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Anomalous heat transport and universality in macroscopic diffusion models

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Abstract

Anomalous diffusion is ubiquitous in nature and relevant for a wide range of applications, including energy transport, especially in bio- and nano-technologies. Numerous approaches have been developed to describe it from a microscopic point of view, and recently, it has been framed within universality classes, characterized by the behaviour of the moments and auto-correlation functions of the transported quantities. It is important to investigate whether such universality applies to macroscopic models. Here, the spectrum of the moments of the solutions of the transport equations is investigated for three continuous PDE models featuring anomalous diffusion. In particular, we consider the transport described by: (i) a generalized diffusion equation with time-dependent diffusion coefficient; (ii) the Porous Medium Equation and (iii) the Telegrapher Equation. For each model, the key features of the source-type solution as well as the analytical results for the moment analysis are revisited and extended via both analytical and numerical approaches. Equivalence of the asymptotic behaviour of the corresponding heat transport is confirmed within the realm of weak anomalous diffusion.

Keywords Transport exponents · Probability distributions · Asymptotic behaviour · PDE

Introduction

The microscopic description of matter that is afforded by statistical mechanics teaches that spatial constraints play a crucial role in a system's response to external forces, and that the related statistical fluctuations are directly connected to dimensionality [1]. A natural challenge for this description is to shed light on anomalous behaviours observed in a variety of low-dimensional systems [2, 3]. More specifically, an outstanding question is how the heat transport properties changes in reduced dimensions [4]. In this framework, atomistic models of classical fluids and crystals as well as particular lattices of particles (e.g. Fermi–Pasta–Ulam chains) show a power-law divergence of the heat conductivity as the one-dimensional system length increases [5]. This is an example of anomalous behaviour: indeed, in three-dimensional systems thermal conduction is well known to converge to a constant that does not depend on the size of the medium, as required by thermodynamics. Similarly, heat transport processes in various low-dimensional systems do not longer follow Fourier's law, which is a diffusion law [6, 7]. Specifically, the solution of Fourier's equation with impulsive initial condition,

$$\frac{\partial f}{\partial t} = k\frac{\partial^2 f}{\partial x^2}, \quad f(0,t) = \delta(x) \tag{1}$$

when normalized, can be seen as a probability density that takes the form of a Gaussian. Moreover, its second moment grows linearly in time, as for any other diffusion process.

Fourier's law describes a well-known macroscopic behaviour, which is fully described once the conductivity, k, and the initial and boundary conditions are given. Transport is

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called anomalous when this is not the case, and it is identified by the deviations from the linear time dependence of the mean-square displacement of the transported quantity, [8]. This kind of transport of heat is under intense investigation, because apparently common in modern science and technology, but it is not fully understood. The macroscopic equations that should describe it, in general, are not established. To understand such a phenomenon, much effort has been devoted to explore its microscopic origin, in the form of statistical mechanical laws (see [6, 9–12] and references therein). A statistical approach is indeed necessary, at the atomic level, because of the intrinsic randomness characterizing microscopic dynamics, and because of the large numbers of such constituents of matter that must be considered. Then, the properties of transport are given by the properties of the probability densities concerning the transported quantities. In particular, the asymptotic behaviour of the probability distribution of the displacement in time $P(|x(t)-x(0)|,t)$ is used to characterize the kind of transport at hand. However, often the latter is not known and the time-asymptotic behaviour of the moments is used instead to investigate the transport properties

$$M_{\mathrm{p}}(t) = \langle |x(t)-x(0)|^{\mathrm{p}} \rangle \sim t^{\eta(\mathrm{p})} \tag{2}$$

which can be accessed numerically or experimentally. Here, $p \in \mathbb{R}$ is the moment order, $\langle\cdot\rangle$ is the mean over a set of trajectories and the function $\eta(p)$ is called spectrum of the moments of the displacement (SMD), and $a(t) \sim b(t)$ means that

$$\lim_{t\to\infty} \frac{a(t)}{b(t)} = r \in \mathbb{R} \tag{3}$$

The exponent $\gamma = \eta(2)$, called transport exponent, is then used to distinguish the various kinds of transport: $\gamma = 1$ corresponds to normal diffusion, $\gamma \neq 1$ to what is known anomalous diffusion. Within the anomalous regime, transport is further classified as subdiffusive for $0 \le \gamma < 1$, superdiffusive for $1 < \gamma < 2$, and ballistic when $\gamma = 2$. The case $\gamma = 0$ does not directly mean that there is no transport: it may just be very slow, *e.g.* the mean-square displacement may grow logarithmically in time [13].

