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We present a new supersymmetric index for three-dimensional $\mathcal{N}=2$ gauge theories defined on $\mathbb{\Sigma}\times S^1$, where $\mathbb{\Sigma}$ is a spindle, with twist or anti-twist for the R-symmetry background gauge field. We start examining general supersymmetric backgrounds of Euclidean new minimal supergravity admitting two Killing spinors of opposite R-charges. We then focus on $\mathbb{\Sigma}\times S^1$ and demostrate how to realise twist and anti-twist. We compute the supersymmetric partition functions on such backgrounds via localization and show that these are captured by a general formula, depending on the type of twist, which unifies and generalises the superconformal and topologically twisted indices.

INTRODUCTION

Localization techniques [1] are a tremendous tool to calculate path-integrals of supersymmetric quantum field theories (SQFTs) on curved manifolds, giving access to a profusion of exact results. This letter focuses on three-dimensional SQFTs, whereof two compelling observables are the partition functions on $S^2\times S^1$ endowed with an R-symmetry background gauge field, with or without flux through the sphere, corresponding to the topologically twisted index [2] and the superconformal index [3, 4], respectively. The large-N limit of the former provides a microscopic interpretation of the entropy of magnetically charged supersymmetric black holes in AdS_4 [5].

The supersymmetric and *accelerating* AdS_4 black hole presented in [6] displays a number of striking features, including a horizon with orbifold singularities and supersymmetry preserved in a novel fashion. In the Euclidean setting, the conformal boundary is $\mathbb{\Sigma}\times S^1$, where $\mathbb{\Sigma}=\mathbb{WCP}^1_{[n_+,n_-]}$ is a *spindle*, *i.e.* a weighted projective space specified by two coprime positive integers $n_\pm$. The entropy of this solution is reproduced by considering a family of Euclidean saddles extending the black holes, where both bulk and boundary metrics are complex [7].

Motivated by these results, in this letter we present two partition functions of $\mathcal{N}=2$ Chern-Simons-matter theories defined on $\mathbb{\Sigma}\times S^1$. Supersymmetry on $\mathbb{\Sigma}$ is preserved by an R-symmetry background gauge field A satisfying only one [8] of the following conditions [9]:

$$\int_{\mathbb{\Sigma}}\frac{\mathrm{d}A}{2\pi}=\frac{1}{2}\left(\frac{1}{n_-}+\frac{\sigma}{n_+}\right)\equiv\frac{\chi_\sigma}{2}\ , \tag{1}$$

with $\sigma=\pm1$ being configurations known as *twist* and *anti-twist*, respectively. Here $\chi_+=\frac{1}{4\pi}\int_{\mathbb{\Sigma}}\sqrt{g}R$ is the orbifold Euler-characteristic of the spindle. We employ the framework of rigid new minimal supergravity and generalize the analysis of [10] to accomodate the intrinsically complex backgrounds of [7]. We extend the localization approach to orbifolds and compute partition functions on $\mathbb{\Sigma}\times S^1$ with both twist and anti-twist. The technical details of the analysis will be spelled out in [11].

GENERAL COMPLEX BACKGROUNDS

We consider a general class of rigid supersymmetric backgrounds of Euclidean new minimal supergravity preserving two Killing spinors $\zeta_\pm$ with R-charges ±1, respectively. The Killing spinor equations (KSEs) are

$$(\nabla_\mu\mp\mathrm{i}A_\mu)\zeta_\pm=-\tfrac{H}{2}\gamma_\mu\zeta_\pm\mp\mathrm{i}V_\mu\zeta_\pm\mp\varepsilon_{\mu\nu\rho}\tfrac{V^\nu}{2}\gamma^\rho\zeta_\pm\ , \tag{2}$$

where A_μ is the R-symmetry background gauge field, V_μ is a globally defined co-closed one-form and H a scalar, a priori all complex-valued. We emphasise that $\zeta_\pm$ are not related by charge conjugation and, differently from previous literature, the metric $g_{\mu\nu}$ can be complex-valued.

Following the conventions of [10], we introduce

$$v=\zeta_+\zeta_-\ ,\quad K^\mu=\zeta_+\gamma^\mu\zeta_-\ ,\quad P^\mu_\pm=\zeta_\pm\gamma^\mu\zeta_\pm/v\ , \tag{3}$$

where $\left(P^\mu_+\right)^*\neq P^\mu_-$. The KSEs imply that K^μ is a complex Killing vector. Thanks to Fierz identities $(K,P_\pm)$ is a canonical complex frame generating the line element

