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Abstract

For a given risk, the well-known classical definition of Value-at-Risk (VaR) does not take into account possible interactions with other observable risks. For this reason, conditional VaRs that capture contagion effects and tail dependence among risks, such as the Co-Value-at-Risk (CoVaR), have been defined and studied in recent literature. In this paper we study conditions that guarantee, in the bivariate setting, the ordering between VaR and CoVaR, allowing to understand which, among the two measures, is more or less conservative than the other. By doing this, we introduce the notion of Probability Equivalent Level of CoVaR-VaR (PELCoV), which is the VaR value of the observable variable for which VaR and CoVaR coincide, and we study some of its properties such as uniqueness and boundedness. In particular, we show that its properties are entirely explained by the copula that describes the dependence between risks, and we provide a list of copulas for which PELCoV is explicitly available, and for which it is or not bounded. A practical applicative example is also presented.

Keywords: Systematic risk, contagion risk measure, value at risk, conditional value at risk, stochastically increasing.

1 Introduction

The study of risk measures to determine capital requirements, the amount of cash that a company should hold in reserve to cover unforeseen losses, has generated considerable attention in the economic and actuarial literature. Capital requirements can be calculated using a value-at-risk approach. Since the Basel Committee on Banking Supervision (BCBS) adopted it in the nineties of the previous century, the value-at-risk (VaR) has become the predominant measure of risk and has been employed in regulatory frameworks of banking (Basel III/IV) and insurance (Solvency II). Representing "the worst expected loss over a given horizon under normal market conditions at a given level of confidence", the popularity of VaR is mainly due to its simplicity and ease of computation. The interested reader may consult Artzner et al. (1999), Jorion (2000), Kupiec (2002) and Denuit et al. (2005) for overviews on the topic.

Formally, given a random risk Y with distribution function F_Y the value-at-risk (VaR) of Y at level $v \in (0,1)$, denoted by $\mathrm{VaR}_v[Y]$, is just the inverse (or quantile function) of F_Y at v, given by

$$\mathrm{VaR}_v[Y] = F_Y^{-1}(v) = \inf\{x : F_Y(x) \geq v\}.$$

However, the value-at-risk approach is focused on the individual risk of an institution and does not capture the systemic risk, which is the risk of failures caused by interactions with other financial institutions (see Bisias et al. (2012) and Benoit et al. (2017) for general surveys on systemic risk). As reported, among others, by Drehmann and Tarashev (2011), Acharya et al. (2017) and Brownlees and Engle (2017), the strong interconnection within the financial system has contributed, especially since the 2007–2009 financial crisis, to the spreading of systemic risk, with adverse consequences in terms of credit supply to the real economy.

One important source of systemic risk is the risk of financial contagion, which occurs when losses in one financial institution spillover to another institution that is linked with the first one (see Glasserman and Young (2016)). As a result, since the first version of the paper by Adrian and Brunnermeier (2016), an active area of research has been devoted to the quantification, management and comparison of contagion risk measures. See, for example, Chen et al. (2013), Girardi and Ergün (2013), Mainik and Schaanning (2014), Sordo et al. (2015), Acharya et al. (2017), Sordo et al. (2018), Das and Fasen-Hartmann (2018a), Ortega-Jiménez et al. (2021) and Dhaene et al. (2022). In particular, Adrian and Brunnermeier (2016) considered a dependence-adjusted version of the VaR called CoVaR that measures the institution risk, conditional on other institutions being in distress. Formally, the co-value-at-risk (CoVaR) of Y at level $v \in (0,1)$, given that X is at level $u \in (0,1)$, denoted by $\mathrm{CoVaR}_{v,u}[Y|X]$, is defined by the VaR of the conditional variable $[Y|X = \mathrm{VaR}_u[X]]$ at level risk v as follows[1]

$$\mathrm{CoVaR}_{v,u}[Y|X] = \mathrm{VaR}_v[Y|X = \mathrm{VaR}_u[X]] = F^{-1}_{Y|X=\mathrm{VaR}_u[X]}(v).$$

Standard financial regulations determine capital requirements based on isolated institutions without considering the risk of systemic institutional failure. However, as pointed out by Acharya (2009), "the goal of prudential regulation should be to ensure the financial stability of the system as a whole, i.e., of an institution not only individually but also as a part of the overall financial system". To address this shortcoming, some authors (including Lehar (2005), Acharya (2009), Huang et al. (2009) and Acharya et al. (2017)) have proposed different prudential regulation models that consider both individual and systemic failure risk. Successful implementation of these models relies on effective methods for monitoring systemic risk. This paper is a step towards achieving this goal by examining a strategy based on the combination of conditional and unconditional VaR to capture the dimension of systemic risk. In particular, we study

[1] A modified definition of CoVaR was given in Girardi and Ergün (2013) and Mainik and Schaanning (2014).

conditions that guarantee the ordering between VaR and CoVaR, allowing us to understand which, among the two measures, is more or less conservative than the other.

For a random variable Y representing a financial risk, this strategy requires the presence of an adequate covariate X (we discuss in Section 3 some guidelines on choosing it). Suppose that, initially, the risk-capital calculation is based on the VaR at level v of Y, for some $v \in (0,1)$. If, given a risk X and $u \in [0,1]$, $\mathrm{CoVaR}_{v,u}[Y|X] < \mathrm{VaR}_v[Y]$ holds, replacing VaR by CoVaR does not make sense as CoVaR contemplates a more optimistic scenario. However, if $\mathrm{CoVaR}_{v,u}[Y|X] > \mathrm{VaR}_v[Y]$ holds, then $\mathrm{VaR}_v[Y]$ underestimates the spillover effect and it is prudent to replace it with $\mathrm{CoVaR}_{v,u}[Y|X]$ as a more conservative provision.

Given $v \in (0,1)$, the point in question here is the existence of a risk-level threshold $u_v \in (0,1)$, as a function of v, such that $\mathrm{CoVaR}_{v,u_v}[Y|X] = \mathrm{VaR}_v[Y]$. Although this u_v might not be unique or even exist, its study is especially interesting in situations where it is unique. As we show in Section 2, we can guarantee its existence and uniqueness under reasonable assumptions on the dependence structure of the vector (X, Y). The primary purpose of this paper is to provide a plausible interpretation of the level u_v and discuss its implications under such assumptions. The findings of this study may enable decision-makers to understand better when "to sound the alarm" and implement adequate response actions.

Another motivation for the study of u_v is the recent work by Li and Wang (2022), who studied a similar balancing point for the equivalence between the value-at-risk and the expected shortfall, called PELVE (Probability Equivalent Level of value-at-risk and expected shortfall). Specifically, for a chosen level ϵ close to 0 and a loss random variable Y, the PELVE is a constant multiplier c such that $\mathrm{ES}_{1-c\epsilon}(Y) = \mathrm{VaR}_{1-\epsilon}(Y)$, where

$$\mathrm{ES}_v[Y] = \frac{1}{1-v}\int_v^1 \mathrm{VaR}_t[Y]dt,$$

is the expected shortfall at level $v \in (0,1)$. The value of PELVE addresses the issue of regulatory capital change if one replaces VaR with ES (see Li and Wang (2022) for details). Generalizations of PELVE have been proposed in Fiori and Rosazza Gianin (2022) and Barczy et al. (2023).

Throughout the paper, we assume the following regularity conditions hold: the random vector (X, Y) has absolutely continuous strictly increasing marginal distribution functions $F_X(\cdot)$ and $F_Y(\cdot)$ and strictly increasing conditional distribution $F_{Y|X=x}(\cdot)$, for all x, on their respective supports. For a random vector satisfying these conditions, we define the probability equivalent level of CoVaR-VaR at risk level v (in short, PELCoV_v) as follows.

Definition 1 *Let (X, Y) be a random vector satisfying regularity conditions. Given $v \in (0,1)$, we say that $u_v \in (0,1)$ is a probability equivalent level of CoVaR-VaR at risk level v ($PELCoV_v$) for X if $CoVaR_{v,u_v}[Y|X] = VaR_v[Y]$.*

We model the dependence structure between X and Y using copulas. If K is the joint distribution of the random vector (X, Y), by the Sklar's theorem, we can write

$$K(x,y) = C\left(F_X(x), F_Y(y)\right),$$

where C is the joint distribution function of the vector-copula (U,V) given by $U = F_X(X)$ and $V = F_Y(Y)$. The function C is the copula and collects the information on the inner dependence of the vector, independently from the marginal behaviour of the components. Under the regularity conditions, C is unique and differentiable (see Nelsen, 2007). For a random vector (X,Y) satisfying regularity conditions and $v \in (0,1)$, we show below that a PELCoV_v only depends on the copula, a result that places our research in the context of other recent studies that model systemic risk using a copula-CoVaR approach, including Reboredo and Ugolini (2015), Bernardi et al. (2017), Jaworski (2017), Karimalis and Nomikos (2018), Jin (2018), Di Clemente (2018) and Zhao (2022), among others.

Decision-makers in finance and insurance are often interested in positive dependence structures among risks. We mean by positive dependence that X and Y are likely to be large or small together. The interested reader can find an introduction to dependence notions in the monograph by Joe (1997) and a recent review of concepts in Navarro et al. (2020). In particular, we focus on the positive regression dependence (PRD) concept introduced by Lehmann (1966), also called stochastically increasing (SI). We first recall the definition of the usual stochastic order. Given two random variables X_1 and X_2, X_1 is said to be smaller than X_2 in the usual stochastic order, denoted by $X_1 \leq_{st} X_2$, if the survival probabilities satisfy

$$\Pr\{X_1 > x\} \leq \Pr\{X_2 > x\}, \, \forall x \in \mathbb{R}. \tag{1}$$

Roughly speaking, X_1 is less likely than X_2 to take on large values. It is straightforward to check the following equivalence,

$$X_1 \leq_{st} X_2 \Longleftrightarrow \text{VaR}_u[X_1] \leq \text{VaR}_u[X_2], \, \forall u \in [0,1]. \tag{2}$$

Further details about the usual stochastic order can be found in Shaked and Shanthikumar (2007). Now we define the SI concept in terms of the $\leq_{st}$ order between conditional distributions. Let (X,Y) be a random vector with copula C. We will say that Y is SI in X, in short $Y \uparrow_{SI} X$, if

$$[Y \mid X = x_1] \leq_{st} [Y \mid X = x_2], \, \forall x_1 \leq x_2. \tag{3}$$

In some sense, if $Y \uparrow_{SI} X$ we expect that Y is likely to take on large values when the conditional random variable X increases. It is clear from (1) that (3) holds if and only if the survival probability $\Pr\{Y > y \mid X = x\}$ is an increasing function of x, for all y. The SI concept is characterized by the copula: $Y \uparrow_{SI} X$ is equivalent to checking that the partial derivative

$$\partial_1 C(u,v) = \Pr\{V \leq v \mid U = u\} \tag{4}$$

is a decreasing function of u, for all v, where (U,V) is defined as above, (see page 41 in Nelsen, 2007). Consequently, the vector (X,Y) is SI if and only if its copula is SI. If $Y \uparrow_{SI} X$ and $X \uparrow_{SI} Y$, we say that the vector (X,Y) is positively dependent through stochastic ordering (PDS). Note that (X,Y) is PDS if and only if its copula C is PDS (see Cai and Wei, 2012). Throughout the paper, we will only consider the case where Y is SI in X in a strict sense, i.e, where the conditional distribution $\Pr\{Y > y \mid X = x\}$ is a strict increasing function of x, for all y. We will denote this property by $Y \uparrow_{SSI} X$ (strict SI property). Note that the SSI property implies that the conditional distribution $[Y \mid X = x]$ varies when $X = x$ does.

The rest of the paper is organized as follows. In Section 2, we show that under some regularity conditions on the copula and a particular dependence structure of the vector, the PELCoV$_v$ exists and is unique. In Section 3, we interpret this notion as a signal that the VaR of Y at level v is "starting to underestimate" the spillover risk effect. The issue is appealing when we have a unique signal for all $v \in (0,1)$, which takes us to study the existence of an upper bound for the set of PELCoV$_v$s, with $v \in (0,1)$. This is why, in Section 4, we focus on the monotonicity and boundedness properties of the PELCoV$_v$. In Section 5, we determine the PELCoV$_v$ and study its boundedness for several parametric families of copulas. In Section 6 we provide an example with real data to illustrate the methodology application.

2 Existence and uniqueness of PELCoV$_v$

In this section we establish conditions to guarantee the existence and uniqueness of the PELCoV$_v$. To begin with, we observe that the relative position of VaR and CoVaR only depends on the copula.

