POLITECNICO DI TORINO
Repository ISTITUZIONALE

Stability of the Faber-Krahn inequality for the short-time Fourier transform

Original
Stability of the Faber-Krahn inequality for the short-time Fourier transform / Gómez, J., Guerra, A., Ramos, J.P.G., Tilli, P.. - In: INVENTIONES MATHEMATICAE. - ISSN 0020-9910. - STAMPA. - 236:2(2024), pp. 779-836. [10.1007/s00222-024-01248-2]

Availability:
This version is available at: 11583/2987315 since: 2024-03-26T12:25:47Z

Publisher:
Springer

Published
DOI:10.1007/s00222-024-01248-2

Terms of use:

This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

Publisher copyright
Springer postprint/Author's Accepted Manuscript

This version of the article has been accepted for publication, after peer review (when applicable) and is subject to Springer Nature's AM terms of use, but is not the Version of Record and does not reflect post-acceptance improvements, or any corrections. The Version of Record is available online at: http://dx.doi.org/10.1007/s00222-024-01248-2

(Article begins on next page)

26 November 2026

Inventiones mathematicae
https://doi.org/10.1007/s00222-024-01248-2

Stability of the Faber-Krahn inequality for the short-time Fourier transform

Jaime Gómez[1] · André Guerra[2] · João P.G. Ramos[1] · Paolo Tilli[3]

Received: 1 August 2023 / Accepted: 20 February 2024
© The Author(s) 2024

Abstract
We prove a sharp quantitative version of the Faber–Krahn inequality for the short-time Fourier transform (STFT). To do so, we consider a deficit $\delta(f;\Omega)$ which measures by how much the STFT of a function $f \in L^2(\mathbb{R})$ fails to be optimally concentrated on an arbitrary set $\Omega \subset \mathbb{R}^2$ of positive, finite measure. We then show that an optimal power of the deficit $\delta(f;\Omega)$ controls both the L^2-distance of f to an appropriate class of Gaussians and the distance of Ω to a ball, through the Fraenkel asymmetry of Ω. Our proof is completely quantitative and hence all constants are explicit. We also establish suitable generalizations of this result in the higher-dimensional context.

1 Introduction

1.1 Main results

Given a function $g \in L^2(\mathbb{R})$ (called the *window*), the *short-time Fourier transform (STFT)* of a function $f \in L^2(\mathbb{R})$ is usually defined as

$$V_g f(x,\omega) = \int_{\mathbb{R}} e^{-2\pi i t\omega} f(t)\overline{g(x-t)}\,\mathrm{d}t. \tag{1.1}$$

✉ J. Gómez
jaime.gomezramirez@epfl.ch

A. Guerra
andre.guerra@eth-its.ethz.ch

J.P.G. Ramos
joao.ramos@epfl.ch

P. Tilli
paolo.tilli@polito.it

1 Mathematics section, EPFL, Lausanne, Switzerland

2 Institute for Theoretical Studies, ETH Zürich, Zürich, Switzerland

3 Dipartimento di Scienze Matematiche, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

Published online: 01 March 2024

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

This transform plays a distinguished role in different areas of mathematics, including time-frequency analysis [19] and signal processing [30], mathematical physics [29], where it is also known as the *coherent state transform*, and semiclassical and microlocal analysis [21, 37].

From the point of view of time-frequency analysis, the STFT is a measure of the "instantaneous frequency" of the signal f at each point, in analogy to what a music score does. As the notion of "instantaneous frequency" is not well-defined for generic signals, due to the uncertainty principle, the STFT can only concentrate a limited amount of its L^2-norm on any set $\Omega \subset \mathbb{R}^2$ with finite Lebesgue measure $|\Omega|$, and finding explicit bounds in terms of $|\Omega|$ is an important issue in time-frequency analysis. For a general window g, this appears to be extremely challenging and only suboptimal bounds have been obtained: we refer the reader to the work of E. Lieb [28] for what is, to our knowledge, the current best result at this level of generality.

For very regular windows, however, the situation improves. In particular, in the relevant case (extensively studied in the literature also in connection with the spectrum of localization operators in the radially symmetric case, see e.g. [1, 9, 18, 35]) where $g = \varphi$ is the *Gaussian* window

$$\varphi(x) = 2^{1/4} e^{-\pi x^2}, \quad x \in \mathbb{R}, \tag{1.2}$$

a complete solution to this concentration problem has recently been given in [33], thus proving a conjecture from [2] (see also [10]). Denoting by $\mathcal{V} f := V_\varphi f$ the STFT with the Gaussian window φ defined in (1.2), the main result of [33] can be stated as follows:

Theorem A ([33]; Faber-Krahn inequality for the STFT) *If $\Omega \subset \mathbb{R}^2$ is a measurable set with finite Lebesgue measure $|\Omega| > 0$, and $f \in L^2(\mathbb{R}) \setminus \{0\}$ is an arbitrary function, then*

$$\frac{\int_\Omega |\mathcal{V} f(x,\omega)|^2 \, \mathrm{d}x \, \mathrm{d}\omega}{\|f\|^2_{L^2(\mathbb{R})}} \leq 1 - e^{-|\Omega|}. \tag{1.3}$$

Moreover, equality is attained if and only if Ω coincides (up to a set of measure zero) with a ball centered at some $z_0 = (x_0, \omega_0) \in \mathbb{R}^2$ and, at the same time, f is a function of the kind

$$f(x) = c\, \varphi_{z_0}(x), \qquad \varphi_{z_0}(x) := e^{2\pi i \omega_0 x} \varphi(x - x_0), \tag{1.4}$$

for some $c \in \mathbb{C} \setminus \{0\}$.

Note that the optimal functions in (1.4) are scalar multiples of the Gaussian window defined in (1.2), translated and modulated according to the center of the ball Ω.

This result, which improves upon Lieb's uncertainty principle [28], has inspired other subsequent works: [36], where a similar result is extended to the case of Wavelet transforms; [4], where Kulikov used techniques inspired by those of [33] to prove some contractivity conjectures; and [14], where R. Frank uses the same circle of

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

ideas to generalize a series of entropy-like inequalities (see also the recent preprint [15]). We refer the reader to [22–25, 27, 31, 34] and the references therein for further closely related work.

In the present paper we investigate the *stability* of Theorem A: given $\Omega \subset \mathbb{R}^2$ and $f \in L^2(\mathbb{R})$ which are almost optimal, in the sense that they almost saturate inequality (1.3), can we infer (and to what extent) that Ω is close to a ball and that f is close to a function of the form (1.4)? To formulate this question precisely, a crucial point is choosing how to measure *almost optimality* as well as *closeness*.

To measure almost optimality in (1.3) for a pair (f, Ω), we will consider the *combined deficit*

$$\delta(f;\Omega) := 1 - \frac{\int_\Omega |\mathcal{V}f(x,\omega)|^2 \,\mathrm{d}x\,\mathrm{d}\omega}{(1-e^{-|\Omega|})\|f\|^2_{L^2(\mathbb{R})}}, \tag{1.5}$$

while we will use the *Fraenkel asymmetry* of $\Omega \subset \mathbb{R}^2$ to measure its distance to a ball:

$$\mathcal{A}(\Omega) := \inf\left\{ \frac{|\Omega \triangle B(x,r)|}{|\Omega|} :\ |B(x,r)| = |\Omega|\,, r>0, x \in \mathbb{R}^2 \right\}. \tag{1.6}$$

The Fraenkel asymmetry is a natural notion of asymmetry and it is often used to formulate the stability of geometric and functional inequalities, such as the isoperimetric inequality [8, 12, 16, 17] or the Faber–Krahn inequality for the Dirichlet Laplacian [5, 7].

Our main result reads as follows:

Theorem 1.1 (Stability of the Faber-Krahn inequality for the STFT) *There is an explicitly computable constant $C > 0$ such that, for all measurable sets $\Omega \subset \mathbb{R}^2$ with finite measure $|\Omega| > 0$ and all functions $f \in L^2(\mathbb{R})\backslash\{0\}$, we have*

$$\min_{z_0 \in \mathbb{C}, |c| = \|f\|_2} \frac{\|f - c\,\varphi_{z_0}\|_2}{\|f\|_2} \le C\left(e^{|\Omega|}\delta(f;\Omega)\right)^{1/2}. \tag{1.7}$$

Moreover, for some explicit constant $K(|\Omega|)$ we also have

$$\mathcal{A}(\Omega) \le K(|\Omega|)\delta(f;\Omega)^{1/2}. \tag{1.8}$$

Remark 1.2 (Sharpness) In Theorem 1.1 the factor $\delta(f;\Omega)^{1/2}$ in (1.7) and (1.8) cannot be replaced by $\delta(f;\Omega)^\beta$, for any $\beta > 1/2$. Similarly, the dependence on $|\Omega|$ in (1.7) is also sharp, in the sense that factor $e^{|\Omega|/2}$ cannot be replaced by $e^{\beta|\Omega|}$ for any $\beta < 1/2$. We refer to Sect. 6 for proofs of these claims.

Remark 1.3 (Higher dimensions) There is a natural generalization of the $STFT$ to functions $f \in L^2(\mathbb{R}^d)$, for any $d \ge 1$. In Sect. 7 we show that a more general version of Theorem 1.1 holds in all dimensions. It is worth noting that, although $\delta(f;\Omega)^{1/2}$ still controls the distance of f to the set of optimizers, there is a dimensional dependence of this estimate on $|\Omega|$.

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

As observed in [33], if the set Ω is fixed and has finite measure, Theorem A (and consequently also Theorem 1.1) can be interpreted in terms of the well-known *localization operator* [6, 9] defined, in terms of the STFT operator $\mathcal{V}\colon L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$ with Gaussian window, by

$$L_\Omega := \mathcal{V}^* 1_\Omega \mathcal{V}, \qquad L_\Omega\colon L^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

This is a positive trace-class operator, hence its norm coincides with its largest eigenvalue

$$\lambda_1(\Omega) := \max_{f \in L^2(\mathbb{R})\setminus\{0\}} \frac{\langle L_\Omega f, f\rangle}{\|f\|^2_{L^2(\mathbb{R})}} = \max_{f \in L^2(\mathbb{R})\setminus\{0\}} \frac{\int_\Omega |\mathcal{V}f(x,\omega)|^2 \,\mathrm{d}x\,\mathrm{d}\omega}{\int_{\mathbb{R}^2} |\mathcal{V}f(x,\omega)|^2 \,\mathrm{d}x\,\mathrm{d}\omega}. \tag{1.9}$$

In particular, due to the arbitrariness of f, (1.3) entails that

$$\lambda_1(\Omega) \le 1 - e^{-|\Omega|}, \tag{1.10}$$

with equality if and only if Ω is a ball, and so we call (1.10) a Faber–Krahn inequality, by analogy with the Dirichlet Laplacian. Clearly, for any fixed Ω, the functions f_Ω that achieve the maximum in (1.9) (i.e. the eigenfunctions of L_Ω associated with its first eigenvalue $\lambda_1(\Omega)$) are those functions whose STFT is optimally concentrated in that particular set Ω. When Ω is a ball, these eigenfunctions are the functions described in (1.4) and appearing also in (1.7): therefore, specifying Theorem 1.1 to the case where $f = f_\Omega$ is the first eigenfunction of L_Ω, normalized so that $\|f_\Omega\|_{L^2} = 1$, we obtain the following stability result for the first eigenvalue and eigenfunction of localization operators:

Corollary 1.4 *Let $\Omega \subset \mathbb{R}^2$ be a measurable set of positive finite Lebesgue measure, and let $\lambda_1(\Omega)$ be the first eigenvalue of the localization operator L_Ω as in (1.9), with unit-norm eigenfunction f_Ω. Then (1.10) holds true, and*

$$\min_{z_0 \in \mathbb{C}, |c|=1} \|f_\Omega - c\,\varphi_{z_0}\|_2 \le C e^{|\Omega|/2} \left(1 - \frac{\lambda_1(\Omega)}{1 - e^{-|\Omega|}}\right)^{1/2}, \tag{1.11}$$

for some universal (explicitly computable) constant C. Moreover, for some explicit constant $K(|\Omega|)$ we also have

$$\mathcal{A}(\Omega) \le K(|\Omega|) \left(1 - \frac{\lambda_1(\Omega)}{1 - e^{-|\Omega|}}\right)^{1/2}.$$

This result is the analogue of the stability results for the Faber–Krahn inequality for the Dirichlet Laplacian [5, 7, 13]. Note, however, that the stability estimate (1.7) is more general than (1.11), because it holds for arbitrary functions $f \in L^2(\mathbb{R})$ which are not assumed to be eigenfunctions of the localization operator L_Ω. Indeed, the results of Theorem 1.1 are stronger than the available stability results for the Faber–Krahn inequality for the Dirichlet Laplacian also in that, contrarily to [5, 7], our proof of Theorem 1.1 is *quantitative* and does not rely on compactness arguments, as in the

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

penalization method [8]. It is for this reason that the constants in estimates (1.7)–(1.8) can be made *explicit*. Note, moreover, that the set Ω in Theorem 1.1 is not assumed to be smooth; in fact, since $\mathcal{V}f$ is essentially an entire function via the Bargmann transform, we can replace Ω with a suitable super-level set of a holomorphic function, which in Sect. 3 we prove to be very well-behaved (we then use the rigidity of the problem to come back from super-level sets of holomorphic functions to the original set Ω).

We saw in Remark 1.2 that (1.7) is sharp, but whether Corollary 1.4 is sharp as well is a more delicate question. To answer it, one would need to either (i) compute the first eigenfunctions of L_Ω for domains Ω close to a ball, or (ii) given a function f close to the Gaussian φ, construct a domain $\Omega_f \subset \mathbb{R}^2$ such that f is the first eigenfunction of L_Ω. Strategy (i) appears rather difficult: to the best of our knowledge, the eigenfunctions of L_Ω are not known even in the simple case where Ω is an ellipse of small eccentricity; see [1, 9]. Implementing strategy (ii) involves tools essentially disjoint from those of this manuscript and so we decided not to address the question of optimality of Corollary 1.4 here; instead, this is one of the main goals of an upcoming work by the third author [35].

To discuss the main ideas behind the proofs of our results, we now briefly recall some facts and background notions from [33], which we shall use throughout the paper. We point out, however, that the proof of Theorem A in [33] cannot be readily adapted to yield quantitative results such as (1.7) or (1.8). Instead, the proof of these inequalities requires a set of new geometric ideas and estimates in the Fock space, which are the core of the present paper and which (often being of a general character, such as Lemma 2.1 or the results in Sect. 3), are of interest on their own.

1.2 Proof strategy in the Bargmann–Fock space

As shown in [33], energy concentration problems for the STFT can be very cleanly formulated (and dealt with) in terms of the Fock space [39], i.e. the Hilbert space $\mathcal{F}^2(\mathbb{C})$ of all holomorphic functions $F\colon \mathbb{C} \to \mathbb{C}$ for which

$$\|F\|_{\mathcal{F}^2} := \left(\int_{\mathbb{C}} |F(z)|^2 \, e^{-\pi|z|^2} \, \mathrm{d}z \right)^{1/2} < \infty,$$

endowed with the natural scalar product

$$\langle F, G \rangle_{\mathcal{F}^2} = \int_{\mathbb{C}} F(z) \overline{G(z)} \, e^{-\pi|z|^2} \, \mathrm{d}z.$$

Here and throughout, $z = x + iy$ and $\mathrm{d}z = \mathrm{d}x \, \mathrm{d}y$ denotes Lebesgue measure on $\mathbb{C}$, always identified with $\mathbb{R}^2$. This Hilbert space is closely connected to the STFT through the Bargmann transform $\mathcal{B}\colon L^2(\mathbb{R}) \to \mathcal{F}^2(\mathbb{C})$, defined for $f \in L^2(\mathbb{R})$ as

$$\mathcal{B}f(z) = 2^{1/4} \int_{\mathbb{R}} f(t) e^{2\pi t z - \pi t^2 - \frac{\pi}{2} z^2} \, \mathrm{d}t, \quad z \in \mathbb{C}, \tag{1.12}$$

see e.g. [19, Sect. 3.4]. The Bargmann transform is a unitary isomorphism which maps the orthonormal basis of Hermite functions on $\mathbb{R}$ onto the orthonormal basis of

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

$\mathcal{F}^2(\mathbb{C})$ given by the normalized monomials

$$e_k(z) = \left(\frac{\pi^k}{k!}\right)^{1/2} z^k, \quad k = 0, 1, 2, \ldots. \tag{1.13}$$

More importantly for us, the definition of $\mathcal{B}$ encodes the crucial property that

$$\mathcal{V}f(x, -\omega) = e^{\pi i x\omega}\mathcal{B}f(z)e^{-\pi|z|^2/2}, \quad z = x + i\omega,$$

which allows us to express the energy concentration in the time-frequency plane in terms of functions in the Fock space, since

$$\frac{\int_\Omega |\mathcal{V}f(x,\omega)|^2 \,\mathrm{d}x\,\mathrm{d}\omega}{\|f\|_{L^2}^2} = \frac{\int_{\Omega'} |\mathcal{B}f(z)|^2\, e^{-\pi|z|^2}\,\mathrm{d}z}{\|\mathcal{B}f\|_{\mathcal{F}^2}^2}, \tag{1.14}$$

where $\Omega' = \{(x,\omega) : (x,-\omega) \in \Omega\}$. In this new setting, the image via $\mathcal{B}$ of the functions φ_{z_0} defined in (1.4) takes the form

$$\mathcal{B}\varphi_{z_0} = F_{z_0}, \qquad F_{z_0}(z) = e^{-\frac{\pi}{2}|z_0|^2} e^{\pi z\overline{z_0}}, \tag{1.15}$$

and therefore Theorem A can be rephrased in terms of the Fock space as follows, cf. [33, Theorem 3.1]:

Theorem B *If $\Omega \subset \mathbb{R}^2$ is a measurable set with positive and finite Lebesgue measure, and if $F \in \mathcal{F}^2(\mathbb{C}) \setminus \{0\}$ is an arbitrary function, then*

$$\frac{\int_\Omega |F(z)|^2\, e^{-\pi|z|^2}\,\mathrm{d}z}{\|F\|_{\mathcal{F}^2}^2} \le 1 - e^{-|\Omega|}. \tag{1.16}$$

Moreover, equality is attained if and only if Ω coincides (up to a set of measure zero) with a ball centered at some $z_0 \in \mathbb{C}$ and, at the same time, $F = cF_{z_0}$ for some $c \in \mathbb{C} \setminus \{0\}$.

Similarly, we can rephrase Theorem 1.1 over the Bargmann–Fock space, as follows:

Theorem 1.5 (Fock space version of Theorem 1.1) *There is an explicitly computable constant $C > 0$ such that, for all measurable sets $\Omega \subset \mathbb{R}^2$ with positive finite measure and all functions $F \in \mathcal{F}^2(\mathbb{C})\setminus\{0\}$, we have*

$$\min_{\substack{|c|=\|F\|_{\mathcal{F}^2},\\ z_0\in\mathbb{C}}} \frac{\left\|F - cF_{z_0}\right\|_{\mathcal{F}^2}}{\|F\|_{\mathcal{F}^2}} \le C\left(e^{|\Omega|}\delta(F;\Omega)\right)^{1/2}, \tag{1.17}$$

where

$$\delta(F;\Omega) := 1 - \frac{\int_\Omega |F(z)|^2 e^{-\pi|z|^2}\,\mathrm{d}z}{(1-e^{-|\Omega|})\|F\|_{\mathcal{F}^2}^2}. \tag{1.18}$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

Moreover, for some universal explicit constant $K(|\Omega|)$ we also have

$$\mathcal{A}(\Omega) \le K(|\Omega|)\delta(F;\Omega)^{1/2}. \tag{1.19}$$

We warn the reader that, in (1.18), we used the same notation as in (1.5) to denote the Fock-counterpart of the deficit. However, no confusion should arise from this conflict of notation, since we always use an upper-case letter to denote elements F of the Fock space, corresponding to elements f of L^2.

We will provide two different proofs of this theorem, based on a careful study of the real analytic function

$$u_F(z) = u(z) := |F(z)|^2\, e^{-\pi|z|^2} \tag{1.20}$$

and the properties of its super-level sets

$$A_t := \{u > t\} = \{z \in \mathbb{C} \colon u(z) > t\}, \tag{1.21}$$

where F is an arbitrary function in $\mathcal{F}^2(\mathbb{C}) \setminus \{0\}$. This study was initiated in [33], where it was proved that the distribution function

$$\mu_F(t) = \mu(t) := |A_t|, \quad t \ge 0 \tag{1.22}$$

is locally absolutely continuous on $(0,\infty)$ and satisfies

$$\mu'(t) \le -\frac{1}{t} \quad \text{for a.e. } t \in (0,T), \qquad T := \max_{z\in\mathbb{C}} u(z), \tag{1.23}$$

from which one readily obtains that

$$\mu(t) \ge \log_+ \frac{T}{t} \quad \text{for all } t > 0, \qquad \text{where } \log_+ x := \max\{0, \log x\}. \tag{1.24}$$

Notice that, when $F = cF_{z_0}$ as in the last part of Theorem B, then $T = |c|^2$ and $\mu(t) = \log_+ T/t$. In [33], (1.23) can be found in the equivalent form

$$u^*(s) + (u^*)'(s) \ge 0, \text{ for almost every } s \ge 0, \tag{1.25}$$

where $u^* \colon \mathbb{R}^+ \to (0,T]$ is the decreasing rearrangement of u, usually defined as

$$u^*(s) := \sup\{t \ge 0 \colon \mu(t) > s\}, \qquad s \ge 0. \tag{1.26}$$

The function u^* is proved to be invertible, with $\mu|_{(0,T]}$ as inverse function (see [4] for a direct usage of (1.23) in this form). This fact enables one to find, for any number $s \ge 0$, a unique super-level set $A_t = A_{u^*(s)}$ of measure s, which is the set where u is most concentrated among all sets of measure s, namely

$$I_F(s) = I(s) := \int_{\{u > u^*(s)\}} u(z)\,\mathrm{d}z \ge \int_\Omega u(z)\,\mathrm{d}z, \quad \text{whenever } |\Omega| = s. \tag{1.27}$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

Based on (1.25), it was proved in [33] that the function $G(\sigma) := I(-\log\sigma)$ is *convex* on $[0, 1]$. Since

$$G(0) = \lim_{s\to\infty} I(s) = \int_{\mathbb{C}} u(z)\,\mathrm{d}z = \|F\|^2_{\mathcal{F}^2}, \qquad G(1) = I(0) = 0,$$

the convexity of G yields the upper bound $G(\sigma) \le \|F\|^2_{\mathcal{F}^2}(1-\sigma)$ or, equivalently,

$$I(s) \le \|F\|^2_{\mathcal{F}^2}(1 - e^{-s}), \tag{1.28}$$

which, combined with (1.27), proves (1.16).

It was then observed in [33] that, if equality holds in (1.16), then by convexity we must have $G(\sigma) \equiv \|F\|^2_{\mathcal{F}^2}(1-\sigma)$ on $[0, 1]$ or, equivalently, $I(s) = \|F\|^2_{\mathcal{F}^2}(1 - e^{-s})$ for every $s \ge 0$, and in particular

$$I'(0) = \|F\|^2_{\mathcal{F}^2}. \tag{1.29}$$

But since $\mathcal{F}^2(\mathbb{C})$ is a Hilbert space with reproducing kernel $K_w(z) = e^{\frac{\pi}{2}|w|^2} F_w(z)$, we have

$$|F(z)|^2 e^{-\pi|z|^2} \le \|F\|^2_{\mathcal{F}^2} \tag{1.30}$$

for all $F \in \mathcal{F}^2(\mathbb{C})$, with equality at some $z = z_0$ if and only if $F = cF_{z_0}$ for some $c \in \mathbb{C}$ (see e.g. [33, Proposition 2.1]). Since in any case $I'(0) = T := \max_{z\in\mathbb{C}} |F(z)|^2 e^{-\pi|z|^2}$, (1.29) shows that equality in (1.16) forces equality (for at least one z) also in (1.30), and this proves the last part of Theorem B.

In the rest of this introduction (and also in Sect. 2) we assume without loss of generality the normalization condition $\|F\|_{\mathcal{F}^2} = 1$. A simple but fundamental observation to both our proofs of Theorem 1.5 is that equality in (1.30) can be precisely quantified: indeed,

$$\min_{\substack{z_0\in\mathbb{C}\\ |c|=1}} \|F - cF_{z_0}\|^2_{\mathcal{F}^2} = 2(1-\sqrt{T}) \le 2(1-T), \tag{1.31}$$

cf. Lemma 2.5 below. Thus, to prove estimate (1.17) in Theorem 1.5, we need to show that the deficit controls $(1-T)$.

Our first proof of Theorem 1.5 is based on a careful study of the area between the graphs of $s \mapsto u^*(s)$ and $s \mapsto e^{-s}$. Consider a parameter $s^* > 0$, defined to be a solution of the equation

$$u^*(s^*) = e^{-s^*}.$$

Such a solution always exists and, as soon as $T < 1$, it is unique. An argument relying on the convexity inequality (1.25) yields

$$\int_0^{s^*} \left(e^{-s} - u^*(s)\right)\mathrm{d}s \le e^{|\Omega|}\delta, \qquad \delta := \delta(F; \{u > u^*(|\Omega|)\}), \tag{1.32}$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

cf. Lemma 2.3. Thus, to prove the desired stability estimate (1.17), by (1.31) and (1.32) it is enough to show that the integral above controls $(1-T)$. In fact, it is not difficult to see that this integral controls $(1-T)$ to a *suboptimal* power, as we have

$$\frac{(1-T)^2}{2} = \int_0^{s_*} (1-s-T)_+ \, \mathrm{d}s \leq \int_0^{s_*} \left(e^{-s} - u^*(s)\right) \mathrm{d}s \leq e^{|\Omega|}\delta. \tag{1.33}$$

Thus, by (1.31), (1.33) already yields a suboptimal form of stability.

