

Reconciling Kozlov's vakonomic method with the traditional non-holonomic method: solution of two benchmark problems

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1 Introduction

The motivation for studying mechanical systems subjected to non-holonomic constraints is determined not only by their mathematical beauty, but also by their relevance in physics [18, 36], e.g. for modeling relativistic fluids; in biomechanics, for describing, e.g., tissue growth and remodeling [32, 33, 53], and animal or cellular motion [4]; and in engineering, especially in robotics [29–31, 40]. The ever-green interest for these systems is the search for a *paradigmatic framework* formalizing non-holonomic mechanics, whose applicability should be as universal as possible. This applies, in particular, to problems exhibiting non-holonomic constraints that are either nonlinear in the generalized velocities of a given system [32–35] or involving the accelerations [16, 17].

A peculiar feature of non-holonomic systems is their intrinsic tendency to involve several aspects of mathematics, such as analysis, differential geometry and numerics, with the aim of providing physically consistent formulations of their evolution. This has led to descriptions of non-holonomic systems different from those of standard analytical mechanics, which are based on the Lagrange-D'Alembert principle, typically “augmented” by the technique of Lagrange multipliers, i.e., the so-called “traditional non-holonomic method” (TNHM) [13, 14, 16, 17]. For example, in the early 80's, Kozlov proposed a method for modeling non-holonomic mechanics, known as “vakonomic method” (VM) [7–10], and standing for mechanics of the “*variational axiomatic kind*” [1], which applies Hamilton's principle of stationary action [13] also to systems subjected to non-holonomic constraints. One of the main features of this method is the capability of describing non-holonomic systems through a fully variational procedure, based on the definition of an appropriate *constrained* Lagrangian function [7–10]. We remark that also other types of variational approaches have been introduced to study systems subjected to non-holonomic constraints (see, e.g., [21, 27, 37, 38, 51, 52]).

Clearly, the introduction of new methods that are, in a sense, groundbreaking, since they question the pillars of the standard variational procedures, requires them to be tested against the more traditional ones. This, indeed, has been done quite intensively by having recourse to widely studied and easy-to-reproduce “benchmark” problems, all encoded in the classical literature of Analytical Mechanics (see e.g. [2, 3, 5, 6, 22, 26, 28, 43–47]). Typical examples are a ball rolling over a moving or fixed surface [2, 3, 5], a skate moving on an inclined or horizontal plane [3, 6, 12], a disk rolling on an inclined plane [1, 14, 22, 43–45], and a two-wheeled carriage moving on a plane [2, 22, 43, 45, 47]. In spite of the fact that these case studies were formulated many years ago, and can be found in textbooks, they remain up-to-date (see e.g. [6, 11, 16–18, 22, 36], and, more recently, [1, 12]), since they are archetypal and, thus, serve as a reference even for much more complex physical situations in which the constraints can be reconducted to those characterizing the original case studies.

The vakonomic method [7–10], used *as is*, can lead, in some cases, to motions that differ from the ones obtained with the TNHM, which is known

to supply solutions observable experimentally for the majority of mechanical problems [1–3, 5, 6, 22]. In particular, as recently shown by Lemos [1], one of such cases is the problem of the “rolling coin”, which describes a rigid coin rolling without slipping on an inclined plane [1, 14], for which the VM is said to lead to unphysical results. Similar problems have also been investigated in [6, 12]. On the other hand, there are problems, e.g. in geometric control theory [4, 28, 48] and in field theory [18, 36], in which it is claimed that the TNHM does not provide physically consistent dynamics, while the VM provides “*interesting results*” [18].

Many authors have compared the VM and the TNHM on the basis of the *variations* introduced to define the associated variational principle [2, 3, 5, 18, 22, 36, 41, 43–45, 48], and in terms of the *geometrical meaning* of such variations [28, 37, 41, 44–46, 48]. Often, one of the purposes of these investigations is to provide conditions that allow to conclude that a solution to a mechanical problem obtained within the TNHM is also a solution of the VM, whereas the converse is, in general, not true (see [3] for a critique on Kozlov's approach). In fact, Lemos' work [1] places itself in this research line, thereby highlighting the discrepancies between the VM and the TNHM for a certain class of problems. Still, in our opinion, Kozlov's idea is worth of being investigated because of its conceptual potential, and, to the best of our understanding, it could be “saved” by performing suitable *modifications* to it (Llibre et al. [12] speak of “*modified vakonomic method*”). Indeed, if our interpretation is correct, it is with this attitude in mind that the works of Ramírez et al. [11] and Llibre et al. [12] have been conceived. From an operative point of view, the modifications to the VM should be able to provide conditions on the variations of the Lagrangian parameters of a given system, and on the associated generalized velocities, in such a way that, under certain working hypotheses, the TNHM solutions can be recovered (see Sect. 2.5). To this end, the variations should satisfy Lagrange-Chetaev's conditions in addition to other conditions specified by the method employed.

To our knowledge, in fact, a possible reconciliation between the vakonomic method and the traditional methodologies is given in a paper by Llibre et al. [12], whose germinal idea was already present in [11], and in which the authors elaborate a modification to the original vakonomic method based on the Hamilton-Suslov variational principle [12, 19, 20] that accounts for the presence of non-vanishing *transpositional relations* [11, 12, 25–27]. Such relations express that, in the presence of a non-holonomic constraint, the variation of a generalized velocity involved in the constraint is generally not equal to the time derivative of the variation of the associated Lagrangian parameter. With this premise, Llibre et al. [12] prove that the equations of motion obtained with their so-called modified vakonomic method (MVM) [12] are equivalent to the ones obtained with the traditional non-holonomic method, up to a “*zero Lebesgue-measure set*” [12]. For their purposes, Llibre et al. [12] propose two *paths*: one is based on the introduction of auxiliary functions that augment the Lagrangian function of the considered problem in a suitable way (see Theorem

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122 2 of [12] and Section 2.2 below), while the other one imposes *a priori* con-
 123 ditions that compel the equivalence of their approach with the TNHM (see
 124 Theorem 3 of [12] and our Corollary 1).

125 Within the context explained above, the scope of this work is:

- 126 (i) To add a methodological note to the paper recently published by Lemos [1]
 127 by proving that the MVM and the TNHM are in agreement with each other
 128 for the “Rolling coin” problem [1, 14].
- 129 (ii) To apply the procedure developed in [11, 12] to a variant of the “Non-
 130 holonomic skate” presented in [12]. Specifically, we consider a non-holonomic
 131 and *electrically charged* skate interacting with an imposed magnetic field,
 132 and we show that, for this problem, if the MVM is formulated according to
 133 the first *path*, it is not equivalent to the TNHM. The second *path* is equiv-
 134 alent to the TNHM, but, *for our problem*, it is inconsistent with the first
 135 *path*. For our purposes, we generalize a benchmark taken from [27].
- 136 (iii) To present a reformulation of the MVM that, on the one hand, retrieves the
 137 equivalence with the TNHM, and, on the other hand, merges the two *paths*
 138 of [12] in a physics-driven way. More in detail, we propose a way to indicate,
 139 on physical grounds (e.g. compliance with evident conservation laws), which
 140 *path* should be followed.

141 It should be noticed that, in the following, our efforts are concentrated on
 142 the reconciliation of the VM with the TNHM for a class of mechanical prob-
 143 lems known to be reliably described by the TNHM, i.e., for which the TNHM
 144 is known to produce solutions consistent with the experiments. In fact, our
 145 interest for such reconciliation does not require that the TNHM should al-
 146 ways be used as a reference, but it resides in the possibility of describing the
 147 dynamics of a given mechanical system through a variational approach that
 148 supplies the same solutions as the TNHM, and, thus, the experimentally ob-
 149 servable motions. In this respect, our work can be seen as a “gym” for testing
 150 the MVM [11, 12], and for understanding whether fundamental results of An-
 151 alytical Mechanics holding in the non-holonomic setting [26] can be extended
 152 to study the symmetries of mechanical systems (see Noether’s theorem and its
 153 extension to non-holonomic problems [49]), and whether, and with which mod-
 154 ifications, the MVM can be imported to field theories (e.g. inelastic processes
 155 in continuum systems [32, 33]). Hence, in this work we are not interested in
 156 comparing the VM and the TNHM, neither on theoretical nor on experimen-
 157 tal bases. Rather, we would just like to present our reformulation of the MVM
 158 of Llibre et al. [12], which enjoys the efficacy and elegance of the Lagrangian
 159 formalism, while providing solutions that are in harmony with the TNHM.

160 As for the structure of this work, Sect. 2 is dedicated to give the reader an
 161 *overview* on the mathematical foundations of the MVM [12]. In particular, in
 162 Sect. 2.1 we establish the geometrical setting used for describing the kinematics
 163 of a non-holonomically constrained system and the need for introducing non-
 164 vanishing “*transpositional relations*” [12, 25–27]; in Sect. 2.2, we have explicit
 165 recourse to the Hamilton-Suslov variational principle [12, 19, 20] that leads us

166 to the dynamic equations for the system under investigation, which are then
 167 closed under the assumption of suitable *solvability conditions* discussed in Sect.
 168 2.3; in Sect. 2.4, we further elaborate on some computational aspects regarding
 169 the MVM, which help us draw the differences and similarities between the
 170 MVM and the TNHM, as expanded in Sects. 2.5 and 2.6. Finally, the “rolling
 171 coin” [1, 14, 28] and the “charged skate” [12, 27] benchmark problems are
 172 examined by means of both the MVM and the TNHM, as seen in Sect. 3
 173 and Sect. 4, respectively. Some geometric remarks related to the analyzed
 174 constraints are summarized in “Appendix A”.

175 2 Modified vakonomic method (MVM)

176 In this section, we summarize the fundamental aspects of the modified vako-
 177 nomic method developed by Llibre et al. [12] that are relevant for our work.
 178 Hereafter, for the sake of brevity, we shall refer to the “modified vakonomic
 179 method” [12] as MVM, in order to distinguish it from the original “vakonomic
 180 method”, referred to as VM, introduced by Kozlov [7–10].

181 2.1 Variations and non-holonomic constraints

182 Following [12], we start our discussion with the presentation of the *Hamilton-*
 183 *Suslov variational principle* [12, 19, 20] for mechanical systems featuring non-
 184 holonomic constraints that fulfill the *Lagrange-Chetaev conditions* [12].

185 Let \mathfrak{M} be a mechanical system whose evolution is characterized by $n \geq 1$,
 186 $n \in \mathbb{N}$, Lagrangian parameters, i.e., n functions of time, collected in the array
 187 $q := (q^1, \dots, q^n) : [t_{\text{in}}, t_{\text{fin}}] \subseteq \mathcal{T} \rightarrow \mathcal{C}$, where $[t_{\text{in}}, t_{\text{fin}}]$ is a closed interval of
 188 the *time line* \mathcal{T} [23], and \mathcal{C} is referred to as the *configuration space* of \mathfrak{M} [13].
 189 The latter is regarded here as an n -dimensional manifold representing the set
 190 of all possible configurations for \mathfrak{M} and, as required in [12], it is assumed to be
 191 smooth in the sequel. In this sense, $q : [t_{\text{in}}, t_{\text{fin}}] \rightarrow \mathcal{C}$ is a one-parameter family
 192 of configurations of \mathcal{C} , thereby parameterizing a curve on this manifold, such
 193 that, for each $t \in [t_{\text{in}}, t_{\text{fin}}]$, ϑ is the unique element of \mathcal{C} satisfying $\vartheta = q(t) \in \mathcal{C}$.

194 For each configuration $\vartheta \in \mathcal{C}$, $T_{\vartheta}\mathcal{C}$ denotes the tangent space of \mathcal{C} at ϑ ,
 195 and $T\mathcal{C} = \cup_{\vartheta \in \mathcal{C}} (\{\vartheta\} \times T_{\vartheta}\mathcal{C})$ is the tangent bundle of \mathcal{C} , so that, for each
 196 pair $(\vartheta, \boldsymbol{\nu}) \in T\mathcal{C}$, $\boldsymbol{\nu} \in T_{\vartheta}\mathcal{C}$ is a realization of the generalized velocity vector
 197 attached at ϑ . Moreover, under the hypothesis that q is differentiable, we
 198 denote by $(q, \dot{q}) :]t_{\text{in}}, t_{\text{fin}}[\rightarrow T\mathcal{C}$ the map such that, for each time $t \in]t_{\text{in}}, t_{\text{fin}}[$,
 199 $\dot{q}(t) \in T_{q(t)}\mathcal{C}$ is the tangent vector to the curve parameterized by q at $q(t)$,
 200 and is equal to the vector of generalized velocities attached at $q(t) \in \mathcal{C}$, i.e.,
 201 there exists $\boldsymbol{\nu} \in T_{q(t)}\mathcal{C}$ such that $\dot{q}(t) = \boldsymbol{\nu}$.

202 We assume that the system \mathfrak{M} is subjected to $m \in \mathbb{N}$, with $m \leq n$,
 203 linearly independent *non-holonomic constraints*, represented by the functions
 204 $\mathcal{V}^{\alpha} : T\mathcal{C} \times \mathcal{T} \rightarrow \mathbb{R}$, for $\alpha = 1, \dots, m$. The presence of such constraints re-
 205 stricts the admissible generalized velocities to the following sub-manifold of

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206 the tangent bundle $T\mathcal{C}$ (see e.g. [2, 39])

$$T\mathcal{C}_c := \{(\vartheta, \boldsymbol{\nu}) \in T\mathcal{C} : \mathcal{V}^\alpha(\vartheta, \boldsymbol{\nu}, t) = 0, \alpha = 1, \dots, m, \forall t \in [t_{\text{in}}, t_{\text{fin}}]\}. \quad (1)$$

207 Moreover, as done in [11, 12], we focus on non-holonomic constraints *linear* in
 208 the generalized velocities. To this end, for each $\alpha = 1, \dots, m$, and for all $\vartheta \in \mathcal{C}$
 209 and $t \in \mathcal{T}$, we introduce a linear map $\mathcal{G}^\alpha(\vartheta, t)[\cdot] : T_\vartheta\mathcal{C} \rightarrow \mathbb{R}$, i.e., an element
 210 of the cotangent space $T_\vartheta^*\mathcal{C}$, represented by

$$\mathcal{G}^\alpha(\vartheta, t)[\cdot] := \sum_{k=1}^n a^{\alpha}_k(\vartheta, t) \mathbf{e}^k(\vartheta)[\cdot], \quad (2)$$

211 with $\mathbf{e}^k(\vartheta)[\cdot]$ denoting the k th basis co-vector of $T_\vartheta^*\mathcal{C}$, and defined such that

$$\mathcal{G}^\alpha(\vartheta, t)[\boldsymbol{\nu}] := \sum_{k=1}^n a^{\alpha}_k(\vartheta, t) \nu^k \equiv \mathcal{V}^\alpha(\vartheta, \boldsymbol{\nu}, t), \quad (3)$$

212 where the identity $\mathbf{e}^k(\vartheta)[\boldsymbol{\nu}] \equiv \nu^k$ is exploited for each $k = 1, \dots, n$, and, for
 213 varying $\alpha = 1, \dots, m$ and $k = 1, \dots, n$, $a^{\alpha}_k : \mathcal{C} \times \mathcal{T} \rightarrow \mathbb{R}$ determines a
 214 collection of given scalar-valued functions.

215 When \mathcal{V}^α is evaluated for $\vartheta = q(t)$ and $\boldsymbol{\nu} = \dot{q}(t)$, we use the notation

$$\mathcal{V}^\alpha(q(t), \dot{q}(t), t) = [\mathcal{V}^\alpha \circ (q, \dot{q}, \tau)](t) =: \hat{\mathcal{V}}^\alpha(t), \quad (4)$$

216 where we have employed the composition of maps to define the scalar-valued
 217 function $\hat{\mathcal{V}}^\alpha : \mathcal{T} \rightarrow \mathbb{R}$, with $\tau : \mathcal{T} \rightarrow \mathcal{T}$ being the *time-identity* function [23]
 218 defined as $\tau(t) := t$. Finally, for all the pairs (q, \dot{q}) that are admissible in the
 219 sense specified by Eq. (1), i.e., $(q, \dot{q}) :]t_{\text{in}}, t_{\text{fin}}[\rightarrow T\mathcal{C}_c$, we can write

$$\hat{\mathcal{V}}^\alpha \equiv \mathcal{V}^\alpha \circ (q, \dot{q}, \tau) = [a^{\alpha}_k \circ (q, \tau)] \dot{q}^k = 0, \quad \alpha = 1, \dots, m. \quad (5)$$

220 In Eq. (5), we have used Einstein's notation to imply a summation over the
 221 repeated index k , and, in the following, we will make large use of such notation,
 222 unless otherwise specified. Note that the *linear independence* of the constraints
 223 is fulfilled by requiring the functions $[a^{\alpha}_k]_{k=1, \dots, n}^{\alpha=1, \dots, m}$ to define a (rectangular)
 224 matrix of rank m .

225 Now that the kinematics of the system \mathfrak{M} is described, we can introduce
 226 the variations of q and \dot{q} as

$$\tilde{Q} : [t_{\text{in}}, t_{\text{fin}}] \times]-\varepsilon_0, +\varepsilon_0[\rightarrow \mathcal{C}, \quad (6a)$$

$$\tilde{V} : [t_{\text{in}}, t_{\text{fin}}] \times]-\varepsilon_0, +\varepsilon_0[\rightarrow T_{\tilde{Q}}\mathcal{C}, \quad (6b)$$

227 with $\varepsilon_0 > 0$ being a smallness parameter, and we require them to be such that

$$\tilde{Q}(t, \varepsilon) \in \mathcal{C}, \quad \tilde{Q}(t, 0) = q(t), \quad \forall t \in [t_{\text{in}}, t_{\text{fin}}], \quad (7a)$$

$$\tilde{V}(t, \varepsilon) \in T_{\tilde{Q}(t, \varepsilon)}\mathcal{C}, \quad \tilde{V}(t, 0) = \dot{q}(t), \quad \forall t \in [t_{\text{in}}, t_{\text{fin}}], \quad (7b)$$

228 where, in Eq. (7b), \dot{q} is prolonged by continuity to t_{in} and t_{fin} .

229 By varying $t \in [t_{\text{in}}, t_{\text{fin}}]$ and $\varepsilon \in]-\varepsilon_0, +\varepsilon_0[$, one can determine the two-
 230 parameter family of tangent spaces $T_{\tilde{Q}}\mathcal{C}$, such that $T_{\tilde{Q}(t,\varepsilon)}\mathcal{C}$ is the element
 231 of this family obtained for fixed $(t, \varepsilon) \in [t_{\text{in}}, t_{\text{fin}}] \times]-\varepsilon_0, +\varepsilon_0[$, and $T_q\mathcal{C} \equiv$
 232 $T_{\tilde{Q}(\cdot, 0)}\mathcal{C}$ is the one-parameter family of tangent spaces identified by $\varepsilon = 0$ and
 233 t varying in $[t_{\text{in}}, t_{\text{fin}}]$.

234 To avoid technicalities, we assume that both \tilde{Q} and \tilde{V} are C^2 maps over
 235 their domain, i.e., $[t_{\text{in}}, t_{\text{fin}}] \times]-\varepsilon_0, +\varepsilon_0[$, and we set

$$\eta(t) := \frac{\partial \tilde{Q}}{\partial \varepsilon}(t, 0), \quad \forall t \in [t_{\text{in}}, t_{\text{fin}}], \quad (8a)$$

$$\zeta(t) := \frac{\partial \tilde{V}}{\partial \varepsilon}(t, 0), \quad \forall t \in [t_{\text{in}}, t_{\text{fin}}], \quad (8b)$$

236 where $\eta : [t_{\text{in}}, t_{\text{fin}}] \rightarrow T_q\mathcal{C}$ and $\zeta : [t_{\text{in}}, t_{\text{fin}}] \rightarrow T_q\mathcal{C}$ are the *first-order variations*
 237 of q and \dot{q} , respectively, and η is such that $\eta(t_{\text{in}}) = \eta(t_{\text{fin}}) = 0$ [12, 13, 23].
 238 Before proceeding, to simplify the notation, in analogy with the maps \tilde{Q} and
 239 \tilde{V} , we introduce the auxiliary map $\tilde{\mathcal{J}} : [t_{\text{in}}, t_{\text{fin}}] \times]-\varepsilon_0, +\varepsilon_0[\rightarrow [t_{\text{in}}, t_{\text{fin}}]$ defined
 240 as $\tilde{\mathcal{J}}(t, \varepsilon) = t$ so that $\partial_\varepsilon \tilde{\mathcal{J}}(t, \varepsilon) = 0$ applies everywhere in its domain.

241 We evaluate now the constraints for the varied configuration and varied
 242 velocity, defined in Eqs. (6a) and (6b), and we set [12]

$$\tilde{\mathcal{V}}^\alpha := \mathcal{V}^\alpha \circ (\tilde{Q}, \tilde{V}, \tilde{\mathcal{J}}) = [a^\alpha_k \circ (\tilde{Q}, \tilde{\mathcal{J}})]\tilde{V}^k. \quad (9)$$

243 Clearly, evaluating $\tilde{\mathcal{V}}^\alpha$ at $\varepsilon = 0$ for each $\alpha = 1, \dots, m$ yields the original set of
 244 constraints, i.e.,

$$\tilde{\mathcal{V}}^\alpha(\cdot, 0) = \mathcal{V}^\alpha \circ (q, \dot{q}, \tau) = [a^\alpha_k \circ (q, \tau)]\dot{q}^k = 0. \quad (10)$$

245 However, following [12], we also require that each $\tilde{\mathcal{V}}^\alpha$, for $\alpha = 1, \dots, m$, satisfies
 246 the corresponding constraint at the first order in ε , i.e., up to orders $o(\varepsilon)$
 247 in the limit $\varepsilon \rightarrow 0$. This requirement, in turn, is expressed by enforcing the set
 248 of conditions:

$$\frac{\partial \tilde{\mathcal{V}}^\alpha}{\partial \varepsilon}(\cdot, 0) = \left[\frac{\partial a^\alpha_i}{\partial q^k} \circ (q, \tau) \right] \eta^k \dot{q}^i + [a^\alpha_k \circ (q, \tau)] \zeta^k = 0, \quad \alpha = 1, \dots, m. \quad (11)$$

249 Furthermore, by adhering to the framework developed in [11, 12], we *hypothe-*
 250 *size* that, in the jargon of [12], all the constraints considered in this work are
 251 *ideal*, in the sense that they satisfy the so-called *Lagrange-Chetaev conditions*
 252 [12, 18, 22], which are expressed in the form

$$\sum_{k=1}^n \left[\frac{\partial \mathcal{V}^\alpha}{\partial \dot{q}^k} \circ (q, \dot{q}, \tau) \right] \eta^k = \sum_{k=1}^n [a^\alpha_k \circ (q, \tau)] \eta^k = 0, \quad \alpha = 1, \dots, m. \quad (12)$$

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253 Note that, since, for each $\alpha = 1, \dots, m$, \mathcal{V}^α is linear in the velocities, Eq. (12)
 254 returns the constraints in the form $\mathcal{V}^\alpha \circ (q, \boldsymbol{\eta}, \tau) = 0$, i.e., with $\boldsymbol{\eta}$ replacing
 255 \dot{q} , thereby prescribing the conditions that must be fulfilled by the first-order
 256 variations η^1, \dots, η^n (see [34, 35]). Thus, by computing the derivative of Eq.
 257 (12) with respect to time, we obtain

$$\begin{aligned} \frac{d}{dt} \left\{ \sum_{k=1}^n \left[\frac{\partial \mathcal{V}^\alpha}{\partial \dot{q}^k} \circ (q, \dot{q}, \tau) \right] \eta^k \right\} &= \frac{d}{dt} \left\{ \sum_{k=1}^n [a^\alpha_k \circ (q, \tau)] \eta^k \right\} \\ &= \left[\frac{\partial a^\alpha_k}{\partial q^i} \circ (q, \tau) \right] \eta^k \dot{q}^i + \left[\frac{\partial a^\alpha_k}{\partial \tau} \circ (q, \tau) \right] \eta^k + [a^\alpha_k \circ (q, \tau)] \dot{\eta}^k = 0. \end{aligned} \quad (13)$$

258 Moreover, by computing the difference between Eq. (11) and (13), and by
 259 changing the indexes appropriately, we obtain, for each $\alpha = 1, \dots, m$, the
 260 relation [12, 16, 17]

$$\begin{aligned} 0 &= \frac{\partial \tilde{\mathcal{V}}^\alpha}{\partial \varepsilon}(\cdot, 0) - \frac{d}{dt} \left\{ \sum_{k=1}^n \left[\frac{\partial \mathcal{V}^\alpha}{\partial \dot{q}^k} \circ (q, \dot{q}, \tau) \right] \eta^k \right\} \\ &= \left[\left(\frac{\partial a^\alpha_i}{\partial q^k} - \frac{\partial a^\alpha_k}{\partial q^i} \right) \circ (q, \tau) \right] \eta^k \dot{q}^i + \left[-\frac{\partial a^\alpha_k}{\partial \tau} \circ (q, \tau) \right] \eta^k \\ &\quad + [a^\alpha_h \circ (q, \tau)] (\zeta^h - \dot{\eta}^h). \end{aligned} \quad (14)$$

261 Equation (14) produces a set of m linear relations between the components of
 262 $\boldsymbol{\zeta} - \dot{\boldsymbol{\eta}}$ and the components of $\boldsymbol{\eta}$. Thus, by following the framework established in
 263 [11, 12], we *hypothesize* (see Remark 1) the existence of a linear transformation
 264 between $\boldsymbol{\zeta} - \dot{\boldsymbol{\eta}}$ and $\boldsymbol{\eta}$ represented by a matrix $\mathbf{W} := [W^h_k]_{k=1, \dots, n}^{h=1, \dots, m}$ such that
 265 [11, 12]

$$\zeta^h - \dot{\eta}^h = \sum_{k=1}^n W^h_k \eta^k, \quad h = 1, \dots, m. \quad (15)$$

266 In the terminology of [11, 12, 25–27, 39], the relations reported in Eq. (15)
 267 are referred to as “*transpositional relations*”, since they express, at the first
 268 order in ε , the fact that, for each $h = 1, \dots, m$, the variation of the time
 269 derivative of the Lagrangian parameter q^h , i.e., ζ^h , is not equal, in general, to
 270 the time derivative of the variation of q^h , i.e., $\dot{\eta}^h$. The statement just given here
 271 contains a deep geometric meaning, and can be formalized quite rigorously by
 272 employing the language of Differential Geometry developed, e.g., in [41]. In
 273 this respect, one should introduce the concept of *second tangent bundle* of \mathcal{C} ,
 274 denoted by $TT\mathcal{C}$, and declare $\boldsymbol{\zeta}$ and $\dot{\boldsymbol{\eta}}$ as elements of $TT\mathcal{C}$, but belonging to
 275 different fibers of $TT\mathcal{C}$. A study of constrained mechanical systems conducted
 276 by having recourse to such concepts of Differential Geometry can be found in
 277 [41], which, in turn, employs the framework established in [42].

