Potential Analysis
https://doi.org/10.1007/s11118-023-10106-4

RESEARCH

Harmonic Bergman Projectors on Homogeneous Trees

Filippo De Mari[1] · Matteo Monti[2] · Maria Vallarino[2]

Received: 10 March 2023 / Accepted: 23 September 2023

Abstract
In this paper we investigate some properties of the harmonic Bergman spaces $\mathcal{A}^p(\sigma)$ on a q-homogeneous tree, where $q \geq 2$, $1 \leq p < \infty$, and σ is a finite measure on the tree with radial decreasing density, hence nondoubling. These spaces were introduced by J. Cohen, F. Colonna, M. Picardello and D. Singman. When $p = 2$ they are reproducing kernel Hilbert spaces and we compute explicitely their reproducing kernel. We then study the boundedness properties of the Bergman projector on $L^p(\sigma)$ for $1 < p < \infty$ and their weak type (1,1) boundedness for radially exponentially decreasing measures on the tree. The weak type (1,1) boundedness is a consequence of the fact that the Bergman kernel satisfies an appropriate integral Hörmander's condition.

Keywords Bergman spaces · Homogeneous trees · Bergman projectors · Calderón-Zygmund theory

Mathematics Subject Classification (2010) 05C05 · 46E22 · 43A85

Introduction

The main focus of this paper is on the projectors associated to the harmonic Bergman spaces on homogeneous trees introduced in [9]. The Bergman spaces $\mathcal{A}^p(\sigma)$, $1 \leq p < \infty$, are

This work is partially supported by the project "Harmonic analysis on continuous and discrete structures" funded by Compagnia di San Paolo (Cup E13C21000270007). Furthermore, the authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

✉ Matteo Monti
matteo.monti@polito.it; mattemonti94@gmail.com

Filippo De Mari
demari@dima.unige.it

Maria Vallarino
maria.vallarino@polito.it

1 Dipartimento di Matematica, Dipartimento di Eccellenza 2023-2027, and MaLGa center, Università di Genova, Via Dodecaneso 35, 16146 Genova, Italy

2 Dipartimento di Scienze Matematiche "Giuseppe Luigi Lagrange", Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

in some ways the harmonic analogues of the classical holomorphic Bergman spaces on the hyperbolic disk, whereby p-integrability is relative to the reference measure σ on the tree, that is a finite measure with radial density with respect to the counting measure, and where harmonicity is relative to the so-called combinatorial Laplacian. The analogy between the hyperbolic disk and the homogeneous tree inspires many ideas behind our constructions (see [6, 8]).

The space $\mathcal{A}^2(\sigma)$ is, as expected, a reproducing kernel Hilbert space (RKHS) and the problem of understanding the associated projectors hinges on the explicit knowledge of the kernel, an information that we derive by introducing a somewhat canonical basis for $\mathcal{A}^2(\sigma)$. The core of this contribution is devoted to proving that, for a prototypical class of measures, the extension of the Bergman projector is bounded on $L^p(\sigma)$ if and only if $p > 1$, and is of weak type (1,1). The results are thus almost faithful reformulations of those that hold true for the holomorphic Bergman spaces on the hyperbolic disk [5, 15, 17, 20, 22], but many of the key ingredients, first and foremost the explicit determination of the reproducing kernel, call for a rather different approach.

Let X be a homogeneous tree. A function on the tree is said to be harmonic if the mean of its values on the neighbors of any vertex coincides with the value at that vertex. J. Cohen, F. Colonna, M. Picardello, and D. Singman introduce harmonic Bergman spaces on homogeneous trees in [9]. They consider a family of reference measures which consists of finite measures that are absolutely continuous with respect to the counting measure and whose Radon-Nikodym derivative σ is a radial strictly positive decreasing function on X. For every $1 \leq p < \infty$, the harmonic Bergman space $\mathcal{A}^p(\sigma)$ is the closed subspace of $L^p(\sigma)$ consisting of harmonic functions. The requirement for the measure σ to be finite is suggested by the fact that the only harmonic function which is p-integrable with respect to the counting measure is the null function.

In the context of the hyperbolic disk, when $p = 2$, the weighted Bergman spaces are RKHS, and the holomorphic Bergman kernel is known as well as the properties of the associated projector. Indeed, the extension of the holomorphic Bergman projector to the weighted L^p-spaces is bounded if and only if $p > 1$, see [17, 20, 22]. Furthermore, it is of weak type $(1, 1)$, see [5, 15]. In our work, first of all, we show that $\mathcal{A}^2(\sigma)$ is a RKHS for every reference measure σ and we provide an explicit formula for the reproducing kernel K_σ in Theorem 11. Since $\mathcal{A}^2(\sigma)$ is closed in $L^2(\sigma)$, there exists an orthogonal projection $P_\sigma : L^2(\sigma) \to \mathcal{A}^2(\sigma)$. We prove that, for a particular class of reference measures, P_σ extends to a bounded operator from $L^p(\sigma)$ to $\mathcal{A}^p(\sigma)$ if and only if $p > 1$. Moreover, we show that P_σ is of weak type (1,1): to do so we use a Calderón-Zygmund decomposition of integrable functions adapted to the measure σ. Notice that the measure σ is not doubling with respect to the standard discrete metric on X, but it turns out to be doubling with respect to the so-called Gromov metric (see Section 4). Hence a Calderón-Zygmund theory in this setting holds, and we show that the Bergman kernel satisfies an integral Hörmander's condition related to such theory, so that it is of weak type (1,1).

The measures we focus on are exponentially decreasing radial measures, i.e. they are exponentially decreasing with respect to the distance from o and can be viewed as natural counterparts of the measures involved in the definition of the standard weighted holomorphic Bergman spaces on the hyperbolic disk. The fact that the extension of the projector to the weighted L^p-spaces is bounded if and only if $p > 1$ follows from the fact that the projector coincides with a particular Toepliz-type operator (see Section 3.4 in [22]).

In the spirit of the results of [3, 4] on the disk, one could investigate the boundedness of the Bergman projectors for general reference measures. In [9–11], the authors introduce and study the optimal measures, a subclass of the reference measures which, roughly speaking, decrease fast as the distance from the origin increases. The exponentially decreasing radial measures are optimal in this sense. The study of the boundedness of the Bergman projector for optimal measures is still work in progress. Another related question is whether the Calderón-Zygmund theory that we develop here could be applied to other operators.

The paper is organized as follows. In the first section we recall the definition of the harmonic Bergman spaces and, for every reference measure, we provide an orthonormal basis of the Hilbert space $\mathcal{A}^2(\sigma)$. The basis plays a fundamental role in Section 2 in the proof of the two formulae for the kernel of the RKHS $\mathcal{A}^2(\sigma)$: the first is a recursive formula, while the second is the explicit formula of the kernel given in Theorem 11. In Section 3 we focus on the exponentially decreasing radial measures and state two results characterizing the boundedness of the extension of a class of Toeplitz-type operators inspired by the operators considered in [22] (see Theorems 14 and 15). As a consequence, in Theorem 17 we show that the extension of the harmonic Bergman projector to the weighted L^p spaces is bounded if and only if $p > 1$. The last section is devoted to the Calderón-Zygmund decomposition of integrable functions (presented in Proposition 30), the formulation of the Hörmander's type condition, see Eq. 32, and the weak type (1,1) boundedness of the Bergman projectors is obtained as byproduct.

In what follows, we shall use the symbol $\simeq$ ($\lesssim$, or $\gtrsim$) between two quantities when the left hand side is equal (smaller than or equal to, or greater than or equal to, respectively) to the right hand side up to multiplication by a (fixed) positive constant. Furthermore we assume the following convention on sums: the sum is null whenever the starting index is greater than the final index. If $Y \subseteq X$, we denote by $\mathbb{1}_Y$ the characteristic function of Y. Finally, we adopt the symbol $\sqcup$ for disjoint unions and $\lfloor x \rfloor$ for the largest integer less than or equal to $x \in \mathbb{R}$.

1 Harmonic Bergman Spaces

1.1 Preliminaries on Homogeneous Trees

We present some preliminary notions and results on homogeneous trees; for more details we refer to [7, 13, 14, 16].

A *graph* is a pair $(X, \mathfrak{E})$, where X is the set of *vertices* and $\mathfrak{E}$ is the family of *unoriented edges*, where an edge is a two-element subset of X. If two vertices u, v in X are joined by an edge, they are called *adjacent* and this is denoted by $u \sim v$. A *tree* is an undirected, connected, cycle-free graph. A q*-homogeneous* tree is a tree in which each vertex has exactly $q+1$ adjacent vertices. With slight abuse, we refer to the set of vertices X as the tree itself. We fix an origin $o \in X$.

From now on we consider a q-homogeneous tree X with $q \geq 2$. Given $u, v \in X$, with $u \neq v$, we denote by $[u, v]$ the unique ordered t-uple $(x_0 = u, x_1, \ldots, x_{t-1} = v) \in X^t$, where $\{x_i, x_{i+1}\} \in \mathfrak{E}$ and all the x_i are distinct. We say that $[u, v]$ is the path starting at u and ending at v. With slight abuse of notation, if $[u, v] = (x_0, \ldots, x_{t-1})$ we write $x_i \in [u, v]$, $i \in \{0, \ldots, t-1\}$. In particular, if u and v are adjacent, both $[u, v]$, $[v, u] \in X^2$ are oriented, unlike the edge $\{u, v\} \in \mathfrak{E}$ which is not. A homogeneous tree X carries a natural distance $d\colon X \times X \to \mathbb{N}$, where for every $u, v \in X$ the distance $d(u, v)$ is the minimal length of a *path* joining u and v. If $v \in X$, then we denote by $S(v, n)$ and $B(v, n)$ the sphere and the ball centered at v with radius $n \in \mathbb{N}$, respectively, i.e.,

$$S(v, n) = \{x \in X\colon d(v, x) = n\} \qquad \text{and} \qquad B(v, n) = \{x \in X\colon d(v, x) \leq n\}.$$

It is straightforward to check that

$$\#S(v,n) = \begin{cases} 1, & n = 0, \\ (q+1)q^{n-1}, & n \neq 0. \end{cases} \tag{1}$$

We call *norm* of a vertex v in X its distance from o, i.e. $|v| = d(o,v)$. We say that a function f on X is *radial* (with respect to o) if its value at a vertex $x \in X$ depends only on $|x|$. If $v \neq o$, then we define the *sector of* v as the subset

$$T_v := \{x \in X : [o,v] \subseteq [o,x]\},$$

and we adopt the convention To $= X$. Moreover, we call *successors* of v the elements of the set $s(v) = \{u \in X : u \sim v, |u| = |v|+1\}$. Evidently,

$$\#s(v) = \begin{cases} q, & \text{if } v \neq o; \\ q+1, & \text{if } v = o. \end{cases}$$

For every $v \neq o$ we call predecessor of v and denote it by $p(v)$ the only neighbor of v which is not a successor of v; it follows that $|p(v)| = |v| - 1$. The vertex o is the only one having no predecessors, and $s(o) = S(o,1)$. This defines the predecessor function $p\colon X \setminus \{o\} \to X$, and, for every positive integer ℓ, its ℓ-fold composition $p^\ell\colon X \setminus B(o,\ell-1) \to X$ is the ℓ-th predecessor function.

1.2 Harmonic Functions and Harmonic Bergman Spaces

Definition 1 Let f be a complex valued function on X. The *combinatorial Laplacian* of f is defined by

$$Lf(v) := f(v) - \frac{1}{q+1}\sum_{u\sim v} f(u), \qquad v \in X.$$

We say that f is *harmonic* on $Y \subseteq X$ if $Lf = 0$ on Y. Equivalently, f is harmonic on Y if

$$f(v) = \frac{1}{q+1}\sum_{u\sim v} f(u), \qquad v \in Y. \tag{2}$$

We shall call a function harmonic if it is harmonic on X.

It is easy to prove that a function is harmonic if and only if for every $v \in X$ and $n \in \mathbb{N}$, we have

$$f(v) = \frac{1}{\#S(v,n)} \sum_{d(v,x)=n} f(x). \tag{3}$$

The harmonicity property Eq. 2 implies a certain rigidity for the function. In particular, the value of a harmonic function at a vertex $y \in X$ "propagates" to every layer of the sector T_y, as showed in the following proposition, which is a modified version of [9, Lemma 4.1]. In that lemma, the authors show that a function which is harmonic and radial on a sector T_y, $y \in X \setminus \{o\}$, is completely described by its values at y and $p(y)$. We consider a harmonic function on the sector T_y, removing the radiality assumption, and we formulate a result for its average on $S(o,n) \cap T_y$, $n \geq |y|$. We omit the proof since it is an easy adaptation of the proof of [9, Lemma 4.1].

Proposition 2 *Let $y \in X \setminus \{o\}$. If $f\colon X \to \mathbb{C}$ is harmonic on T_y, then for every $n \in \mathbb{N}$, $n \geq |y|$, we have*

$$\sum_{\substack{|x|=n \\ x\in T_y}} f(x) = \left(\sum_{j=0}^{n-|y|} q^j\right) f(y) - \left(\sum_{j=0}^{n-|y|-1} q^j\right) f(p(y)). \tag{4}$$

Furthermore, if $f\colon X \to \mathbb{C}$ is radial on T_y and satisfies Eq. 4 for every $n \geq |y|$, then f is harmonic on T_y.