To upgrade (1.33) to an *optimal* estimate, we need to estimate the integral in (1.32) much more precisely, and our approach is to give a precise quantification of the equality cases in (1.24). By passing to the inverse functions we have

$$\int_{e^{-s^*}}^{T} \left(\log\frac{1}{t} - \mu(t)\right) \mathrm{d}t \leq \int_0^{s_*} \left(e^{-s} - u^*(s)\right) \mathrm{d}s, \tag{1.34}$$

cf. (2.50) below, and our proof proceeds by establishing a sharp estimate for the distribution function $\mu(t)$: precisely, there is a universal constant $C > 0$ such that

$$\mu(t) \leq (1 + C(1-T)) \log\frac{T}{t}, \tag{1.35}$$

provided that t and T are sufficiently close to 1 (see Lemma 2.1); in this paper, C always denotes a *universal constant*, which however may change from line to line. Note that, by the suboptimal estimate (1.33), this restriction on t, T does not restrict generality. Establishing (1.35) is the most delicate part of the whole argument, as this estimate relies on a cancellation effect due to analyticity of F. The desired estimate (1.17) then follows by an elementary analysis, after plugging in (1.35) into (1.34) and using again (1.31) and (1.32).

Concerning the stability of the set in (1.19), we note that it is not clear how to quantify inequality (1.27) used in the proof of Theorem B described above, at least for general sets Ω. Nonetheless, since we already have estimate (1.17), we know that u is close to a Gaussian. This allows us to first compare Ω with $A_{u^*(|\Omega|)}$, and then compare $A_{u^*(|\Omega|)}$ with a ball.

The described strategy also works to show the stability of a similar Faber-Krahn inequality for *wavelet transforms* (see [36]), after adapting the current arguments. We plan to address this in a future work.

1.3 The geometry of super-level sets and a variational approach

As mentioned above, we will give two different proofs of Theorem 1.5, the first one having been described in the previous subsection. We now describe our second proof, which is variational in nature and based on the following result, which is of independent interest:

Proposition 1.6 *There are small explicit constants $\delta_0, c > 0$ such that the following holds: for all $F \in \mathcal{F}^2(\mathbb{C})$ such that*

$$e^s \delta(F; A_{u^*(s)}) \leq \delta_0$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

and for all $s < c\log(1/\delta_0)$, *the super-level set*

$$A_{u^*(s)} = \{z \in \mathbb{C} : u(z) > u^*(s)\}$$

has smooth boundary and convex closure.

Proposition 1.6 shows in particular that level sets of u sufficiently close to its maximum can be seen as smooth graphs over a circle, thus they can be deformed to a circle through an appropriate flow. This observation, in turn, allows us to give a variational approach to Theorem 1.5, in the spirit of Fuglede's computation [16] for the quantitative isoperimetric inequality. We refer the reader to [20] for a detailed introduction to variational methods in shape-optimization problems.

To be precise, and comparing with (1.28), for some fixed $s > 0$ we consider the functional

$$\mathcal{K}\colon F \mapsto \frac{I_F(s)}{\|F\|^2_{\mathcal{F}^2}}.$$

We study perturbations of $F_0 \equiv 1$, i.e. we consider $F = 1 + \varepsilon G$ for some small $\varepsilon > 0$. Taking $\Omega = A_{u^*(s)}$, we note that by a formal Taylor expansion we have

$$\begin{aligned}\delta(1+\varepsilon G;\Omega) &= \frac{1}{1-e^{-|\Omega|}}\big(\mathcal{K}[1] - \mathcal{K}[1+\varepsilon G]\big)\\ &\geq \frac{1}{1-e^{-|\Omega|}}\Big(-\frac{\varepsilon^2}{2}\nabla^2\mathcal{K}[1](G,G) + o(\varepsilon^2)\Big),\end{aligned}$$

since $\nabla\mathcal{K}[1](G) = 0$ for all $G \in \mathcal{F}^2(\mathbb{C})$ satisfying the orthogonality conditions

$$\langle 1, G\rangle_{\mathcal{F}^2} = \langle z, G\rangle_{\mathcal{F}^2} = 0,$$

according to Theorem 5.1 and Lemma 5.2. Thus, once the Taylor expansion above has been justified (and this is achieved in Appendix A), we see that for small perturbations of $F_0 \equiv 1$ the deficit is governed by the second variation of $\mathcal{K}$. For stability to hold, this variation ought to be uniformly negative definite, since ε is essentially the left-hand side in (1.17). In Proposition 5.3 we show that, under the above orthogonality conditions, we have

$$\frac{1}{2}\nabla^2\mathcal{K}[1](G,G) \leq -se^{-s}\|G\|^2_{\mathcal{F}^2}. \tag{1.36}$$

This inequality is interesting for several reasons. Firstly, it is sharp, as highlighted by taking $G(z) = z^2$. Secondly, by the suboptimal stability result (1.33), to prove (1.17) it is enough to consider functions with small deficit. Therefore, the above Taylor expansion, combined with (1.36), easily yields the stability estimate (1.17), although with a *suboptimal* dependence of the constant on $|\Omega|$. Finally, the non-degeneracy of $\nabla^2\mathcal{K}$ provided by (1.36), combined once again with the above Taylor expansion, shows that the deficit behaves quadratically near $F_0 \equiv 1$, which leads to a direct proof of the optimality of our estimates, as claimed in Remark 1.2.

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

1.4 Outline

In Sect. 2 we give a first proof of (1.17), following the strategy described in Sect. 1.2 above. In Sect. 3 we study the geometry of the super-level sets of functions with small deficit and, in particular, we prove Proposition 1.6 above. In Sect. 4 we prove the set stability estimate (1.8). Section 5 contains the variational proof described in Sect. 1.3 and in particular the proof of (1.36). In Sect. 6 we prove the claims from Remark 1.2. Finally, in Sect. 7 we extend our results to the higher-dimensional setting, as claimed in Remark 1.3.

2 First proof of the function stability part

The goal of this section is to prove (1.17), by combining a series of new results (potentially of independent interest) valid for arbitrary functions $F \in \mathcal{F}^2$, which for convenience will be assumed to be normalized by

$$\|F\|_{\mathcal{F}^2} = 1. \tag{2.1}$$

In these statements, we will make extensive use of the notation and the background results recalled in Sect. 1.2, concerning the functions $u(z)$, $\mu(t)$ and $u^*(s)$ that can be associated with a given $F \in \mathcal{F}^2$. In particular, as in (1.23), in our statements we will let

$$T := \max_{z\in\mathbb{C}} u(z) = u^*(0) = \max_{z\in\mathbb{C}} |F(z)|^2 e^{-\pi|z|^2}, \tag{2.2}$$

recalling that $T \in [0, 1]$ whenever (2.1) is assumed.

We also note that, since u^* is (by its definition) equimeasurable with u and decreasing, there holds

$$\int_{\{u>u^*(s_0)\}} u(z)\,\mathrm{d}z = \int_0^{s_0} u^*(s)\,\mathrm{d}s \quad \forall s_0 \geq 0. \tag{2.3}$$

Moreover, as recalled in Sect. 1.2, when $F = cF_{z_0}$ (with $|c| = 1$) is one of the optimal functions described in Theorem B, one has $\mu(t) = \log_+ \frac{1}{t}$ or, equivalently, $u^*(s) = e^{-s}$. For this reason, a careful comparison between e^{-s} and $u^*(s)$ (for an arbitrary F satisfying (2.1)) will be the core of the results of this section. Since, when (2.1) holds, letting $s_0 \to \infty$ in (2.3) we have

$$1 = \int_0^\infty u^*(s)\,\mathrm{d}s = \int_0^\infty e^{-s}\,\mathrm{d}s, \tag{2.4}$$

as noted in [27] there exists at least one value $s^* > 0$ for which

$$u^*(s) \quad \begin{cases} \leq e^{-s} & \text{if } s \in [0, s^*] \\ \geq e^{-s} & \text{if } s \geq s^* \end{cases} \tag{2.5}$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

or, equivalently, in terms of the inverse functions, a value $t^* \in (0, T)$ for which

$$\mu(t) \quad \begin{cases} \geq \log \frac{1}{t} & \text{if } t \in (0, t^*] \\ \leq \log \frac{1}{t} & \text{if } t \in [t^*, T] \end{cases} \tag{2.6}$$

(note $\mu(t) = 0$ for $t \geq T$). When $T = 1$ (or, equivalently, if F is one of the optimal functions described in Theorem B, see [33, Proposition 2.1]) and hence $u^*(s) = e^{-s}$, clearly all values of s^* (or t^*) have this property, but when $T < 1$ we will prove in Corollary 2.2 that s^* and t^* are in fact *unique*, with an unexpected universal upper bound on t^* (or lower bound on s^*).

With this background, we are now ready to state and prove the results of this section, starting with a sharp estimate for $\mu(t)$, which shows that (1.24) becomes almost an equality when T is close to 1.

Lemma 2.1 *For every $t_0 \in (0, 1)$, there exists a threshold $T_0 \in (t_0, 1)$ and a constant $C_0 > 0$ with the following property. If $F \in \mathcal{F}^2(\mathbb{C})$ is such that $\|F\|_{\mathcal{F}^2} = 1$ and $T \geq T_0$, then*

$$\mu(t) \leq \left(1 + C_0 \frac{1-T}{T}\right) \log \frac{T}{t} \quad \forall t \in [t_0, T]. \tag{2.7}$$

We note, before proving such a result, that the proof presented below shows that one can choose $C_0 = C/t_0^3$, where C is some universal constant.

Proof Given $t_0 \in (0, 1)$ and F as in the statement, we split the proof into several steps.

STEP I. We may assume that $u(z)$ achieves its absolute maximum T at $z = 0$ and that $F(0)$ is a real number, so that $F(0) = \sqrt{T}$. Expanding F with respect to the orthonormal basis of monomials (1.13), we have

$$\frac{F(z)}{\sqrt{T}} = 1 + R(z), \quad z \in \mathbb{C}, \tag{2.8}$$

where $R(z)$ is the entire function

$$R(z) := \sum_{n=2}^{\infty} \frac{a_n}{\sqrt{T}} \frac{\pi^{n/2} z^n}{\sqrt{n!}}, \quad z \in \mathbb{C} \tag{2.9}$$

The fact that $a_1 = 0$, i.e. $F'(0) = 0$, follows easily from our assumption that $u(z)$ has a critical point at $z = 0$, which by (1.20) forces a critical point for $|F(z)|^2$ and ultimately for $F(z)$. The assumption that $1 = \|F\|^2_{\mathcal{F}^2}$ takes the form $1 = T + \sum_{n=2}^{\infty} |a_n|^2$, which we record in the form

$$\sum_{n=2}^{\infty} \frac{|a_n|^2}{T} = \frac{1-T}{T} =: \delta^2, \tag{2.10}$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

hereby defining δ. In the sequel we will often tacitly assume that δ is small enough, depending only on t_0; in the end, the required smallness of δ will determine the threshold T_0 in the statement of Lemma 2.1.

From (2.9), Cauchy-Schwarz and (2.10) we obtain

$$|R(z)|^2 \le \left(\sum_{n=2}^{\infty} \frac{|a_n|^2}{T}\right)\left(\sum_{n=2}^{\infty} \frac{\pi^n |z|^{2n}}{n!}\right) = \delta^2 \left(e^{\pi|z|^2} - 1 - \pi|z|^2\right). \tag{2.11}$$

In particular, $|R(z)|^2 \le \delta^2 \left(e^{\pi|z|^2} - 1\right)$, hence squaring (2.8) we have

$$\frac{|F(z)|^2}{T} \le 1 + \delta^2 \left(e^{\pi|z|^2} - 1\right) + h(z), \tag{2.12}$$

where $h(z)$ is the real valued harmonic function

$$h(z) := 2\,\mathrm{Re}\, R(z), \qquad z \in \mathbb{C}. \tag{2.13}$$

Step II: Estimates for h. Since $|h(z)| \le 2|R(z)|$, the elementary inequality $e^x - 1 - x \le \frac{x^2}{2} e^x$, written with $x = \pi|z|^2$ and combined with (2.11), implies

$$|h(z)| \le \sqrt{2}\pi\, \delta |z|^2 e^{\frac{\pi|z|^2}{2}}, \qquad \forall z \in \mathbb{C}. \tag{2.14}$$

Differentiating (2.9), and then using Cauchy-Schwarz and (2.10) as in (2.11), we have

$$|R'(z)| \le \sum_{n=2}^{\infty} \frac{|a_n|}{\sqrt{T}} \frac{n\pi^{n/2}|z|^{n-1}}{\sqrt{n!}} \le \delta \left(\sum_{n=2}^{\infty} \frac{n^2 \pi^n |z|^{2(n-1)}}{n!}\right)^{\frac{1}{2}} \le \delta\sqrt{2}\pi |z| e^{\frac{\pi|z|^2}{2}}, \tag{2.15}$$

having used the inequality $\frac{n^2}{n!} \le \frac{2}{(n-2)!}$ in the last passage. Similarly, differentiating (2.9) twice, using Cauchy-Schwarz and estimating the resulting power series, we find

$$|R''(z)| \le \delta \left(\sum_{n=2}^{\infty} \frac{n^2(n-1)^2 \pi^n |z|^{2(n-2)}}{n!}\right)^{\frac{1}{2}} \le \delta C(1 + |z|^2) e^{\frac{\pi|z|^2}{2}}. \tag{2.16}$$

By (2.13) and the Cauchy-Riemann equations $|\nabla h(z)| = 2|R'(z)|$ and $|D^2 h(z)| = 2\sqrt{2}|R''(z)|$, we obtain the following uniform estimates with respect to the angular variable θ for the first and second radial derivatives of $h(re^{i\theta})$:

$$\left|\frac{\partial h(re^{i\theta})}{\partial r}\right| \le \delta 2\sqrt{2}\pi r e^{\frac{\pi r^2}{2}}, \tag{2.17}$$

and

$$\left|\frac{\partial^2 h(re^{i\theta})}{\partial r^2}\right| \le \delta C(1 + r^2) e^{\frac{\pi r^2}{2}}. \tag{2.18}$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

STEP III: Definition of E_σ and $r_\sigma(\theta)$. Assuming $T > t_0$, we consider any $t \in [t_0, T)$ and any complex number $z = re^{i\theta}$ ($r \geq 0$), and we observe that

$$u(re^{i\theta}) > t \quad \Longleftrightarrow \quad \frac{te^{\pi r^2}}{T} < \frac{|F(re^{i\theta})|^2}{T}.$$

Hence, by virtue of (2.12), we obtain the implication

$$u(re^{i\theta}) > t \quad \Longrightarrow \quad g_\theta(r, 1) < 1, \tag{2.19}$$

where, for every fixed $\theta \in [0, 2\pi]$, g_θ is defined as

$$g_\theta(r, \sigma) := e^{\pi r^2}\left(\frac{t}{T} - \delta^2\right) + \delta^2 - \sigma\, h(re^{i\theta}), \quad r \geq 0, \quad \sigma \in [0, 1]. \tag{2.20}$$

The variable $\sigma \in [0, 1]$ plays the role of a parameter that defines the family of planar sets

$$E_\sigma := \left\{re^{i\theta} \in \mathbb{C} \mid g_\theta(r, \sigma) < 1\right\}, \quad \sigma \in [0, 1].$$

Since (2.19) is equivalent to the set inclusion $\{u > t\} \subseteq E_1$, (2.7) will be proved if we show that

$$|E_1| \leq \left(1 + \frac{C\delta^2}{t_0^3}\right) \log \frac{T}{t}. \tag{2.21}$$

The advantage of the parameter σ is that we can easily prove the analogous estimate for E_0 – which is a circle, since $g_\theta(r, 0)$ is independent of θ –, and then show that this estimate is inherited by every E_σ (including E_1), by exploiting a cancellation effect due to the harmonicity of h.

We first show that each set E_σ is *star-shaped* with respect to the origin, by showing that $g_\theta(r, \sigma)$ is increasing in r (for fixed θ and σ). Using (2.17) and assuming e.g. that $\delta^2 + \sqrt{2}\delta \leq t_0/2$, we have from (2.20)

$$\begin{aligned} \frac{\partial g_\theta(r, \sigma)}{\partial r} &= 2\pi r e^{\pi r^2}\left(\frac{t}{T} - \delta^2\right) - \sigma \frac{\partial h(re^{i\theta})}{\partial r} \\ &\geq 2\pi r e^{\pi r^2}\left(t_0 - \delta^2\right) - \delta 2\sqrt{2}\pi r e^{\frac{\pi r^2}{2}} \\ &\geq 2\pi r e^{\pi r^2}\left(t_0 - \delta^2 - \sqrt{2}\delta\right) \geq \pi t_0 r e^{\pi r^2} > 0. \end{aligned} \tag{2.22}$$

Since $g_\theta(0, \sigma) = t/T \geq t_0$, integrating the previous bound we also obtain that

$$g_\theta(r, \sigma) \geq \frac{t_0}{2}\left(1 + e^{\pi r^2}\right), \quad \forall r \geq 0, \quad \forall \sigma \in [0, 1], \tag{2.23}$$

and hence, since $g_\theta(0, \sigma) = t/T < 1$, for every $\sigma \in [0, 1]$ the equation in $r > 0$

$$g_\theta(r, \sigma) = 1 \tag{2.24}$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

has a unique solution $r_\sigma > 0$, which we shall also denote by $r_\sigma(\theta)$ when the dependence of r_σ on the angle θ is to be stressed, as in (2.25) below. Since E_σ is star-shaped, using polar coordinates we can compute its area $|E_\sigma|$ in terms of r_σ, as

$$f(\sigma) := |E_\sigma| = \frac{1}{2}\int_0^{2\pi} r_\sigma(\theta)^2 \, \mathrm{d}\theta, \quad \sigma \in [0,1]. \tag{2.25}$$

Notice that $f(1)$ is the area of E_1 that we want to estimate as in (2.21), while

$$f(0) = \frac{1}{2}\int_0^{2\pi} r_0(\theta)^2 \, \mathrm{d}\theta = \pi r_0^2, \tag{2.26}$$

since when $\sigma = 0$, equation (2.24) simplifies to

$$e^{\pi r_0^2}\left(\frac{t}{T} - \delta^2\right) + \delta^2 = 1, \tag{2.27}$$

so r_0 is independent of θ and E_0 is a ball of radius r_0. Note that the sets E_σ are uniformly bounded, since (2.23) and (2.24) entail that

$$\pi r_\sigma^2 \le \log\frac{2}{t_0}, \quad \forall \sigma \in [0,1]. \tag{2.28}$$

Step IV: Estimates for r'_σ and r''_σ. By (2.22) and the implicit function theorem, r_σ is, for every fixed value of $\theta \in [0, 2\pi]$, a smooth, bounded function of the parameter $\sigma \in [0,1]$. Denoting for simplicity by r'_σ its derivative with respect to σ, we have

$$r'_\sigma = \frac{\partial r_\sigma(\theta)}{\partial \sigma} = -\frac{\frac{\partial g_\theta}{\partial \sigma}(r_\sigma, \sigma)}{\frac{\partial g_\theta}{\partial r}(r_\sigma, \sigma)} = \frac{h(r_\sigma e^{i\theta})}{\frac{\partial g_\theta}{\partial r}(r_\sigma, \sigma)}, \quad \sigma \in [0,1], \tag{2.29}$$

and using (2.14) and (2.22) we find the bound

$$|r'_\sigma| \le \frac{\sqrt{2}\,\delta r_\sigma}{t_0} e^{-\frac{\pi r_\sigma^2}{2}}. \tag{2.30}$$

In particular, this implies that

$$r_\sigma^2 \le 2r_0^2 \quad \forall \sigma \in [0,1], \tag{2.31}$$

since by (2.30) $|r'_\sigma|/r_\sigma \le \sqrt{2}\,\delta/t_0 \le \log\sqrt{2}$ provided δ is small enough, we have for every $\sigma \in [0,1]$

$$\log r_\sigma = \log r_0 + \int_0^\sigma \frac{r'_s}{r_s}\,\mathrm{d}s \le \log r_0 + \sigma \log\sqrt{2} \le \log r_0 + \log\sqrt{2},$$

and (2.31) follows.

Differentiating (2.29) with respect to σ, we have

$$r''_\sigma = \frac{\frac{\partial h(re^{i\theta})}{\partial r} r'_\sigma}{\frac{\partial g_\theta}{\partial r}} - \frac{h(re^{i\theta})\left(\frac{\partial^2 g_\theta}{\partial \sigma \partial r} + \frac{\partial^2 g_\theta}{\partial r^2} r'_\sigma\right)}{\left(\frac{\partial g_\theta}{\partial r}\right)^2}. \tag{2.32}$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

Since by (2.20) and (2.17)

$$\left|\frac{\partial^2 g_\theta}{\partial\sigma\,\partial r}\right| = \left|\frac{\partial h(re^{i\theta})}{\partial r}\right| \le \delta C r e^{\frac{\pi r^2}{2}},$$

while by (2.20) and (2.18)

$$\left|\frac{\partial^2 g_\theta}{\partial r^2}\right| \le (2\pi + 4\pi^2 r^2)e^{\pi r^2}\left(\frac{t}{T} - \delta^2\right) + \sigma\left|\frac{\partial^2 h(re^{i\theta})}{\partial r^2}\right|$$
$$\le \frac{t}{T}(2\pi + 4\pi^2 r^2)e^{\pi r^2} + \delta C(1+r^2)e^{\frac{\pi r^2}{2}} \le C(1+r^2)e^{\pi r^2},$$

from (2.32) and (2.14) we see that

$$|r_\sigma''| \le \frac{\delta 2\sqrt{2}\pi r_\sigma e^{\frac{\pi r_\sigma^2}{2}}}{\pi t_0 r_\sigma e^{\pi r_\sigma^2}}|r_\sigma'| + \frac{\sqrt{2}\pi\delta r_\sigma^2 e^{\frac{\pi r_\sigma^2}{2}}}{(\pi t_0 r_\sigma e^{\pi r_\sigma^2})^2}\left(\delta C r_\sigma e^{\frac{\pi r_\sigma^2}{2}} + C(1+r_\sigma^2)e^{\pi r_\sigma^2}|r_\sigma'|\right)$$
$$\le \frac{\delta C e^{-\frac{\pi r_\sigma^2}{2}}}{t_0}|r_\sigma'| + \frac{\delta C e^{-\frac{3\pi r_\sigma^2}{2}}}{t_0^2}\left(\delta r_\sigma e^{\frac{\pi r_\sigma^2}{2}} + (1+r_\sigma^2)e^{\pi r_\sigma^2}|r_\sigma'|\right).$$

Combining with (2.30),

$$|r_\sigma''| \le \frac{\delta^2 C r_\sigma e^{-\pi r_\sigma^2}}{t_0^2} + \frac{\delta C e^{-\frac{3\pi r_\sigma^2}{2}}}{t_0^2}\left(\delta r_\sigma e^{\frac{\pi r_\sigma^2}{2}} + (1+r_\sigma^2)\frac{\delta C r_\sigma}{t_0}e^{\frac{\pi r_\sigma^2}{2}}\right) \le \frac{\delta^2 C r_\sigma}{t_0^3}. \quad (2.33)$$

Step V: Proof of (2.21). Now, recalling the bounds (2.28) and (2.30), one can differentiate under the integral in (2.25), obtaining

$$f'(\sigma) = \int_0^{2\pi} r_\sigma(\theta)\frac{\partial r_\sigma(\theta)}{\partial\sigma}\,\mathrm{d}\theta. \quad (2.34)$$

Differentiating (2.34) again, and then using (2.33) and (2.30), we obtain the estimate

$$|f''(\sigma)| \le \int_0^{2\pi}\left(|r_\sigma'|^2 + |r_\sigma r_\sigma''|\right)\mathrm{d}\theta \le \frac{C\delta^2}{t_0^3}\int_0^{2\pi} r_\sigma^2\,\mathrm{d}\theta, \quad \sigma\in[0,1].$$

This, combined with (2.31) and recalling (2.26), gives

$$|f''(\sigma)| \le \frac{C\delta^2}{t_0^3}f(0), \quad \forall\sigma\in[0,1]. \quad (2.35)$$

We now claim that $f'(0) = 0$, which is the crucial step of the proof. Indeed, when $\sigma = 0$, we see from (2.27) and (2.22) that r_0 and $\partial g_\theta/\partial r$ are independent of θ, and therefore, by (2.29), when $\sigma = 0$ we may write

$$r_\sigma(\theta)\frac{\partial r_\sigma(\theta)}{\partial\sigma}\bigg|_{\sigma=0} = r_0\frac{h(r_0 e^{i\theta})}{\frac{\partial g_\theta}{\partial r}(r_0,0)} = \phi(r_0)h(r_0 e^{i\theta}),$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

where $\phi(r_0) \neq 0$ depends on r_0 but is independent of θ. Therefore, from (2.34),

$$f'(0) = \int_0^{2\pi} r_\sigma(\theta) \frac{\partial r_\sigma(\theta)}{\partial \sigma}\Big|_{\sigma=0} \mathrm{d}\theta = \phi(r_0) \int_0^{2\pi} h(r_0 e^{i\theta})\, \mathrm{d}\theta.$$

On the other hand, the last integral vanishes, since the mean value theorem applied to the harmonic function h gives

$$\frac{1}{2\pi r_0} \int_0^{2\pi} h(r_0 e^{i\theta})\, \mathrm{d}\theta = h(0) = 0.$$

Hence, as $f'(0) = 0$, we may write, through Taylor's formula,

$$f(s) = f(0) + \frac{f''(\sigma)}{2} s^2 \quad \text{for some } \sigma \in (0, s),$$

and taking $s = 1$ and using (2.35) gives

$$|E_1| = f(1) \leq f(0) + \frac{C\delta^2}{t_0^3} f(0) = \left(1 + \frac{C\delta^2}{t_0^3}\right) f(0). \tag{2.36}$$

Now, as we may assume that $2\delta^2 \leq t_0$, we claim that

$$\pi r_0^2 \leq \left(1 + \frac{2\delta^2}{t_0}\right) \log \frac{T}{t}, \tag{2.37}$$

which, according to (2.27), is equivalent to

$$e^{\pi r_0^2} = \frac{1 - \delta^2}{\frac{t}{T} - \delta^2} \leq \left(\frac{T}{t}\right)^{1 + \frac{2\delta^2}{t_0}}. \tag{2.38}$$

Setting for convenience $\kappa = 1/t_0$ and defining the function

$$\psi(\tau) := \delta^2 + \left(1 - \delta^2 \tau\right) \tau^{2\delta^2 \kappa}, \quad \tau \in [1, \kappa], \tag{2.39}$$

we observe that (2.38) is equivalent to $\psi(T/t) \geq 1$. Since $\psi(1) = 1$, $T/t \in [1, \kappa]$ and ψ is concave (note that $2\delta^2\kappa \leq 1$ by assumption), it suffices to prove that $\psi(\kappa) \geq 1$. Indeed, we have

$$\begin{aligned} \psi(\kappa) &= \delta^2 + \left(1 - \delta^2 \kappa\right) e^{2\delta^2 \kappa \log \kappa} \geq \delta^2 + \left(1 - \delta^2 \kappa\right)(1 + 2\delta^2 \kappa \log \kappa) \\ &= 1 + \delta^2 \left(1 + 2\kappa(1 - \delta^2 \kappa) \log \kappa - \kappa\right) \geq 1 + \delta^2 (1 + \kappa \log \kappa - \kappa), \end{aligned}$$

having used $1 - \delta^2 \kappa \geq \frac{1}{2}$ in the last passage. This shows that $\psi(\kappa) \geq 1$, hence (2.37) is established.

