Robust Coordination of Linear Threshold Dynamics on Directed Weighted Networks

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Robust Coordination of Linear Threshold Dynamics on Directed Weighted Networks

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***Abstract*—We study dynamics in a network of interacting agents updating their binary states according to a time-varying threshold rule. Specifically, agents revise their state asynchronously by comparing the weighted average of the current states of their neighbors in the interaction network with possibly heterogeneous time-varying threshold values. Such thresholds are determined by an exogenous signal representing an external influence field modeling the different agents' biases towards one state with respect to the other one. We prove necessary and sufficient conditions for global stability of consensus equilibria, robustly with respect to the (constant or time-varying) external field. Our results apply to general weighted directed interaction networks and build on super-modularity properties of certain network coordination games whose best response dynamics coincide with the linear threshold dynamics. In particular, we introduce a novel notion of *robust improvement paths* for such games and characterize necessary and sufficient conditions for their existence.**

Index terms: Linear threshold dynamics, coordination games, network games, network robustness, best response dynamics, robust stability.

I. Introduction

Robustness, meant as the ability of a system to maintain its performance under a range of different operating conditions, is undoubtedly a fundamental issue that has long been studied in control [2]. While playing a key role in several domains, robustness and the related notion of resilience have lately become central in multi-agent and network systems, such as infrastructure systems [3]–[6], financial networks [7]–[10], as well as social and economic networks [11]–[14]. In such contexts, robustness is typically presented as the capability of the system to react to localized perturbations by absorbing their effect locally and preventing the global propagation of cascading failures that could prove detrimental for the whole system. A characteristic feature that has been recognized is that the topology of the interconnection pattern is a key factor determining the robustness or fragility of such network systems [15]–[19].

In this paper, we focus on linear threshold dynamics (LTD), a prototypical family of nonlinear network systems first introduced in [20] for fully mixed populations of agents and later extended in various directions [21]–[24]. While LTD can be defined in different ways, their core structure consists of a set of agents identified with nodes of an interaction network that strategically change their binary state (±1) according to a threshold rule. Specifically, agents adopt state $+1$ if and only if the fraction of their neighbors in the interaction network that do so is greater than or equal to a certain exogenous threshold. Various studies of LTD models [21], [25]–[29] have concerned topological conditions guaranteeing or preventing full contagion (i.e., convergence to a configuration where all agents are in state $+1$) starting from an initial condition of relatively few agents in state $+1$. Most of these studies concern random networks of a specific type. A remarkable exception is [21] that introduces the concept of cohesiveness of a subset of nodes in a network, through which one can in principle characterize the extent of a spreading phenomenon.

In the literature, LTD models are typically modeled as closed systems without explicit input or output signals. The basic challenge of this paper is to study LTD intrinsically equipped with an external field modeling a possibly node-specific influence from the external environment. As in the classical LTD models without external field the asymptotic outcomes are always consensus equilibria, our analysis concentrates on when a possibly time-varying external filed can modify this behavior. Precisely, our results are of two types:

- robust stability results showing that the LTD converges to a consensus for every possibly time-varying external field taking values in a certain range;
- control results showing that a suitable control signal is capable of preventing the system from reaching consensus by steering it to a different polarized configuration or by forcing persistent oscillations.

Such behaviors will depend on the topology of the interaction network (building on suitable generalizations of the concept of cohesiveness) and the constraints on the input signal.

LTD can be interpreted as the best response dynamics in a network game whereby agents choose strategically between two states and their payoff is an increasing function of the

Some of the results in the paper appeared in preliminary form in [1].

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number of their neighbors choosing the same state. Such games are known as network coordination games and represent one of the most studied models describing network systems with interactions of strategic complements type [30], [31]. They find numerous applications in modeling social and economic behaviors like the emergence of social norms and conventions or the adoption of new technologies [32]–[36].

Optimal seeding and other intervention problems for network coordination games have been studied in [37] and, in the more general setting of super-modular games, in [38], [39]. Our goal is different in this paper, as we are mainly interested in understanding the resilience of the system against external attacks. Recently, vulnerability of network coordination games against adversarial attacks has been investigated in [40], [41], while [1], [42], [43] study games with a mix of coordinating and anti-coordinating players, and [44] proposes network coordination games as a micro-foundation for community structure in networks.

Our analysis strongly relies on the interpretation of the LTD as a type of best response dynamics of an underlying network coordination game. We then build on super-modularity of such games, i.e., the increasing difference property [45], [46]. Specifically, the convenience for a player to switch from a state to an alternative state is monotone in the fraction of players in their neighborhood already playing the alternative state. Such property continues to hold true under the influence of an external field. A variation of the external field modifies the threshold of the agents, in extreme cases transforming them into stubborn agents, i.e., agents whose best response is always the same state, regardless of her fellow agents' states.

In particular, we study conditions under which a system converges to a consensus equilibrium, independently from the values taken by an external field. As it turns out, two conditions need to be satisfied for such robust stability property to hold true. The first condition, to be referred to as robust indecomposability, is a generalization of the lack of cohesive partitions [32] to parametrized families of heterogeneous network coordination games. It is equivalent (see Theorem 3) to the lack of co-existent equilibria, i.e., equilibria that are not consensus configurations, for any value of the external field within a certain range. On the other hand, the second condition guarantees that the external field is incapable of creating stubborn agents for both states. While the necessity of these two conditions for convergence to a consensus is quite intuitive, the proof of sufficiency is more involved and resides on the possibility to find improvement paths for the game that are robust to modifications of the external field. This is achieved in Theorem 4 that is one of our main results and uses in a crucial way the super-modularity of the game.

The rest of the paper is organized as follows. We report some basic notation in the remaining part of this section. In Section II we present the problem. We introduce the LTD with external field and the fundamental concept of indecomposability (Definition 1). We then state two main results on the asymptotic behavior of such model, Proposition 1 and Theorem 1, and we illustrate the outcomes through a number of examples and simulations. Section III is completely devoted to the analysis of network coordination games, especially the structure of their set of Nash equilibria that play a crucial role in our study of the LTD and the reachability and stability results that we gain from super-modularity properties. Section IV-A contains the core technical part of the paper. In particular, Theorem 4 contains robust reachability and stability results for network coordination games that are the fundamental ingredients to then prove Theorem 1. The paper is completed with a Section of conclusions.

A. Notation

For a finite set $\mathcal{I}$, we consider vector spaces $\mathbb{R}^{\mathcal{I}}$ equipped with the partial order

$$x \leq y \quad \Leftrightarrow \quad x_i \leq y_i, \quad \forall i \in \mathcal{I}\,.$$

We use the notation $x \lneq y$ when $x \leq y$ and $x_i < y_i$ for some i in $\mathcal{I}$. A function $f : \mathbb{R}^{\mathcal{I}} \to \mathbb{R}^{\mathcal{J}}$ is referred to as monotone nondecreasing (nonincreasing) if it preserves (reverses) the partial order $\leq$, i.e., if $f(x) \leq f(y)$ ($f(x) \geq f(y)$) for every $x \leq y$. For a vector x in $\mathbb{R}^{\mathcal{I}}$, $|x|$ in $\mathbb{R}^{\mathcal{I}}$ stands for the vector with entries $(|x|)_i = |x_i|$ for every i in $\mathcal{I}$. The symbol $\mathbf{1}$ indicates a vector with all entries equal to 1.

II. Problem statement and main results

Throughout the paper, we model networks as (finite directed weighted) graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, with set of nodes $\mathcal{V}$, set of directed links $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and weight matrix W in $\mathbb{R}_+^{\mathcal{V}\times\mathcal{V}}$, whose entries are such that $W_{ij} > 0$ if and only if $(i,j) \in \mathcal{E}$. We do not allow for the presence of self-loops, equivalently, we assume that the weight matrix W has zero diagonal. We refer to the graph as undirected in the special case when the weight matrix $W = W'$ is symmetric, so that there is a link (i,j) directed from node i to node j in $\mathcal{E}$ if and only if there is also the reverse link (j,i) directed from node j to node i in $\mathcal{E}$ and both links have the same weight $W_{ij} = W_{ji}$.

For a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ and a subset of nodes $\mathcal{S} \subseteq \mathcal{V}$, we denote by

$$w_i^{\mathcal{S}} = \sum_{j\in\mathcal{S}} W_{ij}$$

the $\mathcal{S}$-*restricted out-degree* of a node i in $\mathcal{V}$. In the special case when $\mathcal{S} = \mathcal{V}$ coincides with the whole node set, we simply refer to $w_i = w_i^{\mathcal{V}}$ as the *out-degree* of a node i in $\mathcal{V}$ and let $w = W\mathbf{1}$ be the vector of out-degrees.

The nodes of the network represent interacting agents. Every agent i in $\mathcal{V}$ is endowed by a binary time-varying state $X_i(t)$. A link (i,j) in $\mathcal{E}$ is meant as directed from its tail node i to its head node j. The presence of a link (i,j) indicates a direct influence of agent j on agent i, with its weight W_{ij} to be interpreted as a measure of such influence. Let $\mathcal{A} = \{\pm 1\}$ be the binary state set of each agent, and let $\mathcal{X} = \mathcal{A}^{\mathcal{V}}$ be the configuration space: a configuration x in $\mathcal{X}$ is a vector whose entries x_i represent the states of the single agents. The constant vectors $x = \pm\mathbf{1}$ will be referred to as *consensus* configurations. On the other hand, we shall refer to every x in $\mathcal{X} \setminus \{\pm\mathbf{1}\}$ as a *co-existent* configuration.

We consider asynchronous time-varying (ATV) *linear threshold dynamics* (LTD) on a network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, whereby agents i in $\mathcal{V}$ update their binary state $X_i(t)$ in $\mathcal{A}$ as described below. For a nonempty set of vectors $\mathcal{H} \subseteq \mathbb{R}^{\mathcal{V}}$, let $h(t)$ in $\mathcal{H}$ for $t \geq 0$ be an exogenous signal modeling a time-varying external field. Let every agent i in $\mathcal{V}$ be equipped with an independent rate-1 Poisson clock.[1] If agent i's clock ticks at some time $t \geq 0$,[2] then agent i modifies her current state $X_i(t^-)$ into a new state $X_i(t)$ such that

$$X_i(t) = \begin{cases} +1 & \text{if} \quad \sum_j W_{ij} X_j(t) + h_i(t) > 0 \\ X_i(t^-) & \text{if} \quad \sum_j W_{ij} X_j(t) + h_i(t) = 0 \\ -1 & \text{if} \quad \sum_j W_{ij} X_j(t) + h_i(t) < 0 \,. \end{cases} \tag{1}$$

The update rule above can be rewritten in the following equivalent way. For i in $\mathcal{V}$ and time $t \geq 0$, let

$$r_i(t) = \frac{1}{2} - \frac{h_i(t)}{2w_i}\,, \tag{2}$$

be a time-varying threshold for agent i. Also, for a configuration x in $\mathcal{X}$, let

$$w_i^-(x) = \sum_{j : x_j = -1} W_{ij}\,, \qquad w_i^+(x) = \sum_{j : x_j = 1} W_{ij}\,, \tag{3}$$

be the aggregate weight of links pointing from agent i to agents in state -1 and, respectively, to those in state $+1$. Then, (1) is equivalent to

$$X_i(t) = \begin{cases} +1 & \text{if} \quad w_i^+(X(t^-)) > w_i r_i(t) \\ X_i(t^-) & \text{if} \quad w_i^+(X(t^-)) = w_i r_i(t) \\ -1 & \text{if} \quad w_i^+(X(t^-)) < w_i r_i(t)\,, \end{cases} \tag{4}$$

i.e., if agent i gets activated at time $t \geq 0$, then: (a) she updates her state $X_i(t)$ to $+1$ if the weighted fraction $w_i^+(X(t^-))/w_i$ of her out-neighbors currently in state $+1$ is above the time-varying threshold $r_i(t)$ (equivalently, if the weighted fraction $w_i^-(X(t^-))/w_i$ of her out-neighbors in state -1 is below the complementary threshold $1 - r_i(t)$); (b) she updates her state $X_i(t)$ to -1 if $w_i^+(X(t^-))/w_i$ is below $r_i(t)$; or (c) she keeps her current state $X_i(t) = X_i(t^-)$ if $w_i^-(X(t^-))/w_i = r_i(t)$.