Lemma 2 *Let (X,Y) be a random vector satisfying regularity conditions with copula C. Given $u, v \in (0,1)$, $CoVaR_{v,u}[Y \mid X] \geq VaR_v[Y]$ (respectively $\leq, =$) if, and only if $\partial_1 C(u,v) \leq v$ (respectively $\geq, =$).*

Proof: Rewriting (4) as $\partial_1 C(u,v) = F_{Y|X=\mathrm{VaR}_u[X]}(\mathrm{VaR}_v[Y])$ the result follows easily. $\square$

Given $v \in (0,1)$, Lemma 2 says that a PELCoV$_v$ only depends on the copula. The next result shows that the continuity of $\partial_1 C(u,v)$ determines the continuity of $CoVaR_{v,u}[Y \mid X]$.

Lemma 3 *Let (X,Y) be a random vector satisfying regularity conditions with copula C. Then, $CoVaR_{v,u}[Y \mid X]$ is continuous in $u \in (0,1)$, for all $v \in (0,1)$ if, and only if, $\partial_1 C(u,v)$ is continuous in $u \in (0,1)$, for all $v \in (0,1)$.*

Proof: We only prove the "if" part (the "only if" part is proved in the same way). Given an $\epsilon > 0$ and $u_0 \in (0,1)$, the inequality

$$\mid CoVaR_{v,u}[Y \mid X] - CoVaR_{v,u_o}[Y \mid X] \mid < \epsilon,$$

is equivalent to

$$\partial_1 C(u, F_Y(y_1)) < v < \partial_1 C(u, F_Y(y_2)), \tag{5}$$

where $y_1 = CoVaR_{v,u_o}[Y \mid X] - \epsilon$ and $y_2 = CoVaR_{v,u_o}[Y \mid X] + \epsilon$. By the assumptions, using that $\partial_1 C(u,v)$ is strict increasing in v,

$$\lim_{u \to u_o} \partial_1 C(u, F_Y(y_1)) = \partial_1 C(u_o, F_Y(y_1)) < v \tag{6}$$

and

$$\lim_{u \to u_o} \partial_1 C(u, F_Y(y_2)) = \partial_1 C(u_o, F_Y(y_2)) > v. \tag{7}$$

From (6) and (7), we see that (5) holds in an interval $\mid u - u_0 \mid < \delta$, with δ small enough. □

Proposition 4 *Let (X, Y) be a random vector satisfying regularity conditions with copula C. Given a risk-level $v \in (0,1)$, there exist $u_v^+, u_v^- \in (0,1)$ such that $CoVaR_{v,u_v^-}[Y|X] \leq VaR_v[Y] \leq CoVaR_{v,u_v^+}[Y|X]$.*

Proof: Let $v \in (0,1)$. Reasoning by contradiction, assume that u_v^+ or u_v^- do not exist. Then, the conditional quantile curve $u \mapsto CoVaR_{v,u}[Y|X]$ lies strictly either above or below the $VaR_v[Y]$. Suppose that the curve lies strictly above (the other case is analogous). From Lemma 2, the inequality $CoVaR_{v,u}[Y|X] > VaR_v[Y]$, for all $u \in (0,1)$, is equivalent to checking that $v > \partial_1 C(u, v)$, for all $u \in (0,1)$. The latter implies that

$$v = \int_0^1 v du > \int_0^1 \partial_1 C(u, v) du = C(1, v) = v, \tag{8}$$

which is a contradiction and concludes the proof. □

We remark that Proposition 4 holds for $0 < v < 1$. This result says that, under regularity conditions, the curves $u \mapsto VaR_v[Y]$ and $u \mapsto CoVaR_{v,u}[Y|X]$ necessarily cut each other. Using (2), this is equivalent to saying that there always exists a range of values of u such that the random variables Y and $[Y|X = VaR_u[X]]$ fail to be stochastically ordered. Consider, for example, $v_1 < v_2$ such that the probability equivalent levels u_1, u_2 verify $u_{v_1} < u_{v_2}$. For $u \in (u_{v_1}, u_{v_2})$ one has $\mathrm{CoVaR}_{v_1,u}[Y|X] > \mathrm{VaR}_{v_1}[Y]$ and $\mathrm{CoVaR}_{v_2,u}[Y|X] < \mathrm{VaR}_{v_2}[Y]$. Thus, by (2), neither $Y \leq_{st} [Y|X = VaR_u[X]]$ nor $Y \geq_{st} [Y|X = VaR_u[X]]$ can hold true. Regarding this issue, we will provide in Section 4 sufficient conditions for the existence of a $u^* \in (0,1)$ such that Y and $[Y|X = VaR_u[X]]$ are stochastically ordered for all $u \in [u^*, 1]$.

The next theorem gives sufficient conditions for existence and uniqueness of $u_v = \mathrm{PELCoV}_v$.

Corollary 5 *Let (X, Y) be a random vector satisfying regularity conditions with a copula C and let $v \in (0,1)$. Then:*
(a) If $\partial_1 C(u, v)$ is continuous in $u \in (0,1)$, there exists at least a $u_v \in (0,1)$ such that $CoVaR_{v,u_v}[Y|X] = VaR_v[Y]$.
ii) If $Y \uparrow_{SSI} X$, then $u_v = PELCoV_v$ is unique.

Proof: The proof of (a) follows from Lemma 3 and Proposition 4 by application of Darboux property to the continuous function $u \mapsto CoVaR_{v,u}[Y|X]$ in the closed interval given by the values u_v^- and u_v^+. To prove (b), note that $Y \uparrow_{SSI} X$ if, and only if, $\mathrm{CoVaR}_{v,u}[Y|X]$ is strictly increasing in u, for all v (see Fernández-Ponce et al., 2011). □

The following example shows that the continuity of $\partial_1 C(u, v)$ in $u \in (0,1)$ is necessary for the existence of u_v.

Example 6 *Let C be a copula having density $c(u,v)$ as follows*

$$c(u,v) = \begin{cases} 2 & if \quad (u-1/2)(v-1/2) > 0, \\ 0 & if \quad (u-1/2)(v-1/2) \leq 0. \end{cases}$$

A straightforward computation shows that

$$\partial_1 C(u,v) = \begin{cases} 2v & if \quad 0 \leq u, v \leq 1/2, \\ 1 & if \quad 0 \leq u \leq 1/2,\ 1/2 < v \leq 1, \\ 0 & if \quad 1/2 < u \leq 1,\ 0 \leq v \leq 1/2, \\ 2v-1 & if \quad 1/2 < u, v \leq 1. \end{cases}$$

By construction, C is absolutely continuous and satisfies the regularity conditions. However, $\partial_1 C(u,v)$ is not continuous in $u \in (0,1)$, for all $v \in (0,1)$. In particular, it is always discontinuous at $u = 0.5$. It is clear that $\partial_1 C(u,v) \neq v$ for all $u, v \in (0,1)$. Hence the $PELCoV_v$ does not exist, for all $v \in (0,1)$.

3 Further discussion on the PELCoV$_v$

In what follows, in order to guarantee existence and uniqueness, we consider a random vector (X,Y) that satisfies the regularity conditions with copula C, such that $Y \uparrow_{SSI} X$ and $\partial_1 C(u,v)$ is continuous in u. To clarify the exposition, we present in Figure 1 some computations[2] of u_v=PELCoV$_v$ for a bivariate normal vector (X,Y) with $\mu_X = 10000$, $\sigma_X = 1000$, $\mu_Y = 40000$, $\sigma_Y = 2500$ and $\rho = 0.9$ (the positive correlation coefficient guarantees that $Y \uparrow_{SSI} X$). Note that u_v only depends on the copula (that is, on ρ). The graphic plots u_{v_i}, $i = 1,2,3$, for $v_1 = 0.4, v_2 = 0.7$ and $v_3 = 0.95$, where u_{v_i} is the abscissa of the intersection point between the conditional quantile curve $u \mapsto CoVaR_{v_i,u}[Y|X]$ and the line $y = VaR_{v_i}[Y]$. Given $v \in [0,1]$, we see from the graphic that

$$u_v = \min_{u \in [0,1]} \left\{ \mathrm{CoVaR}_{v,u'}[Y|X] \geq \mathrm{VaR}_v[Y], \text{for all } u' > u \right\}, \tag{9}$$

representing the lowest risk level u of X from which $\mathrm{CoVaR}_{v,u}[Y|X]$ will be a more conservative reserve position than $\mathrm{VaR}_v[Y]$. For the bivariate normal distribution shown in Figure 1, the PELCoV$_v$ is an increasing function of v and tends to 1 as v goes to 1. However, this is not always the case: the PELCoV$_v$ is not necessarily increasing in v, as we show in Section 5. The existence of an upper bound for the PELCoV$_v$ deserves further discussion. If there exists a $u^* < 1$ such $u_v \leq u^*$ for all v, then

$$\begin{aligned} \mathrm{CoVaR}_{v,u}[Y|X] &\geq \mathrm{CoVaR}_{v,u^*}[Y|X] \\ &\geq \mathrm{CoVaR}_{v,u_v}[Y|X] \\ &= \mathrm{VaR}_v[Y], \end{aligned}$$

[2] Computations are based on formulas (14) to (16).

for all $v \in (0,1)$ and for all $u \in [u^*, 1)$, where the inequalities follow from the fact that $\text{CoVaR}_{v,u}[Y|X]$ is strictly increasing in u (due to $Y \uparrow_{SSI} X$) and the equality is by definition of u_v. Using (2), this can be rewritten as[3]

$$Y \leq_{st} [Y|X = \text{VaR}_u[X]], \forall u \in [u^*, 1). \tag{10}$$

We interpret (10) as follows. Given $v \in (0,1)$, if X exceeds its quantile u^*,

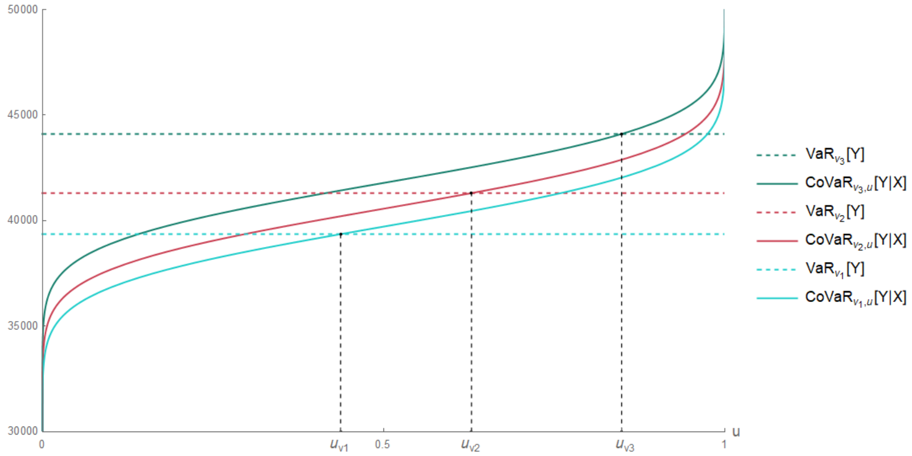

Figure 1: PELCoV_v values for $v_1 = 0.4, v_2 = 0.7$ and $v_3 = 0.95$ for a normal vector (X,Y) with $\mu_X = 10000$, $\sigma_X = 1000$, $\mu_Y = 40000$, $\sigma_Y = 2500$ and $\rho = 0.9$.

$\text{VaR}_v[Y]$ underestimates the risk spillover effect, and it is more prudent to determine reserves in terms of $\text{CoVaR}_{v,u}[Y|X]$, for all $u > u^*$. Returning to the interpretation of (9), the condition $Y \uparrow_{SI} X$ implies that the vector (X,Y) is PQD (positive quadrant dependent, see Nelsen, 2007), that is,

$$\Pr\{Y > \text{VaR}_v[Y] \mid X > \text{VaR}_u[X]\} \geq 1 - v, \quad \text{for all } u, v \in (0,1), \tag{11}$$

where $1 - v = \Pr\{Y > \text{VaR}_v[Y]\}$. This means that the assumption $Y \uparrow_{SI} X$, by itself, ensures that the conditional information that X exceeds its quantile u (for any u) increases the probability of Y exceeding its v quantile. The additional information is that, given $v \in (0,1)$, u_v is the smallest probability-level such that

$$\text{CoVaR}_{v,u}[Y|X] \geq \text{VaR}_v[Y], \forall u \geq u_v. \tag{12}$$

In our approach, this means that if the institution X is becoming riskier and exceeds its u_v quantile, $\text{CoVaR}_{v,u}[Y|X]$ is more conservative than $\text{VaR}_v[Y]$. Thus, u_v acts as a signal or "alert" that indicates that the unconditional VaR is "starting to underestimate" the risk. If we define the regression expected shortfall (RES) at level v as

$$\text{RES}_v[Y|X] = \frac{1}{1-u_v} \int_{u_v}^{1} \text{CoVaR}_{v,u}[Y|X]\, du, \tag{13}$$

[3]Comparisons of the form (10) are discussed in Navarro and Sordo (2018).

it follows from (12) that $\mathrm{RES}_v[Y|X] \geq VaR_v[Y]$. We can interpret (13) as a modified expected shortfall, where the conditional event $\{Y > \mathrm{VaR}_v[Y]\}$ is replaced by $\{X > \mathrm{VaR}_{u_v}[X]\}$, a stress scenario intended to capture the contagion effect[4].