Thus, (2.21) follows by combining (2.36) with (2.26) and (2.37). □

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

Corollary 2.2 (Uniqueness and non-degeneracy of t^*) *If $F \in \mathcal{F}^2(\mathbb{C})$ is such that $\|F\|_{\mathcal{F}^2} = 1$ and $T < 1$, then there is a unique value $t^* \in (0, T)$ satisfying (2.6). Moreover,*

$$t^* \leq \tau^*, \tag{2.40}$$

for some universal constant $\tau^* \in (0, 1)$.

Note that the uniqueness of t^* implies the uniqueness of s^* defined in (2.5), and $t^* = e^{-s^*}$. We also note that there cannot be any universal lower bound on t^*, since $t^* \leq T$ and T can be arbitrarily small.

Proof If (2.6) were true for two distinct values $t_1 < t_2 < T$ of t^*, then we would have $\mu(t) = \log 1/t$ for every $t \in [t_1, t_2]$, whence $\mu'(t) = -1/t$ for every $t \in (t_1, t_2)$. But the proof of [33, Remark 3.5] shows that this happens if and only if the corresponding sets $\{u > t\}$ are balls, $|\nabla u|$ being constant on each boundary $\partial\{u > t\} = \{u = t\}$: this in turn implies that $u(z) = e^{-\pi|z-z_0|^2}$, for some $z_0 \in \mathbb{C}$ and hence $u(z_0) = 1$ (or equivalently that $u^*(s) \equiv e^{-s}$), contradicting to our assumption that $T < 1$.

Now let T_0 and C_0 be the constants provided by Lemma 2.1 when $t_0 = \frac{1}{2}$, and define

$$\tau^* := \max\left\{\frac{1}{2}, T_0, e^{-\frac{1}{C_0}}\right\}.$$

Given F as in our statement, if $t^* \leq 1/2$ then clearly $t^* \leq \tau^*$, and the same is true if $T < T_0$, because certainly $t^* \leq T$. Finally, if $t^* > 1/2$ and $T \geq T_0$, then (2.7) written with $t = t^*$ becomes

$$\log \frac{1}{t^*} = \mu(t^*) \leq (1 + C_0(1 - T)) \log \frac{T}{t^*},$$

which is equivalent to

$$\frac{1}{t^*} \leq \left(\frac{T}{t^*}\right)^{1+C_0(1-T)}, \quad \text{that is,} \quad t^* \leq T^{1+\frac{1}{C_0(1-T)}}.$$

But then, since $T \leq 1$, we obtain

$$t^* \leq T^{\frac{1}{C_0(1-T)}} \leq e^{-\frac{1}{C_0}}$$

and $t^* \leq \tau^*$ also in this case (notice that $x^{\frac{1}{1-x}} \leq e^{-1}$ for every $x \in (0, 1)$). □

We are now ready to start the comparison between $u^*(s)$ and e^{-s}, where the number s^*, uniquely defined by (2.5) if $T < 1$, will play a crucial role. In the next two lemmas, however, it is not necessary to assume that $T < 1$, since when $T = 1$ (and $u^*(s) = e^{-s}$) their claims remain true (though trivial) for all values of s^*.

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

Lemma 2.3 *For every $F \in \mathcal{F}^2(\mathbb{C})$ such that $\|F\|_{\mathcal{F}^2} = 1$ and every $s_0 > 0$, there holds*

$$\frac{(1-T)^2}{2} \le \int_0^{s^*} \big(e^{-s} - u^*(s)\big)\,\mathrm{d}s \le \delta_{s_0} e^{s_0}, \tag{2.41}$$

where T is as in (2.2) and

$$\delta_{s_0} := 1 - \frac{\int_{\{u>u^*(s_0)\}} u(z)\,\mathrm{d}z}{1-e^{-s_0}} = 1 - \frac{\int_0^{s_0} u^*(s)\,\mathrm{d}s}{1-e^{-s_0}}. \tag{2.42}$$

Note that δ_{s_0} coincides with the deficit $\delta(F;\Omega)$ of Theorem 1.5 when $\Omega = \{u > u^*(s_0)\}$ is the super-level set of u, with measure s_0.

Proof Instead of writing explicitly e^{-s}, we will use the notation

$$v^*(s) := e^{-s}, \quad s \ge 0. \tag{2.43}$$

This will be particularly useful in Sect. 7, when we adapt the current proof to higher dimensions.

Since $u^*(x) \le T$ and $v^*(s) \ge 1 - s$, the first inequality in (2.41) follows from

$$\begin{aligned}\int_0^{s^*} \big(v^*(s) - u^*(s)\big)\,\mathrm{d}s &\ge \int_0^{s^*} \big(1-s-T\big)_+\,\mathrm{d}s \\ &= \int_0^{1-T} (1-s-T)\,\mathrm{d}s = \frac{(1-T)^2}{2}.\end{aligned} \tag{2.44}$$

To prove the second inequality, note that $1 - e^{-s_0} = \int_0^{s_0} v^*(s)\,\mathrm{d}s$, and hence we can rewrite (2.42) as

$$\varepsilon := \delta_{s_0} \int_0^{s_0} v^*(s)\,\mathrm{d}s = \int_{s_0}^{\infty} \big(u^*(s) - v^*(s)\big)\,\mathrm{d}s = \int_0^{s_0} \big(v^*(s) - u^*(s)\big)\,\mathrm{d}s. \tag{2.45}$$

The key of the proof is that the ratio

$$r(s) := \frac{u^*(s)}{v^*(s)} \quad \text{is an increasing function on } [0,+\infty), \tag{2.46}$$

as follows immediately from (1.25) since $r(s) = e^s u^*(s)$. In order to implement such an idea, we must now distinguish between some cases:

CASE 1: $s_0 > s^*$. Since $r(s^*) = 1$ and $r(s)$ is increasing by the convexity inequality (1.25), we have from (2.45)

$$\varepsilon = \int_{s_0}^{\infty} u^*(s)\left(1 - \frac{1}{r(s)}\right)\mathrm{d}s \ge \left(1 - \frac{1}{r(s_0)}\right)\int_{s_0}^{\infty} u^*(s)\,\mathrm{d}s.$$

On the other hand, for the same reason,

$$\int_{s^*}^{s_0} \big(u^*(s) - v^*(s)\big)\,\mathrm{d}s = \int_{s^*}^{s_0} u^*(s)\left(1 - \frac{1}{r(s)}\right)\mathrm{d}s \le \left(1 - \frac{1}{r(s_0)}\right)\int_{s^*}^{s_0} u^*(s)\,\mathrm{d}s,$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

which, combined with the previous estimate, gives

$$\int_{s^*}^{s_0} \big(u^*(s) - v^*(s)\big)\,\mathrm{d}s \le \varepsilon\, \frac{\int_{s^*}^{s_0} u^*(s)\,\mathrm{d}s}{\int_{s_0}^{\infty} u^*(s)\,\mathrm{d}s}.$$

Thus, recalling (2.45) and using the last inequality, we find that

$$\int_{s^*}^{\infty} \big(u^*(s) - v^*(s)\big)\,\mathrm{d}s \le \varepsilon + \varepsilon\, \frac{\int_{s^*}^{s_0} u^*(s)\,\mathrm{d}s}{\int_{s_0}^{\infty} u^*(s)\,\mathrm{d}s} = \varepsilon\, \frac{\int_{s^*}^{\infty} u^*(s)\,\mathrm{d}s}{\int_{s_0}^{\infty} u^*(s)\,\mathrm{d}s} \le \frac{\varepsilon}{\int_{s_0}^{\infty} v^*(s)\,\mathrm{d}s}, \tag{2.47}$$

having used (2.4) for the numerator, and the fact that $u^*(s) \ge v^*(s)$ when $s \ge s^*$, for the denominator. Given that clearly $\varepsilon \le \delta_{s_0}$, the second inequality in (2.41) follows immediately since $\int_{s_0}^{\infty} v^*(s)\,\mathrm{d}s = e^{-s_0}$.

Case 2: $s_0 \le s^*$. As $r(s^*) = 1$ and $r(s)$ is increasing, we have from (2.45) again that

$$\varepsilon = \int_0^{s_0} v^*(s)\,(1 - r(s))\,\mathrm{d}s \ge (1 - r(s_0)) \int_0^{s_0} v^*(s)\,\mathrm{d}s.$$

On the other hand, for the same reason,

$$\int_{s_0}^{s^*} \big(v^*(s) - u^*(s)\big)\,\mathrm{d}s = \int_{s_0}^{s^*} v^*(s)\,(1 - r(s))\,\mathrm{d}s \le (1 - r(s_0)) \int_{s_0}^{s^*} v^*(s)\,\mathrm{d}s,$$

which combined with the previous estimate gives

$$\int_{s_0}^{s^*} \big(v^*(s) - u^*(s)\big)\,\mathrm{d}s \le \varepsilon\, \frac{\int_{s_0}^{s^*} v^*(s)\,\mathrm{d}s}{\int_0^{s_0} v^*(s)\,\mathrm{d}s}.$$

Thus, using the last inequality, we find

$$\int_0^{s^*} \big(v^*(s) - u^*(s)\big)\,\mathrm{d}s \le \varepsilon + \varepsilon\, \frac{\int_{s_0}^{s^*} v^*(s)\,\mathrm{d}s}{\int_0^{s_0} v^*(s)\,\mathrm{d}s} = \varepsilon\, \frac{\int_0^{s^*} v^*(s)\,\mathrm{d}s}{\int_0^{s_0} v^*(s)\,\mathrm{d}s} \le \frac{\varepsilon}{\int_0^{s_0} v^*(s)\,\mathrm{d}s} = \delta_{s_0}, \tag{2.48}$$

and the second inequality in (2.41) follows also in this case. □

We are now ready to show that, in (2.41), the first inequality holds in fact in a much stronger form.

Lemma 2.4 *Under the same assumptions as in Lemma 2.3, there holds*

$$1 - T \le C \int_0^{s^*} \big(e^{-s} - u^*(s)\big)\,\mathrm{d}s, \tag{2.49}$$

where $C > 0$ is a universal constant.

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

Proof Passing to the inverse functions, and recalling that $\mu(t)$, restricted to $(0, T)$, is the inverse of $u^*(s)$, we have

$$\int_0^{s^*} \big(e^{-s} - u^*(s)\big)\,\mathrm{d}s \geq \int_0^{s^*} \big(\min\{T, e^{-s}\} - u^*(s)\big)\,\mathrm{d}s = \int_{t^*}^{T} \Big(\log\frac{1}{t} - \mu(t)\Big)\,\mathrm{d}t. \tag{2.50}$$

Observe that, given any universal constant $\tau \in (0, 1)$, in proving (2.49) we may assume (if convenient) that

$$T \geq \tau, \tag{2.51}$$

because otherwise (2.49) would immediately follow from the first inequality in (2.41), as soon as $C \geq 2/(1-\tau)$. In particular, letting T_0 and C_0 be the constants provided by Lemma 2.1 when $t_0 = \tau^*$, where τ^* is the constant obtained in Corollary 2.2, we may assume that $T \geq T_0$, so that (2.7) reads

$$\mu(t) \leq (1 + C_0(1-T)) \log\frac{T}{t} \quad \forall t \in [\tau^*, T]. \tag{2.52}$$

Relying on (2.40), we now use (2.52) to minorize the last integral in (2.50). More precisely, letting $\tau_1 \in [\tau^*, 1)$ denote a universal constant to be chosen later, and further assuming (in addition to $T \geq T_0$) that (2.51) holds also with $\tau = \tau_1$, from (2.40), (2.52) and (2.50) we find

$$\int_0^{s^*} \big(e^{-s} - u^*(s)\big)\,\mathrm{d}s \geq \int_{\tau_1}^{T} \Big(\log\frac{1}{t} - (1 + C_0(1-T)) \log\frac{T}{t}\Big)\,\mathrm{d}t. \tag{2.53}$$

Using $-\log T \geq 1 - T$, for every $t \in (\tau_1, T)$ we have

$$\begin{aligned}\log\frac{1}{t} - (1 + C_0(1-T)) \log\frac{T}{t} &= -\log T - C_0(1-T)\log\frac{T}{t} \\ &\geq 1 - T - C_0(1-T)\log\frac{1}{\tau_1},\end{aligned}$$

and choosing now $\tau_1 \in [\tau^*, 1)$ sufficiently close to 1 in such a way that

$$\varepsilon_1 := 1 - C_0 \log\frac{1}{\tau_1} > 0, \tag{2.54}$$

from (2.53) and the subsequent estimate we obtain

$$\int_0^{s^*} \big(e^{-s} - u^*(s)\big)\,\mathrm{d}s \geq \int_{\tau_1}^{T} (1-T)\left(1 - C_0 \log\frac{1}{\tau_1}\right)\mathrm{d}t \geq \varepsilon_1 (1-T)(T - \tau_1). \tag{2.55}$$

Finally, choosing a larger number $\tau_2 \in (\tau_1, 1)$ and further assuming that (2.51) holds also with $\tau = \tau_2$, we obtain

$$\int_0^{s^*} \big(e^{-s} - u^*(s)\big)\,\mathrm{d}s \geq \varepsilon_1 (\tau_2 - \tau_1)(1 - T)$$

and (2.49) follows, by letting $C^{-1} = \varepsilon_1(\tau_2 - \tau_1)$. □

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

The final ingredient we need is a well-known lemma, whose statement and proof are well-known in the theory of Reproducing Kernel Hilbert spaces. For completeness, we provide its proof here.

Lemma 2.5 *If $F \in \mathcal{F}^2(\mathbb{C})$ and $\|F\|_{\mathcal{F}^2} = 1$, then*

$$\min_{\substack{z_0 \in \mathbb{C} \\ |c|=1}} \|F - cF_{z_0}\|^2_{\mathcal{F}^2} = 2\left(1 - \sqrt{T}\right) \leq 2(1-T). \tag{2.56}$$

Proof Since $\|F_{z_0}\|_{\mathcal{F}^2} = 1$ for every $z_0 \in \mathbb{C}$, for any c with $|c| = 1$ we have

$$\|F - cF_{z_0}\|^2_{\mathcal{F}^2} = 2 - 2\,\mathrm{Re}\left\langle \overline{c}F, F_{z_0}\right\rangle_{\mathcal{F}^2}, \tag{2.57}$$

and, since $\mathcal{F}^2(\mathbb{C})$ is a reproducing kernel Hilbert space with kernel $K_w(z) = e^{\frac{\pi}{2}|w|^2} F_w(z)$, we have $\left\langle F, F_{z_0}\right\rangle_{\mathcal{F}^2} = F(z_0)e^{-\frac{\pi}{2}|z_0|^2}$. Therefore,

$$\|F - cF_{z_0}\|^2_{\mathcal{F}^2} = 2 - 2\,\mathrm{Re}\,\overline{c}F(z_0)e^{-\frac{\pi}{2}|z_0|^2},$$

and choosing the unimodular c that minimizes the last term, we obtain for every z_0

$$\min_{|c|=1} \|F - cF_{z_0}\|^2_{\mathcal{F}^2} = 2 - 2|F(z_0)|e^{-\frac{\pi}{2}|z_0|^2} = 2 - 2\sqrt{u(z_0)}.$$

The equality in (2.56) then follows by minimizing over $z_0 \in \mathbb{C}$, while the inequality is a direct consequence thereof. □

We are now ready to prove (1.17).

Proof of (1.17) By homogeneity, in (1.17) one can assume that $F \in \mathcal{F}^2(\mathbb{C})$ and $\|F\|_{\mathcal{F}^2} = 1$. Then, given Ω as in Theorem 1.5 and letting $s_0 = |\Omega|$, on combining (2.56) with (2.49) and the second inequality in (2.41), one finds

$$\min_{\substack{z_0 \in \mathbb{C} \\ |c|=1}} \|F - cF_{z_0}\|^2_{\mathcal{F}^2} \leq C\delta_{s_0} e^{s_0}, \tag{2.58}$$

where δ_{s_0} is the deficit defined in (2.42), relative to the super-level set $\{u > u^*(s_0)\}$. But (1.27) (rewritten with $s = s_0$) reveals that $\delta_{s_0} \leq \delta(F;\Omega)$, where $\delta(F;\Omega)$ is the deficit relative to Ω as defined in (1.18). Then (1.17) follows from (2.58), taking square roots. □

3 The geometry of super-level sets

In this section we study, for a fixed number $t > 0$, some basic geometric properties of the super-level sets $\{z \in \mathbb{C} : u_F(z) > t\}$. In the proof of Lemma 2.1 we saw that the function $g_\theta(r,\sigma)$, defined in (2.20), is monotone increasing in r and, in particular, its sub-level sets are star-shaped. We will soon see that, by doing a finer analysis, we can prove a stronger version of this result, namely Proposition 1.6.

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

We begin by discussing some useful normalizations that we will use throughout the next sections. Let us first consider the quantity

$$\rho(F) := \min_{z_0\in\mathbb{C}, c\in\mathbb{C}} \frac{\|F - c\cdot F_{z_0}\|_{\mathcal{F}^2}}{\|F\|_{\mathcal{F}^2}}.$$

Without loss of generality, we will assume that

$$\rho(F) = \min_{c\in\mathbb{C}} \frac{\|F - c\|_{\mathcal{F}^2}}{\|F\|_{\mathcal{F}^2}},$$

that is, the closest function to F in $\{cF_{z_0}\}_{z_0\in\mathbb{C}, c\in\mathbb{C}}$ is a multiple of the constant function $F_0 \equiv 1$. This follows by (2.57), since $\rho(F)^2 = \min_{z_0\in\mathbb{C}} \frac{\|F\|^2_{\mathcal{F}^2} - |F(z_0)|^2 e^{-\pi|z_0|^2}}{\|F\|^2_{\mathcal{F}^2}}$. Moreover, we can also assume that

$$F(0) = 1.$$

Now, we note that, by the previous assumptions, we have

$$\begin{aligned}\|F - c\cdot F_{z_0}\|^2_{\mathcal{F}^2} &= \|F\|^2_{\mathcal{F}^2} + |c|^2 - 2\,\mathrm{Re}(\overline{c}F(z_0))e^{-\pi|z_0|^2/2}\\ &\geq \|F\|^2_{\mathcal{F}^2} + |c|^2 - 2|c||F(z_0)|e^{-\pi|z_0|^2/2}\\ &\geq \|F\|^2_{\mathcal{F}^2} - \max_{z_0\in\mathbb{C}} |F(z_0)|^2 e^{-\pi|z_0|^2}.\end{aligned} \tag{3.1}$$

This shows that $\rho(F)$ is attained at $z_0 = 0$ if and only if 0 is a maximum for u_F; hence, our normalization also implies

$$F'(0) = 0.$$

Observe that ρ differs slightly from the distance to the extremizing class used in (1.17), due to the condition on c. However, it is equivalent to this distance: indeed, by Lemma 2.5, we have

$$\rho(F) \leq \min_{z_0\in\mathbb{C}, |c|=\|F\|_{\mathcal{F}^2(\mathbb{C})}} \frac{\|F - c\cdot F_{z_0}\|_{\mathcal{F}^2}}{\|F\|_{\mathcal{F}^2}} = \min_{|c|=\|F\|_{\mathcal{F}^2(\mathbb{C})}} \frac{\|F - c\|_{\mathcal{F}^2}}{\|F\|_{\mathcal{F}^2}} \leq \sqrt{2}\rho(F). \tag{3.2}$$

Lemma 2.5 also shows that

$$2\frac{\|F\|_{\mathcal{F}^2} - 1}{\|F\|_{\mathcal{F}^2}} = 2\left(1 - \frac{|F(0)|}{\|F\|_{\mathcal{F}^2}}\right) = \min_{|c|=\|F\|_{\mathcal{F}^2(\mathbb{C})}} \frac{\|F - c\|^2_{\mathcal{F}^2}}{\|F\|^2_{\mathcal{F}^2}}. \tag{3.3}$$

Note that, by our normalizations, we have $F(0) = 1 \leq \|F\|_{\mathcal{F}^2}$. One the other hand, (3.3), Lemma 2.5, and (2.41) show that

$$1 \leq \|F\|_{\mathcal{F}^2} \leq \frac{2}{2 - C(e^{|\Omega|}\delta(F;\Omega))^{1/2}} \leq 2, \tag{3.4}$$

provided that the deficit is sufficiently small.

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

In addition to $\rho(F)$, it will be convenient to consider the slightly different quantity

$$\varepsilon(F) := \|F - 1\|_{\mathcal{F}^2}.$$

This allows us to write

$$F = 1 + \varepsilon G, \qquad \text{where } \|G\|_{\mathcal{F}^2} = 1 \text{ and } \varepsilon = \varepsilon(F), \tag{3.5}$$

and then the above assumptions are translated as

$$\langle G, 1\rangle_{\mathcal{F}^2} = \langle G, z\rangle_{\mathcal{F}^2} = 0. \tag{3.6}$$

Note that, by (3.4), we can assume that ε is sufficiently small: indeed, if $e^{|\Omega|}\delta(F;\Omega)$ is sufficiently small, then

$$\frac{\varepsilon(F)}{2} \le \frac{\varepsilon(F)}{\|F\|_{\mathcal{F}^2}} = \rho(F) \le \min_{|c|=\|F\|_{\mathcal{F}^2}} \frac{\|F-c\|_{\mathcal{F}^2}}{\|F\|_{\mathcal{F}^2}} \le C\Big(e^{|\Omega|}\delta(F;\Omega)\Big)^{1/4}. \tag{3.7}$$

We are now ready to begin the main part of this section. We begin with a key technical lemma which shows that, above a certain threshold, all level sets of the function u_F behave like those of the standard Gaussian, as long as $\varepsilon(F)$ is sufficiently small.

Lemma 3.1 *Let $F \in \mathcal{F}^2(\mathbb{C})$ satisfy the normalizations in the beginning of this section. There are constants $\varepsilon_0, c_1 > 0$ with the following property: if $\varepsilon(F) \le \varepsilon_0$, then for any $\alpha \in [0, 2\pi]$, the function*

$$\mathcal{G}_\alpha(r) := u_F(re^{i\alpha}) = |F(re^{i\alpha})|^2 e^{-\pi r^2}$$

is strictly decreasing on the interval $\left[0, c_1\sqrt{\log(1/\varepsilon(F))}\right]$.

Proof Without loss of generality we will take $\alpha = 0$. In order to prove the desired assertion, we shall divide our analysis in two cases.

