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(Article begins on next page)

The spindle index from localization

Matteo Inglese,^{1,2} Dario Martelli,^{1,2} and Antonio Pittelli^{1,2,*}

¹*Dipartimento di Matematica, Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italy*

²*INFN, Sezione di Torino, Via Pietro Giuria 1, 10125 Torino, Italy*

We present a new supersymmetric index for three-dimensional $\mathcal{N} = 2$ gauge theories defined on $\Sigma \times S^1$, where Σ is a spindle, with twist or anti-twist for the R -symmetry background gauge field. We start examining general supersymmetric backgrounds of Euclidean new minimal supergravity admitting two Killing spinors of opposite R -charges. We then focus on $\Sigma \times S^1$ and demonstrate how to realise twist and anti-twist. We compute the supersymmetric partition functions on such backgrounds via localization and show that these are captured by a general formula, depending on the type of twist, which unifies and generalises the superconformal and topologically twisted indices.

INTRODUCTION

Localization techniques [1] are a tremendous tool to calculate path-integrals of supersymmetric quantum field theories (SQFTs) on curved manifolds, giving access to a profusion of exact results. This letter focuses on three-dimensional SQFTs, whereof two compelling observables are the partition functions on $S^2 \times S^1$ endowed with an R -symmetry background gauge field, with or without flux through the sphere, corresponding to the topologically twisted index [2] and the superconformal index [3, 4], respectively. The large- N limit of the former provides a microscopic interpretation of the entropy of magnetically charged supersymmetric black holes in AdS_4 [5].

The supersymmetric and *accelerating* AdS_4 black hole presented in [6] displays a number of striking features, including a horizon with orbifold singularities and supersymmetry preserved in a novel fashion. In the Euclidean setting, the conformal boundary is $\Sigma \times S^1$, where $\Sigma = \mathbb{WCP}^1_{[n_+, n_-]}$ is a *spindle*, *i.e.* a weighted projective space specified by two coprime positive integers n_{\pm} . The entropy of this solution is reproduced by considering a family of Euclidean saddles extending the black holes, where both bulk and boundary metrics are complex [7].

Motivated by these results, in this letter we present two partition functions of $\mathcal{N} = 2$ Chern-Simons-matter theories defined on $\Sigma \times S^1$. Supersymmetry on Σ is preserved by an R -symmetry background gauge field A satisfying only one [8] of the following conditions [9]:

$$\int_{\Sigma} \frac{dA}{2\pi} = \frac{1}{2} \left(\frac{1}{n_-} + \frac{\sigma}{n_+} \right) \equiv \frac{\chi_{\sigma}}{2}, \quad (1)$$

with $\sigma = \pm 1$ being configurations known as *twist* and *anti-twist*, respectively. Here $\chi_{\pm} = \frac{1}{4\pi} \int_{\Sigma} \sqrt{g} R$ is the orbifold Euler-characteristic of the spindle. We employ the framework of rigid new minimal supergravity and generalize the analysis of [10] to accommodate the intrinsically complex backgrounds of [7]. We extend the localization approach to orbifolds and compute partition functions on $\Sigma \times S^1$ with both twist and anti-twist. The technical details of the analysis will be spelled out in [11].

GENERAL COMPLEX BACKGROUNDS

We consider a general class of rigid supersymmetric backgrounds of Euclidean new minimal supergravity preserving two Killing spinors ζ_{\pm} with R -charges ± 1 , respectively. The Killing spinor equations (KSEs) are

$$(\nabla_{\mu} \mp iA_{\mu})\zeta_{\pm} = -\frac{H}{2}\gamma_{\mu}\zeta_{\pm} \mp iV_{\mu}\zeta_{\pm} \mp \varepsilon_{\mu\nu\rho}\frac{V^{\nu}}{2}\gamma^{\rho}\zeta_{\pm}, \quad (2)$$

where A_{μ} is the R -symmetry background gauge field, V_{μ} is a globally defined co-closed one-form and H a scalar, a priori all complex-valued. We emphasise that ζ_{\pm} are not related by charge conjugation and, differently from previous literature, the metric $g_{\mu\nu}$ can be complex-valued.