278 To the best of our understanding, the variations taken in the MVM pro-
 279 posed by Llibre et al. [12] live in the intersection between two sets of variations
 280 introduced in a work by Józwiowski&Respondek [22] (based on the paper by

281 Gràcia et al. [48]), i.e., the set of “nonholonomic admissible variations” [22],
 282 defined according to “Chetaev's principle” [22], and the set of “vakonomic ad-
 283 missible variations” [22]. The latter variations are also in agreement with those
 284 presented in the vakonomic approach provided by Lemos [1]. The reason for
 285 requiring that the variations of the MVM belong to the intersection mentioned
 286 above is that we need the consistency with Lagrange-Chetaev's conditions in
 287 order to look for coherence between the TNHMs (based on the fulfillment of
 288 such conditions) and the MVM. More details are reported in “Appendix A”.

289 *Remark 1* (Transpositional relations [11, 12, 26, 27, 39, 50])

290 Granted the approach specified above [11, 12], which, as remarked in [12], places
 291 itself in the school of thought of “Suslov, Voronets, Levi-Civita, Amaldi, . . .” [12], the
 292 presence of non-holonomic constraints renders the operations of “variation” and “time
 293 differentiation” non-commutative in general [16, 17]. More in detail, as discussed
 294 in [11, 12, 26, 39, 50], if a generalized velocity featuring in a given constraint is
 295 taken as *dependent*, i.e., as a function of the other *independent* velocities, then its
 296 corresponding transpositional relation does not vanish. On the contrary, *independent*
 297 velocities produce null transpositional relations [11, 12, 26, 39, 50].

298 To contextualize the introduction of the *square* matrix $W = [W^h_k]_{k=1, \dots, n}^{h=1, \dots, n}$ in
 299 Eq. (15), we briefly review the point of view of Jarzębowska [50], who summa-
 300 rized the discussion of Neimark and Fufaev [26] on the transpositional relations and
 301 the analysis of motion in terms of *quasi-velocities* and *quasi-coordinates*. Hence, by
 302 slightly modifying Jarzębowska's notation [50], we denote here by $\omega^1, \dots, \omega^n$ the n
 303 quasi-velocities of a given mechanical system (they must be as many as the “true”
 304 velocities). Each quasi-velocity ω^h , with $h = 1, \dots, n$, is, in general, a function defined
 305 as $\omega^h := \hat{\omega}^h \circ (q, \dot{q}, \tau)$, which supplies a convenient reformulation of the system's
 306 kinematics. For systems featuring $m \leq n$ non-holonomic constraints ($m \geq 1$), it is
 307 useful to identify the first m quasi-velocities with the functions expressing the con-
 308 straints themselves, i.e., $\omega^\alpha := \hat{\omega}^\alpha \circ (q, \dot{q}, \tau) \equiv \mathcal{V}^\alpha \circ (q, \dot{q}, \tau) = 0$, with $\alpha = 1, \dots, m$
 309 [26, 50] (indeed, this way, the constraints are satisfied as $\omega^1 = 0, \dots, \omega^m = 0$),
 310 while, as remarked in [26], the remaining $n - m$ quasi-velocities can be assigned *ar-*
 311 *bitrarily* through generic functions \mathcal{F}^β , i.e., $\omega^\beta := \hat{\omega}^\beta \circ (q, \dot{q}, \tau) \equiv \mathcal{F}^\beta \circ (q, \dot{q}, \tau)$, with
 312 $\beta = m + 1, \dots, n$, provided that they represent a change of variables in the “space”
 313 of the velocities. Thus, by denoting by $H^h_k := \partial_{\dot{q}^k} \hat{\omega}^h \circ (q, \dot{q}, \tau)$ the coefficients of the
 314 non-singular Jacobian matrix H associated with the quasi-velocities, and adapting
 315 Jarzębowska's procedure [50] to ours, we rewrite Equation (16) of [50] as

$$\zeta^h - \dot{\eta}^h = -[H^{-1}]^h_i \left\{ \frac{\partial \hat{\omega}^i}{\partial \dot{q}^k} \circ (q, \dot{q}, \tau) - \frac{d}{dt} \left[\frac{\partial \hat{\omega}^i}{\partial \dot{q}^k} \circ (q, \dot{q}, \tau) \right] \right\} \eta^k, \quad (16)$$

316 which is obtained by identifying our ζ^h , $\dot{\eta}^h$, and η^k with $\delta \dot{q}_\lambda$, $(\delta q_\lambda)'$, and δq_λ of [50],
 317 respectively, and assuming that the difference $\delta \omega_r - (\delta \pi_r)'$ is null for each $r = 1, \dots, n$
 318 (here $\delta \pi_r$ is the r th form defined by “ $\delta \pi_r := [\partial_{\dot{q}^\sigma} \omega_r] \delta q^\sigma$ ” in [50]). We emphasize that
 319 the latter assumption is not made in [50], although Neimark and Fufaev [26] explain
 320 that it can be done always. At this stage, if in Eq. (16) we set

$$W^h_k := -[H^{-1}]^h_i \left\{ \frac{\partial \hat{\omega}^i}{\partial \dot{q}^k} \circ (q, \dot{q}, \tau) - \frac{d}{dt} \left[\frac{\partial \hat{\omega}^i}{\partial \dot{q}^k} \circ (q, \dot{q}, \tau) \right] \right\}, \quad (17)$$

321 we retrieve Eq. (15). Note that the matrix W obtained this way is related to the one
 322 defined in [50] through $W_J = HWH^{-1}$, where “J” stands for “Jarzębowska”.

If m is the number of nonholonomic constraints linear in the velocities, i.e., if $\omega^\alpha = \hat{\omega}^\alpha \circ (q, \dot{q}, \tau) = [a^\alpha_k \circ (q, \tau)] \dot{q}^k$, as is the case addressed in this work, the terms between braces in Eq. (17), evaluated for $i = \alpha = 1, \dots, m$, become

$$\frac{\partial \hat{\omega}^\alpha}{\partial q^k} \circ (q, \dot{q}, \tau) - \frac{d}{dt} \left[\frac{\partial \hat{\omega}^\alpha}{\partial \dot{q}^k} \circ (q, \dot{q}, \tau) \right] = \left[\left(\frac{\partial a^\alpha_i}{\partial q^k} - \frac{\partial a^\alpha_k}{\partial q^i} \right) \circ (q, \tau) \right] \dot{q}^i - \frac{\partial a^\alpha_k}{\partial \tau} \circ (q, \tau), \quad (18)$$

which, after multiplication by η^k , is the sum of the first two terms on the right-hand side of Eq. (14). In conclusion, in deriving Eq. (15), we have hypothesized the existence of the linear transformation relating $\zeta - \dot{\eta}$ with η because in Eq. (14) the matrix corresponding to $[a^\alpha_h \circ (q, \tau)]$ is rectangular, and, thus, cannot be inverted to obtain Eq. (16) at once.

Remark 2 (Transpositional relations and the “Canonical flip” [22, 41])

A relevant result of the work by Grabowska&Grabowski [41] is their Eq. (2.5), which repropose the transpositional relations discussed by Neimark&Fufaev [26], and establishes a peculiar relationship between the quantities that, in our context, are denoted by ζ and $\dot{\eta}$. In our notation, which follows the one adopted in [26], the transpositional relations could be written as [22, 26, 41]

$$\zeta^h = \dot{\eta}^h + \sum_{\ell, k=1}^n \mathcal{C}^h_{\ell k} \dot{q}^\ell \eta^k, \quad (19)$$

where $\mathcal{C}^h_{\ell k} = -\mathcal{C}^h_{k\ell}$ is a collection of functions that is skew-symmetric in the lower indices, and represents, in local coordinates, the fourth slot of the “Canonical flip” between $TT\mathcal{C}$ and itself [22, 41] (see Appendix A). However, adapted to our framework, in which the constraints are linear in the velocities (cf. Eq. (5)), and are allowed to depend explicitly on time, Eq. (19) is reformulated as

$$\zeta^h = \eta^h + \sum_{k=1}^n \mathcal{C}^h_{0k} \eta^k + \sum_{\ell, k=1}^n \mathcal{C}^h_{\ell k} \dot{q}^\ell \eta^k. \quad (20)$$

This result, in fact, is consistent with the skew-symmetry of the first two terms on the far right-hand side of Eq. (14)¹.

Although obtained in a different context, Eq. (15) (see [11, 12]) can be understood as an equivalent form of the relationship (20) adapted from Grabowska&Grabowski [41], provided the following identification is made:

$$\begin{aligned} \zeta^h - \dot{\eta}^h &= \sum_{k=1}^n W^h_{\ k} \eta^k \equiv \sum_{k=1}^n \mathcal{C}^h_{0k} \eta^k + \sum_{\ell, k=1}^n \mathcal{C}^h_{\ell k} \dot{q}^\ell \eta^k \\ &= \sum_{k=1}^n \{ \mathcal{C}^h_{0k} + \sum_{\ell=1}^n \mathcal{C}^h_{\ell k} \dot{q}^\ell \} \eta^k. \end{aligned} \quad (21)$$

In the remainder of our work, and, in particular, in the presentation of the benchmark problems analyzed below, we will show how relationships similar to Eq. (21) are obtained. Moreover, we will emphasize that the skew-symmetry of the functions $\mathcal{C}^h_{\ell k}$ in the lower indices is respected as a consequence of the structure of the coefficients $W^h_{\ k}$. More specifically, we will present two situations in which Eq. (21) is studied with special care: One case (see Sect. 4.2.2) requires redefining the coefficients $\mathcal{C}^h_{\ell k}$ as functions of q , \dot{q} , and τ (this will involve, in fact, the dependence of these coefficients

¹Whereas the skew-symmetry of the first term is obvious, that of the second one becomes evident if, for each $\alpha = 1, \dots, m$, we introduce an identically null function $a^{\alpha_0} \circ (q, \tau)$, whose only role is to yield the expression $[\partial_k a^{\alpha_0} - \partial_\tau a^{\alpha_k}] \circ (q, \tau)$. Note, however, that this is true because, throughout our work, we *are not* considering constraints affine in the velocities. Indeed, should these constraints be considered, the function $a^{\alpha_0} \circ (q, \tau)$ would not be identically zero, and Eqs. (19) and (20) would change accordingly (see [51] for a discussion on constraints affine in the velocities).

354 on the generalized momenta of the problem); the second case (see Sect. 4.2.3), instead,
 355 requires redefining the generalized velocities \dot{q} by suitably normalizing the generalized
 356 momenta mentioned above.

357 Before closing this section, we remark that one crucial difference between
 358 the MVM and the VM is that, while the MVM is formulated by accounting
 359 explicitly for the *transpositional relations*, the VM is developed without even
 360 introducing such relations [7–12].

361 2.2 Hamilton-Suslov variational principle

362 Given the mechanical system \mathfrak{M} introduced above, and assuming \mathfrak{M} to be
 363 subjected to m non-holonomic constraints of the type specified in Eq. (5),
 364 we denote by $\mathcal{L} : T\mathcal{C} \times [t_{\text{in}}, t_{\text{fin}}] \rightarrow \mathbb{R}$ the Lagrangian function that would
 365 characterize \mathfrak{M} if none of the m imposed constraints were present. Moreover,
 366 we adopt the composition $\hat{\mathcal{L}} := \mathcal{L} \circ (q, \dot{q}, \tau) : [t_{\text{in}}, t_{\text{fin}}] \rightarrow \mathbb{R}$ to rephrase \mathcal{L} as a
 367 function of time. Then, by following [7–10], we introduce also the *constrained*
 368 *Lagrangian function* of \mathfrak{M} as

$$\hat{\mathcal{L}}_c \equiv \mathcal{L}_c \circ (q, \dot{q}, \tau, \lambda) := \mathcal{L} \circ (q, \dot{q}, \tau) - \sum_{\alpha=1}^m \lambda_\alpha [\mathcal{V}^\alpha \circ (q, \dot{q}, \tau)], \quad (22)$$

369 where $\lambda_1, \dots, \lambda_m$ are m Lagrange multipliers associated with the m non-
 370 holonomic constraints. For the sake of a compact notation, such Lagrange
 371 multipliers can be collected in the array $\lambda := (\lambda_1, \dots, \lambda_m)$. Next, we define the
 372 action functional associated with \mathcal{L}_c over the time interval $[t_{\text{in}}, t_{\text{fin}}]$ as [7–10]

$$\begin{aligned} \mathcal{A}_c(q, \lambda) &:= \int_{t_{\text{in}}}^{t_{\text{fin}}} \mathcal{L}_c(q(t), \dot{q}(t), \tau(t), \lambda(t)) dt = \int_{t_{\text{in}}}^{t_{\text{fin}}} \hat{\mathcal{L}}_c(t) dt \\ &= \int_{t_{\text{in}}}^{t_{\text{fin}}} \left\{ \hat{\mathcal{L}}(t) - \sum_{\alpha=1}^m \lambda_\alpha(t) \hat{\mathcal{V}}^\alpha(t) \right\} dt. \end{aligned} \quad (23)$$

373 To perform the variation of the action functional \mathcal{A}_c , and in analogy with
 374 the notation given in Eqs. (6a), (6b), and (9), we write $\tilde{\Lambda}(t, \varepsilon)$ to indicate the
 375 collection of varied Lagrange multipliers, and [7–10, 12]

$$\tilde{\mathcal{L}}(t, \varepsilon) := \mathcal{L}(\tilde{Q}(t, \varepsilon), \tilde{\mathbf{V}}(t, \varepsilon), \tilde{\mathcal{T}}(t, \varepsilon)) \quad (24)$$

376 to indicate the varied Lagrangian function. Note that, for each $\alpha = 1, \dots, n$,
 377 we set $\tilde{\Lambda}_\alpha(t, 0) = \lambda_\alpha(t)$, and we call $\gamma_\alpha(t) := \partial_\varepsilon \tilde{\Lambda}_\alpha(t, 0)$ the first-order variation
 378 [12]. Accordingly, the varied action functional can be written as a function of
 379 ε , and reads [7–10]

$$\tilde{\mathcal{A}}_c(\varepsilon) := \int_{t_{\text{in}}}^{t_{\text{fin}}} \mathcal{L}_c(\tilde{Q}(t, \varepsilon), \tilde{\mathbf{V}}(t, \varepsilon), \tilde{\mathcal{T}}(t, \varepsilon), \tilde{\Lambda}(t, \varepsilon)) dt$$

$$= \int_{t_{\text{in}}}^{t_{\text{fin}}} \left\{ \tilde{\mathcal{L}}(t, \varepsilon) - \sum_{\alpha} \tilde{\Lambda}_{\alpha}(t, \varepsilon) \tilde{\mathcal{V}}^{\alpha}(t, \varepsilon) \right\} dt, \quad (25)$$

where we have simplified the notation by using the convention according to which, in the summations, the index α runs from 1 to m . A similar notation will be employed in the sequel also for summations over indices running from 1 to n . We remark that, in Eqs. (23) and (25), the structure of the Lagrangian function \mathcal{L}_c is taken from [7–10], while the way in which the varied velocity $\tilde{\mathcal{V}}(t, \varepsilon)$ is defined rephrases the definition given in [12]. Finally, for the sake of brevity and for future use, from here on we denote by

$$\sharp^{(1)} := (q, \dot{q}, \tau), \quad \sharp_c^{(1)} := (q, \dot{q}, \tau, \lambda) \quad (26)$$

the lists of arguments of \mathcal{L} and \mathcal{L}_c , respectively, and by

$$\sharp^{(2)} := (q, \dot{q}, \ddot{q}, \tau), \quad \sharp_c^{(2)} := (q, \dot{q}, \ddot{q}, \tau, \lambda, \dot{\lambda}) \quad (27)$$

the lists of arguments of the corresponding Euler-Lagrange operators, i.e.,

$$\mathcal{E}_k \mathcal{L} \circ \sharp^{(2)} := \frac{\partial \mathcal{L}}{\partial q^k} \circ \sharp^{(1)} - \frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{q}^k} \circ \sharp^{(1)} \right], \quad k = 1, \dots, n, \quad (28a)$$

$$\mathcal{E}_k \mathcal{L}_c \circ \sharp_c^{(2)} := \frac{\partial \mathcal{L}_c}{\partial q^k} \circ \sharp_c^{(1)} - \frac{d}{dt} \left[\frac{\partial \mathcal{L}_c}{\partial \dot{q}^k} \circ \sharp_c^{(1)} \right], \quad k = 1, \dots, n. \quad (28b)$$

With all the premises given above, the condition of stationary action at $\varepsilon = 0$ yields [12, 13]

$$\begin{aligned} \tilde{\mathcal{A}}'_c(0) &= \int_{t_{\text{in}}}^{t_{\text{fin}}} \left\{ \sum_k [\mathcal{E}_k \mathcal{L}_c \circ \sharp_c^{(2)}] \eta^k + \sum_{\alpha} \left[\frac{\partial \mathcal{L}_c}{\partial \lambda_{\alpha}} \circ \sharp_c^{(1)} \right] \gamma_{\alpha} \right\} (t) dt \\ &+ \int_{t_{\text{in}}}^{t_{\text{fin}}} \left\{ \sum_h \left[\frac{\partial \mathcal{L}_c}{\partial \dot{q}^h} \circ \sharp_c^{(1)} \right] (\zeta^h - \dot{\eta}^h) \right\} (t) dt = 0. \end{aligned} \quad (29)$$

Equation (29) can be further worked out by employing Eq. (15) to rewrite the differences $\zeta^h - \dot{\eta}^h$ in terms of $\boldsymbol{\eta}$, by introducing the operators [11, 12]

$$\mathcal{D}_k \mathcal{L}_c \circ \sharp_c^{(2)} := \mathcal{E}_k \mathcal{L}_c \circ \sharp_c^{(2)} + \sum_h \left[\frac{\partial \mathcal{L}_c}{\partial \dot{q}^h} \circ \sharp_c^{(1)} \right] W^h{}_k, \quad k = 1, \dots, n, \quad (30)$$

and by noticing that the derivative of \mathcal{L}_c with respect to λ_{α} returns, up to the sign, the α th constraint. These considerations lead to the following compact expression of the condition of stationary action:

$$\tilde{\mathcal{A}}'_c(0) = \int_{t_{\text{in}}}^{t_{\text{fin}}} \left\{ \sum_k [\mathcal{D}_k \mathcal{L}_c \circ \sharp_c^{(2)}] \eta^k + \sum_{\alpha} [-\mathcal{V}^{\alpha} \circ \sharp^{(1)}] \gamma_{\alpha} \right\} (t) dt = 0. \quad (31)$$

Since Eq. (31) must hold for any possible choice of t_{in} and $t_{\text{fin}} > t_{\text{in}}$, we require the integrand to be zero, and, since the latter condition must be fulfilled for arbitrary variations η^1, \dots, η^n and $\gamma_1, \dots, \gamma_m$, we obtain [12]

$$\mathcal{D}_k \mathcal{L}_c \circ \#_c^{(2)} = 0, \quad k = 1, \dots, n, \quad (32a)$$

$$-\mathcal{V}^\alpha \circ \#_c^{(1)} = 0, \quad \alpha = 1, \dots, m. \quad (32b)$$

This is a set of $n + m$ equations in the $n + m$ unknowns given by the n Lagrangian parameters q^1, \dots, q^n and by the m Lagrange multipliers $\lambda_1, \dots, \lambda_m$. We notice that Eq. (32a) are the dynamic equations of the problem, while Eq. (32b) return the constraints.

It must be emphasized, however, that the system of equations just obtained differs from the one of standard Analytical Mechanics for the following facts:

- (i) The dynamic equations (32a) search for functions q^1, \dots, q^n that yield, for each $k = 1, \dots, n$, the vanishing of the quantity $\mathcal{D}_k \mathcal{L}_c$ introduced by Llibre et al. [12], instead of the *standard* Euler-Lagrange operator applied to \mathcal{L}_c , i.e., $\mathcal{E}_k \mathcal{L}_c$. In this respect, it should be noticed that, if one followed Kozlov's vakonomic method [7–10], the dynamic equations would still be formulated in terms of the standard Euler-Lagrange operators applied to \mathcal{L}_c , since the transpositional relations (15) would not be explicitly considered. Exactly at this point the “modification” to the vakonomic dynamics proposed in [11, 12] comes into play. Indeed, Llibre et al. [12] resolve explicitly the transpositional relations in Eq. (15) through the introduction of the matrix W , which, in turn, leads to the definition of the quantity $\mathcal{D}_k \mathcal{L}_c$. From here on, with a slight abuse of notation, we shall refer to this quantity and to other similar ones as “operator”. To conclude, we mention that, in the case of holonomic constraints, Eq. (14) complies with the equalities $\zeta^h = \dot{\eta}^h$ for each $h = 1, \dots, n$, thereby requiring W to be the null matrix.
- (ii) Even though the number of equations (given by the sum of the number of the dynamic equations and of the constraints) equals the number of unknowns, Eqs. (32a) and (32b) are *not* closed, because, for each $k = 1, \dots, n$, the operator $\mathcal{D}_k \mathcal{L}_c$ features the n^2 unknown coefficients of W . To overcome this problem, n^2 auxiliary conditions need to be imposed (see the difference with the approach sketched in Remark 1).
- (iii) In this theory, the equations for the Lagrange multipliers are first order differential equations, whereas they are algebraic in the TNHM.

2.3 Solvability conditions

Equations (32a) and (32b) can be rewritten as

$$\mathcal{D}_k \mathcal{L} \circ \#^{(2)} + \sum_\alpha \dot{\lambda}_\alpha \left[\frac{\partial \mathcal{V}^\alpha}{\partial \dot{q}^k} \circ \#^{(1)} \right] - \sum_\alpha \lambda_\alpha [\mathcal{D}_k \mathcal{V}^\alpha \circ \#^{(2)}] = 0, \quad k = 1, \dots, n, \quad (33a)$$

$$-\mathcal{V}^\alpha \circ \#^{(1)} = 0, \quad \alpha = 1, \dots, m. \quad (33b)$$

As anticipated in Sect. 2.2, we need to assign n^2 additional conditions, denoted hereafter as the *solvability conditions* of the MVM, in order to close the system (33a) and (33b). In particular, due to the properties of the constraints highlighted in Eqs. (14) and (15), mn conditions can be found by substituting Eq. (15) into Eq. (14), which amounts to setting

$$\mathcal{D}_k \mathcal{V}^\alpha \circ \sharp^{(2)} = 0, \quad k = 1, \dots, n, \quad \alpha = 1, \dots, m. \quad (34)$$

The last $n(n - m)$ conditions are assigned via the prescription of *Ansätze*, in which either $n - m$ auxiliary functions satisfying certain conditions [12] are introduced or the fulfillment of some physics-based conditions is assumed.