CASE 1: $1/10 < r < c_1\sqrt{\log(1/\varepsilon)}$. We differentiate the function $\mathcal{G}_0$ in terms of r, which gives us

$$\begin{aligned}\mathcal{G}_0'(r) =& -2\pi r|F(r)|^2 e^{-\pi r^2} + 2\,\mathrm{Re}(F'(r)\overline{F(r)})e^{-\pi r^2}\\ =& -2\pi r(1 + 2\varepsilon\,\mathrm{Re}(G(r)) + \varepsilon^2|G(r)|^2)e^{-\pi r^2}\\ &+ 2\varepsilon\,\mathrm{Re}(G'(r)\overline{(1+\varepsilon G(r))})e^{-\pi r^2},\end{aligned} \tag{3.8}$$

where in the last line we used (3.5). In order to bound the last term we note that, by the Cauchy integral formula,

$$|G'(w)| \le \frac{2}{|w|}\sup_{|z|=2|w|}|G(z)| \le 2\frac{e^{2\pi|w|^2}}{|w|}, \tag{3.9}$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

since $|G(z)| \le e^{\pi|z|^2/2}$, and in addition

$$|1+\varepsilon G(r)| \le \sqrt{2}e^{\frac{\pi}{2}r^2},$$

since $\|1+\varepsilon G\|_{\mathcal{F}^2} \le 2$. Thus, as the ε^2-term in (3.8) is negative, and $|\operatorname{Re} G(r)|e^{-\pi r^2} \le 1$ as $r > 1/10$, we can estimate

$$\mathcal{G}_0'(r) \le -2\pi r(e^{-\pi r^2} - 2\varepsilon) + \frac{8\varepsilon}{r}e^{\frac{3\pi}{2}r^2}.$$

Since $r < c_1\sqrt{\log(1/\varepsilon)}$, we obtain that $e^{\pi r^2} \le e^{\pi c_1^2 \log(1/\varepsilon)} = \varepsilon^{-\pi c_1^2}$. For all ε small enough, we have $e^{-\pi r^2} - 2\varepsilon \ge \varepsilon^{\pi c_1^2} - 2\varepsilon > 0$ provided that $\pi c_1^2 < 1$. Since also $1/10 \le r$, we have

$$\mathcal{G}_0'(r) \le -2\pi r(\varepsilon^{\pi c_1^2} - 2\varepsilon) + 80\varepsilon^{1-\frac{3\pi}{2}c_1^2}.$$

Hence, as long as

$$\frac{5\pi}{2}c_1^2 < 1, \tag{3.10}$$

the term $-2\pi r\varepsilon^{\pi c_1^2}$ dominates over the others. Thus, for sufficiently small ε, we have $\mathcal{G}_0'(r) < 0$.

Case 2: $0 < r \le 1/10$. Notice that this case is more subtle, as $\mathcal{G}_0'(r) \to 0$ when $r \to 0$. We will show that the *second derivative* of $\mathcal{G}_0$ is strictly negative for $r \in (0, 1/10)$: thus the first derivative decreases in $(0, 1)$ and, as $\mathcal{G}_0'(0) = 0$, it follows that $\mathcal{G}_0'(r) < 0$ in this interval, proving the claim.

Starting from (3.8), we compute:

$$\begin{aligned}
\mathcal{G}_0''(r) = &-2\pi(1 + 2\varepsilon\operatorname{Re}(G(r)) + \varepsilon^2|G(r)|^2)e^{-\pi r^2} \\
&- 4\pi\varepsilon r(\operatorname{Re}(G'(r)) + \varepsilon\operatorname{Re}(G'(r)\overline{G(r)}))e^{-\pi r^2} \\
&+ 4\pi^2 r^2(1 + 2\varepsilon\operatorname{Re}(G(r)) + \varepsilon^2|G(r)|^2)e^{-\pi r^2} \\
&+ 2\varepsilon\operatorname{Re}(G''(r)\overline{(1+\varepsilon G(r))})e^{-\pi r^2} + 2\varepsilon^2|G'(r)|^2e^{-\pi r^2} \\
&- 4\pi r\varepsilon\operatorname{Re}(G'(r)\overline{(1+\varepsilon G(r))})e^{-\pi r^2}.
\end{aligned}$$

We now follow the same strategy as in the first case. For $|w| \le 1$, we find the estimates

$$|G'(w)| \le \frac{4\pi}{2\pi}\max_{|z|=2}|G(z)| \le 2e^{2\pi}, \qquad |G''(w)| \le \frac{4\pi}{\pi}\max_{|z|=2}|G(z)| \le 4e^{2\pi}. \tag{3.11}$$

Therefore we have

$$\mathcal{G}_0''(r) \le -2\pi(1 - 2\pi r^2)e^{-\pi r^2} + \varepsilon\mathfrak{h}(\varepsilon),$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

where $\mathfrak{h}\colon \mathbb{R} \to [0,\infty)$ is a smooth function. Since $r < 1/10$, we have $1 - 2\pi r^2 > 0$ and so the first term above is negative. Hence, if ε is sufficiently small, it holds that $\mathcal{G}_0''(r) < 0$ for all $r \in (0, 1/10)$, and the conclusion follows. □

In spite of its simple nature, we can derive several important conclusions from Lemma 3.1, such as the following result.

Lemma 3.2 *Under the same hypotheses of Lemma 3.1, one may find a small constant $c_2 > 0$ such that, for $t > \varepsilon(F)^{c_2}$, the level sets*

$$\{z \in \mathbb{C}\colon u_F(z) > t\}$$

are all star-shaped with respect to the origin. Moreover, for such t, the boundary $\partial\{u_F > t\} = \{u_F = t\}$ is a smooth, closed curve.

Proof Let $c_1 > 0$ be given by Lemma 3.1. We first prove the following assertion: if $|z| > c_1\sqrt{\log(1/\varepsilon)}$ then

$$u_F(z) < 4\varepsilon^{\pi c_1^2}. \tag{3.12}$$

As before, we use the decomposition (3.5) to write

$$u_F(z) = (1 + 2\varepsilon\,\mathrm{Re}(G(z)) + \varepsilon^2|G(z)|^2)e^{-\pi|z|^2}.$$

For $|z| > c_1\sqrt{\log(1/\varepsilon)}$, and ε sufficiently small, since $\|G\|_{\mathcal{F}^2} = 1$ one readily sees that

$$u_F(z) \le \varepsilon^{\pi c_1^2}(1 + 2\varepsilon + \varepsilon^2) < 4\varepsilon^{\pi c_1^2},$$

since we can choose $\pi c_1^2 \le \frac{1}{2}$, cf. (3.10).

We now claim that the conclusion of the lemma holds with $c_2 = \frac{\pi c_1^2}{2}$. If this is not the case, there is $t_0 > \varepsilon^{c_2}$ such that $A_{t_0} := \{z \in \mathbb{C}\colon u_F(z) > t_0\}$ is *not* star-shaped with respect to 0. Thus, there would be a point $w_0 \in A_{t_0}$, such that, for some $r \in (0,1)$, $r \cdot w_0 \notin A_{t_0}$. By (3.12), we must have that

$$|w_0| < c_1\sqrt{\log(1/\varepsilon)}; \tag{3.13}$$

indeed, if $|w_0| > c_1\sqrt{\log(1/\varepsilon)}$ then, by choosing ε even smaller if need be, we would have $u(w_0) < 4\varepsilon^{\pi c_1^2} < \varepsilon^{\pi c_1^2/2} < t_0$, contradicting the fact that $w_0 \in A_{t_0}$. However, (3.13) leads to a contradiction already: if we write $e^{i\alpha_0} = \frac{w_0}{|w_0|}$ then Lemma 3.1 ensures that the function $s \mapsto |F(se^{i\alpha_0})|^2 e^{-\pi s^2}$ is strictly decreasing for $s < c_1\sqrt{\log(1/\varepsilon)}$ and thus we would have

$$t_0 > u_F(r w_0) > u_F(w_0) > t_0,$$

which is a contradiction. Hence A_{t_0} is star-shaped with respect to the origin.

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

The final claim of the lemma, concerning the smoothness of the boundary $\partial\{u_F > t\} = \{u_F = t\}$, follows from the Inverse Function Theorem. Indeed, by (3.12) we see that if z is such that $u_F(z) = t > \varepsilon^{c_2}$ then $|z| < c_1\sqrt{\log(1/\varepsilon)}$, and Lemma 3.1 then guarantees that $\nabla u_F(z) \neq 0$. Thus t is a regular value of u_F and the set $\{u_F = t\}$ is a smooth curve. □

Lemmata 3.1 and 3.2 already show that the super-level sets of u_F are regular and have controlled geometry. We now show that they are in fact *convex*:

Proposition 3.3 *Under the same assumptions as in Lemma 3.1, there are small constants $\varepsilon_0, c_3 > 0$ such that, as long as $\varepsilon(F) \leq \varepsilon_0$ and $s < -c_3 \log(\varepsilon(F))$, the set*

$$A_{u_F^*(s)} := \{z \in \mathbb{C} \colon u_F(z) > u_F^*(s)\}$$

has convex closure.

Proof Choosing ε_0 appropriately, we can apply Lemmas 3.1 and 3.2 to conclude that, for $t > \varepsilon(F)^{c_2}$, the level sets $\{z \in \mathbb{C} \colon u_F(z) > t\}$ are all star-shaped with respect to the origin and have smooth boundary.

We write, for shortness, $u = u_F$ and $u_0 = e^{-\pi|\cdot|^2}$ throughout the rest of this proof. By the triangle inequality and (3.4), we have

$$\begin{aligned} |u - u_0| &= \left|(|F|^2 - 1)e^{-\pi|z|^2}\right| \\ &\leq |F - 1|(|F| + 1)e^{-\pi|z|^2} \leq \varepsilon(F)(\|F\|_{\mathcal{F}^2} + \|1\|_{\mathcal{F}^2}) \leq 3\varepsilon(F) \end{aligned} \tag{3.14}$$

and so

$$\left\{u_0 > u^*(s) + 3\varepsilon(F)\right\} \subset \left\{u > u^*(s)\right\} \subset \left\{u_0 > u^*(s) - 3\varepsilon(F)\right\}. \tag{3.15}$$

This implies that

$$s = |\{u > u^*(s)\}| \geq |\{u_0 > u^*(s) + 3\varepsilon(F)\}| = -\log(u^*(s) + 3\varepsilon(F))$$

or, rearranging,

$$u^*(s) \geq e^{-s} - 3\varepsilon(F), \tag{3.16}$$

In particular, if $e^{-s} > \varepsilon(F)^{c_3}$, then

$$u^*(s) \geq \frac{1}{2}\varepsilon(F)^{c_3} \geq \varepsilon(F)^{c_2} \tag{3.17}$$

provided c_3 and ε_0 are chosen sufficiently small. Thus, for our choice of parameters, the set $A_{u^*(s)}$ is star-shaped and has a smooth boundary.

Arguing similarly to Lemma 3.1 we see that, by further shrinking c_3 if needed, we have

$$\|u - u_0\|_{C^2(A_{u^*(s)})} \leq C_s\varepsilon(F), \tag{3.18}$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

whenever $s \le -c_3 \log(\varepsilon(F))$. Indeed, recalling again (3.5), (3.18) is equivalent to

$$\left\| \left(\mathrm{Re}(G) + \frac{\varepsilon}{2}|G|^2 \right) \cdot u_0 \right\|_{C^2(A_{u^*(s)})} \le \frac{C_s}{2}.$$

Using (3.9), (3.11), a suitable version of the first of those estimates for the second derivative, and (3.14)–(3.17), we see that

$$\left\| \left(\mathrm{Re}(G) + \frac{\varepsilon}{2}|G|^2 \right) \cdot u_0 \right\|_{C^2(A_{u^*(s)})} \le C \sup_{w \in A_{u*(s)}} e^{4\pi |w|^2}. \tag{3.19}$$

If $w \in A_{u^*(s)}$, by (3.15) and similarly to (3.16), we have

$$e^{-\pi |w|^2} \ge u^*(s) - 3\varepsilon(F) \ge \frac{e^{-s}}{2},$$

and hence (3.19) implies (3.18) with $C_s = C \cdot e^{4s}$, where C is an absolute constant.

Let then κ_s denote the curvature of $\partial A_{u^*(s)} = \{u = u^*(s)\}$, thus

$$\kappa_s = -\frac{\nabla^2 u[\nabla u, \nabla u]}{|\nabla u|^3}.$$

For $0 < s < -c_3 \log(\varepsilon(F))$, by (3.15) and (3.17) we have $\{u_0 > \frac{1}{4}\varepsilon(F)^{c_3}\} \supset A_{u^*(s)}$, and hence

$$|\nabla u_0(z)| = 2\pi |z| e^{-\pi |z|^2} \ge C\varepsilon(F)^{c_3} \quad \text{in } \partial A_{u^*(s)}. \tag{3.20}$$

Let us denote by $\tilde{\kappa}_s > 0$ the curvature of the circle $\{u_0 = u^*(s)\}$ and notice that, by (3.17), $\tilde{\kappa}_s \to \infty$ as $s \to 0$. By (3.18) and (3.20), choosing c_3 and ε_0 sufficiently small, we have an estimate

$$|\kappa_s - \tilde{\kappa}_s| \le C C_s \frac{\varepsilon(F)}{\varepsilon(F)^{2c_3}} \le \varepsilon(F)^{\frac{1}{2}},$$

where we used the bound on s in the last inequality. Combining the last two facts, we see that we can choose ε_0 small enough so that $\varepsilon_0^{1/4} \le \tilde{\kappa}_s$ and $\varepsilon_0^{1/2} \le \frac{1}{2}\varepsilon_0^{1/4}$. These choices ensure that

$$\frac{\varepsilon_0^{1/4}}{2} \le \tilde{\kappa}_s - \varepsilon_0^{1/2} \le \kappa_s$$

for all $s < -c_3 \log \varepsilon(F)$. This lower bound implies that $A_{u^*(s)}$ is locally convex. We then use the well-known Tietze–Nakajima theorem (see [32, 38]) which asserts that, as $\overline{A_{u^*(s)}}$ is a closed, connected set, its local convexity implies its convexity, and the assertion is proved. □

Proof of Proposition 1.6 Proposition 1.6 follows immediately from Proposition 3.3 and (3.7), taking $\Omega = A_{u_F^*(s)}$ as usual. □

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

4 Proof of the set stability

In this section we complete the proof of our main Theorem 1.1. As explained in the introduction, it suffices to prove its Fock space analogue, Theorem 1.5.

Proof of Theorems 1.1 and 1.5 Since the stability for the function has already been proved in Sect. 2, it remains to prove stability of the set, i.e. estimate (1.8).

Fix $f \in L^2$ as in the statement of Theorem 1.1, let $F = \mathcal{B}f$, $u_F(z) = |F(z)|^2 e^{-\pi|z|^2}$ and let us write $\delta = \delta(F;\Omega)$ for simplicity. Clearly we may assume that $\delta \leq \delta_0$, for some arbitrarily small constant δ_0. We may also suppose that F is normalized as at the beginning of Sect. 3 and so, as in (3.5), we can write $F = 1 + \varepsilon G$, where $\|G\|_{\mathcal{F}^2} = 1$ satisfies (3.6) and ε satisfies (3.7).

Let $A_\Omega := A_{u_F^*(|\Omega|)}$, as in (1.21). Let $\mathcal{T}$ be any transport map $\mathcal{T}\colon A_\Omega \setminus \Omega \to \Omega \setminus A_\Omega$, that is,

$$1_{\Omega\setminus A_\Omega}(\mathcal{T}(x))\det\nabla\mathcal{T}(x) = 1_{A_\Omega\setminus\Omega}(x),$$

cf. [11, page 12] for details on the existence of such a map. Define

$$B := \left\{x \in A_\Omega \setminus \Omega\colon |\mathcal{T}(x)|^2 - |x|^2 > C_{|\Omega|}\gamma\right\},$$

where $C_{|\Omega|}$, γ are constants to be chosen later. Since $\mathcal{T}$ is a transport map,

$$\int_B (u(z) - u(\mathcal{T}(z)))\,\mathrm{d}z = \int_B u - \int_{\mathcal{T}(B)} u \leq \int_{A_\Omega} u - \int_\Omega u =: d(\Omega). \tag{4.1}$$

In (4.1), the inequality holds by the fact that, for $z \in A_\Omega \setminus \Omega$, $\quad u_F(z) > u_F^*(|\Omega|)$, and the reverse inequality holds for $z \in \Omega \setminus A_\Omega$. Note that from (1.28) we have the bound

$$d(\Omega) \leq \|F\|_{\mathcal{F}^2}^2(1 - e^{-|\Omega|}) - \int_\Omega u = \|F\|_{\mathcal{F}^2}^2(1 - e^{-|\Omega|})\delta \leq 4(1 - e^{-|\Omega|})\delta, \tag{4.2}$$

by (3.4) and the assumption that δ is sufficiently small.

Step I. *Control over B.* In this step, we will show that

$$u(z) - u(\mathcal{T}(z)) \geq 5\gamma \quad \text{for } z \in B, \tag{4.3}$$

after choosing $C_{|\Omega|}$ and γ correctly. To see this, we begin by writing

$$\begin{aligned} u(z) - u(\mathcal{T}(z)) &= e^{-\pi|z|^2} - e^{-\pi|\mathcal{T}(z)|^2} \\ &\quad + 2\varepsilon\left(\mathrm{Re}(G(z)e^{-\pi|z|^2}) - \mathrm{Re}(G(\mathcal{T}(z))e^{-\pi|\mathcal{T}(z)|^2})\right) \\ &\quad + \varepsilon^2\left(|G(z)|^2 e^{-\pi|z|^2} - |G(\mathcal{T}(z))|^2 e^{-\pi|\mathcal{T}(z)|^2}\right). \end{aligned}$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

Since $|G|^2 e^{-\pi|\cdot|^2} \le 1$, we have promptly

$$\begin{aligned} u(z) - u(\mathcal{T}(z)) &\ge e^{-\pi|z|^2} - e^{-\pi|\mathcal{T}(z)|^2} - (4\varepsilon + \varepsilon^2) \\ &= e^{-\pi|z|^2}\left(1 - e^{-\pi(|\mathcal{T}(z)|^2 - |z|^2)}\right) - 4\varepsilon - \varepsilon^2, \end{aligned} \tag{4.4}$$

whenever $z \in B$. Moreover, since $z \in B \subset A_\Omega$, (3.14) shows that

$$e^{-\pi|z|^2} \ge u(z) - 3\varepsilon \ge u^*(|\Omega|) - 3\varepsilon \ge e^{-|\Omega|} - 6\varepsilon > \frac{e^{-|\Omega|}}{2};$$

here we used also

$$e^{-|\Omega|} - 3\varepsilon \le u^*(|\Omega|) \le e^{-|\Omega|} + 3\varepsilon, \tag{4.5}$$

cf. (3.15) and (3.16). If $\pi(|\mathcal{T}(z)|^2 - |z|^2) \ge 1$ then we find

$$u(z) - u(\mathcal{T}(z)) \ge \frac{e^{-|\Omega|}}{2}(1 - e^{-1}) - 4\varepsilon - \varepsilon^2 \ge \frac{e^{-|\Omega|}}{4} - 4\varepsilon - \varepsilon^2.$$

On the other hand, if $\pi(|\mathcal{T}(z)|^2 - |z|^2) \le 1$, from (4.4),

$$u(z) - u(\mathcal{T}(z)) \ge \frac{e^{-|\Omega|}\left(|\mathcal{T}(z)|^2 - |z|^2\right)}{2} - 4\varepsilon - \varepsilon^2 \ge C_{|\Omega|} e^{-|\Omega|}\frac{\gamma}{2} - 4\varepsilon - \varepsilon^2.$$

Choosing $C_{|\Omega|} = 20e^{|\Omega|}$ and $\gamma \ge \varepsilon$, the previous estimates yield the desired (4.3).

STEP II.*Showing that* Ω *is close to* A_Ω. Note the identities

$$|\Omega| - |B| = |\Omega| - |\mathcal{T}(B)| = |\Omega \setminus \mathcal{T}(B)| = |\Omega| - |A_\Omega \setminus \Omega| + |(\Omega \setminus \mathcal{T}(B)) \setminus A_\Omega|,$$

hence

$$\frac{1}{2}|\Omega \Delta A_\Omega| = |A_\Omega \setminus \Omega| = |B| + |(\Omega \setminus \mathcal{T}(B)) \setminus A_\Omega|. \tag{4.6}$$

In this step, we want to estimate both terms on the right-hand side. The estimate for the first term follows by combining (4.1), (4.2) and (4.3):

$$|B| \le \frac{d(\Omega)}{5\gamma} \le \frac{2\delta(1 - e^{-|\Omega|})}{5\gamma}. \tag{4.7}$$

To estimate the second term, note that $\Omega \setminus \mathcal{T}(B)$ is contained in a $C_{|\Omega|}\gamma$-neighborhood of A_Ω; in turn, by (3.15), A_Ω is nested between two concentric balls:

$$\{z : e^{-\pi|z|^2} > u^*(|\Omega|) + 3\varepsilon\} \subset A_\Omega \subset \{z : e^{-\pi|z|^2} > u^*(|\Omega|) - 3\varepsilon\} =: E_\Omega. \tag{4.8}$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

Combining this information with (4.5), and setting $\lambda_\pm = \sqrt{-\pi^{-1}\log(u^*(s)\pm 3\varepsilon)}$ we can estimate

$$\begin{aligned}|(\Omega\setminus\mathcal{T}(B))\setminus A_\Omega| &\le |B_{C_{|\Omega|}\gamma+\lambda_-}\setminus B_{\lambda_+}| = \pi(C_{|\Omega|}\gamma+\lambda_-)^2 + \log(u^*(s)+3\varepsilon)\\ &\le 2\sqrt{\pi}C_{|\Omega|}\gamma\sqrt{-\log(u^*(s)-3\varepsilon)} + \pi(C_{|\Omega|}\gamma)^2\\ &\quad + \log\left(\frac{u^*(s)+3\varepsilon}{u^*(s)-3\varepsilon}\right)\\ &\le 4C_{|\Omega|}\gamma\sqrt{-\log(u^*(s)-3\varepsilon)} + \pi(C_{|\Omega|}\gamma)^2 + C_s\varepsilon\\ &\le 4C_{|\Omega|}\gamma\sqrt{|\Omega|+8\varepsilon e^{|\Omega|}} + \pi C_{|\Omega|}^2\gamma^2 + C_s\gamma\\ &\le 4C_{|\Omega|}\gamma\left(|\Omega|^{1/2} + 4\varepsilon\frac{e^{|\Omega|}}{|\Omega|^{1/2}}\right) + \pi C_{|\Omega|}^2\gamma^2 + C_s\gamma\\ &\le 4C_{|\Omega|}\gamma\left(|\Omega|^{1/2} + 4C\delta^{1/2}\frac{e^{2|\Omega|}}{|\Omega|^{1/2}}\right) + \pi C_{|\Omega|}^2\gamma^2 + C_s\gamma,\end{aligned} \tag{4.9}$$

provided that ε is sufficiently small, depending on $|\Omega|$. Choosing $\varepsilon \le \gamma = C(e^{|\Omega|}\delta)^{1/2}$, where C is the constant provided by Theorem 1.5, and combining (4.6), (4.7) and (4.9), we get

$$|\Omega\Delta A_\Omega| \le C\delta^{1/2},$$

for some new but still explicitly computable constant $C = C(|\Omega|)$.

Step III. *Conclusion.* To conclude, we just need to compare Ω with the ball $S_\Omega := \{z : e^{-\pi|z|^2} \ge e^{-|\Omega|}\}$. By (4.5) and (4.8), we have $S_\Omega \subset E_\Omega$ and

$$|E_\Omega\setminus S_\Omega| \le C\varepsilon \le C(|\Omega|)\delta^{\frac{1}{2}}, \tag{4.10}$$

where we also used (3.7). It follows that

$$\begin{aligned}|S_\Omega\Delta\Omega| &\le |\Omega\setminus E_\Omega| + |E_\Omega\setminus S_\Omega| + |S_\Omega\setminus\Omega|\\ &\le |\Omega\setminus E_\Omega| + |E_\Omega\setminus S_\Omega| + |E_\Omega\setminus\Omega| \le |E_\Omega\Delta\Omega| + C\delta^{1/2}\end{aligned}$$

and so it is enough to bound $|E_\Omega\Delta\Omega|$. We then estimate

$$|E_\Omega\Delta\Omega| = |E_\Omega\setminus\Omega| + |\Omega\setminus E_\Omega| \le |E_\Omega\setminus A_\Omega| + |A_\Omega\setminus\Omega| + |\Omega\setminus A_\Omega| \le C\delta^{1/2},$$

where in the last inequality we estimate $|E_\Omega\setminus A_\Omega|$ as in (4.10) and we also used the estimate from the last step. We have now proved (1.19) and thus also (1.8). □

We remark that, in spite of the sharp exponent of δ in the result above, the asymptotic growth of the constant $K(|\Omega|)$ in (1.8) from the proof above is likely not sharp: as we shall see in Sect. 6, one expects, from the functional stability part, that the sharp

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

growth of the constant should be of the form $\sim e^{|\Omega|/2}$, while the proof above yields $K(|\Omega|) \sim e^{2|\Omega|}$.

Although there is room for improving such a constant with the current methods, it is unlikely that these will suffice in order to upgrade $K(|\Omega|)$ to the aforementioned conjectured optimal growth rate. For that reason, we consider this to be a genuinely interesting problem, which we wish to revisit in a future work.

