A convergence result for a Stefan problem with phase relaxation

A CONVERGENCE RESULT FOR A STEFAN PROBLEM WITH PHASE RELAXATION

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Dedicated to Pierluigi Colli on the occasion of his 65^{th} *birthday*

Abstract. In this paper we consider the model of phase relaxation introduced in [22], where an asymptotic analysis is performed toward an integral formulation of the Stefan problem when the relaxation parameter approaches zero. Assuming the natural physical assumption that the initial condition of the phase is constrained, but taking more general boundary conditions, we prove that the solution of this relaxed model converges in a stronger way to the solution of the classical weak Stefan problem.

1. Introduction

Modelling phase-transition phenomena in a substance attaining two phases (e.g. solid and liquid) in a bounded domain Ω of the space during the time interval $[0,T]$, one is led to the the energy balance equation

$$\frac{\partial}{\partial t}(\theta + \chi) - \Delta\theta = g \qquad \text{in } Q := \Omega \times [0,T], \tag{1.1}$$

where for simplicity we have normalized to 1 all the physical constants. Here the unknowns $\theta = \theta(t,x)$ and $\chi = \chi(t,x)$ stand respectively for the temperature and the phase function: $(1-\chi)/2$ represents the solid concentration of the solid portion, $(1+\chi)/2$ is the concentration of the liquid portion, and $-1 \le \chi \le 1$, so that it is allowed the existence of mushy regions where the substance is a mixture of the solid and liquid parts (cf., e.g., [21, p. 99]). In order to describe the evolution of the system, an equation relating θ and χ is needed. If $\theta = 0$ is the equilibrium temperature at which the two phases can coexist, then we can take the *equilibrium condition of Stefan type* (see, e.g., [21] and the references therein)

$$\chi \in \operatorname{sign}(\theta) \qquad \text{in } Q, \tag{1.2}$$

where sign denotes the multivalued sign graph (i.e. $\operatorname{sign}(r) := -1$ if $r < 0$, $\operatorname{sign}(r) := [-1,1]$ if $r = 0$, $\operatorname{sign}(r) := 1$ if $r > 0$). Problem (1.1)–(1.2) is usually called *Stefan problem*. Notice that (1.2) could be written in the equivalent form

$$\operatorname{sign}^{-1}(\chi) \ni \theta \qquad \text{in } Q, \tag{1.3}$$

sign^{-1} being the inverse relation of the multivalued sign graph ($\operatorname{sign}^{-1}(r) := 0$ if $r \in\,]-1,1[$, $\operatorname{sign}^{-1}(-1) :=]-\infty, 0]$, $\operatorname{sign}^{-1}(1) := [0,\infty[$).

If dynamic supercooling or superheating effects are to be taken into account, then condition (1.3) is usually replaced by the following *relaxation dynamics* for the phase variable χ (cf., e.g.,

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[20, 21] and their references)

$$\varepsilon \frac{\partial \chi}{\partial t} + \operatorname{sign}^{-1}(\chi) \ni \theta \qquad \text{in } Q, \tag{1.4}$$

ε being a small kinetic positive parameter. Alternatively, the relaxation dynamics can also be modeled by the inclusion

$$\varepsilon \frac{\partial \chi}{\partial t} + \chi \in \operatorname{sign}(\theta) \qquad \text{in } Q, \tag{1.5}$$

which is not equivalent to (1.4).

The Stefan problem (1.1)–(1.2) and the Stefan problems with phase relaxation (1.1), (1.4) and (1.1), (1.5) have been extensively studied: see, e.g., [7, 8, 21] for (1.1)–(1.2), [7, 20, 5, 8, 21] for (1.1), (1.4), and [22, 15] for (1.1), (1.5). In particular in [20], uniqueness and existence of (1.1), (1.4), coupled with suitable initial-boundary conditions, are proved in the framework of Sobolev spaces, and the solution of the relaxed problem is shown to converge, in a suitable topology, to the solution of the problem (1.1)–(1.2) as $\varepsilon \searrow 0$. Problem (1.1), (1.5) instead is dealt with in [15] where existence, uniqueness, and asymptotic analysis of the Stefan prblem are studied within the same Sobolev setting. Let us also observe that the Stefan problem with phase relaxation can also be studied taking into account a hyperbolic energy balance yielding a finite speed of propagation for the temperature field (see, e.g., [20, 18, 19, 6, 16, 17]).

Though models (1.1), (1.4) and (1.1), (1.5) are very natural from the analytic point of view, they have some modelling drawbacks. Indeed, as observed in [22], in (1.4) the rate of the phase χ does not depend on χ, because the term sign^{-1} only represents a constraint for the phase, and in (1.5) the phase depends only on the sign of the temperature θ. One would expect instead that the rate of χ decays as χ approaches 1 and that it also decays as θ tends to 0. In order to overcome this modelling issue in [22] the following relaxation dynamics is proposed:

$$\varepsilon \frac{\partial \chi}{\partial t} = \psi(\theta, \chi) \qquad \text{in } Q \tag{1.6}$$

for a suitable class of regular functions $\psi : \mathbb{R}^2 \to \mathbb{R}$ which are increasing in θ, decreasing in χ, and such that $\psi(\theta, \chi) = 0$ if and only if $\chi \in \operatorname{sign}(\theta)$, or more generally

$$\psi(\theta, \chi) = 0 \;\text{ if and only if }\; \chi \in \alpha(\theta), \tag{1.7}$$

where α is a general linearly bounded maximal monotone operator in $\mathbb{R}$, i.e. a continuous increasing graph in $\mathbb{R}^2$ (cf., e.g, [4], however in the next section we will provide all the precise definitions needed in the paper). An example, provided in [22], is

$$\psi(\theta, \chi) = p(\theta^+) \frac{1 - \chi}{2} + p(-\theta^-) \frac{1 - \chi}{2},$$

where $\theta^+ = \max\{\theta, 0\}$, $\theta^- = \max\{-\theta, 0\}$, and $p : \mathbb{R} \longrightarrow [-1, 1]$ is a function such that $\pi_+ = p(\theta^+)$ (respectively $\pi_- = -p - (\theta^-)$) represents the probability of melting a solid particle (respectively "crystallizing a liquid particle") in the unit time, with $p(r)r > 0$ for every $r \neq 0$.

In [22] the model of phase relaxation (1.1), (1.6) is coupled with zero Dirichlet boundary conditions for the temperature, and it is shown that the solution of the relaxed problem (1.1), (1.6) converges in suitable way to a solution of a rather weak formulation of the Stefan problem (1.1), (1.2). To be more precise it is shown that as ε approaches zero along a suitable subsequence, the solution of the relaxed problem converges to a solution of a time-integral formulation of the Stefan problem (1.1), (1.2), and in general this weaker formulation has not a unique solution. The setting adopted in [22] makes the proofs nontrivial and L^1-techniques are needed.

The aim of our present paper is to perform the asymptotic analysis as ε approaches zero of the model of phase relaxation (1.1), (1.6) in the case of a *bounded* graph α (which is *physically very natural*, e.g. $\alpha = \operatorname{sign}$), but assuming *more general* boundary conditions for the temperature θ. In this way we are able to use L^2-techniques, we obtain a stronger convergence along the entire family ε (and not along a subsequence), and we find that the limit problem is actually the unique solution of (1.1), (1.2).