Following the conventions of [10], we introduce

$$v = \zeta_+\zeta_-, \quad K^{\mu} = \zeta_+\gamma^{\mu}\zeta_-, \quad P_{\pm}^{\mu} = \zeta_{\pm}\gamma^{\mu}\zeta_{\pm}/v, \quad (3)$$

where $(P_{+}^{\mu})^* \neq P_{-}^{\mu}$. The KSEs imply that K^{μ} is a complex Killing vector. Thanks to Fierz identities (K, P_{\pm}) is a canonical complex frame generating the line element

$$ds^2 = (K/v)^2 - P_+P_-, \quad (4)$$

with non-zero contractions being

$$\iota_K K = v^2, \quad \iota_{P_-} P_+ = \iota_{P_+} P_- = -2. \quad (5)$$

The KSEs determine the background fields A_{μ}, V_{μ}, H in terms of (K, P_{\pm}) . We are interested in complex backgrounds on a real three-dimensional space \mathcal{M}_3 : since K is a complex Killing vector, it picks out two angular coordinates φ, ψ , thus

$$K = \kappa(\partial_{\psi} + \omega\partial_{\varphi}), \quad (6)$$

where κ and ω are complex constants and φ, ψ are 2π -periodic. By rescaling ζ_{\pm} we can set $\kappa = 1$ without loss of generality. The most general metric on \mathcal{M}_3 invariant under two real Killing vectors $\partial_{\psi}, \partial_{\varphi}$ can be written as

$$ds^2 = f^2 dx^2 + h_{ij} d\psi_i d\psi_j \quad \text{with} \quad \psi_1 = \psi, \psi_2 = \varphi, \quad (7)$$

where the complex-valued functions $f(x)$ and $h_{ij}(x)$, $i, j = 1, 2$ depend only on the coordinate x . In these

coordinates we have

$$\begin{aligned} K &= (h_{11} + \omega h_{12})d\psi + (h_{12} + \omega h_{22})d\varphi, \\ P_{\pm} &= e^{\pm 2i\theta} (\pm f dx + i(\sqrt{h}/v)(-v d\psi + d\varphi)), \\ v^2 &= h_{11} + 2\omega h_{12} + \omega^2 h_{22}, \end{aligned} \quad (8)$$

where $h = \det(h_{ij})$ and $\theta \equiv (\alpha_1\psi + \alpha_2\varphi)/2$, with α_1, α_2 two real constants that we shall discuss momentarily. Defining $A^C \equiv A - \frac{3}{2}V$, the background fields read

$$\begin{aligned} V &= \frac{1}{v} [iHK - \star dK], \\ A^C &= \frac{v^3}{4f\sqrt{h}} \left[\frac{1}{\omega} \left(\frac{h_{11}}{v^2} \right)' d\psi - \left(\frac{h_{22}}{v^2} \right)' d\varphi \right] + d\theta, \end{aligned} \quad (9)$$

where a prime denotes derivative with respect to x . The function H satisfies $\mathcal{L}_K H = 0$ and is otherwise arbitrary; however, it will enter in the localization computation only through the following combinations:

$$\begin{aligned} h_R &\equiv \iota_K V - ivH = -\frac{1}{2v} \star (K \wedge dK), \\ \Phi_R &\equiv \iota_K (A^C + V) - ivH = (\alpha_1 + \omega\alpha_2)/2. \end{aligned} \quad (10)$$