If we stack the agents' states in a vector $X(t)$ in $\mathcal{X}$, then $X(t)$ is a continuous-time inhomogeneous Markov chain on the configuration space $\mathcal{X}$. In the rest of the paper, we focus on the asymptotic behavior of the ATV-LTD $X(t)$ on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ with external field $h(t)$, as described above.

Specifically, we shall determine necessary and sufficient conditions for almost sure convergence (i.e., convergence with probability one) to a consensus configuration. In particular, our main result concerns robust convergence to consensus for ATV-LTD on a graph $\mathcal{G}$ when the external field $h(t)$ is an arbitrary (unknown) signal whose range is a hyper-rectangle in the form

$$\mathcal{H} = \{h : \, h^- \leq h \leq h^+\} = \prod_{i \in \mathcal{V}} \left[h_i^-, h_i^+\right]\,, \tag{5}$$

[1]The assumption that all Poisson clocks have rate 1 is made merely for the sake of simplicity of the exposition. In fact, it is not hard to show that all results in the paper continue to hold true as stated in the more general setting where every agent i's Poisson clock has rate $\lambda_i > 0$.

[2]Observe that, with probability 1, no two agents' clocks will ever tick at the same time t. In fact, the updating mechanism could have been equivalently formulated assuming that updates occur at the ticking of a global rate-$|\mathcal{V}|$ Poisson clock and that each time such global clock ticks one agent is sampled uniformly at random from $\mathcal{V}$ and made update her action according to the threshold rule described in the main text.

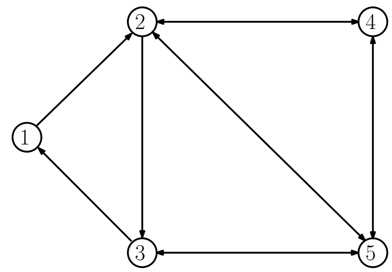

Fig. 1: The network considered in Example 1.

for two (known) vectors h^- and h^+ in $\mathbb{R}^{\mathcal{V}}$ such that $h^- \leq h^+$.

The conditions for robust almost sure convergence to consensus of the ATV-LTD will be determined in terms of graph-theoretic properties of $\mathcal{G}$. In particular, we have the following definition.

Definition 1. *Let h^- and h^+ in $\mathbb{R}^{\mathcal{V}}$ be two vectors such that $h^- \leq h^+$. Then, a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ is (h^-, h^+)-*indecomposable *if for every nontrivial binary partition of the node set*

$$\mathcal{V} = \mathcal{V}_+ \cup \mathcal{V}_-\,, \quad \mathcal{V}_+ \cap \mathcal{V}_- = \emptyset\,, \quad \mathcal{V}_+ \neq \emptyset\,, \quad \mathcal{V}_- \neq \emptyset\,, \tag{6}$$

there exist s in $\{-, +\}$ and a node i in $\mathcal{V}_s$ such that

$$w_i^s + s h_i^s < w_i^{-s}\,, \tag{7}$$

where $w_i^s = w_i^{\mathcal{V}_s}$. In the special case when $h^- = h^+ = h$, we shall more briefly refer to the graph $\mathcal{G}$ as h-indecomposable.

The following example illustrates the notion of indecomposability introduced in Definition 1 above in a simple case.

Example 1. *Consider the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ displayed in Figure 1, with set of nodes $\mathcal{V} = \{1, 2, 3, 4, 5\}$ and weight matrix*

$$W = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

The out-degree vector is then $w = W\mathbf{1} = (1, 3, 2, 2, 3)$.

(i) First, we verify that $\mathcal{G}$ is 0-indecomposable. To see it, first notice that if, for some s in $\{-, +\}$, either $\mathcal{V}_s = \{i\}$ for some i in $\mathcal{V}$ or $\mathcal{V}_s = \{i, j\}$ for some $i \neq j$ in $\mathcal{V}$ such that $(i, j) \notin \mathcal{E}$, then we have $w_i^s = 0 < 1 \leq w_i^{-s}$, so that (7) is satisfied. It is then sufficient to consider binary partitions as in (6) where, $\mathcal{V}_s = \{i, j\}$ for some s in $\{-, +\}$ and $i \neq j$ in $\mathcal{V}$ such that both (i, j) and (j, i) belong to $\mathcal{E}$. This leaves us with four possibilities, corresponding to the four undirected links in the graph: (a) for $\mathcal{V}_s = \{2, 4\}$, we have that $5 \in \mathcal{V}_{-s}$ and $w_5^s = 2 > 1 = w_5^{-s}$ so that (7) is satisfied; for $\mathcal{V}_s = \{2, 5\}$, we have that $4 \in \mathcal{V}_{-s}$ and $w_4^s = 2 > 0 = w_4^{-s}$ so that (7) is satisfied; (c) for both $\mathcal{V}_s = \{3, 5\}$ and $\mathcal{V}_s = \{4, 5\}$, we have that $2 \in \mathcal{V}_{-s}$ and $w_2^s = 2 > 1 = w_2^{-s}$ so that (7) is satisfied.

(ii) Second, we verify that $\mathcal{G}$ is not δ^1-indecomposable. Indeed, let us fix $\mathcal{V}_- = \{2, 4, 5\}$ and $\mathcal{V}_+ = \{1, 3\}$. Then, $w_1^+ + 1 = 1 = w_1^-$, $w_2^- = 2 > 1 = w_2^+$, $w_3^+ = 1 = w_3^-$, $w_4^- = 2 > 0 = w_4^+$, and $w_5^- = 2 > 1 = w_5^+$, so that (7) is violated by every i in $\mathcal{V}_s$ and s in $\{-, +\}$.

(iii) Now, we show that $\mathcal{G}$ is not (h^-, h^+)-indecomposable for $h^- = (0, -1, 0, 0, 0)$ and $h^- = (0, 0, 0, 0, 1)$. Indeed, let us fix $\mathcal{V}_- = \{1, 2, 3\}$ and $\mathcal{V}_+ = \{4, 5\}$. Then, we have that $w_1^- - h_1^- = 1 > 0 = w_1^+$, $w_2^- - h_2^- = 1 + 1 = 2 = w_2^+$, $w_3^- - h_3^- = 1 = w_3^+$, $w_4^+ + h_4^+ = 1 = w_4^-$, and also $w_5^+ + h_5^+ = 1 + 1 = 2 = w_5^-$, so that (7) is violated by every i in $\mathcal{V}_s$ and s in $\{-, +\}$. In contrast, it can be verified that $\mathcal{G}$ is both h^--indecomposable and h^+-indecomposable in this case.

(iv) Finally, let $h^- = 0$ and $h^+ = (0, 2, 0, 0, 2)$. We now show that $\mathcal{G}$ is (h^-, h^+)-indecomposable. To verify that, first notice that if $\mathcal{V}_s = \{i\}$ for some s in $\{-, +\}$ and i in $\mathcal{V}$, then $w_i^s + sh_i^s = sh_i^s < w_i = w_i^{-s}$, so that (7) is satisfied. Similarly, if $1 \in \mathcal{V}_s$ and $2 \in \mathcal{V}_{-s}$, then $w_1^s + sh_1^s = 0 < 1 = w_1^{-s}$. Moreover, if $\{2, 5\} \subseteq \mathcal{V}_s$ for some s in $\{-, +\}$, then (7) is satisfied by every i in $\mathcal{V}_{-s} \cap \{1, 4\}$. This leaves us with four possibilities: (a) $\mathcal{V}_- = \{1, 2, 3\}$ and $\mathcal{V}_+ = \{4, 5\}$; (b) $\mathcal{V}_- = \{1, 2, 4\}$ and $\mathcal{V}_+ = \{3, 5\}$; (c) $\mathcal{V}_- = \{4, 5\}$ and $\mathcal{V}_+ = \{1, 2, 3\}$; (d) $\mathcal{V}_- = \{3, 5\}$ and $\mathcal{V}_+ = \{1, 2, 4\}$. In both cases (a) and (b), we have $w_2^- - h_2^- = 1 < 2 = w_2^+$ so that (7) is satisfied, whereas in both cases (c) and (d), we have $w_5^- - h_5^- = 1 < 2 = w_5^+$ so that (7) is satisfied. Therefore, $\mathcal{G}$ is (h^-, h^+)-indecomposable.

Our first result, stated below, shows that (h^-, h^+)-indecomposability of the graph $\mathcal{G}$ is indeed a necessary condition for robust convergence to a consensus configuration of ATV-LTD when the external field $h(t)$ is a arbitrary signal whose range is the hyper-rectangle (5). First we state a very simple concept: a configuration x^* in $\mathcal{X}$ is called *absorbing* for the ATV-LTD, if $X(t^*) = x^*$ for some $t^* \geq 0$ implies that $X(t) = x^*$ for every $t \geq t^*$.

Proposition 1. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ be a graph. For two vectors h^- and h^+ in $\mathbb{R}^{\mathcal{V}}$ such that $h^- \leq h^+$, let $\mathcal{H}$ be as in (5). If $\mathcal{G}$ is not (h^-, h^+)-indecomposable, then there exist h^* in $\mathcal{H}$ and a co-existent configuration x^* in $\mathcal{X} \setminus \{\pm\mathbf{1}\}$ such that x^* is an absorbing configuration for the ATV-LTD on $\mathcal{G}$ with constant external field $h(t) = h^*$.*

Proof. That the graph $\mathcal{G}$ is not (h^-, h^+)-indecomposable means that there exists a nontrivial binary partition of the node set as in (6) such that

$$w_i^s - w_i^{-s} + sh_i^s \geq 0 \,,$$

for all i in $\mathcal{V}_s$ and s in $\{-, +\}$. The above can be rewritten as

$$s(w_i^+ - w_i^- + h_i^s) \geq 0 \,. \tag{8}$$

Now, let h^* in $\mathbb{R}^{\mathcal{V}}$ be a vector with entries

$$h_i^* = \begin{cases} h_i^- & \text{if} \quad i \in \mathcal{V}_- \\ h_i^+ & \text{if} \quad i \in \mathcal{V}_+ \,, \end{cases}$$

and let x^* in $\mathcal{X}$ be a configuration with entries

$$x_i^* = \begin{cases} -1 & \text{if} \quad i \in \mathcal{V}_- \\ +1 & \text{if} \quad i \in \mathcal{V}_+ \,. \end{cases}$$

Clearly, $h^- \leq h^* \leq h^+$, so that h belongs to $\mathcal{H}$. Moreover, the fact that $\mathcal{V}_- \neq \emptyset \neq \mathcal{V}_+$ and $\mathcal{V}_- \neq \mathcal{V} \neq \mathcal{V}_+$ implies that $x^* \neq \pm\mathbf{1}$ is a co-existent configuration. The inequality in (8) then implies that

$$x_i^* \left(\sum\nolimits_j W_{ij} x_j^* + h_i^* \right) = x_i^* \left(w_i^+ - w_i^- + h_i^* \right) \geq 0 \,, \tag{9}$$

for every i in $\mathcal{V}$. Now, let $X(t)$ evolve according to the LTD on $\mathcal{G}$ with constant external field $h(t) = h^*$ and initial configuration $X(0) = x^*$. It then follows from (1) and (9) that $X(t) = x^*$ for every $t \geq 0$, thus proving the claim. ■

Example 2. *Consider the graph $\mathcal{G}$ shown in Figure 1 and let $h^- = (0, -1, 0, 0, 0)$ and $h^- = (0, 0, 0, 0, 1)$. As verified in Example 1, $\mathcal{G}$ is not (h^-, h^+)-indecomposable, so that Proposition 1 implies the existence of a vector h^* such that $h^- \leq h^* \leq h^+$ and of a co-existent configuration x^* in $\mathcal{X} \setminus \{\pm\mathbf{1}\}$ such that x^* is a fixed point for the LTD on $\mathcal{G}$ with constant external field $h(t) = h^*$. Specifically, in this case we can take $h^* = (0, -1, 0, 0, 1)$ and $x^* = (-1, -1, -1, +1, +1)$.*

While Proposition 1 states that, if the graph $\mathcal{G}$ is not (h^-, h^+)-indecomposable, robust convergence to a consensus configuration is not ensured for the ATV-LTD on $\mathcal{G}$, the following result establishes necessary and sufficient conditions for robust convergence to a consensus configuration when the graph $\mathcal{G}$ is (h^-, h^+)-indecomposable.