To be more precise concerning our results, we assume that α is a bounded maximal monotone operator $\alpha : \mathbb{R} \to \mathscr{P}(\mathbb{R})$ and we supply the system (1.1), (1.6) with the rather general initial-boundary conditions described as follows: letting $\{\Gamma_0, \Gamma_1\}$ be a partition of the boundary of Ω into two measurable sets, we take

$$\theta = \theta_D \qquad \text{on } \Gamma_0 \times [0,T], \tag{1.8}$$

$$\partial_{\mathbf{n}}\theta = -\theta_N \qquad \text{on } \Gamma_1 \times [0,T], \tag{1.9}$$

$$\theta(0,\cdot) + \chi(0,\cdot) = \theta_0 + \chi_0 \qquad \text{in } \Omega, \tag{1.10}$$

where θ_D, θ_N, θ_0, χ_0 are given functions and $\mathbf{n}$ is the outward unit vector normal to the boundary of Ω. If we assume that θ_D is a sufficiently smooth function defined on the cylinder Q, that $\theta_N : \Gamma_1 \times [0,T] \longrightarrow \mathbb{R}$ is regular enough, and that there is a function $u : Q \longrightarrow \mathbb{R}$ such that $u = \Delta u$ in Q, $u = \theta_D$ on $\Gamma_0 \times [0,T]$, and $-\partial_{\mathbf{n}} u = \theta_N$ on $\Gamma_1 \times [0,T]$ and we set $\overline{\theta}_0 := \theta_0 - u(\cdot, 0)$. Hence we rewrite all the equations in the new unknown $\overline{\theta} := \theta - u$ so that problem (1.1), (1.6), (1.8)–(1.10) reads, writing again θ instead of $\overline{\theta}$ for simplicity,

$$\frac{\partial}{\partial t}(\theta + \chi) - \Delta\theta = g - \frac{\partial u}{\partial t} + \Delta u \qquad \text{in } Q, \tag{1.11}$$

$$\varepsilon \frac{\partial \chi}{\partial t} = \psi(\theta + u, \chi) \qquad \text{in } Q, \tag{1.12}$$

$$\theta = 0 \qquad \text{on } \Gamma_0 \times [0,T], \tag{1.13}$$

$$\partial_{\mathbf{n}}\theta = 0 \qquad \text{on } \Gamma_1 \times [0,T], \tag{1.14}$$

$$\theta(0,\cdot) + \chi(0,\cdot) = \theta_0 + \chi_0 \qquad \text{in } \Omega. \tag{1.15}$$

This formulation has the advantage that the boundary conditions for θ are homogeneus and the wider generality is incorporated in the u-terms in the right-hand side of the balance equation and in the non-linearity ψ. As described above, by means of L^2-techinques, we will prove that the only solution of (1.11)–(1.15) converges to the solution of (1.11), (1.13)–(1.15) coupled with

$$\chi \in \alpha(\theta + u) \qquad \text{in } Q$$

as $\varepsilon \searrow 0$ (not only along a suitable subsequence).

The plan of the paper is the following. In Section 2 we list the precise assumptions on the data of the problem and we state our main results. In Section 3 we analyze the relaxed problem (1.11)–(1.15). In the final Section 4 we perform the asymptotic analysis as the relaxation parameter ε goes to zero.

2. Main results

In this section we give the variational formulation of the problems presented in the Introduction and we state our main results.

The set of integers greater than or equal to 1 will be denoted by $\mathbb{N}$. Given $p \in [1, \infty[$, a measure space D, and a real Banach space B, then the space of B-valued functions on D which are p-integrable will be denoted by $L^p(D; B)$; the vector space of essentially bounded B-valued functions on D is denoted by $L^\infty(D; B)$. These spaces will be endowed with their natural norms defined by $\|v\|_{L^p(D;B)} := \left(\int_D \|v(x)\|_B^p \,\mathrm{d}x\right)^{1/p}$ if $p \in [1, \infty[$, and by $\|v\|_{L^\infty(D;B)} := \inf_w \sup_{x \in D} \|w(x)\|_B$, where the infimum is taken over all bounded μ-measurable functions w equal to v μ-almost everywhere, μ being the measure on D. If $p = 2$ and $B = E$ is a Hilbert space then this norm is induced by the inner product $(v_1, v_2)_{L^2(D;E)} = \int_D (v_1(x), v_2(x))_E \,\mathrm{d}x$, where $(\cdot, \cdot)_E$ is the inner product in E. For the theory of integration of vector valued functions we refer, e.g., to [12, Chapter VI]. We will simply set $L^p(D) := L^p(D;\mathbb{R})$ for $p \in [1,\infty]$. If $n \in \mathbb{N}$, the n-dimensional Lebesgue measure of a set $D \subseteq \mathbb{R}^n$ will be denoted by $|D|$. In the following the locutions "almost every" and "almost everywhere" ("a.e.") will always refer to the Lebesgue measure. If $D \subseteq \mathbb{R}^n$ is open, we will make use of the Sobolev space $H^1(D) := \{v \in L^2(D) \ : \ \partial_i v \in L^2(D),\ i = 1,\dots,n\}$ where $\partial_i v$ denotes the partial derivative of v with respect to the i-th variable in the sense of distributions (cf., e.g., [1]). The symbol ∇ will denote the distributional gradient operator so that $H^1(D)$ is a real Hilbert space if it is endowed with the inner product

$$(v_1, v_2)_{H^1(D)} := \int_D v_1(x)v_2(x) + \int_D \nabla v_1(x) \cdot \nabla v_2(x), \qquad v_1, v_2 \in H^1(D), \tag{2.1}$$

which induces the usual norm $\|\cdot\|_{H^1(D)}$. If ∂D is of Lipschitz class and if Γ_0 is open in ∂D, then the restriction operator $C^\infty(\overline{D}) \longrightarrow C(\Gamma_0) : v \longmapsto v|_{\Gamma_0}$ can be uniquely continuosly extended to a linear continuous operator $\gamma_{\Gamma_0} : H^1(D) \longrightarrow L^2(\Gamma_0)$, where Γ_0 is endowed with the $(n-1)$-dimensional surface (Hausdorff) measure (see, e.g., [13, 11]). The notation $v|_{\Gamma_0} := \gamma_{\Gamma_0}(v)$ is commonly used for a function $v \in H^1(D)$. If $a, b \in \mathbb{R}$, $a < b$, we set $L^p(a,b;B) := L^p(]a,b[\,;B)$ for $p \in [1,\infty]$ and, if $B = E$ is a Hilbert space, we define $H^1(a,b;E) := \{f \in L^2(a,b;E) \ : \ f' \in L^2(a,b;E)\}$, where g' denotes the distributional derivative of a function $g : \,]a,b[\, \longrightarrow E$, and the Hilbertian norm defined by $\|f\|^2_{H^1(a,b;E)} := \|f\|^2_{L^2(a,b;E)} + \|f'\|^2_{L^2(a,b;E)}$ is used. For the main properties of the Sobolev space $H^1(a,b;E)$ we refer, e.g., to [4, Appendix]. Now we can present our first set of assumptions.

Assumptions 2.1. The following conditions will be used in the paper.

(H1) $\Omega \subseteq \mathbb{R}^n$ is a bounded open connected set with Lipschitz boundary $\Gamma := \partial\Omega$. Γ_0 and Γ_1 are open subsets of Γ such that $\Gamma_0 \cap \Gamma_1 = \varnothing$. If $\overline{\Gamma}_0$ and $\overline{\Gamma}_1$ denote the closures of Γ_0 and Γ_1 in Γ, then we assume that $\overline{\Gamma}_0 \cup \overline{\Gamma}_1 = \Gamma$ and that $\overline{\Gamma}_0 \cap \overline{\Gamma}_1$ is of Lipschitz class. We define

$$H := L^2(\Omega), \tag{2.2}$$

$$V := H^1_{\Gamma_0}(\Omega) := \{v \in H^1(\Omega) \ : \ v|_{\Gamma_0} = 0\}, \tag{2.3}$$

endowed with their usual inner products, in particular V is endowed with the inner product induced by (2.1). If V' denotes the topological dual space of V, we define the linear continuous operator $\mathsf{A} : V \longrightarrow V'$ by

$${}_{V'}\langle \mathsf{A}v_1, v_2\rangle_V := \int_\Omega \nabla v_1 \cdot \nabla v_2, \quad v_1, v_2 \in V, \tag{2.4}$$

where ${}_{V'}\langle\cdot,\cdot\rangle_V$ denotes the duality between V' and V. The final time of the evolution will be denoted by $T > 0$ and we set $Q := \Omega \times]0,T[$.