Taking $\gamma^1 = \sigma^2$, $\gamma^2 = \sigma^3$, $\gamma^3 = \sigma^1$, with σ^i being the Pauli matrices, in the frame where $e^1 = -f dx$ and

$$e^2 = \sqrt{\frac{h}{h_{11}}} d\varphi, \quad e^3 = \sqrt{h_{11}} (d\psi + \frac{h_{12}}{h_{11}} d\varphi), \quad (11)$$

the Killing spinors satisfying (2) take the form

$$\zeta_+ = e^{i\theta} (u_1, -u_2)^T, \quad \zeta_- = -e^{-i\theta} (u_2, u_1)^T, \quad (12)$$

where T indicates transposition and

$$u_{1,2} = 2^{-1/2} \sqrt{v \mp \omega \sqrt{h/h_{11}}}. \quad (13)$$

$\Sigma \times S^1$ with twist and anti-twist

The results presented so far are purely local and apply to any space \mathcal{M}_3 , including e.g. S^3 . We shall now restrict attention to $\mathcal{M}_3 = \Sigma \times S^1$. We take ψ to parameterize S^1 and $x \in [-1, 1]$, φ as coordinates on Σ . This fixes the behaviour of the functions f, h_{ij} near the poles of Σ , which we denote as $N \equiv \{x = 1\}$ and $S \equiv \{x = -1\}$. By using reparameterization to set for simplicity $f = 1$ and denoting by ϱ_{\pm} the coordinates near each pole, at leading order we have

$$h_{11} \sim h_{11}^{\pm}, \quad h_{22} \sim \frac{1}{n_{\pm}^2} \varrho_{\pm}^2, \quad h_{12} \sim h_{12}^{\pm} \varrho_{\pm}^2, \quad (14)$$

at the north and south poles respectively, where $h_{12}^{\pm}, h_{11}^{\pm}$ are complex constants. By using (9) we find

$$\frac{1}{2\pi} \int_{\Sigma} dA = -\frac{1}{2} \left(\frac{s_+}{n_+} + \frac{s_-}{n_-} \right), \quad (15)$$

where s_{\pm} denote the signs of the function $v/\sqrt{h_{11}}$ at the north and south poles, respectively. Therefore, the type of supersymmetry-preserving twist is completely encoded in the behaviour of the function v . Given a generic metric on $\Sigma \times S^1$ and a parameter ω , we regard the third equation in (8) as a definition of the function v . From this, it follows that generically the function $v/\sqrt{h_{11}}$ has the same sign at both poles, corresponding to the twist case. Instead, the anti-twist is realized if the function $v/\sqrt{h_{11}}$ has opposite sign at the poles. In this case the metric h_{ij} and the parameter ω need be fine-tuned.

Requiring ζ_{\pm} to be (anti-)periodic under $\psi \sim \psi + 2\pi$ implies that $\alpha_1 = n \in \mathbb{Z}$. On Σ the spinors should be defined in two patches $\mathcal{U}_{N/S}$, with the non-singular gauge fields related via gauge transformations, corresponding to different values of α_2 . Specifically, the regular $A_{N/S}$ are obtained taking $\alpha_2 = s_+/n_+$ in \mathcal{U}_N and $\alpha_2 = -s_-/n_-$ in \mathcal{U}_S . Finally, from (13) one can see that at the north and south poles of Σ the spinors behave as

$$\zeta_+^{N/S} \sim (1, -s_{\pm})^T, \quad \zeta_-^{N/S} \sim (s_{\pm}, 1)^T, \quad (16)$$

so that indeed they have the same 2d-chirality for the twist and the opposite 2d-chirality for the anti-twist [8].

By exploiting such formalism we can immediately jump at the localization computation of the partition functions on these backgrounds. Importantly, these will only depend on ω , not on the specific representative metric, with the caveat explained above that for the anti-twist the metric depends on ω . To illustrate these features, it is instructive to consider the explicit background

$$ds^2 = f^2 dx^2 + (1-x^2)(d\varphi - \Omega d\psi)^2 + \beta^2 d\psi^2, \quad (17)$$

where $x \in [-1, 1]$, φ and ψ have 2π -periodicities and

$$f(x) \sim n_{\pm} / \sqrt{2(1 \mp x)}, \quad \text{for } x \rightarrow \pm 1, \quad (18)$$

so that at any constant ψ , the coordinates x, φ describe a spindle, equipped with a generic metric. Although we choose β real and positive for simplicity, the function $f(x)$ and the constants β and Ω can be complex a priori. The frame (K, P_{\pm}) , background fields A, V, H and Killing spinors ζ_{\pm} are then completely determined by our general formulae (8), (9), (12) in terms of the metric functions $f(x), \Omega, \beta$ and the parameter ω .

