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Rational Normal Curves and Hadamard Products

Enrico Carlini, Maria Virginia Catalisano, Giuseppe Favacchio and Elena Guardo

Abstract. Given $r > n$ general hyperplanes in $\mathbb{P}^n$, a star configuration of points is the set of all the n-wise intersection of the hyperplanes. We introduce *contact star configurations*, which are star configurations where all the hyperplanes are osculating to the same rational normal curve. In this paper, we find a relation between this construction and Hadamard products of linear varieties. Moreover, we study the union of contact star configurations on a same conic in $\mathbb{P}^2$, we prove that the union of two contact star configurations has a special h-vector and, in some cases, this is a complete intersection.

Mathematics Subject Classification. 13C40, 13C70, 14M10, 14M99, 14N20.

Keywords. Complete intersection, Hadamard product, star configuration, Gorenstein.

1. Introduction

We say that the hyperplanes in a set $\mathcal{L} = \{\ell_1, \dots, \ell_r\} \subseteq \mathbb{P}^n$, $r \geq n$, *meet properly* if $\ell_{i_1} \cap \cdots \cap \ell_{i_n}$ is a point for any choice of n different indices and $n+1$ hyperplanes are never concurrent. We denote $\ell_{i_1} \cap \cdots \cap \ell_{i_n}$ by $P_{i_1,\dots,i_n}$.

Let $\mathcal{L} = \{\ell_1, \dots, \ell_r\} \subseteq \mathbb{P}^n$ be a set of $r \geq n$ hyperplanes meeting properly. The set of points

$$\mathbb{S}(\mathcal{L}) = \bigcup_{1 \leq i_1 < \dots < i_n \leq r} P_{i_1,\dots,i_n} \subseteq \mathbb{P}^n.$$

is called a *star configuration* of points in $\mathbb{P}^n$ defined by $\mathcal{L}$.

These configurations of points, and their generalizations, have been intensively studied for their algebraic and geometrical properties, see [2,5,8,17,26] for a partial list of papers that have contributed to our understanding them.

Set $S = \mathbb{C}[x_0, \dots, x_n] = \mathbb{C}[\mathbb{P}^n]$, where $\mathbb{C}$ could be replaced by any algebraically closed field of characteristic zero. We recall that the Hilbert

function of a set of points $X \subseteq \mathbb{P}^n$ is the numerical function $H_X : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ defined by

$$H_X(t) = \dim S_t - \dim(I_X)_t,$$

where I_X is the ideal defining X, and the h-vector of a set of points $X \subseteq \mathbb{P}^2$ is the first difference of the Hilbert function of X, that is

$$h_X(t) = H_X(t) - H_X(t-1),$$

where we set $H_X(-1) = 0$.

A star configuration $\mathbb{S}(\mathcal{L})$ defined by a set of r hyperplanes consists of $\binom{r}{n}$ points, and its h-vector is generic, see for instance [17, Theorem 2.6], that means $h_{\mathbb{S}(\mathcal{L})} = \left(1, \ldots, \binom{n-1+i}{n-1}, \ldots, \binom{r-1}{n-1}\right)$. Indeed, the ideal defining $\mathbb{S}(\mathcal{L})$ is minimally generated in degree $r-1$ by all the products of $r-1$ linear forms defining the hyperplanes in $\mathcal{L}$.

We now construct star configurations starting from a rational normal curve γ of $\mathbb{P}^n$. We call them *contact star configurations on* γ, we will not mention γ if it is clear from the context. We recall that an *osculating hyperplane* to the curve γ at a point $P \in \gamma$ is the hyperplane spanned by the length n scheme $nP \cap \gamma$, where nP is a fat point of multiplicity n, (see also Notation 2.1 and Definition 2.4 in [3]).

Definition 1.1. Let $P_1, \ldots, P_r \subseteq \mathbb{P}^n$ be distinct points on a rational normal curve γ of $\mathbb{P}^n$. Denote by $\mathcal{L} = \{\ell_1, \ldots, \ell_r\}$ the set of osculating hyperplanes to γ at $P_1, \ldots, P_r$, respectively. We say that $\mathbb{S}(\mathcal{L})$ is a *contact star configuration* on γ.

Note that, since γ is a rational normal curve, the hyperplanes in $\mathcal{L}$ always meet properly. Indeed, via the nth Veronese embedding $\nu_n : \mathbb{P}^1 \to \mathbb{P}^n$ defined by $\nu_n([\alpha]) = [\alpha^n]$, where $\alpha \in \mathbb{C}[\mathbb{P}^1]_1$, the curve γ is the variety that parameterizes the nth powers of linear forms in two variables. Hence, the points in the hyperplane osculating γ at $[\alpha^n]$ are parametrized by the forms $\alpha \cdot \mathbb{C}[\mathbb{P}^1]_{n-1}$. This implies that the hyperplanes osculating to γ at $[\alpha_1^n], \ldots, [\alpha_n^n]$ meet exactly at the point $[\alpha_1 \cdots \alpha_n]$ (see Remark 3.2 in [3]).

The first motivation to introduce these configurations come from Hadamard products. We show in Sect. 2 that the so called *Hadamard star configurations* are indeed contact star configurations, see Theorem 2.1. This result will give an easy way to explicitly construct examples which only make use of rational points, see in Remark 3.3.

The second motivation is related to their h-vector. A single contact star configuration has a generic h-vector, as any other star configuration. But the behavior of a union of two or more of them deserves further investigation. The homological invariants of a set of points which is a union of star configurations have been studied for instance in [2,27,28]. In the known cases, that require some restrictive assumptions, the h-vector of such a union is always general.

We will mostly focus on $\mathbb{P}^2$, therefore, the contact star configurations are defined by taking lines tangent to an irreducible conic. The study of properties of families of lines tangent to a planar conic is classical in algebraic geometry, see for instance the Cremona's book [11].

We prove, in Sect. 3, that the union of two contact star configurations in $\mathbb{P}^2$, defined by r and s lines, is a complete intersection of type $(r-1, s)$ if either $s = r-1$ or $s = r$; see Theorem 3.1. We also show that, in these cases, the curve of degree s can be chosen to be irreducible.

Moreover, in Sect. 4, we prove that the union of contact star configurations in $\mathbb{P}^2$ defined r and s lines has the same h-vector of two fat points of multiplicities $r-1$ and $s-1$, see Theorem 4.3. We believe that this correspondence with the h-vector of certain scheme of fat points also occurs for a union of three and four contact stars, see Conjecture 4.8. We prove it in some cases, see Theorem 4.5.

In Sect. 5, we apply Theorem 3.1 to the study of a recurring topic in classical projective geometry: polygons circumscribed around an irreducible conic in $\mathbb{P}^2$, see Proposition 5.1, Corollary 5.2 and Proposition 5.3.

Section 6 contains concluding remarks and conjectures for further work.

We will make use of standard tools from linkage theory; see [25] for an overview of the topic and [15–17,22,24] for a partial list of papers which use liaison to study zero-dimensional projective and multiprojective schemes. A well-known result, see [23, Corollary 5.2.19], relates the h-vectors of two arithmetically Cohen–Macaulay schemes in $\mathbb{P}^n$ with the same codimension, that are linked by an arithmetically Gorenstein scheme. In particular, if X, Y are two disjoint sets of reduced points in $\mathbb{P}^2$ and $X \cup Y$ is a complete intersection of type (a, b), then the following formula connects the h-vectors of X, Y and $X \cup Y$:

$$h_{X\cup Y}(t) = h_X(t) + h_Y(a+b-2-t), \quad \text{for any integer } t. \tag{1.1}$$

Since the h-vector of a complete intersection is well known, having the h-vector of X allows us to compute that of Y using the formula above.

2. Hadamard Products

In this section, we show that Hadamard star configurations are contact star configurations. Hadamard products of linear spaces have recently been the subject of study for many interesting properties, see for instance [4,7,8]. We briefly recall some general facts about Hadamard products of linear spaces. Let $P = [a_0 : \cdots : a_n]$ and $Q = [b_0 : \cdots : b_n]$ be two points in $\mathbb{P}^n$. If for some i, we have both $a_i \neq 0$ and $b_i \neq 0$, then we say that the *Hadamard product* of P and Q, denoted $P \star Q$, is defined and we set

$$P \star Q = [a_0 b_0 : \cdots : a_n b_n] \in \mathbb{P}^n.$$

Given two varieties X and Y in $\mathbb{P}^n$, the *Hadamard product* of X and Y, denoted $X \star Y$, is given by

$$X \star Y = \overline{\{P \star Q \mid P \in X, Q \in Y, \quad \text{and} \quad P \star Q \text{ is defined}\}} \subseteq \mathbb{P}^n$$

where the closure is taken with respect to the Zariski topology.