2.3.1 Ansatz 1: approach based on the auxiliary functions

To solve Eqs. (33a) and (33b), Llibre et al. [12] determine univocally the coefficients of W by introducing $n - m$ auxiliary functions $\mathcal{F}^\beta \circ (q, \dot{q}, \tau)$, with $\beta = m + 1, \dots, n$, which yield the following $n(n - m)$ conditions

$$\mathcal{D}_k \mathcal{F}^\beta \circ \sharp^{(2)} = 0, \quad k = 1, \dots, n, \quad \beta = m + 1, \dots, n. \quad (35)$$

As anticipated in Remark 1, the rationale behind the introduction of the set of functions $\{\mathcal{F}^\beta\}_{\beta=m+1}^n$ relies on the concept of quasi-velocities (see e.g. [26, 27, 50]), i.e., a set of functions $\{\omega^h\}_{h=1}^n$ constituting a reparameterization of the velocities $\{\dot{q}^k\}_{k=1}^n$ such that $\omega^h(t) = \hat{\omega}^h(q(t), \dot{q}(t), t)$, for all $h = 1, \dots, n$, and the Jacobian $H^h_k(t) := \partial_{\dot{q}^k} \hat{\omega}^h(q(t), \dot{q}(t), t)$ is non-singular. In fact, for a mechanical system subjected to m constraints on the velocities, it is possible to choose the first m quasi-velocities coincident with the constraints themselves (see Remark 1), and to identify the remaining $n - m$ quasi-velocities with the “arbitrary” functions $\omega^{m+1} \equiv \mathcal{F}^{m+1}, \dots, \omega^n \equiv \mathcal{F}^n$. However, in spite of this arbitrariness, some selection rules are necessary (see [11, 12]). To this end, the identification of the functions $\{\mathcal{F}^\beta\}_{\beta=m+1}^n$ with the corresponding quasi-velocities $\{\omega^\beta\}_{\beta=m+1}^n$ notwithstanding, it is instructive to critically review the method outlined in [12] because it involves conditions on the auxiliary functions that are called for by the variational procedure.

As a starting point, the functions $\mathcal{F}^{m+1}, \dots, \mathcal{F}^n$ must be such that the matrix $H = [H^i_k]_{k=1, \dots, n}^{i=1, \dots, n}$, with components

$$H^i_k = \begin{cases} [\partial_{\dot{q}^k} \mathcal{V}^\alpha] \circ (q, \dot{q}, \tau), & i = \alpha = 1, \dots, m, & k = 1, \dots, n, \\ [\partial_{\dot{q}^k} \mathcal{F}^\beta] \circ (q, \dot{q}, \tau), & i = \beta = m + 1, \dots, n, & k = 1, \dots, n, \end{cases} \quad (36)$$

is *non-singular*, i.e., $\det H \neq 0$, almost everywhere in $T\mathcal{C} \times [t_{\text{in}}, t_{\text{fin}}]$. However, on this point, we would like to emphasize that:

- (i) The conditions in Eq. (35) may be regarded as *gauge conditions* [24] for the problem under study. In fact, through the introduction of $\mathcal{F}^{m+1}, \dots, \mathcal{F}^n$, we

462 can transform *a posteriori* the Lagrangian \mathcal{L}_c in Eq. (22) as [12]

$$\mathcal{L}_c \circ \sharp_c^{(1)} \mapsto \mathcal{L}_c \circ \sharp_c^{(1)} - \sum_{\beta} \kappa_{\beta} [\mathcal{F}^{\beta} \circ \sharp_c^{(1)}] =: \mathcal{L}_L \circ (\sharp_c^{(1)}, \kappa), \quad (37)$$

463 where the summation over β is done from $m + 1$ to n ; $\kappa_{m+1}, \dots, \kappa_n$ are
 464 constant-valued parameters, i.e., $\kappa_{\beta}(t) = \kappa_{\beta 0} \in \mathbb{R}$, for $\beta = m + 1, \dots, n$, and
 465 for all $t \in [t_{\text{in}}, t_{\text{fin}}]$; and $\kappa := (\kappa_{m+1}, \dots, \kappa_n)$ collects all such parameters.
 466 Therefore, by applying the variational procedure described in Sect. 2.2 to the
 467 case in which the Lagrangian is transformed as in Eq. (37), it is necessary
 468 to impose Eq. (35) in order for the dynamic equations associated with \mathcal{L}_L
 469 to be *invariant* with respect to those associated with \mathcal{L}_c , reported in Eq.
 470 (33a). However, this invariance of the dynamic equations is only sufficient
 471 for the equivalence of the descriptions provided by \mathcal{L}_L and \mathcal{L}_c . Indeed, also
 472 the next remark has to be considered.

473 (ii) The transformation in Eq. (37) has to be consistent with the Principle of
 474 Stationary Action [13], in the sense that the Action associated with the
 475 transformed Lagrangian needs to have the same stationary points as the
 476 non-transformed Action. In other words, upon setting

$$\mathcal{A}_L(q, \lambda, \kappa) := \int_{t_{\text{in}}}^{t_{\text{fin}}} \mathcal{L}_L(q(t), \dot{q}(t), t, \lambda(t), \kappa(t)) dt, \quad (38)$$

477 it must hold true that $\mathcal{A}_L(q, \lambda, \kappa) = \mathcal{A}_c(q, \lambda) + C$, where C is a real constant,
 478 i.e., the Actions $\mathcal{A}_L(q, \lambda, \kappa)$ and $\mathcal{A}_c(q, \lambda)$ only differ additively by a constant
 479 term [13]. To this end, the generic function \mathcal{F}^{β} will be chosen either as a
 480 total derivative, e.g. a generalized velocity, or as a function of the generalized
 481 velocities (quasi-velocity) that carries some physical interpretation for the
 482 problem at hand. Note that, in the second case, the associated coefficient κ_{β}
 483 must be zero in order to satisfy the previous hypothesis on \mathcal{A}_L [12]. To see
 484 this, assume that there exists $\bar{\beta} \in \{m + 1, \dots, n\}$ such that $\mathcal{F}^{\bar{\beta}}$ is *not* the
 485 total derivative of a function of the type $f \circ (q, \tau)$, while all other functions \mathcal{F}^{β}
 486 are so, with $\beta \in \{m + 1, \dots, n\} \setminus \{\bar{\beta}\}$. Then, the Action $\mathcal{A}_L(q, \lambda, \kappa)$ becomes

$$\begin{aligned} \mathcal{A}_L(q, \lambda, \kappa) &:= \mathcal{A}_c(q, \lambda) - \sum_{\beta \neq \bar{\beta}} \int_{t_{\text{in}}}^{t_{\text{fin}}} \kappa_{\beta}(t) \mathcal{F}^{\beta}(\sharp_c^{(1)}(t)) dt - \int_{t_{\text{in}}}^{t_{\text{fin}}} \kappa_{\bar{\beta}}(t) \mathcal{F}^{\bar{\beta}}(\sharp_c^{(1)}(t)) dt \\ &= \mathcal{A}_c(q, \lambda) - \underbrace{\sum_{\beta \neq \bar{\beta}} \kappa_{\beta 0} \int_{t_{\text{in}}}^{t_{\text{fin}}} \mathcal{F}^{\beta}(\sharp_c^{(1)}(t)) dt - \kappa_{\bar{\beta} 0} \int_{t_{\text{in}}}^{t_{\text{fin}}} \mathcal{F}^{\bar{\beta}}(\sharp_c^{(1)}(t)) dt}_{=: C} \\ &= \mathcal{A}_c(q, \lambda) + C - \kappa_{\bar{\beta} 0} \int_{t_{\text{in}}}^{t_{\text{fin}}} \mathcal{F}^{\bar{\beta}}(\sharp_c^{(1)}(t)) dt, \end{aligned} \quad (39)$$

487 which means that $\kappa_{\bar{\beta} 0}$ must be zero.

488(iii) Last but not least, the auxiliary functions must be chosen in such a way
 489 that the coefficients of W computed through Eq. (35) must yield *zero trans-*
 490 *positional relations* for those velocities that do not feature in the constraints
 491 or that, if featuring in them, can be varied independently of the other ones
 492 (see Remark 1, and [51]). Operatively, if β^* corresponds to a velocity that
 493 can be chosen as independent or absent in the constraints, one can choose
 494 $\mathcal{F}^{\beta^*} \circ \sharp^{(1)} = \dot{q}^{\beta^*}$. A function of this type complies trivially with the require-
 495 ments (i), (ii), and with the present one, and implies that the β^* th row of
 496 W is identically null.

497 Given these considerations on the choice of the auxiliary functions, Eqs. (34)
 498 and (35) can be rewritten in explicit form as [12]

$$\sum_h \left[\frac{\partial \mathcal{V}^\alpha}{\partial \dot{q}^h} \circ \sharp^{(1)} \right] W^h{}_k = -\mathcal{E}_k \mathcal{V}^\alpha \circ \sharp^{(2)}, \quad k = 1, \dots, n, \quad \alpha = 1, \dots, m, \quad (40a)$$

$$\sum_h \left[\frac{\partial \mathcal{F}^\beta}{\partial \dot{q}^h} \circ \sharp^{(1)} \right] W^h{}_k = -\mathcal{E}_k \mathcal{F}^\beta \circ \sharp^{(2)}, \quad k = 1, \dots, n, \quad \beta = m + 1, \dots, n, \quad (40b)$$

499 so that the coefficients $W^h{}_k$ are obtained by solving (40a) and (40b). Note
 500 that, Eqs. (40a) and (40b) together are equivalent to Eq. (17).

501 In spite of the selection rules mentioned above, we remark that, in the
 502 “charged skate” benchmark studied in Sect. 4, we found that an auxiliary func-
 503 tion that works well in the uncharged case, yields to results that, in our opinion,
 504 are unphysical when the skate is charged and subjected to an interaction with
 505 a magnetic field. According to our calculations, indeed, there occur inconsis-
 506 tencies with the fulfillment of certain conservation laws, that can be related
 507 to the way in which the chosen auxiliary function depends on the velocities
 508 that *are* involved in the constraint. To amend these shortcomings, we suggest
 509 to switch to a different formulation, that we report in the following *Ansatz 2*.

510 2.3.2 Ansatz 2: physics-based conditions

511 In order to close Eqs. (33a) and (33b), we can require the system to respect the
 512 symmetries that may be naturally present in the Lagrangian function \mathcal{L} and
 513 in the constraints. For instance, if we assume the existence of a Lagrangian
 514 parameter $q^{\bar{k}}$ “*ignorable*” [13] in \mathcal{L} , i.e., not explicitly featuring among its
 515 arguments, and such that $\partial_{\dot{q}^{\bar{k}}} \mathcal{V}^\alpha \circ \sharp^{(1)} = 0$ for all $\alpha = 1, \dots, m$, then, by
 516 consistency with classical Analytical Mechanics, expressed by the Lagrange-
 517 D’Alembert method, we may impose the condition

$$\sum_h p_h W^h{}_{\bar{k}} = 0, \quad p_h := \frac{\partial \mathcal{L}}{\partial \dot{q}^h} \circ \sharp^{(1)}, \quad (41)$$

518 to restore the conservation of the generalized momentum $p_{\bar{k}}$ [13].

519 Another scenario could be the one in which all the m constraints, say
 520 $f^1 \circ (q, \tau) = 0, \dots, f^m \circ (q, \tau) = 0$, are holonomic. In this case, it is known that

the equations of motion are $\mathcal{E}_k \mathcal{L} + \sum_{\alpha} \mu_{\alpha} [\partial_{q^k} f^{\alpha} \circ (q, \tau)] = 0$, for $k = 1, \dots, n$ (see, e.g., [13]), and one can put the constraints in “non-holonomic” form by setting $\mathcal{V}^{\alpha} \circ \sharp^{(1)} \equiv d_t[f^{\alpha} \circ (q, \tau)]$, with $\alpha = 1, \dots, m$ and d_t indicating the total time derivative. In this case, however, it automatically applies $\mathcal{E}_k \mathcal{V}^{\alpha} \circ \sharp^{(2)} = 0$, for all $k = 1, \dots, n$, and for all $\alpha = 1, \dots, m$. Thus, if we take $\mathcal{F}^{\beta} \circ \sharp^{(1)} = d_t[g^{\beta} \circ (q, \tau)]$, with $\beta = m + 1, \dots, n$, and $g^{\beta} \circ (q, \tau)$ being arbitrary scalar functions such that the matrix \mathbf{H} in Eq. (36) is non-singular, then \mathbf{W} turns out to be the null matrix, as prescribed by Eqs. (40a) and (40b).

Note that, the conditions in *Ansatz 2* are, in general, not enough to close the problem, so that a combination of the conditions featuring in *Ansatz 1* and *Ansatz 2* is required. From an operative point of view, our *Ansatz 2* suggests to introduce, as shown in *Ansatz 1*, a number ℓ^* of auxiliary functions $\{\mathcal{F}^{\beta}\}_{\beta=m+1}^{m+\ell^*}$, if this is the number of velocities that either do not feature in the constraints or can be chosen as independent. Then, to determine the remaining $n - (m + \ell^*)$ conditions, *Ansatz 2* indicates to apply a criterion that will be formalized in Theorem 1. Indeed, this criterion provides automatically the “physics-based conditions” mentioned at the beginning of this section, also for those cases in which it can be hard to impose them from the outset.

In conclusion, when matrix \mathbf{W} is found, either by *Ansatz 1* or *Ansatz 2*, then the dynamic equations of the “modified vakonomic method” read [11, 12]

$$\mathcal{E}_k \mathcal{L} \circ \sharp^{(2)} + \sum_h p_h W^h_k + \sum_{\alpha} \dot{\lambda}_{\alpha} \left[\frac{\partial \mathcal{V}^{\alpha}}{\partial q^k} \circ \sharp^{(1)} \right] = 0, \quad k = 1, \dots, n, \quad (42a)$$

$$-\mathcal{V}^{\alpha} \circ \sharp^{(1)} = 0, \quad \alpha = 1, \dots, m. \quad (42b)$$

In the presence of generalized forces $\mathcal{Q}_1, \dots, \mathcal{Q}_k$ that cannot be obtained from a scalar potential (Lanczos [13] refers to such forces as “polygenic”), Eq. (42a) acquires, up to the sign, the right-hand side \mathcal{Q}_k , so that the set of Eqs. (42a) and (42b) becomes

$$\mathcal{E}_k \mathcal{L} \circ \sharp^{(2)} + \sum_h p_h W^h_k + \sum_{\alpha} \dot{\lambda}_{\alpha} \left[\frac{\partial \mathcal{V}^{\alpha}}{\partial q^k} \circ \sharp^{(1)} \right] = -\mathcal{Q}_k, \quad k = 1, \dots, n, \quad (43a)$$

$$-\mathcal{V}^{\alpha} \circ \sharp^{(1)} = 0, \quad \alpha = 1, \dots, m. \quad (43b)$$

2.4 Some computational aspects regarding the MVM

In this section, we discuss some computational aspects regarding the solution of the set of Eqs. (42a) and (42b) (or (43a) and (43b)). We remark that having a general methodology for solving numerically the MVM dynamic equations is a key aspect of the theory, since not all problems allow for an analytical solution.

To include problems that involve electromagnetic interactions [24] as well as time-dependent holonomic constraints [13], we assume \mathcal{L} to be of the type

$$\mathcal{L} \circ \sharp^{(1)} = \frac{1}{2} \sum_{h,k=1}^n [G_{hk} \circ (q, \tau)] \dot{q}^h \dot{q}^k + \sum_{h=1}^n [Z_h \circ (q, \tau)] \dot{q}^h + \mathcal{U} \circ (q, \tau), \quad (44)$$

553 where G_{hk} is the generic component of the “metric tensor” associated with the
 554 kinetic energy [13]; Z_h is the h th component of a co-vector field accounting
 555 for Maxwell's (co-)vector potential [24], and for the possible explicit time-
 556 dependence of holonomic constraints; \mathcal{U} is a potential function that collects
 557 both electric and mechanical interactions as well as the contribution to the ki-
 558 netic energy stemming from the possible presence of explicitly time-dependent
 559 holonomic constraints. Moreover, for such type of Lagrangian functions, their
 560 associated k th Euler-Lagrange operator reads

$$\begin{aligned} \mathcal{E}_k \mathcal{L} \circ \sharp^{(2)} = & - [G_{kh} \circ (q, \tau)] \{ \dot{q}^h + [\Gamma^h_{pl} \circ (q, \tau)] \dot{q}^p \dot{q}^l \} - \left[\frac{\partial G_{kh}}{\partial \tau} \circ (q, \tau) \right] \dot{q}^h \\ & + \left[\left(\frac{\partial Z_h}{\partial q^k} - \frac{\partial Z_k}{\partial q^h} \right) \circ (q, \tau) \right] \dot{q}^h + \left(\frac{\partial \mathcal{U}}{\partial q^k} - \frac{\partial Z_k}{\partial \tau} \right) \circ (q, \tau), \end{aligned} \quad (45)$$

561 where $\Gamma^h_{pl} \circ (q, \tau)$ is the generic Christoffel symbol induced by the metric
 562 tensor, i.e. (see, e.g., [6, 15]),

$$\Gamma^h_{pl} := \frac{1}{2} \sum_{r=1}^n G^{hr} \left(\frac{\partial G_{rp}}{\partial q^l} + \frac{\partial G_{rl}}{\partial q^p} - \frac{\partial G_{pl}}{\partial q^r} \right), \quad (46)$$

563 and G^{hr} are the components of the inverse of the metric tensor.

564 Before going further, we notice that the substitution of Eq. (45) into (42a)
 565 renders the latter one of second order in the Lagrangian parameters q^1, \dots, q^n ,
 566 whereas Eq. (42b) is, by definition, of the first order in these variables. Follow-
 567 ing [6], it is convenient to “promote” the constraints to second-order ordinary
 568 differential equations, as to store the constraints in the first non-singular ma-
 569 trix associated with the highest derivatives of q , i.e., the mass matrix. This
 570 can be accomplished by differentiating Eq. (42b) with respect to time, thereby
 571 obtaining

$$\overline{\mathcal{V}^\alpha \circ \sharp^{(1)}} = [a^\alpha_k \circ (q, \tau)] \ddot{q}^k + \left[\frac{\partial a^\alpha_k}{\partial q^h} \circ (q, \tau) \right] \dot{q}^k \dot{q}^h + \left[\frac{\partial a^\alpha_k}{\partial \tau} \circ (q, \tau) \right] \dot{q}^k = 0, \quad (47)$$

572 and by solving the resulting equations together with Eq. (45). For this purpose,
 573 it is convenient to introduce the following notation:

$$M_{kh} := G_{kh} \circ (q, \tau), \quad (48a)$$

$$C^h_{pl} := \Gamma^h_{pl} \circ (q, \tau), \quad (48b)$$

$$\Omega_{kh} := \left[\frac{\partial Z_h}{\partial q^k} - \frac{\partial Z_k}{\partial q^h} \right] \circ (q, \tau) - \frac{\partial G_{kh}}{\partial \tau} \circ (q, \tau), \quad (48c)$$

$$F_k := \left[\frac{\partial \mathcal{U}}{\partial q^k} - \frac{\partial Z_k}{\partial \tau} \right] \circ (q, \tau), \quad (48d)$$

$$A^\alpha_k := a^\alpha_k \circ (q, \tau), \quad (48e)$$

$$\Lambda^\alpha_{kh} := \frac{\partial a^\alpha_k}{\partial q^h} \circ (q, \tau), \quad (48f)$$

$$\Theta^\alpha_k := \frac{\partial a^\alpha_k}{\partial \tau} \circ (q, \tau). \quad (48g)$$

574 Hence, by substituting Eq. (45) in (42a), and changing sign to the resulting
575 expression, Eqs. (42a) and (42b) take on the form

$$M_{kh}\ddot{q}^h + M_{kh}C^h_{pl}\dot{q}^p\dot{q}^l - \Omega_{kh}\dot{q}^h - F_k - p_h W^h_k - \dot{\lambda}_\alpha A^\alpha_k = 0, \quad (49a)$$

$$-A^\alpha_h\ddot{q}^h - \Lambda^\alpha_{pl}\dot{q}^p\dot{q}^l - \Theta^\alpha_h\dot{q}^h = 0. \quad (49b)$$

576 Moreover, Eqs. (49a) and (49b) can be recast in matrix form as

$$\begin{bmatrix} \mathbf{M} & -\mathbf{A}^T \\ -\mathbf{A} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \dot{\lambda} \end{Bmatrix} = \begin{Bmatrix} -\mathbf{M}[\mathbf{C} : \dot{\mathbf{q}} \otimes \dot{\mathbf{q}}] + \Omega\dot{\mathbf{q}} + \mathbf{F} + \mathbf{W}^T\mathbf{p} \\ \Lambda : \dot{\mathbf{q}} \otimes \dot{\mathbf{q}} + \Theta\dot{\mathbf{q}} \end{Bmatrix} \equiv \begin{Bmatrix} \mathbf{b}_q \\ \mathbf{b}_c \end{Bmatrix} + \begin{Bmatrix} \mathbf{W}^T\mathbf{p} \\ \mathbf{0} \end{Bmatrix}, \quad (50)$$

577 where the symbols in Sans Serif font represent the matrices and vectors (in the
578 sense of arrays) associated with the quantities in Eqs. (49a) and (49b).

579 Moreover, we can exploit the saddle-point nature of the system (50) by
580 employing the Schur complement technique [6], and, by doing so, we can recast
581 the system (50) as

$$\ddot{\mathbf{q}} = [\mathbf{M}^{-1} - (\mathbf{M}^{-1}\mathbf{A}^T)\mathbf{S}^{-1}(\mathbf{A}\mathbf{M}^{-1})](\mathbf{b}_q + \mathbf{W}^T\mathbf{p}) - (\mathbf{M}^{-1}\mathbf{A}^T)\mathbf{S}^{-1}\mathbf{b}_c, \quad (51a)$$

$$\dot{\lambda} = -\mathbf{S}^{-1}[(\mathbf{A}\mathbf{M}^{-1})(\mathbf{b}_q + \mathbf{W}^T\mathbf{p}) + \mathbf{b}_c], \quad (51b)$$

582 with $\mathbf{S} := \mathbf{A}\mathbf{M}^{-1}\mathbf{A}^T$ being the Schur complement of the block-wise system.
583 Note that the formalism used in Eq. (51a) is similar to the one adopted in [49].

584 2.5 Differences and similarities with the TNHM

585 In this section, we will acknowledge the differences and the similarities between
586 the MVM and the “traditional non-holonomic method”, in short TNHM, which
587 is based on a generalization of the Lagrange-D'Alembert Principle (see, e.g.,
588 [16, 17]). In particular, let us briefly recall the main results related to dynamic
589 equations characterizing the TNHM [1, 16–18].

590 If we consider a mechanical system described by n Lagrangian parameters
591 and constrained by m non-holonomic constraints $\mathcal{V}^\alpha \circ \#^{(1)} = 0$, with $\alpha =$
592 $1, \dots, m$, then the TNHM is characterized by the $n + m$ equations [1, 3, 6]

$$\mathcal{E}_k \mathcal{L} \circ \#^{(2)} + \sum_{\alpha=1}^m \mu_\alpha \left[\frac{\partial \mathcal{V}^\alpha}{\partial \dot{q}^k} \circ \#^{(1)} \right] = 0, \quad k = 1, \dots, n, \quad (52a)$$

$$-\mathcal{V}^\alpha \circ \#^{(1)} = 0, \quad \alpha = 1, \dots, m, \quad (52b)$$

593 in which μ_α is the α th Lagrange multiplier associated with the α th constraint.

By comparing Eq. (42a) with Eq. (52a) for the case of non-holonomic constraints linear in the generalized velocities, we notice that the quantities $\mathcal{E}_k \mathcal{L} \circ \#^{(2)}$ are the same for both equations, and that the derivatives $\partial_{\dot{q}^k} \mathcal{V}^\alpha \circ \#^{(1)}$ return the coefficients $a^\alpha_k \circ (q, \tau)$ of Eq. (5), for all $\alpha = 1, \dots, m$ and for all $k = 1, \dots, n$. Therefore, the only visual differences between these equations are due to the presence of the quantities $\sum_{h=1}^n p_h W^h_k$ and of $\dot{\lambda}_\alpha$ in lieu of μ_α in Eq. (42a). Moreover, for each $\alpha = 1, \dots, m$, the physical dimensions of the Lagrange multiplier λ_α in Eq. (42a) are equal to the physical dimensions of the corresponding multiplier μ_α multiplied by a characteristic time [32, 33].

