Introducing eigenspaces: semiotic analysis and didactic engineering

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This paper reports part of the author's PhD research, that focuses on eigentheory teaching and learning processes. We describe how the results of a pilot study, based on multimodal semiotic analysis, together with a broader epistemological, didactical and ecological analysis of the mathematical knowledge at stake and the learning context have informed the conception and a priori analysis of activities and methodologies designed within the didactic engineering methodology. These activities have been designed by the author and the teacher of a linear algebra course, with a high number of class attendees, offered in a mechanical engineering degree program in a public University in Italy. We report the multimodal analysis of a fragment of students' activity during the course implementation, to validate the hypotheses guiding their design.

Keywords: Eigentheory, didactic engineering, university mathematics education, semiotic bundle.

Introduction

In the last decades, a growing body of literature has been investigating the obstacles encountered by students when learning linear algebra (Stewart et al., 2019). Among the topics generally covered in a linear algebra course, that of eigentheory has been studied in a number of researches, given its wide applicability in STEM disciplines and the particular difficulties students usually encounter in studying this topic. Thomas & Stewart (2011) investigated eigenvalues and eigenvectors concept development, and provided interesting results regarding the way students understand the equation $Ax = \lambda x$. They showed how students seem to struggle in coordinating the two different processes encapsulated in the two sides of the equation: on the left side there is a matrix multiplied by a vector, while on the right side a multiplication of a scalar by the same vector. Another problem highlighted by the same authors is that the dominant way eigenvectors and eigenvalues are taught is through their algebraic definition and providing an algorithm to compute them. Consequently, the focus is turned too soon to the manipulation of algebraic representations, and in this way "the strong visual, or embodied metaphorical, image of eigenvectors can be obscured by the strength of the formal and symbolic thrust" (p. 186). As detailed in a broad overview provided by Wawro et al. (2019) on the educational research that has focused on eigentheory, other researchers have confirmed Thomas and Stewart's results and detected different reasons that add complexity to the understanding of the topic. Among these, there is the difficulty in coordinating solutions to $Ax = \lambda x$, solutions to the homogeneous systems of equations resulting from $(A - \lambda I)x = 0$, and the null space of the matrix $A - \lambda I$. In general, research on this topic agrees on the fact that integrating a geometrical interpretation of eigenvectors would enhance the understanding of eigentheory. Different researchers, summarized in (Wawro et al., 2019), have shown that the use of dynamic geometric software can help in this direction. Nevertheless, this seems to be rarely done and the algebraic approach continues prevailing.

One of the notions of eigentheory whose understanding has been rarely investigated (Wawro et al., 2019), is that of eigenspace, that is the set of all the eigenvectors associated to an eigenvalue of a linear transformation, together with the null vector. The fact that all the linear combinations of independent eigenvectors related to the same eigenvalue are eigenvectors of the same eigenvalue too, is not trivial at all for students (Salgado & Trigueros, 2015; Wawro et al., 2019). For this reason, there is a high risk that when the definition of eigenspace is given, students assume it, without really grasping the power of the fact that all the eigenvectors associated to an eigenvalue form indeed a vector subspace.

The author's PhD research project faces these issues, trying to investigate teaching and learning processes of eigentheory, in a linear algebra course offered in the first year of a degree in engineering. A pilot study was first realized, followed by the implementation of a teaching-learning sequence designed according to the didactic engineering methodology (Artigue, 2015).

Didactic Engineering: main features

Didactic engineering (DE) has been acknowledged as a research methodology since the early 1980s (Artigue, 2015). As such, it is structured into different phases: preliminary analyses, conception and a priori analysis, realization, observation and data collection, a posteriori analysis and validation. Preliminary analyses lay the foundation for the whole design and take into consideration different aspects that together contribute to the choices shaping the activity conception. The principal dimensions studied at this preliminary stage are the epistemological, the institutional, and the didactical ones. The aim of the institutional analysis is to identify the specificities of the Institution in which the DE takes place, highlighting the conditions and constraint to its implementation, while that of the didactical analysis is to investigate what information previous research has provided about the teaching and learning of the concept at stake. This last dimension is substantially cognitive, and in order to study it, different theoretical frameworks can be integrated into the methodology of DE.