Theorem 1. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ be a graph. For two vectors h^- and h^+ in $\mathcal{R}^{\mathcal{V}}$ such that $h^- \leq h^+$, let $\mathcal{H}$ be as in (5). If $\mathcal{G}$ is (h^-, h^+)-indecomposable, then the ATV-LTD on $\mathcal{G}$ with any external field $h(t) \in \mathcal{H}$ for $t \geq 0$ is such that, for every initial configuration $X(0)$ in $\mathcal{X}$, with probability 1 there exists $t^* \geq 0$ such that*

(i)

$$X(t^*) \in \{\pm\mathbf{1}\}; \tag{10}$$

and

(ii) if $w \geq -h^-$ and $w \geq h^+$, then

$$X(t) \in \{\pm\mathbf{1}\} \,, \qquad \forall t \geq t^* \,; \tag{11}$$

(iii) if $w \geq -ah^{-a}$ and $w \not\geq ah^{-a}$ for some $a = \pm 1$, then

$$X(t) = a\mathbf{1} \,, \qquad \forall t \geq t^* \,. \tag{12}$$

Moreover:

(iv) if $w \not\geq -h^-$ and $w \not\geq h^+$, then there exists a signal $h(t) \in \mathcal{H}$ for $t \geq 0$ such that, for every initial configuration $X(0)$ in $\mathcal{X}$, with probability 1 the ATV-LTD on $\mathcal{G}$ with external field $h(t)$ visits both consensus configurations $+\mathbf{1}$ and $-\mathbf{1}$ infinitely often.

Proof. See Section IV-B. ■

Theorem 1 (i) is to be interpreted as a converse to Proposition 1, as it states that, if the graph $\mathcal{G}$ is (h^-, h^+)-indecomposable, then the set of consensus configurations is reached in finite time with probability 1 from every initial configuration $X(0)$. Theorem 1 (i) and Proposition 1 together guarantee that (h^-, h^+)-indecomposability of the network is a necessary and sufficient condition for global robust reachability of the set of consensus configurations for the ATV-LTD.

Moreover, because of the form of the update rule (1) the condition $w \geq -h^-$ implies that the consensus configuration $+\mathbf{1}$ is absorbing. Symmetrically, the condition $w \geq h^+$ implies that the consensus configuration $-\mathbf{1}$ is absorbing. Hence, point (ii) of Theorem 1 states that if, both $w \geq -h^-$ and $w \geq h^+$, then for every initial configuration $X(0)$ in $\mathcal{X}$, the ATV-LTD gets absorbed in finite time in one of these two consensus configurations. In fact, the probability with which $X(t)$ is absorbed in the consensus configuration $+\mathbf{1}$ rather than in $-\mathbf{1}$ will depend on the initial configuration $X(0)$, the graph $\mathcal{G}$, and the particular external field $h(t)$.

On the other hand, let us consider the case $a = +1$ in point (iii) of Theorem 1 (the case $a = -1$ being completely symmetrical). Then, as discussed above, the condition $w \geq -h^-$ ensures that the consensus configuration $x^* = +\mathbf{1}$ is absorbing. On the other hand, the condition $w \ngeq h^-$ is equivalent to the existence of some agent i in $\mathcal{V}$ such that $w_i < h_i^-$. From the form of the update rule (1), we then deduce that any such agent i will switch her action to $+1$ the first time she gets activated and will stick to $X_i(t) = +1$ ever after. Point (iii) of Theorem 1 then states that with probability 1 all other agents will follow such agent i and switch to state $+1$ in a cascade until the absorbing consensus configuration $+\mathbf{1}$ is reached in finite time.

Finally, in contrast to points (i)-(iii), point (iv) of Theorem 1 does not describe a robust behavior. Rather, the two conditions $w \ngeq -h^-$ and $w \ngeq h^+$ ensure that there exist two (not necessarily distinct) agents i and j in $\mathcal{V}$ such that $w_i < -h_i^-$ and $w_j < h_j^+$, respectively. This implies that agent i will always switch to -1 the first time she gets activated under the external field h^-, while agent j will always switch to $+1$ the first time she gets activated under the external field h^+. Point (ii) of Theorem 1 states that, by manoeuvring the external field $h(t)$ within its range $\mathcal{H}$, one is able to make the system oscillate infinitely often between the two consensus configurations.

The proof of Theorem 1 is one of the main contributions of this paper. The key technical challenges are twofold: on the one hand, we are considering LTD with time-varying external field $h(t)$ and seeking robustness results with respect to $h(t)$, on the other hand, considering weighted directed graphs prevents one from appealing to potential games arguments. We will address these challenging by first focusing on the special case of LTD with constant external field and studying it from a super-modular game theory perspective in Section III. We will then introduce and characterize the key notion of robust improvement path in Section IV-A and finally apply it to prove Theorem 1 in Section IV-B.

Example 3. *Consider once again the graph $\mathcal{G}$ shown in Figure 1 and let $h^+ = (0, 2, 0, 0, 2)$. As is shown in Example 1, $\mathcal{G}$ is $(0, h^+)$-indecomposable. Hence, since $w \geq -h^-$ and $w \geq h^+$, Theorem 1 (ii) implies that the ATV-LTD on $\mathcal{G}$ with any external field $0 \leq h(t) \leq h^+$ for $t \geq 0$ gets absorbed with probability 1 in finite time in a consensus configuration.*

In Figure 3, we simulated the dynamics of $N(t) = \sum_i X_i(t)$ for different initial conditions when the external field is $h(t) = (0, h_2(t), 0, 0, h_5(t))$ with h_2 and h_5 as in Figure 3 (notice that $0 \leq h(t) \leq h^+$). ATV-LTD dynamics get absorbed in consensus configurations.

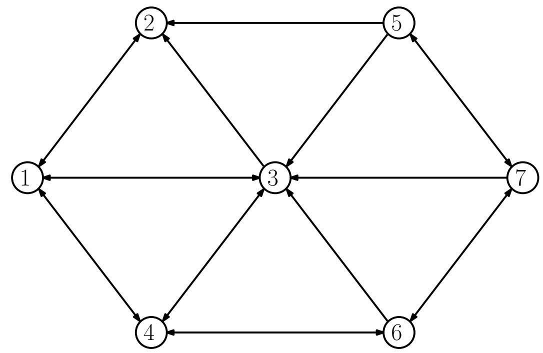

Fig. 2

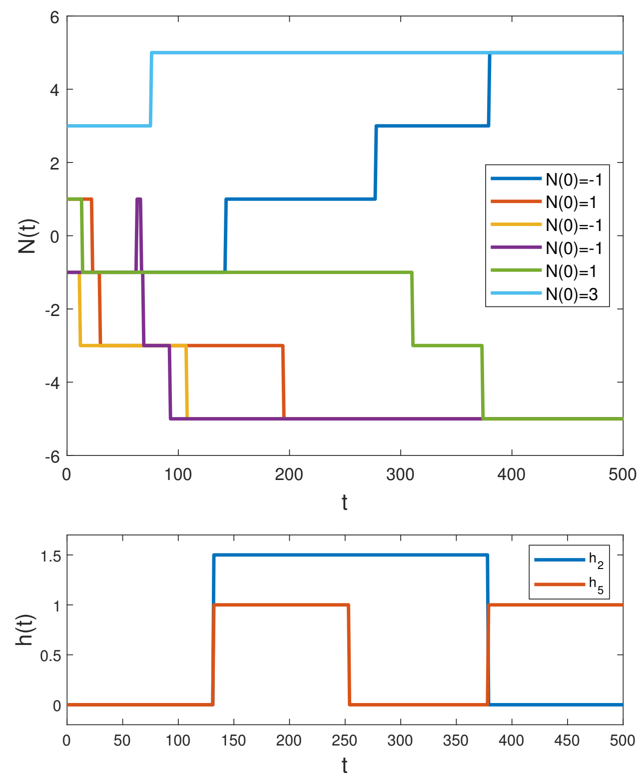

Fig. 3: In the upper panel, dynamics of $N(t) = \sum_i X_i(t)$ for the graph in Figure 1 and $h(t) = (0, h_2(t), 0, 0, h_5(t))$ with h_2 and h_5 as in the lower panel. ATV-LTD get absorbed in consensus configurations for different initial conditions (see Example 3).

Example 4. *Consider the graph $\mathcal{G}$ shown in Figure 2, with node set $\mathcal{V} = \{1, \dots, 7\}$ and out-degree vector $w = (3, 1, 3, 3, 3, 3, 3)$. Let*

$$h^- = (\alpha, 0, \dots, 0)\,, \qquad h^+ = (\beta, 0, \dots, 0)\,,$$

for $\alpha \leq \beta$ in $\mathbb{R}$. Then, for every nontrivial binary partition as in (6), let s in $\{-, +\}$ be such that $1 \in \mathcal{V}_{-s}$. If $2 \in \mathcal{V}_s$, then $w_2^s + sh_2^s = 0 < 1 = w_2^{-s}$, so that (7) is satisfied. On the other hand, for $j = 2, \dots, 6$, if $\{1, \dots, j\} \subseteq \mathcal{V}_{-s}$ and $j + 1 \in \mathcal{V}_s$, then $w_{j+1}^s + sh_{j+1}^s \leq 1 < 2 \leq w_{j+1}^{-s}$, so that (7)

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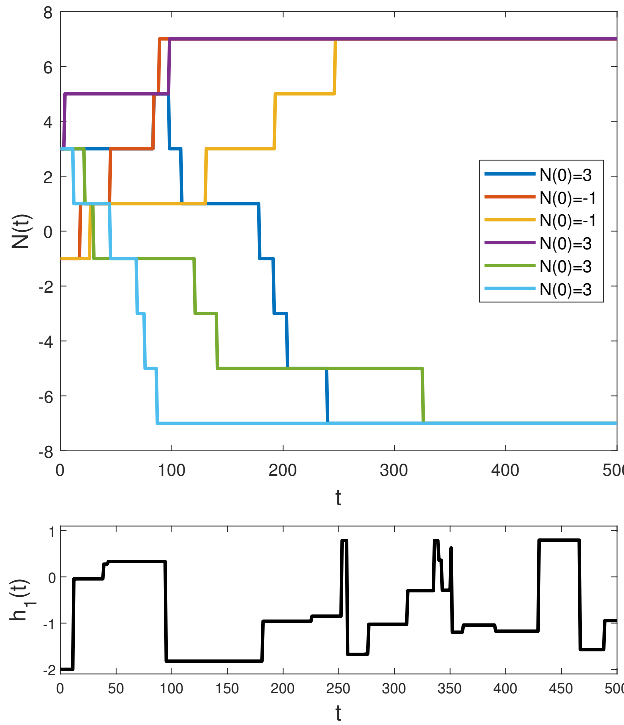

Fig. 4: In the upper panel, dynamics of $N(t) = \sum_i X_i(t)$ for the graph in Figure 2 and $h(t) = (h_1(t), 0, 0, 0, 0)$ with h_1 as in the lower panel. ATV-LTD get absorbed in consensus configurations for different initial conditions (see Example 4 with $\alpha = -2$ and $\beta = 1$).