(H2) We are given

$$\psi : \mathbb{R}^2 \longrightarrow \mathbb{R} \ \text{ Lipschitz continuous with Lipschitz constant } L. \tag{2.5}$$

(H3) For every $\varepsilon > 0$ we are given

$$f_\varepsilon \in L^1(0,T;H) + L^2(0,T;V'), \qquad u_\varepsilon \in L^2(Q), \tag{2.6}$$

where we recall that

$$L^1(0,T;H) + L^2(0,T;V') := \{h = h_1 + h_2 \ : \ h_1 \in L^1(0,T;H),\ h_2 \in L^2(0,T;V')\}$$

endowed with the norm

$$\|h\|_{L^1(0,T;H)+L^2(0,T;V')} := \inf_{h=h_1+h_2} \|h_1\|_{L^1(0,T;H)} + \|h_2\|_{L^2(0,T;V')},$$

where the infimum is taken over all the decompositions $h = h_1 + h_2$ with $h_1 \in L^1(0,T;H)$, $h_2 \in L^2(0,T;V')$.

(H4) For every $\varepsilon > 0$ we are given

$$\theta_{0\varepsilon} \in L^2(\Omega), \qquad \chi_{0\varepsilon} \in L^2(\Omega). \tag{2.7}$$

Remark 2.1. Let us observe that in assumptions (H1), (H2) we do not require that the $(n-1)$-dimensional Hausdorff measure of Γ_0 is strictly positive.

Let us recall that $V \subset H \subset V'$ with dense and compact embeddings, where V' is endowed with its dual norm induced by V and we have identified H with its dual, thus

$$_{V'}\langle e, v\rangle_V = (e, v)_H \qquad \forall e \in H, \quad v \in V.$$

We will also need the following second set of assumptions:

Assumptions 2.2. The following conditions will be used in the paper.

(A1) $\alpha : \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is maximal monotone, i.e. if $D(\alpha) := \{r \in \mathbb{R} \ : \ \alpha(r) \neq \varnothing\}$ then

$$(s_1 - s_2)(r_1 - r_2) \geq 0 \qquad \forall r_1, r_2 \in D(\alpha), \quad s_1 \in \alpha(r_1), s_2 \in \alpha(r_2),$$

and

$$(\sigma - s)(\rho - r) \geq 0, \quad s \in \alpha(r), \quad r \in D(\alpha) \quad \Longrightarrow \quad \sigma \in \alpha(\rho).$$

We also assume that α is "bounded", i.e. there is a constant $M > 0$ such that

$$|s| \leq M \qquad \forall r \in D(\alpha), \quad \forall s \in \alpha(r). \tag{2.8}$$

(A2) $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is the Lipschitz continuous function given in (H2) of Assumptions 2.1 satisfying (2.5) and the following monotonicity condition:

$$\big[\psi(\tau_1, \chi) - \psi(\tau_2, \chi)\big](\tau_1 - \tau_2) \geq 0 \qquad \forall \tau_1, \tau_2, \chi \in \mathbb{R}, \tag{2.9}$$

$$\big[\psi(\tau, \chi_1) - \psi(\tau, \chi_2)\big](\chi_1 - \chi_2) \leq 0 \qquad \forall \chi_1, \chi_2, \tau \in \mathbb{R}, \tag{2.10}$$

i.e. $\psi(\cdot, \chi)$ is increasing for every $\chi \in \mathbb{R}$ and $\psi(\tau, \cdot)$ is decreasing for every $\tau \in \mathbb{R}$.

(A3) We assume the following "compatibility" condition between α and ψ:

$$\psi(\tau, \chi) = 0 \quad \Longleftrightarrow \quad \chi \in \alpha(\tau) \qquad \forall (\tau, \chi) \in \mathbb{R}^2. \tag{2.11}$$

(A4) We are given

$$f \in L^1(0,T;H) + L^2(0,T;V'), \qquad u \in L^2(Q), \tag{2.12}$$

(A5) We are given

$$\theta_0 \in L^2(\Omega), \qquad \chi_0 \in L^\infty(\Omega), \tag{2.13}$$

such that

$$\chi_0(x) \in \alpha\big(\theta_0(x) + u(0,x)\big) \qquad \text{for a.e. } x \in \Omega. \tag{2.14}$$

Remark 2.2. Let us observe that (2.14) is equivalent to condition (3.3) in [22].

Let us recall that under condition (H1) of Assumptions 2.1 and conditions (A1), (A4), (A5) of Assumptions 2.2, it is well-know that the Stefan problem admits a unique solution, i.e. there exists a unique pair $(\theta, \chi) : Q \longrightarrow \mathbb{R}^2$ such that

$$\theta \in L^2(0,T;V) \cap H^1(0,T;H), \tag{2.15}$$
$$\chi \in L^\infty(Q), \tag{2.16}$$
$$\theta + \chi \in H^1(0,T;V') \tag{2.17}$$
$$(\theta + \chi)'(t) + \mathsf{A}\theta(t) = f(t) \qquad \text{in } V', \text{ for a.e. } t \in \,]0,T[, \tag{2.18}$$
$$\chi(t,x) \in \alpha\big(\theta(t,x) + u(t,x)\big) \qquad \text{for a.e. } (t,x) \in Q, \tag{2.19}$$
$$(\theta + \chi)(0) = \theta_0 + \chi_0 \qquad \text{in } V'. \tag{2.20}$$

For a proof we refer, for instance, to [5, 8, 21].

Now we state the weak formulation of the model of phase relaxation (1.11)–(1.15).

Problem $(\mathbf{P}_\varepsilon)$. Assume that $\varepsilon > 0$ and that Assumptions 2.1 are satisfied. Find a pair of functions $(\theta_\varepsilon, \chi_\varepsilon) : Q \longrightarrow \mathbb{R}^2$ satisfying the following conditions:

$$\theta_\varepsilon \in L^2(0,T;V) \cap H^1(0,T;V'), \tag{2.21}$$
$$\chi_\varepsilon \in H^1(0,T;H), \tag{2.22}$$
$$\theta_\varepsilon'(t) + \chi_\varepsilon'(t) + \mathsf{A}\theta_\varepsilon(t) = f_\varepsilon(t) \qquad \text{in } V', \text{ for a.e. } t \in \,]0,T[, \tag{2.23}$$
$$\varepsilon\chi_\varepsilon'(t,x) = \psi\big(\theta_\varepsilon(t,x) + u_\varepsilon(t,x), \chi_\varepsilon(t,x)\big) \qquad \text{for a.e. } (t,x) \in Q, \tag{2.24}$$
$$\theta_\varepsilon(0) = \theta_{0\varepsilon}, \qquad \text{a.e. in } \Omega, \tag{2.25}$$
$$\chi_\varepsilon(0) = \chi_{0\varepsilon}, \qquad \text{a.e. in } \Omega. \tag{2.26}$$

Let us now introduce a general notation which will hold throughout the paper.

Definition 2.1. *For a real Banach space B, and for a function $v \in L^1(0,T;B)$ we define $\widehat{v} : [0,T] \longrightarrow B$ by setting*

$$\widehat{v}(t) := \int_0^t v(s)\,\mathrm{d}s, \qquad t \in [0,T]. \tag{2.27}$$

We also state the following *Baiocchi-Duvaut-Frémond formulation* of the classical Stefan problem (cf. [3, 9, 10]).

Problem $(\mathbf{P})$. Find a pair of functions $(\theta, \chi) : Q \longrightarrow \mathbb{R}^2$ satisfying the following conditions:

$$\widehat{\theta} \in L^\infty(0,T;V) \cap H^1(0,T;H), \tag{2.28}$$
$$\chi \in L^\infty(Q), \tag{2.29}$$
$$\theta(t) + \chi(t) + A\widehat{\theta}(t) = \widehat{f}(t) + \theta_0 + \chi_0 \qquad \text{in } V', \text{ for a.e. } t \in \,]0,T[, \tag{2.30}$$
$$\chi(t,x) \in \alpha\big(\theta(t,x) + u(t,x)\big) \qquad \text{for a.e. } (t,x) \in Q. \tag{2.31}$$

A pair (θ, χ) satisfying (2.28)–(2.31) is also called a *solution of the Stefan problem in the sense of Baiocchi-Duvaut-Frémond.*

Now we state the main results of this paper.