5 An alternative variational approach to the function stability

The purpose of this section is to give a variational proof of the function stability in Theorem 1.5. Fix $s > 0$ and consider the functional

$$\mathcal{K}\colon \mathcal{F}^2(\mathbb{C}) \to \mathbb{R}, \qquad \mathcal{K}[F] := \frac{I_F(s)}{\|F\|^2_{\mathcal{F}^2}},$$

where we recall that $I_F(s)$ is the integral of u_F over its superlevel set of measure s, cf. (1.27). We will prove the following result:

Theorem 5.1 *Fix $s \in (0, \infty)$. There are explicit constants $\varepsilon_0(s), C(s) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, we have*

$$\mathcal{K}[1] - \mathcal{K}[1 + \varepsilon G] \geq C(s)\varepsilon^2,$$

whenever $\|G\|_{\mathcal{F}^2} = 1$ satisfies (3.6).

The proof of Theorem 5.1 is almost independent of the results of Sect. 2, as we will only rely on the suboptimal stability result from Lemma 2.3. This lemma, in turn, does not rely on the other results from that section.

Let us first note that Theorem 5.1 indeed implies the function stability part Theorem 1.5, although without the optimal dependence of the constant on $|\Omega|$.

Alternative proof of* (1.17)*, assuming Theorem 5.1 Without loss of generality, we can assume the normalizations detailed at the beginning of Sect. 3. By the same argument as in (3.7), if the deficit is sufficiently small we see that

$$\|F - 1\|_{\mathcal{F}^2} = \varepsilon(F) = \|F\|_{\mathcal{F}^2}\rho(F) \leq C\big(e^{|\Omega|}\delta(F;\Omega)\big)^{1/4}.$$

where now the last inequality follows by combining Lemma 2.3 with the simple Lemma 2.5, instead of using (1.17). Here, we take $\Omega = A_{u_F^*(t)} = \{u_F > u_F^*(t)\}$. Hence, we can write

$$F = 1 + \varepsilon G, \qquad \|G\|_{\mathcal{F}^2} = 1,$$

where G satisfies (3.6), and we can assume that ε is sufficiently small. Theorem 5.1 then implies that

$$(1 - e^{-|\Omega|})\delta(F;\Omega) = \mathcal{K}[1] - \mathcal{K}[F] \geq C(|\Omega|)\varepsilon^2 = C(|\Omega|)\|F - 1\|^2_{\mathcal{F}^2}.$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

To complete the proof it suffices to note that, by our normalizations, $F(0) = 1 \leq \|F\|_{\mathcal{F}^2}$. Thus

$$\|F - 1\|_{\mathcal{F}^2} \geq \rho(F) \geq 2^{-1/2} \min_{z_0 \in \mathbb{C}, |c| = \|F\|_{\mathcal{F}^2(\mathbb{C})}} \frac{\|F - c \cdot F_{z_0}\|_{\mathcal{F}^2}}{\|F\|_{\mathcal{F}^2}},$$

where the last inequality follows from (3.2). □

The proof of Theorem 5.1 is based on the following technical result:

Lemma 5.2 *There is $\varepsilon_0 = \varepsilon_0(s)$ and a modulus of continuity η, depending only on s, such that*

$$|\mathcal{K}[1 + \varepsilon G] - \mathcal{K}[1]| \leq \left| \frac{\varepsilon^2}{2} \nabla^2 \mathcal{K}[1](G, G) \right| + \eta(\varepsilon)\varepsilon^2$$

for all $0 \leq \varepsilon \leq \varepsilon_0(t)$ and $G \in \mathcal{F}^2(\mathbb{C})$ such that $\|G\|_{\mathcal{F}^2} = 1$ and which satisfy (3.6). Here we have defined

$$\nabla^2 \mathcal{K}[1](G, G) := \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} \mathcal{K}[1 + \varepsilon G]\Big|_{\varepsilon = 0}.$$

The proof of Lemma 5.2 is rather technical and standard, for which reason we moved it to Appendix A. Lemma 5.2 shows that $\mathcal{K}[1 + \varepsilon G] - \mathcal{K}[1]$ is essentially controlled by the second variation of $\mathcal{K}$ at 1, in the direction of G. Since 1 is a local maximum for $\mathcal{K}$, this variation is negative definite, but to prove Theorem 5.1 we need to show that it is *uniformly* negative definite. This is the content of the next proposition, which is the main result of this section.

Proposition 5.3 *For all $G \in \mathcal{F}^2(\mathbb{C})$ such that $\|G\|_{\mathcal{F}^2} = 1$ and which satisfy (3.6), we have*

$$\frac{1}{2} \nabla^2 \mathcal{K}[1](G, G) \leq -se^{-s}.$$

It is clear that Theorem 5.1 is an immediate consequence of the above two results:

Proof of Theorem 5.1 Combining Lemma 5.2 and Proposition 5.3, we have

$$\mathcal{K}[1] - \mathcal{K}[1 + \varepsilon G] \geq -\varepsilon^2 \Big(\frac{1}{2} \nabla^2 \mathcal{K}[1](G, G) + \eta(\varepsilon) \Big) \geq \varepsilon^2 \Big(\frac{C(s)}{2} - \eta(\varepsilon) \Big).$$

The conclusion now follows by choosing $\varepsilon_0 = \varepsilon_0(s)$ even smaller so that $\frac{C(s)}{4} \geq \eta(\varepsilon_0)$. □

The rest of this section is dedicated to the proof of Proposition 5.3. Clearly we first need to compute the second variation of $\mathcal{K}$ and, in order to do so, our strategy is to consider the sets

$$\Omega_\varepsilon := \{u_\varepsilon > u_\varepsilon^*(s)\}, \qquad u_\varepsilon := u_{1+\varepsilon G} = |1 + \varepsilon G|^2 e^{-\pi|\cdot|^2}, \tag{5.1}$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

and to write

$$\Omega_\varepsilon = \Phi_\varepsilon(\Omega_0),$$

for a suitable volume-preserving flow Φ_ε. In order to construct such a flow, we first prove a general lemma which allows us to build a flow that deforms the unit disk into a given family of graphical domains over the unit circle. This type of result is well-known, and we refer the reader for instance to [3, Theorem 3.7] for a more general statement.

Lemma 5.4 *Denote by $D_0 \subset \mathbb{R}^2$ the unit disk, and suppose that we are given a one-parameter family $\{D_\varepsilon\}_{\varepsilon\in[0,\varepsilon_0]}$ of domains, whose boundaries are given by smooth graphs over the unit circle*:

$$\partial D_\varepsilon = \{(1 + g_\varepsilon(\omega))\omega : \omega \in \mathbb{S}^1\}.$$

We assume that the family $\{g_\varepsilon\}_{\varepsilon\in[0,\varepsilon_0]}$ depends smoothly on (ε, ω).

Then there exists a family $\{Y_\varepsilon\}_{\varepsilon\in[0,\varepsilon_0]}$ of smooth vector fields, which depends smoothly on the parameter ε, such that, if Ψ_ε denotes the flow associated with Y_ε, i.e. if

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Psi_\varepsilon = Y_\varepsilon(\Psi_\varepsilon),$$

then $\Psi_\varepsilon(D_0) = D_\varepsilon$. In addition, Y_ε is such that $\mathrm{div}(Y_\varepsilon) = 0$ in a neighbourhood of $\mathbb{S}^1$.

Proof By translating into polar coordinates $r = |z|$ and $\omega = z/|z|$ we see that, if we define a vector field Y_ε locally on a neighbourhood of $\mathbb{S}^1$ by

$$Y_\varepsilon(r, \omega) = \frac{1}{r}(1 + g_\varepsilon(\omega))\partial_\varepsilon g_\varepsilon(\omega)\omega,$$

then Y_ε satisfies $\mathrm{div}(Y_\varepsilon) = 0$ in a neighbourhood of $\mathbb{S}^1$, Moreover, in the same neighbourhood of $\mathbb{S}^1$, we may write the flow Ψ_ε of Y_ε explicitly as

$$\Psi_\varepsilon(r, \omega) = (r^2 + (1 + g_\varepsilon(\omega))^2 - 1)^{\frac{1}{2}}\omega. \tag{5.2}$$

We then extend Ψ_ε from the neighbourhood of $\mathbb{S}^1$ to the whole complex plane, in such a way that $\Psi_\varepsilon(D_0) = D_\varepsilon$ and the map $(\varepsilon, x) \mapsto \Psi_\varepsilon(x)$ is smooth. Taking $\tilde{Y}_\varepsilon$ to be the vector field of the extended version of Ψ_ε, we see that $\tilde{Y}_\varepsilon$ is an extension of Y_ε to the whole space, and moreover, the map $(\varepsilon, x) \mapsto \tilde{Y}_\varepsilon(x)$ is smooth. □

Using the results of Sect. 3 we can readily apply Lemma 5.4 to the sets Ω_ε:

Lemma 5.5 *Let $G \in \mathcal{F}^2(\mathbb{C})$ satisfy (3.6). There is $\varepsilon_0 = \varepsilon_0(s, \|G\|_{\mathcal{F}^2}) > 0$ such that, for all $\varepsilon \in [0, \varepsilon_0]$, there are globally defined smooth vector fields X_ε, with associated flows Φ_ε, such that*

$$\Omega_\varepsilon = \Phi_\varepsilon(\Omega_0).$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

Moreover, X_ε depends smoothly on ε and is divergence-free in a neighborhood of $\partial\Omega_0$. We also have

$$\int_{\partial\Omega_\varepsilon} \langle X_\varepsilon, \nu_\varepsilon \rangle = 0, \tag{5.3}$$

where ν_ε denotes the outward-pointing unit vector field on $\partial\Omega_\varepsilon$.

Proof Up to dilating by a constant (which depends only on s) we can assume that $\Omega_0 = B_1$. Lemma 3.2 shows that, if ε_0 is chosen sufficiently small, the boundaries $\partial\Omega_\varepsilon$ are smooth and the sets Ω_ε are star-shaped with respect to zero, hence they can be written as graphs over $\mathbb{S}^1$:

$$\partial\Omega_\varepsilon = \{(1 + f_\varepsilon(\omega))\,\omega \colon \omega \in \mathbb{S}^1\}. \tag{5.4}$$

We now claim that the function $(\varepsilon, \omega) \mapsto f_\varepsilon(\omega)$ is *smooth* as long as ε is sufficiently small.

Indeed, for fixed ε, the function $\omega \mapsto f_\varepsilon(\omega)$ is smooth, by Lemma 3.2, since it is implicitly defined by $u_\varepsilon((1 + f_\varepsilon(\omega)) \cdot \omega) = u_\varepsilon^*(s)$. Moreover, since ∇u_ε is *bounded* by a constant depending only on s when restricted to $\{u_\varepsilon = u_\varepsilon^*(s)\}$ (this follows, for instance, from the proof of Lemma 3.1), any careful quantification of the proof of the implicit function theorem (cf. [26]) implies that there is a universal $\varepsilon_0(s) > 0$ such that, if $\varepsilon < \varepsilon_0(s)$, then $\varepsilon \mapsto f_\varepsilon(\omega)$ is smooth for any fixed $\omega \in \mathbb{S}^1$. This proves the desired smoothness claim.

By Lemma 5.4, the associated vector fields are explicitly given in a neighbourhood of $\mathbb{S}^1$ by

$$X_\varepsilon(r, \omega) = \frac{1}{r}(1 + f_\varepsilon(\omega))\partial_\varepsilon f_\varepsilon(\omega)\omega, \tag{5.5}$$

and they are divergence-free in a neighbourhood of $\mathbb{S}^1$. Their smoothness then follows from the smoothness of f_ε in ε.

To prove the final claim we note that, since Ω_ε has constant measure equal to s for all $\varepsilon \in [0, \varepsilon_0]$, by a calculation in polar coordinates we see that the function

$$\mathcal{A}(\varepsilon) := \int_{\partial\Omega_0} (1 + f_\varepsilon(\omega))^2 \, \mathrm{d}\mathcal{H}^1(\omega)$$

is constant in the interval $[0, \varepsilon_0]$. Thus,

$$0 = \frac{\mathrm{d}\mathcal{A}(\varepsilon)}{\mathrm{d}\varepsilon} = 2\int_{\partial\Omega_0} \partial_\varepsilon f_\varepsilon(\omega)(1 + f_\varepsilon(\omega)) \, \mathrm{d}\mathcal{H}^1(\omega) = 2\int_{\partial\Omega_0} \langle X_\varepsilon, \nu \rangle \, \mathrm{d}\mathcal{H}^1(\omega),$$

and so the integral above has to vanish; here we used the fact that Ω_0 is a ball. Since $\mathrm{div}(X_\varepsilon) = 0$ in a neighbourhood of $\partial\Omega_0$, and as $\int_{\partial\Omega_0} \langle X_\varepsilon, \nu \rangle \, \mathrm{d}\mathcal{H}^1 = 0$, the divergence theorem shows that, for any Lipschitz Jordan curve γ in the same neighbourhood of $\partial\Omega_0$, we have

$$\int_\gamma \langle X_\varepsilon, \nu_\gamma \rangle \, \mathrm{d}\mathcal{H}^1 = 0,$$

where ν_γ denotes the outward-pointing normal field on γ. Thus (5.3) follows. □

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

Having the previous lemma at our disposal, we can now obtain an explicit formula for $\nabla^2\mathcal{K}[1]$.

Lemma 5.6 *For all $G \in \mathcal{F}^2(\mathbb{C})$ which satisfy (3.6), we have*

$$\frac{1}{2}\nabla^2\mathcal{K}[1](G,G) = \int_{\Omega_0} |G|^2 e^{-\pi|z|^2}\,\mathrm{d}z - \|G\|_{\mathcal{F}^2}^2 \int_{\Omega_0} e^{-\pi|z|^2}\,\mathrm{d}z + e^{-s} \fint_{\partial\Omega_0} |G|^2\,\mathrm{d}\mathcal{H}^1(z). \tag{5.6}$$

Proof Setting $\mathrm{d}\sigma(z) := e^{-\pi|z|^2}\,\mathrm{d}z$ for brevity, let us introduce the auxiliary functions

$$I_\varepsilon := \int_{\Omega_\varepsilon} |1+\varepsilon G|^2\,\mathrm{d}\sigma, \qquad J_\varepsilon := \int_{\mathbb{C}} |1+\varepsilon G|^2\,\mathrm{d}\sigma; \tag{5.7}$$

we also write $K_\varepsilon := \mathcal{K}[1+\varepsilon G]$, where we recall that $\Omega_\varepsilon := \{u_\varepsilon > u_\varepsilon^*(s)\}$, cf. (5.1). We will always take $\varepsilon \le \varepsilon_0$, where ε_0 is as in Lemma 5.5. Here and henceforth, we shall denote derivatives of the quantities K_ε, I_ε, J_ε in the ε variable with primes, that is, K_ε', J_ε', I_ε', etc. With that in mind, we have:

$$\begin{aligned} K_\varepsilon' &= \Big(I_\varepsilon' - I_\varepsilon \frac{J_\varepsilon'}{J_\varepsilon}\Big)\frac{1}{J_\varepsilon}, \\ K_\varepsilon'' &= \Big(I_\varepsilon'' - I_\varepsilon \frac{J_\varepsilon''}{J_\varepsilon}\Big)\frac{1}{J_\varepsilon} - \frac{2J_\varepsilon'}{J_\varepsilon^2}\Big(I_\varepsilon' - I_\varepsilon\frac{J_\varepsilon'}{J_\varepsilon}\Big), \end{aligned} \tag{5.8}$$

and using Reynold's theorem we further compute

$$I_\varepsilon' = 2\int_{\Omega_\varepsilon} \mathrm{Re}(G\cdot\overline{1+\varepsilon G})\,\mathrm{d}\sigma, \qquad J_\varepsilon' = 2\int_{\mathbb{C}} \mathrm{Re}(G\cdot\overline{1+\varepsilon G})\,\mathrm{d}\sigma, \tag{5.9}$$

$$I_\varepsilon'' = 2\int_{\Omega_\varepsilon} |G|^2\,\mathrm{d}\sigma + 2\int_{\partial\Omega_\varepsilon} \mathrm{Re}(G\cdot\overline{1+\varepsilon G})\langle X_\varepsilon, \nu_\varepsilon\rangle e^{-\pi|z|^2}, \qquad J_\varepsilon'' = 2\int_{\mathbb{C}} |G|^2\,\mathrm{d}\sigma. \tag{5.10}$$

Here and in what follows, we write X_ε to be the vector fields built in Lemma 5.5. Note that, to obtain (5.9), we used the fact that u_ε is constant on $\partial\Omega_\varepsilon$, together with the cancelling property (5.3) of the vector fields.

Since $\langle G, 1\rangle_{\mathcal{F}^2} = 0$, Ω_0 is a ball and G is holomorphic, from (5.9) it is easy to see that

$$I_0' = J_0' = 0 \qquad \Longrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{K}[1+\varepsilon G]\Big|_{\varepsilon=0} = K_0' = 0, \tag{5.11}$$

where the implication follows from the first equation in (5.8).

Combining (5.8)–(5.11), we arrive at

$$\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\mathcal{K}[1+\varepsilon G]\Big|_{\varepsilon=0} = 2\Big(\int_{\Omega_0} |G|^2 e^{-\pi|z|^2} - \|G\|_{\mathcal{F}^2}^2 \int_{\Omega_0} e^{-\pi|z|^2}$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

$$+\int_{\partial\Omega_0}\mathrm{Re}(G(z))\langle X_0,\nu\rangle e^{-\pi|z|^2}\Big). \tag{5.12}$$

Since $\partial\Omega_0$ is a circle of radius r_0, where $\pi r_0^2 = s$, our main task is to simplify the last term: specifically, we want to show that

$$\int_{\partial\Omega_0}\mathrm{Re}\,G\langle X_0,\nu\rangle\,\mathrm{d}\mathcal{H}^1 = \fint_{\partial\Omega_0}|G|^2\,\mathrm{d}\mathcal{H}^1. \tag{5.13}$$

In order to prove (5.13), we have to understand how to write X_0 in terms of G on $\partial\Omega_0$.

We first claim that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\mu_\varepsilon(t) = 0, \tag{5.14}$$

where $\mu_\varepsilon(t) := \mu_{1+\varepsilon G}(t) = |\{u_\varepsilon > t\}|$. To prove this claim we build, exactly as in Lemma 5.5, a family of vector fields Y_ε with associated flows Ψ_ε such that $\Psi_\varepsilon(\{u_0 > t\}) = \{u_\varepsilon > t\}$ (note that, by Lemma 3.2, these sets have smooth boundaries, hence we can apply Lemma 5.4). We compute, for $z \in \partial\{u_0 > t\} = \{u_0 = t\}$,

$$t \equiv u_\varepsilon(\Psi_\varepsilon(z)) = \Big(1 + 2\varepsilon\,\mathrm{Re}\,G(z) - 2\pi\varepsilon\langle Y_0(z), z\rangle + O(\varepsilon^2)\Big)e^{-\pi|z|^2}, \tag{5.15}$$

and thus, since the first order term in ε vanishes, we have

$$\mathrm{Re}\,G(z) = \pi\langle Y_0, z\rangle. \tag{5.16}$$

We can now prove (5.14): again by Reynold's formula, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\mu_\varepsilon(t) &= \int_{\partial\{u_0>t\}}\langle Y_0,\nu\rangle \\ &= 2\pi\fint_{\partial\{u_0>t\}}\langle Y_0, z\rangle = 2\fint_{\partial\{u_0>t\}}\mathrm{Re}\,G = 2\,\mathrm{Re}\,G(0) = 0,\end{aligned} \tag{5.17}$$

since $\partial\{u_0 > t\}$ is a circle and $\mathrm{Re}\,G$ is harmonic, where the last equality follows from (3.6).

As we explain in Remark 5.7 below, the function $\varepsilon \mapsto \mu_\varepsilon(t)$ is smooth in ε, whenever ε is sufficiently small, and also smooth in t, for $t \in (\varepsilon^{c_2}, \max u_\varepsilon)$, where c_2 is as in Lemma 3.2. Now let us fix $s > 0$ and recall that $G(0) = 0$. Using the smoothness of $\varepsilon \mapsto \mu_\varepsilon$ first and then the smoothness of μ_0 on a neighbourhood of $u_0^*(s)$, we obtain:

$$\begin{aligned} s &= \mu_\varepsilon(u_\varepsilon^*(s)) \\ &= \mu_0(u_\varepsilon^*(s)) + 2\varepsilon\,\mathrm{Re}(G(0)) + O(\varepsilon^2) \\ &= s + (u_\varepsilon^*(s) - u_0^*(s))\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=u_0^*(s)}\mu_0(t) + O(\varepsilon^2) \\ &= s - \frac{u_\varepsilon^*(s) - u_0^*(s)}{u_0^*(s)} + O(\varepsilon^2),\end{aligned}$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

where we used the fact that $G(0) = 0$ by (3.6). Thus, after rearranging, we find

$$u_\varepsilon^*(s) = (1 + O(\varepsilon^2))e^{-s}.$$

Since Φ_ε is the flow of X_ε, we have

$$\Phi_\varepsilon(z) = \Phi_0(z) + \varepsilon X_0(\Phi_0(z)) + O(\varepsilon^2) = z + \varepsilon X_0(z) + O(\varepsilon^2). \tag{5.18}$$

We now compare the two expansions

$$u_\varepsilon(\Phi_\varepsilon(z)) = (1 + 2\varepsilon \operatorname{Re} G(z) - 2\pi\varepsilon\langle X_0(z), z\rangle + O(\varepsilon^2))e^{-\pi|z|^2},$$
$$u_\varepsilon^*(s) = (1 + O(\varepsilon^2))e^{-s},$$

and we deduce that, on $\partial\{u_0 > u_0^*(s)\} = \partial\Omega_0$, the first order terms in ε must be the same, thus

$$\pi\langle X_0, z\rangle = \operatorname{Re} G(z) \text{ on } \partial\Omega_0. \tag{5.19}$$

Finally, since G is holomorphic and $G(0) = 0$, we have

$$⨍_{\partial\Omega_0} (\operatorname{Re} G)^2 \, \mathrm{d}\mathcal{H}^1 = ⨍_{\partial\Omega_0} \operatorname{Re}(G^2) \, \mathrm{d}\mathcal{H}^1 + ⨍_{\partial\Omega_0} (\operatorname{Im} G)^2 \, \mathrm{d}\mathcal{H}^1 = ⨍_{\partial\Omega_0} (\operatorname{Im} G)^2 \, \mathrm{d}\mathcal{H}^1.$$

Now (5.13) follows by combining this identity with (5.19):

$$\begin{aligned}\int_{\partial\Omega_0} \operatorname{Re} G\langle X_0, \nu\rangle \, \mathrm{d}\mathcal{H}^1 &= 2\pi ⨍_{\partial\Omega_0} \operatorname{Re} G\langle X_0, z\rangle \, \mathrm{d}\mathcal{H}^1 \\ &= 2⨍_{\partial\Omega_0} (\operatorname{Re} G)^2 \, \mathrm{d}\mathcal{H}^1 = ⨍_{\partial\Omega_0} |G|^2 \, \mathrm{d}\mathcal{H}^1,\end{aligned}$$

as wished. □

Proof of Proposition 5.3 Since $G \in \mathcal{F}^2(\mathbb{C})$ satisfies (3.6), we can write

$$G(z) = \sum_{k=2}^{\infty} a_k \left(\frac{\pi^k}{k!}\right)^{\frac{1}{2}} z^k, \qquad \|G\|_{\mathcal{F}^2}^2 = \sum_{k=2}^{\infty} |a_k|^2.$$

It is direct to see that (5.6) can be rewritten using the power series for G as

$$\frac{1}{2}\nabla^2\mathcal{K}[1](G, G) = \sum_{k=2}^{\infty} |a_k|^2 V_k(s), \tag{5.20}$$

where

$$V_k(s) := \frac{\pi^k}{k!} \int_{B(0,\sqrt{\frac{s}{\pi}})} |z|^{2k} e^{-\pi|z|^2} - \int_{B(0,\sqrt{\frac{s}{\pi}})} e^{-\pi|z|^2} + e^{-s}\frac{s^k}{k!}. \tag{5.21}$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

We claim that $V_k(s) \leq 0$ for all $s \geq 0$. This follows by a simple calculus observation:

$$V_k(s) = -\frac{\pi^k}{k!} \int_{\mathbb{C}\setminus B(0,\sqrt{s/\pi})} |z|^{2k} e^{-\pi|z|^2} \, \mathrm{d}z + \left(1 + \frac{s^k}{k!}\right) e^{-s}$$
$$= -\frac{\Gamma(k+1,s)}{k!} + \left(1 + \frac{s^k}{k!}\right) e^{-s} = -\left(\sum_{j=1}^{k-1} \frac{s^j}{j!}\right) e^{-s},$$

where $\Gamma(a,s) := \int_s^\infty r^{a-1} e^{-r} \, \mathrm{d}r$ denotes the upper incomplete Gamma function. In order to conclude the desired bound, notice that $\lim_{k\to\infty} V_k(s) = e^{-s} - 1 < 0$, and, since $V_k(s)$ is *decreasing* in k for $s > 0$ fixed,

$$\inf_{k\geq 2}(-V_k(s)) = -V_2(s) = se^{-s}.$$

The conclusion of Proposition 5.3 follows then directly from (5.20). □

Remark 5.7 In the proof of Lemma 5.6 above we used the fact that the function $(\varepsilon, t) \mapsto \mu_\varepsilon(t)$ is smooth provided that ε is sufficiently small and that $t \in (\varepsilon^{c_2}, \max u_\varepsilon)$, where c_2 is as in Lemma 3.2. This can be seen explicitly as follows: the smoothness in ε follows by Lemma 5.4 and the fact that $\Psi_\varepsilon(\{u_0 > t\}) = \{u_\varepsilon > t\}$. On the other hand, by [33, Lemma 3.2] we have

$$-\partial_t \mu_\varepsilon(t) = \int_{\{u_\varepsilon = t\}} |\nabla u_\varepsilon|^{-1} \, \mathrm{d}\mathcal{H}^1 \text{ for almost every } t \text{ in } (0, \max u_\varepsilon).$$

By the proof of Lemma 3.1 we see that

$$|\nabla u_\varepsilon(z)| \geq C(\varepsilon^{c_2}, \|G\|_{\mathcal{F}^2})|z|, \tag{5.22}$$

hence $\mu_\varepsilon \in C^{0,1}_{\mathrm{loc}}(\varepsilon^{c_2}, \max u_\varepsilon)$. Moreover, the divergence theorem allows us to write

$$-\partial_t \mu_\varepsilon(t) + \partial_t \mu_\varepsilon(t_0) = \int_{\{u_\varepsilon = t\}} |\nabla u_\varepsilon|^{-1} \, \mathrm{d}\mathcal{H}^1 - \int_{\{u_\varepsilon = t_0\}} |\nabla u_\varepsilon|^{-1} \, \mathrm{d}\mathcal{H}^1$$
$$= \int_{\{u_\varepsilon = t\}} \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|^2} \cdot \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|} \, \mathrm{d}\mathcal{H}^1 - \int_{\{u_\varepsilon = t_0\}} \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|^2} \cdot \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|} \, \mathrm{d}\mathcal{H}^1$$
$$= -\int_{\{t_0 > u_\varepsilon > t\}} \operatorname{div}\left(\frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|^2}\right) \mathrm{d}z,$$

for a.e. $t < t_0$. By (5.22), $|\nabla u_\varepsilon|^{-2}$ is bounded and smooth in the set $\{t_0 > u_\varepsilon > t\}$, which shows that $\partial_t \mu_\varepsilon \in C^{0,1}_{\mathrm{loc}}(\varepsilon^{c_2}, \max u_\varepsilon)$. By a straightforward use of the coarea formula, iterating such an argument yields the desired smoothness property of μ_ε. Moreover, we also have that $\partial_t \mu_\varepsilon(t) \leq -\frac{1}{t}$, cf. (1.23), and so by the Implicit Function Theorem the functions $u^*_\varepsilon(s)$ are differentiable in the variable ε for all fixed s, for $\varepsilon < \varepsilon_0(s)$ sufficiently small.