$$\mathrm{d}s^2=(K/v)^2-P_+P_-\ , \tag{4}$$

with non-zero contractions being

$$\iota_K K=v^2\ ,\qquad \iota_{P_-}P_+=\iota_{P_+}P_-=-2\ . \tag{5}$$

The KSEs determine the background fields A_μ,V_μ,H in terms of $(K,P_\pm)$. We are interested in complex backgrounds on a real three-dimensional space $\mathcal{M}_3$: since K is a complex Killing vector, it picks out two angular coordinates φ,ψ, thus

$$K=\kappa\left(\partial_\psi+\omega\partial_\varphi\right)\ , \tag{6}$$

where κ and ω are complex constants and φ,ψ are 2π-periodic. By rescaling $\zeta_\pm$ we can set $\kappa=1$ without loss of generality. The most general metric on $\mathcal{M}_3$ invariant under two real Killing vectors $\partial_\psi,\partial_\varphi$ can be written as

$$\mathrm{d}s^2=f^2\mathrm{d}x^2+h_{ij}\mathrm{d}\psi_i\mathrm{d}\psi_j\quad\text{with}\quad\psi_1=\psi\,,\psi_2=\varphi\,, \tag{7}$$

where the complex-valued functions $f(x)$ and $h_{ij}(x)$, $i,j=1,2$ depend only on the coordinate x. In these coordinates we have

$$\begin{aligned} K &= (h_{11}+\omega h_{12})\mathrm{d}\psi + (h_{12}+\omega h_{22})\mathrm{d}\varphi\,, \\ P_\pm &= \mathrm{e}^{\pm 2\mathrm{i}\theta}\big(\pm f\mathrm{d}x + \mathrm{i}(\sqrt{h}/v)(-\omega\mathrm{d}\psi+\mathrm{d}\varphi)\big)\,, \\ v^2 &= h_{11}+2\omega h_{12}+\omega^2 h_{22}\,, \end{aligned} \tag{8}$$

where $h=\det(h_{ij})$ and $\theta\equiv(\alpha_1\psi+\alpha_2\varphi)/2$, with α_1,α_2 two real constants that we shall discuss momentarily. Defining $A^C\equiv A-\frac{3}{2}V$, the background fields read

$$\begin{aligned} V &= \frac{1}{v}\Big[\mathrm{i}HK - \star\mathrm{d}K\Big]\,, \\ A^C &= \frac{v^3}{4f\sqrt{h}}\Big[\frac{1}{\omega}\Big(\frac{h_{11}}{v^2}\Big)'\mathrm{d}\psi - \Big(\frac{h_{22}}{v^2}\Big)'\mathrm{d}\varphi\Big] + \mathrm{d}\theta\,, \end{aligned} \tag{9}$$

where a prime denotes derivative with respect to x. The function H satisfies $\mathcal{L}_K H=0$ and is otherwise arbitrary; however, it will enter in the localization computation only though the following combinations:

$$\begin{aligned} h_R &\equiv \iota_K V - \mathrm{i}vH = -\frac{1}{2v}\star(K\wedge\mathrm{d}K)\,, \\ \Phi_R &\equiv \iota_K\big(A^C+V\big) - \mathrm{i}vH = (\alpha_1+\omega\alpha_2)/2\,. \end{aligned} \tag{10}$$

Taking $\gamma^1=\sigma^2,\ \gamma^2=\sigma^3,\ \gamma^3=\sigma^1$, with σ^i being the Pauli matrices, in the frame where $e^1=-f\,\mathrm{d}x$ and

$$e^2=\sqrt{\tfrac{h}{h_{11}}}\mathrm{d}\varphi\,,\quad e^3=\sqrt{h_{11}}\big(\mathrm{d}\psi+\tfrac{h_{12}}{h_{11}}\mathrm{d}\varphi\big)\,, \tag{11}$$

the Killing spinors satisfying (2) take the form

$$\zeta_+ = \mathrm{e}^{\mathrm{i}\theta}\big(u_1,-u_2\big)^T\,,\quad \zeta_- = -\mathrm{e}^{-\mathrm{i}\theta}\big(u_2,u_1\big)^T\,, \tag{12}$$

where T indicates transposition and

$$u_{1,2}=2^{-1/2}\sqrt{v\mp\omega\sqrt{h/h_{11}}}\,. \tag{13}$$

$\mathbb{\Sigma}\times S^1$ with twist and anti-twist

The results presented so far are purely local and apply to any space $\mathcal{M}_3$, including e.g. S^3. We shall now restrict attention to $\mathcal{M}_3=\mathbb{\Sigma}\times S^1$. We take ψ to parameterize S^1 and $x\in[-1,1],\varphi$ as coordinates on $\mathbb{\Sigma}$. This fixes the bevaviour of the functions f,h_{ij} near the poles of $\mathbb{\Sigma}$, which we denote as $\mathrm{N}\equiv\{x=1\}$ and $\mathrm{S}\equiv\{x=-1\}$. By using reparameterization to set for simplicity $f=1$ and denoting by $\varrho_\pm$ the coordinates near each pole, at leading order we have