We now return to the initial problem of determining provisions for Y using a VaR-CoVaR-based approach. To choose a good covariate X for Y, we suggest taking into consideration the following criteria.

(a) The dependence structure between X and Y should be easily observable (it determines both u_v and the monotonicity of $\mathrm{CoVaR}_{v,u}[Y|X]$ when u exceeds u_v) but not necessarily too strong. Under comonotonicity, for example, $u_v = v$ and the PELCoV is not helpful. We should also mention that a greater positive dependence between X and Y does not imply a higher PELCoV_v. In general, the PELCoV_v is not monotonic with respect to the concordance ordering of copulas.

(b) Given $v \in (0,1)$, a good "alert" level u_v should be reasonably smaller than v. The higher the PELCoV_v, the less likely the event $[X > \mathrm{VaR}_{u_v}[X]]$ to occur. If the strategy consists of replacing $\mathrm{VaR}_{0.99}[Y]$ with $\mathrm{CoVaR}_{0.99,0.99}[Y|X]$, the alert level $u_v = 0.99$ may come too late to be effective.

(c) Given $v \in (0,1)$, the regression expected shortfall (13) quantifies the impact of X exceeding its u_v-quantile on Y. Therefore, the higher $\mathrm{RES}_v[Y|X]$, the more effective the replacement of $\mathrm{VaR}_v[Y]$ with $\mathrm{CoVaR}_{v,u_v}[Y|X]$ in the sense that it protects the company from a worse scenario.

For example, suppose that Y is normally distributed with parameters μ_Y and σ_Y and we base risk calculations on $\mathrm{VaR}_v[Y]$, for some $0 < v < 1$. It is well-known that

$$\mathrm{VaR}_v[Y] = \mu_Y + \mathrm{VaR}_v[Z]\sigma_Y, \tag{14}$$

where $Z \sim N(0,1)$. Let X be a random variable such that (X,Y) is a bivariate normal vector with correlation coefficient ρ and let μ_X and σ_X be the parameters of X. It is well-known that the conditional distribution $[Y|X=x]$ is a normal distribution with parameters $\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)$ and $\sigma_Y\sqrt{1-\rho^2}$. According to (14) we have

$$\mathrm{CoVaR}_{v,u}[Y|X] = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(\mathrm{VaR}_u[X]-\mu_X) + \mathrm{VaR}_v[Z]\sigma_Y\sqrt{1-\rho^2}, \tag{15}$$

and, by solving the equation $\mathrm{CoVaR}_{v,u}[Y|X] = \mathrm{VaR}_v[Y]$, we obtain

$$u_v = F_Z\left(\frac{1-\sqrt{1-\rho^2}}{\rho}\mathrm{VaR}_v[Z]\right). \tag{16}$$

[4]The risk measure (13) should not be confused with the conditional expected shortfall (or CoES) introduced by Mainik and Schaanning (2014) as

$$\mathrm{CoES}_{v,u}[Y \mid X] = \frac{1}{1-v}\int_v^1 \mathrm{CoVaR}_{t,u}[Y|X]\,dt.$$

While the latter averages the quantiles $F^{-1}_{Y|X=\mathrm{VaR}_u[X]}(t)$ with u fixed and t varying from v on, the regression expected shortfall given by (13) averages the quantiles $F^{-1}_{Y|X=\mathrm{VaR}_u[X]}(v)$, with v fixed and u varying from u_v on.

Now suppose that $v = 0.95$ and $\rho = 0.4$. We can check that $u_{0.95} = 0.6344$. Since the event $[X > \mathrm{VaR}_{0.6344}[X]]$ is likely to happen, we expect the "alarm" to sound often enough to be effective. Combining (13) and (15), we evaluate the impact on Y caused by the contagion effect by

$$\mathrm{RES}_v\,[Y|X] = \mu_Y + k_{\rho,v}\sigma_Y, \tag{17}$$

where $k_{\rho,v} = \rho \mathrm{ES}_{u_v}[Z] + \sqrt{1-\rho^2}\mathrm{VaR}_v[Z]$ (here we have used that $\mathrm{ES}_u[X] = \mu_X + \mathrm{ES}_u[Z]\sigma_X$). It follows from comparing (14) and (17) that the difference $k_{0.4,0.95} - \mathrm{VaR}_{0.95}[Z]$ quantifies the severity of the spillover effect. In this example, $k_{0.4,0.95} = 1.9189 > 1.6448 = \mathrm{VaR}_{0.95}[Z]$, which reasonably justifies the replacement of $\mathrm{VaR}_{0.95}[Y]$ with $\mathrm{CoVaR}_{0.95,0.6344}\,[Y|X]$.

4 Monotonicity and boundedness

As in Section 3, we consider a random vector (X, Y) with copula C that satisfies the regularity conditions, such that $Y \uparrow_{SSI} X$ and $\partial_1 C(u, v)$ is continuous in u for all $v \in (0, 1)$. In this section, we discuss monotonicity and boundedness properties of the PELCoV_v.

4.1 Monotonicity

The growth of the PELCoV_v as a function of v is meaningful and easy to interpret: as the capital requirements increase (moving, for example, from $\mathrm{VaR}_{0.95}[Y]$ to $\mathrm{VaR}_{0.99}[Y]$), the risk-level u for X at which CoVaR becomes more conservative than VaR increases accordingly (that is, $u_{0.95} \leq u_{0.99}$). From Figure 1 and the discussion in Section 3 (where we said that the PELCoV_v of a bivariate normal distribution with $\rho > 0$ is increasing in v), we can think that the map $v \mapsto u_v$ is increasing whenever $Y \uparrow_{SSI} X$. This is a false intuition, as the following example shows.

Example 7 *Let X and W be two independent random variables, $X \sim Exp(1)$ and $W \sim Weibull(0.5, 1)$. The vector (X, Y), where $Y = X + W$, satisfies the regularity conditions (in particular, $Y \uparrow_{SSI} X$). Given $v \in (0, 1)$, the $PELCoV_v$ for this vector is the unique u such that $CoVaR_{v,u}[Y|X] = VaR_v[Y]$. Using that $VaR_v[W + c] = VaR_v[W] + c$ for any constant c, we get $u = F_X\left((VaR_v[Y] - VaR_v[W])\right)$. We can compute numerically these values taking into account that the marginal distribution function of Y is given by*

$$\begin{aligned} F_Y(x) &= \int_0^x F_W(x-y) f_X(y) dy = \int_0^x \left(1 - e^{-\sqrt{x-y}}\right) \cdot e^y dy = \\ &= 1 - e^{-\sqrt{x}} - e^{-x} \cdot DF\left(\frac{1}{2}\right) + e^{-\sqrt{x}} DF\left(\frac{1}{2} - x\right), \end{aligned}$$

where $DF(x)$ is the Dawson's integral $DF(x) = e^{-x^2} \int_0^x e^{-y^2} dy$. Figure 2 plots the $PELCoV_v$ as a function of v. The maximum is attained at $v = v_0 = 0.74816$ and $u^* = u_{v_0} = 0.764213$.

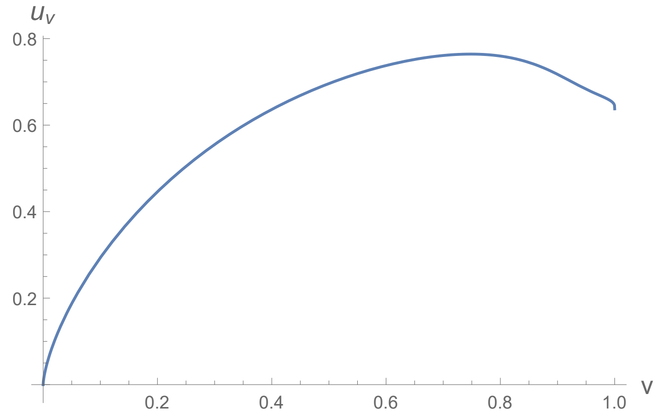

Figure 2: Function $v \mapsto u_v$ for the vector (X, Y), where $Y = X + W$, where $X \sim \text{Exp}(1)$, $W \sim \text{Weibull}(0.5, 1)$ and X and W are independent.

Example 7 shows that the function $v \mapsto u_v$ is not necessarily increasing under the assumption $Y \uparrow_{SSI} X$. It is natural, therefore, to ask for additional conditions under which u_v increases as a function of v. Observe that, by Lemma 2, under the regularity conditions, the PELCoV$_v$ is the unique value $u_v \in (0, 1)$ such that $\partial_1 C(u_v, v) = v$. Assuming differentiability of $\partial_1 C(u, v)$ and using the implicit function theorem we see that

$$\frac{du_v}{dv} = \frac{1 - c(u_v, v)}{\partial_{11} C(u_v, v)}, \tag{18}$$

where $c(u, v) = \frac{\partial^2}{\partial u \partial v} C(u, v)$ is the copula density. In particular, if $\partial_{11} C(u, v) < 0$ for all $u, v \in (0, 1)^2$ (which is equivalent to $Y \uparrow_{SSI} X$), we see that

$$\frac{du_v}{dv} > 0 \text{ if and only if } c(u_v, v) > 1. \tag{19}$$

Direct verification of the growth of u_v as a function of v using (19) requires the analytical expression of u_v. Next we present a sufficient (but not necessary) condition, that does not require it, based on the Bickel and Lehmann (1979) dispersive order. Recall that X_1 is said to be smaller than X_2 in the dispersive order, denoted by $X_1 \leq_{disp} X_2$, if the distances between their VaRs satisfy

$$\text{VaR}_{u_1}[X_1] - \text{VaR}_{u_2}[X_1] \leq \text{VaR}_{u_1}[X_2] - \text{VaR}_{u_2}[X_2] \text{ for all } 0 \leq u_1 < u_2 \leq 1.$$

Although the PELCoV is based only on the copula, we can give sufficient conditions for its monotonicity in terms of both the marginals and copula behaviour.

Proposition 8 *Let (X, Y) be a random vector satisfying regularity conditions with a copula C, such that $Y \uparrow_{SSI} X$ and $\partial_1 C(u, v)$ is continuous in both arguments. If $[Y|X = VaR_u[X]] \leq_{disp} Y$ for all $u \in (0, 1)$, then the PELCoV$_v$ increases with respect to v.*

Proof: Given $v_1 < v_2$, for all $u \in (0,1)$ we have

$$\mathrm{VaR}_{v_2}[Y|X = \mathrm{VaR}_u[X]] - \mathrm{VaR}_{v_1}[Y|X = \mathrm{VaR}_u[X]] \leq \mathrm{VaR}_{v_2}[Y] - \mathrm{VaR}_{v_1}[Y]$$

By Corollary 5, there exists a u_{v_2} such that $CoVaR_{v_2,u_{v_2}}[Y|X] = VaR_{v_2}[Y]$. Therefore, taking $u = u_{v_2}$ in the above inequality it follows that

$$CoVaR_{v_1,u_{v_2}}[Y|X] - VaR_{v_1}[Y] \geq 0, \text{ for all } v_1 < v_2. \tag{20}$$

Since $Y \uparrow_{SII} X$, the function $u \mapsto CoVaR_{v,u}[Y|X]$ is strictly increasing. This fact, together with (20), implies $u_{v_1} \leq u_{v_2}$. □

In Section 3, we said without proof that the PELCoV_v of a bivariate normal distribution with $\rho > 0$ is increasing in v (see Figure 1). Now we can prove it as follows. If (X, Y) is a bivariate normal vector with correlation coefficient ρ, the univariate random variables $[Y|X = F^{-1}(u)]$ and Y are normally distributed with variances $\sigma_Y^2(1-\rho^2)$ and σ_Y^2, respectively. Using that a normal distribution is smaller than another one in the dispersive order if its variance is smaller (see, Table 1.1 in Müller and Stoyan, 2002), the result follows as a consequence of Proposition 8.