In particular, for a variety X in $\mathbb{P}^n$ and a positive integer $r \geq 2$, the rth Hadamard power of X is

$$X^{\star r} = X^{\star(r-1)} \star X,$$

where we define $X^{\star 1} = X$.

When we compute the Hadamard product of X and Y it is often crucial to ensure some condition of generality on X and Y, this is encoded by their points not having too many zero coordinates. For this purpose, we let Δ_i be the set of points of $\mathbb{P}^n$ which have at most $i+1$ non-zero coordinates.

A slightly different definition of Hadamard product is given in Definition 2.15 [7]. If X is a finite set of points in $\mathbb{P}^n$, then the rth *square-free Hadamard product* of $\mathbb{X}$ is

$$X^{\star_r} = \{P_1 \star \cdots \star P_r \mid P_1, \ldots, P_r \in X \text{ distinc points}\}.$$

From [4, Theorem 4.7], it is known that $X^{\star_n}$ is a star configuration of $\binom{m}{n}$ points of $\mathbb{P}^n$, where $X \subseteq \mathbb{P}^n$ is a set of $m > n$ points on a line ℓ such that $\ell \cap \Delta_{n-2} = \emptyset$. The set $X^{\star_n} \subseteq \mathbb{P}^n$ is called an *Hadamard* star configuration.

Let V be a linear space in $\mathbb{P}^n$, for a positive integer r, we consider the subscheme

$$V^{\circ r} = \{P^{\star r} \mid P \in V\} \subseteq \mathbb{P}^n,$$

called the *rth coordinate-wise power* of V. Properties of these schemes have been studied in [13]. In the following theorem, we investigate the case where V is a line. We recall that the linear subspace of dimension d osculating to a rational normal curve γ at P is the linear space spanned by the scheme $(d+1)P \cap \gamma$, see Notation 2.1 and Definition 2.4 in [3].

Theorem 2.1. *Let ℓ be a line in $\mathbb{P}^n$ such that $\ell \cap \Delta_{n-2} = \emptyset$. Then,*

(i) *$\ell^{\circ n} = \{P^{\star n} | P \in \ell\}$ is a rational normal curve;*
(ii) *let $P \in \ell$, then the linear subspace of dimension d osculating to $\ell^{\circ n}$ at $P^{\star n}$ is $P^{\star(n-d)} \star \ell^{\star d}$. In particular, the osculating hyperplane to $\ell^{\circ n}$ at $P^{\star n}$ is $P \star \ell^{\star(n-1)}$;*
(iii) *for each set of n distinct points on ℓ, $P_1, \ldots, P_n \in \ell$, we have*

$$P_1 \star P_2 \star \cdots \star P_n = (P_1 \star \ell^{\star(n-1)}) \cap \cdots \cap (P_n \star \ell^{\star(n-1)}).$$

Proof. The degree of $\ell^{\circ n}$ is n from Corollary 2.8 in [13]. Take a parametrization of the line ℓ, say $P_{ab} = [L_0(a,b) : L_1(a,b) : \cdots : L_n(a,b)] \in \ell$ where the $L_i(a,b)$ are linear forms in the variables a, b. Note that, since $\ell \cap \Delta_{n-2}$ is empty, the forms $L_i(a,b)$ are pairwise not proportional.

(i) The curve $\ell^{\circ n}$ is parametrized by $P_{ab}^{\star n} = [L_0(a,b)^n : L_1(a,b)^n : \cdots : L_n(a,b)^n] \in \ell^{\circ n}$, where the components of $P_{ab}^{\star n}$ are a basis for the forms of degree n in a, b since the $L_i(a,b)$ are pairwise not proportional. Hence, $\ell^{\circ n}$ is a rational normal curve of $\mathbb{P}^n$.
(ii) We will prove item (ii) by induction on d. Let $d = 1$. Now let $P + tQ$ be a point of ℓ, $(t \in \mathbb{C})$, thus the tangent line to $\ell^{\circ n}$ at $P^{\star n}$ is

$$\begin{aligned}\lim_{t\to 0} \langle\, P^{\star n},\ (P+tQ)^{\star n} \,\rangle &= \lim_{t\to 0} \langle\, P^{\star n},\ P^{\star n} + ntP^{\star(n-1)} \star Q + \cdots + t^n Q^{\star n} \,\rangle \\ &= \langle\, P^{\star n},\ P^{\star(n-1)} \star Q \,\rangle = P^{\star(n-1)} \star \ell.\end{aligned}$$

Assume $d > 1$. By the induction hypothesis, the linear space of dimension $d-1$ osculating to $\ell^{\circ n}$ at $P^{\star n}$ is $P^{\star(n-d+1)} \star \ell^{\star(d-1)}$. Let $Q \neq P$ be

a point on ℓ. We have

$$\begin{aligned} & P^{\star(n-d+1)} \star \ell^{\star(d-1)} \\ & \quad = \{P^{\star(n-d+1)} \star (a_1P + b_1Q) \star \ldots \star (a_{d-1}P + b_{d-1}Q) \mid a_i, b_i \in \mathbb{C}\} \\ & \quad = \langle\ P^{\star n},\ P^{\star(n-1)} \star Q,\ P^{\star(n-2)} \star Q^{\star 2}, \ldots, P^{\star(n-d+1)} \star Q^{\star(d-1)} \rangle\ . \end{aligned}$$

Now let again $P + tQ$ be a point of ℓ, $(t \in \mathbb{C})$. The linear space of dimension d osculating to $\ell^{\circ n}$ at $P^{\star n}$ can be obtained by computing the following limit:

$$\lim_{t\to 0} \langle\ P^{\star(n-d+1)} \star \ell^{\star(d-1)},\ (P + tQ)^{\star n}\ \rangle,$$

and this limit, by an easy computation and the equality above, becomes

$$\begin{aligned} & \lim_{t\to 0} \langle\ P^{\star n},\ P^{\star(n-1)} \star Q,\ P^{\star(n-2)} \star Q^{\star 2}, \ldots, P^{\star(n-d+1)} \star Q^{\star(d-1)},\ (P + tQ)^{\star n}\ \rangle \\ & \quad = \lim_{t\to 0} \langle\ P^{\star n},\ P^{\star(n-1)} \star Q, \ldots, P^{\star(n-d+1)} \star Q^{\star(d-1)},\ P^{\star n} + ntP^{\star(n-1)} \star Q \\ & \quad\quad + \cdots + t^n Q^{\star n}\ \rangle \\ & \quad = \lim_{t\to 0} \langle\ P^{\star n},\ P^{\star(n-1)} \star Q, \ldots, P^{\star(n-d+1)} \star Q^{\star(d-1)},\ \binom{n}{d} t^d P^{\star(n-d)} \star Q^{\star d} + \cdots + t^n Q^{\star n}\ \rangle \\ & \quad = \langle\ P^{\star n},\ P^{\star(n-1)} \star Q, \ldots, P^{\star(n-d+1)} \star Q^{\star(d-1)},\ P^{\star(n-d)} \star Q^{\star d}\ \rangle \\ & \quad = P^{\star(n-d)} \star \ell^{\star d}. \end{aligned}$$

(iii) It follows from (ii) and the fact that n different osculating hyperplanes to γ meet properly.

□

Theorem 2.1 shows that a point $P_1 \star \cdots \star P_n \in X^{\star n}$ is the intersection of n hyperplanes that osculate the rational normal curve $\ell^{\circ n}$. Hence, an Hadamard star configuration $X^{\star n}$, constructed from a finite set of points X on a line $\ell \in \mathbb{P}^n$, is a contact star configuration.

In the next remark, we give more details for $n = 2$.