$$h_{11}\sim h_{11}^\pm\,,\qquad h_{22}\sim\tfrac{1}{n_\pm^2}\varrho_\pm^2\,,\qquad h_{12}\sim h_{12}^\pm\varrho_\pm^2\,, \tag{14}$$

at the north and south poles respectively, where $h_{12}^\pm, h_{11}^\pm$ are complex constants. By using (9) we find

$$\frac{1}{2\pi}\int_{\mathbb{\Sigma}}\mathrm{d}A = -\frac{1}{2}\Big(\frac{s_+}{n_+}+\frac{s_-}{n_-}\Big)\,, \tag{15}$$

where $s_\pm$ denote the signs of the function $v/\sqrt{h_{11}}$ at the north and south poles, respectively. Therefore, the type of supersymmetry-preserving twist is completely encoded in the behaviour of the function v. Given a generic metric on $\mathbb{\Sigma}\times S^1$ and a parameter ω, we regard the third equation in (8) as a definition of the function v. From this, it follows that generically the function $v/\sqrt{h_{11}}$ has the same sign at both poles, corresponding to the twist case. Instead, the anti-twist is realized if the function $v/\sqrt{h_{11}}$ has opposite sign at the poles. In this case the metric h_{ij} and the parameter ω need be fine-tuned.

Requiring $\zeta_\pm$ to be (anti-)periodic under $\psi\sim\psi+2\pi$ implies that $\alpha_1=n\in\mathbb{Z}$. On $\mathbb{\Sigma}$ the spinors should be defined in two patches $\mathcal{U}_{\mathrm{N/S}}$, with the non-singular gauge fields related via gauge transformations, corresponding to different values of α_2. Specifically, the regular $A_{\mathrm{N/S}}$ are obtained taking $\alpha_2=s_+/n_+$ in $\mathcal{U}_\mathrm{N}$ and $\alpha_2=-s_-/n_-$ in $\mathcal{U}_\mathrm{S}$. Finally, from (13) one can see that at the north and south poles of $\mathbb{\Sigma}$ the spinors behave as

$$\zeta_+^{\mathrm{N/S}}\sim\big(1,-s_\pm\big)^T\,,\qquad \zeta_-^{\mathrm{N/S}}\sim\big(s_\pm,1\big)^T\,, \tag{16}$$

so that indeed they have the same 2d-chirality for the twist and the opposite 2d-chirality for the anti-twist [8].

By exploiting such formalism we can immediately jump at the localization computation of the partition functions on these backgrounds. Importantly, these will only depend on ω, not on the specific representative metric, with the caveat explained above that for the anti-twist the metric depends on ω. To illustrate these features, it is instructive to consider the explicit background

$$\mathrm{d}s^2=f^2\mathrm{d}x^2+\big(1-x^2\big)\big(\mathrm{d}\varphi-\Omega\mathrm{d}\psi\big)^2+\beta^2\mathrm{d}\psi^2\,, \tag{17}$$

where $x\in[-1,1]$, φ and ψ have 2π-periodicities and

$$f(x)\sim n_\pm/\sqrt{2(1\mp x)}\,,\qquad\text{for}\quad x\to\pm1\,, \tag{18}$$

so that at any constant ψ, the coordinates x,φ describe a spindle, equipped with a generic metric. Although we choose β real and positive for simplicity, the function $f(x)$ and the constants β and Ω can be complex a priori. The frame $(K,P_\pm)$, background fields A,V,H and Killing spinors $\zeta_\pm$ are then completely determined by our general formulae (8), (9), (12) in terms of the metric functions $f(x),\Omega,\beta$ and the parameter ω.

Let us briefly discuss how the two twists are realized for the family of metrics (17). As $v^2=(1-x^2)(\omega-\Omega)^2+\beta^2$, if no relation is imposed between ω, β and Ω, then the R-symmetry background field realizes the *twist*. As a special case, the standard topological twist corresponds to $\omega=\Omega$, yielding $v/\beta=-1$, so that in (1) we have $\sigma=+1$. Conversely, the *anti-twist* is realized by choosing $\Omega=\omega\pm\mathrm{i}\beta$. In this case we can take $v/\beta=x$, so that in (1) we have $\sigma=-1$.