We introduce a technique which allows to extend a function which is harmonic on a ball centered in o to a function harmonic on X. Let $n \in \mathbb{N}$ and g be a function on X which is harmonic on $B(o, n)$. It is easy to see that there are infinitely many ways to extend g to a harmonic function on X which coincides with g on $B(o, n+1)$. As we see next, there is however a unique harmonic function g_n^H on X which is radial when restricted on T_y for every $y \in S(o, n+1)$.

Let $x \in X \setminus B(o, n)$. There exists a unique $y \in S(o, n+1)$ such that $x \in T_y$, and $y = p^{|x|-n-1}(x)$ (where $p^0 = \mathrm{id}_X$). Since we aim to construct g_n^H radial and harmonic on T_y, by Proposition 2 we have that

$$\begin{aligned}
g_n^H(x) &= \frac{1}{\#S(o,|x|)\cap T_y} \sum_{\substack{|z|=|x|, \\ z\in T_y}} g_n^H(z) \\
&= q^{|y|-|x|}\left[\left(\sum_{j=0}^{|x|-|y|} q^j\right) g(y) - \left(\sum_{j=0}^{|x|-|y|-1} q^j\right) g(p(y))\right] \\
&= q^{n+1-|x|}\left[\left(\sum_{j=0}^{|x|-n-1} q^j\right) g(p^{|x|-n-1}(x)) - \left(\sum_{j=0}^{|x|-n-2} q^j\right) g(p^{|x|-n}(x))\right] \\
&= \left(\sum_{j=0}^{|x|-n-1} q^{-j}\right) g(p^{|x|-n-1}(x)) - \left(\sum_{j=1}^{|x|-n-1} q^{-j}\right) g(p^{|x|-n}(x)).
\end{aligned}$$

For simplicity we introduce the notation

$$a_n = \sum_{j=0}^{n} q^{-j} = \frac{q-q^{-n}}{q-1}, \qquad n \in \mathbb{N},$$

and we set $a_{-1} = 0$. Hence

$$g_n^H(x) = \begin{cases} g(x), & |x| \leq n, \\ a_{|x|-n-1}\, g(p^{|x|-n-1}(x)) - \left(a_{|x|-n-1} - 1\right) g(p^{|x|-n}(x)), & |x| > n. \end{cases} \tag{5}$$

The function g_n^H defined above is harmonic on X by Proposition 2 and because

$$X = B(o,n) \cup \bigcup_{y\in S(o,n+1)} T_y.$$

Observe that g_n^H is indeed harmonic on $B(o, n)$ because $a_0 = 1$ and $a_{-1} = 0$, yield $g_n^H = g$ on $B(o, n+1)$, and not only on $B(o, n)$. Furthermore, the extension g_n^H is radial on every sector “starting” from a point in $S(o, n+1)$ by construction (Fig. 1).

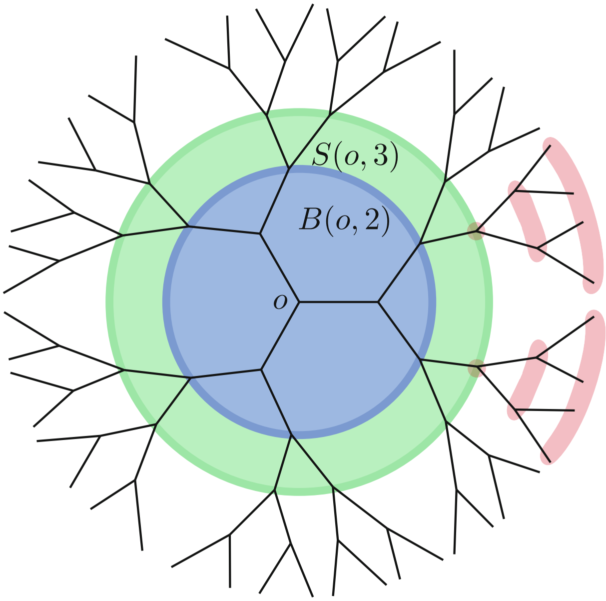

Fig. 1 The function g is harmonic on $B(o, 2)$, that is the set of vertices in the blue area. The function g_2^H is obtained by extending the values of g in $S(o, 3)$ (the green area) along sectors in such a way that g_2^H is harmonic on X and constant on the vertices lying on the same red arc, that is on the "layers" of the sectors

1.3 Harmonic Bergman spaces

Homogeneous trees are classically endowed with the counting measure. The main feature of such measure is the invariance under the group of isometries of the tree. When studying spaces of harmonic functions, this measure is however inadequate because the only harmonic function that is p-summable, $1 \le p < \infty$, with respect to the counting measure is the null function, as we show in the following statement.

Proposition 3 *If f is a harmonic function in $L^p(X)$, $1 \le p < \infty$, then f is the null function.*

Proof Suppose that f is harmonic. We have that

$$\begin{aligned}
\sum_{x\in X} |f(x)|^p =& \sum_{n=0}^{+\infty} \sum_{|x|=n} |f(x)|^p \\
=& \frac{1}{(q+1)^p} \sum_{n=0}^{+\infty} \sum_{|x|=n} \left| \sum_{y\sim x} f(y) \right|^p \\
\le& \frac{(q+1)^{p-1}}{(q+1)^p} \sum_{n=0}^{+\infty} \sum_{|x|=n} \sum_{y\sim x} |f(y)|^p \\
=& \frac{1}{q+1}(q+1)\|f\|^p_{L^p(X)} = \|f\|^p_{L^p(X)} < +\infty,
\end{aligned}$$

since every vertex is neighbor of exactly $q+1$ other vertices. Hence the unique inequality in the computation above is an equality, so that

$$(q+1)^{p-1}\sum_{y\sim x}|f(y)|^p=\left|\sum_{y\sim x}f(y)\right|^p=(q+1)^p|f(x)|^p,$$

which means that $|f|^p$ is harmonic, too. If f is not the null function, then there exists $v\in X$ such that $f(v)\neq 0$. Hence by Eq. 3, we have

$$\sum_{x\in X}|f(x)|^p=\sum_{n=0}^{+\infty}\sum_{d(v,x)=n}|f(x)|^p=|f(v)|^p\sum_{n=0}^{+\infty}\#S(v,n)=+\infty,$$

which is a contradiction. Hence $f=0$. □

Since we are interested in Bergman spaces of harmonic functions, the previous proposition leads to consider finite measures on X. In [9], the authors introduce harmonic Bergman spaces with respect to the following class of measures.

Definition 4 A *reference measure* on X is a finite measure that is absolutely continuous with respect to the counting measure and whose Radon-Nikodym derivative σ is a radial strictly positive decreasing function on X. With slight abuse of notation we denote by σ the reference measure, too. Given a reference measure σ on X for every $p\in[1,\infty)$ the *Bergman space* $\mathcal{A}^p(\sigma)$ is the space of harmonic functions on X such that

$$\|f\|^p_{\mathcal{A}^p(\sigma)}:=\sum_{x\in X}|f(x)|^p\sigma(x)<+\infty.$$

Every Bergman space $\mathcal{A}^p(\sigma)$ is a Banach space and when $p=2$, it is a Hilbert space with the scalar product

$$\langle f,g\rangle_{\mathcal{A}^2(\sigma)}:=\sum_{x\in X}f(x)\overline{g(x)}\sigma(x),\qquad f,g\in\mathcal{A}^2(\sigma). \tag{6}$$

If σ is a reference measure on X, and if we denote by σ_n the value of σ on the sphere $S(0,n)$, and by B_σ the total mass of σ, then by Eq. 1 its value is

$$B_\sigma=\sigma_0+\frac{q+1}{q}\sum_{n=1}^{+\infty}\sigma_n q^n<+\infty.$$

Example 5 Let $\alpha>1$. Interesting examples of reference measures are the exponentially decreasing radial measures, consisting of the measures having density

$$\mu_\alpha(x)=q^{-\alpha|x|},\qquad x\in X.$$

Indeed, μ_α is radial, positive and decreasing. Furthermore, we write B_α in place of B_{μ_α}, namely

$$B_\alpha=1+\frac{q+1}{q}\sum_{n=1}^{+\infty}q^{(1-\alpha)n}=1+\frac{q+1}{q}\frac{q^{1-\alpha}}{1-q^{1-\alpha}}=\frac{q^\alpha+1}{q^\alpha-q}<+\infty. \tag{7}$$

Proposition 6 *For every reference measure σ the measure metric space (X,d,σ) is nondoubling.*

Proof Let σ be a reference measure. For every $n \in \mathbb{N}$, let $v_n \in X$ be such that $|v_n| = 2n$. Then $\max\{\sigma(x) : x \in B(v_n, n)\} = \sigma_n$ and so

$$\sigma(B(v_n, n)) = \sum_{x \in B(v_n, n)} \sigma(x) \le \sigma_n |B(v_n, n)| \lesssim q^n \sigma_n.$$

On the other hand, since $o \in B(v_n, 2n)$, we have

$$\frac{\sigma(B(v_n, 2n))}{\sigma(B(v_n, n))} \gtrsim \frac{\sigma(o)}{q^n \sigma_n} \xrightarrow{n \to \infty} \infty,$$

by the finiteness of σ. This concludes the proof. □

Given a reference measure σ, we introduce the decreasing sequence $(b_n)_{n \in \mathbb{N}}$ which collects some important information on σ. For every $n \in \mathbb{N}$, we define

$$b_n = b_n(\sigma) = \sum_{m=n+1}^{+\infty} \left[\sigma_m a_{m-n-1} \left(\sum_{k=0}^{m-n-1} q^k \right) \right]. \tag{8}$$

The sums are finite because σ is a finite measure on X. We shall use the notation b_n instead of $b_n(\sigma)$ whenever the measure is clear from the context.

The next lemma is a technical result that is very useful in what follows.

Lemma 7 *Let $n \in \mathbb{N}$ and g be a function on X which is harmonic and vanishes on $B(o, n)$. Then there exists a constant $b'_n > 0$ such that for every $f \in \mathcal{A}^2(\sigma)$*

$$\langle f, g_n^H \rangle_{\mathcal{A}^2(\sigma)} = \sum_{|y|=n+1} \left(b_n f(y) - b'_n f(p(y)) \right) \overline{g(y)},$$

where b_n is defined in Eq. 8 and $\langle \cdot, \cdot \rangle_{\mathcal{A}^2(\sigma)}$ in Eq. 6.

Remark 8 The constant b'_n has a structure similar to that of b_n, as can be seen in the proof below, but we are not interested in it.

Proof Observe that, from the fact that $g|_{B(o,n)} = 0$ and Eq. 5, for every $x \in X$ with $|x| > n$ we have $g_n^H(x) = a_{|x|-n-1} g(p^{|x|-n-1}(x))$. Take $f \in \mathcal{A}^2(\sigma)$. Then, by applying Proposition 2 to f, we have

$$\begin{aligned}
\langle f, g_n^H \rangle_{\mathcal{A}^2(\sigma)} &= \sum_{m=n+1}^{+\infty} \sigma_m \sum_{|x|=m} f(x) \overline{g_n^H(x)} \\
&= \sum_{m=n+1}^{+\infty} \sigma_m \sum_{|y|=n+1} \sum_{\substack{|x|=m \\ x \in T_y}} f(x) a_{|x|-n-1} \overline{g(p^{|x|-n-1}(x))} \\
&= \sum_{|y|=n+1} \overline{g(y)} \sum_{m=n+1}^{+\infty} \sigma_m a_{m-n-1} \sum_{\substack{|x|=m \\ x \in T_y}} f(x) \\
&= \sum_{|y|=n+1} \overline{g(y)} \sum_{m=n+1}^{+\infty} \sigma_m a_{m-n-1} \left[\left(\sum_{k=0}^{m-n-1} q^k \right) f(y) - \left(\sum_{k=0}^{m-n-2} q^k \right) f(p(y)) \right] \\
&= \sum_{|y|=n+1} \left(b_n f(y) - b'_n f(p(y)) \right) \overline{g(y)},
\end{aligned}$$

as required. □

1.4 A Canonical Orthonormal Basis of $\mathcal{A}^2(\sigma)$

The goal of this section is the construction of an orthonormal basis of the space $\mathcal{A}^2(\sigma)$.

Let us consider the linear spaces

$$W_v := \Big\{\varphi\colon s(v) \to \mathbb{C}\colon \sum_{z\in s(v)} \varphi(z) = 0\Big\} \simeq \mathbb{C}^{|s(v)|-1} = \begin{cases} \mathbb{C}^q, & v = o, \\ \mathbb{C}^{q-1}, & v \in X \setminus \{o\}. \end{cases}$$

For every $v \in X$ we set $I_v = \{1, \ldots, |s(v)| - 1\}$. For every $v \in X$ we fix an orthonormal basis $\{e_{v,j}\}_{j\in I_v}$ of W_v with respect to to the scalar product

$$\langle \varphi, \psi\rangle_{W_v} = \sum_{y\in s(v)} \varphi(y)\overline{\psi(y)}.$$

Let $v \in X$ and $j \in I_v$. We consider the extension by zero to all of X of $e_{v,j}$, namely,

$$E_{v,j}(x) = \begin{cases} e_{v,j}(x), & x \in s(v); \\ 0, & x \notin s(v). \end{cases}$$

It is immediate to see that $E_{v,j}$ is harmonic on $B(o, |v|)$ and vanishes on $B(o, |v|)$. We denote the harmonic extension of $E_{v,j}$ by $f_{v,j} = (E_{v,j})^H_{|v|}$, namely

$$f_{v,j}(x) = \begin{cases} 0, & \text{if } x \notin T_v \setminus \{v\}, \\ a_{|x|-|v|-1} E_{v,j}(p^{|x|-|v|-1}(x)), & \text{otherwise.} \end{cases} \tag{9}$$

Hence $f_{v,j}$ is harmonic for every $v \in X$ and $j \in I_v$. Furthermore $f_{v,j}$ is bounded, since for every $x \in X$

$$|f_{v,j}(x)| \le (1 - q^{-1})^{-1} \|e_{v,j}\|_\infty.$$

Hence $f_{v,j} \in \mathcal{A}^2(\sigma)$ for every reference measure σ. Notice that, upon setting $f_0(x) \equiv 1$, the family

$$\mathcal{B} = \{f_0\} \cup \{f_{v,j}\colon v \in X,\ j \in I_v\} \subseteq \mathcal{A}^2(\sigma)$$

is independent of the choice of the reference measure σ.