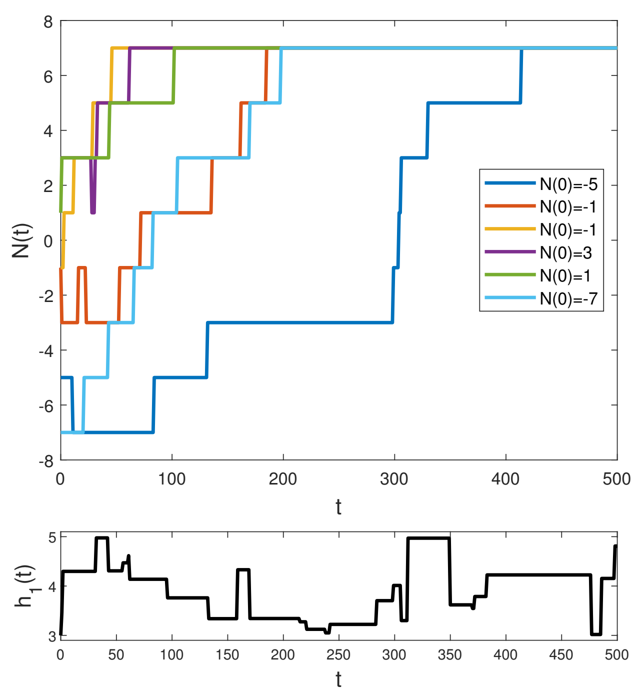

Fig. 5: In the upper panel, dynamics of $N(t) = \sum_i X_i(t)$ for the graph in Figure 2 and $h(t) = (h_1(t), 0, 0, 0, 0)$ with h_1 as in the lower panel. ATV-LTD get absorbed in the consensus configuration $x^* = +\mathbf{1}$ for different initial conditions (see Example 4 with $\alpha = 3$ and $\beta = 5$).

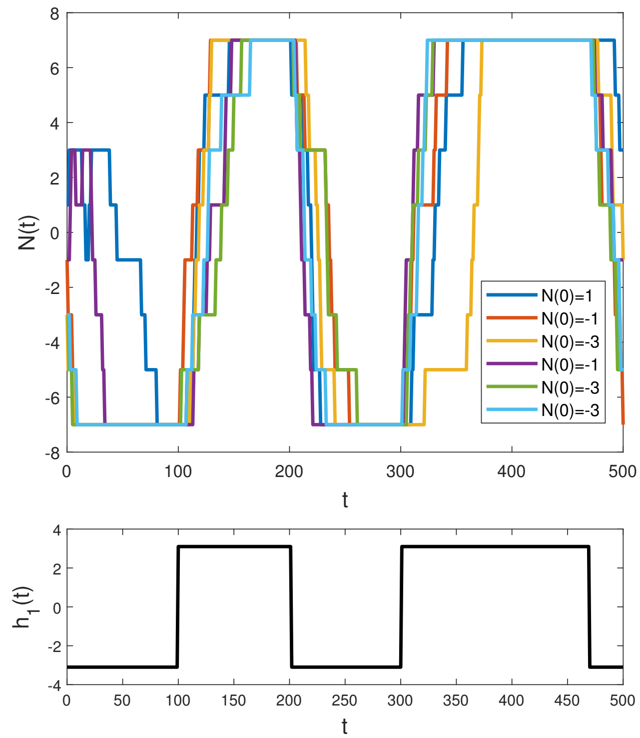

Fig. 6: In the upper panel, dynamics of $N(t) = \sum_i X_i(t)$ for the graph in Figure 2 and $h(t) = (h_1(t), 0, 0, 0, 0)$ with h_1 as in the lower panel. ATV-LTD fluctuate for different initial conditions (see Example 4 with $\alpha = -3.1$ and $\beta = 3.1$).

is satisfied. This proves that $\mathcal{G}$ is (h^-, h^+)-indecomposable for every $\alpha \leq \beta$. Now consider three different cases.

If $-3 \leq \alpha \leq \beta \leq 3$, so that $w \geq -h^-$ and $w \geq h^+$, then Theorem 1 (ii) ensures that, with probability 1, $X(t)$ gets absorbed in finite time in one of the two consensus configurations (see Figure 4).

On the other hand, if $3 < \alpha \leq \beta$, so that $w \geq -h^-$ and $w \ngeq h^-$, then Theorem 1 (iii) ensures that, with probability 1, $X(t)$ gets absorbed in finite time in the consensus configuration $x^* = +\mathbf{1}$ (see Figure 5).

Finally, if $\alpha < -3$ and $\beta > 3$, so that $w \ngeq -h^-$ and $w \ngeq h^+$ then Theorem 1 (iv) ensures that there exists a time-varying signal $h^- \leq h(t) \leq h^+$ such that, with probability 1, $X(t)$ fluctuates forever between the two consensus configurations visiting both of them infinitely many times (see Figure 6).

III. LTD WITH CONSTANT EXTERNAL FIELD

In this section, we study the special case of an LTD with a constant external field. We refer to it as to an *asynchronous linear threshold dynamics* A-LTD. As we shall see, in this special case, the absorbing points of the A-LTD can be interpreted as the (pure strategy Nash) equilibria of an underlying network coordination game and the convergence can be studied purely in terms of the improvement paths of such a game. We shall provide a full characterization of such equilibria and of the asymptotic behavior of the corresponding A-LTD in terms of graph-theoretic properties of the network.

A. LTD and coordination games

We start with the formal introduction of a network coordination game.

Definition 2. *For a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ and a vector h in $\mathbb{R}^{\mathcal{V}}$, the* (network) coordination game *on $\mathcal{G}$ with* external field *h is the game with player set $\mathcal{V}$, whereby every player i in $\mathcal{V}$ has binary action set $\mathcal{A} = \{\pm 1\}$ and utility function $u_i : \mathcal{X} \to \mathbb{R}$ specified by*

$$u_i(x) = x_i \sum_{j \in \mathcal{V}} W_{ij} x_j + h_i x_i \,. \tag{13}$$

Notice that the utility of a player i increases when her neighbors play the same action as her, thus modeling interactions of strategic complements. The external field h_i represents the bias of player i in choosing an action over its alternative. Indeed, the sign of external field h_i determines which is the best action for player i in the absence of any network influences.

As customary in game theory, for a configuration x in $\mathcal{X}$ and a player i in $\mathcal{V}$, we let x_{-i} in $\mathcal{X}_{-i} = \mathcal{A}^{\mathcal{V} \setminus \{i\}}$ stand for the configuration of all players except for player i. We shall then use the common abuse of notation $u_i(x) = u_i(x_i, x_{-i})$ for the utility perceived by player i in configuration x. The *best response* correspondence for a player i in $\mathcal{V}$ is defined as

$$\mathcal{B}_i(x_{-i}) = \operatorname*{argmax}_{x_i \in \mathcal{A}} u_i(x_i, x_{-i}) \,.$$

An action a in $\mathcal{A}$ is *dominant* (*strictly dominant*) for a player i if $a \in \mathcal{B}_i(x_{-i})$ ($\mathcal{B}_i(x_{-i}) = \{a\}$) for every x_{-i} in $\mathcal{X}_{-i}$. A player i having a strictly dominant action a is referred to as an *a-stubborn* agent. A (pure strategy Nash) *equilibrium* is a configuration x^* in $\mathcal{X}$ such that

$$x_i^* \in \mathcal{B}_i(x_{-i}^*) \,, \qquad \forall i \in \mathcal{V} \,.$$

The set of equilibria is denoted by $\mathcal{X}_h^*$. An equilibrium x^* is *strict* if $\mathcal{B}_i(x_{-i}^*) = \{x_i^*\}$ for every i in $\mathcal{V}$.

The following statement gathers a few simple results on coordination games.

Lemma 1. *Consider the coordination game on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ with external field h in $\mathbb{R}^{\mathcal{V}}$. Then, for every i in $\mathcal{V}$,*

(i) the utility function can be written as

$$u_i(x) = x_i \left(h_i + w_i^+(x) - w_i^-(x) \right) \,, \tag{14}$$

where $w_i^+(x)$ and $w_i^-(x)$ are defined as in (3)*;*

(ii) for every configuration x in $\mathcal{X}$

$$x_i \in \mathcal{B}_i(x_{-i}) \qquad \iff \qquad u_i(x) \geq 0 \,; \tag{15}$$

(iii) the best response correspondence has the threshold form

$$\mathcal{B}_i(x_{-i}) = \begin{cases} \{+1\} & \text{if} \quad w_i^+(x) > r_i w_i \\ \{\pm 1\} & \text{if} \quad w_i^+(x) = r_i w_i \\ \{-1\} & \text{if} \quad w_i^+(x) < r_i w_i \,, \end{cases} \tag{16}$$

where $r_i = \frac{1}{2} - \frac{h_i}{2w_i}$ is the threshold of player i;

(iv) action $a = \pm 1$ is a strictly dominant strategy if and only if $a h_i > w_i$.

Proof. (i) This is a consequence of the definitions of $w_i^+(x)$ and $w_i^-(x)$.

(ii) This follows from the equivalent form (14) in (i).

(iii) By substituting the identity $w_i^-(x) = w_i - w_i^+(x)$ into (14), we have that

$$u_i(x) = x_i \left(h_i + 2w_i^+(x) - w_i \right) \,,$$

from which (16) follows directly.

(iv) This follows directly from item (iii). ■

Comparing the form of the best response (16) with the evolution of the ATV-LTD $X(t)$ described in (4), we can notice that jumps occur only when the activated agent i can strictly increase its utility and that the evolution indeed corresponds for i to choose its unique best response action. We now introduce a related classical game theoretic concept, that of improvement path, that exactly captures this phenomenon.

Definition 3. *Given two configurations x and y in $\mathcal{X}$ and a nonnegative integer l, a (length-l) path from x to y is an $(l+1)$-tuple of configurations $(x^{(0)}, x^{(1)}, \dots x^{(l)})$ such that $x^{(0)} = x$, $x^{(l)} = y$, and for every $k = 1, 2, \dots, l$, there exists a player i_k in $\mathcal{V}$ satisfying*

$$x_{-i_k}^{(k)} = x_{-i_k}^{(k-1)} \,, \qquad x_{i_k}^{(k)} \neq x_{i_k}^{(k-1)} \,. \tag{17}$$

The l-tuple $(i_1, i_2, \dots, i_l)$ is referred to as the sequence of active players. The path is called

- monotone *if $x^{(0)} \lneq x^{(1)} \lneq \cdots \lneq x^{(l)}$;*
- anti-monotone *if $x^{(0)} \gneq x^{(1)} \gneq \cdots \gneq x^{(l)}$.*
- *an* improvement path *(*I-path*) if*

$$u_{i_k}(x^{(k)}) > u_{i_k}(x^{(k-1)}) \,, \qquad k = 1, 2, \dots, l \,. \tag{18}$$

In other words, a path is any sequence of configurations such that two consecutive configurations differ in just one component, corresponding to a single agent modifying its own action. In monotone paths, players can modify their actions from -1 to $+1$ only, conversely only modifications from $+1$ to -1 are allowed in anti-monotone paths. In a I-path, the agent modifying its action always increases its utility. Notice that, by convention, the singleton (x) is to be considered as an I-path of length 0 from x to x.

When there exists an I-path from a configuration x to a configuration y, we say that y is *I-reachable* from x and use the notation $x \to y$. If y is reachable from x by a monotone or anti-monotone I-path, we will use the notation $x \uparrow y$ and $x \downarrow y$, respectively. Observe that all these relations are reflexive and transitive.

Definition 4. *A subset of configurations $\mathcal{Y} \subseteq \mathcal{X}$ is called*

- globally I-reachable *if for every $x \in \mathcal{X}$ there exists some configuration $y \in \mathcal{Y}$ that is I-reachable from x;*
- I-invariant *if for every $y \in \mathcal{Y}$ and $z \in \mathcal{X}$ that is I-reachable from y, we have that $z \in \mathcal{Y}$;*
- globally I-stable *if it is globally I-reachable and I-invariant.*

The next result explicitly connects the previous notions to the asymptotic behavior of an A-LTD.