Let us briefly discuss how the two twists are realized for the family of metrics (17). As $v^2 = (1-x^2)(\omega - \Omega)^2 + \beta^2$, if no relation is imposed between ω, β and Ω , then the R -symmetry background field realizes the *twist*. As a special case, the standard topological twist corresponds to $\omega = \Omega$, yielding $v/\beta = -1$, so that in (1) we have $\sigma = +1$. Conversely, the *anti-twist* is realized by choosing $\Omega = \omega \pm i\beta$. In this case we can take $v/\beta = x$, so that in (1) we have $\sigma = -1$.

LOCALIZATION

Let G be a semi-simple Lie group with Lie algebra \mathfrak{g} , with \mathfrak{R}_G a generic representation of G and Ad_G its adjoint representation. For a three-dimensional $\mathcal{N} = 2$ gauge theory, the supersymmetry transformations of a vector multiplet $(\mathcal{A}, \sigma, \lambda, \tilde{\lambda}, D) \in \text{Ad}_G$ and those of a chiral multiplet $(\phi, \psi, F) \in \mathfrak{R}_G$ can be written as a cohomological complex [1]. Solving the BPS equations $\delta\psi = \delta\tilde{\psi} = \delta\lambda = \delta\tilde{\lambda} = 0$ yields the localization locus of classical configurations contributing by Z_{class} to the partition function. Also, this formulation allows for recasting the computation of vector- and chiral-multiplets 1-loop determinants, $Z_{1-L}^{\text{VM}}, Z_{1-L}^{\text{CM}}$, as a cohomological problem. Below we will present the main steps of this procedure, referring to [11] for more details. If $G = G_g \times G_f$, the partition function of a theory on $S^1 \times \Sigma$ with gauge group G_g and flavour group G_f reads

$$Z_{S^1 \times \Sigma}(\omega, u_f, \mathfrak{f}_f) = \sum_{\mathfrak{f}_g \in \Gamma_{\mathfrak{h}_g}} \oint_{\mathcal{C}} \frac{du_g}{|W_g|} \widehat{Z}(u_g, \mathfrak{f}_g | \omega, u_f, \mathfrak{f}_f), \quad (19)$$

where \mathfrak{h}_g is the maximal Cartan-subalgebra of \mathfrak{g}_g , $\Gamma_{\mathfrak{h}_g}$ the corresponding co-root lattice and W_g its Weyl group; while $u_{g,f} \in \mathfrak{h}_{g,f}$ and $\mathfrak{f}_{g,f} \in \Gamma_{\mathfrak{h}_{g,f}}$ denote gauge/flavour holonomies and fluxes, respectively; \widehat{Z} is the product of Z_{class} , Z_{1-L}^{VM} and Z_{1-L}^{CM} and \mathcal{C} is a suitable integration-contour for u_g . The partition function (19) also depends on the spindle data (n_{\pm}, σ) , which we suppressed to not clutter the notation. $Z_{S^1 \times \Sigma}$ is related to the G_f -flavoured Witten-index of the theory quantized on Σ , that is

$$I_{S^1 \times \Sigma} = \text{Tr}_{\mathcal{H}[\Sigma]} \left[e^{-\omega J - \varphi R - \sum_i \varphi_i F_i} \right], \quad (20)$$

where J, R, F_i generate angular momentum, R -symmetry and flavour symmetries, respectively; while $\mathcal{H}[\Sigma]$ is the Hilbert space of states on the spindle, with either twist or anti-twist. We anticipate that the fugacities ω and φ are not independent, but are related by

$$\varphi - \frac{\omega}{4} \chi_{-\sigma} = i\pi n, \quad n \in \mathbb{Z}. \quad (21)$$

For the anti-twist ($\sigma = -1$), taking $n = \pm 1$ so that the spinors are anti-periodic on S^1 , this reproduces the relation found in [7] for the dual accelerating black holes.