Proposition 9 *The family $\mathcal{B}$ is a complete orthogonal system in $\mathcal{A}^2(\sigma)$ for every reference measure σ.*

Proof Fix a reference measure σ. The fact that f_0 is orthogonal to every other function of the family follows from the harmonicity of $f_{v,j}$ and Eq. 3. Indeed

$$\langle f_{v,j}, f_0\rangle_{\mathcal{A}^2(\sigma)} = \sum_{x\in X} f_{v,j}(x)\sigma(x) = \sum_{n=0}^{+\infty} \sigma_n \sum_{|x|=n} f_{v,j}(x) = 0.$$

Let us consider $v, w \in X$ with $v \neq w$. Without loss of generality we may consider two situations: either $T_v \cap T_w = \emptyset$ or $T_v \subsetneq T_w$. In the first case $f_{v,j} \perp f_{w,k}$ for every $j \in I_v$ and $k \in I_w$, because their supports are disjoint. If $T_v \subsetneq T_w$, then we can suppose that $|w| < |v|$. Since $f_{v,j}|_{B(o,|w|+1)} = 0$, from Lemma 7 we have

$$\langle f_{v,j}, f_{w,k}\rangle_{\mathcal{A}^2(\sigma)} = \sum_{|y|=|w|+1} (b_{|w|} f_{v,j}(y) - b'_{|w|} f_{v,j}(p(y)))\overline{E_{w,k}(y)} = 0.$$

It remains to prove the orthogonality in the case $v = w$. Let $j, k \in I_v$ be such that $j \neq k$. We know that $f_{v,k}|_{B(o,|v|)} = 0$, so that by Lemma 7

$$\langle f_{v,j}, f_{v,k}\rangle_{\mathcal{A}^2(\sigma)} = b_{|v|} \sum_{|y|=|v|+1} E_{v,j}(y)\overline{E_{v,k}(y)} = b_{|v|} \sum_{y\in s(v)} e_{v,j}(y)\overline{e_{v,k}(y)} = 0,$$

where we used the fact that $\mathrm{supp}(E_{v,k})$, $\mathrm{supp}(E_{v,j}) \subseteq s(v)$ and the orthogonality of $e_{v,j}$ and $e_{v,k}$ in W_v.

We now show that $\mathcal{B}$ is complete. Take $g \in \mathcal{A}^2(\sigma)$ such that $\langle g, f\rangle_{\mathcal{A}^2(\sigma)} = 0$ for every $f \in \mathcal{B}$. We show that g is the null function in $\mathcal{A}^2(\sigma)$. In particular we prove by induction that $g = 0$ on every $B(o, m)$, $m \in \mathbb{N}$.
We start by observing that $\langle g, f_0\rangle_{\mathcal{A}^2(\sigma)} = 0$ implies $g(o) = 0$. Indeed by Eq. 3

$$0 = \langle g, f_0\rangle_{\mathcal{A}^2(\sigma)} = \sum_{n=0}^{+\infty} \sigma_n \sum_{|x|=n} g(x) = \left(1 + \frac{q+1}{q}\sum_{n=1}^{+\infty} q^n\sigma_n\right) g(o) = B_\sigma g(o). \tag{10}$$

We assume now $g = 0$ on $B(o, m)$ for some $m \in \mathbb{N}$. Let $v \in S(o, m)$. Observe that since g is harmonic and $g(v) = 0$, we have $g|_{s(v)} \in W_v$. Hence for every $j \in I_v$

$$0 = \langle g, f_{v,j}\rangle_{\mathcal{A}^2(\sigma)} = b_m \sum_{y\in s(v)} \overline{e_{v,j}(y)} g(y) \tag{11}$$

and this implies that $g(y) = 0$ for every $y \in s(v)$ and so for every $y \in S(o, m+1)$, namely g vanishes on $B(o, m+1)$. The fact that $g \equiv 0$ follows by induction. □

We now fix a reference measure σ and compute the norm of functions of the family $\mathcal{B}$ in $\mathcal{A}^2(\sigma)$. Evidently, $\|f_0\|^2_{\mathcal{A}^2(\sigma)} = B_\sigma$. Let $v \in X$ and $j \in I_v$. By Eq. 11, we have

$$\|f_{v,j}\|^2_{\mathcal{A}^2(\sigma)} = \langle f_{v,j}, f_{v,j}\rangle_{\mathcal{A}^2(\sigma)} = b_{|v|} \sum_{y\in s(v)} \overline{e_{v,j}(y)} e_{v,j}(y) = b_{|v|}. \tag{12}$$

Hence the norm of $f_{v,j}$ does not depend on j and coincides with the constant in Eq. 8. Hence

$$\mathcal{B}_\sigma = \{B_\sigma^{-\frac{1}{2}} f_0\} \cup \{b_{|v|}^{-\frac{1}{2}} f_{v,j} : v \in X,\ j \in I_v\} \tag{13}$$

is an orthonormal basis of $\mathcal{A}^2(\sigma)$.

2 The reproducing kernel of $\mathcal{A}^2(\sigma)$

In this section we fix a reference measure σ. We show that the Bergman space $\mathcal{A}^2(\sigma)$ is a RKHS and we first obtain a recursive formula for the kernel and we then derive a formula in closed form. Observe that the main ingredient used in the proofs are the harmonic extension and the orthonormal basis defined in the previous section together with the fact that W_v are reproducing kernel Hilbert spaces, too.

Let $z \in X$. We consider the evaluation functional $\Phi_z : \mathcal{A}^2(\sigma) \to \mathbb{C}$ defined by $\Phi_z g = g(z)$. Observe that Φ_z is a bounded operator. Indeed by the Cauchy-Schwarz inequality

$$|\Phi_z g|^2 = |g(z)|^2 \leq \frac{1}{\sigma(z)} \sum_{x\in X} |g(x)|^2\sigma(x) = \frac{\|g\|_2^2}{\sigma(z)}.$$

Thus $\mathcal{A}^2(\sigma)$ is a RKHS, that is for every $z \in X$ there exists $K_z \in \mathcal{A}^2(\sigma)$ such that

$$\langle g, K_z\rangle_{\mathcal{A}^2(\sigma)} = g(z), \qquad g \in \mathcal{A}^2(\sigma).$$

Let $K\colon X \times X \to \mathbb{C}$ be the kernel defined by $K(z,x) := K_z(x)$.

Since $\mathcal{B}_\sigma$ defined in Eq. 13 is an orthonormal basis of $\mathcal{A}^2_\sigma$, for every $z \in X$ we can write

$$K_z = \sum_{f \in \mathcal{B}_\sigma} \langle K_z, f\rangle_{\mathcal{A}^2(\sigma)} f = \sum_{f \in \mathcal{B}_\sigma} \overline{f(z)} f = \frac{1}{B_\sigma} + \sum_{v \in X} \sum_{j \in I_v} \frac{\overline{f_{v,j}(z)} f_{v,j}}{b_{|v|}}. \tag{14}$$

We recall that by Eq. 9, for every $z \in X$

$$\{v \in X\colon f_{v,j}(z) \neq 0 \text{ for some } j \in I_v\} = \begin{cases} \emptyset, & \text{if } v = o;\\ [o, p(z)], & \text{if } v \neq o. \end{cases}$$

Hence for every $z \in X$ the sum in Eq. 14 is finite and the decomposition of K_z holds true pointwise.

Our goal is to compute K_z. To this end, we introduce the auxiliary function $\Gamma\colon X \times X \times X \to \mathbb{R}$ which is a parametrization of the family of reproducing kernels for the spaces $\{W_v\}_{v \in X}$. For every $(v, z, x) \in X \times X \times X$ we set

$$\Gamma(v,z,x) = \begin{cases} 0, & \text{if } \{z,x\} \nsubseteq T_v \setminus \{v\};\\ \frac{\#s(v)-1}{\#s(v)}, & \text{if } \{z,x\} \subseteq T_y \text{ for some } y \in s(v);\\ -\frac{1}{\#s(v)}, & \text{otherwise.} \end{cases}$$

Observe that Γ is symmetric in the second and third variables. Furthermore, $\Gamma(v,z,\cdot)$ is the null function if $z \notin T_v \setminus \{v\}$ and whenever $z \in T_v \setminus \{v\}$ we have $\operatorname{supp}(\Gamma(v,z,\cdot)) = T_v \setminus \{v\}$. Moreover, the values of $\Gamma(v,z,\cdot)$ on $T_v \setminus \{v\}$ are completely determined by the values on $s(v)$, as the value of $\Gamma(v,z,\cdot)$ at $x \in T_v \setminus \{v\}$ is equal to the value at $p^{|x|-|v|-1}(x) \in s(v)$ (see Fig. 2).

We now show that $\Gamma(v,\cdot,\cdot)$ is the reproducing kernel of W_v, namely that for $z \in s(v)$ we have

$$\varphi(z) = \langle \varphi, \Gamma(v,z,\cdot)\rangle_{W_v}, \qquad \varphi \in W_v.$$

First of all $\Gamma(v,z,\cdot) \in W_v$ because

$$\sum_{y \in s(v)} \Gamma(v,z,y) = -(\#s(v)-1)\frac{1}{\#s(v)} + \frac{\#s(v)-1}{\#s(v)} = 0.$$

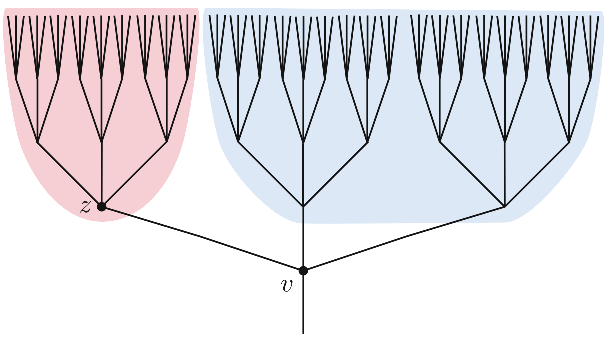

Fig. 2 Partial representation of the function $\Gamma(v,z,\cdot)$ on T_v. The value of $\Gamma(v,z,\cdot)$ at the vertices in the red area is $\frac{\#s(v)-1}{\#s(v)}$, while in the blue area is $-\frac{1}{\#s(v)}$. Clearly, $\Gamma(v,z,v) = 0$

Furthermore,

$$\begin{aligned}\langle \varphi, \Gamma(v,z,\cdot)\rangle_{W_v} &= \frac{\#s(v)-1}{\#s(v)}\varphi(z) - \frac{1}{\#s(v)}\sum_{\substack{y\in s(v)\\ y\neq z}}\varphi(y)\\ &= \frac{\#s(v)-1}{\#s(v)}\varphi(z) + \frac{1}{\#s(v)}\varphi(z) = \varphi(z),\end{aligned}$$

because $\varphi \in W_v$.

It is easy to see that $\Gamma(v,z,\cdot)$ is harmonic on $B(o,|v|)$ so that we can consider the harmonic extension $(\Gamma(v,z,\cdot))^H_{|v|}$, which is bounded by construction. Indeed from the definition of harmonic extension we have for every $x \in T_v \setminus \{v\}$

$$(\Gamma(v,z,\cdot))^H_{|v|}(x) = \left(\sum_{j=0}^{|x|-|v|-1} q^{-j}\right)\Gamma(v,z,p^{|x|-|v|-1}(x)) = a_{|x|-|v|-1}\Gamma(v,z,x), \quad (15)$$

and it vanishes elsewhere. We recall that if $z \notin T_v$, then $\Gamma(v,z,\cdot) = (\Gamma(v,z,\cdot))^H_{|v|}$ is the null function.