Let (X^C, Y^C) denote a random vector with marginals F and G and copula C. Belzunce et al. (2008) say that (X^C, Y^C) is less in the conditional dispersive order[5] than $(X^{C'}, Y^{C'})$ if $[Y^C|X^C = F^{-1}(u)] \leq_{disp} [Y^{C'}|X^{C'} = F^{-1}(u)]$, for all $u \in (0,1)$. Observe that $[Y|X = F^{-1}(u)] \leq_{disp} Y$, for all $u \in (0,1)$, is a particular case for C', the independence copula. This order is also a necessary condition for the growth of u_v as a function of v in some particular models, as we now show.

Proposition 9 *Let $\mathbf{X} = (X, Y)$ be a random vector such that $Y = \psi(X) + W$, where X and W are independent random variables and ψ is a strictly increasing function. The $PELCoV_v$, given by $u_v = F_X\left(\psi^{-1}\left(VaR_v[Y] - VaR_v[W]\right)\right)$, increases with respect to $v \in (0,1)$ if and only if $W \leq_{disp} \psi(X) + W$. Moreover, this last inequality is always true when W has a logconcave density function.*

Proof: First, $Y \uparrow_{SSI} X$ trivially holds. Given $v \in (0,1)$, u_v is determined using the same argument as in Example 7 and the fact that $\psi(\mathrm{VaR}_u[X]) = \mathrm{VaR}_u[\psi(X)]$ for a strictly increasing ψ. Clearly, the growth of u_v as a function of v is equivalent to the growth of $\mathrm{VaR}_v[\psi(X)+W] - \mathrm{VaR}_v[W]$ in v, which is the same as $\psi(X) + W \geq_{disp} W$. By Theorem 3.B.7 in Shaked and Shanthikumar (2007), this inequality is always true when W has a logconcave density function. □

Note that Proposition 9 does not apply to the vector (X, Y) in Example 7 because the density function of the Weibull distribution is not log-concave.

4.2 Boundedness

As we explained in Section 3, the existence of a $u^* < 1$ such $u_v \leq u^*$ for all v, ensures that $\mathrm{VaR}_v[Y]$ will underestimate, for all v, the risk spillover effect as

[5] For further details on multivariate dispersive orderings, see Fernandez-Ponce and Suárez-Llorens (2003) and Arias-Nicolás et al. (2005).

soon as X exceeds its quantile u^*. The aim of this section is to provide conditions to ensure the existence of an upper bound $u^* < 1$ for the set $\{u_v : v \in (0,1)\}$.

If the function that maps v to u_v is increasing, then the supremum of the set $\{u_v : v \in (0,1)\}$ is $\lim_{v\to 1} u_v$. In this case, the existence of an upper bound $u^* < 1$ reduces to the study of this limit. In general, when the function $v \mapsto u_v$ is not monotonic, the supremum may be attained in some $v_0 \in (0,1)$, as in Example 7 (where $v_0 = 0.74816$). The following result establishes conditions under which the existence of an upper bound $u^* < 1$ for the set $\{u_v : v \in (0,1)\}$ can be proved by computing the limits of the PELCoV$_v$ when v tends to 0 and to 1.

Corollary 10 *Let (X, Y) be a random vector satisfying regularity conditions with a copula C, such that $Y \uparrow_{SSI} X$ and $\partial_1 C(u, v)$ is continuous in u. The following conditions are equivalent:*
(1) There exist $u^* < 1$ such that $u_v \leq u^*$ for all $v \in (0,1)$.
(2) $\lim\limits_{v\to 0} u_v < 1$ and $\lim\limits_{v\to 1} u_v < 1$.

Proof: First we show, by contradiction, that there does not exist a $v \in (0,1)$ such that $v = \partial_1 C(1, v)$. Assume that there exists $v^* \in (0,1)$ such that $v^* = \partial_1 C(1, v^*)$. It follows from $Y \uparrow_{SSI} X$ that $\partial_1 C(1, v)$ decreases in u. Then, $v^* < \partial_1 C(u, v^*)$, for all $u \in (0,1)$, which is a contradiction as in (8). Consequently, if the supremum of the set $\{u_v : v \in (0,1)\}$ is attained at some $v_0 \in (0,1)$, then it is necessarily less than one. This implies that a necessary and sufficient condition for the supremum of $\{u_v : v \in (0,1)\}$ to be 1 is $\lim\limits_{v\to 0} u_v = 1$ or $\lim\limits_{v\to 1} u_v = 1$. $\square$

The following theorem gives sufficient conditions for the existence of an upper bound $u^* < 1$ for the set $\{u_v : v \in (0,1)\}$ in terms of the copula density $c(u, v)$.

Theorem 11 *Let (X, Y) be a random vector satisfying regularity conditions with copula C, such that $Y \uparrow_{SSI} X$ and $\partial_1 C(u, v)$ is continuous in u. Let $c(u, v)$ be the copula density. If*
(i) there exists $m < 1$ such that $c(u, v)$ increases in v for all $u \geq m$ and $v \leq u$,
(ii) the limit $\lim\limits_{(u,v)\to(1,1)} c(u, v)$ exists and it is stricty greater than one,
then there exist $u^* < 1$ such that $u_v \leq u^*$ for all $v \in (0,1)$.

Proof: The idea of the proof is to show that the assumptions imply $\lim\limits_{v\to 0} u_v < 1$ and $\lim\limits_{v\to 1} u_v < 1$ and then to apply Corollary 10.

We first show that (ii) implies $\lim\limits_{v\to 1} u_v < 1$. From (ii), there exists $\delta > 0$ such that $c(u, v) > 1$ for all $(u, v) \in (1-\delta, 1)^2$. This implies that $\frac{\int_v^1 c(u,z)dz}{1-v} > 1$ for all $(u, v) \in (1-\delta, 1)^2$ or, equivalently, that

$$\frac{\int_0^v c(u, z)dz}{v} = \frac{\partial_1 C(u, v)}{v} < 1, \text{ for all } (u, v) \text{ in } (1-\delta, 1)^2. \tag{21}$$

Using Lemma 2, it follows that $u_v < 1 - \delta$ for $v > 1 - \delta$ and, consequently, $\lim\limits_{v\to 1} u_v < 1$. Now observe, taking into account the equality in (21), that (i) implies that

$$\frac{\partial_1 C(u, v)}{v} \text{ increases in } v \text{ for all } u \geq m \text{ and } v \leq u. \tag{22}$$

To prove that $\lim_{v\to 0} u_v < 1$, we assume, by contradiction, that $\lim_{v\to 0} u_v = 1$. Then, given $\epsilon > \min(1-m, \delta)$, there exists $\alpha > 0$ such that for $v_1 < \alpha$, $u_{v_1} > 1-\epsilon$. For such v_1 there exists $v_2 > v_1$ such that $1-\delta < v_2 < u_{v_1}$. For these v_1 and v_2 we get

$$1 = \frac{\partial_1 C(u_{v_1}, v_1)}{v_1} \leq \frac{\partial_1 C(u_{v_1}, v_2)}{v_2} < 1,$$

a contradiction, where the equality follows from the definition of PELCoV$_v$, the first inequality follows from (22) and the second (strict) inequality follows from (21). $\square$

5 Parametric examples of PELCoV$_v$s.

In this section, we determine the PELCoV$_v$ and study the existence of an upper bound $u^* < 1$ for several parametric families of copulas. Recall that given a copula $C(u,v)$ satisfying the assumptions of Corollary 5, the PELCoV$_v$ is the unique value $u_v \in (0,1)$ such that $\partial_1 C(u_v, v) = v$. For the existence of an upper bound, it is sufficient to verify the conditions of Theorem 11.

5.1 Families of copulas with bounded PELCoV$_v$s.

The result of this section are summarized in Table 1.

5.1.1 Farlie–Gumbel–Morgenstern family of copulas

Let us consider (X,Y) with the Farlie–Gumbel–Morgenstern copula, given by $C(u,v) = uv(1+\theta(1-u)(1-v))$, for $\theta \in [-1,1]$, $u,v \in [0,1]$. Since $\partial_1 C(u,v) = v + \theta v(2u-1)(1-v)$ if $\theta \in (0,1]$, $Y \uparrow_{SSI} X$. For $\theta > 0$ the copula density $c(u,v) = 1 + \theta(1-2u)(1-2v)$ increases in v for all $u \geq 1/2$. Moreover, $\lim_{(u,v)\to(1,1)} c(u,v) = 1+\theta > 0$ and it follows from Theorem 11 that there exists an upper bound $u^* < 1$ for the PELCoV$_v$s.

Observe that $\partial_1 C(u,v) = v$ if and only if $u = 1/2$. Hence, the PELCoV$_v$ is $u_v = 1/2$ for all $v \in (0,1)$. Moreover, under the SSI condition ($\theta > 0$), $u^* = 1/2$ and $Y \leq_{st} [Y \mid X = x]$ for all $x \geq F^{-1}(1/2)$. Consequently, for all $v \in (0,1)$, VaR$_v[Y]$ underestimates the risk spillover effect whenever X exceeds its median.

5.1.2 Frank family of copulas

The Frank copula $C(u,v) = -\frac{1}{\theta}\ln\left(1 + \frac{(e^{-u\theta}-1)(e^{-v\theta}-1)}{e^{-\theta}-1}\right)$ (see (4.1.5) in Nelsen, 2007) is SSI for $\theta > 0$. The PELCoV$_v$, given by

$$u_v = \frac{1}{\theta}\ln\left(\frac{e^{\theta}\left(1-e^{\theta v}\right)(1-v)}{v\left(e^{\theta v}-e^{\theta}\right)}\right), \tag{23}$$

increases in v. Observe that $v \geq u_v$ if and only if $v \geq \frac{1}{2}$ (see Figure 3). The copula density, given by

$$c(u,v) = \frac{e^{(1+u+v)\theta}\left(e^{\theta}-1\right)\theta}{\left(e^{(u+v)\theta} - e^{\theta}\left(e^{u\theta}+e^{v\theta}-1\right)\right)^2},$$

satisfies the conditions of Theorem 11 for $\theta > 0$ and $m = 1/2$. The upper bound[6] is given by $u^* = \lim_{v\to 1} u_v = \frac{1}{\theta}\ln\left(\frac{e^\theta - 1}{\theta}\right)$.

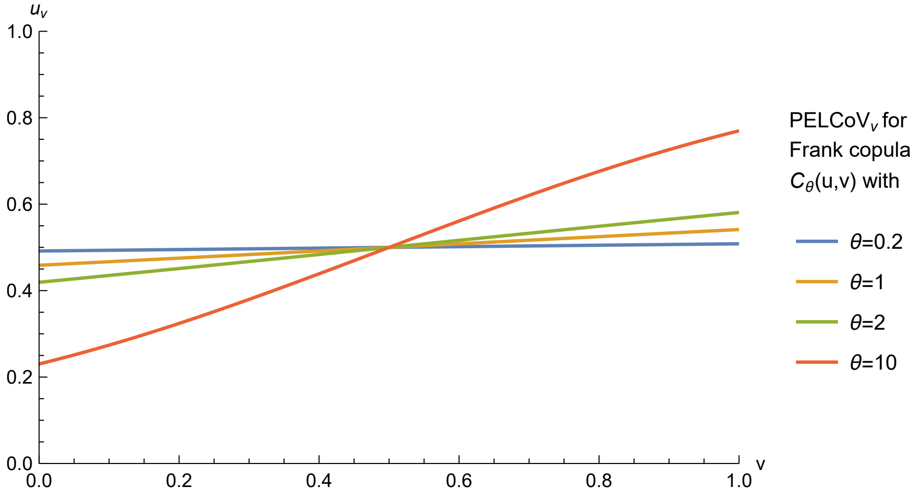

Figure 3: Function $v \mapsto u_v$ for the Frank copula with different values of the dependence parameter.

5.1.3 Clayton family of copulas

Let consider the Clayton copula (see (4.1.1) in Nelsen, 2007), given by $C(u,v)= \left[max(u^{-\theta} + v^{-\theta} - 1, 0)\right]^{-1/\theta}$. This copula is SSI for $\theta > 0$. The copula density is given by

$$c(u,v) = u^{-1-\theta} v^{-1-\theta}\left(u^{-\theta} + v^{-\theta} - 1\right)^{-2-\frac{1}{\theta}}(1+\theta).$$

It can be shown that $c(u,v)$ increases in v if $u > \phi_\theta(v)$, where

$$\phi_\theta(v) = \left(1 - \frac{\theta}{\theta + v^\theta(1+\theta)}\right)^{1/\theta}$$

is a function that increases in v such that

$$\lim_{v\to 1}\phi_\theta(v) = \left(\frac{1+\theta}{1+2\theta}\right)^{1/\theta} = m_\theta.$$

For this m_θ we see that $c(u,v)$ satisfies (i) in Theorem 11. Since

$$\lim_{(u,v)\to(1,1)} c(u,v) = 1+\theta,$$

condition (ii) also holds for $\theta > 0$ and it follows that the PELCoV$_s$s are bounded. The PELCoV$_v$ and the upper bound u^* are given by

$$u_v = \left(\frac{v^{-\theta/(\theta+1)} - 1}{v^{-\theta} - 1}\right)^{1/\theta}; \quad u^* = \lim_{v\to 1} u_v = (1+\theta)^{-\frac{1}{\theta}}.$$

[6]See Appendix A for details on the computation.