Remark 2.2. Consider a line ℓ in $\mathbb{P}^2$ and the respective conic $\ell^{\circ 2}$, let $\mathbb{C}[x, y, z]$ be the coordinate ring of $\mathbb{P}^2$. We have the following facts.

(i) Say ℓ defined by the equation $\alpha x + \beta y - z = 0$, where $\alpha, \beta \neq 0$. Then, from Theorem 2.1(i), we have that $\ell^{\circ 2}$ is a conic. Precisely, one can check that

$$\ell^{\circ 2}: \quad (\alpha^2 x + \beta^2 y - z)^2 - 4\alpha^2\beta^2 xy = 0.$$

(ii) From Theorem 2.1(ii), for each $P \in \ell$, the line $P \star \ell$ is tangent to $\ell^{\circ 2}$ at $P \star P$.

(iii) From Theorem 2.1(iii), for any $P, Q \in \ell$, $P \neq Q$, the two tangent lines to $\ell^{\circ 2}$ through $P \star Q$ are $P \star \ell$ and $Q \star \ell$. Note that this allows us to find an explicit Hadamard decomposition of any point in the plane $\mathbb{P}^2 = \ell \star \ell$. In fact, let $A \in \mathbb{P}^2$, let a and b be the tangent lines to the conic $\ell^{\circ 2}$ through A, and let $P \star P = a \cap \ell^{\circ 2}$, $Q \star Q = b \cap \ell^{\circ 2}$, then $A = P \star Q$.

(iv) For any $P, Q \in \ell$, $P \neq Q$, the line through $P \star P$ and $Q \star Q$ is the polar line of the point $P \star Q$ with respect to $\ell^{\circ 2}$.

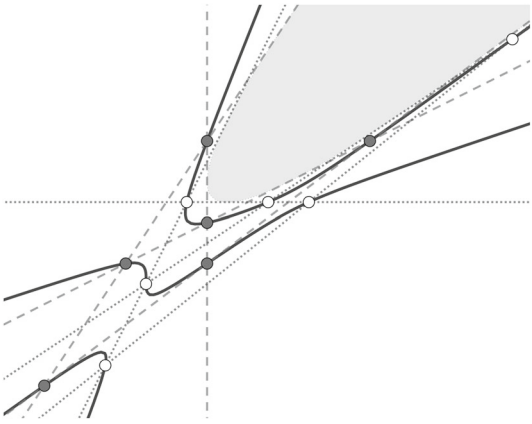


Figure 1. A cubic through 12 points union of 2 contact star configurations

(v) The condition $\ell \cap \Delta_0 = \emptyset$ ensures that ℓ meets the lines $x = 0$, $y = 0$ and $z = 0$ in three distinct points, say P_x, P_y and P_z, respectively. Note that, from the definition of Hadamard product, $P_x * \ell$ is the line $x = 0$ and analogously $P_y * \ell$ is the line $y = 0$ and $P_z * \ell$ is $z = 0$. Then, the conic $\ell^{\circ 2}$ is tangent to the coordinate axes in $P_x * P_x$, $P_y * P_y$ and $P_z * P_z$.

3. Complete Intersections Union of Two Contact Star Configurations in $\mathbb{P}^2$

In this section, γ is an irreducible conic in $\mathbb{P}^2$, and we set $S = \mathbb{C}[x, y, z] = \mathbb{C}[\mathbb{P}^2]$. The main result of this section is the following Theorem. We postpone its proof until page 7, after the development of some special cases.

Theorem 3.1. *Let $X = \mathbb{S}(\mathcal{L})$ and $Y = \mathbb{S}(\mathcal{M})$ be two contact star configurations in $\mathbb{P}^2$ on the same conic, where $\mathcal{L} = \{\ell_1, \dots, \ell_r\}$ and $\mathcal{M} = \{m_1, \dots, m_s\}$ are two disjoint sets of distinct lines. Then,*

(a) *if $s = r - 1$, then the general form in $(I_{X\cup Y})_{r-1}$ is irreducible;*
(b) *if $s = r - 1$, then $X \cup Y$ is a complete intersection of type $(r - 1, r - 1)$;*
(c) *if $s = r$, then the general form in $(I_{X\cup Y})_r$ is irreducible;*
(d) *if $s = r$, then $(I_{X\cup Z})_{r-1} = (I_{X\cup Y})_{r-1}$ where Z denotes a set of $r - 1$ collinear points in Y;*
(e) *if $s = r$, then $X \cup Y$ is a complete intersection of type $(r - 1, r)$.*

Remark 3.2. Theorem 3.1 (e) in particular claims that the 12 points of $X \cup Y$, where $X = \mathbb{S}(\ell_1, \ell_2, \ell_3, \ell_4)$ and $Y = \mathbb{S}(m_1, m_2, m_3, m_4)$ are contact star configurations on the same conic and the 8 lines are distinct, lie on a cubic. This case is pictured in Fig. 1.

Remark 3.3. Combining Theorems 3.1 and 2.1, we are able to explicitly give the coordinates of a (not trivial) complete intersection of rational points in

$\mathbb{P}^2$ of type (a, b), where either $b = a$ or $b = a - 1$. First, we need to fix a rational line ℓ in $\mathbb{P}^2$. Then, we take $a + b + 1$ distinct rational points on ℓ divided in two sets X and Y containing $a+1$ and b points, respectively. Then, $\mathbb{X}^{*2} \cup \mathbb{Y}^{*2}$ is the complete intersection we are looking for. For instance, let ℓ be defined by the linear form $x + y - z$ and $a = b = 3$. We pick on ℓ the following points:

$$X = \{[1:1:2], [1:2:3], [1:3:4], [1:4:5]\}$$
$$\text{and } \ Y = \{[1:-1:0], [1:-2:-1], [1:-3:-2]\}.$$

Then, the set of 9 points

$$\begin{aligned}\mathbb{X}^{*2} \cup \mathbb{Y}^{*2} = &\{[1:2:6], [1:3:8], [1:4:10], [1:6:12], [1:8:15], [1:12:20]\} \\ &\cup \{[1:2:0], [1:3:0], [1:6:2]\}\end{aligned}$$

from Theorem 3.1(b), is a complete intersection of type $(3, 3)$.

Remark 3.4. Note that if $r = s = 2$, then $X \cup Y$ consists of two points and the statements (c), (d), (e) in Theorem 3.1 are trivially true. The statements (a), (b) in the case $r = 3, s = 2$ are also trivial. Indeed, $X \cup Y$ consists of four points in linear general position. The first interesting case occurs when $r = s = 3$ (see Fig. 2A).

In the following lemma, we prove Theorem 3.1 in case $r = s = 3$.

Lemma 3.5. *Let $X = \mathbb{S}(\ell_1, \ell_2, \ell_3)$ and $Y = \mathbb{S}(m_1, m_2, m_3)$ be contact star configurations on the same conic. Then,*

(i) *the general cubic through $X \cup Y$ is irreducible;*
(ii) *a conic containing 5 points of $X \cup Y$ contains $X \cup Y$;*
(iii) *$X \cup Y$ is a complete intersection of a conic and a cubic.*

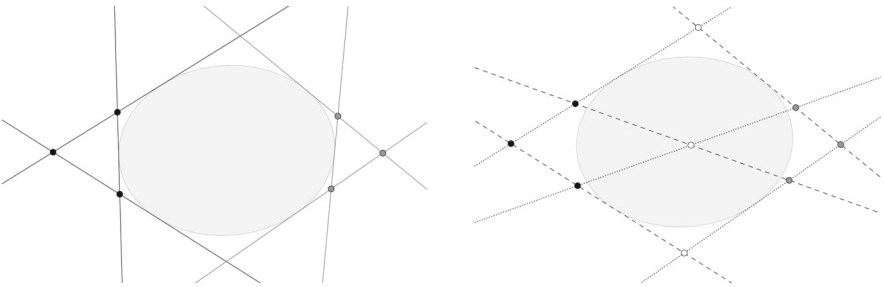
Proof. Since the linear system of the cubics through $X \cup Y$ is not composite with a pencil and it does not have a common component, then by Bertini's Theorem, see for instance [21, Section 5] and [20, Corollary 10.9], the generic cubic of the system is irreducible. To complete the proof, let C_3, C_3' be two cubics union of lines through $X \cup Y$, (see Fig. 2B). The intersection $C_3 \cap C_3'$ consists of 9 points and, by Brianchon's Theorem, see [9, pp. 146-147], the three of them not lying in $X \cup Y$ are on a line (the white circles in Fig. 2B). Thus, by liaison (use formula 1.1), the set $X \cup Y$ is contained in a conic. □

We now prove Theorem 3.1.