Proposition 2. *Let $X(t)$ be an A-LTD on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ with constant external field h in $\mathbb{R}^{\mathcal{V}}$. Consider a subset $\mathcal{Y} \subseteq \mathcal{X}$. Then,*

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(i) $\mathcal{Y}$ *is globally I-reachable if and only if for every initial configuration* $X(0)$ *in* $\mathcal{X}$*, with probability* 1 *there exists* $t^* \geq 0$ *such that* $X(t^*) \in \mathcal{Y}$*;*

(ii) $\mathcal{Y}$ *is I-invariant if and only if for every initial configuration* $X(0)$ *in* $\mathcal{Y}$*,* $X(t) \in \mathcal{Y}$ *for every* $t \geq 0$*;*

(iii) $\mathcal{Y}$ *is globally I-stable if and only if for every initial configuration* $X(0) \in \mathcal{X}$*, with probability* 1 *there exists* $t^* \geq 0$ *such that* $X(t) \in \mathcal{Y}$ *for every* $t \geq t^*$*.*

Proof. (i) It follows from comparing the form of the best response in (16) with the evolution of the ATV-LTD $X(t)$ described in (4) that $X(t)$ can have a transition from a configuration x in $\mathcal{X}$ to another configuration y in $\mathcal{X}$ if and only if there exists i in $\mathcal{V}$ such that $x_{-i} = y_{-i}$, $y_i \neq x_i$, and $\mathcal{B}_i(y_{-i}) = \{y_i\}$. Then, by [47, Theorem 1.2.2], we have

$$\mathbb{P}(\exists t \geq 0 : \; X(t) = y \,|\, X(0) = x) > 0 \Leftrightarrow x \to y$$

This implies that a subset of configurations $\mathcal{Y}$ is globally I-reachable if and only if $\mathbb{P}(\exists t : \; X(t) \in \mathcal{Y} | X(0) = x) > 0$ for every x in $\mathcal{X}$. A standard result in the theory of finite-state Markov chains establishes that this last condition is actually equivalent to the fact that $\mathbb{P}(\exists t : \; X(t) \in \mathcal{Y} | X(0) = x) = 1$ for every $x \in \mathcal{X}$.

(ii) $\mathcal{Y}$ is invariant if and only if for every $y \in \mathcal{Y}$ there is no I-path from y leading outside of $\mathcal{Y}$. By previous considerations, this is equivalent to saying that the process $X(t)$ initialized in $X(0) = y$ can not deterministically leave the set $\mathcal{Y}$.

(iii) This point follows from (i) and (ii). ■

We notice that every equilibrium $x^* \in \mathcal{X}_h^*$, by definition, forms an I-invariant subset, so that it constitutes an absorbing point for the LTD: if $X(0) \in \mathcal{X}_h^*$, then $X(t) = X(0)$ for every t, deterministically.

B. Coordination games as super-modular games

A key property of coordination games is that pure Nash equilibria always exist and form a globally I-reachable subset of configurations. This, together with other properties that will be needed in our future derivations, are consequence of the fact that such games possess the so called *increasing difference property* that is, for every player i, its utility variation when its action changes from -1 to $+1$

$$u_i(1, x_{-i}) - u_i(-1, x_{-i}) = 2\left(\sum_{j \in \mathcal{V}} W_{ij} x_j + h_i\right) \tag{19}$$

is a monotone nondecreasing function of the configuration x_{-i} of the other players. Games with such a property are called super-modular and have received a considerable amount of attention in the literature [30], [31], [45], [46]. A first direct yet crucial consequence of the increasing difference property (19) is the following. For every x_{-i} in $\mathcal{X}_{-i}$, define

$$\mathcal{B}_i^+(x_{-i}) = \max B_i(x_{-i}) \qquad \mathcal{B}_i^-(x_{-i}) = \min B_i(x_{-i})\,.$$

Then, it holds the following.

Lemma 2. *For every player* i *in* $\mathcal{V}$*, both* $\mathcal{B}_i^+(x_{-i})$ *and* $\mathcal{B}_i^-(x_{-i})$ *are monotone nondecreasing in* x_{-i}*.*

To state other consequences of the increasing difference property, we need to introduce some further notation. For two vectors x and y in $\mathbb{R}^{\mathcal{I}}$, the (entry-wise) supremum $x \vee y$ in $\mathbb{R}^{\mathcal{I}}$ and infimum $x \wedge y$ in $\mathbb{R}^{\mathcal{I}}$ have entries, respectively,

$$(x \vee y)_i = \max\{x_i, y_i\}, \quad (x \wedge y)_i = \min\{x_i, y_i\}\,, \quad \forall i \in \mathcal{I}\,.$$

We use the notation $\vee L$ and $\wedge L$ to indicate the supremum and infimum, respectively, of a non empty subset $L \subseteq \mathbb{R}^{\mathcal{I}}$. The next result characterizes properties of monotone and anti-monotone I-paths.

Lemma 3. *For* x, y, z *in* $\mathcal{X}$*, the following relations hold true:*

(i) $x \uparrow y, x \uparrow z \Rightarrow x \uparrow (y \vee z)$

(ii) $x \downarrow y, x \downarrow z \Rightarrow x \downarrow (y \wedge z)$

(iii) $x \uparrow y, x' \geq x \Rightarrow x' \uparrow (y \vee x')$

(iv) $x \downarrow y, x' \leq x \Rightarrow x' \downarrow (y \wedge x')$

(v) $x \to y \Rightarrow x \uparrow y',\; x \downarrow y''$ *for some* $y'' \leq y \leq y'$*.*

Proof. (i) Let $(y^{(0)}, y^{(1)}, \dots y^{(l)})$ and $(z^{(0)}, z^{(1)}, \dots z^{(r)})$ be two monotone I-paths from x to y and z, respectively. Let $(i_1, \dots, i_l)$ and $(j_1, \dots, j_r)$ be the two corresponding sequences of active players. Let $(j_{s_1}, \dots, j_{s_k})$ be the subsequence of $(j_1, \dots, j_r)$ consisting of exactly those players that are not in the sequence $(i_1, \dots, i_l)$. We claim that the sequence $(x^{(0)}, \dots, x^{(l+r)})$ defined by

- $x^{(h)} = y^{(h)}$ for $h = 0, \dots, l$,
- $x^{(h+l)} = x^{(h+l-1)} + 2\delta^{j_{s_h}}$ for $h = 1, \dots, k$

is a monotone I-path from x to $y \vee z$. By construction, it is a monotone path. Moreover, $x^{(h+l-1)} \geq z^{(s_h-1)}$ for every $h = 1, \dots, k$. Since $\{+1\} = \mathcal{B}_{j_{s_h}}(z^{(s_h-1)})$, by Lemma 2, $\{+1\} = \mathcal{B}_{j_{s_h}}(x^{(h-1+l)})$. This implies that it is an I-path.

(ii) The proof is completely analogous to that of (i).

(iii) Let $(x^{(0)}, x^{(1)}, \dots, x^{(l)})$ be a monotone I-path from x to y with set of active players $(i_1, \dots, i_l)$. Consider the subsequence $(i_{s_1}, \dots, i_{s_k})$ of those players for which x and x' coincide. Then, $(x^{(0)} \vee x', x^{(i_{s_1})} \vee x', \dots, x^{(i_{s_k})} \vee x')$ is a monotone I-path from x' to $y \vee x'$. Indeed, notice that, by construction, $x^{(i_{s_k})} \vee x' = x^{(l)} \vee x' = y \vee x'$. We only need to show that it is an I-path. Since $(x^{(i_{s_h})} \vee x')_{-i_{s_h}} \geq x_{-i_{s_h}}^{(i_{s_h})}$ and using the increasing difference property (19) we obtain that

$$\begin{aligned} 0 \;&\leq\; u_{i_{s_h}}(x^{(i_{s_h})}) - u_{i_{s_h}}(x^{(i_{s_h}-1)}) \\ &\leq\; u_{i_{s_h}}(x^{(i_{s_h})} \vee x') - u_{i_{s_h}}(x^{(i_{s_{h-1}})} \vee x')\,. \end{aligned}$$

(iv) The proof is completely analogous to that of (iii).

(v): If $x \uparrow y$, then $y \vee x = y$. If $x \downarrow y$, then $x \geq y$ and $y \vee x = x$. In both cases the result is evident. The general case can be proven by induction on the length of a minimal I-path from x to y. Indeed, by definition of an I-path, for sure we can find an intermediate configuration z for which one of the two possible cases hold: $x \uparrow z \to y$ or $x \downarrow z \to y$. In the first case, using the induction hypothesis $z \uparrow y' \geq y$, we obtain by transitivity that $x \uparrow y' \geq y$. In the second case, using the induction hypothesis $z \uparrow y' \geq y$ and point (iii) with $x \geq z$, we obtain that $x \uparrow (x \vee y') \geq y$. Similarly we prove the other relation. ■

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We introduce two maps $f^+, f^-: \mathcal{X} \to \mathcal{X}$ on the configuration space, respectively defined by

$$\begin{aligned} f^+(x) &= \bigvee\{y \in \mathcal{X} \,|\, x \uparrow y\} \\ f^-(x) &= \bigwedge\{y \in \mathcal{X} \,|\, x \downarrow y\} \end{aligned} \tag{20}$$

for every x in $\mathcal{X}$. Thanks to Lemma 3, we have that $f^+(x)$ ($f^-(x)$) represent the maximal (minimal) configuration that is I-reachable from x by a monotone (anti-monotone) path. Notice that both sets in the righthand side expressions of (20) are nonempty as they contain the configuration x. The following gathers a number of properties relating equilibria of coordination games with the behavior of the maps (20).

Proposition 3. *Consider the coordination game on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ with external field h in $\mathbb{R}^{\mathcal{V}}$. Then,*

(i) f^+ and f^- are monotone nondecreasing maps;

(ii) a configuration x in $\mathcal{X}$ is an equilibrium if and only if

$$f^+(x) = x = f^-(x)\,;$$

(iii) for every configuration x in $\mathcal{X}$, $f^-(f^+(x)) \in \mathcal{X}^*_h$ and $f^+(f^-(x)) \in \mathcal{X}^*_h$ are, respectively, the greatest and least Nash equilibria that are I-reachable from x;

(iv) the set of equilibria $\mathcal{X}^*_h$ is nonempty and

$$\underline{x}^* = f^+(-\mathbf{1}),\ \bar{x}^* = f^-(\mathbf{1})$$

are, respectively, the least and the greatest element in $\mathcal{X}^*_h$;

(v) $\mathcal{X}^*_h$ is globally I-stable.

Proof. (i): It follows from Lemma 3 (i) that $x \uparrow f^+(x)$ for every configuration x in $\mathcal{X}$. If $x' \geq x$, Lemma 3 (iii) yields $x' \uparrow f^+(x) \vee x'$. Therefore $f^+(x') \geq f^+(x) \vee x' \geq f^+(x)$. The proof for f^- is analogous.

(ii) coincides with the definitions of equilibrium.

(iii) Put $\mathcal{X}^+ = \{x \in \mathcal{X} \,|\, f^+(x) = x\}$. We notice that for every $x \in \mathcal{X}$, $f^+(x) \in \mathcal{X}^+$. Moreover, $\mathcal{X}^+$ is closed with respect to anti-monotone I-path. Namely, if $x \in \mathcal{X}^+$ and $x \downarrow y$, then also $y \in \mathcal{X}^+$. To see this, by induction, it is sufficient to prove it when x and y are connected by an anti-monotone I-path of length 1, namely there exists $i \in \mathcal{V}$ such that $y_i < x_i$, $y_{-i} = x_{-i}$, and $\mathcal{B}_i(y) = \{-1\}$. If $y \notin \mathcal{X}^+$, then it would exists $z \in \mathcal{X}$ and a player $j \in \mathcal{V}$ such that $z_j > y_j$, $z_{-j} = y_{-j}$, and $\mathcal{B}_j(y) = \{+1\}$. Evidently $j \neq i$ and from Lemma 3 (iii) applied to $y \uparrow z$ and $x \geq y$ we would obtain $x \uparrow x \vee z$. Since by construction $x \vee z \neq x$, this would imply that $f^+(x) \neq x$ contrarily to the assumption that $x \in \mathcal{X}^+$. Similarly, $\mathcal{X}^- = \{x \in \mathcal{X} \,|\, f^-(x) = x\}$ is closed with respect to monotone I-path. Notice that $\mathcal{X}^*_h = \mathcal{X}^+ \cap \mathcal{X}^-$.