We refer to scale-invariant transport process if the anomalous behaviour of the mean-square displacement (different from the one implied by the Gaussian with linearly growing variance) extends to all moments of the displacement as a linear function of p, which is $\eta(p) = m\,p$. When $\eta(p)$ varies nonlinearly with the moment order p, scale invariance breaks down and transport is said to be strongly anomalous [14]. In the literature, the most commonly investigated strong anomalous scenario is characterized by two regimes, in which the power of t is given by two different expressions, that are linear in the moment order:

$$\eta(p) = \begin{cases} m\,p & \text{for } p \le p_*\,, \\ p-(1-m)p_* & \text{for } p > p_*\,. \end{cases} \tag{4}$$

where m and p_* are given parameters. Such behaviour has been observed in the transport in polygonal billiards [15–17], billiards with infinite horizon [18–20], diffusion in laser-cooled atoms [21], one-dimensional maps [13, 22] and intermittent maps [23, 24], in experiments on the mobility of particles inside living cancer cells [25] and in the motion of particles passively advected by dynamical membranes [26]. It has then been found that such totally different phenomena fall within the same universality class of transport, if transport is dominated by the long ballistic flights that appear in the tails of the corresponding distributions, typically present in anomalous diffusion, while standard diffusion is typically dominated by the short flights, that constitute the bulk of the probability distributions [12, 13, 27, 28]. In brief, a class is defined by the fact that all moments of order larger than 2 have some asymptotic behaviour if the second moment does. The result of Refs. [12, 13, 27, 28] is that many and very different phenomena fall within the same class, corresponding to statistically indistinguishable transport properties.

The universality of transport has been investigated within the realm of microscopic descriptions of transport phenomena, which mainly means in terms of dynamical systems describing classical particle systems, and in terms of stochastic processes. It is interesting to ask whether similar results apply within the set of descriptions given in terms of PDE, used also in the macroscopic world. Indeed, when the number of particles is large, the microscopic approach becomes computationally expensive to simulate; moreover, macroscopic models allow a greater analytical tractability. Therefore, in the present paper we investigate the spectrum of the moments for three different transport laws. In particular, we propose and analyse a generalization of the diffusion equation with time-dependent diffusion coefficient. Then, we turn to the Porous Medium Equation and Telegrapher Equation. The results are then compared in the light of the universality of transport phenomena.

The rest of the paper is structured as follows. In Sect. “Diffusion models and moments analysis”, we present the continuous models for anomalous transport mentioned above, revisiting the moment analysis results present in the literature in the perspective of universality classes of anomalous processes. We then compare the spectrum of moments for the three models exhibiting anomalous diffusion. Our results are further completed and supported by numerical investigation. Finally, Sect. “Conclusions” gives a summary of our main findings and an outlook to future perspectives. The details of part of our calculations are given in two appendices, to simplify the main text.

Diffusion models and moments analysis

In this section, we evaluate the SMD for three models. In this respect, we propose and analyse the time-dependent diffusion coefficient model (TdDC), and two widely studied models featuring anomalous diffusion: the Porous Medium Equation and the Telegrapher Equation. For each system, the key properties of the dynamics are also briefly reviewed.

Time-dependent diffusion coefficient

We consider a modification of the diffusion equation that accounts for a general growth rate, i.e. describing anomalous behaviour. In this respect, we assume a concentrated initial distribution, i.e. $u(x,0) = \delta(x)$, and the spreading to have a Gaussian profile, with diffusion coefficient D, zero mean and mean-square displacement growing as t^γ, with $\gamma \in [0,2]$.

$$u(x,t) = \frac{1}{\sqrt{4\pi D t^\gamma}} \exp\left(-\frac{x^2}{4Dt^\gamma}\right) \tag{5}$$

Via algebraic rearrangements, from Eq. (5) we reconstruct the corresponding transport law, which results in a generalization of the diffusion equation with time-dependent diffusion coefficient (see Appendix A for details)

$$\frac{\partial u}{\partial t} = D t^{\gamma-1} \frac{\partial^2 u}{\partial x^2}. \tag{6}$$

For $\gamma = 1$, Eq. (6) coincides to the diffusion equation, whose source-type solution is represented in Fig. 1B. For $\gamma \neq 1$, it describes anomalous diffusion, see the corresponding solution in Fig. 1A and C. The moments of the displacement for this TdDC model take the form:

$$M_{2\mathrm{n}}(t) = \int_{\mathbb{R}} x^{2\mathrm{n}} u(x,t)dx = (2n-1)!!\,(2D)^{\mathrm{n}} t^{\gamma\mathrm{n}} \tag{7}$$

$$M_{2\mathrm{n}+1}(t) = 0 \tag{8}$$

where $q!!$ denotes the double factorial defined as the product of all positive integers up to q that have the same parity as q. At large times, we therefore get the asymptotic growth estimate $M_{2\mathrm{n}}(t) = O(t^{\gamma\mathrm{n}})$, while the odd moments trivially vanish.