Theorem 2.1. *Assume that $\varepsilon > 0$ and that Assumptions 2.1 hold. Then Problem $(\mathbf{P}_\varepsilon)$ admits a unique solution. Moreover it is well-posed in the sense specified by Proposition 3.1 below.*

Theorem 2.2. *If Assumptions 2.1 and 2.2 are satisfied, then there exists a unique solution* (θ,χ) *of Problem* **(P)**, *and* $(\widehat{\theta},\theta,\chi)$ *is the weak-star limit in* $L^\infty(0,T;V)\times L^2(0,T;H)\times L^\infty(0,T;H)$ *of the sequence* $((\widehat{\theta}_\varepsilon,\theta_\varepsilon,\chi_\varepsilon))_\varepsilon$ *as* $\varepsilon\searrow 0$, *where* $(\theta_\varepsilon,\chi_\varepsilon)$ *is the solution of* $(\mathbf{P}_\varepsilon)$ *and it is assumed that* $\chi_{0\varepsilon}\in L^\infty(\Omega)$ *for every* $\varepsilon>0$ *and*

$$f_\varepsilon \to f \qquad \text{in } L^1(0,T;H)+L^2(0,T;V'), \tag{2.32}$$

$$u_\varepsilon \to u \qquad \text{in } L^2(Q), \tag{2.33}$$

$$\theta_{0\varepsilon} \to \theta_0 \qquad \text{in } H, \tag{2.34}$$

$$\chi_{0\varepsilon} \to \chi_0 \qquad \text{in } L^\infty(\Omega) \tag{2.35}$$

as $\varepsilon\searrow 0$. *Moreover* (θ,χ) *is also the unique solution of the Stefan problem* (2.15)–(2.20).

Remark 2.3. Let us remark that in Theorem 2.2 we have that the whole sequence $(\theta_\varepsilon,\chi_\varepsilon)$ converges to (θ,χ).

Remark 2.4. Since the usual weak formulation of the Stefan problem is stronger than the Baiocchi-Duvaut-Frémond one, from the uniqueness property stated in Theorem 2.2 we deduce that the solution (θ,χ) of **(P)** belongs to $[L^2(0,T;V)\cap L^\infty(0,T;H)]\times L^\infty(Q)$ and satisfies (2.15)–(2.20).

3. The problem with phase relaxation

Let us start by proving a continuous dependence result for Problem $(\mathbf{P}_\varepsilon)$.

Proposition 3.1. *Under Assumptions 2.1 there exists a constant* C_ε, *depending on* T *and on* ε, *such that if*

$$f_{\varepsilon i}\in L^1(0,T;H)+L^2(0,T;V'),\quad u_{\varepsilon i}\in L^2(Q),\quad \theta_{0\varepsilon i}\in H,\quad \chi_{0\varepsilon i}\in H,\qquad i=1,2, \tag{3.1}$$

and if the pair $(\theta_{\varepsilon i},\chi_{\varepsilon i})$ *satisfies* (2.21)–(2.26) *with* θ_ε, χ_ε, f_ε, u_ε, $\theta_{0\varepsilon}$, *and* $\chi_{0\varepsilon}$ *replaced respectively by* $\theta_{\varepsilon i}$, $\chi_{\varepsilon i}$, $f_{\varepsilon i}$, $u_{\varepsilon i}$, $\theta_{0\varepsilon i}$, *and* $\chi_{0\varepsilon i}$, $i=1,2$, *then*

$$\begin{aligned}\|\theta_{\varepsilon 1}(t)-\theta_{\varepsilon 2}(t)\|_H^2+\|\chi_{\varepsilon 1}(t)-\chi_{\varepsilon 2}(t)\|_H^2 &\le C_\varepsilon\left(\|\theta_{0\varepsilon 1}-\theta_{0\varepsilon 2}\|_H^2+\|\chi_{0\varepsilon 1}-\chi_{0\varepsilon 2}\|_H^2\right)\\ &+C_\varepsilon\left(\|f_{\varepsilon 1}-f_{\varepsilon 2}\|^2_{L^1(0,T;H)+L^2(0,T;V')}+\|u_{\varepsilon 1}-u_{\varepsilon 2}\|^2_{L^2(Q)}\right)\end{aligned} \tag{3.2}$$

for every $t\in[0,T]$. *Let us remark that* C_ε *does not depend on* $f_{\varepsilon i}$, $\theta_{0\varepsilon i}$, $\chi_{0\varepsilon i}$, $(\theta_{\varepsilon i},\chi_{\varepsilon i})$, $i=1,2$. *In particular Problem* $(\mathbf{P}_\varepsilon)$ *has at most one solution.*

Proof. Let us set $\widetilde{\theta}_\varepsilon:=\theta_{\varepsilon 1}-\theta_{\varepsilon 2}$, $\widetilde{\chi}_\varepsilon:=\chi_{\varepsilon 1}-\chi_{\varepsilon 2}$, $\widetilde{f}_\varepsilon:=f_{\varepsilon 1}-f_{\varepsilon 2}$, $\widetilde{u}_\varepsilon:=u_{\varepsilon 1}-u_{\varepsilon 2}$, $\widetilde{\theta}_{0\varepsilon}:=\theta_{0\varepsilon 1}-\theta_{0\varepsilon 2}$, and $\widetilde{\chi}_{0\varepsilon}:=\chi_{0\varepsilon 1}-\chi_{0\varepsilon 2}$. Let $f_{\varepsilon k}=f_{\varepsilon kH}+f_{\varepsilon kV}$ be an arbitrary decomposition of $f_{\varepsilon k}$ with $f_{\varepsilon kH}\in L^1(0,T;H)$ and $f_{\varepsilon kV}\in L^2(0,T;V')$ for $k=1,2$, and set $\widetilde{f}_{\varepsilon H}:=f_{\varepsilon 1H}-f_{\varepsilon 2H}$, $\widetilde{f}_{\varepsilon V}:=f_{\varepsilon 1V}-f_{\varepsilon 2V}$.

Moreover for simplicity let us omit the subscript ε throughout the reminder of this proof. Let us fix $t\in[0,T]$ and let us start by testing the difference of the energy balance equations for θ_1 and θ_2 by $\widetilde{\theta}$ and integrate over $[0,t]$, i.e. we consider the difference of the equations (2.23) with θ and χ replaced respectively by θ_i and χ_i, $i=1,2$, we apply it to $\widetilde{\theta}$ and we integrate over $[0,t]$ with $t\in[0,T]$. Using (2.25) we infer that

$$\begin{aligned}&\frac{1}{2}\|\widetilde{\theta}(t)\|_H^2+\int_0^t\!\!\int_\Omega \widetilde{\chi}'(s,x)\widetilde{\theta}(s,x)\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\!\!\int_\Omega|\nabla\widetilde{\theta}(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s\\ &=\frac{1}{2}\|\widetilde{\theta}_0\|_H^2+\int_0^t\!\!\int_\Omega \widetilde{f}_H(s,x)\widetilde{\theta}(s,x)\,\mathrm{d}x\,\mathrm{d}s+\int_0^t {}_{V'}\langle\widetilde{f}_V(s),\widetilde{\theta}(s)\rangle_V\,\mathrm{d}s,\end{aligned} \tag{3.3}$$

therefore using the elementary Young inequality

$$\frac{1}{2}\|\widetilde{\theta}(t)\|_H^2 + \int_0^t \int_\Omega \widetilde{\chi}'(s,x)\widetilde{\theta}(s,x)\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\int_0^t \int_\Omega |\nabla\widetilde{\theta}(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s$$
$$\leq \frac{1}{2}\|\widetilde{\theta}_0\|_H^2 + \frac{1}{2}\int_0^t \|\widetilde{f}_V(s)\|_{V'}^2\,\mathrm{d}s + \int_0^t \|\widetilde{f}_H(s)\|_H\|\widetilde{\theta}(s)\|_H\,\mathrm{d}s + \frac{1}{2}\int_0^t \|\widetilde{\theta}(s)\|_H^2\,\mathrm{d}s. \tag{3.4}$$