Consider now $y = f^-(f^+(x))$. Being in the image of f^-, necessarily $y \in \mathcal{X}^-$. On the other hand, since $f^+(x) \in \mathcal{X}^+$, by the fact that $\mathcal{X}^+$ is closed with respect to anti-monotone I-path, we have that also $y \in \mathcal{X}^+$. Hence y is an equilibrium. The argument for $f^+(f^-(x))$ is completely analogous.

If now y in $\mathcal{X}^*_h$ is any equilibrium reachable from x, namely $x \to y$, by Lemma 3 (v) it follows that $x \uparrow y' \geq y$. By definition of $f^+(x)$ we have that $f^+(x) \geq y' \geq y$. Therefore, by point (i), we have that

$$f^-(f^+(x)) \geq f^-(y) = y\,,$$

with the last equality above following from point (ii). Similarly, we can show that $f^+(f^-(x)) \leq y$. This concludes the proof of point (iii).

(iv) It follows from point (iii) that $\underline{x}^* = f^+(-\mathbf{1}) = f^+(f^-(-\mathbf{1}))$ is a Nash equilibrium. If $x^* \in \mathcal{X}^*_h$ is any other Nash equilibrium, using the monotonicity of f^+ (see point (i)) and the trivial fact that $f^-(x^*) \geq -\mathbf{1}$ we obtain that

$$x^* = f^+(f^-(x^*)) \geq f^+(-\mathbf{1}) = \underline{x}^*$$

An analogous argument proves that $\bar{x}^* = f^-(\mathbf{1})$ is the greatest Nash equilibrium.

(v) By definition, $\mathcal{X}^*_h$ is invariant, while global reachability follows from point (iii). ■

The fact that $\mathcal{X}^*_h$ is globally I-stable implies that the A-LTD $X(t)$ is absorbed in finite time in the set of Nash equilibria of the underlying coordination game. Formally, we have the following result.

Theorem 2. *Let $X(t)$ be the A-LTD on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ with constant external field h in $\mathbb{R}^{\mathcal{V}}$. Then, with probability 1, there exists $t^* \geq 0$ such that*

$$X(t) \in \mathcal{X}^*_h\,, \qquad \forall t \geq t^*\,.$$

Proof. The claim follows directly from Proposition 2 (iii) and Proposition 3 (v). ■

C. Pure Nash equilibria of coordination games

By virtue of Theorem 2, in order to shape our analysis of the asymptotics of the process $X(t)$ and, in particular, determine the conditions that guarantee the convergence to a consensus, we need to analyze the set of Nash equilibria $\mathcal{X}^*_h$. This is done in the remaining part of this section.

We first set some notation. Let

$$\mathcal{X}^\bullet_h = \mathcal{X}^*_h \cap \{\pm\mathbf{1}\}\,, \qquad \mathcal{X}^\circ_h = \mathcal{X}^*_h \setminus \{\pm\mathbf{1}\}\,,$$

indicate, respectively, the subsets of consensus and co-existent equilibria of the coordination game on $\mathcal{G}$ with external field h. We then introduce the following notion.

Definition 5. *A coordination game on a graph $\mathcal{G}$ with external field h is*

- regular *if $|\mathcal{X}^\bullet_h| = 2$;*
- biased *if $|\mathcal{X}^\bullet_h| = 1$; more precisely, for $a = \pm 1$, the coordination game is a-biased if $\mathcal{X}^\bullet_h = \{a\mathbf{1}\}$;*
- frustrated *if $|\mathcal{X}^\bullet_h| = 0$.*

Notice that, in a frustrated coordination game, neither of the consensus configurations is an equilibrium. Since $\mathcal{X}^*_h$ is never empty, a frustrated coordination game always admits at least one co-existent equilibrium. In contrast, when the game is not frustrated (either regular or biased) at least one consensus configuration is an equilibrium. Besides consensus, there might or might not exist co-existent equilibria. To distinguish these cases, the following further classification proves useful.

Definition 6. *A coordination game on a graph $\mathcal{G}$ with external field h is*

- unpolarizable *if $\mathcal{X}_h^\circ = \emptyset$;*
- polarizable *if $\mathcal{X}_h^\circ \neq \emptyset$.*

The set of equilibria of an unpolarizable regular coordination game contains both consensus configurations $\pm\mathbf{1}$ and no other configurations, whereas unpolarizable biased coordination games admit a single (consensus) equilibrium: $x^* = +\mathbf{1}$ in the positively biased case and $x^* = -\mathbf{1}$ in the negatively biased one. On the other hand, polarizable coordination games always admit co-existent equilibria possibly in addition to consensus ones (if they are regular or biased). In the sequel, we shall identify necessary and sufficient conditions for a coordination game to be regular, biased, or frustrated, and for it to be polarizable or unpolarizable.

We now introduce two sets that will play a key role in our analysis:

$$\mathcal{S}_a(h) = \{i \in \mathcal{V} \,|\, ah_i > w_i\}\,, \qquad a = \pm 1\,. \tag{21}$$

By Lemma 1 (iii), $\mathcal{S}_a(h)$ coincides with the set of players for which a is a strictly dominant action, i.e., the set of a-stubborn agents. The following simple result relates the presence of a-stubborn players with that of the consensus equilibrium $-a\mathbf{1}$.

Lemma 4. *Consider a coordination game on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ with external field h and let $\underline{x}^*(h)$ and $\overline{x}^*(h)$ be its least and greatest equilibria, respectively. Then,*

(i) $\underline{x}^*(h) = -\mathbf{1} \;\Leftrightarrow\; \mathcal{S}_{+1}(h) = \emptyset \;\Leftrightarrow\; h \leq w$

(ii) $\overline{x}^*(h) = +\mathbf{1} \;\Leftrightarrow\; \mathcal{S}_{-1}(h) = \emptyset \;\Leftrightarrow\; h \geq -w$

Proof. If $\underline{x}^*(h) = -\mathbf{1}$, then there cannot be $+1$-stubborn players, i.e., $\mathcal{S}_{+1}(h) = \emptyset$, so that $h \leq w$. On the other hand, if $h \leq w$, then, using the threshold form of the best response in Lemma 1 (ii), we deduce that $-\mathbf{1}$ is an equilibrium, namely $\underline{x}^*(h) = -\mathbf{1}$. This proves (i), while (ii) can be proven analogously. ■

In fact, Lemma 4 directly implies the following result.

Proposition 4. *Let $\mathcal{G}$ be a graph with out-degree vector w. Then, the coordination game on $\mathcal{G}$ with external field h is:*

(i) regular if and only if

$$-w \leq h \leq w\,; \tag{22}$$

(ii) a-biased for $a = \pm 1$ if and only if

$$w \geq -ah\,, \qquad w \ngeq ah\,; \tag{23}$$

(iii) frustrated *if and only if*

$$w \ngeq -h\,, \qquad w \ngeq h\,. \tag{24}$$

Remark 1. *The necessary and sufficient conditions in Proposition 4 can be readily interpreted in terms of the presence of stubborn agents, as introduced in Section III-A. In fact, Lemma 4 implies that* (22) *is equivalent to the fact that no player is stubborn,* (23) *is equivalent to the existence of at least one a-stubborn agent but no $-a$-stubborn agents, and* (24) *is equivalent to the existence of both $+1$- and -1-stubborn agents. Hence, Proposition 4 states that a coordination game is regular if and only if there are no stubborn agents, biased if and only if it contains stubborn agents of one type only, and frustrated if it contains stubborn agents of both types.*

In contrast to the relative simplicity of the characterization above, necessary and sufficient conditions for polarizability of coordination games as per Definition 6 are in general more involved and rely on the notion of indecomposability introduced in Definition 1.

Proposition 5. *The coordination game on a graph $\mathcal{G}$ with external field h is unpolarizable if and only if $\mathcal{G}$ is h-indecomposable.*

Proof. Let s in $\{\pm\}$. Given any configuration x^* in $\mathcal{X}$, for any player i such that $x_i^* = s1$, from (14) we can write that

$$\begin{aligned} u_i(x^*) &= s(h_i + w_i^+(x^*) - w_i^-(x^*)) \\ &= s(h_i + sw_i^s(x^*) - sw_i^{-s}(x^*)) \\ &= sh_i + w_i^s(x^*) - w_i^{-s}(x^*)\,. \end{aligned} \tag{25}$$

We now argue as follows. If the coordination game on $\mathcal{G}$ with external field h is polarizable, then there exists an equilibrium $x^* \neq \pm\mathbf{1}$. From (25) and Lemma 1 (i) we derive that, for every s and i such that $x_i^* = s1$,

$$sh_i + w_i^s(x^*) - w_i^{-s}(x^*) \geq 0\,.$$

Let $\mathcal{V}_{x^*}^s$ denote the subset of agents playing action $s1$ in x^*. This implies that relatively to the nontrivial binary partition $\mathcal{V} = \mathcal{V}_{x^*}^+ \cup \mathcal{V}_{x^*}^-$, (7) is violated for every i in $\mathcal{V}_{x^*}^s$ and s in $\{\pm\}$. Hence, $\mathcal{G}$ is not h-indecomposable.

On the other hand, if $\mathcal{G}$ is not h-indecomposable, then by Proposition 1 there exists a co-existent absorbing configuration $x^*$of the LTD on $\mathcal{G}$ with constant external field h. Such x^* in $\mathcal{X}_h^\circ$ is a co-existent equilibrium of the coordination game on $\mathcal{G}$ with external field h, which is then polarizable. ■

Remark 2. *Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ and r in $[0, 1]$, a subset of nodes $\mathcal{S} \subseteq \mathcal{V}$ is called r-*cohesive *[32] if*

$$w_i^{\mathcal{S}} \geq r w_i\,, \qquad \forall i \in \mathcal{S}\,, \tag{26}$$

*and r-*closed *if its complement $\mathcal{V} \setminus \mathcal{S}$ is $(1-r)$-cohesive. Notice that, using the identity $w_i = w_i^s + w_i^{-s}$, condition* (7) *in the special case $h^- = h^+ = h$ can be rewritten as $2w_i^s < w_i - sh_i$. In the special case when players have homogeneous thresholds $r_i = r$ in $[0, 1]$, equivalently when the external field is proportional to the node degree vector, i.e., $h = (1-2r)w$,* (7) *is equivalent to $w_i^s < rw_i$. Hence, in this special case, h-indecomposability of a graph $\mathcal{G}$ is equivalent to the non-existence of nonempty proper subsets of nodes $\mathcal{S}$ that are both r-cohesive and r-closed. In this sense, Proposition 5 generalizes [48, Proposition 9.7] to coordination games on weighted directed graphs with heterogeneous thresholds.*

Example 5. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ be a graph with two nodes $\mathcal{V} = \{1, 2\}$ connected by two directed links of weight $W_{12} = w_1$ and $W_{21} = w_2$, respectively. Consider a coordination game on $\mathcal{G}$ with external field $h = (h_1, h_2)$.*

As illustrated in Figure 7, by Proposition 4, the coordination game is: regular if $|h| \leq w$ (white region); $+1$-biased if $h \geq$

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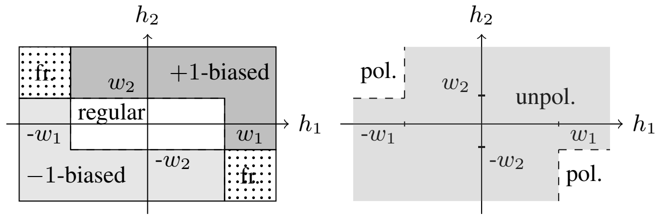

Fig. 7: Classification of two-player coordination games as in Example 5, based on Proposition 4 and 5.

$-w$ and $h \not\leq w$ (dark gray region); -1-biased if $h \leq w$ and $h \not\geq -w$ (light gray region); frustrated if $h_1 > w_1$ and $h_2 < -w_2$ or $h_1 < -w_1$ and $h_2 > w_2$ (dotted region).