LOCALIZATION

Let G be a semi-simple Lie group with Lie algebra $\mathfrak{g}$, with $\mathfrak{R}_{\rm G}$ a generic representation of G and $\mathrm{Ad}_{\rm G}$ its adjoint representation. For a three-dimensional $\mathcal{N}=2$ gauge theory, the supersymmetry transformations of a vector multiplet $(\mathcal{A},\sigma,\lambda,\widetilde{\lambda},D)\in\mathrm{Ad}_{\rm G}$ and those of a chiral multiplet $(\phi,\psi,F)\in\mathfrak{R}_{\rm G}$ can be written as a cohomological complex [1]. Solving the BPS equations $\delta\psi=\delta\widetilde{\psi}=\delta\lambda=\delta\widetilde{\lambda}=0$ yields the localizion locus of classical configurations contributing by $Z_{\rm class}$ to the partition function. Also, this formulation allows for recasting the computation of vector- and chiral-multiplets 1-loop determinants, $Z^{\rm VM}_{\text{1-L}}, Z^{\rm CM}_{\text{1-L}}$, as a cohomological problem. Below we will present the main steps of this procedure, referring to [11] for more details. If $\mathrm{G}=\mathrm{G}_g\times\mathrm{G}_f$, the partition function of a theory on $S^1\times\mathbb{\Sigma}$ with gauge group G_g and flavour group G_f reads

$$Z_{S^1\times\mathbb{\Sigma}}(\omega,u_f,\mathfrak{f}_f)=\sum_{\mathfrak{f}_g\in\Gamma_{\mathfrak{h}_g}}\oint_{\mathcal{C}}\tfrac{\mathrm{d}u_g}{|W_g|}\,\widehat{Z}(u_g,\mathfrak{f}_g|\omega,u_f,\mathfrak{f}_f)\ , \quad (19)$$

where $\mathfrak{h}_g$ is the maximal Cartan-subalgebra of $\mathfrak{g}_g$, $\Gamma_{\mathfrak{h}_g}$ the corresponding co-root lattice and W_g its Weyl group; while $u_{g,f}\in\mathfrak{h}_{g,f}$ and $\mathfrak{f}_{g,f}\in\Gamma_{\mathfrak{h}_{g,f}}$ denote gauge/flavour holonomies and fluxes, respectively; $\widehat{Z}$ is the product of $Z_{\rm class}$, $Z^{\rm VM}_{\text{1-L}}$ and $Z^{\rm CM}_{\text{1-L}}$ and $\mathcal{C}$ is a suitable integration-contour for u_g. The partition function (19) also depends on the spindle data $(n_\pm,\sigma)$, which we suppressed to not clutter the notation. $Z_{S^1\times\mathbb{\Sigma}}$ is related to the G_f-flavoured Witten-index of the theory quantized on $\mathbb{\Sigma}$, that is

$$I_{S^1\times\mathbb{\Sigma}}=\mathrm{Tr}_{\mathscr{H}[\mathbb{\Sigma}]}\left[\mathrm{e}^{-\omega J-\varphi R-\sum_i\varphi_i F_i}\right]\ , \quad (20)$$

where J,R,F_i generate angular momentum, R-symmetry and flavour symmetries, respectively; while $\mathscr{H}[\mathbb{\Sigma}]$ is the Hilbert space of states on the spindle, with either twist or anti-twist. We anticipate that the fugacities ω and φ are not independent, but are related by

$$\varphi-\frac{\omega}{4}\chi_{-\sigma}=\mathrm{i}\pi n\,,\qquad n\in\mathbb{Z}\ . \quad (21)$$

For the anti-twist ($\sigma=-1$), taking $n=\pm1$ so that the spinors are anti-periodic on S^1, this reproduces the relation found in [7] for the dual accelerating black holes.

BPS locus and Chern-Simons term

The vector-multiplet BPS equations read

$$\iota_K(\star\mathcal{F})=-vD-\mathrm{i}h_R\sigma\,,\qquad \iota_K\mathcal{F}=-\mathrm{id}_{\mathcal{A}}(v\sigma)\,, \quad (22)$$

where $\mathcal{F}$ is the field strenght of the gauge field $\mathcal{A}$. Neither solving (22) nor computing $Z_{\rm class}$ or $Z_{\text{1-L}}$ require imposing reality conditions on fields. We restrict for simplicity to Abelian gauge fields, the non-Abelian generalization being straightforward. The BPS locus is parametrized by a gauge field $\mathcal{A}$ obeying $\mathcal{L}_K\mathcal{A}=0$, with flux

$$\mathfrak{f}_G\equiv\frac{1}{2\pi}\int_{\mathbb{\Sigma}}\mathrm{d}\mathcal{A}=\frac{\mathfrak{m}}{n_+n_-}\ , \quad (23)$$

where $\mathfrak{m}\in\mathbb{Z}$ [8] and we can always write

$$\mathfrak{m}=n_+m_--n_-m_+\ ,\quad m_\pm\in\mathbb{Z}\ . \quad (24)$$

The second equation in (22) is solved by

$$\mathcal{A}_\psi+\omega\mathcal{A}_\varphi-\mathrm{i}v\sigma=\Phi_G\ , \quad (25)$$