Proof of Theorem 3.1. We proceed by induction on r. The cases $r \leq 3$ follow by Remark 3.4 and Lemma 3.5. Therefore, assume $r > 3$. Set $X^{(i)} = \mathbb{S}(\mathcal{L} \backslash \{\ell_i\})$, thus $X^{(i)}$ is a star configuration of points defined by $r - 1$ lines.

(a) Consider the sets $X^{(i)} \cup Y$, for $i = 1, \ldots, r-1$. By (e) and by induction, $X^{(i)} \cup Y$ is complete intersection of type $(r-2, r-1)$, thus there exists a curve of degree $r - 2$ through $X^{(i)} \cup Y$, say $C^{(i)}_{r-2}$.

Note that $C^{(i)}_{r-2}$ does not have ℓ_r as a component. In fact, if ℓ_r is a component of $C^{(i)}_{r-2}$, then, by removing ℓ_r, since it does not contain points of Y, we get a curve of degree $r - 3$ through Y. A contradiction,



(A) Case $r = s = 3$.

(B) The nine intersection points of the two cubics.

Figure 2. (**A**) union of two contact star configurations. (**B**) the nine intersection points of two cubics through the union of two contact star configurations on the same conic

since I_Y starts in degree $r-2$.

Since ℓ_r is not a component for $C^{(i)}_{r-2}$, then $C^{(i)}_{r-2}$ meets ℓ_r in exactly $r-2$ points that are $(X \cap \ell_r)\backslash(\ell_i \cap \ell_r)$. From here, it easily follows that the linear system of curves

$$\left\langle \ell_i \cup C^{(i)}_{r-2} \mid i = 1, \ldots, r-1 \right\rangle \tag{3.1}$$

does not have any fixed component and it is not composite with a pencil. Thus, by Bertini's Theorem, the general curve in 3.1 is irreducible.

(b) By (a), since the linear system (3.1) has dimension at least 2, there exist two irreducible curves of degree $r-1$ thorough $X \cup Y$. Since $|X \cup Y| = \binom{r}{2} + \binom{r-1}{2} = (r-1)^2$, we are done.

(c) From (a) the generic curve of degree $r-1$ through $X^{(i)} \cup Y$ is irreducible, say $C^{(i)}_{r-1}$, for each $i = 1, \ldots, r$. Then, the linear system $\left\langle \ell_i \cup C^{(i)}_{r-1} \mid i = 1, \ldots, r \right\rangle$ does not have any fixed component and it is not composite with a pencil. Again for Bertini's Theorem, we are done.

(d) Let $P_{ir} = \ell_i \cap \ell_r$ and $Y^{(i)} = \mathbb{S}(\mathcal{M}\backslash\{m_i\})$, $i = 1, \ldots, r-1$. From (b), the set $X \cup Y^{(i)}$ is a complete intersection of two curves of degree $r-1$, hence

$$\dim_k \left(I_{X \cup Y^{(i)}}\right)_{r-1} = 2$$

and its h-vector is

$$h_{X \cup Y^{(i)}} = (1, 2, 3, \ldots, r-2, r-1, r-2, \ldots, 2, 1).$$

Moreover, $Y^{(i)}\backslash Z = \mathbb{S}(\mathcal{M}\backslash\{m_i, m_r\})$ is a star configuration defined by $r-2$ lines and then its h-vector is

$$h_{Y^{(i)}\backslash Z} = (1, 2, 3, \ldots, r-3).$$

Thus, by liaison, see relation (1.1), we have

$$h_{X \cup (Z\backslash\{P_{ir}\})} = (1, 2, 3, \ldots, r-2, r-1, r-2),$$

then

$$\dim_k \left(I_{X\cup(Z\setminus\{P_{ir}\})}\right)_{r-1} = 2.$$

Since $X \cup (Z\setminus\{P_{ir}\}) \subseteq X \cup Y^{(i)}$, we have

$$\left(I_{X\cup(Z\setminus\{P_{ir}\})}\right)_{r-1} = \left(I_{X\cup Y^{(i)}}\right)_{r-1}. \tag{3.2}$$

Note that to prove that $\dim \left(I_{X\cup(Z\setminus\{P_{ir}\})}\right)_{r-1} = 2$, we can use Lemma 2.2 in [6] instead of liaison.

Let $F \in (I_{X\cup Z})_{r-1}$, then, by (3.2), $F \in (I_{X\cup Y^{(i)}})_{r-1}$ for each $i = 1, \ldots, r-1$. Therefore, $F \in (I_{X\cup Y})_{r-1}$. It follows that $(I_{X\cup Y})_{r-1} = (I_{X\cup Z})_{r-1}$.

(e) Let $F \in (I_{X\cup Y})_{r-1}$ as in the proof in item (d). By item (c), there exists an irreducible form $G \in (I_{X\cup Y})_r$. Since

$$|F \cdot G| = r(r-1) = |X \cup Y|,$$

then $X \cup Y$ is a complete intersection of the curves defined by F and G.

□

A natural question related to Theorem 3.1 arises about the irreducibility of the curve of degree $r-1$ in the case $r = s$. It cannot be guaranteed. Indeed, in the next example, we produce a set of 20 points which is the complete intersection of a quartic and a quintic (i.e., the case $r = s = 5$), where the curve of degree 4 satisfying item (e) of Theorem 3.1 is a union of two conics.

Example 3.6. Let $\mathcal{L} = \{\ell_1, \ldots, \ell_5\}$ be a set of five lines tangent to an irreducible conic γ (the gray parabola in Fig. 3). The contact star configuration $X = \mathbb{S}(\mathcal{L})$ consists of ten points.

We split the ten points into two sets of five points, each contained in an irreducible conic. Then, we consider the quartic union of these two conics (the one dashed and the other dotted in Fig. 3).

Now, we take a new line m tangent to γ, see Fig. 4A. Through each of the four points of intersection of m with the quartic, there is an extra tangent line to γ. Call these tangent liness m_1, m_2, m_3, m_4, see Fig. 4B. The set of points $Y = \mathbb{S}(\{m_1, \ldots, m_4, m\})$ is a contact star configuration and, from Theorem 3.1 (d), $X \cup Y$ is contained in the quartic. Of course this quartic, by construction, is not irreducible.

In order to show that the condition $s = r$ or $s = r-1$ in Theorem 3.1 is also necessary for $X \cup Y$ to be a complete intersection, we prove the following lemma which holds with more general assumptions.

Lemma 3.7. *Let X and Y be two disjoint star configurations defined by r and s lines, $(r \geq s)$, respectively. If $X \cup Y$ is a complete intersection, then either $s = r$ or $s = r-1$.*

Proof. Since $h_X = (1, 2, 3, \ldots, r-1)$ and $h_Y = (1, 2, 3, \ldots, s-1)$, thus, by liaison, see formula (1.1), the h-vector of $X \cup Y$ must be

$$h_{X\cup Y} = (1, 2, 3, \ldots, r-1, s-1, \ldots, 3, 2, 1).$$

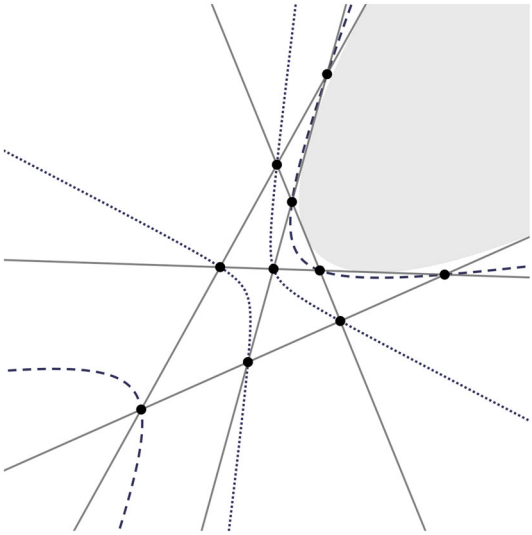
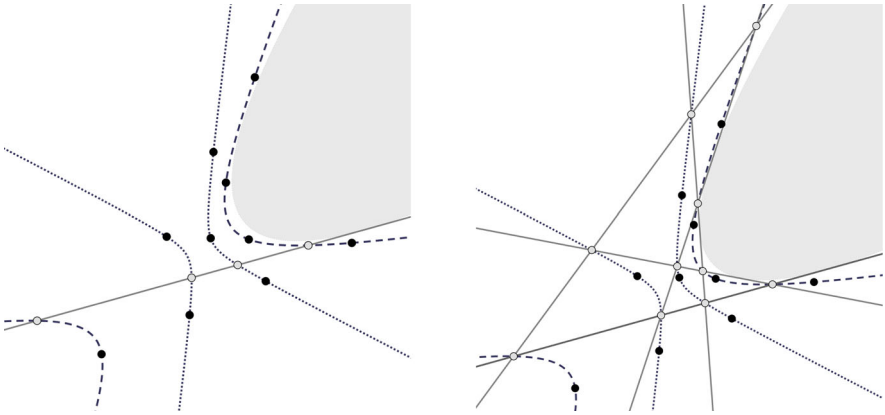