The preliminary analysis guides the phase of conception and a priori analysis, where research hypotheses are made explicit and employed in the design of the activities to implement. Choices made in the conception phase can be made at the macro-level of the design, involving the global project, or at a micro-level, concerning the design of specific activities or situations. The a priori analysis clarifies how the choices made in the conception relate to the preliminary analysis, taking into account its different dimensions.

In the realization phase, the activity or activities are implemented, under the researchers' observations. The type of data collected strongly depends on the theoretical framework accompanying the DE. It is to be expected that the project will not follow exactly the path imagined during the conception phase, so the researcher should carry out an in-vivo analysis during the course of the activities, in order to determine if some teaching variables or choices need to be changed during the course of the project and to make the necessary changes as soon as possible.

A key aspect of this design research methodology is that its validation is internal and realized through a comparison between the a priori and a posteriori analyses. This comparison allows to put to the test the research hypotheses. Again, based on the theory behind the DE, the specific research questions posed and the type of data collected, a posteriori analysis can be conducted using different methodological tools. For example, in the here presented DE, a semiotic lens, namely, the semiotic bundle theory, was used both to enrich the preliminary analysis and conception phase and to realize the a posteriori analysis.

The semiotic bundle

Contributing to a broad strand of research in mathematics education concerned with the semiotic aspects of mathematical practice, Arzarello introduced the notion of semiotic bundle (SB):

> "a system of signs […] that is produced by one or more interacting subjects and that evolves in time. Typically, a semiotic bundle is made of the signs that are produced by a student or by a group of students while solving a problem and/or discussing a mathematical question". (Arzarello et al., 2009, p. 100)

Not only can the collection of signs that are present in the bundle evolve, but also the relationships existing between them can vary as the subject or the interacting subjects produce them. These relationships are analyzed through two types of lenses within this framework: synchronic analysis allows to study the relationships among signs produced simultaneously, while diachronic analysis studies the relationships among semiotic sets activated by the subject (or subjects) in successive moments, thus their evolution. This last kind of analysis has been used also by other theories, but the element of novelty brought by the theory of the semiotic bundle is the chance to use it to observe phenomena considering all the signs which may possibly be produced with different actions that have an intentional character, such as uttering, speaking, writing, drawing, gesticulating, handling an artefact, etc. This means, for example, that the dynamical relationship between gestures and speech can be analyzed, and such an analysis can give important information about the subject's ideas and thinking. In this case, gestures and language are a semiotic bundle, made of two deeply intertwined semiotic sets, of which only the second represents also a semiotic system.

Context, preliminary analyses and research question

The DE presented in this report has been designed for a Linear Algebra and Geometry course offered at the first year of the mechanical engineering degree program at a public Italian University, and taught by Teacher "A". Topics of this course span from basic vector spaces theory (approximately: vector spaces, matrix algebra, linear systems, eigentheory) to Euclidean spaces. A pilot study had been conducted in the Fall term of 2021 with students of the same course taught by Teacher A, and of other two linear algebra courses for different engineering degree courses, taught by different teachers. In the pilot the teachers were not involved. In the week following the classes on eigentheory, taught by the teachers without any researcher's intervention, some students voluntarily attended an extra two-hours tutoring class lead by this paper's author. During the optional activity, students had to work in groups of three or four and re-elaborate the notions of eigentheory encountered in the recently attended lecture, guided by guidelines provided by the researcher. These comprised very open questions such as "How would you explain the concept of eigenvector to someone who has never heard of that before?". Students were allowed to answer in the way they preferred, orally, in written form, with diagrams, etc., and were free to use whatever tools were available to them and encouraged to use other resources that they had encountered, in addition to the book or notes taken during the lessons. For the analysis, all the written protocols of the students, and audio and video recordings of eight of the small groups, were collected. This type of data allowed a multimodal analysis, according to the construct of the semiotic bundle (Arzarello, 2009). After the implementation and analysis of the results of the pilot study, teacher A was involved in the redesign of the module of her course regarding eigentheory for the following year. The DE here described regards this redesign of the course and its implementation in the Fall term of 2022. The results of the pilot study, together with the observation of the whole module on eigentheory as taught by teacher A before our intervention (all classes were video recorded by teacher A), contributed to the preliminary analysis for the DE.