As previously done for the dynamic equations of the MVM in Eq. (50), we can specialize Eqs. (52a) and (52b) for the case of the Lagrangian function in Eq. (44). In particular, by referring to the same notation introduced in Eqs. (48a)–(48g), and by defining the array of Lagrange multipliers of the TNHM, i.e., $\mu := \{\mu_1, \dots, \mu_m\}^T$, Eqs. (52a) and (52b) can be written in matrix form as [6]

$$\begin{bmatrix} \mathbf{M} & -\mathbf{A}^T \\ -\mathbf{A} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\mu} \end{Bmatrix} = \begin{Bmatrix} -\mathbf{M}[\mathbf{C} : \dot{\mathbf{q}} \otimes \dot{\mathbf{q}}] + \Omega \dot{\mathbf{q}} + \mathbf{F} \\ \boldsymbol{\Lambda} : \dot{\mathbf{q}} \otimes \dot{\mathbf{q}} + \Theta \dot{\mathbf{q}} \end{Bmatrix} \equiv \begin{Bmatrix} \mathbf{b}_q \\ \mathbf{b}_c \end{Bmatrix}. \quad (53)$$

Moreover, by defining $\mathbf{S} := \mathbf{A}\mathbf{M}^{-1}\mathbf{A}^T$, the Schur complement technique allows us to invert Eq. (53), which reads as follows:

$$\ddot{\mathbf{q}} = [\mathbf{M}^{-1} - (\mathbf{M}^{-1}\mathbf{A}^T)\mathbf{S}^{-1}(\mathbf{A}\mathbf{M}^{-1})]\mathbf{b}_q - (\mathbf{M}^{-1}\mathbf{A}^T)\mathbf{S}^{-1}\mathbf{b}_c, \quad (54a)$$

$$\boldsymbol{\mu} = -\mathbf{S}^{-1}[(\mathbf{A}\mathbf{M}^{-1})\mathbf{b}_q + \mathbf{b}_c]. \quad (54b)$$

Note that, both for the MVM and for the TNHM, we obtain the same block-wise matrix on the left-hand side of Eqs. (50) and (53). However, the right-hand side of these equations is different since $\mathbf{W}^T \mathbf{p}$ features in Eq. (50), only, as a result of the followed procedure.

At this stage, we find it convenient to formalize the results obtained so far by providing a definition of *equivalence* between the “modified vakonomic method” (MVM) and the “traditional non-holonomic method” (TNHM).

Definition 1 (Equivalence between MVM and TNHM)

Given the same set of initial conditions on the generalized coordinates and velocities, we say that the MVM, i.e., Eqs. (42a) and (42b), and the TNHM, i.e., Eqs. (52a) and (52b), are *equivalent*, if they return the same solution \mathbf{q} .

Theorem 1 (Characterization of the equivalence between MVM and TNHM)

Let us consider a mechanical system described by a Lagrangian function of the type given in Eq. (44) and by the set of non-holonomic constraints featuring in Eq. (5), which are linear in the generalized velocities. Then, the MVM, i.e., Eqs. (51a) and (51b), and the TNHM, i.e., Eqs. (54a) and (54b), are equivalent if, and only if, the following conditions are met

$$[\mathbf{M}^{-1} - (\mathbf{M}^{-1}\mathbf{A}^T)\mathbf{S}^{-1}(\mathbf{A}\mathbf{M}^{-1})]\mathbf{W}^T \mathbf{p} = \mathbf{0}, \quad (55a)$$

$$\dot{\lambda} = \mu - (S^{-1}AM^{-1})W^T\mathbf{p}. \quad (55b)$$

628 *Proof* If the MVM and the TNHM predict the same motion, which implies that the
 629 collection $q = (q^1, \dots, q^n)$, represented by the array \mathbf{q} , satisfies Eqs. (51a) and (51b)
 630 as well as Eqs. (54a) and (54b) at the same time, then subtracting Eq. (54a) from
 631 Eq. (51a), and (54b) from Eq. (51b) yields Eqs. (55a) and (55b).

632 Conversely, if Eqs. (55a) and (55b) hold true, then Eq. (51a) becomes identical
 633 to Eq. (54a), and it is possible to establish a univocally determined relationship
 634 between the multipliers of the two methods, i.e., $\dot{\lambda}$ and μ , thereby predicting the
 635 same solution. \square

636 As stated in Theorem 1, the fulfillment of Eqs. (55a) and (55b) provides
 637 the equivalence between the MVM and the TNHM. Moreover, it contributes
 638 to the understanding of whether a problem formulated with the vakonomic
 639 method yields, in a general context, the same solutions as the TNHM. In this
 640 sense, Theorem 1 seems to give an affirmative answer, since it prescribes the
 641 conditions under which the vakonomic method, modified as indicated by the
 642 MVM of [11, 12], returns the same results obtained within the TNHM.

643 We emphasize that, granted the equivalence between the MVM and the
 644 TNHM, the identification between the Lagrange multipliers in Eq. (55b) is
 645 similar to the ones originally presented in [12]. Note also that, even though it is
 646 possible to *formally* find relations as in Eq. (55b), they alone are not sufficient,
 647 in general, to guarantee the equivalence between the TNHM and the MVM.

648 **Corollary 1** (Sufficient condition for the equivalence between MVM and TNHM)
 649 *Given a mechanical system of the type addressed in Theorem 1, the MVM is*
 650 *equivalent to the TNHM, if it holds true that*

$$W^T\mathbf{p} = \mathbf{0}, \quad (56a)$$

$$\dot{\lambda} = \mu. \quad (56b)$$

651 *Proof* If the condition $W^T\mathbf{p} = \mathbf{0}$ applies, then Eq. (55a) is trivially satisfied, and,
 652 thus, the MVM is equivalent to the TNHM. In addition, Eq. (56b) follows directly
 653 from Eq. (55b). \square

654 We emphasize that our Theorem 1 is a reinterpretation of Theorem 2 in [12],
 655 while our Corollary 1, and, by extension, also our *Ansatz 2*, aim to reinterpret
 656 Theorem 3 of Llibre et al. [12] and Section 3.2 of Ramírez et al. [11], as a
 657 sufficient condition for Theorem 1.

658 *Remark 3* (On the role of the Lagrangian, its generalized momenta \mathbf{p} , and W)
 659 The condition (56a) requires that the array of the generalized momenta \mathbf{p} belongs
 660 to the kernel of the matrix W^T , and it amounts to requiring that the second term
 661 on the left-hand side of Eq. (42a) vanishes identically, i.e., $\sum_{h=1}^n p_h W^h_k = 0$, for
 662 $k = 1, \dots, n$. For a given Lagrangian function \mathcal{L} , which identifies the components
 663 $p_h = \partial_{\dot{q}^h} \mathcal{L} \circ \sharp^{(1)}$ of \mathbf{p} , whether or not \mathbf{p} belongs to the kernel of W^T depends both on

664 W itself, and, thus, in general, on the conditions imposed to determine the coefficients
 665 of W , and on the form of p . The conditions on W , in fact, can be obtained by
 666 following the *Ansatz 1* and/or the *Ansatz 2*. On the other hand, for a given matrix W ,
 667 determined e.g. through the assignment of suitable functions $\mathcal{F}^{m+1}, \dots, \mathcal{F}^n$ (we recall
 668 that the constraints provide indications on the adequacy of such functions, which
 669 are none other than additional quasi-velocities), the array of momenta stemming
 670 from a given \mathcal{L} does not necessarily belong to the kernel of W^T , depending on the
 671 interactions accounted for by \mathcal{L} . Indeed, the Lagrangian function addressed in our
 672 work, specified in Eq. (44), is more general than the ones typically considered in the
 673 context of vakonomic mechanics, since it accounts for interactions that do not allow
 674 to write it as the sum of a kinetic energy plus a potential depending solely on the
 675 Lagrangian parameters and possibly on time. This will become clearer below, when
 676 we shall analyze a modification of the classical benchmark of the non-holonomic
 677 skate, by including magnetic interactions that yield the presence in the Lagrangian
 678 function of a term linear in the generalized velocities. In fact, this generalization
 679 leads to momenta that are affine in the velocities and that, because of their nature,
 680 cannot belong to the kernel of W^T , and do not trivially satisfy Theorem 1. This issue,
 681 however, will be discussed in Sect. 2.6, and specialized to the case of the “charged
 682 skate” in Sect. 4.2.1.

683 Before closing this section, we deem it worthwhile to emphasize that, to
 684 the best of our understanding, the MVM proposed by Llibre et al. [12] could be
 685 viewed as a *variational version* of the method based on the quasi-velocities
 686 (see e.g. [26, 50]), in which the variational form of Newton's second law of
 687 dynamics is obtained by employing D'Alembert principle, but admitting that
 688 the time derivative of the variations are different from the variations of the
 689 velocities. In this respect, the sufficient conditions for the equivalence between
 690 the MVM and the TNHM, i.e., Eqs. (55a) and (55b), are a consequence of the
 691 method of the quasi-velocities.

692 2.6 The case of momenta linear in the velocities

693 In Eqs. (43a) and (43b) of Sect. 2.4, we have shown the equations of the MVM
 694 in the presence of “polygenic forces” [13]. In fact, it is possible to obtain the
 695 same form also in the case in which the interaction that leads to the term
 696 $\sum_{h=1}^n [Z_h \circ (q, \tau)] \dot{q}^h$ in the Lagrangian function of Eq. (44) is formally treated
 697 as a force of this type (although this force is not polygenic *per se*). Indeed, by
 698 making the identification

$$-Q_k \equiv - \left[\left(\frac{\partial Z_h}{\partial q^k} - \frac{\partial Z_k}{\partial q^h} \right) \circ (q, \tau) \right] \dot{q}^h + \frac{\partial Z_k}{\partial \tau} \circ (q, \tau), \quad (57)$$

699 and, thus, consistently omitting the term $\sum_{h=1}^n [Z_h \circ (q, \tau)] \dot{q}^h$ in the Lagrangian
 700 function of Eq. (44), i.e., redefining $\mathcal{L} \circ \sharp^{(1)}$ as

$$\mathcal{L}_0 \circ \sharp^{(1)} = \frac{1}{2} \sum_{h,k=1}^n [G_{hk} \circ (q, \tau)] \dot{q}^h \dot{q}^k + \mathcal{U} \circ (q, \tau), \quad (58)$$

Equations (42a) and (42b) are recast in the form

$$\mathcal{E}_k \mathcal{L}_0 \circ \sharp^{(2)} + \sum_h p_{0h} W^h_k + \sum_\alpha \dot{\lambda}_\alpha \left[\frac{\partial \mathcal{V}^\alpha}{\partial \dot{q}^k} \circ \sharp^{(1)} \right] = -\mathcal{Q}_k, \quad k = 1, \dots, n, \quad (59a)$$

$$-\mathcal{V}^\alpha \circ \sharp^{(1)} = 0, \quad \alpha = 1, \dots, m, \quad (59b)$$

with $p_{0h} := \partial_{\dot{q}^h} \mathcal{L}_0 \circ \sharp^{(1)} = \sum_{\ell=1}^n [G_{h\ell} \circ (q, \tau)] \dot{q}^\ell$. This last result, which stipulates that the momenta are linear functions of the velocities, has deep repercussions on the fulfillment of conditions (55a) and (55b), with \mathbf{p}_0 substituting \mathbf{p} . More details on this issue will be discussed in Remark 8 of Sect. 4.2.1 for the specific case of the “charged skate”.

3 The “rolling coin” benchmark

In this section, we compute analytically the MVM equations of motion for the benchmark problem of the “rolling coin”, recently presented in [1, 14], and adopted as a *benchmark* for comparing the VM and the TNHM in [22, 43–45].

The mechanical system that we are considering is composed by a coin, hereafter denoted by \mathfrak{C} , idealized as a bi-dimensional rigid disk of radius R and mass m , that can roll without slipping over a plane inclined of an angle $\alpha \in]0, \pi/2[$ with respect to a horizontal plane.

As done in [1, 14], we specify a coordinate system $\{O, (x, y, z)\}$ in which the y -axis is parallel to the inclined plane and is aligned along the direction of steepest descent; the z -axis “enters” the inclined plane and is orthogonal to it; the x -axis is such that the x -, y -, and z -axis form a right-handed triad, whose origin O is a point having the same z coordinate as the center of mass of \mathfrak{C} , hereafter denoted by G . Moreover, similarly to [14], we introduce the coordinates of G , i.e., $(x_G, y_G, 0)$; the angle of rolling, ϕ , taken clockwise from the z -axis; and the angle θ between the y -axis and the axis along which rolling occurs. Note that x_G , y_G , ϕ , and θ are the Lagrangian parameters of the mechanical system under study, and are thus intended as functions of time, defined over the interval of observation $[t_{\text{in}}, t_{\text{fin}}]$.

From here on, the identification $q \equiv (q^1, q^2, q^3, q^4) = (x_G, y_G, \phi, \theta)$ is made, which implies that the system satisfies automatically two holonomic constraints: one requires G to experience only motions parallel to the inclined plane; the other one prescribes that the coin \mathfrak{C} remains orthogonal to the inclined plane during its entire motion.

If gravity is the only interaction accounted for, the Lagrangian function of the mechanical system under study reads [1, 14, 26]

$$\mathcal{L} \circ (q, \dot{q}) = \underbrace{\frac{1}{2} m [(\dot{q}^1)^2 + (\dot{q}^2)^2 + \frac{1}{2} R^2 (\dot{q}^3)^2 + \frac{1}{4} R^2 (\dot{q}^4)^2]}_{\mathcal{K} \circ \dot{q}} + \underbrace{(mg \sin \alpha) q^2}_{\mathcal{U} \circ q}, \quad (60)$$

733 where \mathcal{K} is the kinetic energy; \mathcal{U} is the gravitational potential; and g denotes
734 the magnitude of the gravitational acceleration.

735 The assumption that the coin “*rolls without slipping*” [1, 14] determines
736 $m = 2$ non-holonomic constraints given by [22, 26, 43–45, 47]

$$\mathcal{V}^1 \circ (q, \dot{q}) := \dot{q}^1 - R\dot{q}^3 \sin q^4 = 0, \quad (61a)$$

$$\mathcal{V}^2 \circ (q, \dot{q}) := \dot{q}^2 - R\dot{q}^3 \cos q^4 = 0. \quad (61b)$$

737 Note that, by indicating with $\boldsymbol{\tau} = \sin\theta \mathbf{e}_x + \cos\theta \mathbf{e}_y$ the unit vector defining the
738 direction of rolling, Eqs. (61a) and (61b) express, in terms of the Lagrangian
739 parameters of the model and their derivatives, the fact that the velocity of
740 the coin's center of mass, \mathbf{v}_G , satisfies the condition $\mathbf{v}_G = R\dot{\phi}\boldsymbol{\tau}$, given in the
741 physical space.

742 3.1 The traditional non-holonomic approach

743 In the benchmark that we are analyzing, the dynamic equations of the TNHM,
744 namely Eqs. (52a) and (52b), admit the expressions [1, 14, 22, 43–45, 47]

$$-m\ddot{q}^1 + \mu_1 = 0, \quad (62a)$$

$$mg \sin\alpha - m\ddot{q}^2 + \mu_2 = 0, \quad (62b)$$

$$-\frac{1}{2}mR^2\ddot{q}^3 - \mu_1 R \sin q^4 - \mu_2 R \cos q^4 = 0, \quad (62c)$$

$$-\frac{1}{4}mR^2\ddot{q}^4 = 0. \quad (62d)$$

745 As shown in [1], Eqs. (62a)–(62d) can be solved analytically by exploiting the
746 direct integrability of Eq. (62d), so we will not further investigate this solution.

747 3.2 The “modified vakonomic” approach

748 In this section, we study the benchmark problem introduced above by means
749 of the MVM [12], thereby determining the motion from the dynamic equations
750 (42a) and (42b) in compliance with the *solvability conditions* (40a) and (40b).

751 By applying the Euler-Lagrange operators defined in Eq. (28a) to the
752 Lagrangian function in Eq. (60), we obtain

$$\mathcal{E}_1 \mathcal{L} \circ \sharp^{(2)} = -m\ddot{q}^1, \quad (63a)$$

$$\mathcal{E}_2 \mathcal{L} \circ \sharp^{(2)} = -m\ddot{q}^2 + mg \sin\alpha, \quad (63b)$$

$$\mathcal{E}_3 \mathcal{L} \circ \sharp^{(2)} = -\frac{1}{2}mR^2\ddot{q}^3, \quad (63c)$$

$$\mathcal{E}_4 \mathcal{L} \circ \sharp^{(2)} = -\frac{1}{4}mR^2\ddot{q}^4. \quad (63d)$$

753 Furthermore, we write explicitly the differential operators in Eq. (34) for the
754 expressions of the two constraints in (61a) and (61b), respectively. Hence, for

755 $\alpha = 1$, we have

$$\mathcal{D}_1 \mathcal{V}^1 \circ \#^{(2)} = W^1_1 - (R \sin q^4) W^3_1 = 0, \quad (64a)$$

$$\mathcal{D}_2 \mathcal{V}^1 \circ \#^{(2)} = W^1_2 - (R \sin q^4) W^3_2 = 0, \quad (64b)$$

$$\mathcal{D}_3 \mathcal{V}^1 \circ \#^{(2)} = W^1_3 - (R \sin q^4) W^3_3 + (R \cos q^4) \dot{q}^4 = 0, \quad (64c)$$

$$\mathcal{D}_4 \mathcal{V}^1 \circ \#^{(2)} = W^1_4 - (R \sin q^4) W^3_4 - (R \cos q^4) \dot{q}^3 = 0, \quad (64d)$$

756 while, for $\alpha = 2$, we obtain

$$\mathcal{D}_1 \mathcal{V}^2 \circ \#^{(2)} = W^2_1 - (R \cos q^4) W^3_1 = 0, \quad (65a)$$

$$\mathcal{D}_2 \mathcal{V}^2 \circ \#^{(2)} = W^2_2 - (R \cos q^4) W^3_2 = 0, \quad (65b)$$

$$\mathcal{D}_3 \mathcal{V}^2 \circ \#^{(2)} = W^2_3 - (R \cos q^4) W^3_3 - (R \sin q^4) \dot{q}^4 = 0, \quad (65c)$$

$$\mathcal{D}_4 \mathcal{V}^2 \circ \#^{(2)} = W^2_4 - (R \cos q^4) W^3_4 + (R \sin q^4) \dot{q}^3 = 0. \quad (65d)$$

757 By choosing the arbitrary *auxiliary functions* as [11, 12]

$$\mathcal{F}^3 \circ \#^{(1)} := \dot{q}^3, \quad (66a)$$

$$\mathcal{F}^4 \circ \#^{(1)} := \dot{q}^4, \quad (66b)$$

758 Equation (35), for $\beta = 3$ and $\beta = 4$, implies that

$$\mathcal{D}_k \mathcal{F}^3 \circ \#^{(2)} = W^3_k = 0, \quad k = 1, \dots, 4, \quad (67a)$$

$$\mathcal{D}_k \mathcal{F}^4 \circ \#^{(2)} = W^4_k = 0, \quad k = 1, \dots, 4. \quad (67b)$$

759 We can represent the coefficients W^h_k in a more compact way by assembling
760 the 4×4 matrix W that, by having recourse to Eqs. (64a)–(64d), (65a)–(65d),
761 and (67a) and (67b), is given by

$$W = \begin{bmatrix} 0 & 0 & -(R \cos q^4) \dot{q}^4 & (R \cos q^4) \dot{q}^3 \\ 0 & 0 & (R \sin q^4) \dot{q}^4 & -(R \sin q^4) \dot{q}^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (68)$$

762 Our choice of the auxiliary functions in Eqs. (66a) and (66b) is done in com-
763 pliance with Remark 1, since, by inspection of the constraints (61a) and (61b),
764 the generalized velocities \dot{q}^3 and \dot{q}^4 are *independent*, thereby leading to the
765 vanishing of the corresponding transpositional relations $\zeta^3 - \eta^3$ and $\zeta^4 - \eta^4$. In
766 fact, this choice is a *strong* way of guaranteeing this result because the condi-
767 tions (67a) and (67b), implied by the selected auxiliary functions, ensure that
768 the third and fourth row of W are null.

769 Finally, if we substitute Eqs. (63a)–(63d) and the coefficients collected in
770 Eq. (68) into the Euler-Lagrange equations of the MVM written in (42a), we

771 can recast Eqs. (42a) and (42b) in the form

$$m\ddot{q}^1 - \dot{\lambda}_1 = 0, \quad (69a)$$

$$m\ddot{q}^2 - \dot{\lambda}_2 = mg \sin \alpha, \quad (69b)$$

$$\frac{1}{2}mR^2\ddot{q}^3 + \dot{\lambda}_1 R \sin q^4 + \dot{\lambda}_2 R \cos q^4 = -mR(\dot{q}^1 \cos q^4 - \dot{q}^2 \sin q^4)\dot{q}^4, \quad (69c)$$

$$\frac{1}{4}mR^2\ddot{q}^4 = mR(\dot{q}^1 \cos q^4 - \dot{q}^2 \sin q^4)\dot{q}^3, \quad (69d)$$

$$\dot{q}^1 = R\dot{q}^3 \sin q^4, \quad (69e)$$

$$\dot{q}^2 = R\dot{q}^3 \cos q^4. \quad (69f)$$

772 Before going further, we notice that the quantity $\dot{q}^1 \cos q^4 - \dot{q}^2 \sin q^4$, which
 773 features both in Eqs. (69c) and (69d), is none other than the component along
 774 the axis \mathbf{e}_z of the cross product $\mathbf{v}_G \times \boldsymbol{\tau}$, which is null because the velocity of the
 775 center of mass of the coin has to be parallel to the unit vector $\boldsymbol{\tau}$. In particular,
 776 it should be noticed that this condition is naturally satisfied by the constraint
 777 of rolling without slipping, but it is, in fact, more general than the latter, since
 778 it expresses a merely geometric fact. In conclusion, it applies that [26]

$$(\mathbf{v}_G \times \boldsymbol{\tau}) \cdot \mathbf{e}_z = \dot{q}^1 \cos q^4 - \dot{q}^2 \sin q^4 = 0. \quad (70)$$

779 By virtue of this result, Eqs. (69a)–(69f) become

$$m\ddot{q}^1 - \dot{\lambda}_1 = 0, \quad (71a)$$

$$m\ddot{q}^2 - \dot{\lambda}_2 = mg \sin \alpha, \quad (71b)$$

$$\frac{1}{2}mR^2\ddot{q}^3 + \dot{\lambda}_1 R \sin q^4 + \dot{\lambda}_2 R \cos q^4 = 0, \quad (71c)$$

$$\frac{1}{4}mR^2\ddot{q}^4 = 0. \quad (71d)$$

$$\dot{q}^1 = R\dot{q}^3 \sin q^4, \quad (71e)$$

$$\dot{q}^2 = R\dot{q}^3 \cos q^4. \quad (71f)$$

780 By making the identifications $\mu_\alpha \equiv \dot{\lambda}_\alpha$, for $\alpha = 1, \dots, m$, the set of equations
 781 (71a)–(71d) is equal to the set of dynamic equations obtained within the
 782 TNHM, i.e. Eqs. (62a)–(62d). This shows that, in agreement with [12], the
 783 MVM is indeed *equivalent* with the TNHM for the “rolling coin” problem.

784 *Remark 4* (The MVM and the TNHM for the “rolling coin” problem)

785 It should be noticed that the equivalence between the MVM and the TNHM, and the
 786 identification between their respective Lagrange multipliers, could have been proven
 787 in advance since, in the case considered above, the matrix W in Eq. (68) and the
 788 array of momenta \mathbf{p} satisfy the hypothesis of Corollary 1, namely $W^T \mathbf{p} = \mathbf{0}$.

789 *Remark 5* (“Canonical flip” [22, 41] for the “rolling coin” problem)

790 Looking at the structure of the matrix W featuring in Eq. (68), and computing the
791 quantities $\zeta^h - \dot{\eta}^h$, with $h = 1, \dots, n$, once as $\zeta^h - \dot{\eta}^h = \sum_{k=1}^n W^h_{\ell k} \eta^k$ and once as
792 $\zeta^h - \dot{\eta}^h = \sum_{\ell, k=1}^n \mathcal{C}^h_{\ell k} \dot{q}^\ell \eta^k$ (cf. Eq. (21)), we can choose

$$\mathcal{C}^1_{43} \dot{q}^4 = W^1_3 = -(R \cos q^4) \dot{q}^4 \quad \Rightarrow \quad \mathcal{C}^1_{43} \equiv \hat{\mathcal{C}}^1_{43} \circ q = -R \cos q^4, \quad (72a)$$

$$\mathcal{C}^1_{34} \dot{q}^3 = W^1_4 = +(R \cos q^4) \dot{q}^3 \quad \Rightarrow \quad \mathcal{C}^1_{34} \equiv \hat{\mathcal{C}}^1_{34} \circ q = +R \cos q^4, \quad (72b)$$

$$\mathcal{C}^2_{43} \dot{q}^4 = W^2_3 = +(R \sin q^4) \dot{q}^4 \quad \Rightarrow \quad \mathcal{C}^2_{43} \equiv \hat{\mathcal{C}}^2_{43} \circ q = +R \sin q^4, \quad (72c)$$

$$\mathcal{C}^2_{34} \dot{q}^3 = W^2_4 = -(R \sin q^4) \dot{q}^3 \quad \Rightarrow \quad \mathcal{C}^2_{34} \equiv \hat{\mathcal{C}}^2_{34} \circ q = -R \sin q^4, \quad (72d)$$

793 while all the other entries of $\mathcal{C}^h_{\ell k}$ can be set equal to zero. We notice that, in the case
794 studied in this remark, the nonzero entries of $\mathcal{C}^h_{\ell k}$, reported in Eqs. (72a)–(72d),
795 depend on q , and, in particular, on q^4 , only.