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

It is important to note that (3.6) is crucial as a normalization for the above proof to work: as a matter of fact, many of the cancellations in the proof of Lemma 5.6 only appeared since $G(0) = \langle G, 1\rangle = 0$. Moreover, if $\langle G, z\rangle \neq 0$, it could happen that $\nabla^1 \mathcal{K}[1](G, G) = 0$, which would cause the proof of sharp stability to collapse.

The argument implicit in the reduction to (3.6) is hence a vital part of the proof: heuristically, it plays the pivotal role of providing us with a single point z_0 – which, through translations, may be assumed to be the origin – for which one can compare the level sets of the functions u_ε to balls centered at z_0. The fact that z_0 is given by the point where each u_ε attains its maximum allows thus for a connection between the analytic and geometric natures of the problem, highlighting further the importance of the aforementioned reduction.

6 Sharpness of the stability estimates

In this short section we prove the sharpness claimed in Remark 1.2 concerning the estimates in Theorem 1.1. We will see that the variational approach of the previous section is quite useful in this regard. The following is the key proposition we require:

Proposition 6.1 *Let $s > 0$ be a fixed positive real number. For each $\varepsilon > 0$ sufficiently small there is a constant $C > 0$ and sequences $\{\Omega_\varepsilon\}_\varepsilon$ and $\{\tilde{F}_\varepsilon\}_\varepsilon \subset \mathcal{F}^2(\mathbb{C})$ with $\|\tilde{F}_\varepsilon\|_{\mathcal{F}^2} = 1, \quad \forall\, \varepsilon > 0$, and such that*:

(i) Ω_0 *a ball and* $|\Omega_\varepsilon| = s$;
(ii) $\inf_{c, z_0 \in \mathbb{C}} \|\tilde{F}_\varepsilon - c \cdot F_{z_0}\|_{\mathcal{F}^2} \geq \frac{\varepsilon}{C}$;
(iii) *the deficit satisfies* $\delta(\tilde{F}_\varepsilon; \Omega_\varepsilon) \leq C \frac{s e^{-s}}{1 - e^{-s}} \varepsilon^2$.

Proof Let $F_\varepsilon(z) = 1 + \varepsilon z^2$ and as usual let us write $u_\varepsilon(z) := |F_\varepsilon(z)|^2 e^{-\pi |z|^2}$. Consider, as in (5.1), the domains

$$\Omega_\varepsilon = \{z \in \mathbb{C} \colon u_\varepsilon(z) > u_\varepsilon^*(s)\},$$

where $s > 0$ is fixed. We then have

$$\begin{aligned}(1 - e^{-s})\delta(F_\varepsilon; \Omega_\varepsilon) &= \mathcal{K}[1] - \mathcal{K}[F_\varepsilon] \\ &\leq -\frac{\varepsilon^2}{2} \nabla^2 \mathcal{K}[1](z^2, z^2) + \varepsilon^2 \eta(\varepsilon) = \frac{2 s e^{-s}}{\pi^2} \varepsilon^2 + \eta(\varepsilon) \varepsilon^2,\end{aligned} \tag{6.1}$$

where we used Lemma 5.2 to pass to the second line and also (5.6) in the last equality. Now note that taking ε sufficiently small yields the desired upper bound if we choose $\tilde{F}_\varepsilon = \frac{F_\varepsilon}{\|F_\varepsilon\|_{\mathcal{F}^2}}$. For the lower bound on $\|\tilde{F}_\varepsilon - c F_{z_0}\|_{\mathcal{F}^2}$, we recall from (3.1) that

$$\|\tilde{F}_\varepsilon - c \cdot F_{z_0}\|_{\mathcal{F}^2}^2 \geq 1 - \max_{z_0 \in \mathbb{C}} |\tilde{F}_\varepsilon(z_0)|^2 e^{-\pi |z_0|^2}.$$

In order to finish, we only need to show that the only global maximum of $|F_\varepsilon(z)|^2 e^{-\pi |z|^2}$ occurs at $z = 0$, which is equivalent to showing that

$$(1 + 2\varepsilon(x^2 - y^2) + \varepsilon^2 |z|^4) < e^{\pi |z|^2},$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

for each $z \in \mathbb{C} \setminus \{0\}$. As $1 + \pi|z|^2 + \frac{\pi^2}{2}|z|^4 < e^{\pi|z|^2}$, this inequality is true if $\varepsilon < \frac{\pi}{4}$. Thus, for such ε,

$$\|\tilde{F}_\varepsilon - cF_{z_0}\|_{\mathcal{F}^2}^2 \geq 1 - \frac{1}{1 + \frac{2}{\pi^2}\varepsilon^2} \geq \frac{\varepsilon^2}{\pi^2},$$

which concludes the proof. □

We are now ready to prove the claims in Remark 1.2.

Corollary 6.2 *The following assertions hold*:

(i) *The factor* $\delta(f;\Omega)^{1/2}$ *cannot be replaced by* $\delta(f;\Omega)^\beta$, *for any* $\beta > 1/2$, *in* (1.7) *and* (1.8);

(ii) *There is no* $c \in (0,1)$ *such that*, *for all measurable sets* $\Omega \subset \mathbb{C}$ *of finite measure*, *we have*

$$\min_{z_0 \in \mathbb{C}, |c| = \|f\|_2} \frac{\|f - c\,\varphi_{z_0}\|_2}{\|f\|_2} \leq C\left(e^{c|\Omega|}\delta(f;\Omega)\right)^{1/2}.$$

Proof Notice that (ii) follows directly from the statement of Proposition 6.1 by taking $s \to \infty$, so we just have to prove (i). The fact that one cannot improve the exponent in (1.7) follows directly from Proposition 6.1 above.

To see that one cannot improve the exponent in (1.8) we argue as follows. For the domains Ω_ε built in Proposition 6.1, we may use Lemma 5.5 to write $\Omega_\varepsilon = \Phi_\varepsilon(\Omega_0)$, provided that ε is small enough. As we saw in (5.18) we may write

$$\Phi_\varepsilon(z) = z + \varepsilon X_0(z) + O(\varepsilon^2), \qquad \text{where } X_0(z) = h_0(z)z,$$

for some scalar function $h_0 \colon \mathbb{C}\setminus\{0\} \to \mathbb{R}$. Indeed, that X_0 has this form follows from its explicit formula (5.5) in Lemma 5.5. Since $\pi\langle X_0(z), z\rangle = \mathrm{Re}(z^2)$ on $\partial\Omega_0$ by (5.19), we have

$$h_0(z) = \frac{\mathrm{Re}(z^2)}{\pi|z|^2} = \frac{\cos(2\theta)}{\pi}$$

for $z = r(\Omega_0)e^{i\theta} \in \partial\Omega_0$, where $r(\Omega_0)$ denotes the radius of the ball Ω_0. Hence,

$$\begin{aligned}|\Omega_\varepsilon \triangle \Omega_0| \geq |\Omega_\varepsilon \setminus \Omega_0| &\geq \left|\left\{z = re^{i\theta} : r(\Omega_0) < r < r(\Omega_0) + \varepsilon\frac{\cos(2\theta)}{\pi} - C\varepsilon^2\right\}\right| \\ &> c\,r(\Omega_0)^2\varepsilon,\end{aligned}$$

which concludes the proof. □

7 Generalizations to higher dimensions

In this section we will provide the generalization of Theorems 1.1 and 1.5 to higher dimensions $d \geq 1$. We believe that the results of Sects. 5 and 6 also have higher-dimensional counterparts, but as the focus of this paper is on the 1-dimensional case we do not elaborate further on this.

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

Given a window function $g \in L^2(\mathbb{R}^d)$, the STFT of a function $f \in L^2(\mathbb{R}^d)$ is defined as

$$V_g f(x,\omega) := \int_{\mathbb{R}^d} e^{-2\pi i y\cdot\omega} f(y)\overline{g(x-y)}\,\mathrm{d}y, \quad x,\omega \in \mathbb{R}^d,$$

coherently with (1.1). As in dimension 1, we will only be interested in the case where $g(x) = \varphi(x)$ is the standard Gaussian window defined as

$$\varphi(x) := 2^{d/4} e^{-\pi|x|^2}, \quad x \in \mathbb{R}^d, \tag{7.1}$$

so as before we set $\mathcal{V} f := V_\varphi f$. Note that (7.1) reduces to (1.2) when $d = 1$.

The d-dimensional version of Theorem A, proved in [33], can be stated as follows.

Theorem C ([33]; Faber-Krahn inequality for the STFT in dimension d) *If $\Omega \subset \mathbb{R}^{2d}$ is a measurable set with finite Lebesgue measure $|\Omega| > 0$, and $f \in L^2(\mathbb{R}^d) \setminus \{0\}$ is an arbitrary function, then*

$$\frac{\int_\Omega |\mathcal{V} f(x,\omega)|^2\,\mathrm{d}x\,\mathrm{d}\omega}{\|f\|^2_{L^2(\mathbb{R}^d)}} \le \int_0^{|\Omega|} e^{-(d!\,s)^{\frac{1}{d}}}\,\mathrm{d}s. \tag{7.2}$$

Moreover, equality is attained if and only if Ω coincides (up to a set of measure zero) with a ball centered at some $z_0 = (x_0,\omega_0) \in \mathbb{R}^{2d}$ and, at the same time, for some $c \in \mathbb{C}\setminus\{0\}$

$$f(x) = c\,\varphi_{z_0}(x), \qquad \varphi_{z_0}(x) := e^{2\pi i \omega_0\cdot x}\varphi(x - x_0), \tag{7.3}$$

where φ is the Gaussian defined in (7.1).

We point out that in [33, Theorem 4.1] the right hand side of (7.2) is expressed in terms of the incomplete Gamma function and implicit constants depending on d, whereas the present formulation (which is equivalent but more explicit) is taken from [34] (see the remark after Theorem 2.3 therein).

It appears from (7.2) that, in dimension $d \ge 1$, the function

$$v^*(s) := e^{-(d!\,s)^{\frac{1}{d}}}, \quad s \ge 0, \tag{7.4}$$

plays a crucial role, since when $d = 1$, $v^*(s) = e^{-s}$ and the right hand side of (7.2) reduces to $1 - e^{-|\Omega|}$, as in (1.16). To state a stability result in dimension d, we must modify the deficit δ defined in (1.5) to suit the right hand side of (7.2), so we let

$$\delta(f;\Omega) := 1 - \frac{\int_\Omega |\mathcal{V} f(x,\omega)|^2\,\mathrm{d}x\,\mathrm{d}\omega}{\|f\|^2_{L^2(\mathbb{R}^d)}\int_0^{|\Omega|} v^*(s)\,\mathrm{d}s}, \tag{7.5}$$

which once again reduces to (1.5) when $d = 1$. Redefining the asymmetry index $\mathcal{A}(\Omega)$ by simply replacing $\mathbb{R}^2$ with $\mathbb{R}^{2d}$ in (1.6), our extension of Theorem 1.1 to dimension d can be stated as follows.

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

Theorem 7.1 (Stability of the FK inequality for the STFT in dimension d) *There is an explicitly computable constant $C = C(d) > 0$ such that, for all measurable sets $\Omega \subset \mathbb{R}^{2d}$ with finite measure $|\Omega| > 0$ and all functions $f \in L^2(\mathbb{R}^d)\backslash\{0\}$, we have*

$$\min_{z_0 \in \mathbb{C}^d, |c| = \|f\|_2} \frac{\|f - c\,\varphi_{z_0}\|_2}{\|f\|_2} \leq C \left(\frac{\delta(f;\Omega)}{\int_{|\Omega|}^{\infty} e^{-(d!\,s)^{\frac{1}{d}}}\,\mathrm{d}s} \right)^{\frac{1}{2}}. \tag{7.6}$$

Moreover, for some explicit constant $K = K(d, |\Omega|)$ we also have

$$\mathcal{A}(\Omega) \leq K\,\delta(f;\Omega)^{1/2}. \tag{7.7}$$

As in the case of Theorem 1.1, the first step is to translate the problem into the Fock space $\mathcal{F}^2(\mathbb{C}^d)$, now defined as the Hilbert space of all holomorphic functions $F\colon \mathbb{C}^d \to \mathbb{C}$ such that

$$\|F\|_{\mathcal{F}^2} := \left(\int_{\mathbb{C}^d} |F(z)|^2\, e^{-\pi|z|^2}\,\mathrm{d}z \right)^{1/2} < \infty,$$

with its induced inner product. An orthonormal basis – that reduces to (1.13) when $d = 1$ – is given, using multi-index notation, by the normalized monomials

$$e_\alpha(z) = (\pi^{|\alpha|}/\alpha!)^{1/2}\, z^\alpha, \quad \alpha \in \mathbb{N}^d, \quad z \in \mathbb{C}^d, \tag{7.8}$$

while the reproducing kernels are the functions $K_w(z) = e^{\frac{\pi}{2}|w|^2} F_w(z)$, where, in analogy to (1.15),

$$F_{z_0}(z) = e^{-\frac{\pi}{2}|z_0|^2} e^{\pi z \cdot \overline{z_0}}. \tag{7.9}$$

The Bargmann transform is now an unitary operator from $L^2(\mathbb{R}^d)$ onto $\mathcal{F}^2(\mathbb{C}^d)$, defined as in (1.12), with $\mathbb{R}^d$ and $\mathbb{C}^d$ in place of $\mathbb{R}$ and $\mathbb{C}$, and the multi-index notation being adopted. Moreover, the functions F_{z_0} in (7.9) are, much as in (1.15), the Bargmann transforms of the optimal functions φ_{z_0} defined in (7.3). In this setting, since by an identity similar to (1.14) the concentration of a function f on Ω can still be expressed in terms of its Bargmann transform, one can rephrase Theorem 7.1 in terms of Fock spaces.

Theorem 7.2 (Fock space version of Theorem 7.1) *There is a computable constant $C = C(d) > 0$ such that, for all measurable sets $\Omega \subset \mathbb{R}^{2d}$ with finite measure $|\Omega| > 0$ and all functions $F \in \mathcal{F}^2(\mathbb{C}^d)\backslash\{0\}$, we have*

$$\min_{\substack{|c| = \|F\|_{\mathcal{F}^2}, \\ z_0 \in \mathbb{C}^d}} \frac{\|F - cF_{z_0}\|_{\mathcal{F}^2}}{\|F\|_{\mathcal{F}^2}} \leq C \left(\frac{\delta(F;\Omega)}{\int_{|\Omega|}^{\infty} e^{-(d!\,s)^{\frac{1}{d}}}\,\mathrm{d}s} \right)^{\frac{1}{2}}, \tag{7.10}$$

where

$$\delta(F;\Omega) := 1 - \frac{\int_\Omega |F(z)|^2 e^{-\pi|z|^2}\,\mathrm{d}z}{\|F\|^2_{\mathcal{F}^2} \int_0^{|\Omega|} v^*(s)\,\mathrm{d}s}. \tag{7.11}$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

Moreover, for some explicit constant $K = K(d, |\Omega|)$ we also have

$$\mathcal{A}(\Omega) \leq K\delta(F;\Omega)^{1/2}. \tag{7.12}$$

We point out that (7.10) reduces to (1.17), when $d = 1$ and $v^*(s) = e^{-s}$.

The proof of (7.10) can be obtained by arguments similar to those given in Sect. 2, where every result has a suitable analogue in dimension d. Therefore, we limit ourselves to describing the relevant, and not always trivial, changes that are necessary to adapt Sect. 2 to dimension d.

We start with the necessary background results from [33], which we discussed in Sect. 1.2 only in dimension one. We warn the reader that in [33] some numerical constants were written in terms of $\boldsymbol{\omega}_{2d}$, the volume of the unit ball in $\mathbb{R}^{2d}$: here, in (7.13) and (7.14), we write them explicitly using the fact that $\boldsymbol{\omega}_{2d} = \pi^d/d!$, as done in [34].

Given $F \in \mathcal{F}^2(\mathbb{C}^d)$, the function u and its super-level sets A_t are defined as in (1.20) and (1.21), now and henceforth with $z \in \mathbb{C}^d$. The distribution function $\mu(t)$ is defined as in (1.22), with the adopted convention that $|\cdot|$ denotes Lebesgue measure in $\mathbb{R}^{2d}$, but with (1.23) being replaced (see [33, §4]) by

$$\mu'(t) \leq -\frac{d\,\mu(t)^{1-1/d}}{(d\,!)^{\frac{1}{d}}\,t} \quad \text{for a.e. } t \in (0,T), \qquad T := \max_{z\in\mathbb{C}^d} u(z), \tag{7.13}$$

while (1.24) becomes

$$\mu(t) \geq \frac{1}{d!}\left(\log_+ \frac{T}{t}\right)^d \quad \text{for all } t > 0.$$

Similarly, the decreasing rearrangement $u^*(s)$ (i.e., the inverse function of $\mu(t)$) is defined exactly as in (1.26), but now with (1.25) being replaced by

$$(u^*)'(s) + \frac{(d\,!)^{\frac{1}{d}}\,u^*(s)}{d\;s^{1-\frac{1}{d}}} \geq 0, \quad \text{for a.e. } s > 0. \tag{7.14}$$

These changes are natural in dimension d, since when F equals one of the optimal functions defined in (7.9), we have $u(z) = e^{-\pi|z-z_0|^2}$ and its distribution function is

$$\mu(t) = \frac{1}{d!}\left(\log_+ \frac{1}{t}\right)^d, \quad t > 0, \tag{7.15}$$

as the explicit volume of the unit ball is $\boldsymbol{\omega}_{2d} = \pi^d/d!$. Note that, for the particular μ in (7.15), (7.13) is an equality. Moreover, if in (7.15) we let $\mu(t) = s > 0$ and solve for t, the resulting inverse function is just the function $v^*(s)$ defined in (7.4), much as e^{-s} is the inverse of $\log_+ \frac{1}{t}$ when $d = 1$. In particular, note that (7.14) becomes an equality when $u^* = v^*$.

Finally, the fact, expressed by (1.27), that super-level sets maximize the concentration under a volume constraint, is clearly still valid, as so is (1.30), with equality if and only if F is a multiple of some F_{z_0}.

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

For what concerns Sect. 2, beside obvious changes such as $\mathbb{C}$ being replaced with $\mathbb{C}^d$ and analogous changes, a general rule is that e^{-s} should always be replaced by $v^*(s)$, e.g. in (2.4) and (2.5), and log with $(\log)^d/d!$, e.g. in (2.6). Accordingly, the claim of Lemma 2.1 becomes

$$\mu(t) \le \frac{1}{d!}\,(1 + C_0(1-T))\,(\log T/t)^d \qquad \forall t \in [t_0, T],$$

where now the underlying constants may depend on the dimension d. The proof follows the same pattern, with some changes being necessary, which we now indicate. Replacing n by $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ and adopting the multi-index notation, (2.9) becomes

$$R(z) := \sum_{|\alpha| \ge 2} \frac{a_\alpha}{\sqrt{T}}\,\frac{\pi^{|\alpha|/2} z^\alpha}{\sqrt{\alpha!}}, \qquad z = (z_1, \dots, z_d) \in \mathbb{C}^d, \tag{7.16}$$

where we used the basis defined in (7.8). Accordingly, (2.10) changes into

$$\sum_{|\alpha| \ge 2} \frac{|a_\alpha|^2}{T} = \frac{1-T}{T} =: \delta^2. \tag{7.17}$$

Some additional caution is needed in order to estimate the subsequent powers series. The outcome of (2.11) is unchanged, now with $z \in \mathbb{C}^d$, but after Cauchy–Schwarz one faces the multivariate power series

$$\sum_{|\alpha| \ge 2} \frac{\pi^{|\alpha|}\,\left|z^{2\alpha}\right|}{\alpha!} = \sum_{|\alpha| \ge 2} \frac{\pi^{|\alpha|}\,\left|z_1^{2\alpha_1} \cdots z_d^{2\alpha_d}\right|}{\alpha_1! \cdots \alpha_d!} = e^{\pi|z|^2} - 1 - \pi|z|^2. \tag{7.18}$$

Subtler changes are needed in Step II. After defining $h(z)$ as in (2.13) and obtaining (2.14), one replaces (2.15) by

$$\left|\frac{\partial R(z)}{\partial z_i}\right| \le \delta\sqrt{2}\pi|z|e^{\frac{\pi|z|^2}{2}}, \qquad z \in \mathbb{C}^d, \quad 1 \le i \le d. \tag{7.19}$$

For instance, letting $e_1 = (1, 0, \dots, 0) \in \mathbb{N}^d$, differentiating (7.16), and then using Cauchy-Schwarz and (7.17) one obtains

$$\left|\frac{\partial R(z)}{\partial z_1}\right| \le \sum_{|\alpha| \ge 2} \frac{|a_\alpha|}{\sqrt{T}}\,\frac{\pi^{|\alpha|/2}\alpha_1\,\left|z^{\alpha - e_1}\right|}{\sqrt{\alpha!}} \le \delta\left(\sum_{|\alpha| \ge 2} \frac{\pi^{|\alpha|}\alpha_1^2\,\left|z^{2(\alpha-e_1)}\right|}{\alpha!}\right)^{\frac{1}{2}}. \tag{7.20}$$

Focussing on multi-indices α of a given size $k \ge 2$, we have

$$\sum_{|\alpha|=k} \frac{\pi^{|\alpha|}\alpha_1^2\,\left|z^{2(\alpha-e_1)}\right|}{\alpha!} = \sum_{\substack{|\alpha|=k \\ \alpha_1 \ge 1}} \frac{\pi^k \alpha_1\,\left|z^{2(\alpha-e_1)}\right|}{(\alpha_1-1)!\alpha_2! \cdots \alpha_d!} = \sum_{|\beta|=k-1} \frac{\pi^k(1+\beta_1)\,\left|z^{2\beta}\right|}{\beta_1! \cdots \beta_d!}.$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

Since $1+\beta_1 \le k \le 2(k-1)$, we have, from the multinomial theorem,

$$\sum_{|\alpha|=k} \frac{\pi^{|\alpha|}\alpha_1^2 \left|z^{2(\alpha-e_1)}\right|}{\alpha!} \le \frac{2\pi^k}{(k-2)!} \sum_{|\beta|=k-1} \frac{(k-1)!\left|z^{2\beta}\right|}{\beta_1!\cdots\beta_d!}$$
$$= \frac{2\pi^k}{(k-2)!}(|z_1|^2+\cdots|z_d|^2)^{k-1}$$

and, since $|z_1|^2+\cdots|z_d|^2 = |z|^2$, summing over $k \ge 2$ we obtain

$$\sum_{|\alpha|\ge 2} \frac{\pi^{|\alpha|}\alpha_1^2 \left|z^{2(\alpha-e_1)}\right|}{\alpha!} \le 2\sum_{k=2}^{\infty} \frac{\pi^k |z|^{2(k-1)}}{(k-2)!} = 2\pi^2|z|^2 e^{\pi|z|^2},$$

which combined with (7.20) proves (7.19) when $i=1$ (when $i>1$ the proof being the same). The analogue of (2.16) now reads

$$\left|\frac{\partial^2 R(z)}{\partial z_i \partial z_j}\right| \le C\delta(1+|z|^2)e^{\frac{\pi|z|^2}{2}}, \quad z\in\mathbb{C}^d, \quad 1\le i,j\le d. \tag{7.21}$$

For instance, differentiating (7.16) twice, then using Cauchy–Schwarz and (7.17), one finds

$$\left|\frac{\partial^2 R(z)}{\partial z_1 \partial z_2}\right| \le \delta\left(\sum_{|\alpha|\ge 2} \frac{\pi^{|\alpha|}\alpha_1^2\alpha_2^2 \left|z^{2(\alpha-e_1-e_2)}\right|}{\alpha!}\right)^{\frac{1}{2}}, \tag{7.22}$$

where $e_2 = (0,1,0,\ldots,0)\in\mathbb{N}^d$. Since the sum can be restricted to those multi-indices α where $\alpha_1\ge 1$ and $\alpha_2\ge 1$ (which imply that $|\alpha|\ge 2$), letting $\beta = \alpha - e_1 - e_2$ we have

$$\sum_{|\alpha|\ge 2} \frac{\pi^{|\alpha|}\alpha_1^2\alpha_2^2 \left|z^{2(\alpha-e_1-e_2)}\right|}{\alpha!} = \sum_{\beta\in\mathbb{N}^d} \frac{\pi^{2+|\beta|}(1+\beta_1)(1+\beta_2)\left|z^{2\beta}\right|}{\beta!}$$
$$= \pi^2 S(z_1)S(z_2)\prod_{j=2}^{d}\left(\sum_{\beta_j=0}^{\infty}\frac{\pi^{\beta_j}|z_j|^{2\beta_j}}{\beta_j!}\right)$$
$$= \pi^2 S(z_1)S(z_2)e^{\pi(|z_3|^2+\cdots+|z_d|^2)},$$

where

$$S(z_i) := \sum_{\beta_i=0}^{\infty} \frac{\pi^{\beta_i}(1+\beta_i)|z_i|^{2\beta_i}}{\beta_i!} < \pi(1+|z_i|^2)e^{\pi|z_i|^2}.$$

Plugging these estimates into (7.22), one obtains (7.21) when $i=1$ and $j=2$, and hence also for all $i\ne j$, by the same argument. Finally, the case where $i=j$ can be treated similarly, by a suitable modification of (2.16).