Before our intervention, the module on eigentheory of the course followed the sequence traditionally used in Italian linear algebra courses: the definition of eigenvectors and eigenvalues, as those vectors v and scalars λ for which $Av = \lambda v$ stands, was given; some mainly algebraic examples were shown by the teacher (this is the case for teacher A, who also used to present a few geometrical examples, but it rarely happens in other courses); the procedure for computing eigenvalues and eigenvectors of a given matrix was shown and algebraically justified; the definition of eigenspace was presented; the theorems regarding diagonalization were introduced. Our analysis of the most commonly used Italian books and of notes of some other linear algebra courses taught in the same University where we conducted our research, ensures us that this teaching sequence is dominant in Italian universities. Descriptions of curricula and courses found in the literature seem to confirm that the same path is dominant in other countries too. The raison d'être of eigentheory is only hinted at the beginning of the course, when the teacher explains that eigenvalues and related notions are useful for diagonalizing matrices, but this rationale is soon lost by the students, also because in the following introduction of definitions and theorems, it is no longer taken into consideration. Furthermore, this is only the mathematical raison d'être of the knowledge at hand, whereas the reason why it is taught in an engineering course is not addressed at all. This problem affects the entire linear algebra course and is shared by mathematical courses for engineering and other STEM degree programs in Italian universities. In such a course structure, all the difficulties highlighted in the existing literature are likely to arise. In addition, the geometric interpretation, which, as confirmed by the literature, would help overcome these difficulties, although presented by the teacher with some examples, is completely absent in students' practices, which consist solely of algebraic computations to find eigenvalues, eigenvectors and eigenspaces.

Analysis of part of the pilot study

The analysis with the SB lens of students' activities in the pilot study allowed us to expand the epistemological and didactical analysis. Due to space constraints, we will present here only some results concerning students' conceptualization of eigenspaces. In the small-group out-of-lecture activity, most of the groups struggled in understanding what an eigenspace is, by trying to interpret its definition. Almost all the students, after reading the main eigentheory related definitions, began to review their class notes, including examples of transformations and related "eigen-objects" provided by the teacher. They struggled to understand why those indicated by the teacher were the transformation's eigenvalues, vectors and spaces, likely because they had not a clear idea of what these were. Nevertheless, a very interesting aspect emerged from two of the groups whose activity we have recorded. Students in these groups, before even reading on their notes the definition of subspace, began to wonder how many eigenvalues could exist for the same direction. The need to answer this question arose from the desire of better grasping what eigenvalues and eigenvectors are. In one particular group we observed how the attempt to tackle this issue, brought to an autonomous conceptualization of what eigenspaces are. When these students later encountered the definition in their lecture notes, they could make sense of it, due to their prior spontaneous inquiry. It is particularly interesting the way the evolution of the semiotic resources used by students played an important role in this process of discovery. The student who initiated the inquiry in the group starts considering the type of representation he is more confident with, namely the geometric one, that he represents in an embodied way, using gestures (in Fig. 1a and 1b the student shows to his peers that an eigenvector is a vector whose image gets "stretched" but remains in the same direction). Then, to align with the discourse of his fellow group members, he gradually shifts to using written diagrams (Fig. 1c) and then to symbolic formulas to ultimately arrive to an almost correct abstract formulation of the fact that any multiple of an eigenvector is also an eigenvector related to the same eigenvalue. This excerpt of students' activity is described and analyzed in greater detail in Piroi (2023). Students in this example, however, do not investigate the case where there are independent eigenvectors related to the same eigenvalue, most likely limited by the choice of a two-dimensional geometric example, and this limits their deep understanding of what an eigenspace is. The crucial point of this example, however, is that students naturally inquire about how many different eigenvalues might correspond to a given direction and, subsequently, what all the eigenvectors related to a given eigenvalue are. This exploration leads them to understand, with the stated limitation regarding dimension, that all eigenvectors related to a same eigenvalue form a vector subspace. This confirms the few results present in the literature about students' difficulties in understanding eigenspaces, but further suggests that letting students actively explore examples of geometrical transformations, they are likely to mobilize more visual and dynamic semiotic resources, such as gestures, and thus to visualize the eigenspaces formed by all the eigenvectors related to an eigenvalue. This might help them make sense of the definition of eigenspace, when they encounter it.