Figure 3. Example 3.6: The star configuration X and the not irreducible quartic through X



(A) The line m intersecting the quartic in four points.

(B) The contact star configuration Y.

Figure 4. Construction of the set of points $X \cup Y$ on a reduced quartic

Since $X \cup Y$ is a complete intersection, then, to ensure the symmetry, we get $s = r$ or $s = r - 1$. □

Theorem 3.1 together with Lemma 3.7 give the following result.

Theorem 3.8. *Let $X = \mathbb{S}(\ell_1, \ldots, \ell_r)$ and $Y = \mathbb{S}(m_1, \ldots, m_s)$ be two contact star configurations on the same conic. Then, $X \cup Y$ is a complete intersection if and only if either $s = r$ or $s = r - 1$.*

4. The h-Vector of a Union of Contact Star Configurations in $\mathbb{P}^2$

In this section, we work in $\mathbb{P}^2$, so γ is always an irreducible conic, and we set $S = \mathbb{C}[x, y, z] = \mathbb{C}[\mathbb{P}^2]$. The main result of this section, Theorem 4.3, shows that a union of two contact star configurations on a conic γ has the same h-vector as a scheme of two fat points in $\mathbb{P}^2$. This point of view will allow us to make further considerations on the h-vector of more than two contact star configurations on a conic γ, see Theorem 4.5.

The h-vector of two fat points is well known, several papers investigate it in a more general setting, see for instance [14, Theorem 1.5 and Example 1.6] and also [10,12,18,19] just to cite some of them. Recall that if the multiplicities of the two points are m and n, with $m \geq n$, then the h-vector is

$$(1, 2, \ldots, m-1, m, n, n-1, \ldots, 2, 1). \tag{4.1}$$

To prove the claimed result, we need the following lemma.

Lemma 4.1. *Let $Y = \mathbb{S}(m_1, \ldots, m_s)$ be a star configuration of s lines, and let $m_{s+1}, \ldots, m_{s+t}$ be t further lines, $t \geq 1$. Consider the star configuration $X = \mathbb{S}(m_1, \ldots, m_s, m_{s+1}, \ldots, m_{s+t})$. Then, the h-vector of $X\backslash Y$ is*

$$h_{X\backslash Y} = (1, \ldots, t-1, \underbrace{t, t, \ldots, t}_{s}).$$

Proof. Since the h-vector of X is $(1, 2, \ldots, s+t-1)$ and $X\backslash Y$ is contained in a curve of degree t, that is $m_{t+1} \cup \cdots \cup m_{t+s}$, then $h_{X\backslash Y} \leq (1, \ldots, t-1, \underbrace{t, t, \ldots, t}_{s}) \leq h_X$. Since,

$$|X\backslash Y| = \binom{s+t}{2} - \binom{s}{2} = \binom{t}{2} + st = 1 + \cdots + t - 1 + \underbrace{t + t + \cdots + t}_{s},$$

we are done. □

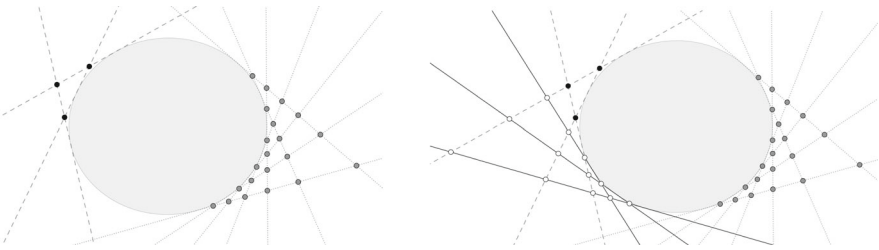
The next example shows how we compute the h-vector of a union of two contact star configurations in $\mathbb{P}^2$.

Example 4.2. Let $X = \mathbb{S}(\ell_1, \ldots, \ell_7)$ and $Y = \mathbb{S}(m_1, m_2, m_3)$ be two contact star configurations on a conic γ, where all the lines are distinct. Figure 5A gives a representation of this case.

To compute the h-vector of $X \cup Y$, we consider three further lines m_4, m_5, m_6 tangent to γ. Let $Y' = \mathbb{S}(m_1, \ldots, m_6) \supseteq Y$, see Fig. 5B.

From Theorem 3.1(b), $X \cup Y'$ is a complete intersection of type $(6, 6)$, hence the h-vector of $X \cup Y'$ is $h_{X\cup Y'} = (1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1)$. By Lemma 3.7, the h-vector of $Y'\backslash Y$ is $h_{Y'\backslash Y} = (1, 2, 3, 3, 3)$. Thus, by formula (1.1), we get

t	0	1	2	3	4	5	6	7	8	9	10
$h_{X\cup Y'}(t)$	1	2	3	4	5	6	5	4	3	2	1
$h_{Y'\backslash Y}(10-t)$							3	3	3	2	1
$h_{X\cup Y}(t)$	1	2	3	4	5	6	2	1			



(A) The set $X \cup Y$. **(B)** The set $X \cup Y'$.

Figure 5. Example 4.2

which shows that $X \cup Y$ has the same h-vector as a scheme of two fat points of multiplicity 2 and 6, see (4.1).

Now we state the general theorem.

Theorem 4.3. *Let $X = \mathbb{S}(\ell_1, \dots, \ell_r)$ and $Y = \mathbb{S}(m_1, \dots, m_s)$ be two contact star configurations on a conic. Let P, Q be two distinct points in $\mathbb{P}^2$. Then,*

$$h_{X \cup Y} = h_{(r-1)P+(s-1)Q}.$$

Proof. Assume $s \leq r$. If either $s = r$ or $s = r-1$, then the statement follows from Theorem 3.1. In fact, in both the cases, $X \cup Y$ is a complete intersection and it has the required h-vector.

Therefore, assume $s < r-1$ and let $t = r-s-1$. Consider t further lines, $m_{s+1}, \dots, m_{s+t}$, tangent to γ, and denote by $Y' = \mathbb{S}(m_1, \dots, m_{s+t})$. From Theorem 3.1 (b), $X \cup Y'$ is a complete intersection of type $(t+s, t+s)$ and its h-vector is

$$h_{X \cup Y'} = (1, 2, \dots, t+s-1, t+s, t+s-1, \dots, 2, 1).$$

By Lemma 3.7, the h-vector of $Y' \backslash Y$ is

$$h_Z = (1, 2, \dots, t-1, \underbrace{t, t, \dots, t}_{s}).$$

By formula (1.1), we get

$$\begin{aligned} h_{X \cup Y} &= (1, 2, \dots, t+s-1, t+s, s-1, s-2, \dots, 2, 1) = \\ &= (1, 2, \dots, r-2, r-1, s-1, s-2, \dots, 2, 1) \end{aligned}$$

which is the h-vector of $(r-1)P + (s-1)Q$. □

Theorem 3.1 allows us to compute the h-vector of a union of three contact star configurations, on the same conic in the special case described in Theorem 4.5. The proof of Theorem 4.5 requires the following well-known result about the Hilbert function of a scheme of three fat points not lying on a line (see for instance [10, Theorem 3.1]).