where $\Phi_{\rm G}$ is an arbitrary complex constant (in each patch $\mathcal{U}_{\rm N/S}$). The first equation in (22) yields the auxiliary field D. The Abelian Chern-Simons term contributes to $Z_{\rm class}$ via $Z_{\rm CS}=\mathrm{e}^{-S_{\rm CS}}$, with [12]

$$S_{\rm CS}=\frac{\mathrm{i\,k}}{4\pi}\int(\mathcal{A}\wedge\mathcal{F}+2\mathrm{i}\star D\sigma)=2\pi\mathrm{i\,k}\,\mathfrak{f}_{\rm G}u\ , \quad (26)$$

evaluated on BPS configurations solving (22), where

$$u\equiv\Big(\mathcal{A}^+_\psi+\mathcal{A}^-_\psi-\mathrm{i}v^+\sigma^+-\mathrm{i}v^-\sigma^-\Big)/2\ , \quad (27)$$

with the $\pm$ superscripts denoting quantities evaluated at the N/S poles of $\mathbb{\Sigma}$. The differential operators acting on a chiral-multiplet field $\phi\in\mathfrak{R}_{\rm G}$ with R-charge q^ϕ_R are

$$\begin{aligned}L_K&=\mathcal{L}_K-\mathrm{i}\,q^\phi_R\,\Phi_R\ ,\\ L_{P_\pm}&=\mathcal{L}_{P_\pm}-\mathrm{i}\,q^\phi_R\,\iota_{P_\pm}\big(A^C+V\big)-\mathrm{i}\,\iota_{P_\pm}\mathcal{A}\ ,\end{aligned} \quad (28)$$

and the chiral-multiplet BPS equations read

$$\delta^2\phi=(L_K+\mathcal{G}_{\Phi_{\rm G}})\phi=0\ ,\qquad F+\mathrm{i}L_{P_-}\phi=0\ , \quad (29)$$

where for a chiral-multiplet scalar field $\phi\in\mathfrak{R}_{\rm G}$ we have

$$\mathcal{G}_{\Phi_{\rm G}}\phi=-\mathrm{i}\Phi_{\rm G}\circ_{\mathfrak{R}_{\rm G}}\phi\ , \quad (30)$$

where $\circ_{\mathfrak{R}_{\rm G}}$ indicates the action of Φ_G according to the representation $\mathfrak{R}_{\rm G}$. For arbitrary q^ϕ_R and Φ_G, regularity of the solutions to the first equation in (29) implies $\phi=0$, while $F=0$ follows from the second. Similarly, $\widetilde{\phi}=\widetilde{F}=0$ for an anti-chiral multiplet. Taking $\mathcal{A}$ and σ to be Hermitian, (25) splits in two real equations determining $\mathcal{A}_\psi$ and σ in terms of $\mathcal{A}_\varphi$, which is constrained by the first equation in (22). We impose no reality conditions on the auxiliary field D.

1-loop determinants

Using a set of cohomological variables, the 1-loop determinant of a chiral multiplet of R-charge r in $\mathfrak{R}_{\rm G}$ can be cast in the form

$$Z^{\rm CM}_{\text{1-L}}=\frac{\det_{\mathrm{Ker}L_{P_+}}(L_K+\mathcal{G}_{\Phi_{\rm G}})}{\det_{\mathrm{Ker}L_{P_-}}(L_K+\mathcal{G}_{\Phi_{\rm G}})}\ , \quad (31)$$

while the 1-loop determinant of a vector multiplet, $Z^{\rm VM}_{\text{1-L}}$, includes the contribution of BRST-ghosts compatible with supersymmetry [1]. However, a standard argument implies that formally $Z^{\rm VM}_{\text{1-L}} = Z^{\rm CM}_{\text{1-L}}(r=2)|_{\mathfrak{R}_{\rm G}=\mathrm{Ad}_G}$ [2], so in the following we shall focus on chiral multiplets.

The modes $\mathscr{F}^{\pm} \in L^2\big[\Sigma \times S^1\big]$ contributing to (31) obey the linear ODEs $L_{P_\pm}\mathscr{F}^{\pm} = 0$ and are eigenfunctions of the operator $L_K + \mathcal{G}_{\Phi_{\rm G}}$. The functional determinant of the latter is an infinite product of eigenvalues that, after regularization, explicitly provides (31) in terms of special functions [11]. In this letter we shall sketch an alternative route to the same result, namely extracting the eigenvalues from an *equivariant orbifold index theorem.*