Figure 4 plots the function $v \mapsto u_v$ for different values of θ.

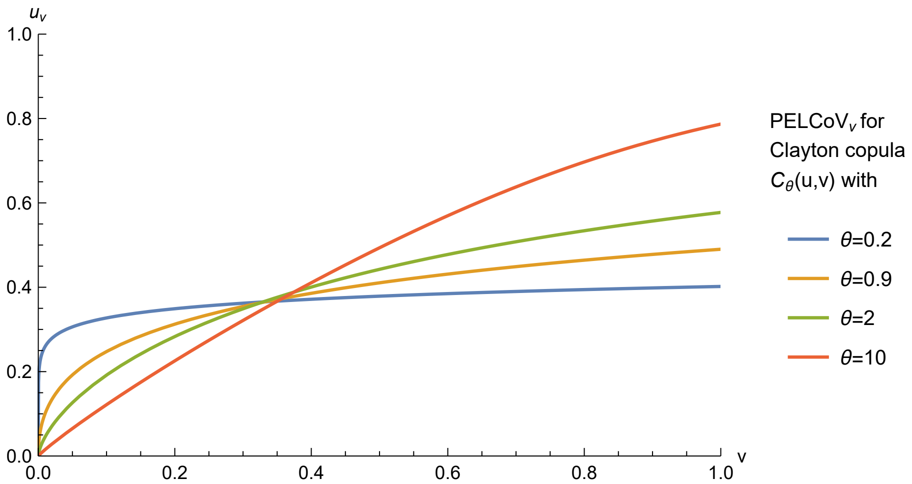

Figure 4: Function $v \mapsto u_v$ for the Clayton copula with different values of the dependence parameter.

5.1.4 Ali-Mikhail-Haq family of copulas

Let us consider the Ali-Mikhail-Haq copula (see (4.3.1) in Nelsen, 2007), given by $C(u,v) = \frac{uv}{(1-\theta(1-u)(1-v))}$ for $\theta \in [-1,1]$. The copula is SSI for $\theta \in (0,1]$. Given $\theta \in (0,1)$, it can be shown that the copula density, given by

$$c(u,v) = \frac{1+\theta(u+v+uv-2+(1-u)(1-v)\theta)}{(1-(1-u)(1-v)\theta)^3},$$

increases in v for $u \geq \frac{2}{3}$. Moreover, for $\theta > 0$

$$\lim_{(u,v)\to(1,1)} c(u,v) = 1+\theta > 1,$$

and it follows from Theorem 11 that there exists an upper bound u^* for the PELCoV$_v$s. It can be checked that

$$u_v = \frac{\theta(1-v)-1+\sqrt{1-\theta(1-v)}}{\theta(1-v)}; \qquad u^* = \lim_{v\to 1} u_v = \frac{1}{2}.$$

Figure 5 plots the function $v \mapsto u_v$ for different values of θ.

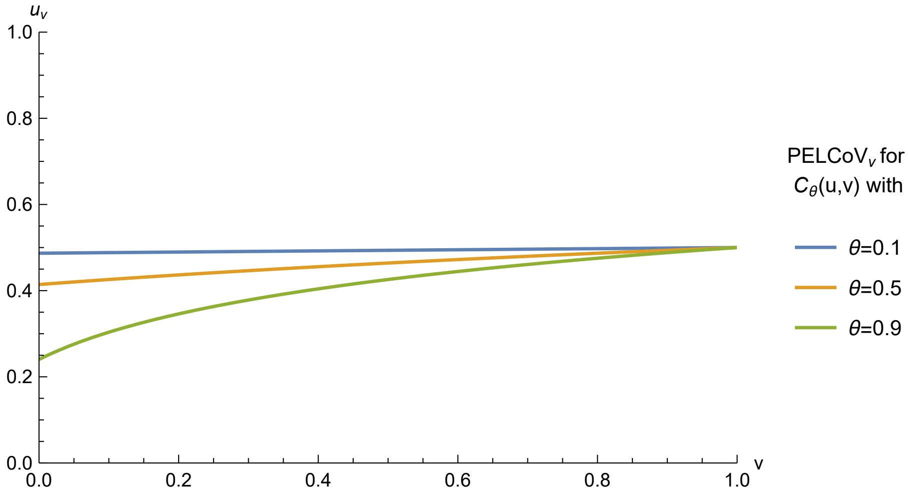

Figure 5: Function $v \mapsto u_v$ for the Ali-Mikhail-Haq copula with different values of the dependence parameter.

5.1.5 Other families of copulas

For the following families of copulas, the conditions (i) and (ii) in Theorem 11 and the bound u^* have been numerically investigated.

(a) Family of copulas (4.1.13) in Nelsen (2007). The copula

$$C(u,v) = \exp\left(1 - \left[(1-\ln u)^\theta + (1-\ln v)^\theta - 1\right]^{1/\theta}\right)$$

is SSI for $\theta > 1$. The function $v \mapsto u_v$ increases in v and the limit $u^* = \lim_{v \to 1} u_v$ exists for all $\theta \in (1, 1000)$ (see Figure 6).

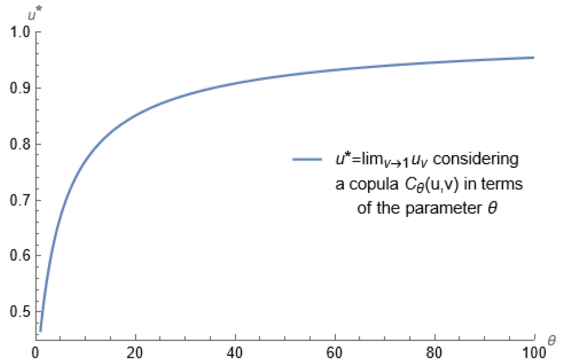

Figure 6: Function $\theta \mapsto u^* = \lim_{v \to 1} u_v$ for the copula (4.1.13) in Nelsen (2007).

(b) Family of copulas (4.1.17) in Nelsen (2007). The copula

$$C(u,v) = \left(1 + \frac{\left[(1+u)^{-\theta} - 1\right]\left[(1+v)^{-\theta} - 1\right]}{2^{-\theta} - 1}\right)^{-1/\theta} - 1, \quad \theta \neq 0,$$

is SSI for $\theta > -1$ ($\theta \neq 0$). The PELCoV$_v$ increases in v and the limit $u^* = \lim_{v \to 1} u_v$ exits for all $\theta \in (-1, 1000) \setminus \{0\}$ (see Figure 7).

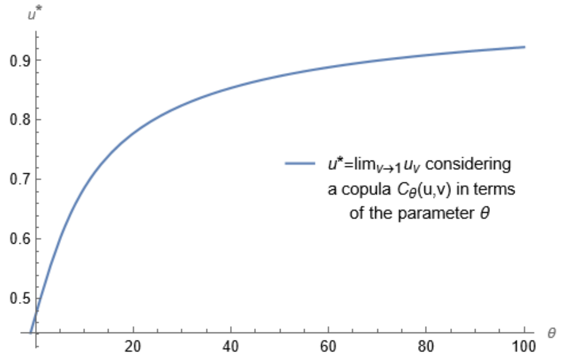

Figure 7: Function $\theta \mapsto u^* = \lim_{v \to 1} u_v$ for the copula (4.1.17) in Nelsen (2007).

(c) Family of copulas 19 in Table 4.1 in Nelsen (2007). The copula

$$C(u, v) = \frac{\theta}{\ln\left(e^{\theta/u} + e^{\theta/v} - e^{\theta}\right)}, \quad \theta > 0,$$

is SSI. The PELCoV$_v$ increases in v and the limit $u^* = \lim_{v \to 1} u_v$ exits for all $\theta \in (0, 1000)$ (see Figure 8).

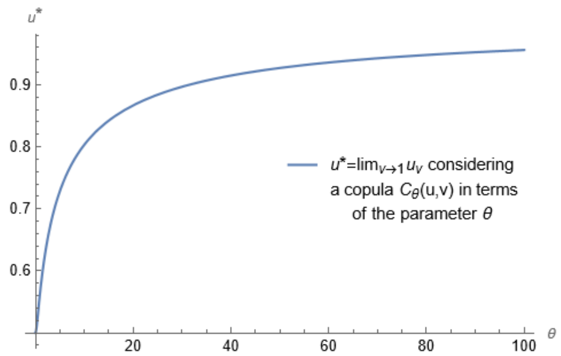

Figure 8: Function $\theta \mapsto u^* = \lim_{v \to 1} u_v$ for the copula 19 in Table 4.1 in Nelsen (2007).

Copulas with bounded PELCoV	**PELCoV**	**Bound**
$(X, \psi(X)+W)$ with ψ strictly increasing and X, W independent.	$u_v = F_X\left(\psi^{-1}\left(F^{-1}_{\psi(X)+W}(v) - F^{-1}_W(v)\right)\right)$	If W has a logconcave density function
FGM copula: for $\theta \in (0,1]$ $C(u,v) = uv(1+\theta(1-u)(1-v))$	$u_v = 1/2$	$u^* = 1/2$
Frank copula: for $\theta > 0$, $C(u,v) = -\frac{1}{\theta}\ln\left(1 + \frac{\left(e^{-u\theta}-1\right)\left(e^{-v\theta}-1\right)}{e^{-\theta}-1}\right)$	$u_v = \frac{1}{\theta}\ln\left(\frac{e^{\theta}\left(1-e^{\theta v}\right)(1-v)}{v(e^{\theta v}-e^{\theta})}\right)$	$u^* = \frac{1}{\theta}\ln\left(\frac{e^{\theta}-1}{\theta}\right)$
Clayton copula: for $\theta > 0$, $C(u,v) = \left[max(u^{-\theta}+v^{-\theta}-1,0)\right]^{-1/\theta}$	$u_v = \left(\frac{v^{-\theta/(\theta+1)}-1}{v^{-\theta}-1}\right)^{1/\theta}$	$u^*(1+\theta)^{-\frac{1}{\theta}}$
AMH copula: for $\theta \in (0,1]$, $C(u,v) = \frac{uv}{(1-\theta(1-u)(1-v))}$	$u_v = \frac{\theta(1-v)-1+\sqrt{1-\theta(1-v)}}{\theta(1-v)}$	$u^* = 1/2$
Copula 13 in Table 4.1 in Nelsen (2007): for $\theta > 0$, $C(u,v) = \exp\left(1 - \left[(1-\ln u)^{\theta} + (1-\ln v)^{\theta} - 1\right]^{1/\theta}\right)$	Numerically computed	Corollary 10 applies. Numerically computed
Copula 17 in Table 4.1 in Nelsen (2007): for $\theta > 0$ $C(u,v) = \left(1 + \frac{\left[(1+u)^{-\theta}-1\right]\left[(1+v)^{-\theta}-1\right]}{2^{-\theta}-1}\right)^{-1/\theta} - 1$	Numerically computed.	Corollary 10 applies. Numerically computed.
Copula 19 in Table 4.1 in Nelsen (2007): for $\theta > 0$, $C(u,v) = \frac{\theta}{\ln\left(e^{\theta/u}+e^{\theta/v}-e^{\theta}\right)}$	Numerically computed.	Corollary 10 applies. Numerically computed.

Table 1: PELCoVs and their bounds for vectors (X, Y) such that $Y \uparrow_{SSI} X$.

5.2 Families of copulas with unbounded PELCoV$_v$s.

This section shows several families of copulas such that the PELCoV$_v$s do not have an upper bound. We start with the gaussian family, which has appeared before in several examples joining a bivariate random vector. To be consistent with such examples, for the gaussian family we also consider the marginals.

5.2.1 Gaussian family of copulas

Let $\mathbf{X} = (X, Y) \sim \mathbf{N}(\mu, \mathbf{\Sigma})$ be a bivariate normal random vector with $\mu = (\mu_1, \mu_2)$ and $\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$. For $\rho > 0$ the vector is SSI. We saw in Section 4 that the PELCoV$_v$ increases in terms of v, with u_v given by (16).