Proposition 10 *Let $z \in X$ and $[o,z] = \{v_t\}_{t=0}^{|z|}$. The kernel K_z is*

$$K_z = \begin{cases} \frac{1}{B_\sigma}, & \text{if } z = o,\\ K_o + \frac{1}{b_0}(\Gamma(o,z,\cdot))^H_0, & \text{if } |z| = 1,\\ -\frac{1}{q}K_{v_{m-2}} + \frac{q+1}{q}K_{v_{m-1}} + \frac{1}{b_{m-1}}(\Gamma(v_{m-1},z,\cdot))^H_{m-1}, & \text{if } |z| = m > 1.\end{cases}$$

Proof Since the measure σ is finite and the constant functions are harmonic, $K_o = \frac{1}{B_\sigma} \in \mathcal{A}^2(\sigma)$. The reproducing property follows from the computations in Eq. 10. Now we observe that for every $v,\ z \in X$ such that $z \in T_v$ and $g \in \mathcal{A}^2(\sigma)$, by Lemma 7 and $\operatorname{supp}(\Gamma(v,z,\cdot)) = T_v \setminus \{v\}$, we have

$$\begin{aligned}\langle g, (\Gamma(v,z,\cdot))^H_{|v|}\rangle_{\mathcal{A}^2(\sigma)} &= \sum_{|y|=|v|+1}(b_{|v|}g(y) - b'_{|v|}g(p(y)))\Gamma(v,z,y)\\ &= b_{|v|}\sum_{y\in s(v)} g(y)\Gamma(v,z,y) - b'_{|v|}g(v)\sum_{y\in s(v)}\Gamma(v,z,y)\\ &= b_{|v|}\sum_{y\in s(v)} g(y)\Gamma(v,z,y), \end{aligned} \quad (16)$$

where we used $\Gamma(v,z,\cdot)|_{s(v)} \in W_v$.
We now consider the case when $|z| = 1$. The function $K_z \in \mathcal{A}^2(\sigma)$ because it is sum of functions in $\mathcal{A}^2(\sigma)$. We prove the reproducing property. For $g \in \mathcal{A}^2(\sigma)$, by the reproducing formula of K_o and Eq. 16 with $v = o$,

$$\begin{aligned}\langle g, K_z\rangle_{\mathcal{A}^2(\sigma)} &= g(o) + \frac{1}{b_0}\langle g, (\Gamma(o,z,\cdot))^H_0\rangle_{\mathcal{A}^2(\sigma)}\\ &= g(o) + \sum_{|y|=1} g(y)\Gamma(o,z,y)\\ &= g(o) + \frac{q}{q+1}g(z) - \frac{1}{q+1}\sum_{\substack{|y|=1\\ y\neq z}} g(y) = g(z),\end{aligned}$$

where we used that g is harmonic at o.
It remains to consider the case when $|z| = m > 1$. We have $K_z \in \mathcal{A}^2(\sigma)$ since it is the sum of bounded and harmonic functions. For $g \in \mathcal{A}^2(\sigma)$ by induction on m and Eq. 16 with $v = v_{m-1}$ we have

$$\begin{aligned}
\langle g, K_z\rangle_{\mathcal{A}^2(\sigma)} &= -\frac{1}{q}g(v_{m-2}) + \frac{q+1}{q}g(v_{m-1}) + \frac{1}{b_{m-1}}\langle g, (\Gamma(v_{m-1}, z, \cdot))^H_{m-1}\rangle_{\mathcal{A}^2(\sigma)} \\
&= -\frac{1}{q}g(v_{m-2}) + \frac{q+1}{q}g(v_{m-1}) + \sum_{y\in s(v_{m-1})} \Gamma(v_{m-1}, z, y)g(y) \\
&= -\frac{1}{q}g(v_{m-2}) + \frac{1}{q}\sum_{y\sim v_{m-1}} g(y) + \frac{q-1}{q}g(z) - \frac{1}{q}\sum_{\substack{y\in s(v_{m-1})\\ y\neq z}} g(y) = g(z),
\end{aligned}$$

where we used the fact that g is harmonic at v_{m-1}. □

In Proposition 10 the kernel K_z is expressed through a two-step recursive formula. We want to find an explicit formula for K_z.

Theorem 11 *For every* $(z, x) \in X \times X$

$$K(z, x) = \frac{1}{B_\sigma} + \frac{q^2}{(q-1)^2}\sum_{v\in X}\frac{1}{b_{|v|}}\Gamma(v, z, x)(1 - q^{|v|-|z|})(1 - q^{|v|-|x|}). \tag{17}$$

Proof Let $z \in X$ and $[o, z] = \{v_t\}_{t=0}^{|z|}$. We start by proving that

$$K_z(x) = \frac{1}{B_\sigma} + \sum_{t=0}^{|z|-1} a_{|z|-t-1}\frac{1}{b_t}(\Gamma(v_t, v_{t+1}, x))^H_t, \qquad x \in X. \tag{18}$$

The case $z = o$ follows trivially from Proposition 10 and the convention on sums stated in the Introduction. We prove Eq. 18 by induction on $m = |z| \geq 1$. The case $m = 1$ directly follows from Proposition 10, too. Let $m \in \mathbb{N}$, $m > 1$ and $z \in X$, with $|z| = m$, and suppose that Eq. 18 holds for every vertex in $B(o, m-1)$. Hence by Proposition 10 we have

$$\begin{aligned}
K_z &= -\frac{1}{q}K_{v_{m-2}} + \frac{q+1}{q}K_{v_{m-1}} + \frac{1}{b_{m-1}}(\Gamma(v_{m-1}, z, \cdot))^H_{m-1} \\
&= -\frac{1}{q}\left[\frac{1}{B_\sigma} + \sum_{t=0}^{m-3} a_{m-t-3}\frac{1}{b_t}(\Gamma(v_t, v_{t+1}, \cdot))^H_t\right] \\
&\quad + \frac{q+1}{q}\left[\frac{1}{B_\sigma} + \sum_{t=0}^{m-2} a_{m-t-2}\frac{1}{b_t}(\Gamma(v_t, v_{t+1}, \cdot))^H_t\right] + \frac{1}{b_{m-1}}(\Gamma(v_{m-1}, z, \cdot))^H_{m-1} \\
&= \frac{1}{B_\sigma} + \sum_{t=0}^{m-2}\left(\frac{q+1}{q}a_{m-t-2} - \frac{1}{q}a_{m-t-3}\right)\frac{1}{b_t}(\Gamma(v_t, v_{t+1}, \cdot))^H_t \\
&\quad + \frac{1}{b_{m-1}}(\Gamma(v_{m-1}, z, \cdot))^H_{m-1} \\
&= \frac{1}{B_\sigma} + \sum_{t=0}^{m-1} a_{m-t-1}\frac{1}{b_t}(\Gamma(v_t, v_{t+1}, \cdot))^H_t,
\end{aligned}$$

where we used $(q+1)a_{n-1} - a_{n-2} = qa_n$. Hence we proved Eq. 18 by induction. Since $\mathrm{supp}((\Gamma(v_t, v_{t+1}, \cdot))_t^H) = T_{v_t} \setminus \{v_t\}$, we have that the t-th term of the sum in Eq. 18 does not vanish if and only if $x \in T_{v_t} \setminus \{v_t\}$, that is when $v_t \in [o, x]$, and hence by Eq. 15, we have

$$\begin{aligned} K(z,x) &= \frac{1}{B_\sigma} + \sum_{v \in X} \frac{1}{b_{|v|}} a_{|z|-|v|-1} a_{|x|-|v|-1} \Gamma(v, z, x) \\ &= \frac{1}{B_\sigma} + \frac{q^2}{(q-1)^2} \sum_{v \in X} \frac{1}{b_{|v|}} (1 - q^{|v|-|z|})(1 - q^{|v|-|x|}) \Gamma(v, z, x). \end{aligned}$$

□

Remark 12 The confluent of two vertices $z, x \in X$ is the common vertex of $[o, x]$ and $[o, z]$ farthest from o, denoted by $z \wedge x$. It is possible to see that the value of the kernel K at $(z, x) \in X \times X$ depends only on the values of $|x|$, $|z|$ and $|z \wedge x|$. Indeed, from Eq. 17 and the fact that $\Gamma(v, z, x)$ does not vanish if and only if $v \in [o, z] \cap [o, x] = [o, z \wedge x]$, we have

$$\begin{aligned} K_z(x) &= \frac{1}{B_\sigma} + \frac{q^2}{(q-1)^2} \sum_{t=0}^{|z \wedge x|} \frac{1}{b_t} a_{|z|-t-1} a_{|x|-t-1} (1 - q^{t-|z|})(1 - q^{t-|x|}) \\ &= \frac{1}{B_\sigma} + \frac{q^2}{(q-1)^2} \sum_{t=0}^{|z \wedge x|} \frac{1}{b_t} \Gamma(v_t, z, x)(1 - q^{t-|z|})(1 - q^{t-|x|}), \end{aligned} \tag{19}$$

where $\{v_t\}_{t=0}^{|z|} = [o, z]$. Furthermore, it is clear that K is symmetric, that is $K(z, x) = K(x, z)$.

In the following sections we restrict our attention to the family of the exponentially decreasing radial measures μ_α, $\alpha > 1$, defined in Example 5.

We shall use the notation L^p_α and $\mathcal{A}^p_\alpha$ for the Lebesgue and Bergman spaces with respect to μ_α, respectively. Furthermore, we denote by $K_\alpha : X \times X \to \mathbb{R}$ the reproducing kernel of $\mathcal{A}^2_\alpha$. It will be useful to keep track of the weight in the constants introduced in Eq. 8, so we denote them by $b_{\alpha,n}$. In particular observe that in this case there is a relation between the constants: for every $n \in \mathbb{N}$

$$\begin{aligned} b_{\alpha,n} &= \sum_{m=n+1}^{+\infty} \left[q^{-\alpha m} \left(\sum_{k=0}^{m-n-1} q^k \right) \left(\sum_{j=0}^{m-n-1} q^{-j} \right) \right] \\ &= \sum_{\ell=1}^{+\infty} \left[q^{-\alpha(\ell+n)} \left(\sum_{k=0}^{\ell-1} q^k \right) \left(\sum_{j=0}^{\ell-1} q^{-j} \right) \right] = q^{-\alpha n} b_{\alpha,0}. \end{aligned} \tag{20}$$

Furthermore we set $B_\alpha = \mu_\alpha(X)$.

Now we show that the kernel K_α satisfies an integral condition which will be formalized in Section 4, see Eq. 32.

Proposition 13 *The following holds*

$$\sup_{v \in X \setminus \{o\}} \sup_{x, y \in T_v} \sum_{z \in X \setminus T_v} |K_\alpha(z, x) - K_\alpha(z, y)| q^{-\alpha|z|} < +\infty. \tag{21}$$

Proof Let $v \in X \setminus \{o\}$. We start by proving that Eq. 21 holds for $y = v$. Consider $x \in T_v$ and observe that if $z \in X \setminus T_v$, then $z \wedge x = z \wedge v$ and $\Gamma(u, z, x) = \Gamma(u, z, v)$ for every $u \in [o, z \wedge v]$. Hence, if we put $[o, z \wedge v] = \{u_t\}_{t=0}^{|z\wedge v|}$, then from Eq. 19 we have

$$\begin{aligned}K_\alpha(z, x) - K_\alpha(z, v) &= \frac{q^2 b_{\alpha,0}}{(q-1)^2} \sum_{t=0}^{|z\wedge v|} q^{\alpha t}\Gamma(u_t, z, v)(1 - q^{t-|z|})(q^{t-|v|} - q^{t-|x|}) \\ &= \frac{q^2 b_{\alpha,0}}{(q-1)^2} \sum_{t=0}^{|z\wedge v|} q^{(1+\alpha)t}\Gamma(u_t, z, v)(1 - q^{t-|z|})(q^{-|v|} - q^{-|x|}).\end{aligned}$$

Then, since $|\Gamma(u_t, z, v)| < 1$ and $0 < q^{-|v|} - q^{-|x|} \leq q^{-|v|}$, we have

$$\sum_{z\in X\setminus T_v} |K_\alpha(z, x) - K_\alpha(z, v)|q^{-\alpha|z|} \lesssim \sum_{z\in X\setminus T_v} \sum_{t=0}^{|z\wedge v|} q^{(1+\alpha)t} q^{-|v|} q^{-\alpha|z|}.$$

Observe that, since $z \in X \setminus T_v$, then $|z \wedge v| \in \{0, \ldots, |v| - 1\}$. Then, for every $\ell \in \{0, \ldots, |v| - 1\}$, we put

$$Y_\ell = \{z \in X\colon |z \wedge v| = \ell\} = T_{u_\ell} \setminus T_{u_{\ell+1}}.$$

Since μ_α is radial,

$$\mu_\alpha(Y_\ell) = \mu_\alpha(u_\ell) + (\#s(u_\ell) - 1)\mu_\alpha(T_{u_{\ell+1}}) = \begin{cases} \frac{1}{1-q^{1-\alpha}}, & \text{if } \ell = 0; \\ q^{-\alpha\ell}\,\frac{1-q^{-\alpha}}{1-q^{1-\alpha}}, & \text{if } 1 \leq \ell < |v|. \end{cases}$$

Hence we have that

$$\begin{aligned}\sum_{z\in X\setminus T_v} |K_\alpha(z, x) - K_\alpha(z, v)|q^{-\alpha|z|} &\lesssim \sum_{\ell=0}^{|v|-1} q^{(1+\alpha)\ell} q^{-|v|} \sum_{z\in Y_\ell} q^{-\alpha|z|} \\ &\lesssim \sum_{\ell=0}^{|v|-1} q^{(1+\alpha)\ell} q^{-|v|} q^{-\alpha\ell} \\ &\simeq \sum_{\ell=0}^{|v|-1} q^{\ell-|v|} \leq \frac{q}{q-1}.\end{aligned}$$

Finally, by the triangular inequality we have Eq. 21 for every $y \in T_v$. □

3 Boundedness of the Bergman projector on L^p_α

In this section we study the boundedness properties of the extension of the Bergman projector to L^p_α spaces. For the class of exponentially decreasing radial measures we are able to prove that the extension of the Bergman projector to the relative weighted L^p-space is bounded if and only if $p > 1$ (see Theorem 17).