In general $\mathrm{G} = \mathrm{G}_g \times \mathrm{G}_f$ and $\mathfrak{g} = \mathfrak{g}_g \oplus \mathfrak{g}_f$, so that $\mathfrak{R}_{\rm G} = \mathfrak{R}_g \otimes \mathfrak{R}_f$ and the chiral multiplet in $\mathfrak{R}_{\rm G}$ is coupled to dynamical vector-multiplets in $\mathrm{Ad}_{\mathrm{G}_g}$ and background vector-multiplets in $\mathrm{Ad}_{\mathrm{G}_f}$, providing flavour fugacities. For both the twist and the anti-twist the results are conveniently expressed via the following variables:

$$
\begin{aligned}
p_+ &= m_+ - \sigma\, r/2 \ , \qquad p_- = m_- + r/2 \ , \\
\mathfrak{b} &= 1 + \sigma \lfloor \sigma\, p_+/n_+ \rfloor + \lfloor -p_-/n_- \rfloor \ , \\
\mathfrak{c} &= \mathrm{mod}\,(-p_-, n_-)/n_- - \sigma\, \mathrm{mod}\,(\sigma\, p_+, n_+)/n_+ \ , \\
\gamma &= -n/2 + \omega\, \chi_{-\sigma}/4 \ , \qquad q = \exp\,(2\pi\mathrm{i}\,\omega) \ , \\
y &= q^{\mathfrak{c}/2} \mathrm{e}^{2\pi\mathrm{i}(r\gamma - u)} \ ,
\end{aligned} \tag{32}
$$

where $\lfloor \mathrm{x} \rfloor$ is the floor of x, namely the greatest integer less than or equal to x; while $\mathrm{mod}\,(\mathrm{x},\mathrm{y})$ is the reminder of the integer division of x by y. Identifying $\omega = \mathrm{i}\omega/(2\pi)$ and $\gamma = \mathrm{i}\varphi/(2\pi)$ reproduces (21). For a general G, one makes the replacements $m_\pm \to \rho(m_\pm)$ and $u \to \rho(u)$, where $\rho = \rho_g + \rho_f$ is the weight of $\mathfrak{R}_{\rm G}$.

Equivariant orbifold index theorem

The 1-loop determinant on $\Sigma \times S^1$ is obtained from the equivariant index of the operator L_{P_-} with respect to the group action $g = \exp\left(-\mathrm{i}\epsilon\, \delta^2\right)$ with equivariant parameter ϵ [1]. By inspection, $g = g_\Sigma\, g_{S^1}$, with $g_{S^1} \in U(1)$ acting freely on $\Sigma \times S^1$; then, the full index $I^\sigma_{\Sigma\times S^1}$ is the product of I^σ_Σ and the sum of irreducible characters of $U(1)$ [13].

For the twist, $L_{P_-}|_\Sigma = \overline{\partial}_L$ at both poles of the spindle, where $\overline{\partial}_L$ is the Dolbeault operator twisted by the holomorphic line orbibundle

$$
L = \mathcal{O}(n_- p_+ - n_+ p_-) = \mathcal{O}(-\mathfrak{m} - \tfrac{r}{2}(n_+ + \sigma n_-)) \ , \tag{33}
$$

over Σ, which requires $r \in 2\,\mathbb{Z}$ for L to be well-defined. Notice that, if $\Sigma = S^2$, then $r \in \mathbb{Z}$ for the topological twist, while r is not quantized on the superconformal index background. For the anti-twist, $L_{P_-}|_\Sigma = \partial_L$ at the north pole and $L_{P_-}|_\Sigma = \overline{\partial}_L$ at the south pole of Σ. The different behaviour of L_{P_-} for twist and anti-twist implies a different equivariant action on $T_{\rm N,S}\Sigma$, making σ appear in the final result.

In general, the index takes the form of a sum of equivariant characteristic classes of an *associated orbifold* over the fixed-point set of a group action [14]. For the spindle, such a set comprises of n_+ copies of the north pole and n_- copies of the south poles, giving

$$
I^\sigma_\Sigma = \frac{1}{n_+} \sum_{j=0}^{n_+-1} \frac{\omega_+^{-jp_+} q_+^{-p_+}}{1-\omega_+^{j\sigma} q_+^\sigma} + \frac{1}{n_-} \sum_{j=0}^{n_--1} \frac{\omega_-^{-jp_-} q_-^{-p_-}}{1-\omega_-^{-j} q_-^{-1}} \ , \tag{34}
$$

where $q_\pm = q^{1/n_\pm}$ and $\omega_\pm = \mathrm{e}^{2\pi\mathrm{i}/n_\pm}$. The parameters $q_\pm$ encode the linear $U(1)$ action near the north and south-pole of Σ, respectively, with the denominators arising from the action on the complexified tangent space and the numerators corresponding to the equivariant Chern characters of the line bundle L.