BPS locus and Chern-Simons term

The vector-multiplet BPS equations read

$$\iota_K(\star \mathcal{F}) = -vD - ih_R \sigma, \quad \iota_K \mathcal{F} = -\text{id}_{\mathcal{A}}(v\sigma), \quad (22)$$

where \mathcal{F} is the field strength of the gauge field \mathcal{A} . Neither solving (22) nor computing Z_{class} or Z_{1-L} require imposing reality conditions on fields. We restrict for simplicity

to Abelian gauge fields, the non-Abelian generalization being straightforward. The BPS locus is parametrized by a gauge field \mathcal{A} obeying $\mathcal{L}_K \mathcal{A} = 0$, with flux

$$\mathfrak{f}_G \equiv \frac{1}{2\pi} \int_{\Sigma} d\mathcal{A} = \frac{\mathfrak{m}}{n_+ n_-}, \quad (23)$$

where $\mathfrak{m} \in \mathbb{Z}$ [8] and we can always write

$$\mathfrak{m} = n_+ m_- - n_- m_+, \quad m_{\pm} \in \mathbb{Z}. \quad (24)$$

The second equation in (22) is solved by

$$\mathcal{A}_{\psi} + \omega \mathcal{A}_{\varphi} - iv\sigma = \Phi_G, \quad (25)$$

where Φ_G is an arbitrary complex constant (in each patch $\mathcal{U}_{N/S}$). The first equation in (22) yields the auxiliary field D . The Abelian Chern-Simons term contributes to Z_{class} via $Z_{\text{CS}} = e^{-S_{\text{CS}}}$, with [12]

$$S_{\text{CS}} = \frac{ik}{4\pi} \int (\mathcal{A} \wedge \mathcal{F} + 2i \star D\sigma) = 2\pi i k \mathfrak{f}_G u, \quad (26)$$

evaluated on BPS configurations solving (22), where

$$u \equiv \left(\mathcal{A}_{\psi}^+ + \mathcal{A}_{\psi}^- - iv^+ \sigma^+ - iv^- \sigma^- \right) / 2, \quad (27)$$

with the \pm superscripts denoting quantities evaluated at the N/S poles of Σ . The differential operators acting on a chiral-multiplet field $\phi \in \mathfrak{R}_G$ with R -charge q_R^{ϕ} are

$$\begin{aligned} L_K &= \mathcal{L}_K - i q_R^{\phi} \Phi_R, \\ L_{P_{\pm}} &= \mathcal{L}_{P_{\pm}} - i q_R^{\phi} \iota_{P_{\pm}} (A^C + V) - i \iota_{P_{\pm}} \mathcal{A}, \end{aligned} \quad (28)$$

and the chiral-multiplet BPS equations read

$$\delta^2 \phi = (L_K + \mathcal{G}_{\Phi_G}) \phi = 0, \quad F + i L_{P_-} \phi = 0, \quad (29)$$

where for a chiral-multiplet scalar field $\phi \in \mathfrak{R}_G$ we have

$$\mathcal{G}_{\Phi_G} \phi = -i \Phi_G \circ_{\mathfrak{R}_G} \phi, \quad (30)$$

where $\circ_{\mathfrak{R}_G}$ indicates the action of Φ_G according to the representation \mathfrak{R}_G . For arbitrary q_R^{ϕ} and Φ_G , regularity of the solutions to the first equation in (29) implies $\phi = 0$, while $F = 0$ follows from the second. Similarly, $\tilde{\phi} = \tilde{F} = 0$ for an anti-chiral multiplet. Taking \mathcal{A} and σ to be Hermitian, (25) splits in two real equations determining \mathcal{A}_{ψ} and σ in terms of \mathcal{A}_{φ} , which is constrained by the first equation in (22). We impose no reality conditions on the auxiliary field D .