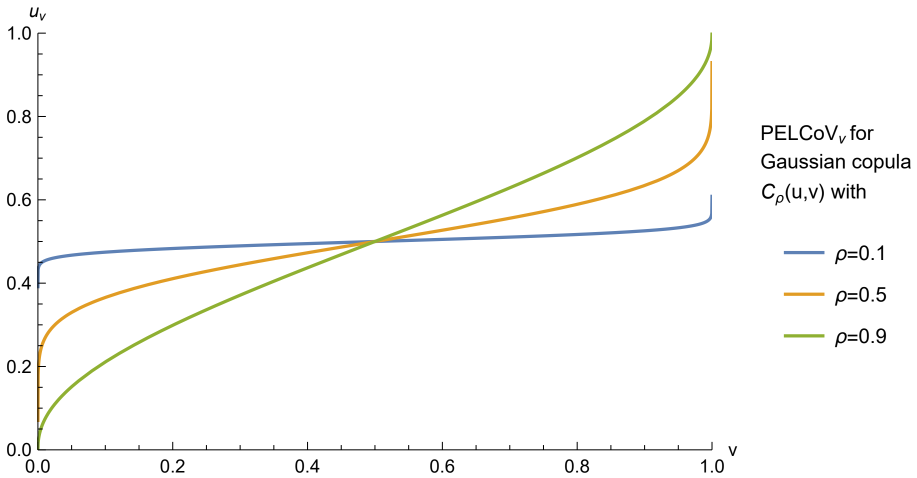

Figure 9: Function $v \mapsto u_v$ for the Gaussian copula with different values of the dependence parameter.

Observe that $u_v < v$ for all $v \in (\frac{1}{2}, 1)$ (see Figure 9). Moreover, since $\lim_{v\to 0} u_v = 0$ and $\lim_{v\to 1} u_v = 1$, it follows from Corollary 10, that there does not exist a $u^* < 1$ such that $u_v \leq u^*$ for all $v \in (0, 1)$.

5.2.2 Gumbel Hougard family of copulas

The Gumbel Hougard family of copulas (see 4.1.4 in Nelsen, 2007) has the form $C(u,v) = \exp\left(-\left[(-\ln u)^\theta + (-\ln v)^\theta\right]^{1/\theta}\right)$. It is SSI for $\theta \geq 1$. The density is given by

$$c(u,v) = \frac{e^{-\left((-\ln[u])^\theta + (-\ln[v])^\theta\right)^{\frac{1}{\theta}}} (\ln[v]\ln[u])^{\theta-1} h(u,v)}{uv}$$

where

$$h(u,v) = \left(\theta - 1 + \left((-\ln[u])^\theta + (-\ln[\mathrm{v}])^\theta\right)^{\frac{1}{\theta}}\right)\left((-\ln[u])^\theta + (-\ln[v])^\theta\right)^{-2+\frac{1}{\theta}}.$$

Since $\lim_{v\to 1} c(u,v) = 0$, for all $u \in (0,1)$, $\lim_{u\to 1} c(u,v) = 0$, for all $v \in (0,1)$ and $\lim_{u\to 1} c(u,u) = \infty$, we conclude that $\lim\limits_{(u,v)\to(1,1)} c(u,v)$ does not exist and Theorem 11 does not apply. The plot of u_v as a function of u is obtained numerically for different values of $\theta \geq 1$. Figure 10 shows that there does not exist an upper bound $u^* < 1$ for the sequence of $u_v,\ v \in (0,1)$.

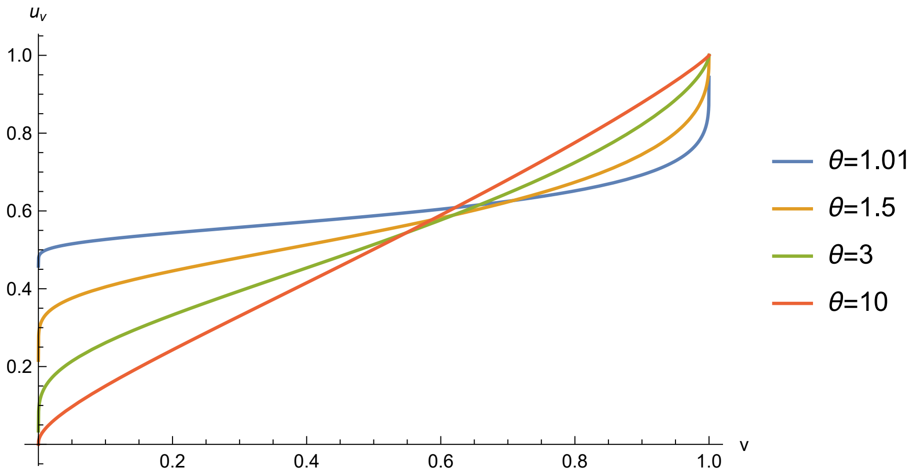

Figure 10: Function $v \mapsto u_v$ for the Gumbel Hougard copula with different values of the dependence parameter.

5.3 Other families of copulas

For the following families of copulas, the property SSI and the PELCoVs have been numerically investigated for all $\theta \in (1, 1000)$. In general, for these copulas, there does not exist a $u^* < 1$ such that $u_v \leq u^*$ for all $v \in (0,1)$.

(a) The family of copulas 4.1.6 in Nelsen (2007), given by $C(u,v) = 1 - \left[(1-u)^\theta + (1-v)^\theta - (1-u)^\theta(1-v)^\theta\right]^{1/\theta}$, for $\theta > 1$, see Figure 11.

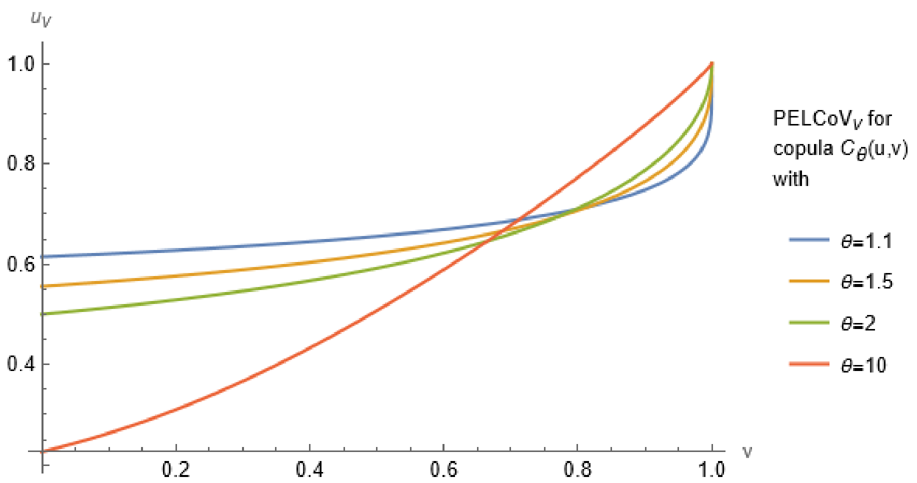

Figure 11: Function $v \mapsto u_v$ for the copula 4.1.6 in Nelsen (2007) with different values of the dependence parameter.

(b) The family of copulas 4.1.12 in Nelsen (2007) given, for $\theta \geq 1$, by $C(u,v) = \left(1 + \left[\left(u^{-1}-1\right)^{\theta} + \left(v^{-1}-1\right)^{\theta}\right]^{1/\theta}\right)^{-1}$, see Figure 12.

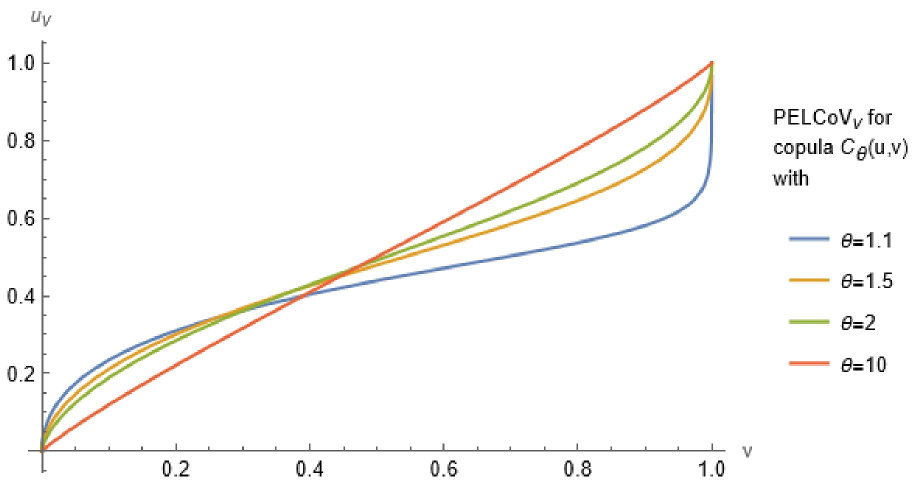

Figure 12: Function $v \mapsto u_v$ for the copula 4.1.12 in Nelsen (2007) with different values in the dependence parameter.

(c) The family of copulas 4.1.14 in Nelsen (2007), given, for $\theta \geq 1$, by $C(u,v) = \left(1 + \left[\left(u^{-1/\theta}-1\right)^{\theta} + \left(v^{-1/\theta}-1\right)^{\theta}\right]^{1/\theta}\right)^{-\theta}$, see Figure 13.

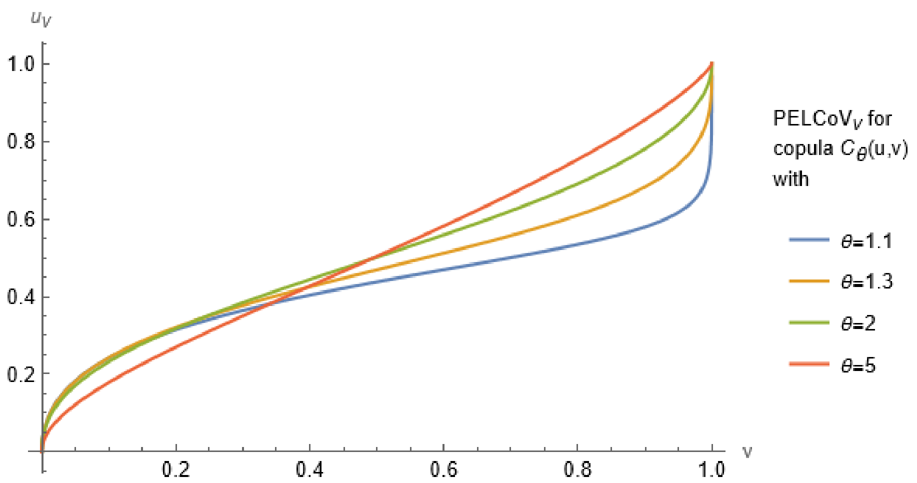

Figure 13: Function $v \mapsto u_v$ for the copula 4.1.14 in Nelsen (2007) with different values of the dependence parameter.

6 Example

This section provides a real-data-based example to illustrate the methodology. We consider the negative log-returns (or losses) of stock prices of two companies that operate in highly dependent sectors: International Consolidated Airlines Group S.A. (or simply, IAG), a holding company for several airlines including British Airways, Iberia and Air Europa, and Melia Hotels International S.A. (or MHI) a Spanish chain that operates more than 400 hotels throughout more than 40 countries. The example shows that the VaR of MHI asset log returns underestimated the risk compared to CoVaR (conditional on VaR for IAG asset log returns) during periods of the COVID-19 financial crisis.

6.1 The model

The copula models discussed so far assume a dependence structure constant over time. However, empirical studies of multivariate time series have revealed the existence of time-varying dependence among asset returns. We use the copula time series model introduced by Patton (2006b) to capture a possible dynamic dependence structure. This model assumes that the functional form of the copula remains fixed over the sample while the parameters vary according to some evolution equation. In order to estimate the parameters in this equation, we first need to specify the marginal distributions for the asset log returns. This model has been widely used to study the co-movements of asset returns in the framework of time-varying dependence structures (see, for example, Patton (2006a) Reboredo (2011, 2013) and Ji et al. (2019)).