Porous Medium Equation

The Porous Medium Equation (PME) is an emblematic example of nonlinear partial differential equation and describes density-dependent diffusivity

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u^{\mathrm{m}}}{\partial x^2} = \frac{\partial}{\partial x}\left(m u^{\mathrm{m}-1}\frac{\partial u}{\partial x}\right), \quad m > 1. \tag{9}$$

In order to revisit PME in the context of anomalous diffusion, we analytically derive the relation between the exponent m and the transport exponent $\gamma = \eta(2)$. This yields:

$$m = (2-\gamma)/\gamma, \tag{10}$$

as explained in detail in Appendix B. In this respect, Eq. (9) thus describes normal diffusion for $\gamma = 1$, which is $m = 1$, and anomalous diffusion for $\gamma \neq 1$, which is $m \neq 1$. Many applications have been motivating for a thorough study of its mathematical theory, such as the gas flow through a porous medium [29], plasma radiation [30] as well as population dynamics characterized by self-avoiding diffusion [31].

The source-type solution of the PME was proposed by Zel'dovich [32], Kompaneets and Barenblatt [33], and keeping in mind Eq. (10), it explicitly reads:

$$u(x,t) = t^{-\alpha}(C - kx^2 t^{-\gamma})_+^{\frac{1}{m-1}}, \tag{11}$$

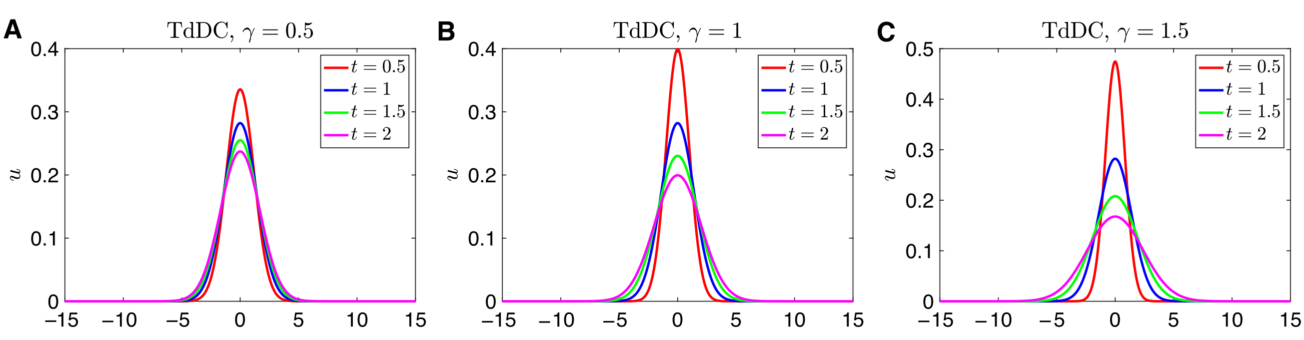

Fig. 1 The TdDC source-type solution for $D = 1$, resulting in **A** subdiffusive behaviour obtained for $\gamma = 0.5$, **B** normal diffusion obtained for $\gamma = 1$, **C** superdiffusive behaviour obtained for $\gamma = 1.5$

where $m \in (1, \infty)$, and

$$\alpha = \frac{d}{d(m-1)+2}, \quad k = \frac{(m-1)\alpha}{2md}, \quad \gamma = \frac{2\alpha}{d}, \quad (\cdot)_+ = \max\{\cdot, 0\} \tag{12}$$

d being the space dimension ($d = 1$ in our case). The latter is known as ZKB solution.

For $m > 1$, the ZKB solution has peculiar features that distinguish it from the diffusion equation. Indeed, it propagates with finite speed with free boundary at distance

$$|x| = \sqrt{\frac{C}{k}}\, t^{\frac{\gamma}{2}}$$

which separates the regions where the solution is positive from those where it vanishes, see Fig. 2A. Conversely, the diffusion equation is characterized by infinite speed of propagation. Since $m > 1$ corresponds to $\gamma < 1$, we find a subdiffusive process. It can be proved that in the $m \to 1$ limit, the ZKB solution tends to the fundamental solution of the diffusion equation [34], *i.e.*

$$\lim_{m \to 1} u(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left(\frac{-x^2}{4t}\right)$$

The ZKB solution thus tends to the TdDC solution with $\gamma = D = 1$ (Fig. 1B).

The extension of the PME to $m < 1$ is also called the fast diffusion equation (FDE) and admits the ZKB solution which, unlike the case with $m > 1$, exhibits power-like tails at infinity, see Fig. 2B. Since $m < 1$ corresponds to $\gamma > 1$, this choice reflects a superdiffusive process.

We proceed with the moment analysis, recalling the following result relative to the case $m > 1$ [34].

