \times]-\varepsilon_0, +\varepsilon_0[\to \mathbb{R}$, such that $\tilde{\mathcal{V}}^\alpha(t,\varepsilon) := \mathcal{V}^\alpha(\tilde{Q}(t,\varepsilon), \tilde{\boldsymbol{V}}(t,\varepsilon))$, and $\tilde{\mathcal{V}}^\alpha(t,0) \equiv \mathcal{V}^\alpha(q(t), \dot{q}(t)) = 0$, for all $\alpha = 1, \ldots, m$ and for all $t \in [t_{\text{in}}, t_{\text{fin}}]$ (see Eqs. (9) and (10)), we write

$$\tilde{\mathcal{V}}^\alpha(t,\varepsilon) = \underbrace{\tilde{\mathcal{V}}^\alpha(t,0)}_{=0} + \partial_\varepsilon\tilde{\mathcal{V}}^\alpha(t,0)\varepsilon + o(\varepsilon), \qquad \varepsilon \to 0, \tag{A6}$$

and we set $\partial_\varepsilon\tilde{\mathcal{V}}^\alpha(t,0) = 0$, thereby obtaining Eq. (11), i.e.,

$$0 = \partial_\varepsilon\tilde{\mathcal{V}}^\alpha(t,0) = \frac{\partial\mathcal{V}^\alpha}{\partial q^k}(q(t), \dot{q}(t))\eta^k(t) + \frac{\partial\mathcal{V}^\alpha}{\partial\dot{q}^k}(q(t), \dot{q}(t))\zeta^k(t). \tag{A7}$$

Equation (A7), which characterizes the vakonomic variations, can be also worked out further as

$$\begin{aligned}0 &= \left[\frac{\partial \mathcal{V}^\alpha}{\partial q^k}\circ(q,\dot q)\right]\eta^k + \left[\frac{\partial \mathcal{V}^\alpha}{\partial \dot q^k}\circ(q,\dot q)\right][\zeta^k-\dot\eta^k] + \left[\frac{\partial \mathcal{V}^\alpha}{\partial \dot q^k}\circ(q,\dot q)\right]\dot\eta^k\\ &= \left[\mathscr{E}_k\mathcal{V}^\alpha\circ(q,\dot q,\ddot q)\right]\eta^k + \left[\frac{\partial \mathcal{V}^\alpha}{\partial \dot q^k}\circ(q,\dot q)\right][\zeta^k-\dot\eta^k] + \frac{\mathrm{d}}{\mathrm{d}t}\left\{\left[\frac{\partial \mathcal{V}^\alpha}{\partial \dot q^k}\circ(q,\dot q)\right]\eta^k\right\}\\ &= \left[\mathscr{E}_k\mathcal{V}^\alpha\circ(q,\dot q,\ddot q)\right]\eta^k + \left[\frac{\partial \mathcal{V}^\alpha}{\partial \dot q^k}\circ(q,\dot q)\right][\zeta^k-\dot\eta^k], \end{aligned}\tag{A8}$$

with

$$\mathscr{E}_k\mathcal{V}^\alpha\circ(q,\dot q,\ddot q) := \frac{\partial \mathcal{V}^\alpha}{\partial q^k}\circ(q,\dot q) - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{V}^\alpha}{\partial \dot q^k}\circ(q,\dot q)\right],\tag{A9}$$

and the last summand of Eq. (A8) being null by virtue of Lagrange-Chetaev's conditions (cf. Eq. (A8) with Eq. (14): They are the same for non-holonomic constraints linear in the velocities and not depending explicitly on time).

With all the premises specified above, the set of the variations characterizing the MVM can be defined as:

$$\begin{aligned}\mathscr{W}_{\mathrm{c}}^{\mathrm{MVM}} :=\{&\mathrm{T}_\varepsilon\tilde\Sigma(t,0)\in TT\mathscr{C}|\ (q(t),\dot q(t))\in T\mathscr{C}_{\mathrm{c}},\ \ (q(t),\boldsymbol{\eta}(t))\in T\mathscr{C}_{\mathrm{c}},\\ &\boldsymbol{\zeta}(t)|\ \partial_\varepsilon\tilde{\mathcal{V}}^\alpha(t,0)=0\}.\end{aligned}\tag{A10}$$

Note that, as written in Sect. 2.1, in Eq. (A8) we *guess* (this is, in fact, an educated guess, since we are aware of the existence of the transpositional relations) that $\zeta^k-\dot\eta^k$ can be expressed in terms of the linear transformation $\zeta^k-\dot\eta^k = W^k{}_\ell\eta^\ell$, in which $[W^k{}_\ell]^{k=1,\dots,n}_{\ell=1,\dots,n}$ are the coefficients of a *partially* unknown matrix W. Indeed, if the matrix made by the derivatives $\partial_{\dot q^k}\mathcal{V}^\alpha\circ(q,\dot q)$ were square (it is rectangular, here), the coefficients of W could be determined from Eq. (A8) by solving for $\zeta^k-\dot\eta^k$. However, this matrix can be made square by introducing suitable quasi-velocities, as is done in the standard procedure leading to the transpositional relations (see e.g. [26, 50] and cf. Eq. (17)). This is, to our understanding, the philosophy underlying the approach proposed by Llibre et al. [12], whose main objective is to put the non-holonomic constraints in the variational framework, but with variations that *are not* only vakonomic, but both vakonomic and complying with Lagrange-Chetaev's conditions.

Finally, if the further identification $W^k{}_\ell \equiv \mathcal{C}^k{}_{h\ell}\dot q^h$ is made, where $\mathcal{C}^k{}_{h\ell}$ are the coefficients of the third-order tensor field defined in [22, 41], or in [26] (see Remark 2), and that characterizes the transpositional relations, then, in the formalism of [22], $\mathrm{T}_\varepsilon\tilde\Sigma(t,0)$ can be rewritten as

$$\mathrm{T}_\varepsilon\tilde\Sigma(t,0) = (q^a(t),\dot q^b(t),\eta^c(t),\underbrace{\dot\eta^d(t)+\mathcal{C}^d{}_{k\ell}(q(t))\dot q^k(t)\eta^\ell(t)}_{=\zeta^d(t)}).\tag{A11}$$