Proposition 4.4. (The Hilbert function of three fat points) *Let $Z = m_1P_1 + m_2P_2 + m_3P_3$ be a scheme of three general fat points of multiplicity $m_1 \geq m_2 \geq m_3 \geq 1$, respectively. Then,*

$$H_Z(d) = \begin{cases} d+1+H'_Z(d-1) & if\ 0 \le d \le m_1+m_2-2 \\ \deg(Z) & if\ d \ge m_1+m_2-1 \end{cases}$$

where $Z' = (m_1-1)P_1 + (m_2-1)P_2 + m_3P_3$.

Theorem 4.5. *Let* $X = \mathbb{S}(\ell_1 \dots, \ell_r)$, $Y = \mathbb{S}(m_1, \dots, m_s)$ *and* $W = \mathbb{S}(n_1, \dots, n_t)$ *be contact star configurations on a conic, where* $t \ge r \ge s \ge 2$, *and the* $r+s+t$ *lines are distinct. Let* $Z = (t-1)P_1 + (r-1)P_2 + (s-1)P_3$ *be a scheme of three general fat points. If* $t \in \{r+s-1, r+s, r+s+1\}$, *then*

$$\begin{aligned} h_{X\cup Y\cup W} &= h_Z \\ &= (1, \dots, t-2,\ t-1, \underbrace{r+s-2,\ r+s-4, \dots, r-s+2}_{s-1}, \\ &\quad \underbrace{r-s, r-s-1, r-s-2, \dots}_{r-s}). \end{aligned} \tag{4.2}$$

Proof. We prove the theorem for $t = r+s$. The other two cases can be proved similarly.

First we compute the h-vector of $X \cup Y \cup W$.

Note that $X \cup Y$ is contained in the contact star configuration $T = \mathbb{S}(m_1, \dots, m_r, n_1, \dots, n_s)$. Then, $T \cup W$, by Theorem 3.1 (d), is a complete intersection of type $(r+s-1, r+s)$, and so

$$h_{T\cup W} = (1, \dots, r+s-2,\ r+s-1,\ r+s-1,\ r+s-2, \dots, 1).$$

Since the set $T \backslash (X \cup Y)$ is a complete intersection of type (r, s), then its h-vector is

$$(1, \dots, s-1, \underbrace{s, \dots, s}_{r-s+1}, s-1, \dots, 1).$$

Hence, by formula (1.1), we get

$$\begin{aligned} h_{X\cup Y\cup W} &= (1, \dots, r+s-2,\ r+s-1, \underbrace{r+s-2,\ r+s-4, \dots, r-s+2}_{s-1}, \\ &\quad \underbrace{r-s, r-s-1, r-s-2, \dots}_{r-s}). \end{aligned} \tag{4.3}$$

Observe that, for $s = 2$, we get

$$h_{X\cup Y\cup W} = (1, \dots, r,\ r+1,\ r, \underbrace{r-2, r-3, \dots}_{r-2}),$$

for $r = s$, we have

$$h_{X\cup Y\cup W} = (1, \dots, 2r-2,\ 2r-1,\ \underbrace{2r-2,\ 2r-4, \dots, 2}_{r-1}),$$

and for $r = s = 2$

$$h_{X\cup Y\cup W} = (1,\ 2,\ 3,\ 2).$$

We will prove the theorem by induction on $2r+2s-3$, that is, on the sum of the multiplicities of the three fat points. If $2r+2s-3 = 5$, then

$Z = 3P_1 + P_2 + P_3$, whose h-vector is $(1,\ 2,\ 3,\ 2)$, so the statement is proved for $(r, s) = (2, 2)$.

Assume $2r + 2s - 3 \geq 7$, and recall that $Z = (r+s-1)P_1 + (r-1)P_2 + (s-1)P_3$ ($r \geq s \geq 2$). By Proposition 4.4, we have

$$H_Z(d) = \begin{cases} d+1+H_{Z'}(d-1) & \text{if } 0 \leq d \leq (r+s-1)+(r-1)-2 \\ \deg(Z) & \text{if } d \geq (r+s-1)+(r-1)-1 \end{cases}$$

where $Z' = (r+s-2)P_1 + (r-2)P_2 + (s-1)P_3$. Hence, the h-vector of Z is

$$h_Z(d) = \begin{cases} 1+h_{Z'}(d-1) & \text{if } 0 \leq d \leq (r+s-1)+(r-1)-2 \\ \deg(Z)-d-H_{Z'}(d-2) & \text{if } d = (r+s-1)+(r-1)-1 \\ 0 & \text{if } d \geq (r+s-1)+(r-1) \end{cases}. \tag{4.4}$$

Now, we will compute $H_{Z'}(d-2)$ for $d = (r+s-1)+(r-1)-1 = 2r+s-3$.

For $r > s$, we have $r+s-2 > r-2 \geq s-1$, and $d-2 = 2r+s-5 = (r+s-2)+(r-2)-1$, which is the sum of the two highest multiplicities. Hence, by Proposition 4.4, we get $H_{Z'}(2r+s-5) = \deg(Z')$.

If $r = s$, we have $Z' = (2r-2)P_1 + (r-2)P_2 + (r-1)P_3$ and we need to compute $H_{Z'}(3r-5)$. Since the line P_1P_3 is a fixed component for the curves of degree $3r-5$ through Z', we have

$$\dim(I_{Z'})_{3r-5} = \dim(I_{Z''})_{3r-6},$$

where $Z'' = (2r-3)P_1 + (r-2)P_2 + (r-2)P_3$. Since the scheme Z'' gives independent conditions to the curve of degree $3r-6$ (see again Proposition 4.4), hence $H_{Z''}(3r-6) = \deg(Z'')$. It follows that

$$\begin{aligned} H_{Z'}(3r-5) &= \tbinom{3r-3}{2} - \dim(I_{Z'})_{3r-5} = \tbinom{3r-3}{2} - \dim(I_{Z''})_{3r-6} \\ &= \tbinom{3r-3}{2} - \tbinom{3r-4}{2} + \deg(Z'') = 3r^2 - 5r + 1. \end{aligned}$$

Thus, for $d = 2r+s-3$, we have

$$\deg(Z) - d - H_{Z'}(d-2) = \begin{cases} \deg(Z) - (2r+s-3) - \deg(Z') = 1 & \text{if } r > s \\ \deg(Z) - (3r-3) - (3r^2-5r+1) = 2 & \text{if } r = s \end{cases}.$$

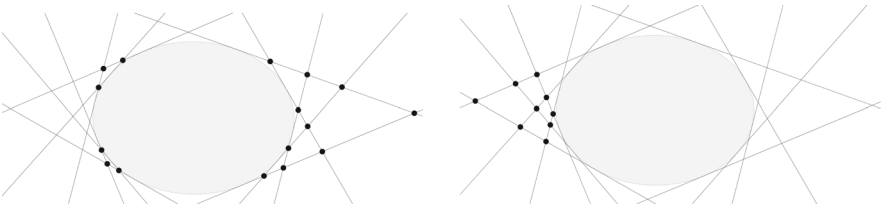
From this equality and from (4.4), we get

$$h_Z(d) = \begin{cases} 1+h_{Z'}(d-1) & \text{if } 0 \leq d \leq 2r+s-4 \\ 1 & \text{if } d = 2r+s-3 \text{ and } r > s \\ 2 & \text{if } d = 3r-3 \text{ and } r = s \\ 0 & \text{if } d \geq 2r+s-2 \end{cases}. \tag{4.5}$$

Now, if $r > s$, from the inductive hypothesis, by substituting r with $r-1$ in formula (4.2), we get the h-vector of $Z' = (r+s-2)P_1 + (r-2)P_2 + (s-1)P_3$, that is,

$$h_{Z'} = (1, \ldots, r+s-3,\ r+s-2, \underbrace{r+s-3,\ r+s-5, \ldots, r-s+1}_{s-1},$$
$$\underbrace{r-s-1, r-s-2, \ldots)}_{r-s-1}.$$

In case $r = s$, we have $Z' = (2r-2)P_1 + (r-1)P_3 + (r-2)P_2$, (note that $2r-2 \geq r-1 \geq r-2$). Let $s' = r-1$. With this notation $Z' =$



(A) The union of three contact star configurations on a conic.