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

Then, by (2.13) and the Cauchy–Riemann equations, it is easy to see that (7.19) and (7.21) provide bounds for the gradient $\nabla h(z)$ and the Hessian $D^2h(z)$: in particular, in polar coordinates $r\omega \in \mathbb{R}^{2d}$, one has the following bounds for the radial derivatives

$$\left|\frac{\partial h(r\omega)}{\partial r}\right| \le \delta C r e^{\frac{\pi r^2}{2}}, \qquad \left|\frac{\partial^2 h(r\omega)}{\partial r^2}\right| \le \delta C(1+r^2)e^{\frac{\pi r^2}{2}}, \quad r \ge 0, \quad \omega \in \mathbb{S}^{2d-1}, \tag{7.23}$$

which replace (2.17) and (2.18). The rest of the proof requires only minor changes, such as the systematic usage of polar coordinates $r\omega \in \mathbb{R}^{2d}$ (instead of $re^{i\theta}$) with $r \ge 0$ and $\omega \in \mathbb{S}^{2d-1}$, as in (7.23). In particular, in (2.24) and in the sequel, $r_\sigma = r_\sigma(\theta)$ becomes $r_\sigma = r_\sigma(\omega)$. Also integrals should be changed accordingly, e.g., (2.25) now becomes

$$f(\sigma) := |E_\sigma| = \frac{1}{2d}\int_{\mathbb{S}^{2d-1}} r_\sigma(\omega)^{2d}\,\mathrm{d}S(\omega), \quad \sigma \in [0,1],$$

with the obvious related changes, e.g. in (2.34), while (2.26) becomes

$$f(0) = \frac{1}{2d}\int_{\mathbb{S}^{2d-1}} r_0(\omega)^{2d}\,\mathrm{d}S(\omega) = |B(0,r_0)| = \frac{\pi^d}{d!}r_0^{2d}.$$

Corollary 2.2 is unchanged and has a similar proof, where one of course should replace log with $(\log)^d/d!$ as already mentioned.

The claim (2.41) of Lemma (2.3) must be rewritten as

$$\frac{(1-T)^{d+1}}{(d+1)!} \le \int_0^{s^*}\big(v^*(s)-u^*(s)\big)\,\mathrm{d}s \le \frac{\delta_{s_0}}{\int_{s_0}^\infty v^*(s)\,\mathrm{d}s}, \tag{7.24}$$

with the proof, after rewritten in terms of $v^*(s)$ remaining valid almost ad litteram to prove (7.24), the only necessary changes being the following. For the first inequality in (7.24), in (2.44) one should use, instead of $v^*(s) \ge 1-s$ as when $d=1$ and $v^*(s) = e^{-s}$, the similar inequality

$$v^*(s) = e^{-(d!\,s)^{\frac{1}{d}}} \ge 1-(d!\,s)^{\frac{1}{d}},$$

and change (2.44) into

$$\begin{aligned}\int_0^{s^*}\big(v^*(s)-u^*(s)\big)\,\mathrm{d}s &\ge \int_0^{s^*}\Big(1-(d!\,s)^{\frac{1}{d}}-T\Big)_+\,\mathrm{d}s\\ &= \int_0^{\frac{(1-T)^d}{d!}}\Big(1-(d!\,s)^{\frac{1}{d}}-T\Big)\,\mathrm{d}s,\end{aligned}$$

which after a routine computation yields the first inequality in (7.24).

Then the proof goes on unaltered, except that now (2.46) follows from (7.14) and (7.4), rather than (1.25) and (2.43), and in CASE 1 one arrives at (2.47). Since $\varepsilon \le \delta_{s_0}$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

by virtue of (2.45), (2.47) proves (7.24). Finally, CASE 2 requires no changes, since (2.48) already implies (7.24).

The claim of Lemma 2.4 now becomes

$$1 - T \leq C \int_0^{s^*} \big(v^*(s) - e^*(s)\big)\, \mathrm{d}s. \tag{7.25}$$

In the proof, the first inequality in (7.24) can now be used to justify, in a similar way, why on proving (7.25) one can freely assume (2.51) if needed. Thus, replacing also $\log \frac{1}{t}$ by the right hand side of (7.15), and rewriting (2.52) as

$$\mu(t) \leq \frac{1}{d!}(1 + C_0(1-T)) \left(\log \frac{T}{t}\right)^d \quad \text{for all } t \in [\tau^*, T],$$

one obtains the following version of (2.53):

$$d! \int_0^{s^*} \big(v^*(s) - u^*(s)\big)\, \mathrm{d}s \geq \int_{\tau_1}^{T} \left(\left(\log \frac{1}{t}\right)^d - (1 + C_0(1-T)) \left(\log \frac{T}{t}\right)^d\right) \mathrm{d}t. \tag{7.26}$$

Then, using $a^d - b^d \geq (a-b)a^{d-1}$ with the choice $a = \log \frac{1}{t}$ and $b = \log \frac{T}{t}$, since $a - b = -\log T$ and $-\log T \geq 1 - T$ we can replace the minorization after (2.53) by

$$\begin{aligned}
&\left(\log \frac{1}{t}\right)^d - (1 + C_0(1-T)) \left(\log \frac{T}{t}\right)^d \\
&\quad \geq (-\log T) \left(\log \frac{1}{t}\right)^{d-1} - C_0(1-T) \left(\log \frac{T}{t}\right)^d \\
&\quad \geq (1-T) \left(\log \frac{1}{t}\right)^{d-1} - C_0(1-T) \left(\log \frac{1}{t}\right)^d \\
&\quad \geq (1-T) \left(\log \frac{1}{t}\right)^{d-1} \left(1 - C_0 \log \frac{1}{\tau_1}\right),
\end{aligned}$$

for all $t \in [\tau_1, T]$. Finally, fixing $\tau_1 \in (\tau^*, 1)$ in analogy to (2.54), from (7.26) and the previous estimate, in place of (2.55) now one obtains

$$d! \int_0^{s^*} \big(v^*(s) - u^*(s)\big)\, \mathrm{d}s \geq \varepsilon_1 (1-T) \int_{\tau_1}^{T} \left(\log \frac{1}{t}\right)^{d-1} \mathrm{d}t.$$

As explained after (2.55), one can proceed by further assuming $T \geq \tau_2 > \tau_1$, now obtaining

$$d! \int_0^{s^*} \big(v^*(s) - u^*(s)\big)\, \mathrm{d}s \geq \varepsilon_1 (1-T) \int_{\tau_1}^{\tau_2} \left(\log \frac{1}{t}\right)^{d-1} \mathrm{d}t,$$

which proves (7.25).

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

Lemma 2.5, as is well known, remains valid, with the obvious notational changes and the reproducing kernels described before (7.9). As a consequence, the proof of (7.10) can be completed essentially as the proof of (1.17), replacing (2.58) by

$$\min_{\substack{z_0\in\mathbb{C}^d\\ |c|=1}} \|F - cF_{z_0}\|^2_{\mathcal{F}^2(\mathbb{C}^d)} \le C\frac{\delta_{s_0}}{\int_{s_0}^{\infty} v^*(s)\,\mathrm{d}s}.$$

Finally, the proof of the set stability remains virtually unchanged. This finishes the proof of Theorem 7.1.

We now discuss the sharpness of the estimate in Theorem 7.1, in analogy to the discussions of Sects. 5 and 6, and we explain how to adapt the arguments of these sections to the case of general dimension; as before, we keep the notation from these sections.

The first observation to be made is that Lemmas 5.4 and 5.5 hold after the obvious changes have been made, with essentially identical proofs. As in Sect. 5, we wish to compute the second variation $\partial_\varepsilon^2 \mathcal{K}[1+\varepsilon G]|_{\varepsilon=0}$, where G now satisfies that

$$\langle G, 1\rangle = \langle G, z_i\rangle = 0, \quad i = 1,\dots,d.$$

We note that the same argument as in the proof of Lemma 5.6 implies that (5.12) still holds when passing to the higher-dimensional case. Hence, we only need to compute $\langle X_0, \nu\rangle$, where X_0 is defined as the vector field associated with the flows Φ_ε at $\varepsilon = 0$ in the analogue of Lemma 5.5, and ν denotes the unit normal at $\partial\Omega_0$.

In order to do so, we adapt the proof of Lemma 5.6: first, note that equations (5.15)–(5.17) hold in the exact same way also in the higher-dimensional case. Moreover, we also note that

$$\begin{aligned} s &= \mu_\varepsilon(u_\varepsilon^*(s)) \\ &= \mu_0(u_\varepsilon^*(s)) + O(\varepsilon^2) \\ &= s - (u_\varepsilon^*(s) - u_0^*(s))\frac{d\cdot s^{1-1/d}}{(d!)^{1/d}u_0^*(s)} + O(\varepsilon^2), \end{aligned}$$

where the last equality simply follows by differentiating (7.15) with respect to t and evaluating at $t = u_0^*(s) = v^*(s)$. We hence conclude once again that

$$\begin{aligned} u_\varepsilon(\Phi_\varepsilon(z)) &= (1 + 2\varepsilon\,\mathrm{Re}\,G(z) - 2\pi\varepsilon\langle X_0(z), z\rangle + O(\varepsilon^2))e^{-\pi|z|^2}, \\ u_\varepsilon^*(s) &= \left(1 + \frac{(d!)^{1/d}}{d\cdot s^{1-1/d}}O(\varepsilon^2)\right)u_0^*(s). \end{aligned}$$

Again, since $\Phi_\varepsilon(\{u_0 = u_0^*(s)\}) = \{u_\varepsilon = u_\varepsilon^*(s)\}$ by definition, if one looks at $z \in \{u_0 = u_0^*(s)\}$ and compares the expansion in ε of $u_\varepsilon(\Phi_\varepsilon(z)) = u_\varepsilon^*(s)$, one arrives at

$$\mathrm{Re}(G) = \pi\langle X_0, z\rangle = \pi\cdot r(\Omega_0)\langle X_0, \nu\rangle \text{ on } \partial\Omega_0, \tag{7.27}$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

where $r(\Omega_0)$ denotes the radius of Ω_0, which by the definition of v^* in (7.4), is given by

$$r(\Omega_0) = \left(\frac{(d!s)^{1/d}}{\pi} \right)^{1/2}.$$

Hence,

$$\begin{aligned}
\int_{\partial\Omega_0} \mathrm{Re}(G)\langle X_0, \nu\rangle e^{-\pi|z|^2}\, \mathrm{d}\mathcal{H}^{2d-1} &= \frac{\mathcal{H}^{2d-1}(\partial\Omega_0)\cdot e^{-\pi r(\Omega_0)^2}}{\pi\cdot r(\Omega_0)} \fint_{\partial\Omega_0} \mathrm{Re}(G)^2\, \mathrm{d}\mathcal{H}^{2d-1} \\
&= \frac{\mathcal{H}^{2d-1}(\partial\Omega_0)\cdot e^{-\pi r(\Omega_0)^2}}{2\pi\cdot r(\Omega_0)} \fint_{\partial\Omega_0} |G|^2\, \mathrm{d}\mathcal{H}^{2d-1} \\
&= e^{-(d!s)^{1/d}} \frac{d\cdot s^{1-1/d}}{(d!)^{1/d}} \fint_{\partial\Omega_0} |G|^2\, \mathrm{d}\mathcal{H}^{2d-1},
\end{aligned}$$

where the second identity may be justified by the fact that Ω_0 is a ball centered at 0 and G, $\overline{G}$ are harmonic functions with $G(0) = 0$. Thus, as in Sect. 5, this shows that

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\mathcal{K}[1+\varepsilon G]\Big|_{\varepsilon=0} &= \int_{\Omega_0} |G|^2 e^{-\pi|z|^2} - \|G\|_{\mathcal{F}^2}^2 \int_{\Omega_0} e^{-\pi|z|^2} \\
&\quad + e^{-(d!s)^{1/d}} \frac{d\cdot s^{1-1/d}}{(d!)^{1/d}} \fint_{\partial\Omega_0} |G|^2. \qquad (7.28)
\end{aligned}$$

We now wish to compute the right-hand side of (7.28) for

$$G(z) = \sum_{i=1}^{d} z_i^2 + \sum_{1\le i<j\le d} \sqrt{2} z_i z_j.$$

In order to do so, we note that each monomial in the definition of G is orthogonal to every other monomial not just over $\mathbb{C}^d$ but in fact over any ball centered at the origin. Thus we have

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\mathcal{K}[1+\varepsilon G]\Big|_{\varepsilon=0} &= \int_{\Omega_0} |z|^4 e^{-\pi|z|^2} - \|G\|_{\mathcal{F}^2}^2 \int_{\Omega_0} e^{-\pi|z|^2} \\
&\quad + e^{-(d!s)^{1/d}} \cdot \frac{d\cdot s^{1-1/d}}{(d!)^{1/d}} \fint_{\partial\Omega_0} |z|^4. \qquad (7.29)
\end{aligned}$$

In order to further analyze (7.29), note the identity

$$\begin{aligned}
\int_{\Omega_0} |G|^2 e^{-\pi|z|^2} - \|G\|_{\mathcal{F}^2}^2 \int_{\Omega_0} e^{-\pi|z|^2} &= -\int_{\mathbb{C}^d\setminus\Omega_0} |G|^2 e^{-\pi|z|^2} \\
&\quad + \|G\|_{\mathcal{F}^2}^2 \int_{\mathbb{C}^d\setminus\Omega_0} e^{-\pi|z|^2}.
\end{aligned}$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

For our choice of G, we can explicitly evaluate these integrals: indeed, a routine computation implies that

$$\int_{\mathbb{C}^d\setminus\Omega_0} |z|^4 e^{-\pi|z|^2}\,\mathrm{d}z = \frac{\Gamma(d+2,\pi r(\Omega_0)^2)}{\pi^2\Gamma(d)},$$

while $\|G\|_{\mathcal{F}^2}^2 = \frac{d(d+1)}{\pi^2}$, which implies that

$$\|G\|_{\mathcal{F}^2}^2 \int_{\mathbb{C}^d\setminus\Omega_0} e^{-\pi|z|^2}\,\mathrm{d}z = \frac{d(d+1)\Gamma(d,\pi r(\Omega_0)^2)}{\pi^2\Gamma(d)}.$$

Hence, by using that $\Gamma(k,x) = (k-1)!e^{-x}\left(\sum_{j=0}^{k-1}\frac{x^j}{j!}\right)$, we conclude that

$$\int_{\Omega_0} |z|^4 e^{-\pi|z|^2}\,\mathrm{d}z - \|G\|_{\mathcal{F}^2}^2 \int_{\Omega_0} e^{-\pi|z|^2}\,\mathrm{d}z = -\frac{ds\cdot e^{-(d!s)^{1/d}}}{\pi^2}\left(1+d+(d!s)^{1/d}\right).$$

Finally, the last term in (7.29) may be explicitly computed to be $\frac{d\cdot s}{\pi^2}e^{-(d!s)^{1/d}}(d!s)^{1/d}$. Thus, plugging these into (7.29), we obtain

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\mathcal{K}[1+\varepsilon G]\bigg|_{\varepsilon=0} = -e^{-(d!s)^{1/d}}\cdot d(d+1)\frac{s}{\pi^2}.$$

A straightforward adaptation of the arguments from Sect. 6 shows the desired sharpness of the exponent, as well as the stability of the order of growth of the constant in (7.10). That is, we are able to obtain the following result:

Corollary 7.3 *The following assertions hold*:

(i) *The factor* $\delta(f;\Omega)^{1/2}$ *cannot be replaced by* $\delta(f;\Omega)^{\beta}$, *for any* $\beta > 1/2$, *in* (7.6);
(ii) *There is no* $c \in (0,(d!)^{1/d})$ *such that, for all measurable sets* $\Omega\subset\mathbb{C}^d$ *of finite measure, we have*

$$\min_{z_0\in\mathbb{C}^d,|c|=\|f\|_2} \frac{\|f-c\,\varphi_{z_0}\|_2}{\|f\|_2} \le C\left(e^{c|\Omega|^{1/d}}\delta(f;\Omega)\right)^{1/2}.$$

Elementary computations reveal that the denominator on the right-hand side of (7.10) behaves as

$$\int_{|\Omega|}^{\infty} e^{-(d!s)^{1/d}}\,\mathrm{d}s \approx C_d|\Omega|^{\frac{d-1}{d}}e^{-(d!|\Omega|)^{1/d}} \quad \text{as } |\Omega|\to\infty,$$

for some explicitly computable constant $C_d > 0$. Thus, Corollary 7.3(ii) yields, indeed, the desired optimal dependence of (7.10) on $|\Omega|$.

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

Appendix A: Proof of Lemma 5.2

This appendix is dedicated to the proof of the technical Lemma 5.2. Before proceeding with the main part of the proof it is convenient to establish some auxiliary estimates for the vector fields X_ε, with flows Φ_ε, which were constructed in Lemma 5.5.

Lemma A.1 *Let $G \in \mathcal{F}^2(\mathbb{C})$ satisfy (3.6). Let Φ_ε, X_ε, Ω_ε be as in Lemma 5.5. There is $\varepsilon_0 = \varepsilon_0(s, \|G\|_{\mathcal{F}^2}) > 0$ and a modulus of continuity $\eta\colon [0,\infty) \to [0,\infty)$, depending on s, such that, when $\varepsilon \le \varepsilon_0$,*

$$\|X_\varepsilon - X_0\|_{C^1} \le \eta(\varepsilon)\|G\|_{\mathcal{F}^2}, \tag{A.1}$$

$$\|\Phi_\varepsilon - \Phi_0\|_{C^1} \le \eta(\varepsilon)\|G\|_{\mathcal{F}^2}, \tag{A.2}$$

$$|\mathcal{H}^1(\partial\Omega_\varepsilon) - \mathcal{H}^1(\partial\Omega_0)| \le \eta(\varepsilon)\|G\|_{\mathcal{F}^2}, \tag{A.3}$$

$$|\Omega_\varepsilon \triangle \Omega_0| \le \eta(\varepsilon)\|G\|_{\mathcal{F}^2}, \tag{A.4}$$

$$\sigma(\Omega_\varepsilon \triangle \Omega_0) \le \eta(\varepsilon)\|G\|_{\mathcal{F}^2}. \tag{A.5}$$

We note that in (A.5), as in the proof of Lemma 5.6, we denote for simplicity $d\sigma(z) := e^{-\pi|z|^2}\,dz$.

Proof Let f_ε be as in (5.4). We begin by proving that

$$\|f_\varepsilon\|_{C^2(\partial\Omega_0)} \le \eta(\varepsilon)\|G\|_{\mathcal{F}^2}, \tag{A.6}$$

whenever ε is sufficiently small.

We first argue exactly as in Proposition 3.3. More precisely, we write $u_0 = e^{-\pi|\cdot|^2}$, and so, as in (3.15), we have

$$\{u_0 > u_\varepsilon^*(s) + 3\varepsilon\} \subset \{u_\varepsilon > u_\varepsilon^*(s)\} \subset \{u_0 > u_\varepsilon^*(s) - 3\varepsilon\}. \tag{A.7}$$

Thus we see that

$$|u_\varepsilon^*(s) - e^{-s}| \le 3\varepsilon. \tag{A.8}$$

If $z \in \partial\Omega_\varepsilon = \{u_\varepsilon = u_\varepsilon^*(s)\}$, we have

$$\begin{aligned} s - 8e^s\varepsilon \le s - \log(1 + 8\varepsilon e^s) &\le -\log(u_\varepsilon^*(s) + 3\varepsilon) \\ &\le \pi|z|^2 \\ &\le -\log(u_\varepsilon^*(s) - 3\varepsilon) \le s + 8e^s\varepsilon \end{aligned} \tag{A.9}$$

for all ε sufficiently small (depending on s). In particular, we may take $z \in \partial\Omega_\varepsilon$ and so $z = (1 + f_\varepsilon(\omega))\omega$ for some $\omega \in \partial\Omega_0$, thus $\pi|z|^2 = \pi|\omega|^2(1 + f_\varepsilon(\omega))^2 = s(1 + f_\varepsilon(\omega))^2$, and the last inequalities yield

$$1 - 8e^s s^{-1}\varepsilon \le (1 + f_\varepsilon(\omega))^2 \le 1 + 8e^s s^{-1}\varepsilon.$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

Since $1 + f_\varepsilon(\omega) \geq 0$, this easily implies the C^0-estimate

$$|f_\varepsilon(\omega)| \leq 4e^s s^{-1}\varepsilon. \tag{A.10}$$

We now prove the C^1-estimate for f_ε on $\partial\Omega_0$; in order to simplify a bit the notation, we assume that $\partial\Omega_0 = \mathbb{S}^1$ and so we write $\omega = e^{i\theta}$. Then $u_\varepsilon(\Phi_\varepsilon(\omega)) = u_\varepsilon^*(s)$ implies

$$\begin{aligned} 0 &= \partial_\theta(u_\varepsilon^*(s)) = \partial_\theta\left(u_\varepsilon(\Phi_\varepsilon(e^{i\theta}))\right) \\ &= \left(2\varepsilon\,\mathrm{Re}[G'(\Phi_\varepsilon)\partial_\theta(\Phi_\varepsilon(e^{i\theta}))] + 2\varepsilon^2\,\mathrm{Re}[G'(\Phi_\varepsilon)\partial_\theta(\Phi_\varepsilon(e^{i\theta}))\,\overline{G(\Phi_\varepsilon)}]\right)e^{-\pi|\Phi_\varepsilon|^2} \\ &\quad + \left(1 + 2\varepsilon\,\mathrm{Re}(G(\Phi_\varepsilon)) + \varepsilon^2|G(\Phi_\varepsilon)|^2\right)(-\pi\,\partial_\theta|\Phi_\varepsilon(e^{i\theta})|^2)e^{-\pi|\Phi_\varepsilon|^2}, \end{aligned} \tag{A.11}$$

which can be rewritten as

$$|1+\varepsilon G(\Phi_\varepsilon)|^2\partial_\theta|\Phi_\varepsilon(e^{i\theta})|^2 = \frac{2\varepsilon}{\pi}\,\mathrm{Re}\left(G'(\Phi_\varepsilon)\partial_\theta(\Phi_\varepsilon(e^{i\theta}))\left(1+\varepsilon\overline{G(\Phi_\varepsilon)}\right)\right). \tag{A.12}$$

By the explicit formula for Φ_ε, cf. (5.2) and (5.5), we have

$$\begin{aligned} \partial_\theta(\Phi_\varepsilon(e^{i\theta})) &= ie^{i\theta}(1 + f_\varepsilon(e^{i\theta})) + ie^{2i\theta}f_\varepsilon'(e^{i\theta}), \\ \partial_\theta|\Phi_\varepsilon(e^{i\theta})|^2 &= 2ie^{i\theta}(1 + f_\varepsilon(e^{i\theta}))f_\varepsilon'(e^{i\theta}). \end{aligned} \tag{A.13}$$

We now argue essentially as in Lemma 3.1. More precisely, we note that if ε is sufficiently small (depending on $\|G\|_{\mathcal{F}^2}$) then

$$\frac{1}{2} \leq |1 + \varepsilon G(\Phi_\varepsilon)| \leq \frac{3}{2}, \qquad \frac{1}{2} \leq |1 + f_\varepsilon(e^{i\theta})| \leq \frac{3}{2},$$

where the last bounds follow from (A.10). Thus, taking absolute values in (A.12) and then inserting identities (A.13), we obtain

$$\begin{aligned} |f_\varepsilon'(e^{i\theta})| &\leq C\varepsilon|\partial_\theta(\Phi_\varepsilon(e^{i\theta}))||G'(\Phi_\varepsilon)|\,(1 + \varepsilon|G(\Phi_\varepsilon)|) \\ &\leq C\varepsilon\|G\|_{\mathcal{F}^2}\left(1 + |f_\varepsilon'(e^{i\theta})| + \varepsilon\|G\|_{\mathcal{F}^2}\right). \end{aligned} \tag{A.14}$$

Here, to pass to the last line, we also used the estimate $|G'(\Phi_\varepsilon)| \leq C\|G\|_{\mathcal{F}^2}$ which follows from the Cauchy integral formula and the C^0-estimate (A.10), exactly as in (3.9). By choosing ε sufficiently small, we can absorb the term $|f_\varepsilon'(e^{i\theta})|$ on the right-hand side of (A.14) into its left-hand side, and so we obtain

$$|f_\varepsilon'(e^{i\theta})| \leq \eta(\varepsilon)\|G\|_{\mathcal{F}^2}. \tag{A.15}$$

By differentiating (A.11) once more with respect to s, repeating the argument above and using the bound (A.15), one likewise obtains $|f_\varepsilon''(e^{i\theta})| \leq \eta(\varepsilon)\|G\|_{\mathcal{F}^2}$, as claimed in (A.6). Since the details are essentially the same and bear no real insight, we omit them.