796 3.3 Analytical solution

797 In this section, we are interested in finding an analytical solution to the dy-
798 namical equations previously obtained within the MVM, i.e., Eqs. (71a)–(71d).
799 To this end, we introduce the following set of initial conditions, i.e., at $t = t_{\text{in}}$,
800 for the generalized coordinates and velocities:

$$q^1(t_{\text{in}}) = q^1_{\text{in}}, \quad q^2(t_{\text{in}}) = q^2_{\text{in}}, \quad q^3(t_{\text{in}}) = q^3_{\text{in}}, \quad q^4(t_{\text{in}}) = q^4_{\text{in}}, \quad (73a)$$

$$\dot{q}^1(t_{\text{in}}) = \nu^1_{\text{in}}, \quad \dot{q}^2(t_{\text{in}}) = \nu^2_{\text{in}}, \quad \dot{q}^3(t_{\text{in}}) = \nu^3_{\text{in}}, \quad \dot{q}^4(t_{\text{in}}) = \nu^4_{\text{in}}, \quad (73b)$$

801 where their specified values q^k_{in} and ν^k_{in} , for $k = 1, \dots, 4$, should satisfy the con-
802 straints in Eqs. (69e) and (69f), both for physical and for numerical consistency
803 [6]. Furthermore, we introduce the initial values for the multipliers

$$\lambda_1(t_{\text{in}}) = \lambda_{1\text{in}}, \quad \lambda_2(t_{\text{in}}) = \lambda_{2\text{in}}. \quad (74)$$

804 By taking inspiration from the solution strategy employed by Lemos [1],
805 from the direct integration of Eq. (71d) we obtain that q^4 is *affine* in time,
806 that is

$$q^4(t) = \Omega[t - t_{\text{in}}] + q^4_{\text{in}} = \Omega t + \theta_0, \quad \forall t \in [t_{\text{in}}, t_{\text{fin}}], \quad (75)$$

807 where $\Omega := \dot{q}^4(t) \equiv \dot{q}^4(t_{\text{in}}) = \nu^4_{\text{in}}$ for all $t \in [t_{\text{in}}, t_{\text{fin}}]$, and, for a better read-
808 ability, we write $\theta_0 := q^4_{\text{in}} - \Omega t_{\text{in}}$. Note that the angular velocity $\dot{q}^4(t) = \Omega$ is
809 constant in time.

810 We can exploit the *saddle-point nature* of the system of equations under
811 study by decoupling the dynamic equations (71a)–(71d) from the constraints,
812 i.e., (71e) and (71f), so that the latter ones can be computed *a posteriori*.
813 Hence, by substituting the constraints in (71e) and (71f) into Eqs. (71a) and

814 (71b), we obtain

$$\dot{\lambda}_1 = m\ddot{q}^3 R \sin q^4 + m\dot{q}^3 \dot{q}^4 R \cos q^4, \quad (76a)$$

$$\dot{\lambda}_2 = m\ddot{q}^3 R \cos q^4 - m\dot{q}^3 \dot{q}^4 R \sin q^4 - mg \sin \alpha. \quad (76b)$$

815 With the expressions obtained in Eqs. (76a) and (76b) for the two time deriva-
816 tives of the Lagrange multipliers, and with the expression of q^4 in Eq. (75), we
817 can re-frame Eq. (71c) as follows

$$\ddot{q}^3(t) = \frac{2g \sin \alpha}{3R} \cos(\Omega t + \theta_0). \quad (77)$$

818 In the following, we introduce, for each $k = 1, \dots, 4$ and for each $\alpha = 1, 2$,
819 the notations $q^k(t) \equiv \hat{q}^k(t; \Omega)$ and $\lambda_\alpha(t) \equiv \hat{\lambda}_\alpha(t; \Omega)$ in order to emphasize the
820 dependence of the solution of the system under study on the parameter Ω . In
821 particular, depending on the value of Ω , we can distinguish two cases.

822 **Case 1** $\Omega \neq 0$

823 By assuming $\Omega \neq 0$, the right-hand side of Eq. (77) represents an oscillatory
824 forcing term with angular frequency Ω and initial phase θ_0 . By integrating Eq.
825 (77), we obtain

$$q^3(t) \equiv \hat{q}^3(t; \Omega) = \hat{q}_{\text{osc}}^3(t; \Omega) + \nu_{\text{in}}^3[t - t_{\text{in}}] + q_{\text{in}}^3, \quad (78)$$

826 where we have introduced the auxiliary notation

$$\begin{aligned} \hat{q}_{\text{osc}}^3(t; \Omega) &:= \Phi_s(\Omega) \{ \sin(\Omega t) - \sin(\Omega t_{\text{in}}) \} - \Phi_c(\Omega) \{ \cos(\Omega t) - \cos(\Omega t_{\text{in}}) \} \\ &\quad - \Omega[t - t_{\text{in}}] \{ \Phi_c(\Omega) \sin(\Omega t_{\text{in}}) + \Phi_s(\Omega) \cos(\Omega t_{\text{in}}) \}, \end{aligned} \quad (79a)$$

$$\Phi(\Omega) := \frac{2g \sin \alpha}{3R\Omega^2}, \quad \Phi_c(\Omega) := \Phi(\Omega) \cos \theta_0, \quad \Phi_s(\Omega) := \Phi(\Omega) \sin \theta_0. \quad (79b)$$

827 Note that, by virtue of Eq. (78), the Lagrangian parameter $q^3(t)$ features
828 an oscillatory contribution, given by Eq. (79a), which is characterized by the
829 amplitude $\Phi(\Omega)$ in Equation (79b), and a contribution that is affine in time,
830 i.e., $\nu_{\text{in}}^3[t - t_{\text{in}}] + q_{\text{in}}^3$. Moreover, by differentiating in time Eq. (79a), one obtains

$$\begin{aligned} \frac{\partial \hat{q}_{\text{osc}}^3}{\partial t}(t; \Omega) &= \Omega \{ \Phi_s(\Omega) [\cos(\Omega t) - \cos(\Omega t_{\text{in}})] + \Phi_c(\Omega) [\sin(\Omega t) - \sin(\Omega t_{\text{in}})] \} \\ &= \Omega \Phi(\Omega) [\sin(\Omega t + \theta_0) - \sin(\Omega t_{\text{in}} + \theta_0)]. \end{aligned} \quad (80)$$

831 Thus, by integrating Eqs. (71e) and (71f), $q^1(t)$ and $q^2(t)$ read as

$$\begin{aligned} q^1(t) \equiv \hat{q}^1(t; \Omega) &= -\frac{1}{4} R \Phi(\Omega) [\sin(2\Omega t + 2\theta_0) - \sin(2\Omega t_{\text{in}} + 2\theta_0)] \\ &\quad - R \left[\frac{\nu_{\text{in}}^3}{\Omega} - \Phi(\Omega) \sin(\Omega t_{\text{in}} + \theta_0) \right] \{ \cos(\Omega t + \theta_0) - \cos(\Omega t_{\text{in}} + \theta_0) \} \end{aligned}$$

$$+ \frac{1}{2}R\Omega\Phi(\Omega)[t - t_{\text{in}}] + q_{\text{in}}^1, \quad (81a)$$

$$q^2(t) \equiv \hat{q}^2(t; \Omega) = -\frac{1}{4}R\Phi(\Omega)[\cos(2\Omega t + 2\theta_0) - \cos(2\Omega t_{\text{in}} + 2\theta_0)] \\ + R\left[\frac{\nu_{\text{in}}^3}{\Omega} - \Phi(\Omega)\sin(\Omega t_{\text{in}} + \theta_0)\right]\{\sin(\Omega t + \theta_0) - \sin(\Omega t_{\text{in}} + \theta_0)\} \\ + q_{\text{in}}^2. \quad (81b)$$

832 In light of the calculations above, we conclude that, for $\Omega \neq 0$, $q^1(t)$ and
833 $q^2(t)$ exhibit two oscillatory contributions: one with angular frequency Ω and
834 the other one with angular frequency 2Ω . In particular, $q^1(t)$ is unbounded in
835 the limit $t \rightarrow +\infty$, whereas $q^2(t)$ is bounded. Therefore, the center of mass of
836 the coin moves indefinitely along the x -axis and, in addition, will never reach
837 the end of the inclined plane along the y -axis.

838 Finally, we compute the Lagrange multipliers *a posteriori* by integrating
839 Eqs. (76a) and (76b) in light of the relations obtained for $q^3(t)$ and $q^4(t)$, i.e.,
840 Eqs. (75) and (80). By doing this, we obtain

$$\lambda_1(t) \equiv \hat{\lambda}_1(t; \Omega) = -\frac{1}{2}mR\Omega\Phi(\Omega)[\cos(2\Omega t + 2\theta_0) - \cos(2\Omega t_{\text{in}} + 2\theta_0)] \\ + mR[\nu_{\text{in}}^3 - \Omega\Phi(\Omega)\sin(\Omega t_{\text{in}} + \theta_0)]\{\sin(\Omega t + \theta_0) - \sin(\Omega t_{\text{in}} + \theta_0)\} \\ + \lambda_{1\text{in}}, \quad (82a)$$

$$\lambda_2(t) \equiv \hat{\lambda}_2(t; \Omega) = \frac{1}{2}mR\Omega\Phi(\Omega)[\sin(2\Omega t + 2\theta_0) - \sin(2\Omega t_{\text{in}} + 2\theta_0)] \\ + mR[\nu_{\text{in}}^3 - \Omega\Phi(\Omega)\sin(\Omega t_{\text{in}} + \theta_0)]\{\cos(\Omega t + \theta_0) - \cos(\Omega t_{\text{in}} + \theta_0)\} \\ - (mg\sin\alpha)[t - t_{\text{in}}] + \lambda_{2\text{in}}. \quad (82b)$$

841 **Case 2** $\Omega = 0$

842 In the case $\Omega = 0$, we find that $q^4(t) = \theta_0 \equiv q_{\text{in}}^4$ for all $t \in [t_{\text{in}}, t_{\text{fin}}]$, and
843 that the right-hand side of Eq. (77) is constant in time, thereby returning a
844 uniformly accelerated angular motion for q^3 . Thus, by integrating Eq. (77) two
845 times, we obtain

$$q^3(t) \equiv \hat{q}^3(t; 0) = \frac{1}{2}\xi[t - t_{\text{in}}]^2 + \nu_{\text{in}}^3[t - t_{\text{in}}] + q_{\text{in}}^3, \quad \xi := \frac{2g\sin\alpha\cos\theta_0}{3R}, \quad (83)$$

846 where ξ represent the angular acceleration associated with q^3 .

847 Since the evolution in time of q^3 is known, we can deduce the evolution of
848 q^1 and q^2 by substituting Eq. (83) in the constraints, i.e., Eqs. (71e) and (71f).
849 By integrating the resulting expressions, we get

$$q^1(t) = \frac{1}{2}\xi(R\sin\theta_0)[t - t_{\text{in}}]^2 + \underbrace{\nu_{\text{in}}^3(R\sin\theta_0)}_{\equiv \nu_{\text{in}}^1}[t - t_{\text{in}}] + q_{\text{in}}^1, \quad (84a)$$

$$q^2(t) = \frac{1}{2}\xi(R\cos\theta_0)[t - t_{\text{in}}]^2 + \underbrace{\nu_{\text{in}}^3(R\cos\theta_0)}_{\equiv \nu_{\text{in}}^2}[t - t_{\text{in}}] + q_{\text{in}}^2. \quad (84b)$$

850 Finally, we can compute λ_1 and λ_2 *a posteriori* by integrating Eqs. (76a)
851 and (76b) as follows:

$$\lambda_1(t) = m\xi(R \sin\theta_0)[t - t_{\text{in}}] + \lambda_{1\text{in}}, \quad (85a)$$

$$\lambda_2(t) = \{m\xi(R \cos\theta_0) - mg \sin\alpha\}[t - t_{\text{in}}] + \lambda_{2\text{in}}. \quad (85b)$$

852 We conclude noting that, for $\Omega = 0$, the Lagrange multipliers $\lambda_1(t)$ and
853 $\lambda_2(t)$ are *affine* in time, whereas their time derivatives are constants. This
854 means that, since $\dot{\lambda}_1(t)$ and $\dot{\lambda}_2(t)$ are equal to the Lagrange multipliers $\mu_1(t)$
855 and $\mu_2(t)$, respectively, of the TNHM (see Corollary 1 and Remark 4), and
856 since the latter ones measure the magnitude of the reaction forces associated
857 with the imposed constraints, we can say that the reaction forces are constants.

858 *Remark 6* (Continuity of the third Lagrangian parameter with respect to Ω)
859 From Case 1 and Case 2, it follows that the solution of Eq. (77) is defined as

$$q^3(t) \equiv \hat{q}^3(t; \Omega) = \begin{cases} \hat{q}_{\text{osc}}^3(t; \Omega) + \nu_{\text{in}}^3[t - t_{\text{in}}] + q_{\text{in}}^3, & \text{if } \Omega \neq 0, \\ \frac{1}{2}\xi[t - t_{\text{in}}]^2 + \nu_{\text{in}}^3[t - t_{\text{in}}] + q_{\text{in}}^3, & \text{if } \Omega = 0, \end{cases} \quad (86)$$

860 where $\hat{q}_{\text{osc}}^3(t; \Omega)$ is reported in Eq. (79a). In particular, we emphasize that $\hat{q}^3(t, \cdot)$ is
861 continuous with respect to Ω uniformly in $[t_{\text{in}}, t_{\text{fin}}]$, since it holds that

$$\lim_{\Omega \rightarrow 0} \hat{q}^3(t; \Omega) = \hat{q}^3(t; 0), \quad t \in [t_{\text{in}}, t_{\text{fin}}]. \quad (87)$$

862 3.4 Numerical simulations

863 In this section, we present the graphical results of the “Rolling coin” problem
864 with the purpose of visualizing that the MVM is equivalent, for the considered
865 problem, to the TNHM.

866 Hence, we numerically solve the system of Eqs. (71a)–(71f), put in a more
867 general form as in Eq. (50), over the interval $[t_{\text{in}}, t_{\text{fin}}] \equiv [0, T]$. To this end, we
868 introduce the set of *normalized* Lagrangian parameters

$$\mathbf{q}^1 := q^1/R, \quad \mathbf{q}^2 := q^2/R, \quad \mathbf{q}^3 := q^3, \quad \mathbf{q}^4 := q^4, \quad (88)$$

869 and the set of *normalized* Lagrange multipliers as

$$\lambda_1 := \lambda_1/(mR), \quad \lambda_2 := \lambda_2/(mR). \quad (89)$$

870 Note that the normalization of a given Lagrangian parameter is necessary only
871 when it represents a translational kinematic descriptor, and not an angular
872 one, while both the normalized Lagrange multipliers “absorb” the coin’s mass
873 m and radius R . As a consequence of (88) and (89), Eqs. (71a)–(71f) are thus
874 normalized as follows:

$$\ddot{\mathbf{q}}^1 - \dot{\lambda}_1 = 0, \quad (90a)$$

$$\ddot{q}^2 - \dot{\lambda}_2 = \frac{g \sin \alpha}{R}, \quad (90b)$$

$$\frac{1}{2} \ddot{q}^3 + \dot{\lambda}_1 \sin q^4 + \dot{\lambda}_2 \cos q^4 = 0, \quad (90c)$$

$$\frac{1}{4} \ddot{q}^4 = 0, \quad (90d)$$

$$\dot{q}^1 = \dot{q}^3 \sin q^4, \quad (90e)$$

$$\dot{q}^2 = \dot{q}^3 \cos q^4. \quad (90f)$$

875 Finally, Eqs. (90a)–(90f) are solved numerically using the “trapezium rule” in
 876 the case in which the physical and numerical parameters are the ones reported
 877 in Table 1.

Parameter	Value	Units	Description
R	11.625	mm	Radius of the coin
m	7.5	g	Mass of the coin
g	9.81	m/s ²	Gravitational acceleration
α	$\pi/6$	rad	Angle of the inclined plane
T	2	s	Total time of simulation
Δt	10^{-4}	s	Time-step

Table 1: Physical and numerical parameters chosen for the simulation of a standard “1 Euro” coin rolling down an infinitely long inclined plane

878 *Parametric sweep of Ω*

879 The first scenario we examine is the case of a coin, initially tilted towards
 880 the direction of steepest descent of the plane, i.e., $q_{\text{in}}^4 = 0$, that rolls without
 881 slipping from the top of the inclined plane. Moreover, we consider the initial
 882 angular velocity $\nu_{\text{in}}^4 \equiv \Omega$, and we study its effect on the evolution of the system.
 883 Then, we initialize the remaining generalized coordinates as $q_{\text{in}}^1 = 0$, $q_{\text{in}}^2 = 0$
 884 and $q_{\text{in}}^3 = 0$, and the Lagrange multipliers as $\lambda_{1 \text{ in}} = 0$ and $\lambda_{2 \text{ in}} = 0$. Further,
 885 we assume that $\nu_{\text{in}}^3 = 0$, which implies that the initial velocities ν_{in}^1 and ν_{in}^2 are
 886 zero too, since they have to satisfy the constraints in Eqs. (61a) and (61b) at
 887 time $t = t_{\text{in}} = 0$.

888 In Fig. 1, we show the trajectories of the center of mass of the coin for
 889 different values of Ω , e.g. $\Omega \in \{\pi/2, \pi, 2\pi\}$ rad/s. We observe that gyroscopic
 890 effects arise when non-vanishing values of Ω are considered, as predicted by the
 891 analytical solution of the problem (see e.g. Section 3.3) [1, 14]. These effects,
 892 which are produced by the action of the constraints, are due to an oscillatory
 893 contribution characterized by angular frequency 2Ω , since, if we specialize the
 894 analytical solution to the case under examination, the coordinates of the center
 895 of mass of the coin read

$$q^1(t) \equiv \hat{q}^1(t; \Omega) = -\frac{1}{4} R \Phi(\Omega) \sin(2\Omega t) + \frac{1}{2} R \Omega \Phi(\Omega) t, \quad (91a)$$

$$q^2(t) \equiv \hat{q}^2(t; \Omega) = -\frac{1}{4} R \Phi(\Omega) [\cos(2\Omega t) - 1]. \quad (91b)$$

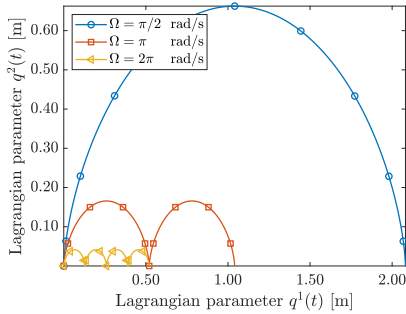


Figure 1: Trajectories of the center of mass of the coin for different values of $\Omega \in \{\pi/2, \pi, 2\pi\}$ rad/s simulated from $t_{\text{in}} = 0$ s to $t_{\text{fin}} = 2$ s.

896 Note that, by expanding Eqs. (91a) and (91b) in a neighborhood of $\Omega = 0$,
 897 and for a fixed time $t \in [t_{\text{in}}, t_{\text{fin}}]$, we obtain, in the limit $\Omega \rightarrow 0$, that

$$q^1(t) \equiv \hat{q}^1(t; \Omega) = \frac{1}{3}R\Phi(\Omega)\Omega^3 t^3 + o(\Omega) = \frac{2}{9}g \sin\alpha \Omega t^3 + o(\Omega), \quad (92a)$$

$$q^2(t) \equiv \hat{q}^2(t; \Omega) = \frac{1}{2}R\Phi(\Omega)\Omega^2 t^2 + o(1) = \frac{1}{3}g \sin\alpha t^2 + o(1). \quad (92b)$$

898 Therefore, we find that $\hat{q}^1(t; 0) = \lim_{\Omega \rightarrow 0} \hat{q}^1(t; \Omega)$, and that the leading term
 899 of $\hat{q}^2(t; \Omega)$ does not depend on Ω and grows quadratically in time in the
 900 considered neighborhood.

901 In conclusion, if $\Omega = 0$, the coin rolls downwards indefinitely and reaches
 902 the end of the inclined plane, when the latter one is finite. On the other hand,
 903 when $\Omega \neq 0$, the center of mass of the coin exhibits a behavior similar to
 904 the one of a “yo-yo”, since it oscillates back and forth in the direction of the
 905 y -axis, while moving indefinitely along the x -axis, and the amplitude of the
 906 oscillations depends on Ω . Moreover, even for inclined planes whose slope is
 907 finite, yet sufficiently longer than the amplitude of the y -oscillations, there
 908 exist values of Ω such that the coin does not reach the end of the slope.

909 In Fig. 2, we have reported four plots describing the qualitative and quanti-
 910 tative behavior of the Lagrange multipliers λ_1 and λ_2 in the case in which Ω is
 911 either π rad/s or 2π rad/s. In particular, in Fig. 2a–2c, we introduce ρ_1 and ρ_2
 912 as the “reactive forces” associated to q^1 and q^2 , respectively, and ρ_3 the “reac-
 913 tive moment” associated to q^3 , so that the corresponding pairs of “Lagrangian
 914 parameter – reactive force” could be compared. Note that, the expressions of
 915 ρ_1 , ρ_2 and ρ_3 follow from Eqs. (71a)–(71c) and read

$$\rho_1 := \dot{\lambda}_1, \quad \rho_2 := \dot{\lambda}_2, \quad \rho_3 := -\dot{\lambda}_1 R \sin q^4 - \dot{\lambda}_2 R \cos q^4, \quad (93)$$

916 while, by virtue of Eq. (71d), the “reactive moment” associated to q^4 , i.e., ρ_4 ,
 917 is identically zero.

918 In Fig. 2a we observe that in the instants of time in which the reactive
 919 force ρ_1 vanishes, the Lagrangian parameter q^1 exhibits, in the same instants,

920 inflection points. This phenomenon is a direct consequence of Eq. (71e). More-
 921 over, as one can see in Fig. 2b,c, for the instants of time in which q^2 or q^3
 922 admit a local maximum, their corresponding reactive force and moment, i.e.,
 923 ρ_2 and ρ_3 , feature a local minimum, and vice versa. Indeed, this is in com-
 924 pliance with the analytical expressions of the Lagrange multipliers $\lambda_1(t)$ and
 925 $\lambda_2(t)$ featuring in Eqs. (82a) and (82b), and reported in Figure 2d, since, in
 926 the case we are considering, they read

$$\lambda_1(t) \equiv \hat{\lambda}_1(t; \Omega) = -\frac{1}{2}mR\Omega\Phi(\Omega)[\cos(2\Omega t) - 1], \quad (94a)$$

$$\lambda_2(t) \equiv \hat{\lambda}_2(t; \Omega) = \frac{1}{2}mR\Omega\Phi(\Omega)\sin(2\Omega t) - (mg\sin\alpha)t, \quad (94b)$$

927 which, in turn, produce the following reactions

$$\rho_1(t) = \frac{2}{3}mg\sin\alpha\sin(2\Omega t), \quad (95a)$$

$$\rho_2(t) = \frac{2}{3}mg\sin\alpha\cos(2\Omega t) - mg\sin\alpha, \quad (95b)$$

$$\rho_3(t) = \frac{1}{3}mgR\sin\alpha\cos(\Omega t). \quad (95c)$$

928 Therefore, by comparing the expression in Eqs. (95a)–(95c) with Eqs. (91a),
 929 (91b) and (78), we conclude that:

- 930 (i) The constraint \mathcal{V}^1 has the effect to change the curvature of the trajectory
 931 of the center of mass of the coin, i.e., the second time derivative of $q^1(t)$.
 932 (ii) The constraint \mathcal{V}^2 produces a reaction force on q^2 opposing to the motion
 933 that, in turn, determines the oscillatory behavior of the center of mass of
 934 the coin along the y -axis.