Proposition 1 *Let be u the ZKB solution of PME. If for some* $n \geq 1$ *we have* $M_{2n}(t) < \infty$, *then the moment* $M_{2n}(t)$ *is finite for every* $t \geq 0$ *and we have the growth estimate for large times*:

$$M_{2n}(t) = O(t^{n\gamma}), \qquad \text{where } \gamma = \frac{2}{d(m-1)+2} < 1. \tag{13}$$

Trivially, $M_{2n+1} = 0$, for all $n \geq 1$. These results will be numerically extended for $m < 1$ in the next section.

Telegrapher Equation

The Telegrapher Equation can be obtained from microscopic dynamics in various fashions. For instance, one may consider the following one-dimensional stochastic process. Let us consider a particle placed at the origin $x = 0$ at $t = 0$ that moves with some fixed constant speed s such as, for instance, the transport of energy due to electromagnetic radiation, or photons [35–37]. The initial movement direction is equally chosen with probability 1/2. The process then follows a homogeneous Poisson process of rate λ. In particular, when a Poisson process of rate λ occurs, the particle instantaneously changes movement direction maintaining the same speed s. When the next event occurs, the orientation changes again, independently of the previous one. This random movement has been first studied by Goldstein [38] and Kac [39]. Their main result shows that the density $u(x, t)$, $x \in [-st, st]$, $t \geq 0$ follows the following hyperbolic PDE

$$\frac{\partial^2 u}{\partial t^2} + 2\lambda \frac{\partial u}{\partial t} - s^2 \frac{\partial^2 u}{\partial x^2} = 0, \tag{14}$$

which is called Telegrapher Equation. The source-type solution is obtained completing Eq. (14) with the following initial conditions

$$u(x,0) = \delta(x), \qquad \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0. \tag{15}$$

One then obtains (see [40, 41])

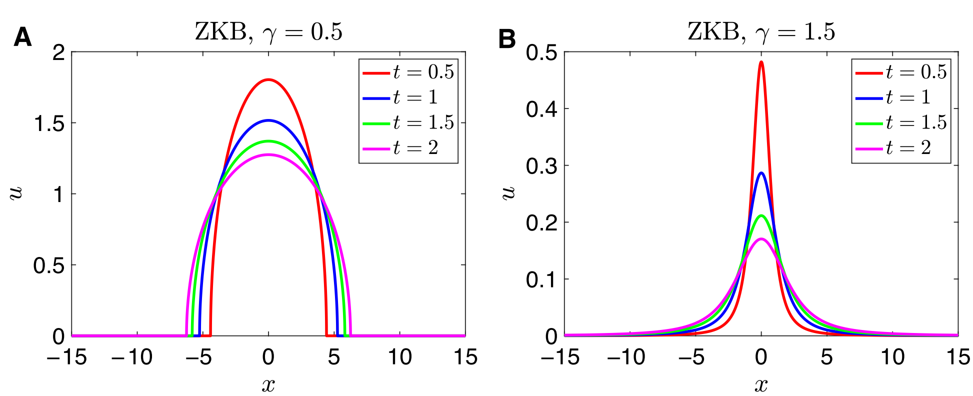

Fig. 2 The ZKB solution of the PME, resulting in **A** subdiffusive behaviour obtained for $m > 1$ (i.e. $\gamma = 0.5$), **B** superdiffusive behaviour obtained for $m < 1$ (i.e. $\gamma = 1.5$)

$$u(x,t) = \frac{e^{-\lambda t}}{2}[\delta(st-x)+\delta(st+x)] + \frac{e^{-\lambda t}}{2s}\left[\lambda I_0\left(\frac{\lambda}{s}\sqrt{s^2t^2-x^2}\right) + \frac{\partial}{\partial t} I_0\left(\frac{\lambda}{s}\sqrt{s^2t^2-x^2}\right)\right] H(st-|x|), \quad (16)$$

where $H(x)$ is the Heaviside function

$$H(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$

and $I_0(z)$ is the modified Bessel function of zero order defined as

$$I_0(z) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left(\frac{z}{2}\right)^{2k} \quad (17)$$

See Fig. 3 for representative evolution of the TE solution.

The moments of the displacement for this density function are explicitly given in [42] as

$$M_{2n}(t) = e^{-\lambda t} s^{2n} 2^{n-1/2} \lambda^{-n+1/2} t^{n+1/2} \Gamma\left(n+\frac{1}{2}\right)[I_{n+1/2}(\lambda t) + I_{n-1/2}(\lambda t)],$$
$$M_{2n+1}(t) = 0, \quad (18)$$

where $I_\nu(z)$ is the modified Bessel function of order ν

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(k+\nu+1)}\left(\frac{z}{2}\right)^{2k+\nu},$$

and $\Gamma(x)$ is the Euler gamma-function.