We denote the negative log returns of IAG as the variable X_t and the negative log returns of MHI as the variable Y_t. Since log-returns may present the characteristics of autocorrelation and conditional heterocedasticity, two ARMA-GARCH models are employed to describe their marginal distributions. The analyzes described in Section 6.3 support the following models for the two marginal series. The marginal distribution of IAG is specified as an MA(1)-GARCH$(1,1)$ model with skew t-innovations:

$$\begin{aligned}
X_t &= \eta_t - \theta_1 \eta_{t-1} \\
\eta_t &= \sigma_{x,t} a_t, \quad \sigma_{x,t}^2 = \omega_x + \alpha_x \eta_{t-1}^2 + \beta_x \sigma_{x,t-1}^2, \\
\sqrt{\frac{v_x}{\sigma_{x,t}^2 (v_x - 2)}}\eta_t &\sim \text{iid } t_{v_x}
\end{aligned}$$

where a_t is a sequence of iid random variables with mean 0 and variance 1. Similarly, the marginal distribution of MHI is specified as a AR(5)-GARCH $(1,1)$ model with skew t-innovations as follows:

$$\begin{aligned}
Y_t &= \phi_4 Y_{t-4} + \phi_5 Y_{t-5} + \varepsilon_t \\
\varepsilon_t &= \sigma_{y,t} b_t, \quad \sigma_{y,t}^2 = \omega_y + \alpha_y \varepsilon_{t-1}^2 + \beta_y \sigma_{y,t-1}^2, \\
\sqrt{\frac{v_y}{\sigma_{y,t}^2 (v_y - 2)}}\varepsilon_t &\sim \text{iid } t_{v_y}
\end{aligned}$$

where b_t is a sequence of iid random variables with mean 0 and variance 1.

Denote by $\boldsymbol{F}_t(\bullet; \boldsymbol{\lambda}_c)$ the conditional distribution function of the bivariate time serie $\boldsymbol{X}_t = (X_t, Y_t)$, given the information set at time $t-1$. By applying Sklar's theorem to the joint conditional distribution function, we have

$$\boldsymbol{F}_t(x, y; \boldsymbol{\lambda}) = C_t\left(F_t\left(x; \lambda_1\right), G_t\left(y; \lambda_2\right); \lambda_c\right), \tag{24}$$

where λ_1 and λ_2 are the parameters for the marginal conditional distributions, λ_c are the parameters for the conditional copula and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_c)$ are the parameters for the joint conditional distribution. For the applicability of the copula models studied in this paper, we considered a model with bounded PELCoV (the Frank model) and another with unbounded PELCoV (the Gaussian model) to perform the fits. The goodness-of-fit test for bivariate copulas based on White's information matrix equality (White (1982)) gave a p-value equal to 0.12 for the Frank copula and that based on Kendall process, as proposed by Wang and Wells (2000), gave a p-value equal to 0.14 for the Gaussian copula. Although AIC showed that the Gaussian performed better than the Frank copula, we do not have enough evidence to reject these models at usual significance levels under the assumption of time-invariant dependence.

Following Patton (2006b), for each model, we specify the time-varying copula as follows. For the Gaussian copula, the linear correlation coefficient ρ_t evolves according to an ARMA(1,10)-type process:

$$\rho_t = \Lambda_1\left(\nu_0 + \nu_1 \rho_{t-1} + \nu_2 \frac{1}{10}\sum_{j=1}^{10} \Phi^{-1}\left(u_{t-j}\right) \cdot \Phi^{-1}\left(v_{t-j}\right)\right)$$

where Φ^{-1} is the inverse of the standard normal distribution function, $\Lambda_1(x) = \left(1 - e^{-x}\right) / \left(1 + e^{-x}\right)$ is the modified logistic transformation, designed to keep ρ_t in $(-1, 1)$ and $\lambda_c = (\nu_0, \nu_1, \nu_2)$. The model includes ρ_{t-1} as a regressor to capture any persistency in the dependence parameter and incorporates the intuition that the correlation should increase (decrease) if the transformed marginals have the same (opposite) sign. For the Frank copula, the driving mechanism for the dependence parameter is:

$$\theta_t = \Lambda_2 \left(\omega_0 + \omega_1 \theta_{t-1} + \omega_2 \frac{1}{10} \sum_{j=1}^{10} |u_{t-j} - v_{t-j}| \right)$$

where $\Lambda_2(x) = (1 + e^{-x})$ and $\lambda_c = (\omega_0, \omega_1, \omega_2)$. The choice of Λ_2 ensures that C_t is SSI. The dependence parameter equation includes θ_{t-1} as a regressor and a forcing variable based on the closeness of the most recent u_t and v_t.

6.2 Estimation and testing

The set of parameters λ_c of the copula is estimated by maximum likelihood, where the log-likelihood function of a random sample $(x_t, y_t)_{t=1}^n$ is, according to (24), given by

$$l(\boldsymbol{\lambda}) = \sum_{t=1}^{n} \left\{ \log f_t\left(x_t; \lambda_1\right) + \log g_t\left(y_t; \lambda_2\right) + \log c_t\left(F_t\left(x_t; \lambda_1\right), G_t\left(y_t; \lambda_2\right); \lambda_c\right) \right\},$$

where f_t, g_t and c_t are the marginal conditional density functions and the conditional copula density function, respectively. We use a two-stage maximum likelihood estimation procedure based on maximizing the parameters separately for the marginals and the copula, see Joe and Xu (1996). At the first stage only the parameters from the marginal distributions are estimated. At the second stage, the dependence parameter is estimated from the copula likelihood by solving the following:

$$\hat{\lambda}_c = \arg\max_{\lambda_c} \sum_{t=1}^{n} \log c_t\left(\hat{u}_t, \hat{v}_t; \lambda_c\right),$$

where $\hat{u}_t = F_t(x_t; \hat{\lambda}_1)$ and $\hat{v}_t = G_t(y_t; \hat{\lambda}_2)$ are pseudo-sample observations from the copula. Under standard conditions, this procedure gives consistent and asymptotically normal estimates (see Joe (1997)).

6.3 Data analysis

We used adjusted weekly closing prices for IAG and MHI obtained from the public website *http://es.finance.yahoo.com* covering seven years, from January 8, 2016, to December 30, 2022, yielding $n = 363$ observations for each group. We considered negative log returns computed as $r_t = \log(p_{t-1}/p_t)$, where p_t and p_{t-1} are stock prices at weeks t and $t-1$, respectively. Table 2 provides descriptive statistics for log returns series. The Shapiro-Wilk test strongly rejected the normality, and the assets log returns exhibit a positive excess kurtosis. The empirical correlation coefficient is 0.7034.

The parameters of the ARMA(p,q)-tGARCH(r,s) models were empirically determined by identifying the optimal models among alternative values using the Akaike Information Criterion (AIC). The Jarque-Vera test strongly rejected the normality of residuals. We used the Ljung-Box test of the squared standardized residuals to test the validity of the volatility equation. The sample autocorrelation function (ACF) and p-values of Kolmogorov-Smirnov and Ljung-Box tests suggest that the models are adequate. Table 3 provides the parameter estimates for the marginal models. Figure 14 shows the scatterplot of the empirical copula $(F_t(x_t; \hat{\lambda}_1), G_t(y_t; \hat{\lambda}_2))$.

	MHI	IAG
Mean	0.002081	0.003845
Std dev	0.057062	0.076719
Max	0.327792	0.513617
Min	-0.345877	-0.369389
Skewness	-0.020238	1.017383
Kurtosis	8.145324	10.414437
Shapiro-Wilk p-val	0.00000	0.000000
Pearson's r	0.7034985	
number observed	363	

Table 2: Descriptive statistics for log returns. Ljung–Box test for the squared returns is computed with 20 lags.

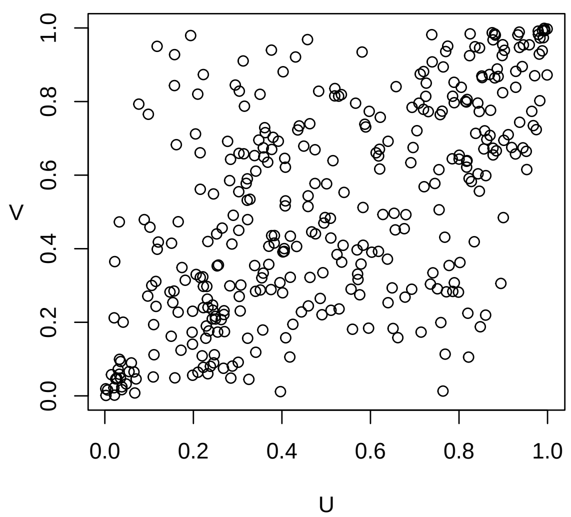

Figure 14: Scatter-plot of the empirical copula of the bivariate time series (X_t, Y_t), given by $(F_t(x_t; \hat{\lambda}_1), G_t(y_t; \hat{\lambda}_2))$.

Mean equation			
MHI		IAG	
$\hat{\phi}_4$ $\hat{\phi}_5$	-0.1604 (0.0520) 0.0893 (0.0521)	$\hat{\theta}_1$	0.1668(0.0481)
Variance equation			
$\hat{\alpha}_x$ $\hat{\beta}_x$ Ljung-Box R^2 p-val Jarque-Bera p-val	0.1129(0.03869) 0.8649 (0.03999) 0.88279 0	$\hat{\alpha}_y$ $\hat{\beta}_y$	0.1129 (0.03869) 0.8649 (0.03999) 0.8827 0
Marginal models			
d.f. $\hat{u}_x$ KS test p -val Ljung-Box (RS) p-val	2.96 0.5598 0.39	d.f. $\hat{u}_y$	3.01 0.4357 0.53

Table 3: Maximum likelihood estimates with asymptotic standard errors in parentheses of the parameters of the marginal distribution models for MHI and IAG log returns. Ljung–Box test for the squared residuals (R^2) and for standardized residuals (RS) are computed with 20 lags. Jarque-Vera tests the normality of residuals.

6.4 Results

Given the bivariate time series (X_t, Y_t), where X_t and Y_t describe the losses (negative log returns) of IAG and MHI, respectively, we have obtained the time-varying PELCoV$_v$ for the series X_t, $v \in (0,1)$, under different dynamic copula models. Figure 15 plots the time series X_t together with the $u_v(t)$-quantiles for different values of v ($v_1 = 0.99, v_2 = 0.95$ and $v_3 = 0.8$). Given t, the $u_v(t)$-quantile is $F^{-1}_{X_t}(u_v(t))$ (called PELCoV$_v$-IAG-quantile in the plots). The left-hand plots show the quantiles associated with the PELCoV$_v$s under the fitted Gaussian copula and the right-hand plots show the quantiles associated to the PELCoV$_v$s and to their upper bounds u^*_t under the fitted Frank copula. The upper bounds depicted in the plots are given by $F^{-1}_{X_t}(u^*_t)$, where u^*_t is obtained from equation (23), where θ is replaced with θ_t and v takes the probability levels $0.99, 0.95$ and 0.80, respectively. If X_t exceeds $F^{-1}_{X_t}(u^*_t)$, it is more prudent to determine reserves in terms of CoVaR for all $u_t > u^*_t$. The plots allow us to compare, for each t, the actual IAG loss (in terms of negative log returns) and the minimum loss above which CoVaR$_{v_i,u}$ [MHI|IAG] had been a more prudential reserve provision than VaR$_v$[MHI].

X_t was always closer to its u_v-quantile (whatever the probability level u) under the Frank copula than the Gaussian copula. All the plots, except one (the plot in the top left corner), show that VaR$_v$[MHI] underestimated the risk compared to CoVaR$_{v_i,u}$ [MHI|IAG] during the pandemic period[7]. The COVID-19 pandemic significantly impacted many sectors of the economy, especially in the airlines and hotel industry, and materialized as a significant drop in the demand for passenger flights and accommodations. In these circumstances, it is not surprising that the value at risk underestimated the spillover effect.

[7]Reversing the roles between MHI and IAG leads to a similar result in this period.