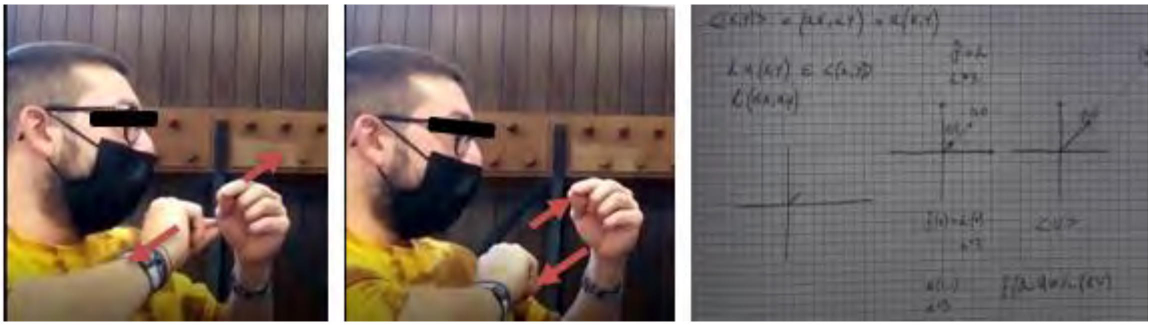

Figure 1 (a,b,c): Evolution of the signs produced by the group of students

Thus, the pilot study not only allowed us to perform a fine-grained analysis of students' difficulties in understanding eigenvalues and eigenvectors. It also revealed that the proposed activity actually helped students make sense of these concepts. It is when the students start collectively reinterpreting the examples given by the teacher, mobilizing different semiotic resources, such as written and oral language, drawings and gestures, that they begin to create and share an effective conception of eigenvectors, etc. The work in small groups allowed the students to mobilize different semiotic resources for the sake of communication, but those resources became fundamental tools for comprehension itself. This aspect is evident in the presented extract. As a matter of fact, educational research has widely demonstrated that collective active learning is beneficial to students' learning also in Engineering courses (Prince, 2004). Nevertheless, the considered learning context has a number of constraints that highly hinder the implementation of collective activities. Firstly, the course was attended by more than 200 students and the physical disposition of the desks in the classrooms, lined up in long rows, does not allow working easily in small groups, making it also very difficult for the teacher to walk around the desks to observe the students' interactions. Moreover, the thick curriculum and on the other side the little time allocated to the course, urge the teacher to follow the standard teaching sequence.

Taking into account the epistemological, didactical and ecological dimensions of the mathematical knowledge to be taught and of the specific learning context, we formulate the following research question: What kind of mathematical-didactic proposal, compatible with the learning context described with constraints such as the very high number of students attending the classes, can be implemented to foster the understanding of eigentheory? To answer this question, we will describe some choices made in the redesign of the course and will analyze part of the data collected during the implementation using the SB lens. The results will answer the research question and validate the research hypotheses generated during the conception and a priori analysis phase.

Design, implementation and analysis

Considering the various aspects that emerged in the preliminary analysis, we redesigned the eigentheory module, devoting a significant portion of the lecture to students' collective exploration of eigentheory concepts. We designed a series of tasks to be progressively proposed to students, that could support their understanding of eigenvectors, etc., avoiding the difficulties highlighted by the literature and emerged in our pilot study. We addressed the institutional constraints that hindered the implementation of collective activities by introducing a technological tool, namely the online notice board 'Padlet'. Through this platform, the teacher could assign problems to be solved on the spot by pairs or small groups of students during the lectures. Then, at the end of the time allocated for solving the problem, the groups could anonymously post a picture of their solution on the shared padlet. The teacher would project the padlet so that everyone could see the answers posted by others. She would also be able to check, as solutions were posted, if there were any particularly interesting solutions to present to the class later, as a bridge between the groups activity and what she would explain immediately afterwards. For the a posteriori analysis, we not only had all the pictures posted by the students in the padlet at our disposal, but we also audio and video recorded three randomly chosen small groups during all the activities about eigentheory conducted in the classes. This type of data was necessary to analyze students' activity according to the multimodal paradigm (Radford et al., 2017) at the core of the SB framework.