935 **Effect of a non-zero θ_0**

936 The second scenario we examine for the “rolling coin” problem concerns how
 937 considering non-zero values of $\theta_0 \equiv q_{\text{in}}^4$, for the fixed value of $\Omega = 2\pi$ rad/s,
 938 affects the overall motion of the system. In particular, we will compare the
 939 solutions when the chosen value of θ_0 belongs to the set $\{0, \pi/2\}$ (see Fig.
 940 3). Note that, the two cases above describe the situations in which the initial
 941 direction of rolling, indicated by the unit vector $\boldsymbol{\tau}$, is either aligned with the
 942 direction of steepest descent ($\theta_0 = 0$) or is orthogonal to it ($\theta_0 = \pi/2$). As in
 943 the study above, we initialize the remaining generalized coordinates as $q_{\text{in}}^1 = 0$,
 944 $q_{\text{in}}^2 = 0$ and $q_{\text{in}}^3 = 0$, and the Lagrange multipliers as $\lambda_{1\text{in}} = 0$ and $\lambda_{2\text{in}} = 0$.

945 In Fig. 3, we observe that in the case in which $\theta_0 = 0$ rad, the trajectory
 946 of the center of mass of the coin features an oscillatory motion that makes the
 947 center of mass move in the positive direction of the x -axis without ever going
 948 backward in the same direction. Instead, for $\theta_0 = \pi/2$ rad, the center of mass of
 949 the coin experiences an oscillatory motion that moves the coin initially in the
 950 negative direction of the x -axis. Since this behavior is periodic in the motion,
 951 a knot is produced in the trajectory of the center of mass when $t \simeq 1$ s.

952 Finally, from Fig. 4, we have compared, as done previously in Fig. 2, the
 953 pairs of “Lagrangian parameters – Reactive forces” that are obtained when Ω is

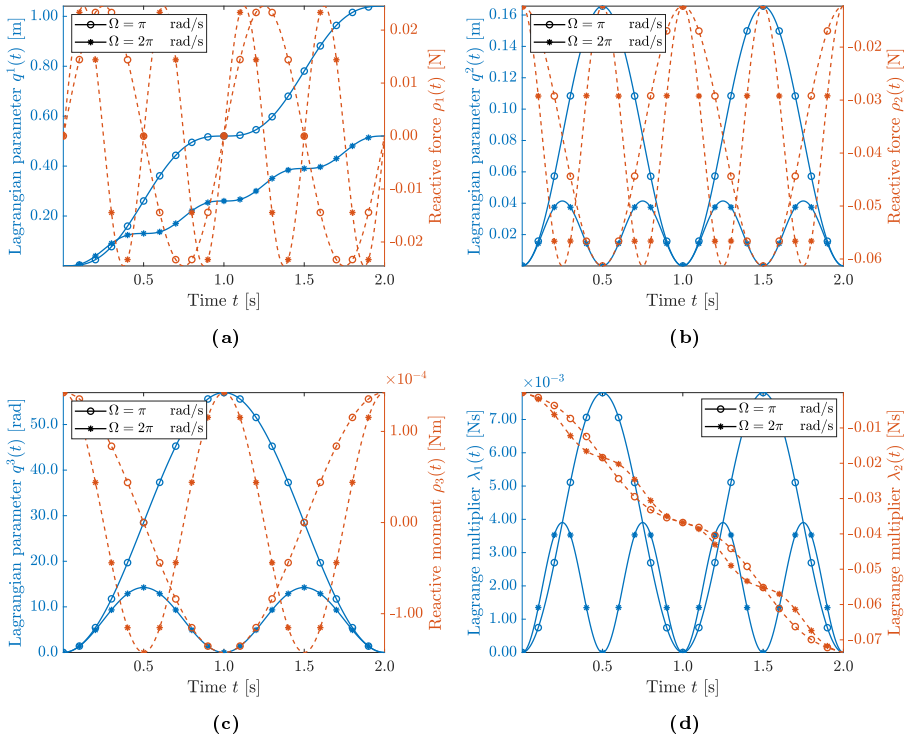


Figure 2: In 2a–2c the pairs “Lagrangian parameter – Reactive force” are represented by continuous and dotted lines, respectively. In Figure 2d, $\lambda_1(t)$ and $\lambda_2(t)$ are represented with a continuous and a dotted line, respectively.

954 fixed at 2π rad/s and $\theta_0 \in \{0, \pi/2\}$ rad. In this case, by looking at Fig. 4a–4c,
 955 we can draw similar conclusions to what we draw for the parametric study on
 956 Ω . Nevertheless, we acknowledge the fact that, when $\theta_0 \neq 0$, then the reactive
 957 forces ρ_1 and ρ_2 are determined by an apposition of oscillatory motion with
 958 different angular frequencies (see Fig. 4a–4b). On the other hand, as shown in
 959 Fig. 4c, ρ_3 is defined by only one cosinusoidal contribution that is shifted by
 960 the presence of a non-zero θ_0 . Indeed, these phenomena are explained by the
 961 analytical expressions for the three generalized reactive forces, which, in the
 962 case of $t_{\text{in}} = 0$ and $\theta_0 \neq 0$, read as follows:

$$\rho_1(t) = \frac{2}{3}mg \sin\alpha \left[\sin(2\Omega t + 2\theta_0) - \sin\theta_0 \cos(\Omega t + \theta_0) \right], \quad (96a)$$

$$\rho_2(t) = \frac{2}{3}mg \sin\alpha \left[\cos(2\Omega t + 2\theta_0) + \sin\theta_0 \sin(\Omega t + \theta_0) \right] - mg \sin\alpha, \quad (96b)$$

$$\rho_3(t) = \frac{1}{3}mg \sin\alpha R \cos(\Omega t). \quad (96c)$$

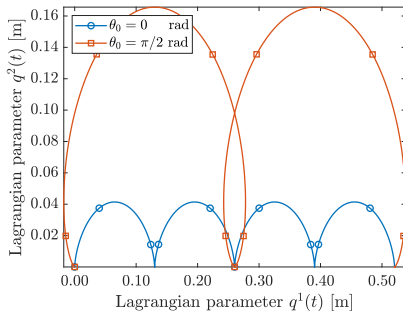


Figure 3: Trajectories of the center of mass of the coin for $\theta_0 \in \{0, \pi/2\}$ rad which are simulated from $t_{\text{in}} = 0$ s to $t_{\text{fin}} = 2$ s.

963 Note that the presence of non-zero θ_0 provides the same “reactive moment”
 964 $\rho_3(t)$, i.e., Eq. (96c), that we had in the case with $\theta_0 = 0$, i.e., Eq. (95c),
 965 thereby manifesting the independence of $\rho_3(t)$ with respect to θ_0 .

966 4 The “charged skate” benchmark

967 In this section, we study the dynamic equations for the benchmark problem
 968 of the “non-holonomic skate” (see e.g. [2, 5, 6, 12, 26, 28, 43, 44, 47]), which
 969 we have modified by introducing an electromagnetic interaction (see also [27]).
 970 Also in this case, we employ the “modified vakonomic method” (MVM) with
 971 the objective of showing how an interaction of this type can “break” the equiv-
 972 alence between the MVM and the TNHM, thereby violating the hypothesis of
 973 Theorem 1. In particular, the role of the *Ansätze*, needed for the closure of
 974 Eqs. (42a) and (42b), is investigated in the sequel.

975 We consider a three-dimensional rigid skate \mathfrak{S} , shaped as a rectangular
 976 parallelepiped having mass m , length ℓ , and cross section of area σ^2 , that slides
 977 over an inclined plane of an angle α with respect to a horizontal plane. We
 978 assume that the skate is electrically charged, with volumetric charge density
 979 e distributed uniformly in \mathfrak{S} , so that the total electric charge of the skate
 980 is $Q := e\ell\sigma^2$. Moreover, we let the skate interact with a field of magnetic
 981 induction associated with the (co-)vector potential \mathbf{A} [24].

982 A coordinate system $\{O, (x, y, z)\}$ is prescribed such that the x -axis is
 983 parallel to the inclined plane and is aligned with the direction of steepest
 984 descent; the z -axis “exits” the inclined plane and is orthogonal to it; the y -axis
 985 is such that the x -, y - and z -axis form a right-handed triad; the origin O
 986 is fixed at the same z -coordinate as the center of mass of the skate, hereafter
 987 denoted by G . Furthermore, we denote by $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and $\{e^x, e^y, e^z\}$ the
 988 basis unit vectors and co-vectors, respectively, associated with the coordinate
 989 system $\{O, (x, y, z)\}$.

990 By introducing $(x_G, y_G, 0)$ as the coordinates of G , and θ as the angle
 991 between the x -axis and the axis of the skate, we choose x_G, y_G and θ as the

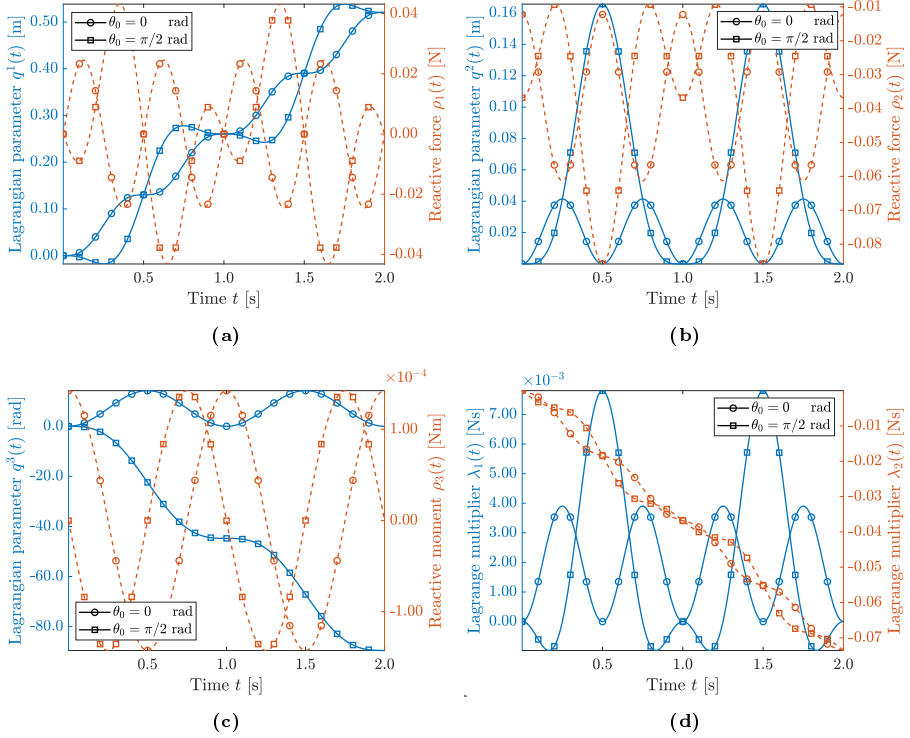


Figure 4: In 4a, 4b and 4c the pairs “Lagrangian parameter – Reactive force” are represented by continuous and dotted lines, respectively. In Fig. 4d, $\lambda_1(t)$ and $\lambda_2(t)$ are represented with a continuous and a dotted line, respectively.

$n = 3$ Lagrangian parameters of the mechanical system under study, thereby leading to the identification $q \equiv (q^1, q^2, q^3) \equiv (x_G, y_G, \theta)$ [12].

In the following, we will only consider the case in which the magnetic induction field $\mathbf{B} = \text{curl}\mathbf{A}$ is homogeneous in space and orthogonal to the inclined plane, i.e., $\mathbf{B}(x, y, z, t) = B_0(t)\mathbf{e}_z$, so that the resulting Lorentz force, acting on the skate, is parallel to the inclined plane itself. To this end, we design the (co-)vector potential to be $\mathbf{A}(x, y, z, t) = [-\frac{1}{2}B_0(t)y]\mathbf{e}^x + [\frac{1}{2}B_0(t)x]\mathbf{e}^y$.

Given the premises above, and if, in addition to the considered magnetic interactions, we also account for the gravitational interaction, then the Lagrangian of the skate reads [5, 6, 12, 43]

$$\begin{aligned} \mathcal{L} \circ (q, \dot{q}, \tau) = & \underbrace{\frac{1}{2}m \left[(\dot{q}^1)^2 + (\dot{q}^2)^2 + \frac{\ell^2 + \sigma^2}{12} (\dot{q}^3)^2 \right]}_{\mathcal{K} \circ \dot{q}} + \underbrace{(mg \sin \alpha) q^1}_{\mathcal{U}_g \circ q} \\ & + \underbrace{\left[-\frac{QB_0}{2} q^2 \right] \dot{q}^1 + \left[\frac{QB_0}{2} q^1 \right] \dot{q}^2 + \left[\frac{QB_0(\ell^2 + \sigma^2)}{24} \right] \dot{q}^3}_{\mathcal{U}_{m \circ (q, \dot{q}, \tau)}}, \end{aligned} \quad (97)$$

1002 where \mathcal{K} is the kinetic energy; \mathcal{U}_g and \mathcal{U}_m are the gravitational and magnetic
 1003 potential functions, respectively; g is the magnitude of the gravity acceleration
 1004 vector, and $B_0 \equiv B_0 \circ \tau$ is the magnitude of the magnetic induction field.

1005 By denoting by $\mathbf{n} = \cos\theta\mathbf{e}_x + \sin\theta\mathbf{e}_y$ the unit vector aligned with the
 1006 skate's axis, we assume the velocity of the center of mass of the skate, \mathbf{v}_G , to
 1007 remain always parallel to \mathbf{n} . This condition, written in the physical space as
 1008 $\mathbf{v}_G \times \mathbf{n} = \mathbf{0}$, prescribes the non-holonomic constraint to be [5, 6, 12, 43–45]

$$\mathcal{V}^1 \circ (q, \dot{q}) := (\sin q^3)\dot{q}^1 - (\cos q^3)\dot{q}^2 = 0. \quad (98)$$

1009 Note that the absence of \dot{q}^3 in the constraint (98) is sufficient to conclude that
 1010 the transpositional relation $\zeta^3 - \eta^3$ must be zero, i.e., $\zeta^3 - \eta^3 = \sum_{h=1}^3 W^3_h \eta^h =$
 1011 0 . In turn, this means that the coefficients W^3_1 , W^3_2 , and W^3_3 must be either
 1012 identically zero or such that their combination with η^1 , η^2 , and η^3 is zero
 1013 because of Lagrange-Chetaev's conditions (12).

1014 4.1 The traditional non-holonomic approach

1015 In this section, we compute the dynamic equations produced by the TNHM for
 1016 the “charged skate” problem. Hence, we specify the Euler-Lagrange operators
 1017 in Eq. (28a) for the Lagrangian in Eq. (97), i.e.,

$$\mathcal{E}_1 \mathcal{L} \circ \sharp^{(2)} = -m\ddot{q}^1 + QB_0\dot{q}^2 + \frac{1}{2}Q\dot{B}_0q^2 + mg \sin\alpha, \quad (99a)$$

$$\mathcal{E}_2 \mathcal{L} \circ \sharp^{(2)} = -m\ddot{q}^2 - QB_0\dot{q}^1 - \frac{1}{2}Q\dot{B}_0q^1, \quad (99b)$$

$$\mathcal{E}_3 \mathcal{L} \circ \sharp^{(2)} = -\frac{1}{12}m(\ell^2 + \sigma^2)\ddot{q}^3 - \frac{1}{24}Q\dot{B}_0(\ell^2 + \sigma^2), \quad (99c)$$

1018 and, by substituting Eqs. (99a)–(99c) as well as the constraint (98) into Eqs.
 1019 (52a) and (52b), we find the TNHM dynamic equations for the “charged skate”
 1020 problem to be as follows:

$$m\ddot{q}^1 - \mu_1 \sin q^3 = mg \sin\alpha + QB_0\dot{q}^2 + \frac{1}{2}Q\dot{B}_0q^2, \quad (100a)$$

$$m\ddot{q}^2 + \mu_1 \cos q^3 = -QB_0\dot{q}^1 - \frac{1}{2}Q\dot{B}_0q^1, \quad (100b)$$

$$\frac{1}{12}m(\ell^2 + \sigma^2)\ddot{q}^3 = -\frac{1}{24}Q(\ell^2 + \sigma^2)\dot{B}_0, \quad (100c)$$

$$(\sin q^3)\dot{q}^1 - (\cos q^3)\dot{q}^2 = 0. \quad (100d)$$

1021 We notice that Eqs. (100a)–(100c) can be recast in a more suggestive form
 1022 by rewriting them in terms of the canonical momenta:

$$\underbrace{\frac{d}{dt} \left[m\dot{q}^1 - \frac{1}{2}QB_0q^2 \right]}_{=:p_1} = \mu_1 \sin q^3 + mg \sin\alpha + \frac{1}{2}QB_0\dot{q}^2, \quad (101a)$$

$$\frac{d}{dt} \underbrace{\left[m\dot{q}^2 + \frac{1}{2}QB_0\dot{q}^1 \right]}_{=:p_2} = -\mu_1 \cos q^3 - \frac{1}{2}QB_0\dot{q}^1, \quad (101b)$$

$$\frac{d}{dt} \underbrace{\left[\frac{1}{12}m[\ell^2 + \sigma^2]\dot{q}^3 + \frac{1}{24}Q[\ell^2 + \sigma^2]B_0 \right]}_{=:p_3} = 0. \quad (101c)$$

1023 Note also that, since the Lagrangian parameter q^3 is “*ignorable*” [13], Eq. (101c)
 1024 is equivalent to state the conservation of the generalized momentum conjugated
 1025 with q^3 , which, thus, turns out to be an integral of the motion, i.e.,

$$p_3(t) = \frac{\partial \mathcal{L}}{\partial \dot{q}^3} (\#^{(1)}(t)) = \frac{1}{12}m[\ell^2 + \sigma^2]\dot{q}^3(t) + \frac{1}{24}[\ell^2 + \sigma^2]QB_0(t) = C, \quad (102)$$

1026 with C being an integration constant.

1027 4.2 The “modified vakonomic” approach

1028 Now, we apply the procedure reported in Sect. 2 to the problem under study.
 1029 Hence, we solve Eqs. (42a) and (42b) in compliance with the *solvability condi-*
 1030 *tions* (40a) and (40b), for the case in which the Lagrangian function and the
 1031 constraint are of the type specified in Eqs. (97) and (98), respectively.

1032 To this end, we express the conditions in Eq. (34), for $k \in \{1, 2, 3\}$ and
 1033 with $\alpha = 1$, in the case in which the considered constraint is the one reported
 1034 in Eq. (98):

$$\mathcal{D}_1 \mathcal{V}^1 \circ \#^{(2)} = (\sin q^3)W^1_1 - (\cos q^3)W^2_1 - (\cos q^3)\dot{q}^3 = 0, \quad (103a)$$

$$\mathcal{D}_2 \mathcal{V}^1 \circ \#^{(2)} = (\sin q^3)W^1_2 - (\cos q^3)W^2_2 - (\sin q^3)\dot{q}^3 = 0, \quad (103b)$$

$$\mathcal{D}_3 \mathcal{V}^1 \circ \#^{(2)} = (\sin q^3)W^1_3 - (\cos q^3)W^2_3 + (\cos q^3)\dot{q}^1 + (\sin q^3)\dot{q}^2 = 0. \quad (103c)$$

1035 To find all the coefficients W^h_k , we proceed as follows:

1036 4.2.1 Case A: Llibre et al.’s auxiliary functions [12]

1037 In the sequel, we take the same auxiliary functions as those suggested by Llibre
 1038 et al. [12] for the case of the “non-holonomic skate” without magnetic field (see
 1039 also [27] for comparison). However, we flip their order from the one reported
 1040 in [12], thereby writing

$$\mathcal{F}^2 \circ \#^{(1)} := \dot{q}^3, \quad (104a)$$

$$\mathcal{F}^3 \circ \#^{(1)} := (\cos q^3)\dot{q}^1 + (\sin q^3)\dot{q}^2, \quad (104b)$$

1041 to emphasize that the choice of these functions must respect the criterion given
 1042 in *Ansatz 1*. In particular, this means that they must respect the transposi-
 1043 tional relation $\zeta^3 - \eta^3 = \sum_{h=1}^3 W^3_h \eta^h = 0$, and, indeed, $\mathcal{F}^2 \circ \#^{(1)} := \dot{q}^3$ ensures

1044 the fulfillment of this condition in *strong* way, as can be seen by the equations

$$\mathcal{D}_1 \mathcal{F}^2 \circ \sharp^{(2)} = W^3_1 = 0, \quad (105a)$$

$$\mathcal{D}_2 \mathcal{F}^2 \circ \sharp^{(2)} = W^3_2 = 0, \quad (105b)$$

$$\mathcal{D}_3 \mathcal{F}^2 \circ \sharp^{(2)} = W^3_3 = 0. \quad (105c)$$

1045 Conversely, the auxiliary function in Eq. (104b) leads to the conditions

$$\mathcal{D}_1 \mathcal{F}^3 \circ \sharp^{(2)} = (\cos q^3) W^1_1 + (\sin q^3) W^2_1 + (\sin q^3) \dot{q}^3 = 0, \quad (106a)$$

$$\mathcal{D}_2 \mathcal{F}^3 \circ \sharp^{(2)} = (\cos q^3) W^1_2 + (\sin q^3) W^2_2 - (\cos q^3) \dot{q}^3 = 0, \quad (106b)$$

$$\mathcal{D}_3 \mathcal{F}^3 \circ \sharp^{(2)} = (\cos q^3) W^1_3 + (\sin q^3) W^2_3 - \underbrace{[(\sin q^3) \dot{q}^1 - (\cos q^3) \dot{q}^2]}_{=0} = 0. \quad (106c)$$

1046 We remark that, as anticipated in the *Ansatz 1* of Sect. 2.3, Eq. (104b) identifies \mathcal{F}^3 with the component of the projection of the vector \mathbf{v}_G onto the unit vector
1047 \mathbf{n} , namely $\mathcal{F}^3 \circ \sharp^{(1)} \equiv \mathbf{v}_G \cdot \mathbf{n}$, which is the geometric interpretation of a
1048 quasi-velocity.
1049

1050 By solving the linear algebraic system in Eqs. (103a)–(103c), (105a)–(105c),
1051 and (106a)–(106c) for the unknown coefficients W^h_k , we have [12]

$$W = \begin{bmatrix} 0 & \dot{q}^3 & -\dot{q}^2 \\ -\dot{q}^3 & 0 & \dot{q}^1 \\ 0 & 0 & 0 \end{bmatrix}. \quad (107)$$

1052 Hence, by substituting Eqs. (99a)–(99c) and (107) into Eqs. (42a) and (42b),
1053 we obtain that the dynamic equations returned by the MVM for the “charged
1054 skate” problem read

$$m\ddot{q}^1 - \dot{\lambda}_1 \sin q^3 = mg \sin \alpha - m\dot{q}^2 \dot{q}^3 + QB_0 \dot{q}^2 + \frac{1}{2} Q \dot{B}_0 q^2 - \frac{1}{2} QB_0 q^1 \dot{q}^3, \quad (108a)$$

$$m\ddot{q}^2 + \dot{\lambda}_1 \cos q^3 = m\dot{q}^1 \dot{q}^3 - QB_0 \dot{q}^1 - \frac{1}{2} Q \dot{B}_0 q^1 - \frac{1}{2} QB_0 q^2 \dot{q}^3, \quad (108b)$$

$$\frac{1}{12} m(\ell^2 + \sigma^2) \ddot{q}^3 = -\frac{1}{24} Q(\ell^2 + \sigma^2) \dot{B}_0 + \frac{1}{2} QB_0(q^1 \dot{q}^1 + q^2 \dot{q}^2), \quad (108c)$$

$$(\sin q^3) \dot{q}^1 - (\cos q^3) \dot{q}^2 = 0. \quad (108d)$$

1055 For ease of comparison with the TNHM, we rewrite Eqs. (108a)–(108c) by
1056 highlighting the canonical momenta, i.e.,

$$\frac{d}{dt} \underbrace{\left[m\dot{q}^1 - \frac{1}{2} QB_0 q^2 \right]}_{=: p_1} = \dot{\lambda}_1 \sin q^3 + mg \sin \alpha + \frac{1}{2} QB_0 \dot{q}^2 - p_2 \dot{q}^3, \quad (109a)$$

$$\frac{d}{dt} \underbrace{\left[m\dot{q}^2 + \frac{1}{2}QB_0q^1 \right]}_{=:p_2} = -\dot{\lambda}_1 \cos q^3 - \frac{1}{2}QB_0\dot{q}^1 + p_1\dot{q}^3, \quad (109b)$$

$$\frac{d}{dt} \underbrace{\left[\frac{1}{12}m[\ell^2 + \sigma^2]\dot{q}^3 + \frac{1}{24}Q[\ell^2 + \sigma^2]B_0 \right]}_{=:p_3} = \underbrace{\frac{1}{2}QB_0[q^1\dot{q}^1 + q^2\dot{q}^2]}_{=-p_1\dot{q}^2 + p_2\dot{q}^1}. \quad (109c)$$

1057 From Eq. (109c), we notice that the generalized momentum p_3 is not conserved
 1058 with the functions \mathcal{F}^2 and \mathcal{F}^3 of Eqs. (104a) and (104b) characterizing *Ansatz*
 1059 *1* for this problem. The criticality with this choice of functions is that they
 1060 do not guarantee the equivalence between the MVM and the TNHM, i.e., the
 1061 characterizing condition of Theorem 1 in Eq. (55a) is not satisfied. Indeed,
 1062 such condition would require the vanishing of the quantities

$$(\cos q^3)(W^T \mathbf{p})_1 + (\sin q^3)(W^T \mathbf{p})_2 \quad \text{and} \quad (W^T \mathbf{p})_3, \quad (110)$$

1063 which, in the case considered, i.e., with W as in Eq. (107), are instead

$$(\cos q^3)(W^T \mathbf{p})_1 + (\sin q^3)(W^T \mathbf{p})_2 = -\frac{1}{2}QB_0[q^1 \cos q^3 + q^2 \sin q^3]\dot{q}^3 \neq 0, \quad (111a)$$

$$(W^T \mathbf{p})_3 = \frac{1}{2}QB_0[q^1\dot{q}^1 + q^2\dot{q}^2] \neq 0, \quad (111b)$$

1064 where the right-hand sides of Eqs. (111a) and (111b) stem from the terms
 1065 making the momenta *affine* in the velocities.