By recalling that $r \in 2\,\mathbb{Z}$, we can resum (34) into

$$
I^\sigma_\Sigma = \frac{q^{-\sigma\lfloor \sigma\, p_+/n_+ \rfloor}}{1-q^\sigma} + \frac{q^{\lfloor -p_-/n_- \rfloor}}{1-q^{-1}} \ , \tag{35}
$$

which is valid for both types of twists, simply choosing the sign of σ. It is interesting to discuss the case $\sigma = +1$, which is perhaps more familiar in the mathematics literature. In this case the fraction in (35) simplifies in a polynomial, that can be written as

$$
I^{+1}_\Sigma = \begin{cases} q^{-\lfloor p_+/n_+ \rfloor} + \dots + q^{\lfloor -p_-/n_- \rfloor} \ , \\ -q^{-\lfloor (p_- -1)/n_- \rfloor} - \dots - q^{\lfloor -(p_+ +1)/n_+ \rfloor} \ , \end{cases} \tag{36}
$$

with first and second line holding for $\mathfrak{b} \geq 0$ and $\mathfrak{b} \leq -1$, respectively. The expansions above match those of the equivariant index of $\overline{\partial}_L$, counting the $\mathfrak{b}$ holomorphic sections of L, and agree with the Kawasaki-Riemann-Roch theorem [15] in the non-equivariant limit:

$$
\lim_{q\to 1} I^{+1}_\Sigma = \mathfrak{b} = \deg(L) + 1 \ . \tag{37}
$$

Including the contribution of g_{S^1} and fugacities for flavour and gauge symmetries yields

$$
I^\sigma_{\Sigma\times S^1} = \sum_{k\in\mathbb{Z}} \mathrm{e}^{-2ik\epsilon} y^{-1} \left(\frac{q^{(1-\mathfrak{b})/2}}{1-q^\sigma} - \frac{q^{(1+\mathfrak{b})/2}}{1-q} \right) \ , \tag{38}
$$

with $\mathfrak{b}, y, q$ reported in (32). By Taylor-expanding in q the two fractions in (38) we obtain an infinite set of eigenvalues for each pole of the spindle. These can then be converted into infinite products in a standard way. Remarkably, after suitable regularization, the final result for the 1-loop determinant can be written as single formula valid for both twists:

$$
Z^{\rm CM}_{\text{1-L}} = (-y)^{\frac{1-\sigma-2\mathfrak{b}}{4}} q^{\frac{(1-\sigma)(\mathfrak{b}-1)}{8}} \frac{\left(q^{\frac{1}{2}(1+\mathfrak{b})} y^{-1}; q\right)_\infty}{\left(q^{\frac{\sigma}{2}(1-\mathfrak{b})} y^{-\sigma}; q\right)_\infty} \ , \tag{39}
$$

where $(z;q)_n$ is the n-th q-Pochhammer symbol. For the twist $\sigma = +1$ and (39) simplifies as the finite product

$$
Z^{\rm CM}_{\text{1-L}} = (-y)^{-\mathfrak{b}/2} \left(q^{\frac{1}{2}(1-\mathfrak{b})} y^{-1}; q\right)^{-1}_{\mathfrak{b}} \ , \tag{40}
$$

with the $\mathfrak{b}$ factors appearing in the q-Pochammer symbol above precisely corresponding to the contributions of the $\mathfrak{b}$ sections of the line bundle L, counted by (36). For either choice of twist, invariance under large gauge transformations, corresponding to integer shifts of u, induces a 1/2-shift of the Chern-Simons level k. After root decomposition, the 1-loop determinant of a vector multiplet turns out to be independent of the R-symmetry twist and, up to a regularization-dependent sign, it reads

$$Z_{\text{1-L}}^{\text{VM}} = \prod_{\alpha>0}\prod_{I=\pm}\left(\mathrm{z}^{-\alpha/2} - q^{\frac{\alpha_+}{2n_+}+\frac{\alpha_-}{2n_-}-\left\lfloor\frac{\alpha_I}{n_I}\right\rfloor}\mathrm{z}^{\alpha/2}\right)^{\mu_I} \times q^{\frac{1}{8}(\mu_- - \mu_+)\frac{\alpha(\mathfrak{m})}{n_+ n_-}}\,, \tag{41}$$

where $\alpha_\pm = \alpha(m_\pm)$, $\mathrm{z}^\alpha = \mathrm{e}^{2\pi \mathrm{i}\alpha(u)}$, α is the weight of the adjoint representation, while

$$\mu_I \equiv \begin{cases} 1 & \text{if} \quad \alpha_I/n_I \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}\,. \tag{42}$$