In analogy with the operators studied by Zhu in Section 3.4 of [22], we introduce two families of operators. For any real parameters a, b and for $c > 1$, we define the integral operators

$$S_{a,b,c} f(z) = q^{-a|z|} \sum_{x\in X} |K_c(z, x)| f(x) q^{-b|x|},$$

$$T_{a,b,c}f(z) = q^{-a|z|} \sum_{x \in X} K_c(z,x) f(x) q^{-b|x|}.$$

We prove two results that imply the boundedness properties of the Bergman projectors. Theorem 14 is devoted to the study of the boundedness of $S_{a,b,c}$ and $T_{a,b,c}$ on weighted L^p-spaces for $p > 1$; the case $p = 1$ needs different arguments and for this reason is treated apart in Theorem 15. The two theorems are the analogues of Theorem 3.11 and Theorem 3.12 in [22], respectively. The proofs of both theorems are postponed to Section 3.1.

Theorem 14 *Let $\alpha > 1$, $c > 1$ and $1 < p < \infty$. The following conditions are equivalent:*

(i) the operator $S_{a,b,c}$ is bounded on L^p_α;
(ii) the operator $T_{a,b,c}$ is bounded on L^p_α;
(iii) the parameters satisfy

$$c \le a + b, \qquad -pa < \alpha - 1 < p\,(b-1)\,.$$

Theorem 15 *Let $\alpha > 1$ and $c > 1$. The following conditions are equivalent:*

(i) the operator $S_{a,b,c}$ is bounded on L^1_α;
(ii) the operator $T_{a,b,c}$ is bounded on L^1_α;
(iii) the parameters either satisfy

$$c = a + b, \qquad -a < \alpha - 1 < b - 1,$$

or satisfy

$$c < a + b, \qquad -a < \alpha - 1 \le b - 1.$$

We state a corollary which is simply a reformulation of the previous theorems when $c = a + b$.

Corollary 16 *Let $1 \le p < \infty$ and $\alpha > 1$. If $a, b \in \mathbb{R}$ are such that $a + b > 1$, then the following conditions are equivalent:*

(i) the operator $S_{a,b,a+b}$ is bounded on L^p_α;
(ii) the operator $T_{a,b,a+b}$ is bounded on L^p_α;
(iii) the parameters satisfy

$$-pa < \alpha - 1 < p\,(b-1)\,.$$

Let $\beta > 1$. Since $\mathcal{A}^2_\beta$ is a closed subspace of L^2_β, there exists an orthogonal projection $P_\beta \colon L^2_\beta \to \mathcal{A}^2_\beta$. Observe that by the reproducing property of $K_{\beta,z} = K_\beta(z,\cdot)$, $z \in X$, we can write the projection $P_\beta f$ of $f \in L^2_\beta$ as follows

$$P_\beta f(z) = \langle P_\beta f, K_{\beta,z}\rangle_{\mathcal{A}^2_\beta} = \langle f, P_\beta K_{\beta,z}\rangle_{L^2_\beta} = \langle f, K_{\beta,z}\rangle_{L^2_\beta},$$

where we used the orthogonality of P_β. Hence we can rewrite P_β as the integral operator on L^2_β associated to the reproducing kernel K_β, that is

$$P_\beta f(z) = \sum_{x \in X} K_\beta(z,x) f(x) q^{-\beta|x|}, \qquad f \in L^2_\beta,\ z \in X.$$

Since μ_β is finite, $L^p_\beta \subseteq L^2_\beta$ whenever $p \ge 2$. It is then natural to investigate whether the restriction of P_β to L^p_β is bounded. Furthermore, when $1 \le p < 2$ one has $L^2_\beta \subsetneq L^p_\beta$ and we shall study whether P_β admits a bounded extension to L^p_β. A more general question that we want to answer is whether the integral operator $\mathcal{K}^\alpha_\beta$, $\alpha > 1$, with kernel $K_\beta(z,x)q^{(\alpha-\beta)|x|}$ with respect to the measure μ_α, that is

$$\mathcal{K}^\alpha_\beta f(z) = \sum_{x\in X} K_\beta(z,x)f(x)q^{(\alpha-\beta)|x|}q^{-\alpha|x|}, \qquad f \in L^p_\alpha \cap L^2_\beta,\ z \in X, \tag{22}$$

extends to a bounded operator from L^p_α to $\mathcal{A}^p_\alpha$. The following result answers the above questions.

Theorem 17 *Let $1 \le p < \infty$, $\alpha, \beta > 1$. The operator $\mathcal{K}^\alpha_\beta$ extends to a bounded operator from L^p_α to $\mathcal{A}^p_\alpha$ if and only if*

$$p(\beta - 1) > \alpha - 1.$$

In particular, P_α is bounded on L^p_α if and only if $p > 1$.

Proof It is sufficient to observe that from Eq. 22, $\mathcal{K}^\alpha_\beta = T_{0,\beta,\beta}$ on $L^p_\alpha \cap L^2_\beta$. Hence, the result follows from Corollary 16. □

Remark 18 It is worthwhile observing that the unboundedness of P_α on L^1_α may be seen directly with the following example. We make use of Lemma 23 that will be proved in the next subsection.
For every $n \in \mathbb{N}$, we fix a vertex v_n in $S(o,n)$, and define

$$f_n(x) = \mathbb{1}_{\{v_n\}}(x)q^{\alpha|x|}, \qquad x \in X.$$

Clearly, $\|f_n\|_{L^1_\alpha} = 1$ and $f_n \in L^2_\alpha$. Hence, $P_\alpha f_n(z) = K_\alpha(z, v_n)$ and by Lemma 23

$$\|P_\alpha f_n\|_{L^1_\alpha} = \sum_{z\in X} |K_\alpha(z, v_n)|q^{-\alpha|z|} \gtrsim |v_n| = n.$$

This shows that P_α does not admit a bounded extension to L^1_α.

As a direct application of Theorem 17, we deduce the following result on the dual of Bergman spaces.

Corollary 19 *Let $1 < p < \infty$ and $\alpha > 1$. Then*

$$(\mathcal{A}^p_\alpha)^* = \mathcal{A}^{p'}_\alpha,$$

where $1 < p' < \infty$ is such that $\frac{1}{p} + \frac{1}{p'} = 1$, with equivalent norms under the pairing

$$\langle f, g\rangle_{\mathcal{A}^p_\alpha \times \mathcal{A}^{p'}_\alpha} = \sum_{z\in X} f(z)g(z)q^{-\alpha|z|} \qquad f \in \mathcal{A}^p_\alpha,\ g \in \mathcal{A}^{p'}_\alpha. \tag{23}$$

Proof Let $g \in \mathcal{A}^{p'}_\alpha$. By Hölder inequality we have that

$$|\langle f, g\rangle_{\mathcal{A}^p_\alpha \times \mathcal{A}^{p'}_\alpha}| \le \|g\|_{\mathcal{A}^{p'}_\alpha}\|f\|_{\mathcal{A}^p_\alpha},$$

for every $f \in \mathcal{A}^p_\alpha$ so that g defines a functional in $(\mathcal{A}^p_\alpha)^*$. Conversely, for $\Phi \in (\mathcal{A}^p_\alpha)^*$, by the Hahn-Banach theorem, there exists $\tilde{\Phi} \in (L^p_\alpha)^*$ such that $\tilde{\Phi}|_{\mathcal{A}^p_\alpha} = \Phi$ and $\|\Phi\|_{(\mathcal{A}^p_\alpha)^*} \ge \|\tilde{\Phi}\|_{(L^p_\alpha)^*}$. Then, there exists $h \in L^{p'}_\alpha$ such that

$$\Phi(f) = \tilde{\Phi}(f) = \langle f, h\rangle_{L^p_\alpha \times L^{p'}_\alpha},$$

for every $f \in \mathcal{A}_\alpha^p$. By the orthogonality of P_α and Theorem 17,

$$\Phi(f) = \langle P_\alpha f, P_\alpha h\rangle_{\mathcal{A}_\alpha^p \times \mathcal{A}_\alpha^{p'}} = \langle f, P_\alpha h\rangle_{\mathcal{A}_\alpha^p \times \mathcal{A}_\alpha^{p'}}.$$

Hence Φ corresponds to $P_\alpha h \in \mathcal{A}_\alpha^2$ under the pairing Eq. 23. □

3.1 Proofs of Theorems 14 and 15

This subsection is devoted to the proofs of Theorems 14 and 15, splitting them up in various steps. In both statements it is obvious that (i) implies (ii). For the rest of the section α, a, b, c denote real parameters with $c > 1$.

3.1.1 Proof that (ii) implies (iii)

In this subsection we suppose that the operator $T_{a,b,c}$ is bounded on L_α^p and we deduce necessary conditions on the parameters a, b, c, α.

Proposition 20 *Let $1 \leq p < \infty$. If $T_{a,b,c} f \in L_\alpha^p$ for every $f \in L_\alpha^p$, then $-pa < \alpha - 1$.*

Proof Consider, for $x \in X$, $f(x) = q^{-R|x|}$ with $R \in \mathbb{R}$ such that

$$R > \max\left\{\frac{1-\alpha}{p}, 1-b\right\}.$$

Since $Rp > 1 - \alpha$ we have that $f \in L_\alpha^p$ and for every $z \in X$

$$\begin{aligned} T_{a,b,c} f(z) &= q^{-a|z|} \sum_{x \in X} K_c(z, x) q^{-(b+R)|x|} \\ &= q^{-a|z|} \sum_{n=0}^{+\infty} q^{-(b+R)n} \sum_{|x|=n} K_c(z, x) \\ &= q^{-a|z|} \sum_{n=0}^{+\infty} q^{-(b+R)n} \# S(o, n) K_c(z, o) \end{aligned}$$

by Eq. 3 applied to the harmonic function $K_c(z, \cdot)$. Hence, since $R > 1 - b$,

$$T_{a,b,c} f(z) = q^{-a|z|} \frac{1}{B_c} \left[1 + \frac{q+1}{q} \sum_{n=1}^{+\infty} q^{(-b-R+1)n}\right] = \frac{B_{b+R}}{B_c} q^{-a|z|}, \qquad z \in X.$$

Now observe that $T_{a,b,c} f \in L_\alpha^p$ implies

$$\sum_{z \in X} q^{-(ap+\alpha)|z|} = 1 + \frac{q+1}{q} \sum_{n=1}^{+\infty} q^{(1-ap-\alpha)n} < +\infty,$$

which holds if and only if $-pa < \alpha - 1$, as required. □

From now on we write, for $1 \leq p < \infty$

$$\|e_{v,j}\|_p = \left(\sum_{y \in s(v)} |e_{v,j}(y)|^p\right)^{1/p}, \qquad v \in X,\ j \in I_v.$$

Proposition 21 *Let* $1 \leq p < \infty$. *If* $T_{a,b,c}$ *is bounded on* L^p_α, *then* $a + b \geq c$.

Proof Fix $R \in \mathbb{R}$ such that

$$R > \max\left\{\frac{1-\alpha}{p}, c - b\right\}.$$

For every $v \in X \setminus \{o\}$ and $j \in I_v$, we define $g_{v,j}(x) = f_{v,j}(x)q^{-R|x|}$, $x \in X$, where $f_{v,j} \in \mathcal{B}$ are defined in Eq. 9. Since $R > \frac{1-\alpha}{p}$, we have that $g_{v,j} \in L^p_\alpha$. Thus,

$$\begin{aligned} T_{a,b,c}g_{v,j}(z) &= q^{-a|z|}\sum_{x\in X} K_c(z,x) f_{v,j}(x)q^{-(b+R)|x|} \\ &= q^{-a|z|}\langle f_{v,j}, K_{c,z}\rangle_{L^2_{b+R}} \end{aligned}$$

since $K_{c,z} \in L^2_c \subseteq L^2_{b+R}$ because $R > c - b$. Now we use the decomposition Eq. 14 of $K_{c,z}$ on the orthonormal basis of $\mathcal{A}^2_c$ and obtain

$$\begin{aligned} \langle K_{c,z}, f_{v,j}\rangle_{L^2_{b+R}} &= \langle \frac{1}{B_c} + \sum_{u\in X}\sum_{k\in I_u} \frac{\overline{f_{u,k}(z)} f_{u,k}}{b_{c,|u|}}, f_{v,j}\rangle_{L^2_{b+R}} \\ &= \frac{\overline{f_{v,j}(z)}}{b_{c,|v|}}\langle f_{v,j}, f_{v,j}\rangle_{L^2_{b+R}} \\ &= \frac{b_{b+R,|v|}}{b_{c,|v|}}\overline{f_{v,j}(z)}, \end{aligned}$$

where we use the orthogonality of $\mathcal{B}$ and Eq. 12. The norm of $T_{a,b,c}g_{v,j}$ in L^p_α is

$$\begin{aligned} \|T_{a,b,c}g_{v,j}\|^p_{L^p_\alpha} &= \left(\frac{b_{b+R,|v|}}{b_{c,|v|}}\right)^p \sum_{z\in X} |f_{v,j}(z)|^p q^{-(ap+\alpha)|z|} \\ &= \left(\frac{b_{b+R,|v|}}{b_{c,|v|}}\right)^p \sum_{n=0}^{+\infty} q^{-(ap+\alpha)n} \sum_{|z|=n} |f_{v,j}(z)|^p. \end{aligned}$$