(B) The nine points in $T\setminus(X\cup Y)$ on a grid.

Figure 6. Example 4.7

$(r+s'-1)P_1+(r-1)P_3+(s'-1)P_2$. By applying the inductive hypothesis and then by substituting s' with $r-1$, we get

$$\begin{aligned} h_{Z'} &= (1,\dots,r+s'-2,\ r+s'-1,\underbrace{r+s'-2,\ r+s'-4,\dots r-s'+2}_{s'-1},\\ &\quad r-s',r-s'-1,r-s'-2,\dots,1)\\ &= (1,\dots,2r-3,\ 2r-2,\underbrace{2r-3,\ 2r-5,\dots,3}_{r-2},1). \end{aligned}$$

By (4.3) and (4.5), the conclusion follows. □

Remark 4.6. Note that this theorem gives a non-algorithmic formula for the h-vector of three fat points of multiplicities m_1,m_2,m_3 when $m_1=m_2+m_3$ or $m_1=m_2+m_3\pm 1$.

We illustrate the case $r=3,s=3,t=6$ in the following example.

Example 4.7. Let $X=\mathbb{S}(\ell_1,\ell_2,\ell_3)$, $Y=\mathbb{S}(m_1,m_2,m_3)$ and $W=\mathbb{S}(n_1,\dots,n_6)$ be contact star configurations on the same conic, see Fig. 6A. Set $T=\mathbb{S}(\ell_1,\ell_2,\ell_3,m_1,m_2,m_3)$, see Fig. 6B.

Then,

$$\begin{aligned} h_{W\cup T} &= (1,2,3,4,5,5,4,3,2,1),\\ h_{T\setminus(X\cup Y)} &= (1,2,3,2,1), \end{aligned}$$

and, by liaison, $h_{X\cup Y\cup W}=(1,2,3,4,5,4,2)$, which is the h-vector of three fat points of multiplicity $2,2,5$.

Theorem 4.5 and experiments using CoCoA [1] suggest the following conjecture.

Conjecture 4.8. *The h-vector of a scheme of $s\le 4$ general fat points of multiplicities m_i, $(i=1,\dots,s)$ is equal to the h-vector of the union of s contact star configurations defined by m_i+1 lines tangent to the same conic.*

The next example shows that the conjecture does not hold for $s=5$.

Example 4.9. The h-vector of five general fat points of multiplicity 2 in $\mathbb{P}^2$ is

$$(1,2,3,4,4,1)$$

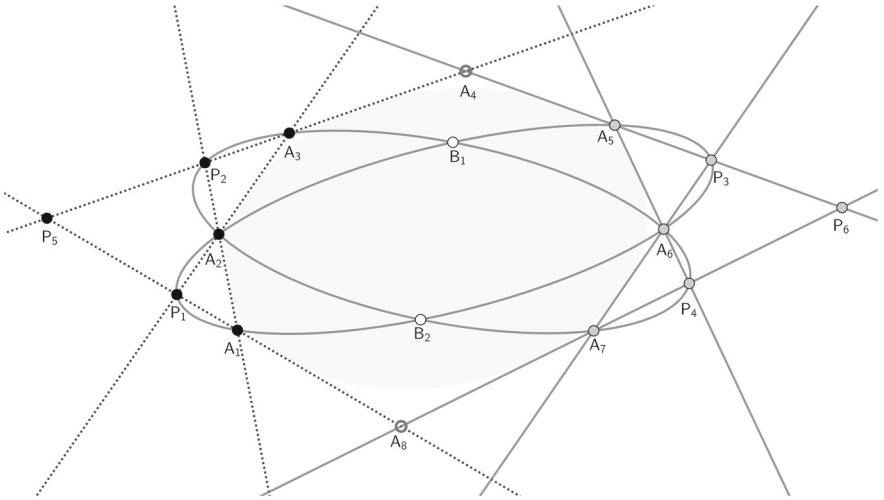


Figure 7. The case described in Proposition 5.1

but we checked with CoCoA [1] that the h-vector of five general contact star configurations on a conic defined by three lines is

$$(1, 2, 3, 4, 5).$$

5. Applications of Theorem 3.1 to Polygons

As an application of Theorem 3.1, we get a result that in a certain sense extends the Brianchon's Theorem to an octagon circumscribed to a conic.

Proposition 5.1. *Let $A_1, \dots, A_8$ be the vertices of an octagon and let ℓ_{ij} be the line A_iA_j. Set $P_1 = \ell_{18} \cap \ell_{23}$ and $P_2 = \ell_{12} \cap \ell_{34}$ and let γ_1 and γ_2 be the conics through A_1, A_2, A_5, A_6, P_1 and A_2, A_3, A_6, A_7, P_2, respectively. Let $\gamma_1 \cap \gamma_2 = \{A_2, A_6, B_1, B_2\}$. If the octagon circumscribes a conic γ, then the points A_4, A_8, B_1, B_2 are on a line. (See Fig. 7).*

Proof. Note that, by Theorem 3.1 (d), the point $P_3 = \ell_{45} \cap \ell_{67}$ belongs to the conic γ_1. In fact, A_1, A_2, P_1 and A_5, A_6, P_3 are two contact star configurations on the same conic and then a complete intersection of type $(2, 3)$. Analogously, the point $P_4 = \ell_{56} \cap \ell_{78}$ belongs to the conic γ_2. Moreover, observe that the points $A_1, A_2, A_3, P_1, P_2, P_5 = \ell_{18} \cap \ell_{34}$, and the points $A_5, A_6, A_7, P_3, P_4, P_6 = \ell_{45} \cap \ell_{78}$ are two contact star configurations each defined by four lines tangent to the same conic, hence by Theorem 3.1 (d) these 12 points are a complete intersection of type $(3, 4)$, thus their h-vector is $(1, 2, 3, 3, 2, 1)$.
Now, consider the two quartics $\gamma_1 \cup \ell_{34} \cup \ell_{78}$ and $\gamma_2 \cup \ell_{18} \cup \ell_{45}$. This two quartics meet in a complete intersection of 16 points, that consists of the 12

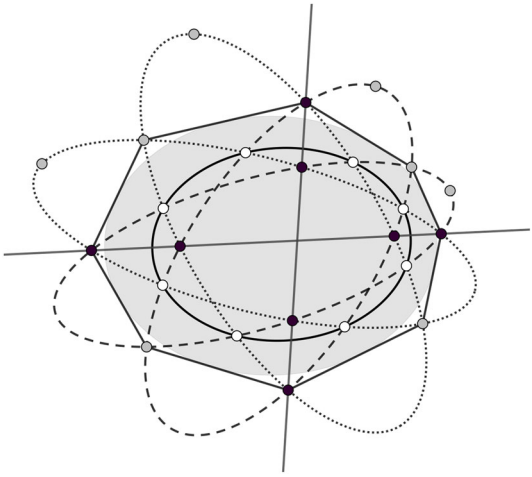


Figure 8. The case described in Corollary 5.2

points described above and the points A_4, A_8, B_1, B_2. By relation (1.1), we get

	0 1 2 3 4 5 6
The h -vector of the 16 points	1 2 3 4 3 2 1
The h -vector of the 12 points	1 2 3 3 2 1
The h -vector of $\{A_4, A_8, B_1, B_2\}$	1 1 1 1

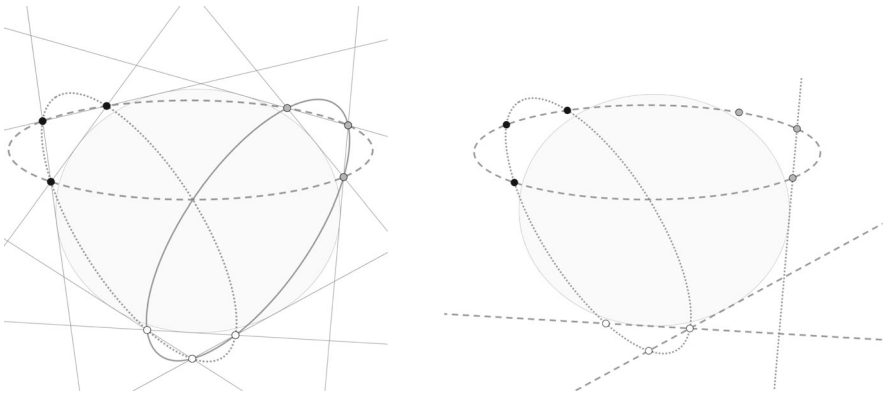
The table above shows that the four points are collinear. □