In particular, Eq. (18) provides the second moment as

$$M_2(t) = \frac{s^2 t}{\lambda} - \frac{s^2}{2\lambda^2}(1-e^{-2\lambda t}) \quad (19)$$

in agreement with [41]. From the perspective of the spread rate, Eq. (19) classifies the TE as a normal diffusive process, i.e. $M_2(t) \sim t$, as $t \to \infty$, because the last term in the r.h.s. of Eq. (19) is asymptotically negligible. Taking the limit $t \to \infty$ (or $\lambda \to \infty$, or both $t, \lambda \to \infty$) for fixed s and n the following result holds:

$$M_{2n}(t) \sim \left(\frac{s^2 t}{\lambda}\right)^n (2n-1)!! \quad (20)$$

From Eq. (20), we derive $M_{2n}(t) = O(t^n)$ as $t \to \infty$, for fixed s, λ and n.

Table 1 now summarizes the analytical results concerning the SMD for the three different models we have considered.

Preliminary numerical investigation suggests that the theoretical result on the asymptotic behaviour of the PME model can be extended also to the $m > 1$ case. Figure 4 finally shows the spectrum of the moments of displacement for the considered models.

In this investigation, adjusting the parameters of the three different models so that the second moments have same

Table 1 SMD, where γ denotes the transport exponent of the mean-square displacement

Model	$\eta(p)$	Valid for	References
TdDC	$\frac{p\gamma}{2}$	$\gamma \in [0,2]$, $p = 2n$, $n \geq 1$	
PME	$\frac{p\gamma}{2}$	$\gamma \in (0,1)$, $p = 2n$, $n \geq 1$	[34]
TE	$\frac{p\gamma}{2}$	$\gamma = 1$, $p = 2n$, $n \geq 1$	[35]

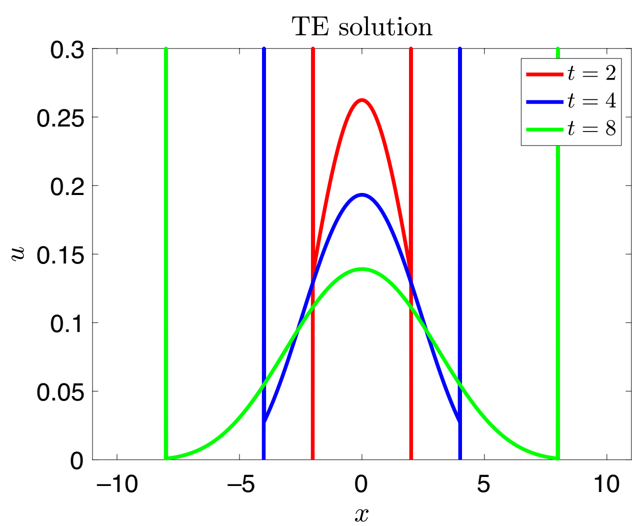

Fig. 3 The TE solution obtained for $\lambda = 1$, $s = 1$. The vertical lines represent the Dirac δ-functions at the travelling front of the distribution

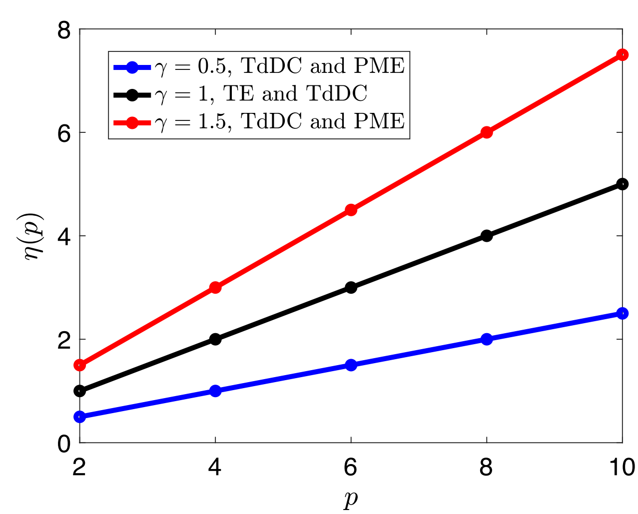

Fig. 4 Spectrum of the moments of displacement $\eta(p)$ for **i** TdDC and PME, both obtained for $\gamma = 0.5$; **ii** TE and TdDC, obtained for $\gamma = 1$; **iii** TdDC and PME, obtained for $\gamma = 1.5$

asymptotic behaviour, we obtain the asymptotic agreement of the remaining moments. Therefore, a certain kind of universality applies to the PDEs considered here as models of transport. In case of anomalous transport, $\gamma \neq 1$, we only have scale free, or weak anomalous transport, since $\eta(p) = p\gamma/2$ is linear in p. The Telegrapher Equation, however, has a single transport regime: normal diffusion. Hence, it falls in a corresponding universality class for $\gamma = 1$ only. Interestingly, its asymptotic behaviour is statistically indistinguishable from the solution of the diffusion equation, although it is characterized by a propagating front with finite speed, while the diffusion equation is characterized by infinitely fast propagation. The fact is that the difference between the two distributions becomes negligible over a wider and wider interval of the real line, as t grows.