Although the classical contribution (26) only depends on the total gauge-field flux $\mathfrak{m}$, the 1-loop determinants depend a priori on $m_\pm$ through $\mathfrak{b}$, $\mathfrak{c}$, defined in (32), and through $\alpha_\pm$. Taking $a_\pm \in \mathbb{Z}$ such that $a_- n_+ - n_- a_+ = 1$, we can parameterise $m_\pm = (a_\pm + t\, n_\pm)\mathfrak{m}$, where $t \in \mathbb{Z}$. Since both $\mathfrak{b}$ and $\mathfrak{c}$ are independent of t, $Z_{\text{1-L}}^{\text{CM}}$ depends only on $\mathfrak{m}$; similarly, one can check that $Z_{\text{1-L}}^{\text{VM}}$ depends only on $\mathfrak{m}$. Thus, in the complete partition function (19) we have to sum only over the gauge flux $\mathfrak{m} \in \mathbb{Z}$. Finally, taking $n_+ = n_- = 1$ reduces the full partition function to the topologically twisted index [2] and the superconformal index [4] upon setting $\sigma = \pm 1$, respectively.

DISCUSSION

In this letter we demonstrated that three-dimensional $\mathcal{N} = 2$ SQFTs can be defined on $\mathbb{\Sigma} \times S^1$, endowed with both types of supersymmetry-preserving twists and that the corresponding partition functions give rise to two novel indices. These can be expressed by a single formula, generalizing and unifying the superconformal and topologically twisted indices. We therefore refer to this as to the *spindle index*. Many more details and applications will be discussed in [11]. The expression we found for the 1-loop determinant (39) resembles the supersymmetric observables involving vortex defects computed in [16] and it would be interesting to investigate further their relationship. We anticipate that the large-N limit of the spindle index should reproduce the entropy functions associated to the supersymmetric and accelerating AdS$_4$ black holes [6, 17]. More generally, it should reproduce the entropy functions presented in [18], valid for an extensive class of three-dimensional $\mathcal{N} = 2$ theories with gravity duals. We expect that gravitational blocks [19], whose gluing yields these entropy functions, should arise in the large-N limit of the single fixed-point contributions to the orbifold equivariant index discussed here. Our findings suggest that some observables of SQFTs compactified on spindles and other orbifolds $\mathbb{M}_p$ can be computed via localization. In particular, it would be interesting to compute orbifold partition functions of SQFTs on $M_{d-p} \times \mathbb{M}_p$ and to prove the large-N gravitational block formulas conjectured in [20, 21].

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[1] V. Pestun, Commun. Math. Phys. **313**, 71 (2012).
[2] F. Benini and A. Zaffaroni, JHEP **07**, 127.
[3] Y. Imamura and S. Yokoyama, JHEP **04**, 007.
[4] A. Kapustin and B. Willett, (2011).
[5] F. Benini, K. Hristov, and A. Zaffaroni, JHEP **05**, 054.
[6] P. Ferrero, J. P. Gauntlett, J. M. P. Ipiña, D. Martelli, and J. Sparks, Phys. Rev. D **104**, 046007 (2021).
[7] D. Cassani, J. P. Gauntlett, D. Martelli, and J. Sparks, Phys. Rev. D **104**, 086005 (2021).
[8] P. Ferrero, J. P. Gauntlett, and J. Sparks, JHEP **01**, 102.
[9] We adopt the standard convention in which A enters the KSE as in (2), whereas in [8] $A_{\text{there}} = 2\,A_{\text{here}}$.
[10] C. Closset, T. T. Dumitrescu, G. Festuccia, and Z. Komargodski, JHEP **05**, 017.
[11] M. Inglese, D. Martelli, and A. Pittelli, (2023).
[12] $\int \mathcal{A} \wedge \mathcal{F}$ is evaluated by extending $\mathcal{A}$ to a gauge field $\hat{\mathcal{A}}$ on $\mathcal{M}_4$ =Disk$\times\mathbb{\Sigma}$ and computing $\int_{\mathcal{M}_4} \hat{\mathcal{F}} \wedge \hat{\mathcal{F}}$.
[13] M. F. Atiyah, *Elliptic Operators and Compact Groups*, Vol. 401 (Springer-Verlag, Berline, Germany, 1974).
[14] M. Vergne, Duke Mathematical Journal **82**, 637 (1996).
[15] C. Closset, H. Kim, and B. Willett, JHEP **11**, 004.
[16] K. Hosomichi, S. Lee, and T. Okuda, JHEP **01**, 033.
[17] P. Ferrero, M. Inglese, D. Martelli, and J. Sparks, Phys. Rev. D **105**, 126001 (2022).
[18] A. Boido, J. P. Gauntlett, D. Martelli, and J. Sparks, Phys. Rev. Lett. **130**, 091603 (2023).
[19] F. Benini, K. Hristov, and A. Zaffaroni, Phys. Lett. **B771**, 462 (2017).
[20] F. Faedo and D. Martelli, JHEP **02**, 101.
[21] F. Faedo, A. Fontanarossa, and D. Martelli, Lett. Math. Phys. **113**, 51 (2023).