1066 Hence, if we take the same auxiliary functions as in [12] for the problem at
 1067 hand, then, although the transpositional relations are maintained, the MVM
 1068 is *not equivalent* to the TNHM, since Eq. (55a) of Theorem 1 is not satisfied,
 1069 and a conservation law, which should exist, is lost.

1070 *Remark 7* (“Canonical flip” [22, 41] for the Case A of the “charged skate”)
 1071 Similarly to what has been done in Remark 5, the matrix W in Eq. (107) allows to
 1072 choose the coefficients $\mathcal{C}^h_{\ell k}$ of the associated “Canonical flip” as follows

$$\mathcal{C}^1_{32}\dot{q}^3 = W^1_2 = +\dot{q}^3 \quad \Rightarrow \quad \mathcal{C}^1_{32} \equiv \hat{\mathcal{C}}^1_{32} \circ q = +1, \quad (112a)$$

$$\mathcal{C}^1_{23}\dot{q}^2 = W^1_3 = -\dot{q}^2 \quad \Rightarrow \quad \mathcal{C}^1_{23} \equiv \hat{\mathcal{C}}^1_{23} \circ q = -1, \quad (112b)$$

$$\mathcal{C}^2_{31}\dot{q}^3 = W^2_1 = -\dot{q}^3 \quad \Rightarrow \quad \mathcal{C}^2_{31} \equiv \hat{\mathcal{C}}^2_{31} \circ q = -1, \quad (112c)$$

$$\mathcal{C}^2_{13}\dot{q}^1 = W^2_3 = +\dot{q}^1 \quad \Rightarrow \quad \mathcal{C}^2_{13} \equiv \hat{\mathcal{C}}^2_{13} \circ q = +1. \quad (112d)$$

1073 As for the “rolling coin”, all the other entries of $\mathcal{C}^h_{\ell k}$ can be set equal to zero.

1074 *Remark 8* (The “charged skate” in the case of momenta linear in the velocities)
 1075 The fact that, with the matrix W of Eq. (107), the results (111a) and (111b) spoil the
 1076 fulfillment of Theorem 1, and make, thus, the MVM not equivalent to the TNHM,
 1077 is a direct consequence of the last three summands of the Lagrangian function (97),
 1078 which correspond to $\sum_{h=1}^3 [Z_h \circ (q, \tau)]^h \dot{q}^h$, and render the momenta affine (rather than

linear) in the generalized velocities. However, if we follow the approach presented in Sect. 2.6, so that we deal with Eqs. (59a) and (59b), the dynamic equations (108a)–(108d) for the charged non-holonomic skate with the magnetic interactions regarded as (pseudo-)polygenic [13] become

$$m\ddot{q}^1 - \dot{\lambda}_1 \sin q^3 = \underbrace{-m\dot{q}^2 \dot{q}^3}_{=(W^T \mathbf{p}_0)_1} + \underbrace{[QB_0 \dot{q}^2 + \frac{1}{2}Q\dot{B}_0 \dot{q}^2]}_{=\Omega_1} + mg \sin \alpha, \quad (113a)$$

$$m\ddot{q}^2 + \dot{\lambda}_1 \cos q^3 = \underbrace{+m\dot{q}^1 \dot{q}^3}_{=(W^T \mathbf{p}_0)_2} + \underbrace{[-QB_0 \dot{q}^1 - \frac{1}{2}Q\dot{B}_0 \dot{q}^1]}_{=\Omega_2}, \quad (113b)$$

$$\frac{1}{12}m(\ell^2 + \sigma^2)\ddot{q}^3 = \underbrace{0}_{=(W^T \mathbf{p}_0)_3} + \underbrace{[-\frac{1}{24}Q(\ell^2 + \sigma^2)\dot{B}_0]}_{=\Omega_3}, \quad (113c)$$

$$(\sin q^3)\dot{q}^1 - (\cos q^3)\dot{q}^2 = 0. \quad (113d)$$

Accordingly, the condition (55a) of Theorem 1, instead of Eqs. (111a) and (111b), produces

$$\begin{aligned} (\cos q^3)(W^T \mathbf{p}_0)_1 + (\sin q^3)(W^T \mathbf{p}_0)_2 &= -(\cos q^3)m\dot{q}^2 \dot{q}^3 + (\sin q^3)m\dot{q}^3 \dot{q}^1 \\ &= m\dot{q}^3[(\sin q^3)\dot{q}^1 - (\cos q^3)\dot{q}^2] = 0, \end{aligned} \quad (114a)$$

$$(W^T \mathbf{p}_0)_3 = -\dot{q}^2 m \dot{q}^1 + \dot{q}^1 m \dot{q}^2 = 0, \quad (114b)$$

with the right-hand side of Eq. (114a) being null by virtue of the constraint. Therefore, within the present formulation, Eq. (55a) of Theorem 1 is automatically satisfied, and Eq. (55b) follows by working out the terms $\dot{\lambda}_1 \sin q^3 - m\dot{q}^2 \dot{q}^3$ and $-\dot{\lambda}_1 \cos q^3 + m\dot{q}^1 \dot{q}^3$.

4.2.2 Case B: Direct use of Theorem 1

In addition to what has been done in Remark 8, we may also reason in a different way, which constitutes the core of *Ansatz 2*. Specifically, to compute W , we adopt the conditions supplied in Eqs. (103a)–(103c) and (105a)–(105c), which stem from the constraint and from the use of $\mathcal{F}^2 \circ \sharp^{(1)} = \dot{q}^3$, respectively, and, in lieu of introducing \mathcal{F}^3 , we invoke directly Theorem 1, thereby requiring the vanishing of the quantities in Eq. (110):

$$\begin{aligned} (\cos q^3)(W^T \mathbf{p})_1 + (\sin q^3)(W^T \mathbf{p})_2 \\ = p_1 \cos q^3 W^1_1 + p_1 \sin q^3 W^1_2 + p_2 \cos q^3 W^2_1 + p_2 \sin q^3 W^2_2 = 0, \end{aligned} \quad (115a)$$

$$(W^T \mathbf{p})_3 = p_1 W^1_3 + p_2 W^2_3 = 0. \quad (115b)$$

This guarantees the equivalence between the MVM and the TNHM.

It is important to remark that the conditions (115a) and (115b) amount to requiring that the vector associated with $W^T \mathbf{p}$ lies on the plane on which the skate's motion takes place and is *orthogonal* to the skate's axis. Thus, the plane projection of this vector is orthogonal also to the velocity of the skate's center of mass \mathbf{v}_G , so that the force $W^T \mathbf{p}$ produces no power on \mathbf{v}_G . Moreover,

by virtue of Eq. (115b), $W^T \mathbf{p}$ produces no source/sink of momentum for p_3 , so that it does not spoil its conservation.

The eight conditions (103a)–(103c), (105a)–(105c), (115a) and (115b) permit to determine the nine coefficients of the resulting matrix W up to an arbitrary function ϱ , i.e.,

$$W^1_1 = \varrho, \quad (116a)$$

$$W^1_2 = \frac{p_2 \dot{q}^3}{\sin q^3 [p_1 \cos q^3 + p_2 \sin q^3]} - \varrho \cot q^3, \quad (116b)$$

$$W^1_3 = -p_2 \frac{\dot{q}^1 \cos q^3 + \dot{q}^2 \sin q^3}{p_1 \cos q^3 + p_2 \sin q^3}, \quad (116c)$$

$$W^2_1 = -\dot{q}^3 + \varrho \tan q^3, \quad (116d)$$

$$W^2_2 = -\dot{q}^3 \frac{p_1 \sin q^3 - p_2 \cos q^3}{p_1 \cos q^3 + p_2 \sin q^3} - \varrho, \quad (116e)$$

$$W^2_3 = p_1 \frac{\dot{q}^1 \cos q^3 + \dot{q}^2 \sin q^3}{p_1 \cos q^3 + p_2 \sin q^3}, \quad (116f)$$

$$W^3_1 = W^3_2 = W^3_3 = 0. \quad (116g)$$

Hence, the equations of motion (42a) and (42b) take on the form

$$\dot{p}_1 = \dot{\lambda}_1 \sin q^3 + mg \sin \alpha + \frac{1}{2} Q B_0 \dot{q}^2 - p_2 \dot{q}^3 + \varrho [p_1 + p_2 \tan q^3], \quad (117a)$$

$$\dot{p}_2 = -\dot{\lambda}_1 \cos q^3 - \frac{1}{2} Q B_0 \dot{q}^1 + p_2 \dot{q}^3 \cot q^3 - \varrho [p_2 + p_1 \cot q^3], \quad (117b)$$

$$\dot{p}_3 = 0. \quad (117c)$$

Thus, as predicted by Theorem 1, the MVM is equivalent to the TNHM, provided the following identification of the Lagrange multipliers is made:

$$\mu_1 \equiv \dot{\lambda}_1 - p_2 \dot{q}^3 \csc q^3 + \varrho [p_1 \csc q^3 + p_2 \sec q^3]. \quad (118)$$

Note that, because of the equivalence between the two methods, the present reformulation of the MVM conserves the momentum p_3 , that is, Eq. (117c) is identical to Eq. (101c). In this respect, it should also be noticed that, even though Eqs. (117a)–(117c) apparently depend on ϱ , which is still unknown, this dependence is not effective. Indeed, if the MVM has to be equivalent to the TNHM, this dependence cancels out by virtue of Eq. (55a) of Theorem 1 when the equations of motion are put in the form (51a). This means that, if the equivalence between the MVM and the TNHM is maintained, the determination of the motion does not require the complete knowledge of W and, thus, of ϱ . However, as predicted by Eq. (51b), the Lagrange multiplier λ_1 does depend on ϱ , thereby yielding a one-parameter family of solutions that are *all* equivalent to those obtained by the TNHM and lead to one, and only one, μ_1 . Clearly, as anticipated in *Ansatz 2*, ϱ can be determined through a physics-based condition, or by adopting Corollary 1, which requires $W^T \mathbf{p} = 0$. In particular, the

1124 latter case retrieves the condition pointed out by Ramírez and Sadovskaia [11],
 1125 and, later, by Llibre et al. [12] (see, in particular, Theorem 3 of [12]).

1126 Before closing this section, we find it convenient to summarize the results
 1127 that we deem particularly noteworthy in the following Remarks.

1128 *Remark 9* (Consequences of the adopted methodology)

1129 *Transpositional relations.* Given the matrix W , whose coefficients are reported in Eqs.
 1130 (116a)–(116g), the transpositional relations characterizing the problem at hand are

$$\zeta^1 - \dot{\eta}^1 = \frac{p_2 \dot{q}^3}{\sin q^3 [p_1 \cos q^3 + p_2 \sin q^3]} \eta^2 - \frac{p_2 \dot{q}^2}{\sin q^3 [p_1 \cos q^3 + p_2 \sin q^3]} \eta^3, \quad (119a)$$

$$\zeta^2 - \dot{\eta}^2 = -\dot{q}^3 \eta^1 + \frac{p_2 \cos q^3 - p_1 \sin q^3}{p_1 \cos q^3 + p_2 \sin q^3} \dot{q}^3 \eta^2 + \frac{\dot{q}^1 \cos q^3 + \dot{q}^2 \sin q^3}{p_1 \cos q^3 + p_2 \sin q^3} p_1 \eta^3, \quad (119b)$$

$$\zeta^3 - \dot{\eta}^3 = 0. \quad (119c)$$

1131 It is interesting to notice that the unknown parameter ϱ featuring in some coefficients
 1132 of W , i.e., in Eqs. (116a), (116b), (116d) and (116e), does not enter the transpositional
 1133 relations in Eqs. (119a)–(119c) because of the *Lagrange-Chetaev condition* in
 1134 Eq. (12).

1135 *Indetermination of the matrix W .* Our formulation of the MVM according to *Ansatz*
 1136 *2*, which renounces to one of the auxiliary functions, and invokes directly Theorem 1,
 1137 does not determine univocally all the 9 entries of W . This is testified by the independent
 1138 unknown ϱ , and follows from the fact that the conditions delivered by Eq. (55a)
 1139 are only two (difference between the total number of Lagrangian parameters, i.e., 3,
 1140 and number of constraints, i.e., 1). Even though this could seem to be a deficiency of
 1141 our approach with respect to the one developed in [12], one can assign or determine
 1142 ϱ through other conditions, e.g. physically inspired, as suggested by *Ansatz 2*.

1143 *Remark 10* (“Canonical flip” [22, 41] for the Case B of the “charged skate”)

1144 Although the case analyzed here shares some similarities with Remark 7, some
 1145 changes arise, which are worth of being investigated. This time, we start looking at
 1146 Eqs. (119a)–(119c), and we compare them with the relationships (21). To this end,
 1147 we notice that Eqs. (119a) and (119b) can be rewritten as

$$\zeta^1 - \dot{\eta}^1 = \frac{p_2 \csc q^3}{p_1 \cos q^3 + p_2 \sin q^3} \dot{q}^3 \eta^2 - \frac{p_2 \csc q^3}{p_1 \cos q^3 + p_2 \sin q^3} \dot{q}^2 \eta^3, \quad (120a)$$

$$\begin{aligned} \zeta^2 - \dot{\eta}^2 = & -\frac{p_1 \cos q^3}{p_1 \cos q^3 + p_2 \sin q^3} \dot{q}^3 \eta^1 + \frac{p_1 \cos q^3}{p_1 \cos q^3 + p_2 \sin q^3} \dot{q}^1 \eta^3 \\ & - \frac{p_1 \sin q^3}{p_1 \cos q^3 + p_2 \sin q^3} \dot{q}^3 \eta^2 + \frac{p_1 \sin q^3}{p_1 \cos q^3 + p_2 \sin q^3} \dot{q}^2 \eta^3, \end{aligned} \quad (120b)$$

1148 where, to obtain Eq. (120b), we have made use of the Lagrange-Chetaev condition
 1149 $(\sin q^3) \eta^1 - (\cos q^3) \eta^2 = 0$. Hence, a direct inspection yields

$$\mathcal{C}^1_{32} \equiv \hat{\mathcal{C}}^1_{32} \circ (q, p) = \frac{p_2 \csc q^3}{p_1 \cos q^3 + p_2 \sin q^3} = -\mathcal{C}^1_{23} \equiv -\hat{\mathcal{C}}^1_{23} \circ (q, p), \quad (121a)$$

$$\mathcal{C}^2_{31} \equiv \hat{\mathcal{C}}^2_{31} \circ (q, p) = -\frac{p_1 \cos q^3}{p_1 \cos q^3 + p_2 \sin q^3} = -\mathcal{C}^2_{13} \equiv -\hat{\mathcal{C}}^2_{13} \circ (q, p), \quad (121b)$$

$$\mathcal{C}^2_{32} \equiv \hat{\mathcal{C}}^2_{32} \circ (q, p) = -\frac{p_1 \sin q^3}{p_1 \cos q^3 + p_2 \sin q^3} = -\mathcal{C}^2_{23} \equiv -\hat{\mathcal{C}}^2_{23} \circ (q, p), \quad (121c)$$

1150 whereas all the other coefficients of $\mathcal{C}^h_{\ell k}$ can be set equal to zero. It should be noticed
 1151 that, in the present situation, the nonzero coefficients reported in Eqs. (121a)–(121c)
 1152 are functions of *both* the Lagrangian parameters q and of the generalized momenta
 1153 p . This result is, in fact, a consequence of the Lagrangian function featuring a term
 1154 linear in the velocities, which, in turn, renders the momenta affine functions of the
 1155 velocities themselves, and is in harmony with the functional dependence prescribed
 1156 in [11, 12] for the entries of the matrix W . We emphasize that, in Eqs. (121a)–
 1157 (121c), the identifications of the functions $\mathcal{C}^h_{\ell k}$ is *purely* formal, since we have not
 1158 determined them through the assignment of a set of quasi-velocities.

1159 *Remark 11* (Absence of electromagnetic interactions [12])

1160 If the interaction with the magnetic field is switched off (i.e., if $B_0(t) = 0$ is zero
 1161 at all times), Eqs. (109a)–(109c), as well as Eqs. (117a)–(117c) upon setting $\varrho = 0$,
 1162 simplify to the case already addressed in [12]. In particular, we obtain the same
 1163 dynamic equations found by Llibre et al. [12] by employing W as in Eq. (107) for the
 1164 “non-holonomic skate” problem, i.e., [12]

$$\dot{p}_1 \equiv m\dot{q}^1 = -m\dot{q}^2\dot{q}^3 + mg \sin \alpha + \dot{\lambda}_1 \sin q^3, \quad (122a)$$

$$\dot{p}_2 \equiv m\dot{q}^2 = m\dot{q}^1\dot{q}^3 - \dot{\lambda}_1 \cos q^3, \quad (122b)$$

$$\dot{p}_3 = 0. \quad (122c)$$

1165 Moreover, the matrix W with coefficients in Eqs. (116a)–(116g) trivially reduces to
 1166 the one in Eq. (107). Hence, by removing the magnetic interaction, the equivalence
 1167 between the MVM and the TNHMs is restored again for Case A, while, in Case B, the
 1168 equivalence was already present even with the magnetic interaction. In both cases,
 1169 the assumptions $B_0 \equiv 0$ and $\varrho = 0$ induce the same relation between the Lagrange
 1170 multipliers of the two methods, which reads

$$\mu_1 \equiv \dot{\lambda}_1 - m\dot{q}^2\dot{q}^3 \csc q^3 = \dot{\lambda}_1 - m[\dot{q}^1 \cos q^3 + \dot{q}^2 \sin q^3]\dot{q}^3, \quad (123)$$

1171 where the last equality is obtained by working out $\csc q^3$ and employing the constraint.
 1172 Finally, the absence of the magnetic interaction allows the transversal relations
 1173 to simplify to the ones found in [12], i.e.,

$$\zeta^1 - \dot{\eta}^1 = \dot{q}^3 \eta^2 - \dot{q}^2 \eta^3, \quad \zeta^2 - \dot{\eta}^2 = \dot{q}^1 \eta^3 - \dot{q}^3 \eta^1, \quad \zeta^3 - \dot{\eta}^3 = 0. \quad (124)$$

1174 4.2.3 Case C: New formulation of the constraint

1175 Looking at the relationship connecting the generalized velocities and the mo-
 1176 menta, the constraint (98), the expression of the functions \mathcal{F}^2 and \mathcal{F}^3 supplied
 1177 by Llibre et al. [12], and the matrix W of Eq. (107), we notice that there ex-
 1178 ists a sort of “natural pattern” among all these characteristic features of the
 1179 considered problem. Indeed, by referring to Case A, both in Eq. (98) and in
 1180 \mathcal{F}^3 there appear the same *constrained velocities*, whereas \mathcal{F}^2 involves the “un-
 1181 constrained velocity” \dot{q}^3 . Moreover, the structure of W is such that: its first
 1182 2×2 block is skew-symmetric in \dot{q}^3 ; its last column features the components

1183 of the vector $\mathbf{e}_3 \times \mathbf{v}_G$, which has zero projection onto the skate's unit vector
 1184 \mathbf{n} , so that the mixed product $(\mathbf{e}_3 \times \mathbf{v}_G) \cdot \mathbf{n}$ vanishes, thereby returning the
 1185 constraint; and its last row is null. These results render the MVM equivalent
 1186 to the TNHM in the absence of the magnetic field, as highlighted in Remark
 1187 11, although they lead to a loss of equivalence in the presence of the magnetic
 1188 field, as made evident in Eqs. (109a)–(109c) and in the subsequent discussion.
 1189 It was indeed this broken equivalence, and the need for restoring it, that made
 1190 us “re-design” W to obtain the matrix in Eqs. (116a)–(116g) under the guid-
 1191 ance of Theorem 1. However, this required to renounce to the “natural pattern”
 1192 mentioned above. Yet, this pattern can be recovered by redefining the con-
 1193 straint in such a way that the *effective velocities* of the skate, i.e., p_1/m and
 1194 p_2/m (or $(M^{-1}\mathbf{p})_1$ and $(M^{-1}\mathbf{p})_2$, in matrix notation), rather than \dot{q}^1 and \dot{q}^2 ,
 1195 are constrained. Clearly, this amounts to modifying the original problem, but
 1196 in a still physically sound way, so as to account for the velocity shift induced
 1197 by the magnetic field by passing from a constraint linear in the velocities \dot{q}^1
 1198 and \dot{q}^2 to one *affine* in these velocities (see Eq. (125) below).

1199 By virtue of the discussion above, and setting $\chi := \frac{QB_0}{2m}$, we introduce

$$\begin{aligned} \mathcal{V}_{\text{new}}^1 \circ \#^{(1)} &= (\sin q^3) \frac{p_1}{m} - (\cos q^3) \frac{p_2}{m} \\ &= \sin q^3 [\dot{q}^1 - \chi q^2] - \cos q^3 [\dot{q}^2 + \chi q^1] \\ &= (\sin q^3) \dot{q}^1 - (\cos q^3) \dot{q}^2 - \chi [(\sin q^3) q^2 + (\cos q^3) q^1] = 0. \end{aligned} \quad (125)$$

1200 The new constraint expressed in Eq. (125) suggests that the “natural pattern”
 1201 discussed above can be recovered by defining also the auxiliary functions in
 1202 terms of the effective velocities, i.e.,

$$\mathcal{F}_{\text{new}}^2 \circ \#^{(1)} = \underbrace{\dot{q}^3 + \chi}_{=: 12p_3/m[\ell^2 + \sigma^2]}, \quad (126a)$$

$$\mathcal{F}_{\text{new}}^3 \circ \#^{(1)} = \underbrace{\cos q^3 [\dot{q}^1 - \chi q^2]}_{=: p_1/m} + \underbrace{\sin q^3 [\dot{q}^2 + \chi q^1]}_{=: p_2/m}. \quad (126b)$$

1203 Accordingly, the new solvability conditions read

$$\mathcal{D}_1 \mathcal{V}_{\text{new}}^1 \circ \#^{(2)} = (\sin q^3) W^1_1 - (\cos q^3) W^2_1 - (\cos q^3) [\dot{q}^3 + \chi] = 0, \quad (127a)$$

$$\mathcal{D}_2 \mathcal{V}_{\text{new}}^1 \circ \#^{(2)} = (\sin q^3) W^1_2 - (\cos q^3) W^2_2 - (\sin q^3) [\dot{q}^3 + \chi] = 0, \quad (127b)$$

$$\mathcal{D}_3 \mathcal{V}_{\text{new}}^1 \circ \#^{(2)} = (\sin q^3) W^1_3 - (\cos q^3) W^2_3 + (\cos q^3) \frac{p_1}{m} + (\sin q^3) \frac{p_2}{m} = 0, \quad (127c)$$

$$\mathcal{D}_1 \mathcal{F}_{\text{new}}^2 \circ \#^{(2)} = W^3_1 = 0, \quad (127d)$$

$$\mathcal{D}_2 \mathcal{F}_{\text{new}}^2 \circ \#^{(2)} = W^3_2 = 0, \quad (127e)$$

$$\mathcal{D}_3 \mathcal{F}_{\text{new}}^2 \circ \#^{(2)} = W^3_3 = 0, \quad (127f)$$

$$\mathcal{D}_1 \mathcal{F}_{\text{new}}^3 \circ \#^{(2)} = (\cos q^3) W^1_1 + (\sin q^3) W^2_1 + (\sin q^3) [\dot{q}^3 + \chi] = 0, \quad (127g)$$

$$\mathcal{D}_2 \mathcal{F}_{\text{new}}^3 \circ \#^{(2)} = (\cos q^3) W^1_2 + (\sin q^3) W^2_2 - (\cos q^3) [\dot{q}^3 + \chi] = 0, \quad (127\text{h})$$

$$\mathcal{D}_3 \mathcal{F}_{\text{new}}^3 \circ \#^{(2)} = (\cos q^3) W^1_3 + (\sin q^3) W^2_3 - \underbrace{(\sin q^3) \frac{p_1}{m} + (\cos q^3) \frac{p_2}{m}}_{=0} = 0. \quad (127\text{i})$$

1204 Finally, by virtue of these results, the new matrix W_{new} , which solves Eqs.
1205 (127a)–(127i), takes on the form

$$W_{\text{new}} = \begin{bmatrix} 0 & (\dot{q}^3 + \chi) - (\dot{q}^2 + \chi q^1) \\ -(\dot{q}^3 + \chi) & 0 & (\dot{q}^1 - \chi q^2) \\ 0 & 0 & 0 \end{bmatrix}, \quad (128)$$

1206 which corresponds to *shifting* the original matrix W_{old} in Eq. (107) by a matrix

$$W_{\text{mag}} = \chi \begin{bmatrix} 0 & 1 & -q^1 \\ -1 & 0 & -q^2 \\ 0 & 0 & 0 \end{bmatrix}, \quad (129)$$

1207 which accounts for the interaction of the skate with the magnetic field, i.e.,
1208 $W_{\text{new}} = W_{\text{old}} + W_{\text{mag}}$.