As for eigenspaces, we did not present the definition; instead, we let students explore geometrical examples where the question of "how many eigenvectors related to the same eigenvalue can exist" is likely to arise. In the second part of the first class on eigentheory we proposed a series of known bi and tri-dimensional geometrical transformations (rotations, reflections, projections) and asked the groups of students to find their eigenvectors and eigenvalues. We chose the examples so that eigenspaces of dimension one (lines) and two (planes) would appear. The hypothesis behind the choice of this task was that students are expected to notice that all vectors on a line or on a plane -

depending on the example- are eigenvectors with respect to the same eigenvalue, and thus to visualize the set of eigenvectors related to a same eigenvalue as a vector subspace. In the following activity, students had to answer the question, expressed in more general and formal terms: "If I have two eigenvectors that are independent of each other but related to the same eigenvalue, are all their linear combinations also eigenvectors?". By answering to this question, students were expected to transpose in algebraic form the geometrical ideas encountered in the previous task and to discover the generalizability of the statement for all vector spaces. We will present here a fragment of a couple of students' activities related to the task "consider the reflection with respect to a plane passing through the origin, and find, if any, its eigenvectors and eigenvalues".

At the beginning, they start drawing the transformation on their notebooks. They reason a bit on the drawing but struggle to visualize the three-dimensional transformation and thus which are the vectors that keep the same direction when transformed. Then, they start representing the situation with their hands, gesturing in the air (Fig. 2a). Doing so, they immediately realize that all the vectors staying on the student on the right's hand (representing the plane of reflection), remain fixed by the transformation, that is they stay on the same direction and keep the same magnitude, meaning that they are all eigenvectors related to the eigenvalue 1. They easily visualize that this property is shared by all the vectors staying on the plane. Later they start looking for other different eigenvectors. They state that all the vectors lying in the plane perpendicular to the reflection plane (represented in Fig. 2b by the student's left hand) are eigenvectors related to -1. They do not seem very sure about it, and explore various similar configurations by moving their hands. At one point, one of the students (Fig. 2 c and d) begins representing the vectors with his right index finger and, by moving it in different directions, realizes that all and only the vectors lying on the line passing through the origin and perpendicular to the reflection plane (represented by his left hand) are eigenvectors related to -1.

Figure 2 (a,b,c,d): key gestures performed by the students while solving the task

At the end of this excerpt, when the two students have to summarize what they have found, they realize that in the two explored cases (the previous task required them to find the "eigen-objects" of the reflection in the plane with respect of a line), the set of eigenvectors related to an eigenvalue always formed a line or a plane. For space reasons, we cannot continue the analysis, but in the following group activity they are able to generalize this finding, proving algebraically that the linear combinations of eigenvectors with the same eigenvalue are always eigenvectors of the same value.

Conclusions

The analysis of the video recordings allowed us to observe how students' mobilization of multimodal, dynamical, resources, such as their gestures, enabled them to visualize that the set of eigenvectors related to the same eigenvalue form a vector subspace. In the following activity, which unfortunately we do not have the space to analyze here, they were able to switch to a different semiotic register, the algebraic one, to generalize this statement and form a correct conceptualization of eigenspace. The choice of letting students explore geometrical transformations where they could visualize eigenspaces of dimension one or three helped overcome the difficulties highlighted by Wawro and colleagues (2019) and observed also in our pilot study. The SB lens was used to analyze all the excerpts of activity from the three video-recorded groups, in order to validate the hypotheses guiding the redesign of the course and of the tasks assigned to the students as small groups activities in the classes on eigentheory. Moreover, additional data were collected, including questionnaires, in order to validate (or not) other research hypotheses regarding the students' attitudes toward the new didactical organization of the course module. The analysis of these aspects will be reported elsewhere.

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