Since $\operatorname{supp}(f_{v,j}) \subseteq T_v \setminus \{v\}$, the sum of $|f_{v,j}|^p$ on the sphere $S(o,n)$ vanishes for every $n \leq |v|$. If $n > |v|$, then the sum is on $S(o,n) \cap T_v$ and if $z \in T_v$ is such that $|z| = n$ then $p^{|z|-|v|-1}(z)$ is the unique vertex in $s(v)$ such that z lies in its sector. Hence by Eq. 9 we have

$$\begin{aligned} \sum_{|z|=n} |f_{v,j}(z)|^p &= \sum_{\substack{|z|=n \\ z\in T_v}} |e_{v,j}(p^{|z|-|v|-1}(z))|^p a^p_{n-|p^{|z|-|v|-1}(z)|} \\ &= a^p_{n-|v|-1} \sum_{\substack{|z|=n \\ z\in T_v}} |e_{v,j}(p^{|z|-|v|-1}(z))|^p \\ &= a^p_{n-|v|-1} q^{n-|v|-1} \sum_{y\in s(v)} |e_{v,j}(y)|^p \\ &= a^p_{n-|v|-1} q^{n-|v|-1} \|e_{v,j}\|^p_p. \end{aligned}$$

For simplicity, for every $s \in \mathbb{R}$ and $1 \leq p < \infty$, we put

$$C(s,p) := \sum_{m=1}^{+\infty} q^{(1-s)m-1} a^p_{m-1},$$

which is finite whenever $s > 1$. The above computation yields

$$
\begin{aligned}
\|T_{a,b,c}g_{v,j}\|_{L^p_\alpha}^p &= \left(\frac{b_{b+R,|v|}}{b_{c,|v|}}\right)^p \sum_{n=|v|+1}^{+\infty} q^{-(ap+\alpha)n} a_{n-|v|-1}^p q^{n-|v|-1} \|e_{v,j}\|_p^p \\
&= \|e_{v,j}\|_p^p \left(\frac{b_{b+R,|v|}}{b_{c,|v|}}\right)^p \sum_{m=1}^{+\infty} q^{-(ap+\alpha)(m+|v|)} a_{m-1}^p q^{m-1} \\
&= \|e_{v,j}\|_p^p \left(\frac{b_{b+R,|v|}}{b_{c,|v|}}\right)^p q^{-(ap+\alpha)|v|} \sum_{m=1}^{+\infty} q^{(1-(ap+\alpha))m-1} a_{m-1}^p \\
&= \|e_{v,j}\|_p^p \, C(ap+\alpha, p) \left(\frac{b_{b+R,|v|}}{b_{c,|v|}}\right)^p q^{-(ap+\alpha)|v|},
\end{aligned}
$$

where $C(ap+\alpha, p)$ converges because $ap+\alpha > 1$, by Proposition 20. Furthermore,

$$
\begin{aligned}
\|g_{v,j}\|_{L^p_\alpha}^p &= \sum_{x\in X} |f_{v,j}(x)|^p q^{-(Rp+\alpha)|x|} \\
&= \sum_{n=0}^{+\infty} q^{-(Rp+\alpha)n} \sum_{|x|=n} |f_{v,j}(x)|^p \\
&= \sum_{n=|v|+1}^{+\infty} q^{-(Rp+\alpha)n} a_{n-|v|-1}^p q^{n-|v|-1} \|e_{v,j}\|_p^p \\
&= \|e_{v,j}\|_p^p q^{-(Rp+\alpha)|v|} \sum_{m=1}^{+\infty} q^{(1-(Rp+\alpha))m-1} a_{m-1}^p \\
&= \|e_{v,j}\|_p^p \, C(Rp+\alpha, p) q^{-(Rp+\alpha)|v|},
\end{aligned}
$$

where $C(Rp+\alpha, p) \to 1$ when $R \to +\infty$. From the boundedness of $T_{a,b,c}$ and by Eq. 20, it follows that for every $v \in X \setminus \{o\}$:

$$
\begin{aligned}
\frac{\|T_{a,b,c}g_{v,j}\|_{L^p_\alpha}^p}{\|g_{v,j}\|_{L^p_\alpha}^p} &\simeq \left(\frac{b_{b+R,|v|}}{b_{c,|v|}}\right)^p q^{-(ap+\alpha-Rp-\alpha)|v|} \\
&\simeq q^{-p(R+b-c)|v|} q^{-(ap-Rp)|v|} \\
&= q^{(c-a-b)p|v|},
\end{aligned}
$$

which is bounded if and only if $c \leq a+b$. □

Proposition 22 *Let $1 < p < \infty$. If $T_{a,b,c}$ is bounded on L^p_α, then $\alpha - 1 < p(b-1)$.*

Proof The boundedness of $T_{a,b,c}$ on L^p_α is equivalent to the boundedness of the adjoint operator $T^*_{a,b,c}$ on $L^{p'}_\alpha$. It is easy to see that

$$
T^*_{a,b,c}g(x) = q^{-(b-\alpha)|x|} \sum_{z\in X} K_c(x,z) g(z) q^{-(a+\alpha)|z|} = T_{b-\alpha, a+\alpha, c}\, g(x), \qquad g \in L^{p'}_\alpha.
$$

Hence, the fact that $T^*_{a,b,c}$ is bounded on $L^{p'}_\alpha$ implies, by Proposition 20, that $-p'(b-\alpha) < \alpha - 1$, that is $\alpha - 1 < p(b-1)$. □

Propositions 20, 21, 22 show that (ii) implies (iii) in Theorem 14. Now we focus on the case $p = 1$, and we prove that (ii) implies (iii) in Theorem 15.

Lemma 23 *Let $\alpha > 1$. Then:*

$$\sum_{z\in X} |K_\alpha(x,z)|q^{-\alpha|z|} \gtrsim |x|, \qquad x \in X.$$

Proof The case $x = o$ is trivial. For every $x \in X \setminus \{o\}$, we put $\{v_t\}_{t=0}^{|x|} = [o, x]$. Then, by Eq. 19

$$\begin{aligned}
\sum_{z\in X}|K_\alpha(x,z)|q^{-\alpha|z|} &\geq \sum_{t=1}^{|x|} |K_\alpha(x, v_t)|q^{-\alpha t} \\
&= \sum_{t=1}^{|x|}\left(\frac{1}{B_\alpha} + \frac{q^2}{(q+1)^2}\sum_{v\in X}\frac{1}{b_{\alpha,|v|}}\Gamma(v, v_t, x)(1-q^{|v|-t})(1-q^{|v|-|x|})\right)q^{-\alpha t} \\
&\gtrsim b_{\alpha,o}^{-1}\sum_{t=1}^{|x|}\sum_{v\in X} q^{\alpha(|v|-t)}\Gamma(v, v_t, x)(1-q^{|v|-t})(1-q^{|v|-|x|}) \\
&= \sum_{t=1}^{|x|}\sum_{\ell=0}^{t-1} q^{\alpha(\ell-t)}\Gamma(v_\ell, v_t, x)(1-q^{\ell-t})(1-q^{\ell-|x|}) \\
&\gtrsim \sum_{t=1}^{|x|}\sum_{\ell=0}^{t-1} q^{\alpha(\ell-t)} \simeq \sum_{t=1}^{|x|} q^{-\alpha t}q^{\alpha t} = |x|,
\end{aligned}$$

where we used the fact that $\mathrm{supp}(\Gamma(\cdot, v_t, x)) = [o, v_{t-1}] = [v_0, v_{t-1}]$ and the function is greater than or equal to $\frac{q-1}{q}$ there. $\square$

Proposition 24 *If $T_{a,b,c}$ is bounded on L^1_α, then*

$$\begin{aligned}
&\alpha < b, \quad \textit{when } c = a+b; \\
&\alpha \leq b, \quad \textit{when } c < a+b.
\end{aligned}$$

Proof From Proposition 21, if $T_{a,b,c}$ is bounded on L^1_α, then $c \leq a + b$. The boundedness of $T_{a,b,c}$ on L^1_α implies the boundedness of the adjoint operator $T^*_{a,b,c}$ on L^∞_α which is given by

$$T^*_{a,b,c}g(x) = q^{-(b-\alpha)|x|}\sum_{z\in X} K_c(x,z)g(z)q^{-(a+\alpha)|z|}, \quad g \in L^\infty_\alpha.$$

In particular, by Eq. 3

$$\begin{aligned}
T^*_{a,b,c}\mathbb{1}_X(x) &= q^{-(b-\alpha)|x|}\sum_{z\in X} K_c(x,z)q^{-(a+\alpha)|z|} \\
&= q^{-(b-\alpha)|x|}\sum_{n=0}^{+\infty} q^{-(a+\alpha)n}\sum_{|z|=n} K_c(x,z) \\
&= q^{-(b-\alpha)|x|}\frac{1}{B_c}\sum_{n=0}^{+\infty}\#S(o,n)q^{-(a+\alpha)n} = \frac{B_{a+\alpha}}{B_c}q^{-(b-\alpha)|x|},
\end{aligned}$$

which belongs to L^∞_α if and only if $\alpha \leq b$.

Suppose now that $a+b=c$. We know that $\alpha \leq b$ and we want to prove that $\alpha < b$. Suppose by contradiction that $\alpha = b$. For every $x \in X$ define

$$g_x(z) = \begin{cases} |K_c(z,x)| K_c(z,x)^{-1}, & \text{if } K_c(z,x) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|g_x\|_{L^\infty_\alpha} = 1$ and

$$T^*_{a,b,c} g_x(x) = \sum_{z \in X} |K_c(x,z)| q^{-c|z|} \gtrsim |x|,$$

by Lemma 23. Thus $T^*_{a,b,c}$ is unbounded on L^∞_α and consequently $T_{a,b,c}$ is unbounded on L^1_α for $\alpha = b$ and $c = a+b$. □

Propositions 20, 21, 24 show that (ii) implies (iii) in Theorem 15.

3.1.2 Proof that (iii) implies (i)

We start by stating a technical lemma, which will be useful both in Propositions 26 and 27, that are devoted to prove that (iii) implies (i) in the case $p > 1$ and $p = 1$, respectively.

Lemma 25 *Let $\beta, \gamma > 1$. There exist $C_1, C_2 > 0$ depending only on β and γ such that*

$$\sum_{x \in X} |K_\gamma(z,x)| q^{-\beta|x|} \leq \begin{cases} C_1(1 + q^{-(\beta-\gamma)|z|}), & \text{if } \gamma \neq \beta, \\ C_2(1+|z|), & \text{if } \gamma = \beta. \end{cases}$$

Proof Let $z \in X$ and $\{v_j\}_{j=0}^{|z|} = [o,z]$. We start by applying Eqs. 19 and 20 to the kernel K_γ, obtaining

$$\begin{aligned} &\sum_{x \in X} |K_\gamma(z,x)| q^{-\beta|x|} \\ &\quad = \sum_{x \in X} \left| \frac{1}{B_\gamma} + \frac{q^2 b_{\gamma,0}^{-1}}{(q-1)^2} \sum_{j=0}^{|z \wedge x|} q^{\gamma j} \Gamma(v_j, z, x)(1 - q^{j-|z|})(1 - q^{j-|x|}) \right| q^{-\beta|x|} \\ &\quad \leq \frac{B_\beta}{B_\gamma} + \frac{q^2 b_{\gamma,0}^{-1}}{(q-1)^2} \sum_{x \in X} \sum_{j=0}^{|z \wedge x|} q^{\gamma j} \left| \Gamma(v_j, z, x) \right| (1 - q^{j-|z|})(1 - q^{j-|x|}) q^{-\beta|x|} \\ &\quad \leq \frac{B_\beta}{B_\gamma} + \frac{q^2 b_{\gamma,0}^{-1}}{(q-1)^2} \sum_{x \in X} q^{-\beta|x|} \sum_{j=0}^{|z \wedge x|} q^{\gamma j}, \end{aligned}$$

where we used that $|\Gamma| < 1$ and $|z|, |x| > j$. Now observe that, for every $x \in X$ and $0 \leq \ell < |z|$, $|z \wedge x| = \ell$ is equivalent to $x \in T_{v_\ell} \setminus T_{v_{\ell+1}}$ and $z \wedge x = z$ if and only if $x \in T_z$. Furthermore, we have that

$$|S(o,m) \cap T_{v_\ell} \setminus T_{v_{\ell+1}}| = \begin{cases} 0, & \text{if } m < \ell; \\ 1, & \text{if } m = \ell; \\ (q-1) q^{m-\ell-1}, & \text{if } m > \ell. \end{cases}$$

Hence, we have

$$\begin{aligned}\sum_{x\in X} q^{-\beta|x|}\sum_{j=0}^{|z\wedge x|} q^{\gamma j} &= \sum_{\ell=0}^{|z|-1}\sum_{x\in T_{v_\ell}\setminus T_{v_{\ell+1}}} q^{-\beta|x|}\sum_{j=0}^{\ell} q^{\gamma j} + \sum_{x\in Tz} q^{-\beta|x|}\sum_{j=0}^{|z|} q^{\gamma j}\\ &\simeq \sum_{\ell=0}^{|z|-1}\left(\sum_{j=0}^{\ell} q^{\gamma j}\right)\sum_{m=\ell}^{+\infty} q^{m-\ell}q^{-\beta m} + \sum_{n=|z|}^{+\infty} q^{n-|z|}q^{-\beta n}\sum_{j=0}^{|z|} q^{\gamma j}\\ &\simeq \sum_{\ell=0}^{|z|} q^{(\gamma-1)\ell}\sum_{m=\ell}^{+\infty} q^{(1-\beta)m} \simeq \sum_{\ell=0}^{|z|} q^{(\gamma-\beta)\ell},\end{aligned}$$

where we used that $|T_z \cap S(o,n)| = q^{n-|z|}$ when $n \geq |z|$ and $\beta > 1$. This proves that there exist $C_1, C_2 > 0$ (depending on γ and β) such that the thesis holds true. $\square$