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

Having (A.6) at our disposal, estimate (A.1) follows easily from the explicit formula for the fields X_ε in (5.5) and (A.2) follows immediately from (A.1) by ODE theory; alternatively, one may also argue this directly from the explicit form of the underlying vector fields in Lemma 5.4.

To prove the remaining estimates we return to (A.7) and (A.8) to see that $\Omega_\varepsilon = \{u_\varepsilon > u_\varepsilon^*(s)\}$ is nested between two balls:

$$B_\varepsilon^1 := \{u_0 > e^{-s} + 4\varepsilon\} \subset \Omega_\varepsilon \subset \{u_0 > e^{-s} - 4\varepsilon\} =: B_\varepsilon^2.$$

Proposition 3.3 ensures that, by choosing ε_0 sufficiently small, Ω_ε is convex, hence

$$\mathcal{H}^1(\partial B_\varepsilon^1) \le \mathcal{H}^1(\partial\Omega_\varepsilon) \le \mathcal{H}^1(\partial B_\varepsilon^2).$$

Let us denote by R_ε^1, R_ε^2 the radii of B_ε^1, B_ε^2 respectively. If we choose ε_0 sufficiently small, depending on s, then it is easy to see that

$$s - 8e^s\varepsilon \le \pi(R_\varepsilon^1)^2 \le \pi(R_\varepsilon^2)^2 \le s + 8e^s\varepsilon, \tag{A.16}$$

similarly to (A.9). This immediately yields

$$4\pi(s - 8e^s\varepsilon) \le \mathcal{H}^1(\partial B_\varepsilon^1)^2 \le \mathcal{H}^1(\partial B_\varepsilon^2)^2 \le 4\pi(s + 8e^s\varepsilon),$$

and, as $\mathcal{H}^1(\partial\Omega_0)^2 = 4\pi s$, (A.3) follows. Similarly, for (A.4), we have

$$|\Omega_\varepsilon \triangle \Omega_0| \le |B_\varepsilon^2 \setminus B_\varepsilon^1| \le \pi\left((R_\varepsilon^2)^2 - (R_\varepsilon^1)^2\right) \le 16e^s\varepsilon.$$

Finally, (A.5) follows from (A.4) through $\sigma(\Omega_\varepsilon \triangle \Omega_0) \le |\Omega_\varepsilon \triangle \Omega_0|$. □

Proof of Lemma 5.2 Returning to the beginning of the proof of Lemma 5.6, and in particular to (5.11), we can write, using Taylor's theorem,

$$K_\varepsilon = K_0 + \frac{\varepsilon^2}{2}K_0'' + \int_0^\varepsilon (\varepsilon - s)(K_s'' - K_0'')\,\mathrm{d}s,$$

and thus our task is to show that, for all $\varepsilon \in [0, \varepsilon_0]$, we have

$$|K_\varepsilon'' - K_0''| \le \eta(\varepsilon) \tag{A.17}$$

for a suitable modulus of continuity η. We recall that, as in the proof of Lemma 5.6, primes denote derivatives with respect to ε, and we also recall the definition of I_ε and J_ε from (5.7).

Although in the statement of the lemma we assumed that $\|G\|_{\mathcal{F}^2} = 1$, for the sake of clarity we will still write $\|G\|_{\mathcal{F}^2}$ explicitly in our estimates. Since $\langle G, 1\rangle_{\mathcal{F}^2} = 0$ by (3.6), we have

$$J_\varepsilon - J_0 = \|1 + \varepsilon G\|_{\mathcal{F}^2}^2 - \|1\|_{\mathcal{F}^2}^2 = \varepsilon^2\|G\|_{\mathcal{F}^2}^2 = \varepsilon^2, \tag{A.18}$$

and so $J_\varepsilon \ge J_0 = 1$. By (5.8), the function K_ε'', seen as a function of $(I_\varepsilon, I_\varepsilon', I_\varepsilon'', J_\varepsilon, J_\varepsilon', J_\varepsilon'')$, is smooth in the set $\{J_\varepsilon > 0\}$ and therefore there is a constant C such that

$$\left|K_\varepsilon'' - K_0''\right| \le C(|I_\varepsilon - I_0| + |I_\varepsilon' - I_0'| + |I_\varepsilon'' - I_0''| + |J_\varepsilon - J_0| + |J_\varepsilon' - J_0'|). \tag{A.19}$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

since $J_\varepsilon'' = J_0''$ is independent of ε, cf. (5.10). It now suffices to estimate each of the terms on the right-hand side of (A.19). In (A.18), we have already estimated $|J_\varepsilon - J_0|$ and, similarly, we have

$$|J_\varepsilon' - J_0'| = 2\int_{\mathbb{C}} \langle G, \varepsilon G\rangle \,\mathrm{d}\sigma = 2\varepsilon \|G\|_{\mathcal{F}^2}^2, \tag{A.20}$$

where we used the identities in (5.9). Thus it remains to estimate the first three terms in (A.19).

For the first term, again since $\langle G, 1\rangle_{\mathcal{F}^2} = 0$, we estimate using the fundamental theorem of calculus:

$$|I_\varepsilon - I_0| \le \varepsilon \sup_\varepsilon |I_\varepsilon'| \le 2\varepsilon |\langle G, 1 + \varepsilon G\rangle_{\mathcal{F}^2}| = 2\varepsilon^2 \|G\|_{\mathcal{F}^2}^2. \tag{A.21}$$

For the second term, with the help of (A.5), we estimate

$$\begin{aligned} |I_\varepsilon' - I_0'| &= 2\left|\langle G, (1+\varepsilon G)1_{\Omega_\varepsilon} - 1_{\Omega_0}\rangle_{\mathcal{F}^2}\right| \\ &\le 2\|G\|_{\mathcal{F}^2}\sigma(\Omega_\varepsilon \Delta \Omega_0) + 2\varepsilon \|G\|_{\mathcal{F}^2}^2 \\ &\le \eta(\varepsilon)\|G\|_{\mathcal{F}^2}^2. \end{aligned} \tag{A.22}$$

Finally, we arrive at the third term:

$$\begin{aligned} \frac{1}{2}(I_\varepsilon'' - I_0'') = &\int_{\mathbb{C}} |G|^2 (1_{\Omega_\varepsilon} - 1_{\Omega_0})\,\mathrm{d}\sigma + \int_{\partial\Omega_\varepsilon} \langle G, 1+\varepsilon G\rangle \langle X_\varepsilon, \nu_\varepsilon\rangle e^{-\pi|z|^2}\,\mathrm{d}\mathcal{H}^1(z) \\ &- \int_{\partial\Omega_0} \langle G, 1\rangle \langle X_0, \nu_0\rangle e^{-\pi|z|^2}\,\mathrm{d}\mathcal{H}^1(z). \end{aligned}$$

The first term on the right-hand side is easily estimated using (A.4):

$$\int_{\mathbb{C}} |G|^2 (1_{\Omega_\varepsilon} - 1_{\Omega_0})\,\mathrm{d}\sigma \le \|G\|_{\mathcal{F}^2}^2 |\Omega_\varepsilon \Delta \Omega_0| \le \eta(\varepsilon)\|G\|_{\mathcal{F}^2}^3.$$

For the last two terms, we write

$$\begin{aligned} &\int_{\partial\Omega_\varepsilon} \langle G, 1+\varepsilon G\rangle \langle X_\varepsilon, \nu_\varepsilon\rangle e^{-\pi|z|^2}\,\mathrm{d}\mathcal{H}^1(z) - \int_{\partial\Omega_0} \langle G, 1\rangle \langle X_0, \nu_0\rangle e^{-\pi|z|^2}\,\mathrm{d}\mathcal{H}^1(z) \\ &\quad = A_1 + A_2 + A_3, \end{aligned}$$

where

$$\begin{aligned} A_1 &:= \int_{\partial\Omega_\varepsilon} \langle G, 1+\varepsilon G\rangle e^{-\pi|z|^2} (\langle X_\varepsilon, \nu_\varepsilon\rangle - \langle X_0, \nu_0\rangle)\,\mathrm{d}\mathcal{H}^1(z), \\ A_2 &:= \int_{\partial\Omega_\varepsilon} \varepsilon |G|^2 \langle X_0, \nu_0\rangle e^{-\pi|z|^2}\,\mathrm{d}\mathcal{H}^1(z), \\ A_3 &:= \int_{\partial\Omega_\varepsilon} \langle G, 1\rangle e^{-\pi|z|^2} \langle X_0, \nu_0\rangle - \int_{\partial\Omega_0} \langle G, 1\rangle e^{-\pi|z|^2} \langle X_0, \nu_0\rangle\,\mathrm{d}\mathcal{H}^1(z). \end{aligned}$$

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

Here, we have extended ν_ε to a neighbourhood of $\partial\Omega_0$ as $\nu_\varepsilon = -\frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}$. With this definition, it follows from a computation similar to that of Lemma 3.1 that $\|\nu_\varepsilon - \nu_0\|_{L^\infty} \le \tilde{C}(s)\varepsilon\|G\|_{\mathcal{F}^2}$ in a neighbourhood of $\partial\Omega_0$. Thus, on $\partial\Omega_\varepsilon$,

$$\begin{aligned}|\langle X_\varepsilon, \nu_\varepsilon\rangle - \langle X_0, \nu_0\rangle| &\le |\langle X_\varepsilon, \nu_\varepsilon - \nu_0\rangle| + |\langle X_\varepsilon - X_0, \nu_0\rangle| \\ &\le C(s)\varepsilon\|G\|_{\mathcal{F}^2} + \eta(\varepsilon)\|G\|_{\mathcal{F}^2}\end{aligned}$$

and so, using (A.1) and (A.3), we estimate

$$\begin{aligned}|A_1| &\le C(s)\varepsilon\|G\|_{\mathcal{F}^2}\int_{\partial\Omega_\varepsilon} |G|(1+\varepsilon|G|)e^{-\pi|z|^2}\,\mathrm{d}\mathcal{H}^1(z) \\ &\le C(s)\eta(\varepsilon)(\|G\|_{\mathcal{F}^2}^2 + \|G\|_{\mathcal{F}^2}^3).\end{aligned}$$

Similarly, by (A.1) and (5.19) we have

$$|A_2| \le \varepsilon\|G\|_{\mathcal{F}^2}^2\int_{\partial\Omega_\varepsilon}\langle X_0, \nu_0\rangle\,\mathrm{d}\mathcal{H}^1 \le C(s)\varepsilon\|G\|_{\mathcal{F}^2}^2.$$

Finally, we note that

$$\begin{aligned}&\int_{\partial\Omega_\varepsilon}\langle G, 1\rangle e^{-\pi|z|^2}\langle X_0, \nu_0\rangle\,\mathrm{d}\mathcal{H}^1(z) \\ &\quad= \int_{\partial\Omega_0}|\nabla\Phi_\varepsilon(\omega)|\langle G(\Phi_\varepsilon(\omega)), 1\rangle\langle X_0(\Phi_\varepsilon(\omega)), \nu_0(\Phi_\varepsilon(\omega))\rangle e^{-\pi|\Phi_\varepsilon(\omega)|^2}\,\mathrm{d}\mathcal{H}^1(\omega),\end{aligned}$$

and so we have

$$\begin{aligned}A_3 = \frac{2}{\mathcal{H}^1(\partial\Omega_0)}\int_{\partial\Omega_0}&|\nabla\Phi_\varepsilon|\operatorname{Re} G(\Phi_\varepsilon)\langle X_0(\Phi_\varepsilon), \nu_0(\Phi_\varepsilon)\rangle e^{-\pi|\Phi_\varepsilon|^2} \\ &- \operatorname{Re} G\langle X_0, \nu\rangle e^{-\pi|\cdot|^2}\,\mathrm{d}\mathcal{H}^1.\end{aligned}$$

By (A.2) we have $||\nabla\Phi_\varepsilon| - 1| \le \eta(\varepsilon)\|G\|_{\mathcal{F}^2}$, and so it suffices to estimate the function

$$g(z) := \operatorname{Re} G(z)\langle X_0(z), z/|z|\rangle e^{-\pi|z|^2}.$$

We have

$$\begin{aligned}|\nabla g(z)| &\le (|G'(z)||X_0(z)| + |G(z)||\nabla X_0(z)| + C(s)|G(z)||X_0(z)| \\ &\quad + 2\pi|z||G(z)||X_0(z)|)e^{-\pi|z|^2} \\ &\le \|G\|_{\mathcal{F}^2}(C(s)|G(z)| + 2\pi|z||G(z)|)e^{-\pi|z|^2} \le C(s)\|G\|_{\mathcal{F}^2}^2,\end{aligned}$$

where to pass to the second line we used the definition of X_0 from (5.5) (together with the smoothness of $\varepsilon \mapsto f_\varepsilon$ proved in Lemma 5.5) and the fact that G is in the

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

Fock space (together with $G(0) = G'(0) = 0$, as done in (2.15) for the function $R(z)$). Hence, using again (A.2), and as $\Phi_0(\omega) = \omega$, we have

$$|g(\Phi_\varepsilon(\omega)) - g(\omega)| \leq \|\nabla g\|_{L^\infty} |\Phi_\varepsilon(\omega) - \omega| \leq C(s)\eta(\varepsilon)\|G\|^3_{\mathcal{F}^2},$$

which yields immediately

$$|A_3| \leq \eta(\varepsilon)\|G\|^2_{\mathcal{F}^2}.$$

Hence, combining the last estimates, and up to replacing η with a new modulus of continuity, we have

$$|I''_\varepsilon - I''_0| \leq \eta(\varepsilon)\|G\|^2_{\mathcal{F}^2}. \tag{A.23}$$

The desired decay (A.17) now follows by combining (A.18)–(A.23). □

Acknowledgements A.G. was supported by Dr. Max Rössler, the Walter Haefner Foundation and the ETH Zürich Foundation. J.P.G.R. acknowledges financial support of the European Research Council under the Grant Agreement No. 721675 "Regularity and Stability in Partial Differential Equations (RSPDE)".

Funding Open access funding provided by EPFL Lausanne.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. Abreu, L.D., Dörfler, M.: An inverse problem for localization operators. Inverse Probl. **28**(11) (2012)
2. Abreu, L.D., Speckbacher, M.: Donoho-Logan large sieve principles for modulation and polyanalytic Fock spaces. Bull. Sci. Math. **171**, 103032 (2021)
3. Acerbi, E., Fusco, N., Morini, M.: Minimality via second variation for a nonlocal isoperimetric problem. Commun. Math. Phys. **322**, 515–557 (2013)
4. Aleksei, K.: Functionals with extrema at reproducing kernels. Geom. Funct. Anal. **32**(4), 938–949 (2022)
5. Allen, M., Kriventsov, D., Neumayer, R.: Sharp quantitative Faber-Krahn inequalities and the Alt-Caffarelli-Friedman monotonicity formula. Ars Inveniendi Analytica, 49 (2023)
6. Berezin, F.: Wick and anti-Wick operator symbols. Math. USSR Sb. **15**(4), 577–606 (1971)
7. Brasco, L., De Philippis, G., Velichkov, B.: Faber–Krahn inequalities in sharp quantitative form. Duke Math. J. **164**(9), 1777–1831 (2015)
8. Cicalese, M., Leonardi, G.P.: A selection principle for the sharp quantitative isoperimetric inequality. Arch. Ration. Mech. Anal. **206**(2), 617–643 (2012)
9. Daubechies, I.: Time-frequency localization operators: a geometric phase space approach. IEEE Trans. Inf. Theory **34**(4), 605–612 (1988)
10. Donoho, D., Stark, P.: Uncertainty principles and signal recovery. SIAM J. Appl. Math. **49**(3), 906–931 (1989)

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

11. Figalli, A., Glaudo, F.: An Invitation to Optimal Transport, Wasserstein Distances, and Gradient Flows, vol. 8. EMS Press (2021)
12. Figalli, A., Maggi, F., Pratelli, A.: A mass transportation approach to quantitative isoperimetric inequalities. Invent. Math. **182**, 167–211 (2010)
13. Fleschler, I., Tolsa, X., Villa, M.: Faber-Krahn inequalities, the alt-Caffarelli-Friedman formula, and Carleson's ε^2 conjecture in higher dimensions (2023). ArXiv preprint arXiv:2306.06187
14. Frank, R.: Sharp inequalities for coherent states and their optimizers. Adv. Nonlinear Stud. **23**(1), 20220050 (2023)
15. Frank, R., Nicola, F., Tilli, P.: The generalized Wehrl entropy bound in quantitative form (2023). ArXiv preprint arXiv:2307.14089
16. Fuglede, B.: Stability in the isoperimetric problem for convex or nearly spherical domains in R^n. Trans. Am. Math. Soc. **314**(2), 619 (1989)
17. Fusco, N., Maggi, F., Pratelli, A.: The sharp quantitative isoperimetric inequality. Ann. Math. **168**, 941–980 (2008)
18. Galbis, A.: Norm estimates for selfadjoint Toeplitz operators on the Fock space. Complex Anal. Oper. Theory **16** (2022)
19. Gröchenig, K.: Foundations of Time–Frequency Analysis. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, Cambridge (2001)
20. Henrot, A., Pierre, M.: Variation et Optimisation de Formes. Mathématiques & Applications, vol. 48. Springer, Berlin (2005)
21. Hörmander, L.: The Analysis of Linear Partial Differential Operators I. Classics in Mathematics. Springer, Berlin (2003)
22. Kalaj, D.: Contraction property of certain classes of log $\mathcal{M}-$ subharmonic functions in the unit ball in $\mathbb{R}^n$ (2022). ArXiv preprint arXiv:2207.02054
23. Kalaj, D.: Contraction property of differential operator on Fock space (2022). ArXiv preprint arXiv:2207.13606
24. Kalaj, D., Melentijević, P.: Weighted contractivity for derivatives of functions in the Bergman space on the unit disk (2023). ArXiv preprint arXiv:2302.13424
25. Kalaj, D., Ramos, J.P.G.: A Faber-Krahn type inequality for log-subharmonic functions in the hyperbolic ball (2023). ArXiv preprint arXiv:2303.08069
26. Krantz, S.G., Parks, H.R.: The Implicit Function Theorem: History, Theory, and Applications. Springer, Berlin (2002)
27. Kulikov, A., Nicola, F., Ortega-Cerdà, J., Tilli, P.: Monotonicity theorems for subharmonic functions on manifolds (2022). ArXiv preprint arXiv:2212.14008
28. Lieb, E.: Integral bounds for radar ambiguity functions and Wigner distributions. J. Math. Phys. **31**(3), 594–599 (1990)
29. Lieb, E., Loss, M.: Analysis, vol. 14. Am. Math. Soc., Providence (2001)
30. Mallat, S.: A Wavelet Tour of Signal Processing. The Sparse Way. Elsevier, Amsterdam (2009)
31. Melentijević, P.: Hypercontractive inequalities for weighted Bergman spaces (2022). ArXiv preprint arXiv:2211.03655
32. Nakajima, S.: Über konvexe kurven und flächen. Tohoku Math. J. **29**, 227–230 (1928)
33. Nicola, F., Tilli, P.: The Faber–Krahn inequality for the short-time Fourier transform. Invent. Math. **230**(1), 1–30 (2022)
34. Nicola, F., Tilli, P.: The norm of time-frequency localization operators. Trans. Am. Math. Soc. **376**(10), 7353–7375 (2023)
35. Ramos, J.P.G.: Inverse problems for eigenfunctions of localization operators. In preparation (2023)
36. Ramos, J.P.G., Tilli, P.: A Faber–Krahn inequality for wavelet transforms. Bull. Lond. Math. Soc. **55**(4), 2018–2034 (2023)
37. Tataru, D.: Phase space transforms and microlocal analysis. In: Phase Space Analysis of Partial Differential Equations. Vol. II, pp. 505–524. Pubbl. Cent. Ric. Mat. Ennio De Giorgi, Scuola Norm. Sup., Pisa (2004)
38. Tietze, H.: Über konvexheit im kleinen und im großen und über gewisse den punkten einer menge zugeordnete dimensionszahlen. Math. Z. **28**(1), 697–707 (1928)
39. Zhu, K.: Analysis on Fock Spaces. Graduate Texts in Mathematics, vol. 263. Springer, Boston (2012)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Content courtesy of Springer Nature, terms of use apply. Rights reserved.

Terms and Conditions

Springer Nature journal content, brought to you courtesy of Springer Nature Customer Service Center GmbH ("Springer Nature").

Springer Nature supports a reasonable amount of sharing of research papers by authors, subscribers and authorised users ("Users"), for small-scale personal, non-commercial use provided that all copyright, trade and service marks and other proprietary notices are maintained. By accessing, sharing, receiving or otherwise using the Springer Nature journal content you agree to these terms of use ("Terms"). For these purposes, Springer Nature considers academic use (by researchers and students) to be non-commercial.

These Terms are supplementary and will apply in addition to any applicable website terms and conditions, a relevant site licence or a personal subscription. These Terms will prevail over any conflict or ambiguity with regards to the relevant terms, a site licence or a personal subscription (to the extent of the conflict or ambiguity only). For Creative Commons-licensed articles, the terms of the Creative Commons license used will apply.

We collect and use personal data to provide access to the Springer Nature journal content. We may also use these personal data internally within ResearchGate and Springer Nature and as agreed share it, in an anonymised way, for purposes of tracking, analysis and reporting. We will not otherwise disclose your personal data outside the ResearchGate or the Springer Nature group of companies unless we have your permission as detailed in the Privacy Policy.

While Users may use the Springer Nature journal content for small scale, personal non-commercial use, it is important to note that Users may not:

1. use such content for the purpose of providing other users with access on a regular or large scale basis or as a means to circumvent access control;
2. use such content where to do so would be considered a criminal or statutory offence in any jurisdiction, or gives rise to civil liability, or is otherwise unlawful;
3. falsely or misleadingly imply or suggest endorsement, approval , sponsorship, or association unless explicitly agreed to by Springer Nature in writing;
4. use bots or other automated methods to access the content or redirect messages
5. override any security feature or exclusionary protocol; or
6. share the content in order to create substitute for Springer Nature products or services or a systematic database of Springer Nature journal content.

In line with the restriction against commercial use, Springer Nature does not permit the creation of a product or service that creates revenue, royalties, rent or income from our content or its inclusion as part of a paid for service or for other commercial gain. Springer Nature journal content cannot be used for inter-library loans and librarians may not upload Springer Nature journal content on a large scale into their, or any other, institutional repository.

These terms of use are reviewed regularly and may be amended at any time. Springer Nature is not obligated to publish any information or content on this website and may remove it or features or functionality at our sole discretion, at any time with or without notice. Springer Nature may revoke this licence to you at any time and remove access to any copies of the Springer Nature journal content which have been saved.

To the fullest extent permitted by law, Springer Nature makes no warranties, representations or guarantees to Users, either express or implied with respect to the Springer nature journal content and all parties disclaim and waive any implied warranties or warranties imposed by law, including merchantability or fitness for any particular purpose.

Please note that these rights do not automatically extend to content, data or other material published by Springer Nature that may be licensed from third parties.

If you would like to use or distribute our Springer Nature journal content to a wider audience or on a regular basis or in any other manner not expressly permitted by these Terms, please contact Springer Nature at

onlineservice@springernature.com