1-loop determinants

Using a set of cohomological variables, the 1-loop determinant of a chiral multiplet of R -charge r in \mathfrak{R}_G can be cast in the form

$$Z_{1-L}^{\text{CM}} = \frac{\det_{\text{Ker} L_{P_+}} (L_K + \mathcal{G}_{\Phi_G})}{\det_{\text{Ker} L_{P_-}} (L_K + \mathcal{G}_{\Phi_G})}, \quad (31)$$

while the 1-loop determinant of a vector multiplet, Z_{1-L}^{VM} , includes the contribution of BRST-ghosts compatible with supersymmetry [1]. However, a standard argument implies that formally $Z_{1-L}^{\text{VM}} = Z_{1-L}^{\text{CM}}(r=2)|_{\mathfrak{R}_G = \text{Ad}_G}$ [2], so in the following we shall focus on chiral multiplets.

The modes $\mathcal{F}^\pm \in L^2[\Sigma \times S^1]$ contributing to (31) obey the linear ODEs $L_{P_\pm} \mathcal{F}^\pm = 0$ and are eigenfunctions of the operator $L_K + \mathcal{G}_{\Phi_G}$. The functional determinant of the latter is an infinite product of eigenvalues that, after regularization, explicitly provides (31) in terms of special functions [11]. In this letter we shall sketch an alternative route to the same result, namely extracting the eigenvalues from an *equivariant orbifold index theorem*.

In general $G = G_g \times G_f$ and $\mathfrak{g} = \mathfrak{g}_g \oplus \mathfrak{g}_f$, so that $\mathfrak{R}_G = \mathfrak{R}_g \otimes \mathfrak{R}_f$ and the chiral multiplet in \mathfrak{R}_G is coupled to dynamical vector-multiplets in Ad_{G_g} and background vector-multiplets in Ad_{G_f} , providing flavour fugacities. For both the twist and the anti-twist the results are conveniently expressed via the following variables:

$$\begin{aligned} p_+ &= m_+ - \sigma r/2, & p_- &= m_- + r/2, \\ \mathfrak{b} &= 1 + \sigma[\sigma p_+/n_+] + \lfloor -p_-/n_- \rfloor, \\ \mathfrak{c} &= \text{mod}(-p_-, n_-)/n_- - \sigma \text{mod}(\sigma p_+, n_+)/n_+, \\ \gamma &= -n/2 + \omega \chi_{-\sigma}/4, & q &= \exp(2\pi i \omega), \\ y &= q^{\mathfrak{c}/2} e^{2\pi i(r\gamma - u)}, \end{aligned} \quad (32)$$

where $\lfloor x \rfloor$ is the floor of x , namely the greatest integer less than or equal to x ; while $\text{mod}(x, y)$ is the remainder of the integer division of x by y . Identifying $\omega = i\omega/(2\pi)$ and $\gamma = i\varphi/(2\pi)$ reproduces (21). For a general G , one makes the replacements $m_\pm \rightarrow \rho(m_\pm)$ and $u \rightarrow \rho(u)$, where $\rho = \rho_g + \rho_f$ is the weight of \mathfrak{R}_G .

Equivariant orbifold index theorem

The 1-loop determinant on $\Sigma \times S^1$ is obtained from the equivariant index of the operator L_{P_-} with respect to the group action $g = \exp(-i\epsilon \delta^2)$ with equivariant parameter ϵ [1]. By inspection, $g = g_\Sigma g_{S^1}$, with $g_{S^1} \in U(1)$ acting freely on $\Sigma \times S^1$; then, the full index $I_{\Sigma \times S^1}^\sigma$ is the product of I_Σ^σ and the sum of irreducible characters of $U(1)$ [13].

For the twist, $L_{P_-}|_\Sigma = \bar{\partial}_L$ at both poles of the spindle, where $\bar{\partial}_L$ is the Dolbeault operator twisted by the holomorphic line orbifold

$$L = \mathcal{O}(n_- p_+ - n_+ p_-) = \mathcal{O}(-\mathfrak{m} - \frac{r}{2}(n_+ + \sigma n_-)), \quad (33)$$

over Σ , which requires $r \in 2\mathbb{Z}$ for L to be well-defined. Notice that, if $\Sigma = S^2$, then $r \in \mathbb{Z}$ for the topological twist, while r is not quantized on the superconformal index background. For the anti-twist, $L_{P_-}|_\Sigma = \partial_L$ at the north pole and $L_{P_-}|_\Sigma = \bar{\partial}_L$ at the south pole of Σ . The different behaviour of L_{P_-} for twist and anti-twist implies a different equivariant action on $T_{N,S}\Sigma$, making σ appear in the final result.