Conclusions

The present study consists in a new analysis of the asymptotic behaviour of the moments of the normalized solutions of macroscopic transport equations, meant to investigate the universality that is observed in statistical mechanics, at the microscopic level [12]. In this perspective, we have analysed the SMD for three different continuous models exhibiting anomalous diffusion, as possible macroscopic models of the anomalous heat transport evidenced by microscopic investigations. First, we have presented and analysed a generalized diffusion equation with a time-dependent diffusion coefficient that is directly linked to the growth rate property of the density spreading. We have then turned to the Porous Medium Equation and the Telegrapher Equation, revisiting the known key features of their dynamics as well as the theoretical results on the asymptotic moment analysis. Also, we have analytically obtained an explicit relation between the power appearing in the Porous Medium Equation and the spreading rate of the source-type ZKB solution. Preliminary numerical investigation has then suggested that the SMD valid in the subdiffusive regime, i.e. $\eta(p) = \frac{p\gamma}{2}$ for $\gamma \in (0, 1)$, can be extended to the superdiffusive scenario, i.e. it holds also for $\gamma > 1$.

We have found that a certain degree of universality applies to the PDEs considered here. If the parameters of the three different models are fixed so that their second moments have same asymptotic behaviour, the asymptotic behaviour of the remaining moments are also equal. The Telegrapher Equation transport is always diffusive and hence can be considered for the class with $\gamma = 1$ only. The fact that quite different transport models asymptotically behave in statistically indistinguishable fashions is striking also in the case of our PDEs. In fact, we find in the same class systems that propagate with finite speed and systems that propagate with infinite speed, in quite different ways. This is explained by the fact that the difference between the solutions of the different transport equations turn negligible over a wider and wider interval of the real line, as t grows. At the same time, the asymptotic regimes often are the relevant ones.

This study can be developed in different directions. First of all, a deeper numerical study would be needed to strength our observation about the SMD for the Porous Medium Equation for $\gamma > 1$. Furthermore, it would be interesting to extend the comparison of the asymptotic time behaviour of $\eta(p)$ to other continuous models for anomalous transport such as fractional diffusion equations.

Appendix A: Time-dependent diffusion coefficient

We here derive a modification of the diffusion equation that accounts for anomalous spreading. To this purpose, we introduce the following Gaussian distribution (Eq. (5) in the main text) whose mean-square displacement grows as t^γ, for $\gamma \in [0, 2]$,

$$u(x,t) = \frac{1}{\sqrt{4\pi D t^\gamma}} \exp\left(-\frac{x^2}{4Dt^\gamma}\right) \tag{A1}$$

Basic algebra provides its time derivative

$$\frac{\partial u}{\partial t} = \frac{\gamma}{\sqrt{\pi}(4D)^{\frac{3}{2}}}\left(\frac{x^2}{t^{1+\frac{3\gamma}{2}}} - \frac{2D}{t^{1+\frac{\gamma}{2}}}\right)\exp\left(\frac{-x^2}{4Dt^\gamma}\right) \tag{A2}$$

as well as the first and second derivatives w.r.t. the spatial variable x

$$\frac{\partial u}{\partial x} = \frac{-2\,x}{\sqrt{\pi}(4D)^{\frac{3}{2}} t^{\frac{3\gamma}{2}}} \exp\left(\frac{-x^2}{4Dt^\gamma}\right) \tag{A3}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\sqrt{\pi}(4D)^{\frac{3}{2}}}\left(\frac{x^2}{Dt^{\frac{5\gamma}{2}}} - \frac{2}{t^{\frac{3\gamma}{2}}}\right)\exp\left(\frac{-x^2}{4Dt^\gamma}\right) \tag{A4}$$

The equality $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$ can be immediately checked for $\gamma = 1$. Similarly, for $\gamma \neq 1$, we impose

$$\frac{\partial u}{\partial t} = a\left(\frac{\partial u}{\partial x}\right) + b\left(\frac{\partial^2 u}{\partial x^2}\right) \tag{A5}$$

Substituting Equations (A2), (A3) and (A4) into Eq. (A5), it turns out that the latter is verified for $a = 0$ and $b = \frac{D}{t^{1-\gamma}}$ and takes the following form (Eq. (6) in the main text)

$$\frac{\partial u}{\partial t} = Dt^{\gamma-1}\frac{\partial^2 u}{\partial x^2}$$