Proposition 26 *Let $1 < p < \infty$. If $a + b \geq c > 1$ and $-pa < \alpha - 1 < p(b-1)$, then $S_{a,b,c}$ is bounded on L^p_α.*

Proof We set

$$H(z,x) = |K_c(z,x)|q^{-a|z|}q^{-(b-\alpha)|x|},$$

so that the operator $S_{a,b,c}$ becomes

$$S_{a,b,c}f(z) = \sum_{x\in X} H(z,x)f(x)q^{-\alpha|x|}.$$

Our purpose is to apply Schur's test (see Theorem 3.6 in [22]) to the integral operator with positive kernel $H\colon X\times X \to [0,+\infty)$. To do so, we have to show that there exists a positive function h on X such that

$$\sum_{z\in X} H(z,x)h(z)^{p'}q^{-\alpha|z|} \lesssim h(x)^{p'}, \qquad \sum_{x\in X} H(z,x)h(x)^{p}q^{-\alpha|x|} \lesssim h(z)^{p}. \tag{24}$$

Observe that the two inequalities assumed for α are equivalent to

$$-\frac{a+\alpha-1}{p} < \frac{a}{p'}, \qquad -\frac{b-1}{p'} < \frac{b-\alpha}{p}.$$

Hence, since $a + b > 1$, it is possible to choose an element

$$\gamma \in \left(-\frac{b-1}{p'}, \frac{a}{p'}\right) \cap \left(-\frac{a+\alpha-1}{p}, \frac{b-\alpha}{p}\right) \neq \emptyset. \tag{25}$$

We want to show that $h(x) = q^{-\gamma|x|}$ satisfies Eq. 24. Let $z \in X$. We suppose $\gamma \neq \frac{c-b}{p'}$. We can apply Lemma 25 since $b + \gamma p' > 1$ by Eq. 25, obtaining

$$\begin{aligned}\sum_{x\in X} H(z,x)h(x)^{p'}q^{-\alpha|x|} &= q^{-a|z|}\sum_{x\in X}|K_c(z,x)|q^{-(b+\gamma p')|x|}\\ &\lesssim q^{-a|z|}(1 + q^{-(b+\gamma p'-c)|z|})\\ &\lesssim q^{-\gamma p'|z|} = h(z)^{p'},\end{aligned}$$

where we used $a + b - c \geq 0$ and $a > \gamma p'$. Similarly, when $\gamma = \frac{c-b}{p'}$ we can apply again Lemma 25 and conclude by using $a > \gamma p'$. On the other hand, we have that if $\gamma \neq \frac{c-a-\alpha}{p}$, by $a+\gamma p+\alpha>0$ and by Lemma 25,

$$\begin{aligned}\sum_{z\in X} H(z,x)h(z)^p q^{-\alpha|z|} &= q^{-(b-\alpha)|x|}\sum_{z\in X}|K_c(z,x)|q^{-(a+\gamma p+\alpha)|z|}\\ &\lesssim q^{-(b-\alpha)|x|}(1+q^{-(a+\gamma p+\alpha-c)|z|})\\ &\lesssim q^{-\gamma p|z|} = h(z)^p,\end{aligned}$$

since $a+b\geq c$ and, by Eq. 25, $b-\alpha>\gamma p$. Similarly when $\gamma=\frac{c-a-\alpha}{p}$.

In conclusion, Eq. 24 holds and by Schur's test the operator $S_{a,b,c}$ is bounded on $L^p_\alpha(X)$. □

Notice that Proposition 26 shows that (iii) implies (i) in Theorem 14.

Proposition 27 *If $a+b\geq c$ and*

$$\begin{aligned} &-a<\alpha-1<b-1, \quad \textit{when } c=a+b;\\ &-a<\alpha-1\leq b-1, \quad \textit{when } c<a+b,\end{aligned}$$

then $S_{a,b,c}$ is bounded on L^1_α.

Proof Let $f\in L^1_\alpha$. We suppose $c\neq a+\alpha$ and we observe that, since $a+\alpha>1$, by Lemma 25

$$\begin{aligned}\|S_{a,b,c}f\|_{L^1_\alpha} &= \sum_{z\in X}\left|\sum_{x\in X}|K_c(z,x)|f(x)q^{-b|x|}\right|q^{-(a+\alpha)|z|}\\ &\leq \sum_{x\in X}|f(x)|q^{-b|x|}\sum_{z\in X}|K_c(z,x)|q^{-(a+\alpha)|z|}\\ &\lesssim \sum_{x\in X}|f(x)|q^{-b|x|}(1+q^{-(a+\alpha-c)|x|})\\ &\lesssim \sum_{x\in X}|f(x)|q^{-\alpha|x|} = \|f\|_{L^1_\alpha},\end{aligned}$$

where we used the fact that $a+b-c\geq 0$ and $b\geq\alpha$. The case $c=a+\alpha$ follows similarly using again Lemma 25 and $b>\alpha$. Hence, $S_{a,b,c}$ is bounded on L^1_α. □

Proposition 27 shows that (iii) implies (i) in Theorem 15.

4 Calderón-Zygmund Decomposition

In this section, we discuss a Calderón-Zygmund decomposition of functions in L^1_α and we formulate the integral Hörmander's condition for kernels on the tree which guarantees the weak type (1,1) boundedness of integral operators which are bounded on L^2_α . As byproduct, we have that P_α is of weak type (1,1) for every $\alpha>1$.

By Proposition 6 the measure metric space (X,d,μ_α) is nondoubling. We now introduce the Gromov distance ρ, see [2, 18], and show that the measure metric space (X,ρ,μ_α) is doubling. For every $u,v\in X$ define

$$\rho(v,u)=\begin{cases}0, & \text{if } u=v;\\ e^{-|v\wedge u|}, & \text{if } v\neq u.\end{cases}$$

For every $v \in X$, observe that if $u \in X \setminus \{v\}$ then $\rho(v,u) = e^{-|v \wedge u|} \in [e^{-|v|}, 1]$ and $|v \wedge u| = -\log(\rho(v,u))$, that is

$$u \in T_{p^{|v|+\log(\rho(v,u))}(v)} \setminus T_{p^{|v|+\log(\rho(v,u))-1}(v)}.$$

Thus, the nontrivial balls with respect to ρ centred at v are sectors of the tree. More in general, we have

$$B_\rho(v,r) := \{u \in X : \rho(v,u) < r\} = \begin{cases} \{v\}, & \text{if } 0 < r \leq e^{-|v|}, \\ T_{p^{|v|+\lfloor \log r \rfloor}(v)}, & \text{if } e^{-|v|} < r \leq 1, \\ X, & \text{if } r > 1. \end{cases} \tag{26}$$

Observe that in the special case $v = o$ we have that $B_\rho(o,r) = \{o\}$ if $0 < r \leq 1$ and $B_\rho(o,r) = X$ for every $r > 1$. Hence every vertex v is the center of exactly $|v| + 2$ balls.

Proposition 28 *For every $\alpha > 1$ the measure metric space (X, ρ, μ_α) is globally doubling with doubling constant*

$$D_\alpha = \max\left\{q^\alpha + 1, \frac{q^\alpha + 1}{q^\alpha - q}\right\},$$

that is

$$\mu_\alpha(B_\rho(v, 2r)) \leq D_\alpha \mu_\alpha(B_\rho(v,r)), \qquad v \in X,\ r > 0.$$

Proof Let $\alpha > 1$. We start by observing that for every $u \in X \setminus \{o\}$

$$\mu_\alpha(T_u) = \sum_{\ell=0}^{+\infty} q^\ell q^{-\alpha(\ell+|u|)} = q^{-\alpha|u|} \frac{1}{1 - q^{1-\alpha}}.$$

Let $0 < r \leq 1$. Observe that if $\{x\} := x - \lfloor x \rfloor \in [0,1)$, then

$$\lfloor \log(2r) \rfloor = \begin{cases} \lfloor \log r \rfloor, & \text{if } 0 \leq \{\log r\} < 1 - \log 2, \\ 1 + \lfloor \log r \rfloor, & \text{if } 1 - \log 2 \leq \{\log r\} < 1. \end{cases}$$

Hence whenever $B_\rho(v,r) = \{v\}$ we have that $B_\rho(v,2r) \in \{\{v\}, T_v\}$, and if $B_\rho(v,r) = T_u$ for some $u \in X \setminus \{o\}$ then $B_\rho(v,2r) \in \{T_u, T_{p(u)}\}$.

If $v \in X \setminus \{o\}$, then

$$\frac{\mu_\alpha(T_v)}{\mu_\alpha(\{v\})} = \frac{q^{-\alpha|v|}(1 - q^{1-\alpha})^{-1}}{q^{-\alpha|v|}} = \frac{1}{1 - q^{1-\alpha}}. \tag{27}$$

If $|v| > 1$, then

$$\frac{\mu_\alpha(T_{p(v)})}{\mu_\alpha(T_v)} = \frac{q^{-\alpha(|v|-1)}(1 - q^{1-\alpha})^{-1}}{q^{-\alpha|v|}(1 - q^{1-\alpha})^{-1}} = q^\alpha. \tag{28}$$

If $|v| = 1$, then

$$\frac{\mu_\alpha(X)}{\mu_\alpha(T_v)} = \frac{(1 + q^{-\alpha})(1 - q^{1-\alpha})^{-1}}{q^{-\alpha}(1 - q^{1-\alpha})^{-1}} = q^\alpha + 1. \tag{29}$$

Finally, we consider the case $v = o$. In this case, it is sufficient to check that, by Eq. 7

$$\frac{\mu_\alpha(X)}{\mu_\alpha(\{o\})} = \frac{1 + q^{-\alpha}}{1 - q^{1-\alpha}} = \frac{q^\alpha + 1}{q^\alpha - q}.$$

Hence (X, ρ, μ_α) is doubling with constant $D_\alpha = \max\{q^\alpha + 1, (q^\alpha + 1)/(q^\alpha - q)\}$. □

As a consequence of Proposition 28 in this setting one can develop a classical Calderón-Zygmund theory using the balls of the Gromov metric, i.e. using sectors (see [12, 21]). Our argument is inspired by [15, Theorem 1.1], where a similar construction is developed in the setting of the hyperbolic disk (see also the Whitney decomposition in [1]). Since it is not difficult to construct an explicit decomposition algorithm for sectors and then describe the associated Calderón-Zygmund decomposition of integrable functions, we think that it is worthwhile discussing this construction in detail, as we do next.

We start with a preliminary geometrical result that allows us to obtain an infinite family of partitions of a sector. In particular, the partition at a given scale is a refinement of the partition at the previous scale, and the measure of a partitioning set is comparable with the measure of the set which contains it in the previous partition.

Lemma 29 *Let $v \in X \setminus \{o\}$. For every $m \in \mathbb{N}$, there exists $I_m \in \mathbb{N}$ and sets $Q_{k,m} \subseteq T_v$ for every $k \in \mathcal{I}_m := \{0, \dots, I_m\}$ such that*

(i) $Q_{k,m} \cap Q_{k',m} = \emptyset$ *for every* $k \neq k'$;
(ii) *the sector* T_v *is the disjoint union of the sets* $Q_{k,m}$, $k \in \mathcal{I}_m$;
(iii) *the partition at scale* $m > 0$ *is a refinement of the partition at scale* $m-1$, *that is, for every* $k' \in \mathcal{I}_{m-1}$ *there exists* $\mathcal{I}_{m,k'} \subseteq \mathcal{I}_m$ *such that*

$$Q_{k',m-1} = \bigsqcup_{k \in \mathcal{I}_{m,k'}} Q_{k,m};$$

(iv) *for every* $k \in \mathcal{I}_m$ *and* $k' \in \mathcal{I}_{m-1}$ *for which* $Q_{k,m} \subseteq Q_{k',m-1}$, *we have*

$$\mu_\alpha(Q_{k,m}) \le \mu_\alpha(Q_{k',m-1}) \le D_\alpha \mu_\alpha(Q_{k,m}).$$

Observe that in (iv) the constant D_α can be replaced by $\max\{q^\alpha, (1-q^{1-\alpha})^{-1}\}$, because we focus only on T_v.

Proof For every $m \in \mathbb{N}$ we set

$$I_m = \frac{q^{m+1} - q}{q-1}.$$

We label the vertices of T_v in such a way that $v_0 = v$ and $s(v_k) = \{v_{qk+\ell} \colon \ell \in \{1, \dots, q\}\}$ for every $k \in \mathbb{N}$. Since $\mathcal{I}_0 = \{0\}$ it is sufficient to set $Q_{0,0} = T_v$. Then for every $m \in \mathbb{N} \setminus \{0\}$ we set

$$\begin{aligned} Q_{k,m} &:= \{v_k\}, && \text{if } k \in \mathcal{I}_{m-1}, \\ Q_{k,m} &:= T_{v_k}, && \text{if } k \in \mathcal{I}_m \setminus \mathcal{I}_{m-1}. \end{aligned}$$

In this way, (i), (ii), and (iii) easily follow by construction. Finally, (iv) follows from Eqs. rapportosettorevertice, rapportosettori, and rapportosettoretutto, and the fact that

$$Q_{k',m-1} \in \begin{cases} \{T_v, T_{p(v)}\}, & \text{if } Q_{k,m} = T_v; \\ \{\{v\}, T_v\}, & \text{if } Q_{k,m} = \{v\}. \end{cases}$$

□

The previous result leads to a Calderón-Zygmund decomposition for integrable functions on the tree at a level $t \in \mathbb{R}^+$ sufficiently large w.r.t the L^1_α-norm of the function.