Exploiting equation (2.24) for the phase relaxation and the Lipschitz continuity (2.5) of ψ, we find $C_1 > 0$ depending on ε, but independent of θ_i, χ_i, u_i, θ_{0i}, χ_{0i}, f_i, such that (omitting for simplicity the integration variables s and x in some lines)

$$\int_0^t \int_\Omega \widetilde{\chi}'(s,x)\widetilde{\theta}(s,x)\,\mathrm{d}x\,\mathrm{d}s$$
$$= \frac{1}{\varepsilon}\int_0^t \int_\Omega \big[\psi(\theta_1+u_1,\chi_1) - \psi(\theta_2+u_2,\chi_2)\big]\widetilde{\theta}\,\mathrm{d}x\,\mathrm{d}s$$
$$\geq -\int_0^t \int_\Omega \frac{L}{\varepsilon}\left(|\widetilde{\theta}+\widetilde{u}| + |\widetilde{\chi}|\right)|\widetilde{\theta}|\,\mathrm{d}x\,\mathrm{d}s$$
$$\geq -\int_0^t \int_\Omega \frac{L}{\varepsilon}\left(|\widetilde{\theta}|^2 + |\widetilde{u}||\widetilde{\theta}| + |\widetilde{\chi}||\widetilde{\theta}|\right)\mathrm{d}x\,\mathrm{d}s$$
$$\geq -C_1\int_0^t \int_\Omega (|\widetilde{\theta}|^2 + |\widetilde{u}|^2 + |\widetilde{\chi}|^2)\,\mathrm{d}x\,\mathrm{d}s. \tag{3.5}$$

Now let us multiply the equation (2.24) for the phase relaxation by $\widetilde{\chi}$, and integrate it over $\Omega \times [0,t]$. Thanks to (2.26) and to the Lipschitz continuity (2.5) of ψ, we deduce that

$$\frac{1}{2}\|\widetilde{\chi}(t)\|_H^2 = \frac{1}{2}\|\widetilde{\chi}_0\|_H^2$$
$$+ \frac{1}{\varepsilon}\int_0^t \int_\Omega \big[\psi(\theta_1(s,x)+u_1(s,x),\chi_1(s,x)) - \psi(\theta_2(s,x)+u_2(s,x),\chi_2(s,x))\big]\widetilde{\chi}(s,x)\,\mathrm{d}x\,\mathrm{d}s$$
$$\cdot \leq \frac{1}{2}\|\widetilde{\chi}_0\|_H^2 + \frac{L}{\varepsilon}\int_0^t \int_\Omega \left(|\widetilde{\theta}+\widetilde{u}| + |\widetilde{\chi}|\right)|\widetilde{\chi}|\,\mathrm{d}x\,\mathrm{d}s$$
$$\leq \frac{1}{2}\|\widetilde{\chi}_0\|_H^2 + \frac{L}{\varepsilon}\int_0^t \int_\Omega \left(|\widetilde{\theta}||\widetilde{\chi}| + |\widetilde{u}||\widetilde{\chi}| + |\widetilde{\chi}|^2\right)\mathrm{d}x\,\mathrm{d}s. \tag{3.6}$$

Summing (3.4) and (3.6), taking into account of (3.5), and using the elementary Young inequality, we obtain that there exists a constant C_2 depending on ε, but independent of θ_i, χ_i, u_i, θ_{0i}, χ_{0i}, f_i, such that

$$\|\widetilde{\theta}(t)\|_H^2 + \int_0^t \int_\Omega |\nabla\widetilde{\theta}(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s + \|\widetilde{\chi}(t)\|_H^2$$
$$\leq C_2\left(\|\widetilde{\theta}_0\|_H^2 + \|\widetilde{\chi}_0\|_H^2 + \int_0^t \|\widetilde{f}_V(s)\|_{V'}^2\,\mathrm{d}s + \int_0^t \|\widetilde{f}_H(s)\|_H\|\widetilde{\theta}(s)\|_H\,\mathrm{d}s + \int_0^t \|\widetilde{u}(s)\|_H^2\,\mathrm{d}s\right)$$
$$+ C_2\left(\int_0^t \|\widetilde{\theta}(s)\|_H^2\,\mathrm{d}s + \int_0^t \|\widetilde{\chi}(s)\|_H^2\,\mathrm{d}s\right).$$

Thus an application of a generalized version of the Gronwall Lemma (cf. [2, Theorem 2.1]), yields (3.2). □

Now we can conclude the proof of Theorem 2.1.

Proof of Theorem 2.1. For simplicity let us omit the subscript ε. Let us define $\Sigma := \{h \in H^1(0,T;H) \ : \ h(0) = \chi_0\}$ so that Σ is a complete metric space when it is endowed with the metric induced by the norm of $H^1(0,T;H)$. Fix $X \in \Sigma$. Then, thanks to a standard result for parabolic equations (cf., e.g., [3, Theorem 3.2]), there exists a unique $\theta_X \in L^2(0,T;V) \cap H^1(0,T;V')$ such that

$$\theta_X' + \mathsf{A}\theta_X = f - X' \qquad \text{in } V', \text{ for a.e. } t \in]0,T[, \tag{3.7}$$

$$\theta_X(0) = \theta_0, \qquad \text{a.e. in } \Omega. \tag{3.8}$$

Now define $\chi : Q \longrightarrow \mathbb{R}$ by

$$\chi(t,x) := \chi_0(x) + \frac{1}{\varepsilon}\int_0^t \psi(\theta_X(s,x) + u(s,x), X(s,x))\,\mathrm{d}s, \qquad t \in [0,T]\,,\ x \in \Omega. \tag{3.9}$$

Using (3.9), the Lipschitz continuity of ψ, and the fact that θ_X, X and u belong to $L^2(0,T;H)$, it is immediately seen that $\chi \in \Sigma$. Hence we can define the operator $\mathsf{S} : \Sigma \longrightarrow \Sigma$ associating to X the unique χ satisfying (3.7)–(3.9). We have that χ is a fixed point of S if and only if (θ_χ, χ) is a solution to Problem $(\mathbf{P}_\varepsilon)$. We are going to apply the shrinking fixed point theorem. For $i = 1, 2$, fix $X_i \in \Sigma$ and let $\theta_i \in L^2(0,T;V) \cap H^1(0,T;V')$ be the unique function such that (3.7)–(3.8) hold with θ_X and X replaced by θ_i and X_i. Set $\chi_i := \mathsf{S}(X_i)$ and define $\widetilde{X} := X_1 - X_2$, $\widetilde{\theta} := \theta_1 - \theta_2$, $\widetilde{\chi} := \chi_1 - \chi_2$. Let $t \in [0,T]$ be fixed. Let us integrate in time the difference of equations (3.7) for $i = 1, 2$ and test it by $\widetilde{\theta}$. Integrating the result over $]0,t[$, applying the Young inequality, and observing that $\widetilde{X}(0) = 0$, we infer that

$$\frac{1}{2}\int_0^t \int_\Omega |\widetilde{\theta}(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\int_\Omega \left|\int_0^t \nabla\widetilde{\theta}(s,x)\,\mathrm{d}s\right|^2 \mathrm{d}x \le \frac{1}{2}\int_0^t \int_\Omega |\widetilde{X}(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s. \tag{3.10}$$

Therefore, using (3.9) and the Lipschitz continuity (2.5) of ψ, we get

$$\begin{aligned} &\int_0^t \int_\Omega |\widetilde{\chi}'(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s \\ &= \frac{1}{\varepsilon^2}\int_0^t \int_\Omega \left|\psi\big(\theta_1(s,x) + u(s,x), X_1(s,x)\big) - \psi\big(\theta_2(s,x) + u(s,x), X_2(s,x)\big)\right|^2 \mathrm{d}x\,\mathrm{d}s \\ &\le \frac{L^2}{\varepsilon^2}\int_0^t \int_\Omega \big(|\widetilde{\theta}(s,x)|^2 + |\widetilde{X}(s,x)|^2\big)\,\mathrm{d}x\,\mathrm{d}s \le \frac{2L^2}{\varepsilon^2}\int_0^t \int_\Omega |\widetilde{X}(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{3.11}$$