Appendix B: Insight on the anomalous diffusion properties of PME

Let us consider a ZKB solution with a front that moves at a given speed, i.e. $x^2 \sim t^\gamma$,

$$u(x,t) = t^{-\alpha}(C - kx^2 t^{-\gamma})_+^{\nu} \tag{B6}$$

where the constants α, ν allow generality and will be related to γ, C and k as well as to the power m appearing in $\frac{\partial u}{\partial t} = \frac{\partial^2 u^{\mathrm{m}}}{\partial x^2}$. Simple algebra provides the time derivative of $u(x,t)$ as

$$\frac{\partial u}{\partial t} = t^{-\alpha-1}(C - kx^2 t^{-\gamma})^{\nu-1}\left[k(\nu\gamma+\alpha)x^2 t^{-\gamma} - \alpha C\right] \tag{B7}$$

as well as the first and second derivatives w.r.t. the spatial variable x

$$\frac{\partial u}{\partial x} = -2k\nu t^{-\alpha-\gamma} x (C - k\,x^2\,t^{-\gamma})^{\nu-1} \tag{B8}$$

$$\frac{\partial^2 u}{\partial x^2} = -2k\nu t^{-\alpha-\gamma}(C - kx^2 t^{-\gamma})^{\nu-2}[C - k(2\nu-1)x^2 t^{-\gamma}] \tag{B9}$$

We then evaluate

$$\begin{aligned}\frac{\partial^2 u^{\mathrm{m}}}{\partial x^2} =& (m-1)mu^{\mathrm{m}-2}\left(\frac{\partial u}{\partial x}\right)^2 + mu^{\mathrm{m}-1}\frac{\partial^2 u}{\partial x^2}\\ =& t^{-\alpha\mathrm{m}-\gamma}(C - kx^2 t^{-\gamma})^{\nu\mathrm{m}-2}[2k^2\nu m(2\nu m-1)x^2 t^{-\gamma} - 2k\nu mC]\end{aligned} \tag{B10}$$

obtained by replacing expressions given in Eqs. (B8) and (B9).

By equating the expressions in Eqs. (B7) and (B10), we then observe that the law of evolution of the PME is given for the following parameter relations

$$\alpha = \frac{1-\gamma}{m-1}, \quad \nu = \frac{1}{m-1}, \quad k = \frac{1-\gamma}{2m}, \quad m = \frac{2-\gamma}{\gamma} \tag{B11}$$

It is immediate to check that the parameter relations in Eq. (B11) are equivalent to those presented in the main text in Eq. (12).

To sum up, we have explicitly derived the relation between the exponent appearing in the PME and the transport feature of its solution. The resulting diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u^{\frac{2-\gamma}{\gamma}}}{\partial x^2} \tag{B12}$$

will then describe normal diffusion for $\gamma = 1$ and anomalous diffusion for $\gamma \neq 1$.

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References

1. Lepri S, Livi R, Politi A. Thermal transport in low dimensions. Lect Notes Phys. 2016;921:1–37.
2. Zhmakin AI. Non-fourier heat conduction: from phase-lag models to relativistic and quantum transport. Berlin: Springer; 2023.
3. Giberti C, Rondoni L, Vernia C. O (n) fluctuations and lattice distortions in 1-dimensional systems. Front Phys. 2019;7:180.
4. Livi R, Lepri S. Heat in one dimension. Nature. 2003;421(6921):327–327.
5. Livi R. Heat transport in one dimension. J Stat Mech: Theory Exp. 2020;2020(3):034001.
6. Metzler R, Klafter J. The random walk's guide to anomalous diffusion: a fractional dynamics approach. Phys Rep. 2000;339(1):1–77.
7. Lepri S, Livi R, Politi A. Thermal conduction in classical low-dimensional lattices. Phys Rep. 2003;377:1.
8. Hoffmann KH, Prehl J. Anomalous transport: foundations and applications. edited by rainer klages, günter radons, and igor m. sokolov copyright 2008 wiley-vch verlag gmbh & co. kgaa, weinheim isbn: 978-3-527-40722-4. Anomalous Transport: Foundations and Applications, 2008;397.
9. Rieder Z, Lebowitz JL, Lieb E. Properties of a harmonic crystal in a stationary nonequilibrium state. J Math Phys. 1967;8:1073.
10. Klages R, Radons G, Sokolov IM. Anomalous transport: foundations and applications 2008.
11. Dematteis G, Rondoni L, Proment D, De Vita F, Onorato M. Coexistence of ballistic and fourier regimes in the β fermi-pasta-ulam-tsingou lattice. Phys Rev Lett. 2020;125(2):024101.
12. Vollmer J, Rondoni L, Tayyab M, Giberti C, Mejía-Monasterio C. Displacement autocorrelation functions for strong anomalous diffusion: a scaling form, universal behavior, and corrections to scaling. Phys Rev Res. 2021;3:013067. https://doi.org/10.1103/PhysRevResearch.3.013067.
13. Salari L, Rondoni L, Giberti C, Klages R. A simple non-chaotic map generating subdiffusive, diffusive, and superdiffusive dynamics. Chaos. 2015;25:073113. https://doi.org/10.1063/1.4926621.
14. Castiglione P, Mazzino A, Muratore-Ginanneschi P, Vulpiani A. On strong anomalous diffusion. Physica D. 1999;134(1):75–93.