On the other hand, Proposition 5 ensures that the coordination game on $\mathcal{G}$ is unpolarizable if and only if one of the following holds true: (i) $h < w$, (ii) $h > -w$, (iii) $|h_1| < w_1$, (iv) $|h_2| < w_2$, as illustrated in Figure 7. Notice that, for the special case of only two-players, the coordination game is polarizable if and only if it is frustrated.

Biased unpolarizable coordination games can be characterized in an equivalent simpler form.

Proposition 6. *The coordination game on a graph $\mathcal{G}$ with external field h has the unique equilibrium $x^* = a\mathbf{1}$ for a in $\{\pm 1\}$ if and only if the following conditions are both satisfied:*

(a) $w \not\geq ah$;

(b) every non-empty subset $\mathcal{R} \subseteq \mathcal{V} \setminus \mathcal{S}_a(h)$ contains some node i such that

$$w_i^{\mathcal{R}} < w_i^{\mathcal{V}\setminus\mathcal{R}} + ah_i\,. \tag{27}$$

Proof. (Only if) Assume that the coordination game is a-biased and unpolarizable. Then, condition (a) follows from Proposition 4 (ii). Also, this implies that $\mathcal{S}_a(h) \neq \emptyset$. To prove (b), assume by contradiction that there exists a non-empty subset $\mathcal{R} \subseteq \mathcal{V} \setminus \mathcal{S}_a(h)$ such that

$$w_i^{\mathcal{R}} \geq w_i^{\mathcal{V}\setminus\mathcal{R}} + ah_i\,, \tag{28}$$

for every i in $\mathcal{R}$ and let x in $\mathcal{X}$ be a configuration such that $x_i = a$ for every i in $\mathcal{V} \setminus \mathcal{R}$ and $x_i = -a$ for every i in $\mathcal{R}$. Notice that (28) and (25) imply that $u_i(x) \geq 0$, so that, by Lemma 1 (i), $-a = x_i \in \mathcal{B}_i(x_{-i})$, for every i in $\mathcal{R} = \mathcal{V}_x^{-a}$. If $a = +1$ ($a = -1$), this implies that there are no monotone (anti-monotone) I-paths of positive length starting at x, so that in particular $f^a(x) = x$. Then, by Proposition 3 (iii), we get that

$$x^* = f^{-a}(x) = f^{-a}(f^a(x))$$

is an equilibrium. Now, notice that on the one hand $x_i^* = -a$ for every i in $\mathcal{R}$ (since $x_i = -a$ and $x^* = f^{-a}(x)$), on the other hand $x_i^* = a$ for every i in $\mathcal{S}_a(h)$ (since those are stubborn players). Hence, x^* is a co-existent equilibrium, thus contradicting the assumption that the game is unpolarizable. Therefore, if the coordination game is a-biased and unpolarizable, both conditions (a) and (b) must be satisfied.

(If) Given any player i in $\mathcal{V} \setminus \mathcal{S}_a(h)$, from the application of (27) with $\mathcal{R} = \{i\}$ we obtain that

$$ah_i + w_i \geq ah_i + w_i^{\mathcal{V}\setminus\mathcal{R}} > w_i^{\mathcal{R}} \geq 0\,.$$

Since $ah_i + w_i > ah_i - w_i \geq 0$ for every i in $\mathcal{S}_a(h)$, we deduce that $ah + w \geq 0$. Together with assumption (a), by Proposition 4 (ii), this yields that the game is a-biased. We finally prove that the game is unpolarizable. By contradiction, suppose there exists a co-existent equilibrium x^*. Necessarily $x_i^* = a$ for every i in $\mathcal{S}_a(h)$. Put $\mathcal{R} = \mathcal{V}_{x^*}^{-a}$ and notice that (25) yields

$$\begin{aligned} 0 &\leq u_i(x^*) \\ &= -ah_i + w_i^{-a}(x^*) - w_i^a(x^*) \\ &= -ah_i + w_i^{\mathcal{R}} - w_i^{\mathcal{V}\setminus\mathcal{R}} \end{aligned}$$

for every i in $\mathcal{R}$ thus contradicting condition (b). The proof is then complete. ■

Remark 3. *In the special case $h = (1-2r)w$ considered in Remark 2, i.e., when players have homogeneous thresholds $r_i = r$ in $[0,1]$, (27) is equivalent to $w_i^{\mathcal{R}} < (1-r)w_i$, so condition (b) of Proposition 6 reduces to the non-existence of $(1-r)$-cohesive subsets of $\mathcal{V} \setminus \mathcal{S}_a(h)$. In the literature [32], such property is referred to as the set $\mathcal{V} \setminus \mathcal{S}_a(h)$ being uniformly not $(1-r)$-cohesive. In this sense, Proposition 6 generalizes [48, Proposition 9.8] to coordination games with heterogeneous thresholds.*

We conclude this section with the statement below, gathering some results on global I-stability of consensus equilibria for coordination games that directly follow from the analysis just developed.

Corollary 1. *For a graph $\mathcal{G}$ with out-degree vector w, consider the coordination game on it with external field h. Assume that $\mathcal{G}$ is h-indecomposable and that $|h| \leq w$. Then, $\mathcal{X}_h^* = \{\pm\mathbf{1}\}$.*

Proof. Proposition 4 (i) implies that when $|h| \leq w$, the game is regular so that the two consensus configurations $-\mathbf{1}$ and $+\mathbf{1}$ are both equilibria. Since $\mathcal{G}$ is h-indecomposable, Proposition 5 guarantees that the game is unpolarizable. Then, the set of equilibria is $\mathcal{X}_h^* = \{\pm\mathbf{1}\}$. ■

If we combine Corollary 1 and Theorem 2, we obtain the following result, that provides sufficient conditions for the A-LTD with constant external field h in $\mathbb{R}^{\mathcal{V}}$ to be absorbed in finite time in a consensus configuration.

Corollary 2. *Let $X(t)$ be the A-LTD on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ with constant external field h in $\mathbb{R}^{\mathcal{V}}$. Assume that $\mathcal{G}$ is h-indecomposable and that $|h| \leq w$. Then, with probability 1, there exists $t^* \geq 0$ such that*

$$X(t) \in \{\pm\mathbf{1}\}\,, \qquad \forall t \geq t^*\,.$$

The generalization of such result to a time-varying external field $h(t)$ is not straightforward. The analysis of the general case is carried on in the next section and leads to the proof of Theorem 1.

IV. Robust stability

In this section, we first introduce and characterize robust versions of the notions introduced in Definitions 5 and 6 and we generalize the results in Section III-C. We then combine the robust analysis of the set of Nash equilibria of coordination games with the reachability and stability properties of super-modular games proved in Section III-B. This is done in Theorem 4 and will pave the way to the proof of Theorem 1 on the asymptotic behavior of the ATV-LTD.

A. Robustness network coordination

For a graph $\mathcal{G}=(\mathcal{V},\mathcal{E},W)$ and a set of vectors $\mathcal{H}\subseteq\mathbb{R}^{\mathcal{V}}$, we say that a property of the coordination game on $\mathcal{G}$ is satisfied *$\mathcal{H}$-robustly* if it is satisfied *for every* external field h in $\mathcal{H}$. In what follows we concentrate on the special case when, for two vectors h^- and h^+ in $\mathbb{R}^{\mathcal{V}}$ such that $h^-\leq h^+$, the set $\mathcal{H}$ is the hyper-rectangle in (5). In this case, verifying that certain properties are $\mathcal{H}$-robustly satisfied can be significantly simplified with respect to checking the property for every single value of h in $\mathcal{H}$. Below we report results in this sense, starting with the following robust version of Proposition 4.

Corollary 3. *Let $\mathcal{G}$ be a graph with out-degree vector w and let $\mathcal{H}$ be as in (5) for two vectors $h^-\leq h^+$. Then, the coordination game on $\mathcal{G}$ is:*

(i) $\mathcal{H}$-robustly regular if and only if

$$w\geq h^+\,,\qquad w\geq -h^-\,;\tag{29}$$

(ii) $\mathcal{H}$-robustly a-biased for an action $a=\pm1$ if and only if

$$w\geq -ah^{-a}\,,\qquad w\ngeq ah^{-a}\,;\tag{30}$$

(iii) $\mathcal{H}$-robustly frustrated if and only if

$$w\ngeq -h^+\,,\qquad w\ngeq h^-\,.$$

Another interesting property is the robust unpolarizability, to be interpreted as the resilience of a coordination game against getting co-existent equilibria. By virtue of Proposition 5, this can equivalently be expressed as a robust indecomposability of the graph $\mathcal{G}$. However, it is useful to reformulate this in a form analogous to Definition 1, as in the following result.

Theorem 3. *Let $\mathcal{G}=(\mathcal{V},\mathcal{E},W)$ be a graph and let $\mathcal{H}$ be as in (5) for two vectors $h^-\leq h^+$. Then, the following conditions are equivalent:*

(a) the coordination game on $\mathcal{G}$ is $\mathcal{H}$-robustly unpolarizable;
(b) $\mathcal{G}$ is $\mathcal{H}$-robustly indecomposable;
(c) $\mathcal{G}$ is (h^-,h^+)-indecomposable.

Proof. Clearly, equivalence between conditions (a) and (b) directly follows from Proposition 5. We shall now prove equivalence between conditions (a) and (c).

First, assume that the coordination game on $\mathcal{G}$ is not $\mathcal{H}$-robustly unpolarizable, i.e., there exists an external field h in $\mathcal{H}$ such that the set of equilibria $\mathcal{X}^*_h$ contains a co-existent configuration $x^*\neq\pm\mathbf{1}$. Notice that $h_i^-\leq h_i\leq h_i^+$ implies that $x_i^*h_i\leq s_ih^{s_i}$, where $s_i=\mathrm{sgn}(x_i^*)$, for every player i in $\mathcal{V}$. Then, Lemma 1 (i) and (ii) imply that

$$\begin{aligned}0&\leq u_i(x^*)\\&=x_i^*h_i+x_i^*w_i^+(x^*)-x_i^*w_i^-(x^*)\\&\leq s_ih^{s_i}+w_i^{s_i}-w_i^{-s_i}\,.\end{aligned}$$

Hence, there exists no node i in $\mathcal{V}$ satisfying (7) for the nontrivial binary partition $\mathcal{V}=\mathcal{V}^+_{x^*}\cup\mathcal{V}^-_{x^*}$. This proves that condition (c) is not satisfied.

On the other hand, if condition (c) is not satisfied, then Proposition 1 implies that there exist h^* in $\mathcal{H}$ and a co-existent configuration x^* in $\mathcal{X}\setminus\{\pm\mathbf{1}\}$ such that x^* is an absorbing configuration for the A-LTD on $\mathcal{G}$ with constant external field h^*. Because of the equivalence (ii) in Proposition 2, it follows that $x^*\in\mathcal{X}^\circ_{h^*}$, so that the coordination game on $\mathcal{G}$ with external field h^* is polarizable, hence condition (a) is not satisfied. ∎

We now focus on the stability results in Corollary 1, for which, besides their straightforward robust generalization, some deeper consequences can be derived as reported below. These results build on the stability properties gathered in Proposition 3 and will in turn prove instrumental for the analysis carried on in next section.

Theorem 4. *Let $\mathcal{G}=(\mathcal{V},\mathcal{E},W)$ be a graph of order $n=|\mathcal{V}|$ and let $\mathcal{H}$ be as in (5) for two vectors $h^-\leq h^+$. Assume that $\mathcal{G}$ is $\mathcal{H}$-robustly indecomposable. Then,*

(i) for every configuration x in $\mathcal{X}$ there exists an $\mathcal{H}$-robust I-path from x to $\{\pm\mathbf{1}\}$ of length at most n. In particular, $\{\pm\mathbf{1}\}$ is $\mathcal{H}$-robustly globally I-reachable;
(ii) if condition (29) holds true, then $\{\pm\mathbf{1}\}$ is $\mathcal{H}$-robustly globally I-stable;
(iii) if there exists an action $a=\pm1$ such that (30) holds true, then there exists an $\mathcal{H}$-robust I-path from every configuration x in $\mathcal{X}$ to $a\mathbf{1}$. In particular, in this case, the set $\{a\mathbf{1}\}$ is $\mathcal{H}$-robustly globally I-stable.