In general, the index takes the form of a sum of equivariant characteristic classes of an *associated orbifold* over the fixed-point set of a group action [14]. For the spindle, such a set comprises of n_+ copies of the north pole and n_- copies of the south poles, giving

$$I_\Sigma^\sigma = \frac{1}{n_+} \sum_{j=0}^{n_+-1} \frac{\omega_+^{-jp_+} q_+^{-p_+}}{1 - \omega_+^j q_+^\sigma} + \frac{1}{n_-} \sum_{j=0}^{n_- -1} \frac{\omega_-^{-jp_-} q_-^{-p_-}}{1 - \omega_-^j q_-^{-\sigma}}, \quad (34)$$

where $q_\pm = q^{1/n_\pm}$ and $\omega_\pm = e^{2\pi i/n_\pm}$. The parameters q_\pm encode the linear $U(1)$ action near the north and south-pole of Σ , respectively, with the denominators arising from the action on the complexified tangent space and the numerators corresponding to the equivariant Chern characters of the line bundle L .

By recalling that $r \in 2\mathbb{Z}$, we can resum (34) into

$$I_\Sigma^\sigma = \frac{q^{-\sigma[\sigma p_+/n_+]}}{1 - q^\sigma} + \frac{q^{\lfloor -p_-/n_- \rfloor}}{1 - q^{-1}}, \quad (35)$$

which is valid for both types of twists, simply choosing the sign of σ . It is interesting to discuss the case $\sigma = +1$, which is perhaps more familiar in the mathematics literature. In this case the fraction in (35) simplifies in a polynomial, that can be written as

$$I_\Sigma^{+1} = \begin{cases} q^{-\lfloor p_+/n_+ \rfloor} + \dots + q^{\lfloor -p_-/n_- \rfloor}, \\ -q^{-\lfloor (p_- - 1)/n_- \rfloor} - \dots - q^{\lfloor -(p_+ + 1)/n_+ \rfloor}, \end{cases} \quad (36)$$

with first and second line holding for $\mathfrak{b} \geq 0$ and $\mathfrak{b} \leq -1$, respectively. The expansions above match those of the equivariant index of $\bar{\partial}_L$, counting the \mathfrak{b} holomorphic sections of L , and agree with the Kawasaki-Riemann-Roch theorem [15] in the non-equivariant limit:

$$\lim_{q \rightarrow 1} I_\Sigma^{+1} = \mathfrak{b} = \text{deg}(L) + 1. \quad (37)$$

Including the contribution of g_{S^1} and fugacities for flavour and gauge symmetries yields

$$I_{\Sigma \times S^1}^\sigma = \sum_{k \in \mathbb{Z}} e^{-2ik\epsilon} y^{-1} \left(\frac{q^{(1-\mathfrak{b})/2}}{1 - q^\sigma} - \frac{q^{(1+\mathfrak{b})/2}}{1 - q} \right), \quad (38)$$

with \mathfrak{b}, y, q reported in (32). By Taylor-expanding in q the two fractions in (38) we obtain an infinite set of eigenvalues for each pole of the spindle. These can then be converted into infinite products in a standard way. Remarkably, after suitable regularization, the final result for the 1-loop determinant can be written as single formula valid for both twists:

$$Z_{1-L}^{\text{CM}} = (-y)^{\frac{1-\sigma-2\mathfrak{b}}{4}} q^{\frac{(1-\sigma)(\mathfrak{b}-1)}{8}} \frac{\left(q^{\frac{1}{2}(1+\mathfrak{b})} y^{-1}; q \right)_\infty}{\left(q^{\frac{\sigma}{2}(1-\mathfrak{b})} y^{-\sigma}; q \right)_\infty}, \quad (39)$$

where $(z; q)_n$ is the n -th q -Pochhammer symbol. For the twist $\sigma = +1$ and (39) simplifies as the finite product