Corollary 5.2. *Let $A_1, \ldots, A_8$ be the vertices of an octagon and let ℓ_{ij} be the line A_iA_j. Set $P_1 = \ell_{18} \cap \ell_{23}, P_2 = \ell_{12} \cap \ell_{34}, P_3 = \ell_{23} \cap \ell_{45}, P_4 = \ell_{34} \cap \ell_{56}$ and let γ_i be the conic through $A_i, A_{i+1}, A_{4+i}, A_{5+i}, P_i$, $i = 1, \ldots 4$ ($A_9 = A_1$). If the octagon circumscribes a conic γ, then the eight points $(\gamma_1 \cap \gamma_3) \cup (\gamma_2 \cap \gamma_4)$ are on a conic. (See Fig. 8).*

Proof. The quartics $\gamma_1 \cup \gamma_4$ and $\gamma_2 \cup \gamma_3$ meet in 16 points, in black and white in figure 5.2. Eight of them, the black dots, that is $(\gamma_1 \cap \gamma_2) \cup (\gamma_3 \cap \gamma_4)$, by Proposition 5.1, lie on two lines. Therefore, the h-vector of these eight points must be $(1, 2, 2, 2, 1)$. By formula (1.1), the residual points lie on a conic. □

Recall that, by Theorem 3.1(e), the six points in the union of two contact star configurations each defined by three lines tangent to the same conic, are contained in a conic. An interesting case occurs considering three such contact star configurations, see Proposition 5.3, which can also be translated into a property of a polygon of nine sides circumscribing a conic.

Proposition 5.3. *Let $X_1 = \mathbb{S}(\ell_1, \ell_2, \ell_3), X_2 = \mathbb{S}(m_1, m_2, m_3), X_3 = \mathbb{S}(n_1, n_2, n_3)$ be contact star configurations on a conic. Let γ_{ij} be the conic containing $X_i \cup X_j$. Then, $\gamma_{12}, \gamma_{13}, \gamma_{23}$ meet in a point. See Fig. 9A.*



(A) Three conics meeting in a point.

(B) The cubic and the quartic.

Figure 9. Proposition 5.3

Proof. Consider the cubic $\gamma_{12} \cup n_1$ and the quartic $\gamma_{13} \cup m_1 \cup m_2$ (respectively, dotted and dashed in Fig. 9B). The cubic and the quartic meet in 12 points (twice in $m_1 \cap m_2$). Six of these points, precisely $X_1 \cup \mathbb{S}(n_1, m_1, m_2)$, lie on a conic, by Theorem 3.1(e). Therefore, also the residual six points lie on a conic (by formula 1.1) that is, by Theorem 3.1(d), the conic γ_{23}. □

6. Further Directions

According to our computations, it should be possible to extend some of the results in this paper to higher dimensional spaces. We state the following conjecture.

Conjecture 6.1. *Let $X := \mathbb{S}(\ell_1, \dots, \ell_r)$ and $Y := \mathbb{S}(m_1, \dots, m_s)$ be two contact star configurations in $\mathbb{P}^n$, where $r \geq s$ and all the hyperplanes are distinct. Then, the h-vector of $X \cup Y$ is*

$$h_{X\cup Y} = \left(1, \binom{n}{n-1}, \dots, \binom{r-1}{n-1}, \binom{s-1}{n-1}, \dots, \binom{n}{n-1}, 1\right).$$

In particular, if either $s = r$ or $s = r - 1$ then $X \cup Y$ is a Gorenstein set of points.

If Conjecture 6.1 is true, then it is possible to generalize Remark 3.3 in order to construct Gorenstein sets of rational points in $\mathbb{P}^n$ with a special h-vector. We show the procedure in the following example.

Example 6.2. Let $S = \mathbb{C}[x, y, z, t] = \mathbb{C}[\mathbb{P}^3]$ and let $\ell \subseteq \mathbb{P}^3$ be the line in $\mathbb{P}^3$ defined by the ideal $(z - x - y, t - x + y)$. Consider two sets of four points on ℓ

$$X = \{[1, 1, 2, 0], [1, 2, 3, 1], [1, 3, 4, 2], [1, 4, 5, 3]\},$$
$$Y = \{[1, -1, 0, -2], [1, -2, -1, -3], [1, -3, -2, -4], [1, -4, -3, -5]\}.$$

Then,

$$\begin{aligned}\mathbb{X}^{\star^3} \cup \mathbb{Y}^{\star^3} = &\{[1,6,24,0],[1,8,30,0],[1,12,40,0],[1,24,60,6]\}\cup\\ &\{[1,-6,0,-24],[1,-8,0,-30],[1,-12,0,-40],[1,-24,-6,-60]\}.\end{aligned}$$

According to CoCoA, the set of eight points $\mathbb{X}^{\star^3} \cup \mathbb{Y}^{\star^3} \subseteq \mathbb{P}^3$ is in fact Gorenstein and its h-vector is $(1,3,3,1)$.

On the other hand, it is interesting to ask if contact star configurations need to be constructed on rational normal curves. Of course, one can extend the definition by taking, for instance, high contact linear spaces to some other irreducible curve or surface and, with some assumptions of generality, again get a star configuration. However, we do not know if this construction on a variety of a different kind will lead to configurations with special properties either from the point of view of the h-vector or something else. Therefore, we ask if the converse of Conjecture 6.1 is also true.

Question 6.3. *Let $X{:=}\mathbb{S}(\ell_1,\dots,\ell_r)$ and $Y{:=}\mathbb{S}(m_1,\dots,m_s)$, where $r \geq s$, be two star configurations in $\mathbb{P}^n$ defined by distinct hyperplanes. Suppose that the h-vector of $X \cup Y$ is*

$$\left(1, \binom{n}{n-1}, \dots, \binom{r}{n-1}, \binom{s}{n-1}, \dots, \binom{n}{n-1}, 1\right).$$

Then, are X and Y two contact star configurations on the same rational normal curve?

A similar question can be asked in the case of a Gorenstein set of points.

Question 6.4. *Let $X{:=}\mathbb{S}(\ell_1,\dots,\ell_r)$ and $Y{:=}\mathbb{S}(m_1,\dots,m_s)$, where $r \geq s$, be two star configurations in $\mathbb{P}^n$ defined by distinct hyperplanes. Suppose that $X \cup Y$ is a Gorenstein set of points in $\mathbb{P}^n$. Then, are X and Y two contact star configurations on the same rational normal curve with either $s = r$ or $s = r-1$?*

In the next proposition, we positively answer Question 6.4 in $\mathbb{P}^2$ for the case $r = s = 3$.

Proposition 6.5. *Let X and Y be two star configurations, both defined by 3 distinct lines. Let $X \cup Y$ be a complete intersection of type $(2,3)$. Then, X and Y are contact star configurations on the same conic γ.*

Proof. Let denote $X = \mathbb{S}(\ell_1,\ell_2,\ell_3)$ and $Y = \mathbb{S}(m_1,m_2,m_3)$. Set $P_{ij}{:=}\ell_i \cap \ell_j$ and $Q_{ij}{:=}m_i \cap m_j$. Let denote by p_{ij} and q_{ij} the lines dual to P_{ij} and Q_{ij} and by L_i and M_j the points dual to the lines ℓ_i and m_j. By hypothesis there is a conic $\mathfrak{c}$ passing thorough the six points in $X \cup Y$. Then, the lines p_{ij} and q_{ij} are tangent to the conic $\mathfrak{c}^\vee$ dual to $\mathfrak{c}$. Then, $\{L_1,L_2,L_3\}$ and $\{M_1,M_2,M_3\}$ are $\mathfrak{c}^\vee$-contact star configurations. Hence, from Theorem 3.1 (e), there is a conic $\gamma^\vee$ passing through $\{L_1,L_2,L_3,M_1,M_2,M_3\}$. This proves that X and Y are contact star configurations on a conic γ, that is, the conic dual to $\gamma^\vee$. □

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