1209 We notice that, since the new constraint $\mathcal{V}_{\text{new}}^1$ in Eq. (125) is the sum of
1210 the original constraint in Eq. (98) and of an additional term not involving
1211 the generalized velocities, the condition in Eq. (55a), which guarantees the
1212 equivalence between the MVM and the TNHMs, requires the vanishing of the
1213 same terms displayed in Eq. (110). Indeed, by computing the product $W_{\text{new}}^T \mathbf{p}$,
1214 we observe that the equivalence is satisfied, since the following identities hold:

$$(\cos q^3)(W_{\text{new}}^T \mathbf{p})_1 + (\sin q^3)(W_{\text{new}}^T \mathbf{p})_2 = \frac{12p_3}{\ell^2 + \sigma^2} \left[\sin q^3 \frac{p_1}{m} - \cos q^3 \frac{p_2}{m} \right] = 0, \quad (130\text{a})$$

$$(W_{\text{new}}^T \mathbf{p})_3 = -\frac{p_2}{m} p_1 + \frac{p_1}{m} p_2 = 0. \quad (130\text{b})$$

1215 Moreover, the dynamic equations returned by the MVM with the new formu-
1216 lation of both the constraint in Eq. (125) and the auxiliary functions in Eqs.
1217 (126a) and (126b) read

$$\dot{p}_1 = \dot{\lambda}_1 \sin q^3 + mg \sin \alpha + \frac{1}{2} Q B_0 \dot{q}^2 - p_2 (\dot{q}^3 + \chi), \quad (131\text{a})$$

$$\dot{p}_2 = -\dot{\lambda}_1 \cos q^3 - \frac{1}{2} Q B_0 \dot{q}^1 + p_1 (\dot{q}^3 + \chi), \quad (131\text{b})$$

$$\dot{p}_3 = 0. \quad (131\text{c})$$

1218 Finally, the relation between the Lagrange multipliers of the MVM and the
1219 TNHMs is a direct consequence of Eqs. (131a) and (131b), and, following the

1220 same procedure as in Eq. (118), it holds that

$$\mu_1 \equiv \dot{\lambda}_1 - (p_1 \cos q^3 + p_2 \sin q^3) (q^3 + \chi). \quad (132)$$

1221 *Remark 12* (“Canonical flip” [22, 41] for the Case C of the “charged skate”)
 1222 Looking at Eq. (128), it is immediate to notice that, in spite of the presence of the
 1223 factor χ , the structure of the “new” matrix W_{new} is the same as the one obtained
 1224 for the Case A (see Eq. (107)), with the sole difference that the entries of W_{new}
 1225 coincide with the components of the momenta of the theory, p , normalized by the
 1226 corresponding components of the mass matrix. A direct consequence of this fact is
 1227 that the coefficients $\mathcal{C}^h_{\ell k}$ of the “Canonical flip” characterizing this version of the
 1228 considered problem are exactly those determined in Remark 7, provided that the
 1229 transpositional relations are written as

$$\zeta^1 - \dot{\eta}^1 = + (q^3 + \chi)\eta^2 - (q^2 + \chi q^1)\eta^3 = + \underbrace{\frac{12 p_3}{m[\ell^2 + \sigma^2]}}_{=:\pi^3} \eta^2 - \underbrace{\frac{p_2}{m}}_{=:\pi^2} \eta^3, \quad (133a)$$

$$\zeta^2 - \dot{\eta}^2 = - (q^3 + \chi)\eta^1 + (q^1 - \chi q^2)\eta^3 = - \underbrace{\frac{12 p_3}{m[\ell^2 + \sigma^2]}}_{=:\pi^3} \eta^1 + \underbrace{\frac{p_1}{m}}_{=:\pi^1} \eta^3, \quad (133b)$$

$$\zeta^3 - \dot{\eta}^3 = 0, \quad (133c)$$

1230 where π^1 , π^2 , and π^3 are the normalized momenta, and can be interpreted as the
 1231 *effective velocities* of the problem at hand (in the sense that they are the physical
 1232 quantities, having physical dimensions of velocities, that are effectively constrained).

1233 5 Conclusions

1234 In this work, we have employed the “modified vakonomic method” (MVM),
 1235 introduced by Llibre et al. [12], to achieve four main results that, in our opinion,
 1236 may deepen the understanding of vakonomic mechanics:

- 1237 (i) We have shown that, for the “rolling coin” problem, the TNHM and MVM
 1238 are equivalent. By doing so, we have proven that the methodology outlined
 1239 in [12], which, however, was not adopted therein for this problem, allows
 1240 to reconcile the vakonomic approach with the traditional one, at least for
 1241 the considered case. This result, in fact, confirms the statement given by
 1242 Lemos [1], who spoke of “*complete inequivalence between the vakonomic and*
 1243 *the non-holonomic method*”, but it contextually indicates how to overcome
 1244 this inconsistency by using the procedure developed in [12].
- 1245 (ii) After testing the methodological power of the MVM with the “rolling
 1246 coin”, we have considered another widely studied problem, namely, the
 1247 “non-holonomic skate”, and we have elaborated a variant of it obtained by
 1248 assuming the skate to be electrically charged and exposed to an imposed,

1249 homogeneous magnetic field (we have, in fact, modified the framework out-
 1250 lined in [27]). We emphasize, in this respect, that, whereas Llibre et al. [12]
 1251 studied the uncharged “non-holonomic skate” by using their MVM, we have
 1252 studied the variant to this problem described above in two steps. First, we
 1253 have employed the MVM *as is* (i.e., as formulated in [12]), and we have seen
 1254 that, according to our results, it *is not* equivalent to the TNHM as long as
 1255 the auxiliary functions of the MVM are taken like in [12]. However, our sec-
 1256 ond step was to show that, by following the rationale supplied by Theorem
 1257 1, the equivalence can be restored through a suitable set of conditions on
 1258 W entering the transpositional relations.

1259 (iii) Theorem 1, in fact, is the result that states the characterizing conditions
 1260 by which the MVM can always be made consistently equivalent to the
 1261 TNHM. This, however, requires to further modify the MVM according to
 1262 the physical motivations given in *Ansatz 2*. Although the germinal idea of
 1263 this modification is already present in [11] (cf. the condition $W^T p = 0$, in
 1264 our notation) and in [12] (cf. the unnumbered equation in the Remark 19
 1265 of [12] with our Eqs. (55a) and (55b)), we have reinterpreted and expanded
 1266 it to formulate a new criterion, which we may call “M²VM”, where “M²”
 1267 stands for “Modified Modified”. The reason for this, as remarked in Sect. 4,
 1268 is that one can formulate problems for which it is not straightforward to
 1269 find a full set of auxiliary functions guaranteeing the equivalence between
 1270 the MVM and the TNHM. Therefore, in our opinion, the approach outlined
 1271 in [12] should be augmented by invoking Theorem 1.

1272 (iv) In Sect. 4.2.3, we have highlighted the existence of a “natural pattern”, and
 1273 we have discussed how this “pattern” is not respected when the charged
 1274 non-holohomic skate is subjected to the classical constraint (98), and the
 1275 auxiliary functions supplied in Eqs. (104a) and (104b) are used (cf. [12]).
 1276 Thus, we have proposed a variant of the constraint in which the constrained
 1277 velocities are the *effective ones*, i.e., those obtained by dividing the momenta
 1278 p_1 and p_2 by the mass, and p_3 by the corresponding coefficient of the mass
 1279 matrix, and we have re-defined the auxiliary functions in terms of such
 1280 velocities. As reported in Sect. 4.2.3, this restores the pattern and allows to
 1281 apply the MVM by Llibre et al. [12] *as is*. In addition, this way of looking at
 1282 a given mechanical problem could lead to a more physical conception of the
 1283 constraints which may result, for instance, in the passage from a constraint
 1284 linear in the velocities to one that is affine in them.

1285 We are aware of studies (see, e.g., [18, 28, 36]) suggesting that, for some
 1286 problems of field theory or geometric control theory, the vakonomic method
 1287 leads to results that are sometimes believed to be physically remarkable in
 1288 comparison with those obtained with the traditional non-holonomic method.
 1289 This belief, in our opinion, promotes further investigations on the vakonomic
 1290 method, with its generalizations, and on its relationship with the TNHM. In
 1291 this respect, it is important to emphasize that the aim of our work is not to
 1292 affirm that the TNHM should always be taken as reference for all types of
 1293 problems, and that, thus, other methods (e.g. the MVM) should always be

made equivalent to it. However, we are saying that, if there are physical reasons—even very subtle ones—hinting that the MVM can be made equivalent to the TNHM, then this equivalence must be accounted for. Indeed, following this philosophy, we highlighted, through Theorem 1, the possibility of applying the variational approach of Libre et al. [12] to a class of problems that the scientific community regards as correctly described by the TNHM.

It should be noted that, by promoting the further study of the *vakonomic procedure*, we intend the study of *generalized variational procedures* that well-suit the non-holonomic setting and, when expected, should be consistent with the TNHM. In fact, to our knowledge, there is no experimental evidence implying that the VM is better than the TNHM in modeling classical mechanical systems subjected to non-holonomic constraints (see e.g. [3, 5]), but this discussion is out of the scope of our work. In light of the previous sentence, our interest for the MVM relies on the possibility of having a fully-variational procedure specialized to non-holonomic systems, which, while being consistent with the TNHM, allows to exploit the Lagrangian formalism.

As a last remark, we would like to emphasize that our work served, essentially, two main purposes. The first one was to satisfy our need to test the MVM in its original formulation by [12] against both standard (see Sect. 3) and non-standard (see Sect. 4) problems in order to see whether or not the consistency with the TNHM holds. Exactly in this spirit, we view our work as an instructive “exercise” on the MVM. The second purpose, instead, was to propose some *modifications* of the MVM that ensure both the consistency with the TNHM (see Theorem 1) and the fulfillment of the appropriate transversitional relations (see Remark 1). Given the results above, future investigations could make use of the insight gained in this work on the MVM to look for symmetries in complicated systems (see [49] for a review on Noether's theorem in non-holonomic systems), and for extending the MVM to the continuum setting as a tool for approaching the rheonomy and non-holonomy of the inelastic processes [32, 33].

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Statements and Declarations

• Competing interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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1496 Appendix A

1497 The scope of this appendix is to put the variations characterizing the
1498 MVM within the context of Differential Geometry. We start combining
1499 Józwiowski&Respondek’s formalism [22], which implements the framework
1500 established by Gràcia et al. [48], with ours. In doing this, however, for the sake
1501 of simplicity, we consider here only constraints of the type $\mathcal{V}^\alpha \circ (q, \dot{q}) = [a^\alpha_k \circ$
1502 $q] \dot{q}^k = 0$, i.e., that do not depend explicitly on time. We denote by $\gamma := (q, \dot{q}) :$
1503 $[t_{\text{in}}, t_{\text{fin}}] \rightarrow T\mathcal{C}$ a “path” [22], identified with the “tangent lift” [22] of q at t ,
1504 i.e., $\gamma(t) \equiv T_t q(t) = (q(t), \dot{q}(t))$. In terms of $\tilde{Q} : [t_{\text{in}}, t_{\text{fin}}] \times] - \varepsilon_0, +\varepsilon_0[\rightarrow \mathcal{C}$ (see
1505 Eq. (6a)), we also define $\tilde{\Gamma} := (\tilde{Q}, \partial_t \tilde{Q}) \equiv T_t \tilde{Q} : [t_{\text{in}}, t_{\text{fin}}] \times] - \varepsilon_0, +\varepsilon_0[\rightarrow T\mathcal{C}$,
1506 so that the chain of equalities

$$\gamma(t) = (q(t), \dot{q}(t)) = (\tilde{Q}(t, 0), \partial_t \tilde{Q}(t, 0)) = T_t \tilde{Q}(t, 0) = \tilde{\Gamma}(t, 0) \quad (\text{A1})$$

1507 applies. Moreover, by differentiating \tilde{Q} with respect to ε , and evaluating the
1508 result at $\varepsilon = 0$, we obtain, for each $t \in [t_{\text{in}}, t_{\text{fin}}]$, the quantity (see [22])

$$\xi(t) := T_\varepsilon \tilde{Q}(t, 0) = (\tilde{Q}(t, 0), \partial_\varepsilon \tilde{Q}(t, 0)) \equiv (q(t), \boldsymbol{\eta}(t)), \quad (\text{A2})$$

1509 in which the identification $\boldsymbol{\eta}(t) \equiv \partial_\varepsilon \tilde{Q}(t, 0) \in T_{q(t)}\mathcal{C}$ has been made by virtue
1510 of Eq. (8a). Looking at the structure of $\xi(t)$, we notice that the second tangent

1511 bundle formalism arises naturally by iterating the operation of tangent lift,
1512 i.e., by computing the tangent lift of ξ , which yields

$$T_t \xi(t) = (q(t), \boldsymbol{\eta}(t), \dot{q}(t), \dot{\boldsymbol{\eta}}(t)) \in TT\mathcal{C}. \quad (\text{A3})$$

1513 Finally, following [22], we compute the quantity

$$\begin{aligned} T_\varepsilon \tilde{\Gamma}(t, 0) &= T_\varepsilon T_t \tilde{Q}(t, 0) = (q(t), \dot{q}(t), \partial_\varepsilon \tilde{Q}(t, 0), \partial_\varepsilon \partial_t \tilde{Q}(t, 0)) \\ &= (q(t), \dot{q}(t), \boldsymbol{\eta}(t), \dot{\boldsymbol{\eta}}(t)) \in TT\mathcal{C}, \end{aligned} \quad (\text{A4})$$

1514 thereby recovering the identity $T_\varepsilon \tilde{\Gamma}(t, 0) = \kappa_{\mathcal{C}}(T_t \xi(t))$, where the map $\kappa_{\mathcal{C}} :$
1515 $TT\mathcal{C} \rightarrow TT\mathcal{C}$ is referred to as “canonical flip” in [22, 41], since it changes
1516 the position of the entries of the second and third slots of the elements of
1517 $TT\mathcal{C}$. In the jargon adopted in [22, 41], the quantity $T_\varepsilon \tilde{\Gamma}(t, 0)$ is referred to as
1518 “natural variation” and is also denoted by $\delta_\xi \gamma(t) \equiv T_\varepsilon \tilde{\Gamma}(t, 0)$ to highlight that
1519 the natural variation is *generated* by $\xi(t)$ through the canonical flip.

1520 In analogy with the reasoning leading to Eq. (A1), and with the purpose of
1521 putting the approach of Llibre et al. [12] in the language used in [22], we also
1522 define the pair $\tilde{\Sigma} := (\tilde{Q}, \tilde{V}) : [t_{\text{in}}, t_{\text{fin}}] \times]-\varepsilon_0, +\varepsilon_0[\rightarrow T\mathcal{C}$, where we require \tilde{V}
1523 to satisfy the condition $\tilde{V}(t, 0) = \dot{q}(t)$, for $t \in [t_{\text{in}}, t_{\text{fin}}]$ (see Eq. (6b)). Hence,
1524 we compute $T_\varepsilon \tilde{\Sigma}(t, 0)$, thereby obtaining

$$\begin{aligned} T_\varepsilon \tilde{\Sigma}(t, 0) &= (\tilde{Q}(t, 0), \tilde{V}(t, 0), \partial_\varepsilon \tilde{Q}(t, 0), \partial_\varepsilon \tilde{V}(t, 0)) \\ &= (q(t), \dot{q}(t), \boldsymbol{\eta}(t), \boldsymbol{\zeta}(t)) \in TT\mathcal{C}, \quad \text{with } \boldsymbol{\zeta}(t) := \partial_\varepsilon \tilde{V}(t, 0). \end{aligned} \quad (\text{A5})$$

1525 The quantity $T_\varepsilon \tilde{\Sigma}(t, 0)$ in Eq. (A5) defines a generic variation. To specify
1526 such variation for the MVM, we need to impose the following conditions:
1527 (a) The path (q, \dot{q}) must satisfy the m given constraints $\mathcal{V}^\alpha(q(t), \dot{q}(t)) = 0$,
1528 with $\alpha = 1, \dots, m$, at all times, so that the pair $(q(t), \dot{q}(t))$ belongs to $T\mathcal{C}_c$,
1529 defined in Eq. (1). (b) The pair $(q, \boldsymbol{\eta}) \in T\mathcal{C}$ is compelled to fulfill Lagrange-
1530 Chetaev's conditions (12), which here becomes $[\partial_{\dot{q}} \mathcal{V}^\alpha(q(t), \dot{q}(t))] \boldsymbol{\eta}(t) \equiv$
1531 $[\partial_{\dot{q}^k} \mathcal{V}^\alpha(q(t), \dot{q}(t))] \eta^k(t) = 0$, for $\alpha = 1, \dots, m$. (c) As specified in [12], for every
1532 $t \in [t_{\text{in}}, t_{\text{fin}}]$, the vector $\boldsymbol{\zeta}(t)$ is required to satisfy the constraints *at the first*
1533 *order in ε* . Thus, upon defining the functions $\tilde{\mathcal{V}}^\alpha : [t_{\text{in}}, t_{\text{fin}}] \times]-\varepsilon_0, +\varepsilon_0[\rightarrow \mathbb{R}$,
1534 such that $\tilde{\mathcal{V}}^\alpha(t, \varepsilon) := \mathcal{V}^\alpha(\tilde{Q}(t, \varepsilon), \tilde{V}(t, \varepsilon))$, and $\tilde{\mathcal{V}}^\alpha(t, 0) \equiv \mathcal{V}^\alpha(q(t), \dot{q}(t)) = 0$, for
1535 all $\alpha = 1, \dots, m$ and for all $t \in [t_{\text{in}}, t_{\text{fin}}]$ (see Eqs. (9) and (10)), we write

$$\tilde{\mathcal{V}}^\alpha(t, \varepsilon) = \underbrace{\tilde{\mathcal{V}}^\alpha(t, 0)}_{=0} + \partial_\varepsilon \tilde{\mathcal{V}}^\alpha(t, 0) \varepsilon + o(\varepsilon), \quad \varepsilon \rightarrow 0, \quad (\text{A6})$$

1536 and we set $\partial_\varepsilon \tilde{\mathcal{V}}^\alpha(t, 0) = 0$, thereby obtaining Eq. (11), i.e.,

$$0 = \partial_\varepsilon \tilde{\mathcal{V}}^\alpha(t, 0) = \frac{\partial \mathcal{V}^\alpha}{\partial q^k}(q(t), \dot{q}(t)) \eta^k(t) + \frac{\partial \mathcal{V}^\alpha}{\partial \dot{q}^k}(q(t), \dot{q}(t)) \zeta^k(t). \quad (\text{A7})$$

Equation (A7), which characterizes the vakonomic variations, can be also worked out further as

$$\begin{aligned}
 0 &= \left[\frac{\partial \mathcal{V}^\alpha}{\partial q^k} \circ (q, \dot{q}) \right] \eta^k + \left[\frac{\partial \mathcal{V}^\alpha}{\partial \dot{q}^k} \circ (q, \dot{q}) \right] [\zeta^k - \dot{\eta}^k] + \left[\frac{\partial \mathcal{V}^\alpha}{\partial \dot{q}^k} \circ (q, \dot{q}) \right] \dot{\eta}^k \\
 &= \left[\mathcal{E}_k \mathcal{V}^\alpha \circ (q, \dot{q}, \ddot{q}) \right] \eta^k + \left[\frac{\partial \mathcal{V}^\alpha}{\partial \dot{q}^k} \circ (q, \dot{q}) \right] [\zeta^k - \dot{\eta}^k] + \frac{d}{dt} \left\{ \left[\frac{\partial \mathcal{V}^\alpha}{\partial \dot{q}^k} \circ (q, \dot{q}) \right] \eta^k \right\} \\
 &= \left[\mathcal{E}_k \mathcal{V}^\alpha \circ (q, \dot{q}, \ddot{q}) \right] \eta^k + \left[\frac{\partial \mathcal{V}^\alpha}{\partial \dot{q}^k} \circ (q, \dot{q}) \right] [\zeta^k - \dot{\eta}^k], \tag{A8}
 \end{aligned}$$

with

$$\mathcal{E}_k \mathcal{V}^\alpha \circ (q, \dot{q}, \ddot{q}) := \frac{\partial \mathcal{V}^\alpha}{\partial q^k} \circ (q, \dot{q}) - \frac{d}{dt} \left[\frac{\partial \mathcal{V}^\alpha}{\partial \dot{q}^k} \circ (q, \dot{q}) \right], \tag{A9}$$

and the last summand of Eq. (A8) being null by virtue of Lagrange-Chetaev's conditions (cf. Eq. (A8) with Eq. (14): They are the same for non-holonomic constraints linear in the velocities and not depending explicitly on time).

With all the premises specified above, the set of the variations characterizing the MVM can be defined as:

$$\begin{aligned}
 \mathcal{W}_c^{\text{MVM}} &:= \{ T_\varepsilon \tilde{\Sigma}(t, 0) \in TT\mathcal{C} \mid (q(t), \dot{q}(t)) \in T\mathcal{C}, \quad (q(t), \boldsymbol{\eta}(t)) \in T\mathcal{C}_c, \\
 &\quad \zeta(t) \mid \partial_\varepsilon \tilde{\mathcal{V}}^\alpha(t, 0) = 0 \}. \tag{A10}
 \end{aligned}$$

Note that, as written in Sect. 2.1, in Eq. (A8) we *guess* (this is, in fact, an educated guess, since we are aware of the existence of the transpositional relations) that $\zeta^k - \dot{\eta}^k$ can be expressed in terms of the linear transformation $\zeta^k - \dot{\eta}^k = W^k_\ell \eta^\ell$, in which $[W^k_\ell]_{\ell=1, \dots, n}^{k=1, \dots, n}$ are the coefficients of a *partially* unknown matrix W . Indeed, if the matrix made by the derivatives $\partial_{\dot{q}^k} \mathcal{V}^\alpha \circ (q, \dot{q})$ were square (it is rectangular, here), the coefficients of W could be determined from Eq. (A8) by solving for $\zeta^k - \dot{\eta}^k$. However, this matrix can be made square by introducing suitable quasi-velocities, as is done in the standard procedure leading to the transpositional relations (see e.g. [26, 50] and cf. Eq. (17)). This is, to our understanding, the philosophy underlying the approach proposed by Llibre et al. [12], whose main objective is to put the non-holonomic constraints in the variational framework, but with variations that *are not* only vakonomic, but both vakonomic and complying with Lagrange-Chetaev's conditions.

Finally, if the further identification $W^k_\ell \equiv \mathcal{C}^k_{h\ell} \dot{q}^h$ is made, where $\mathcal{C}^k_{h\ell}$ are the coefficients of the third-order tensor field defined in [22, 41], or in [26] (see Remark 2), and that characterizes the transpositional relations, then, in the formalism of [22], $T_\varepsilon \tilde{\Sigma}(t, 0)$ can be rewritten as

$$T_\varepsilon \tilde{\Sigma}(t, 0) = (q^a(t), \dot{q}^b(t), \eta^c(t), \underbrace{\dot{\eta}^d(t) + \mathcal{C}^d_{k\ell}(q(t)) \dot{q}^k(t) \eta^\ell(t)}_{=\zeta^d(t)}). \tag{A11}$$