On the other hand

$$\int_0^t \int_\Omega |\widetilde{X}(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s \le \int_0^t \int_\Omega s\int_0^s |\widetilde{X}'(r,x)|^2\,\mathrm{d}r\,\mathrm{d}x\,\mathrm{d}s \le \int_0^t \int_\Omega t\int_0^s |\widetilde{X}'(r,x)|^2\,\mathrm{d}r\,\mathrm{d}x\,\mathrm{d}s,$$

hence

$$\int_0^t \int_\Omega |\widetilde{\chi}'(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s \le \frac{2tL^2}{\varepsilon^2}\int_0^t \int_0^s \int_\Omega |\widetilde{X}'(r,x)|^2\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}s, \tag{3.12}$$

so that there exists a constant C independent of X_1 and X_2 such that

$$\|\widetilde{\chi}\|^2_{H^1(0,t;H)} \le C\int_0^t \|\widetilde{X}\|^2_{H^1(0,s;H)}\,\mathrm{d}s. \tag{3.13}$$

This entails that $\|\mathsf{S}^n(\widetilde{X})\|_{H^1(0,t;H)} \le ((CT)^n/n!)^{1/2}\|\widetilde{X}\|_{H^1(0,t;H)}$ for every $n \in \mathbb{N}$, therefore the iterated mapping S^n is a strict contraction for n sufficiently large, and consequently S admits a unique fixed point in Σ, which leads to the solution we are looking for. □

4. Asymptotic behavior

Throughout this section we will assume the non restrictive condition that $\varepsilon < 1$.

Let us start by stating the following easy consequence of the assumptions on the function ψ, as already observed in [22, formula (1.12)].

Lemma 4.1. *Under the Assumptions 2.1 and 2.2 we have that*

$$\psi(\tau,\chi) > 0 \quad \iff \quad \chi < \inf \alpha(\tau), \tag{4.1}$$

$$\psi(\tau,\chi) < 0 \quad \iff \quad \chi > \sup \alpha(\tau), \tag{4.2}$$

for every $(\tau,\chi) \in \mathbb{R}^2$.

Now we prove that if the initial datum χ_0 is constrained by α^{-1} (cf. (2.14)), then the solution χ_ε of (2.24) is uniformly bounded on Q.

Lemma 4.2. *Under the Assumptions 2.1 and 2.2, if we are given* $\chi_\varepsilon \in H^1(0,T;H)$, $\theta_\varepsilon \in L^2(Q)$, *and* $\eta_\varepsilon > 0$ *such that*

$$|\chi_\varepsilon(0,x)| \le |\chi_0(x)| + \eta_\varepsilon \qquad \textit{for a.e. } x \in \Omega \tag{4.3}$$

and

$$\varepsilon\chi_\varepsilon'(t,x) = \psi\big(\theta_\varepsilon(t,x) + u_\varepsilon(t,x), \chi_\varepsilon(t,x)\big) \qquad \textit{for a.e. } (t,x) \in Q, \tag{4.4}$$

then

$$|\chi_\varepsilon(t,x)| \le M + \eta_\varepsilon \qquad \textit{for a.e. } (t,x) \in Q, \tag{4.5}$$

where we recall that M *is defined in condition (A1) of Assumptions 2.2, so that* $M \ge \sup\{\alpha(\tau) : \tau \in D(\alpha)\}$.

Proof. Since χ_ε and χ_ε' belong to $L^2(Q)$, if $\varphi \in C^\infty(0,T)$ has compact support and if $z \in L^2(\Omega)$, by the Fubini theorem we have that

$$\begin{aligned}
&\int_\Omega z(x) \int_0^T (\chi_\varepsilon(t,x)\varphi'(t) + \chi_\varepsilon'(t,x)\varphi(t)) \,\mathrm{d}t \,\mathrm{d}x \\
&= \int_0^T \int_\Omega z(x)(\chi_\varepsilon(t,x)\varphi'(t) + \chi_\varepsilon'(t,x)\varphi(t)) \,\mathrm{d}x \,\mathrm{d}t \\
&= \int_0^T (z, \chi_\varepsilon(t))_H \varphi'(t) \,\mathrm{d}t + \int_0^T (z, \chi_\varepsilon'(t))_H \varphi(t) \,\mathrm{d}t.
\end{aligned}$$

From the previous chain of equalities, recalling that $\chi_\varepsilon \in H^1(0,T;H)$ and using again the Fubini theorem, we infer that

$$\begin{aligned}
&\int_\Omega z(x) \int_0^T (\chi_\varepsilon(t,x)\varphi'(t) + \chi_\varepsilon'(t,x)\varphi(t)) \,\mathrm{d}t \,\mathrm{d}x \\
&= \int_0^T \Big(z, \chi_\varepsilon(0) + \int_0^t \chi_\varepsilon'(s) \,\mathrm{d}s\Big)_H \varphi'(t) \,\mathrm{d}t + \int_0^T (z, \chi_\varepsilon'(t))_H \varphi(t) \,\mathrm{d}t \\
&= (z, \chi_\varepsilon(0))_H \int_0^T \varphi'(t) \,\mathrm{d}t + \int_0^T \int_0^t (z, \chi_\varepsilon'(s))_H \varphi'(t) \,\mathrm{d}s \,\mathrm{d}t + \int_0^T (z, \chi_\varepsilon'(t))_H \varphi(t) \,\mathrm{d}t \\
&= \int_0^T (z, \chi_\varepsilon'(s))_H \int_s^T \varphi'(t) \,\mathrm{d}t \,\mathrm{d}s + \int_0^T (z, \chi_\varepsilon'(t))_H \varphi(t) \,\mathrm{d}t \\
&= -\int_0^T (z, \chi_\varepsilon'(s))_H \varphi(s) \,\mathrm{d}s + \int_0^T (z, \chi_\varepsilon'(t))_H \varphi(t) \,\mathrm{d}t = 0,
\end{aligned}$$

whence, by the arbitrariness of z, we infer that $\int_0^T (\chi_\varepsilon'(t,x)\varphi(t) + \chi_\varepsilon(t,x)\varphi'(t)) \,\mathrm{d}t = 0$ for a.e. $x \in \Omega$, i.e. that $\chi_\varepsilon'(\cdot,x)$ is the distributional derivative of $\chi_\varepsilon(\cdot,x)$ for a.e. $x \in \Omega$. Since

$\chi_\varepsilon'(\cdot,x) \in L^1(0,T)$ for a.e. $x \in \Omega$, we obtain that there exists a measurable set $A \subseteq \Omega$ such that $|\Omega \setminus A| = 0$ and

$$\chi_\varepsilon(t,x) = \chi_\varepsilon(0,x) + \int_0^t \chi_\varepsilon'(s,x)\,\mathrm{d}s \qquad \forall t \in [0,T]\,, \qquad \forall x \in A.$$

It follows that for every $x \in A$ the function $\chi_\varepsilon(\cdot,x)$ is absolutely continuous from $[0,T]$ into $\mathbb{R}$. It is not restrictive to assume that $\chi_0(x) \in \alpha(\theta_0(x)+u(0,x))$ for every $x \in A$, so that $|\chi_0(x)| \le M$ for every $x \in A$. Therefore $|\chi_\varepsilon(0,x)| \le M + \eta_\varepsilon$ for every $x \in A$. Let us fix $x \in A$ and prove that $|\chi_\varepsilon(t,x)| \le M + \eta_\varepsilon$ for every $t \in [0,T]$. Indeed, if this were not true, there would exist $t_0 \in \,]0,T[$ such that $|\chi_\varepsilon(t_0,x)| > M + \eta_\varepsilon$. Let us first assume that $\chi_\varepsilon(t_0,x) > M + \eta_\varepsilon$. Then, by continuity, there exists $a_0 \in [0,t_0[$ such that $\chi_\varepsilon(a_0,x) = M + \eta_\varepsilon$, and $\chi_\varepsilon(t,x) > M + \eta_\varepsilon$ for every $t \in \,]a_0,t_0]$. In particular $\chi_\varepsilon(t,x) > \sup\{\alpha(r) \ : \ r \in D(\alpha)\}$, hence $\chi_\varepsilon(t,x) > \sup\{\alpha\big(\theta_\varepsilon(t,x)+u_\varepsilon(t,x), \chi_\varepsilon(t,x)\big)\}$ for every $t \in \,]a_0,t_0]$, so that $\psi\big(\theta_\varepsilon(t,x)+u_\varepsilon(t,x), \chi_\varepsilon(t,x)\big) < 0$ for every $t \in \,]a_0,t_0]$ by (4.2). It follows that $\chi_\varepsilon'(t,x) < 0$ for a.e. $t \in \,]a_0,t_0]$, therefore, as $\chi_\varepsilon(\cdot,x)$ is absolutely continuous, we infer that $\chi_\varepsilon(\cdot,x)$ is decreasing on $]a_0,t_0]$, a contradiction. An analogous argument can be used in the case $\chi_\varepsilon(t_0,x) < -M - \eta_\varepsilon$. ☐

We need the following auxiliary lemma, where we make use of the notation (2.27) introduced in Definition 2.1: $\widehat{v}(t) = \int_0^t v(s)\,\mathrm{d}s$, for $t \in [0,T]$, $v \in L^1(0,T;B)$, and a Banach space B.