Proposition 30 *Let $f \in L^1_\alpha$ and $t > \|f\|_{L^1_\alpha}/\mu_\alpha(X)$. There exist two families $\mathcal{Q}$ and $\mathcal{F}$ of disjoint sets of the form $Q_{k,m}$ such that, if we denote by Ω and F the disjoint union of all the sets in $\mathcal{Q}$ and $\mathcal{F}$, respectively, the following properties hold:*

(i) $X = \Omega \sqcup F$;
(ii) $|f(z)| \leq t$ *for every* $z \in F$;
(iii) *there exist* $g, b\colon X \to \mathbb{C}$ *and* $C > 0$ *such that* $f = g + b$, $\operatorname{supp} b \subseteq \Omega$, *and* $\|g\|_{L^2_\alpha}^2 \lesssim t\|f\|_{L^1_\alpha}$. *Moreover, if we set* $b_Q = b\mathbb{1}_Q$ *for every* $Q \in \mathcal{Q}$, *then*

$$\sum_{z\in Q} b_Q(z)q^{-\alpha|z|} = 0, \qquad \sum_{Q\in\mathcal{Q}} \|b_Q\|_{L^1_\alpha} \leq C\|f\|_{L^1_\alpha}, \qquad Q \in \mathcal{Q}.$$

Proof For every $v \in S(o,1)$ we consider the decomposition of the sector T_v given by Lemma 29. We define two families of subsets $\mathcal{Q}_v$ and $\mathcal{F}_v$ following the steps below. Starting from $Q_{k,m} = Q_{0,0} = T_v$:

1) if

$$\frac{1}{\mu_\alpha(Q_{k,m})} \sum_{z\in Q_{k,m}} |f(z)|q^{-\alpha|z|} > t,$$

then we put $Q_{k,m} \in \mathcal{Q}_v$ and we stop. Otherwise,
2a) if $\#Q_{k,m} = 1$ then $Q_{k,m} \in \mathcal{F}_v$ and we stop;
2b) if $\#Q_{k,m} > 1$ then for each set in the family

$$Q_{k,m+1} \cup \{Q_{kq+j,m+1}\colon j \in 1,\dots q\}$$

we repeat the procedure, starting from 1).

We define

$$\mathcal{Q} := \begin{cases} \bigsqcup_{v\in S(o,1)} \mathcal{Q}_v, & \text{if } |f(o)| \leq t; \\ \{o\} \cup \bigsqcup_{v\in S(o,1)} \mathcal{Q}_v, & \text{otherwise,} \end{cases} \qquad \mathcal{F} := \begin{cases} \{o\} \cup \bigsqcup_{v\in S(o,1)} \mathcal{F}_v, & \text{if } |f(o)| \leq t; \\ \bigsqcup_{v\in S(o,1)} \mathcal{F}_v, & \text{otherwise.} \end{cases}$$

We denote by Ω and F the (disjoint) union of all the subsets in $\mathcal{Q}$ and $\mathcal{F}$, respectively. The sets Ω and F clearly satisfy (i) and (ii). We prove that, for every $Q \in \mathcal{Q}$,

$$t < \frac{1}{\mu_\alpha(Q)} \sum_{z\in Q} |f(z)|q^{-\alpha|z|} \leq C_\alpha t, \qquad Q \in \mathcal{Q}. \tag{30}$$

For every $Q \in \mathcal{Q}$ we put

$$\tilde{Q} = \begin{cases} X, & \text{if } Q = \{o\} \text{ or } Q = Q_{0,0} \in \mathcal{Q}_v,\, v \in S(o,1) \\ Q_{k',m-1} & \text{if } Q = Q_{k,m} \in \mathcal{Q}_v,\, m > 0,\, v \in S(o,1), \end{cases}$$

where k' is defined in (iv) of Lemma 29. Observe that $\tilde{Q} \notin \mathcal{Q}$ and that, by Proposition 28, $\mu_\alpha(\tilde{Q}) \leq C_\alpha \mu_\alpha(Q)$. Then we have that

$$\frac{1}{\mu_\alpha(Q)} \sum_{z\in Q} |f(z)|q^{-\alpha|z|} \leq \frac{\mu_\alpha(\tilde{Q})}{\mu_\alpha(Q)} \frac{1}{\mu_\alpha(\tilde{Q})} \sum_{z\in\tilde{Q}} |f(z)|q^{-\alpha|z|} \leq C_\alpha t,$$

which gives Eq. 30. It is easy to see that

$$\mu_\alpha(\Omega) \leq \frac{1}{t} \sum_{Q\in\mathcal{Q}} \sum_{x\in Q} \frac{1}{\mu_\alpha(Q)} \left(\sum_{z\in Q} |f(z)|q^{-\alpha|z|} \right) q^{-\alpha|x|} \leq \frac{\|f\|_{L^1_\alpha}}{t}. \tag{31}$$

We now define $b = f - g$, where

$$g(z) = \begin{cases} f(z), & z \in F; \\ \frac{1}{\mu_\alpha(Q)} \sum_{x \in Q} f(x) q^{-\alpha|x|}, & z \in Q. \end{cases}$$

It is obvious that $\operatorname{supp} b \subseteq \Omega$. We show next that $\|g\|^2_{L^2_\alpha} \leq (1 + C^2_\alpha) t \|f\|_{L^1_\alpha}$. Indeed, by Eq. 30,

$$\begin{aligned} \|g\|^2_{L^2_\alpha} &= \sum_{z \in F} |g(z)|^2 q^{-\alpha|z|} + \sum_{z \in \Omega} |g(z)|^2 q^{-\alpha|z|} \\ &= \sum_{z \in F} |f(z)|^2 q^{-\alpha|z|} + \sum_{Q \in \mathcal{Q}} \sum_{z \in Q} \left| \frac{1}{\mu_\alpha(Q)} \sum_{x \in Q} f(x) q^{-\alpha|x|} \right|^2 q^{-\alpha|z|} \\ &\leq \sum_{z \in F} t|f(z)| q^{-\alpha|z|} + \mu_\alpha(\Omega) C^2_\alpha t^2 \leq (1 + C^2_\alpha) t \|f\|_{L^1_\alpha} < +\infty, \end{aligned}$$

where we used Eq. 31. The fact that $b_Q = b\mathbb{1}_Q$, $Q \in \mathcal{Q}$, has vanishing mean on Q follows by construction. Furthermore, since $|b(z)| \leq |f(z)| + |g(z)|$ we have

$$\begin{aligned} \sum_{Q \in \mathcal{Q}} \sum_{z \in Q} |b_Q(z)| q^{-\alpha|z|} &\leq \sum_{z \in \Omega} |f(z)| q^{-\alpha|z|} + \sum_{Q \in \mathcal{Q}} \sum_{z \in Q} |g(z)| q^{-\alpha|z|} \\ &\leq \|f\|_{L^1_\alpha} + \mu_\alpha(\Omega) C_\alpha t \lesssim \|f\|_{L^1_\alpha}, \end{aligned}$$

by Eq. 31. □

In the doubling measure metric space (X, ρ, μ_α), the standard integral Hörmander's condition (see [19] and formula (10) Ch.I in [21]) for a kernel $K : X \times X \to \mathbb{C}$ is

$$\sup_{v \in X, r > 0} \sup_{x, y \in B_\rho(v, r)} \int_{X \setminus B_\rho(v, 2r)} |K(z, x) - K(z, y)| \mu_\alpha(z) < +\infty.$$

Thanks to the shape of the balls, see Eq. 26, it is equivalent to

$$\sup_{v \in X \setminus \{o\}} \sup_{x, y \in T_v} \sum_{z \in X \setminus T_v} |K(z, x) - K(z, y)| q^{-\alpha|z|} < +\infty. \tag{32}$$

Notice that this is precisely what is proved to hold in Proposition 13 for the Bergman kernel K_α. We then have the following boundedness result for integral operators (see Theorem 3 Ch.I [21]).

Theorem 31 *Fix $\alpha > 1$ and let $K : X \times X \to \mathbb{C}$ be a kernel satisfying the Hörmander's condition Eq. 32 with respect to μ_α. If the integral operator defined on functions $f \in L^2_\alpha$ by*

$$\mathcal{K} f(z) = \sum_{x \in X} K(z, x) f(x) q^{-\alpha|x|}$$

is bounded on L^2_α, then $\mathcal{K}$ is of weak type (1,1). Furthermore, $\mathcal{K}$ admits a bounded extension $\mathcal{K}$ on L^p_α, for every $1 < p < 2$.

The following result is obtained as byproduct of Proposition 13 and Theorem 31. It is a discrete counterpart of the result for (unweighted and holomorphic) Bergman spaces on the hyperbolic disk obtained in [15].

Corollary 32 *The Bergman projector P_α is of weak type $(1, 1)$, for every $\alpha > 1$.*

Acknowledgements The authors are grateful to Marco M. Peloso for suggesting references on the weak type (1,1) boundedness of the Bergman projector in the case of the hyperbolic disk, and to Matteo Levi for useful comments.

Author Contributions Filippo De Mari, Matteo Monti, and Maria Vallarino wrote the main manuscript text. All authors reviewed the manuscript.

Funding Open access funding provided by Politecnico di Torino within the CRUI-CARE Agreement. Open access funding provided by Politecnico di Torino within the CRUI-CARE Agreement.

Declarations

Competing Interests The authors declare no competing interests.

References

1. Arcozzi, N., Rochberg, R., Sawyer, E.: Carleson measures for analytic Besov spaces. Rev. Mat. Iberoamericana **18**(2), 443–510 (2002)
2. Arcozzi, N., Rochberg, R., Sawyer, E., Wick, B.: Potential theory on trees, graphs and Ahlfors-regular metric spaces. Potential Anal. **41**(2), 317–366 (2014)
3. Békollé, D.: Inégalité à poids pour le projecteur de Bergman dans la boule unité de $\mathbb{C}^n$. Studia Math. **71**(3), 305–323 (1982)
4. Békollé, D., Bonami, A.: Inégalités à poids pour le noyau de Bergman. C. R. Acad. Sci. Paris Sér. A-B **286**(18), A775–A778 (1978)
5. Békollé, D., Bonami, A., Garrigós, G., Nana, C., Peloso, M.M., Ricci, F.: Lecture notes on Bergman projectors in tube domains over cones: an analytic and geometric viewpoint. IMHOTEP J. Afr. Math. Pures Appl. **5**, (2004)
6. Boiko, T., Woess, W.: Moments of Riesz measures on Poincaré disk and homogeneous tree-a comparative study. Expo. Math. **33**(3), 353–374 (2015)
7. Cartier, P.: Harmonic analysis on trees. In: Proceedings of Symposia in Pure Mathematics, vol. 26, pp. 419–424 (1973)
8. Cohen, J.M., Colonna, F.: Embeddings of trees in the hyperbolic disk. Complex Variables Theory Appl. **24**(3–4), 311–335 (1994)
9. Cohen, J.M., Colonna, F., Picardello, M., Singman, D.: Bergman spaces and Carleson measures on homogeneous isotropic trees. Potential Anal. **44**, 05 (2016)
10. Cohen, J.M., Colonna, F., Picardello, M., Singman, D.: Fractal functions with no radial limits in Bergman spaces on trees. Hokkaido Math. J. **47**, 269–289 (2018)
11. Cohen, J.M., Colonna, F., Picardello, M., Singman, D.: Carleson measures for non-negative subharmonic functions on homogeneous trees. Potential Anal. **52**(1), 41–67 (2020)
12. Coifman, R.R., Weiss, G.: Extensions of Hardy spaces and their use in analysis. Bull. Amer. Math. Soc. **83**(4), 569–645 (1977)
13. Cowling, M., Meda, S., Setti, A.G.: An overview of harmonic analysis on the group of isometries of a homogeneous tree. Expo. Math. **16**(5), 385–423 (1998)
14. Cowling, M., Meda, S., Setti, A.G.: Estimates for functions of the Laplace operator on homogeneous trees. Trans. Amer. Math. Soc. **352**, 01 (2000)
15. Deng, Y., Huang, L., Zhao, T., Zheng, D.: Bergman projection and Bergman spaces. J. Operator Theory **46**(1), 3–24 (2001)

16. Figà-Talamanca, A., Nebbia, C.: Harmonic analysis and representation theory for groups acting on homogenous trees, vol. 162. Cambridge University Press, (1991)
17. Forelli, F., Rudin, W.: Projections on spaces of holomorphic functions in balls. Indiana Univ. Math. J. **24**(6), 593–602 (1974)
18. Gromov, M.: Metric structures for Riemannian and non-Riemannian spaces. Modern Birkhäuser Classics. Birkhäuser Boston Inc, Boston, MA (2007)
19. Hörmander, L.: Estimates for translation invariant operators in L^p spaces. Acta Math. **104**, 93–140 (1960)
20. Stein, E.M.: Singular integrals and estimates for the Cauchy-Riemann equations. Bull. Amer. Math. Soc. **79**(2), 440–445 (1973)
21. Stein, E.M.: Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. Princeton Mathematical Series, vol. 43. Princeton University Press, Princeton, NJ (1993)
22. Zhu, K.: Operator Theory in Function Spaces. American Mathematical Society, Mathematical surveys and monographs (2007)