15. Sanders DP, Larralde H. Occurrence of normal and anomalous diffusion in polygonal billiard channels. Phys Rev E. 2006;73:026205. https://doi.org/10.1103/PhysRevE.73.026205.
16. Jepps OG, Rondoni L. Thermodynamics and complexity of simple transport phenomena. J Phys A: Math Gen. 2006;39:1311. https://doi.org/10.1088/0305-4470/39/6/007.
17. Orchard J, Rondoni L, Mejía-Monasterio C, Frascoli F. Diffusion and escape from polygonal channels: extreme values and geometric effects. J Stat Mech: Theory Exper. 2021;2021(7):073208.
18. Armstead DN, Hunt BR, Ott E. Anomalous diffusion in infinite horizon billiards. Phys Rev E. 2003;67:021110. https://doi.org/10.1103/PhysRevE.67.021110.
19. Schmiedeberg M, Stark H. Superdiffusion in a honeycomb billiard. Phys Rev E. 2006;73:031113. https://doi.org/10.1103/PhysRevE.73.031113.
20. Courbage M, Edelman M, Fathi SMS, Zaslavsky GM. Problem of transport in billiards with infinite horizon. Phys Rev E. 2008;77:036203. https://doi.org/10.1103/PhysRevE.77.036203.
21. Aghion E, Kessler DA, Barkai E. Large fluctuations for spatial diffusion of cold atoms. Phys Rev Lett. 2017;118:260601. https://doi.org/10.1103/PhysRevLett.118.260601.
22. Pikovsky AS. Statistical properties of dynamically generated anomalous diffusion. Phys Rev A. 1991;43:3146–8. https://doi.org/10.1103/PhysRevA.43.3146.
23. Castiglione P, Mazzino A, Muratore-Ganannneschi P, Vulpiani A. On strong anomalous diffusion. Physica D. 1999;134:75. https://doi.org/10.1016/S0167-2789(99)00031-7.
24. Artuso R, Cristadoro G. Anomalous transport: a deterministic approach. Phys Rev Lett. 2003;90:244101. https://doi.org/10.1103/PhysRevLett.90.244101.
25. Gal N, Weihs D. Experimental evidence of strong anomalous diffusion in living cells. Phys Rev E. 2010;81:020903. https://doi.org/10.1103/PhysRevE.81.020903.
26. Sneppen K, Jensen MH. Multidiffusion in critical dynamics of strings and membranes. Phys Rev E. 1994;49:919–22. https://doi.org/10.1103/PhysRevE.49.919.
27. Vezzani A, Barkai E, Burioni R. Single-big-jump principle in physical modeling. Phys Rev E. 2019;100(1):012108.
28. Giberti C, Rondoni L, Tayyab M, Vollmer J. Equivalence of position-position auto-correlations in the slicer map and the lévy-lorentz gas. Nonlinearity. 2019;32(6):2302.
29. Leibenzon LS. The motion of a gas in a porous medium. Acad. Sciences URSS, Moscow 1953;2, –101929891930.
30. Zel'dovich Y, Raizer Y. Physics of shock waves and high-temperature hydrodynamic phenomena 1966.
31. Gurtin ME, MacCamy RC. On the diffusion of biological populations. Math Biosci. 1977;33(1–2):35–49.
32. Zel'dovich YB, Kompaneets AS. Towards a theory of heat conduction with thermal conductivity depending on the temperature. Collection of papers dedicated to 70th Anniversary of A. F. Ioffe, 1950;61–72.
33. Barenblatt GI. On some unsteady motions of a liquid or a gas in a porous medium. Prikl Mat Mekh. 1952;16(1):67–78.
34. Vázquez JL. The porous medium equation: mathematical theory 2007.
35. Kolesnik A. Moment analysis of the telegraph random process. Buletinul Academiei de Ştiinţe a Moldovei Matematica. 2012;68(1):90–107.
36. Dunkel J, Hänggi P. Relativistic brownian motion. Phys Rep. 2009;471(1):1–73.
37. O'Hara P, Rondoni L. Brownian motion in minkowski space. Entropy. 2015;17(6):3581–94.
38. Goldstein S. On diffusion by discontinuous movements, and on the telegraph equation. Quart J Mech Appl Math. 1951;4(2):129–56.
39. Kac M. A stochastic model related to the telegrapher's equation. Rocky Mount J Math. 1974;4(3):497–509.
40. Pinsky MA. Lectures on random evolution 1991.
41. Orsingher E. Probability law, flow function, maximum distribution of wave-governed random motions and their connections with kirchoff's laws. Stoch Process Appl. 1990;34(1):49–66.
42. Iacus S, Yoshida N. Estimation for the discretely observed telegraph process. Theory Prob Math Stat. 2009;78:37–47.