Lemma 4.3. *Under the hypothesis (H1) in Assumptions 2.1, if $F \in L^1(0,T;H) + L^2(0,T;V')$, $e_0 \in H$, $v \in L^2(0,T;V) \cap L^\infty(0,T;H)$, and $\delta > 0$, then, recalling notation* (2.27)*, we have that*

$$\int_0^t {}_{V'}\langle \widehat{F}(s), v(s)\rangle_V \,\mathrm{d}s \le \delta\left(1 + t + \frac{t^2}{2}\right)\|v\|^2_{L^2(0,t;H)} + \delta\left(\|\nabla\widehat{v}(t)\|^2_{H^n} + \int_0^t \|\nabla\widehat{v}(s)\|^2_{H^n}\,\mathrm{d}s\right) + \frac{1+t}{4\delta}\|F\|^2_{L^1(0,T;H)+L^2(0,T;V')} \tag{4.6}$$

and

$$\int_0^t {}_{V'}\langle e_0, v(s)\rangle_V \,\mathrm{d}s \le \delta\|v\|^2_{L^2(0,t;H)} + \frac{t}{4\delta}\|e_0\|^2_H \tag{4.7}$$

for every $t \in [0,T]$.

Proof. Let $F_1 \in L^1(0,T;H)$ and $F_2 \in L^2(0,T;V')$ be arbitrarily taken so that $F = F_1 + F_2$. We have that

$$\begin{aligned}
\int_0^t {}_{V'}\langle \widehat{F}_1(s), v(s)\rangle_V \,\mathrm{d}s &\le \int_0^t \|\widehat{F}_1(s)\|_H \|v(s)\|_H \,\mathrm{d}s \\
&\le \delta\|v\|^2_{L^2(0,t;H)} + \frac{1}{4\delta}\|\widehat{F}_1\|^2_{L^2(0,t;H)} \\
&= \delta\|v\|^2_{L^2(0,t;H)} + \frac{1}{4\delta}\int_0^t \left\|\int_0^s F_1(r)\,\mathrm{d}r\right\|^2_H \mathrm{d}s \\
&\le \delta\|v\|^2_{L^2(0,t;H)} + \frac{1}{4\delta}\int_0^t \left(\int_0^s \|F_1(r)\|_H\,\mathrm{d}r\right)^2 \mathrm{d}s \\
&\le \delta\|v\|^2_{L^2(0,t;H)} + \frac{1}{4\delta} t\|F_1\|^2_{L^1(0,T;H)}.
\end{aligned} \tag{4.8}$$

Let us observe that for any Banach space B we have

$$\|\widehat{v}(t)\|^2_B = \left\|\int_0^t v(s)\,\mathrm{d}s\right\|^2_B \le \left(\int_0^t \|v(s)\|_B\,\mathrm{d}s\right)^2 \le t\|u\|^2_{L^2(0,t;B)}, \tag{4.9}$$

therefore, integrating by parts and applying Young inequality, we find that

$$
\begin{aligned}
&\int_0^t {}_{V'}\langle \widehat{F}_2(s), v(s)\rangle_V \,\mathrm{d}s \\
&= {}_{V'}\langle \widehat{F}_2(t), \widehat{v}(t)\rangle_V - \int_0^t {}_{V'}\langle F_2(s), \widehat{v}(s)\rangle_V \,\mathrm{d}s \\
&\le \|\widehat{F}_2(t)\|_{V'}\|\widehat{v}(t)\|_V + \int_0^t \|F_2(s)\|_{V'}\|\widehat{v}(s)\|_V \,\mathrm{d}s \\
&= \|\widehat{F}_2(t)\|_{V'}\left(\|\widehat{v}(t)\|_H^2 + \|\nabla\widehat{v}(t)\|_{H^n}^2\right)^{1/2} + \int_0^t \|F_2(s)\|_{V'}\left(\|\widehat{v}(s)\|_H^2 + \|\nabla\widehat{v}(s)\|_{H^n}^2\right)^{1/2} \mathrm{d}s \\
&\le \delta\left(t\|v\|_{L^2(0,t;H)}^2 + \|\nabla\widehat{v}(t)\|_{H^n}^2\right) + \frac{1}{4\delta}\|\widehat{F}_2(t)\|_{V'}^2 \\
&\quad + \delta\int_0^t \left(s\|v\|_{L^2(0,s;H)}^2 + \|\nabla\widehat{v}(s)\|_{H^n}^2\right)\mathrm{d}s + \frac{1}{4\delta}\|F_2\|_{L^2(0,T;V')}^2 \\
&\le \delta\left(t\|v\|_{L^2(0,t;H)}^2 + \|\nabla\widehat{v}(t)\|_{H^n}^2\right) + \frac{t}{4\delta}\|F_2\|_{L^2(0,t;V')}^2 \\
&\quad + \delta(t^2/2)\|v\|_{L^2(0,t;H)}^2 + \delta\int_0^t \|\nabla\widehat{v}(s)\|_{H^n}^2\,\mathrm{d}s + \frac{1}{4\delta}\|F_2\|_{L^2(0,T;V')}^2, \qquad (4.10)
\end{aligned}
$$

thus (4.6) follows from (4.8), (4.10), and from the elementary inequality $a^2 + b^2 \le (|a| + |b|)^2$, holding for $a, b \in \mathbb{R}$. Finally estimate (4.7) is a consequence of (4.9) and of formula

$$\int_0^t {}_{V'}\langle e_0, v(s)\rangle_V \,\mathrm{d}s = {}_{V'}\langle e_0, \widehat{v}(t)\rangle_V \le \|e_0\|_H\|\widehat{v}(t)\|_H.$$

□

We can now deduce the estimate for the temperature θ.

Lemma 4.4. *Under the assumptions of Theorem 2.2, there exists a constant C_1 independent of ε, but depending on T, Ω, α, ψ, f, u, θ_0, χ_0, such that if $(\theta_\varepsilon, \chi_\varepsilon)$ is the only solution of Problem $(\mathbf{P}_\varepsilon)$, then, recalling notation* (2.27),

$$\|\theta_\varepsilon\|_{L^2(0,T;H)} + \|\widehat{\theta}_\varepsilon\|_{L^\infty(0,T;V)} + \varepsilon^{1/2}\|\theta_\varepsilon\|_{L^2(0,T;V)} \le C_1. \qquad (4.11)$$

Proof. We will tacitly use the convergences (2.32)–(2.35). Let us fix $t \in [0,T]$. First we integrate the energy balance equation (2.23) with respect to time over $[0,s]$ with $s \in [0,t]$, and test it by $\theta_\varepsilon(s)$. After a further integration over $[0,t]$, and recalling (2.27), we get

$$
\begin{aligned}
&\|\theta_\varepsilon\|_{L^2(0,t;H)}^2 + \int_0^t\int_\Omega \chi_\varepsilon(s,x)\theta_\varepsilon(s,x)\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\int_\Omega |\nabla\widehat{\theta}_\varepsilon(t,x)|^2\,\mathrm{d}x \\
&= \int_0^t {}_{V'}\langle \theta_{\varepsilon 0} + \chi_{\varepsilon 0} + \widehat{f}_\varepsilon(s), \theta_\varepsilon(s)\rangle_V \,\mathrm{d}s, \qquad (4.12)
\end{aligned}
$$

therefore using Lemma 4.3 we infer that there exists a constant K_1 depending on $\|\theta_0\|_H$, $\|\chi_0\|_H$, $\|f\|_{L^1(0,T;H)+L^2(0,T;V')}$, and T, but independent of ε, such that

$$
\begin{aligned}
&\frac{1}{2}\|\theta_\varepsilon\|_{L^2(0,t;H)}^2 + \int_0^t\int_\Omega \chi_\varepsilon(s,x)\theta_\varepsilon(s,x)\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{4}\int_\Omega |\nabla\widehat{\theta}_\varepsilon(t,x)|^2\,\mathrm{d}x \\
&\le K_1 + K_1\int_0^t\int_\Omega |\nabla\widehat{\theta}_\varepsilon(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s. \qquad (4.13)
\end{aligned}
$$