Proof. (i) In the proof, we will make use of the maps f^+ and f^- defined in (20) for different values of the vector h and this dependence will be captured in the notation $f^+(\cdot,h)$ and $f^-(\cdot,h)$.

Given an arbitrary configuration x in $\mathcal{X}$, consider the external field h^x in $\mathcal{H}$ with entries

$$h_i^x=h_i^{x_i}\,,\qquad i\in\mathcal{V}\,,$$

and let

$$\underline{x}=f^+(f^-(x,h^x),h^x)\,,\qquad \overline{x}=f^-(f^+(x,h^x),h^x)\,.$$

By Proposition 3 (iii), the above are, respectively, the least and greatest equilibria of the coordination game on $\mathcal{G}$ with external field h^x that are I-reachable from configuration x. As the graph $\mathcal{G}$ is $\mathcal{H}$-robustly indecomposable, it follows from Theorem 3 that the coordination game on $\mathcal{G}$ is $\mathcal{H}$-robustly unpolarizable. Since $h^x\in\mathcal{H}$, this implies that the coordination game on $\mathcal{G}$ with external field h^x is unpolarizable, so that its equilibria $\underline{x}$ and $\overline{x}$ are both consensus configurations. Since clearly $\underline{x}\leq\overline{x}$, there are three possible alternative cases:

(a) $\underline{x}=\overline{x}=+\mathbf{1}$;
(b) $-\mathbf{1}=\underline{x}<\overline{x}=+\mathbf{1}$;
(c) $\underline{x}=\overline{x}=-\mathbf{1}$.

In both cases (a) and (b) we have

$$+\mathbf{1}=\overline{x}=f^-(f^+(x,h^x),h^x)=f^+(x,h^x)\,.$$

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This implies that there exists a monotone path γ from x to $+\mathbf{1}$ that is an I-path for the coordination game on $\mathcal{G}$ with external field h^x. Since in any monotone path only players originally playing action -1 can get activated, and since $h_i^- = h_i^x$ for every such player, we have that γ is also an I-path for the coordination game on $\mathcal{G}$ with external field h^-. A direct monotonicity argument then shows that γ is also a monotone I-path for the coordination game on $\mathcal{G}$ with any external field h in $\mathcal{H}$.

Similarly, in both cases (b) and (c) we have

$$-\mathbf{1} = \underline{x} = f^-(f^+(x, h^x), h^x) = f^-(x, h^x)\,,$$

so that by an argument completely analogous to the one developed above we can find an $\mathcal{H}$-robust anti-monotone I-path from x to $-\mathbf{1}$. The proof of point (i) is then completed by the observation that the length of monotone and anti-monotone paths is never larger than n.

(ii) By Corollary 3 (i), we have that $\mathcal{X}_h^* = \{\pm\mathbf{1}\}$ for every $h \in \mathcal{H}$. Result then follows from Proposition 3 (v).

(iii) By Corollary 3 (ii), we have that $\mathcal{X}_h^* = \{a\mathbf{1}\}$ for every $h \in \mathcal{H}$. The same argument used in (i) to prove the existence of an $\mathcal{H}$-robust I-path to $\{\pm\mathbf{1}\}$ leads to the existence of an $\mathcal{H}$-robust I-path from every configuration x in $\mathcal{X}$ to $a\mathbf{1}$. We complete the proof using again Proposition 3 (v). ■

B. *Proof of Theorem 1*

We now apply the results of Section IV-A to prove Theorem 1. Recall that the ATV-LTD $X(t)$ on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ with external field $h(t)$ is a continuous-time inhomegeneous Markov chain $X(t)$, whereby agents i in $\mathcal{V}$ get activated at the ticking of independent rate-1 Poisson clocks and, when activated at time $t \geq 0$, they modify their state according to the update rule (1).

We shall denote by $\Lambda(t)$ in $\mathbb{R}^{\mathcal{X}\times\mathcal{X}}$ the transition rate matrix of the continuous-time Markov chain $X(t)$, whose entries $\Lambda_{xy}(t)$ stand for the transition rates from configuration x in $\mathcal{X}$ to configuration y in $\mathcal{X}$ at time $t \geq 0$. Notice that $\Lambda(t)$ depends on the external field $h(t)$ and for this reason it is time-varying. We have that $\Lambda_{xy}(t) = 0$ whenever x and y differ in more than one entry, reflecting the fact that with probability 1 no two agents will modify their action simultaneously. On the other hand, if there exists i in $\mathcal{V}$ such that $x_{-i} = y_{-i}$ and $x_i \neq y_i$, then

$$\Lambda_{xy}(t) = \begin{cases} 1 & \text{if} \quad y_i(\sum_j W_{ij}x_i + h_i(t)) > 0 \\ 0 & \text{if} \quad y_i(\sum_j W_{ij}x_i + h_i(t)) \leq 0\,. \end{cases}$$

Finally, the diagonal entries of $\Lambda(t)$ are nonpositive and such that every row sum is zero, i.e., $\Lambda_{xx}(t) = -\sum_{y\neq x}\Lambda_{xy}(t)$.

Notice that the form of the update rule (1) of the ATV-LTD implies that the following uniform bounds hold true for the transition rates of $X(t)$ at any time $t \geq 0$:

$$\Lambda_{xy}(t) > 0 \;\Rightarrow\; \Lambda_{xy}(t) \geq 1\,, \qquad \forall x, y \in \mathcal{X}\,, \tag{31}$$

$$\sum_{y\neq x}\Lambda_{xy}(t) \leq n\,, \qquad \forall x \in \mathcal{X}\,, \tag{32}$$

where we recall that $n = |\mathcal{V}|$ is the number of agents.

(i) For every initial profile $X(0) = x^{(0)}$, Theorem 4 (i) guarantees the existence of an $\mathcal{H}$-robust I-path $(x^{(0)}, x^{(1)}, \ldots x^{(l)})$ of length $l \leq n$ from $x^{(0)}$ to the set of consensus configurations $\{\pm\mathbf{1}\}$. Consider now the discrete-time jump chain [47, p. 87] associated to $X(t)$, defined by

$$Y(k) = X(T_k)\,, \qquad k = 0, 1, \ldots$$

where $0 = T_0 < T_1 < T_2 < \ldots$ are the random times when the value of $X(t)$ changes. Using (31) and (32), we can estimate the probability that the ATV-LTD follows this path at some time as

$$\mathbb{P}\left(Y_{s+1} = x^{(1)}, \ldots, Y_{s+l} = x^{(l)} \,\middle|\, Y_s = x^{(0)}\right) \geq 1/n^l \geq 1/n^n$$

for every $s \geq 0$. This implies that

$$\mathbb{P}(Y_{s+k} \notin \{\pm\mathbf{1}\}\,\forall k = 1, \ldots, n \,|\, Y_s = x^{(0)}) \leq 1 - 1/n^n\,.$$

A standard induction argument now yields that, for every initial condition $x^{(0)}$ in $\mathcal{X}$ and for every $h = 1, 2, \ldots,$

$$\mathbb{P}(Y_s \neq \pm\mathbf{1}\,\forall s = 0, \ldots, hn \,|\, X(0) = x^{(0)}) \leq (1 - 1/n^n)^h\,.$$

and thus also

$$\mathbb{P}(Y_s \neq \pm\mathbf{1}\,\forall s = 0, \ldots, hn) \leq (1 - 1/n^n)^h\,.$$

Let now $T_{\pm\mathbf{1}} = \inf\{t \geq 0 :\; X(t) \in \{\pm\mathbf{1}\}\}$ be the (possibly infinite) first time that $X(t)$ is a consensus configuration. Then,

$$\begin{aligned} \mathbb{P}(T_{\pm\mathbf{1}} < +\infty) &= 1 - \lim_{h\to+\infty}\mathbb{P}(Y_s \neq \pm\mathbf{1}\,\forall s = 0, \ldots, hn) \\ &\geq 1 - \lim_{h\to+\infty}(1 - 1/n^n)^h \\ &= 1\,, \end{aligned}$$

thus proving that, with probability 1, the set of consensus configurations $\{\pm\mathbf{1}\}$ is reached in finite time.

(ii) It follows from Corollary 3 (i) that the coordination game on $\mathcal{G}$ is $\mathcal{H}$-robustly regular, namely both consensus configurations are equilibria for every $h \in \mathcal{H}$. This implies that $\Lambda_{xy}(t) = 0$ for every $x \in \{\pm\mathbf{1}\}$, $y \neq x$, and $t \geq 0$. This yields (ii).

(iii) Applying Theorem 4 (iii) and argueing as in the proof of point (i) we show that the all-a configuration $a\mathbf{1}$ is reached in finite time with probability a, i.e., $T_{a\mathbf{1}} = \inf\{t \geq 0 :\; X(t) = a\mathbf{1}\}$ satisfies

$$\mathbb{P}(T_{a\mathbf{1}} < +\infty) = 1\,.$$

Since $a\mathbf{1}$ is an equilibrium for every $h \in \mathcal{H}$, as in the proof of (ii) we obtain that $a\mathbf{1}$ is an absorbing point for the ATV-LTD.

(iv) By assumption, there exist two players i and j in $\mathcal{V}$ such that $w_i < h_i^+$ and $w_j < -h_j^-$. Since $i \in \mathcal{S}_+(h^+)$ and $\mathcal{G}$ is $\mathcal{H}$-robustly indecomposable, Corollary 1 implies that $\mathcal{X}_{h^+}^* = \{+\mathbf{1}\}$ is globally I-stable for the coordination game on $\mathcal{G}$ with external field h^+. This implies that, for every $\tau > 0$, the ATV-LTD on $\mathcal{G}$ with an external field $h(t)$ such that $h(t) = h^+$ for all t in $[0, \tau)$, is such that

$$\alpha_+ = \min_{x\in\mathcal{X}}\mathbb{P}(X(\tau) = +\mathbf{1}|(X(0) = x)) > 0\,.$$

Analogously, since $j \in \mathcal{S}_-(h^-)$ and $\mathcal{G}$ is $\mathcal{H}$-robustly indecomposable, we get that the coordination dynamics with an external field that is constant $h(t) = h^-$ in the interval $[0, \tau)$ is such that

$$\alpha_- = \min_{x\in\mathcal{X}} \mathbb{P}(X(\tau) = -\mathbf{1}|(X(0) = x)) > 0\,.$$

For the ATV-LTD with periodic piece-wise constant external field defined as follows

$$h(t) = \begin{cases} h^+ & \text{if} \quad 2k\tau \le t < (2k+1)\tau \quad & k \in \mathbb{Z}_+ \\ h^- & \text{if} \quad (2k+1)\tau \le t < (2k+2)\tau \quad & k \in \mathbb{Z}_+\,, \end{cases}$$

we then have that

$$\mathbb{P}(X((2k{+}1)\tau) = +\mathbf{1},\, X((2k{+}2)\tau) = -\mathbf{1}|X(2k\tau) = x) \ge \alpha\,,$$

for every $k \in \mathbb{Z}_+$ and $x \in \mathcal{X}$, where

$$\alpha = \alpha_+\alpha_- > 0\,.$$

It then follows that with probability 1 there exist infinitely many nonnegative integer values of k such that

$$X((2k+1)\tau) = +\mathbf{1}\,, \qquad X((2k+2)\tau) = -\mathbf{1}\,,$$

thus proving that $X(t)$ keeps fluctuating forever. ■

V. Conclusion

We have studied asynchronous time-varying linear threshold dynamics on general weighted directed graphs of interacting agents, equipped with an external field modeling exogenous interventions or individual biases towards specific actions. We have proved necessary and sufficient conditions for global stability of consensus equilibria, robustly with respect to the (constant or time-varying) external field.

A key step in our analysis has consisted in the introduction of novel robust notions of improvement and best response paths. Our analysis has strongly relied on super-modularity of coordination games, but also their peculiar threshold structure of best response correspondences. Extension of such concepts and results to more general super-modular games is a challenging problem that deserves further investigation.

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