$$Z_{1-L}^{\text{CM}} = (-y)^{-\mathfrak{b}/2} \left(q^{\frac{1}{2}(1-\mathfrak{b})} y^{-1}; q \right)_\mathfrak{b}^{-1}, \quad (40)$$

with the \mathfrak{b} factors appearing in the q -Pochhammer symbol above precisely corresponding to the contributions of the \mathfrak{b} sections of the line bundle L , counted by (36). For either choice of twist, invariance under large gauge transformations, corresponding to integer shifts of u , induces a $1/2$ -shift of the Chern-Simons level k . After root decomposition, the 1-loop determinant of a vector multiplet turns out to be independent of the R -symmetry twist and, up to a regularization-dependent sign, it reads

$$Z_{1-L}^{\text{VM}} = \prod_{\alpha>0} \prod_{I=\pm} \left(z^{-\alpha/2} - q^{\frac{\alpha_+}{2n_+} + \frac{\alpha_-}{2n_-} - \lfloor \frac{\alpha_I}{n_I} \rfloor} z^{\alpha/2} \right)^{\mu_I} \times q^{\frac{1}{8}(\mu_- - \mu_+) \frac{\alpha(\mathfrak{m})}{n_+ n_-}}, \quad (41)$$

where $\alpha_{\pm} = \alpha(m_{\pm})$, $z^{\alpha} = e^{2\pi i \alpha(u)}$, α is the weight of the adjoint representation, while

$$\mu_I \equiv \begin{cases} 1 & \text{if } \alpha_I/n_I \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}. \quad (42)$$

Although the classical contribution (26) only depends on the total gauge-field flux \mathfrak{m} , the 1-loop determinants depend a priori on m_{\pm} through \mathfrak{b} , \mathfrak{c} , defined in (32), and through α_{\pm} . Taking $a_{\pm} \in \mathbb{Z}$ such that $a_- n_+ - n_- a_+ = 1$, we can parameterise $m_{\pm} = (a_{\pm} + t n_{\pm})\mathfrak{m}$, where $t \in \mathbb{Z}$. Since both \mathfrak{b} and \mathfrak{c} are independent of t , Z_{1-L}^{CM} depends only on \mathfrak{m} ; similarly, one can check that Z_{1-L}^{VM} depends only on \mathfrak{m} . Thus, in the complete partition function (19) we have to sum only over the gauge flux $\mathfrak{m} \in \mathbb{Z}$. Finally, taking $n_+ = n_- = 1$ reduces the full partition function to the topologically twisted index [2] and the superconformal index [4] upon setting $\sigma = \pm 1$, respectively.

DISCUSSION

In this letter we demonstrated that three-dimensional $\mathcal{N} = 2$ SQFTs can be defined on $\Sigma \times S^1$, endowed with both types of supersymmetry-preserving twists and that the corresponding partition functions give rise to two novel indices. These can be expressed by a single formula, generalizing and unifying the superconformal and topologically twisted indices. We therefore refer to this as to the *spindle index*. Many more details and applications will be discussed in [11]. The expression we found for the 1-loop determinant (39) resembles the supersymmetric observables involving vortex defects computed in [16] and it would be interesting to investigate further their relationship. We anticipate that the large- N limit of the spindle index should reproduce the entropy functions associated to the supersymmetric and accelerating

AdS₄ black holes [6, 17]. More generally, it should reproduce the entropy functions presented in [18], valid for an extensive class of three-dimensional $\mathcal{N} = 2$ theories with gravity duals. We expect that gravitational blocks [19], whose gluing yields these entropy functions, should arise in the large- N limit of the single fixed-point contributions to the orbifold equivariant index discussed here. Our findings suggest that some observables of SQFTs compactified on spindles and other orbifolds \mathbb{M}_p can be computed via localization. In particular, it would be interesting to compute orbifold partition functions of SQFTs on $M_{d-p} \times \mathbb{M}_p$ and to prove the large- N gravitational block formulas conjectured in [20, 21].

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* apittelli88@gmail.com

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