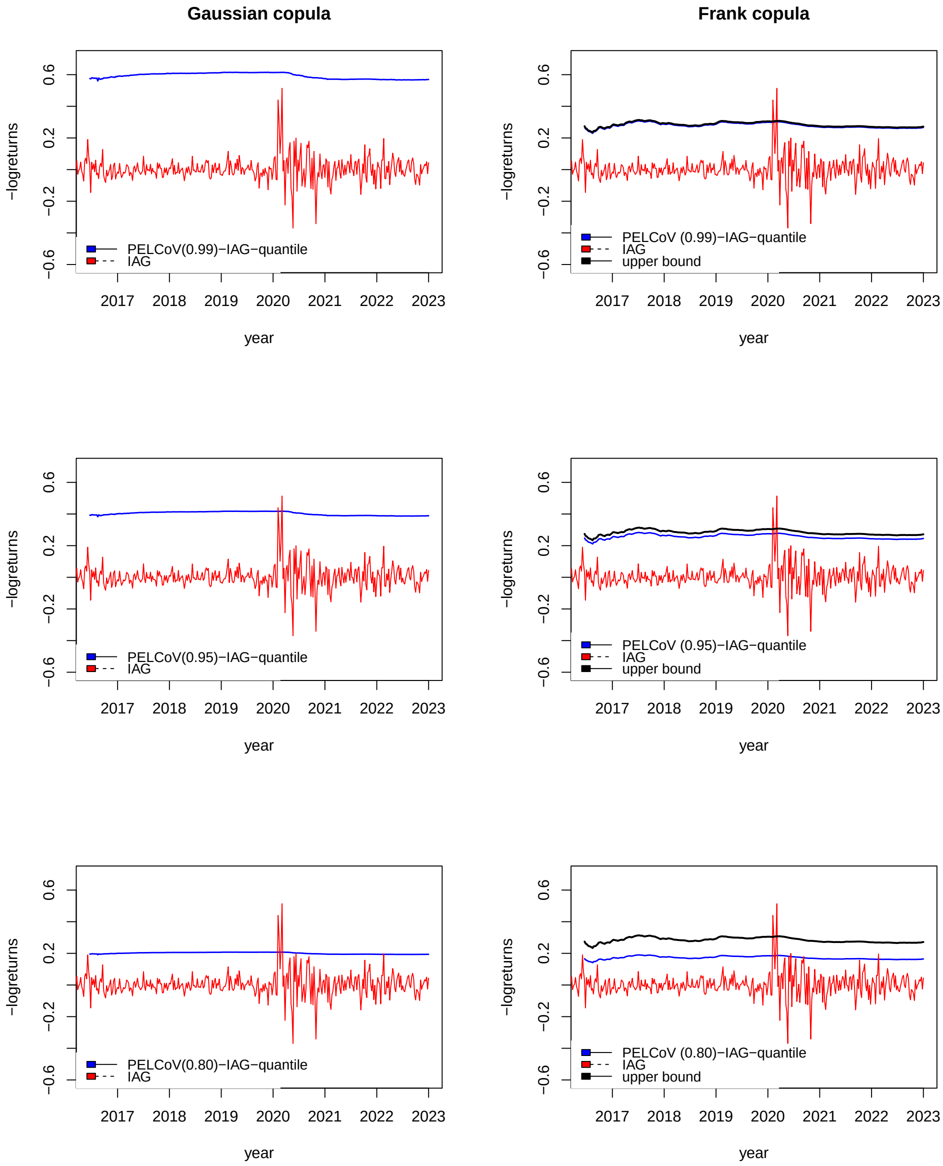

Figure 15: Given the bivariate time series (X_t, Y_t), the left-hand plots show the time series X_t (log-returns of IAG) together with their u_v-quantiles for $v = 0.99, 0.95$ and 0.80, respectively, under the assumption of a time-varying Gaussian copula. The right-hand plots show the time series X_t, together with their u_v-quantiles for $v = 0.99, 0.95$ and 0.80, respectively, and with their upper bounds (given by the u^*-quantiles) under the assumption of a time-varying Frank copula.

7 ~~Conclusions and further work~~ Discussion

We have explored a method for monitoring systemic risk that can be used to determine provisions in the framework of a prudential regulation model. In particular, we have provided conditions that guarantee the ordering between VaR and CoVaR, allowing us to understand which, among the two measures, is more or less conservative than the other.

The prudential regulation strategy described is based on determining the PELCoV in order to decide whether to replace the VaR by the CoVaR. It is important to remark that the PELCoV acts as a signal that indicates that the unconditional VaR underestimated spillover effects, but the PELCoV itself is not aimed to capture such effect. In case of extremal dependence (comonotonicity), since $Y = F_Y^{-1}(F_X(X))$, the PELCoV u_v is equal to v. Therefore, when considering families of copulas where the convergence to comonotonicity is determined by the convergence of a parameter, we would see that the PELCoV converges to v in terms of the dependence parameter. However, there are cases where such convergence is attained from below and others from above, which means that the PELCoV can be either increasing or decreasing in terms of the dependence parameter for different copulas and different values of v. For instance, considering the Gaussian, Gumbel Hoougard or Frank families of copulas, the PELCoV increases in their dependence parameter when $v > \frac{1}{2}$ and decreases if $v < \frac{1}{2}$. On the other hand, the PELCoV is constant for the Farlie–Gumbel–Morgenstern and decreases in the dependence parameter considering the Ali-Mikhail-Haq family of copulas. The different behaviours of the PELCoV considering these families of copulas serve to illustrate that the monotonicity of the PELCoV is influenced by the dependence structure itself and the considered level of risk v. That is, even if the PELCoV is uniquely determined by the copula, its size is not an indicator of any kind of the intensity of the dependence.

An important problem that remains to be addressed concerns the non-parametric inference on the PELCoV. Reasoning as in Li and Wang (2022), a natural candidate to estimate the PELCoV, given $v \in (0,1)$, is the u that solves the equation

$$\widehat{\mathrm{CoVaR}}_{v,u}\,[Y|X] = \widehat{\mathrm{VaR}}_v[Y], \tag{25}$$

where $\widehat{\mathrm{VaR}}_v[Y]$ and $\widehat{\mathrm{CoVaR}}_{v,u}$ are the empirical VaR and CoVaR, respectively. The empirical VaR is the sample quantile, given by $\widehat{\mathrm{VaR}}_v[Y] = X_{[i]}$, for $v \in \left(\frac{i-1}{n}, \frac{i}{n}\right], i = 1, ..., n$, where $X_{[1]} \leq X_{[2]} \leq ... \leq X_{[n]}$, are the order statistics of the data $X_1, ..., X_n$. Regarding CoVaR, there are in the literature different ways to estimate it. It can be obtained, for example, from models with time-varying second moments, by using maximum likelihood estimation or, as in Adrian and Brunnermeier (2016), using quantile regression. Unfortunately, the curve $u \mapsto \mathrm{CoVaR}_{v,u}(Y|X)$ is not, in general, strictly increasing. Fixed v, the equation $\mathrm{CoVaR}_{v,u}\,[Y|X] = \mathrm{VaR}_v[Y]$ may not have a solution or have more than one and we cannot guarantee the existence or uniqueness of the solution of equation (25).

An alternative way to estimate the PELCoV is by solving the equation $\partial_1 \widehat{C(u,v)} = u$. The most straightforward way to estimate $\partial_1 C(u,v)$ would be by means of the Markov kernel, as in Fuchs and Trutschnig (2020). However, as before, we need additional assumptions to ensure that the curve $u \mapsto \partial_1 C(u,v)$ is strictly increasing. This is our theoretical SSI property, an assumption whose empirical verification is open to further study.

As suggested by one of the referees, in a similar way to the Probability Equivalent Level between VaR and CoVaR (PELCoV) defined and studied here, a probability equivalent level could be also considered between the risk measures ES (Expected Shortfall) and CoES (Conditional Expected Shortfall); see, e.g., Sordo et al. (2018), for their definition and applications. Despite its analogous interpretation as an alert level, there is a significant difference between the new concept and the PELCoV: a probability equivalent level between ES and CoES does not only depend on the copula, but also on the marginals. Another difference is that, under the SSI assumption, we interpret the existence of an upper bound for this probability equivalent level in terms of the increasing convex order (instead of the stochastic order, as in the case of the PELCoV) between the random variables $[Y|X = \mathrm{VaR}_u[X]]$ and Y. This will be the topic for future research.

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A Appendix: additional computations

This appendix gives the details in the computation of $u^* = \lim_{v\to 1} u_v$ for the Frank family of copulas, whose density function is given by

$$c(u,v) = \frac{e^{(1+u+v)\theta}\left(e^{\theta}-1\right)\theta}{\left(e^{(u+v)\theta} - e^{\theta}\left(e^{u\theta}+e^{v\theta}-1\right)\right)^2}.$$

To find the analytical form of the PELCoV_v and the corresponding upper bound, let us first note that

$$\partial_1 C(u,v) = \frac{e^{\theta}\left(1-e^{\theta v}\right)}{e^{\theta(u+v)} + e^{\theta}\left(1-e^{\theta u}-e^{\theta v}\right)},$$

and that $\partial_1 C(u,v) > 0$ for $\theta > 0$, so that the copula is SSI for $\theta > 0$. Moreover, we can see that $\partial_1 C(u,v) \leq v$ if and only if

$$\frac{e^{\theta}\left(1-e^{\theta v}\right)}{v} \leqslant e^{\theta u}e^{\theta v} + e^{\theta} - e^{\theta u}e^{\theta} - e^{\theta(1+v)},$$

that is, if $\frac{e^{\theta}\left(1-e^{\theta v}\right)}{v} - e^{\theta} + e^{\theta(1+v)} \leqslant e^{\theta u}\left(e^{\theta v}-e^{\theta}\right)$, which holds when

$$\frac{1}{\theta}\ln\left(\frac{e^{\theta}\left(1-e^{\theta v}\right) - ve^{\theta} + ve^{\theta(1+v)}}{v\left(e^{\theta v}-e^{\theta}\right)}\right) \geq u.$$

Therefore, the PELCoV_v, u_v, is given by:

$$u_v = \frac{1}{\theta}\ln\left(\frac{e^{\theta}\left(1-e^{\theta v}\right)(1-v)}{v\left(e^{\theta v}-e^{\theta}\right)}\right).$$

Then, for $\theta > 0$, u_v increases in v if $\frac{\left(1-e^{\theta v}\right)(1-v)}{v\left(e^{\theta v}-e^{\theta}\right)}$ increases in v. Considering $f(v) = \left(1-e^{\theta v}\right)(1-v)$ and $g(v) = v\left(e^{\theta v}-e^{\theta}\right)$, by L'Hôpital's Rule of Monotonicity (Anderson et al., 1997), it would suffice to observe that $\frac{f'(v)}{g'(v)}$ increases in $v \in (0,1)$. To this aim, note that, given $\theta > 0$,

$$\frac{d}{dv}\left(\frac{f'(v)}{g'(v)}\right) = \frac{e^{v\theta}\theta\left(2+e^{v\theta}\theta+v\theta+e^{\theta}(-2+\theta-v\theta)\right)}{\left(e^{\theta}-e^{v\theta}(1+v\theta)\right)^2}$$

is positive for all $v \in (0,1)$ if

$$h(v) = 2 + e^{v\theta}\theta + v\theta + e^{\theta}(-2+\theta-v\theta)$$

is positive for all $v \in (0,1)$. One can easily check that $h(v)$ is continuous and derivable and that $h'(v) = \theta - e^{\theta}\theta + e^{\theta v}\theta^2$, which is zero for $v_0 = \frac{1}{\theta}\ln\left(\frac{e^{\theta}-1}{\theta}\right)$.

Since $h''(v_0) = \left(e^\theta - 1\right)\theta^2 > 0$, v_0 is a minimum. Therefore, it suffices to verify that $h(0)$, $h(1)$ and $h(v_0)$ are positive when $\theta > 0$. It is easy to see that $h(0) = h(1) = 2 + e^\theta(\theta - 2) + \theta > 0$ and that $h(v_0) = 1 + e^\theta(\theta - 1) + \left(e^\theta - 1\right)\ln\left(\frac{\theta}{e^\theta - 1}\right)$. As $\ln(x) > 1 - x$ for all $x \in (0,1)$, letting $x = \frac{\theta}{e^\theta - 1}$, one gets that

$$h(v_0) > 1 + e^\theta(\theta - 1) + \left(e^\theta - 1\right)\left(1 - \frac{\theta}{e^\theta - 1}\right) = \left(e^\theta - 1\right)\theta.$$

Thus, for the Frank copula, u_v is increasing with respect to v. In order to find u^*, one should only consider the limit when v tends to 1. Using continuity and L'Hôpital's result, as $u_v = \frac{1}{\theta}\ln\left(\frac{e^\theta\left(1 - e^{\theta v}\right)(1 - v)}{v\left(e^{\theta v} - e^\theta\right)}\right)$, it follows

$$\begin{aligned}
\lim_{v \to 1} u_v &= \frac{1}{\theta} \cdot \ln\left(\lim_{v \to 1} \frac{e^\theta\left(1 + e^{v\theta}(-1 + \theta - v\theta)\right)}{e^\theta - e^{v\theta}(1 + v\theta)}\right) \\
&= \frac{1}{\theta}\ln\left(\frac{e^\theta\left(1 - e^\theta\right)}{e^\theta(1 - 1 - \theta)}\right) \\
&= \frac{1}{\theta}\ln\left(\frac{e^\theta - 1}{\theta}\right).
\end{aligned}$$

Therefore, $u^* = \frac{1}{\theta}\ln\left(\frac{e^\theta - 1}{\theta}\right)$.