Let us recall that $f_\varepsilon = f_{\varepsilon 1} + f_{\varepsilon 2}$ with $f_{\varepsilon 1} \in L^1(0,T;H)$ and $f_{\varepsilon 2} \in L^2(0,T;V')$. We test by $\varepsilon\theta_\varepsilon$ the energy balance equation (2.23) and integrate over $[0,t]$. Thanks to (2.25), we infer that

$$\begin{aligned}&\frac{\varepsilon}{2}\|\theta_\varepsilon(t)\|_H^2 + \varepsilon\int_0^t\int_\Omega \chi'_\varepsilon(s,x)\theta_\varepsilon(s,x)\,\mathrm{d}x\,\mathrm{d}s + \varepsilon\int_0^t\int_\Omega |\nabla\theta_\varepsilon(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s\\ &= \frac{\varepsilon}{2}\|\theta_{0\varepsilon}\|_H^2 + \varepsilon\int_0^t (f_{\varepsilon 1}(s),\theta_\varepsilon(s))_H\,\mathrm{d}s + \varepsilon\int_0^t {}_{V'}\langle f_{\varepsilon 2}(s),\theta_\varepsilon(s)\rangle_V\,\mathrm{d}s \qquad (4.14)\end{aligned}$$

therefore, recalling that $\varepsilon < 1$, several applications of Young and Hölder inequlities yield

$$\begin{aligned}&\frac{\varepsilon}{2}\|\theta_\varepsilon(t)\|_H^2 + \varepsilon\int_0^t\int_\Omega \chi'_\varepsilon(s,x)\theta_\varepsilon(s,x)\,\mathrm{d}x\,\mathrm{d}s + \frac{\varepsilon}{2}\int_0^t\int_\Omega |\nabla\theta_\varepsilon(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s\\ &\le K_2 + K_2\int_0^t \|f_{\varepsilon 1}(s)\|_H\varepsilon^{1/2}\|\theta_\varepsilon(s)\|_H\,\mathrm{d}s + K_2\int_0^t \varepsilon\|\theta_\varepsilon(s)\|_H^2\,\mathrm{d}s \qquad (4.15)\end{aligned}$$

for some $K_2 > 0$ depending on θ_0, on f, but independent of ε. Thanks to equation (2.24) for the phase relaxation and to the monotonicity (2.9) of ψ in the first variable, we can write (omitting in some lines the integration variable (s,x)):

$$\begin{aligned}&\varepsilon\int_0^t\int_\Omega \chi'_\varepsilon(s,x)\theta_\varepsilon(s,x)\,\mathrm{d}x\,\mathrm{d}s\\ &= \int_0^t\int_\Omega \psi(\theta_\varepsilon+u_\varepsilon,\chi_\varepsilon)\theta_\varepsilon\,\mathrm{d}x\,\mathrm{d}s\\ &= \int_0^t\int_\Omega \psi(\theta_\varepsilon+u_\varepsilon,\chi_\varepsilon)(\theta_\varepsilon+u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_\Omega \psi(\theta_\varepsilon+u_\varepsilon,\chi_\varepsilon)u_\varepsilon\,\mathrm{d}x\,\mathrm{d}s\\ &= \int_0^t\int_\Omega \big[\psi(\theta_\varepsilon+u_\varepsilon,\chi_\varepsilon)-\psi(0,\chi_\varepsilon)\big](\theta_\varepsilon+u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s\\ &\quad + \int_0^t\int_\Omega \psi(0,\chi_\varepsilon)(\theta_\varepsilon+u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_\Omega \psi(\theta_\varepsilon+u_\varepsilon,\chi_\varepsilon)u_\varepsilon\,\mathrm{d}x\,\mathrm{d}s\\ &\ge \int_0^t\int_\Omega \psi(0,\chi_\varepsilon)(\theta_\varepsilon+u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_\Omega \psi(\theta_\varepsilon+u_\varepsilon,\chi_\varepsilon)u_\varepsilon\,\mathrm{d}x\,\mathrm{d}s. \qquad (4.16)\end{aligned}$$

On the other hand, recalling the Lipschitz continuity (2.5) of ψ, we get that (we still omit the integration variable (s,x)):

$$\begin{aligned}&\int_0^t\int_\Omega \psi(0,\chi_\varepsilon)(\theta_\varepsilon+u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_\Omega \psi(\theta_\varepsilon+u_\varepsilon,\chi_\varepsilon)u_\varepsilon\,\mathrm{d}x\,\mathrm{d}s\\ &= \int_0^t\int_\Omega \big[\psi(0,\chi_\varepsilon)-\psi(\theta_\varepsilon+u_\varepsilon,\chi_\varepsilon)\big]u_\varepsilon\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega \psi(0,\chi_\varepsilon)\theta_\varepsilon\,\mathrm{d}x\,\mathrm{d}s\\ &= \int_0^t\int_\Omega \big[\psi(0,\chi_\varepsilon)-\psi(\theta_\varepsilon+u_\varepsilon,\chi_\varepsilon)\big]u_\varepsilon\,\mathrm{d}x\,\mathrm{d}s\\ &\quad + \int_0^t\int_\Omega \big[\psi(0,\chi_\varepsilon)-\psi(0,0)\big]\theta_\varepsilon\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega \psi(0,0)\theta_\varepsilon\,\mathrm{d}x\,\mathrm{d}s\\ &\ge -\int_0^t\int_\Omega L|\theta_\varepsilon+u_\varepsilon||u_\varepsilon|\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_\Omega L|\chi_\varepsilon||\theta_\varepsilon|\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_\Omega |\psi(0,0)||\theta_\varepsilon|\,\mathrm{d}x\,\mathrm{d}s. \qquad (4.17)\end{aligned}$$

Let us observe that thanks to (2.35) and to Lemma 4.2, we have that there exists $M_1 > 0$ (depending on M) such that

$$\|\chi_\varepsilon\|_\infty \le M_1 \qquad (4.18)$$

for every $\varepsilon < 1$. Therefore, collecting together (4.16)–(4.17), and using the elementary Young inequality, we infer that there exists a constant $K_3 > 0$ depending only on T, $|\Omega|$, L, $|\psi(0,0)|$, $\|u\|_{L^2(0,T;H)}$, and M, such that

$$\varepsilon \int_0^t \int_\Omega \chi_\varepsilon'(s,x)\theta_\varepsilon(s,x)\,\mathrm{d}x\,\mathrm{d}s \geq -\,K_3 - \frac{1}{8}\|\theta_\varepsilon\|^2_{L^2(0,t;H)}. \tag{4.19}$$

Using again the boundedness of $\|\chi_\varepsilon\|_\infty$ and the elementary Young inequality we also have that

$$\int_0^t \int_\Omega \chi_\varepsilon(t,x)\theta(t,x)\,\mathrm{d}x\,\mathrm{d}t \geq -2M_1^2 t|\Omega| - \frac{1}{8}\|\theta_\varepsilon\|^2_{L^2(0,t;H)}. \tag{4.20}$$

Therefore adding (4.19) and (4.20), and taking into account of (4.15) and (4.16), we find a constant K with the same dependencies of K_1, K_2, K_3, but independent of ε, such that

$$\begin{aligned}&\frac{1}{4}\|\theta_\varepsilon\|^2_{L^2(0,t;H)} + \frac{1}{4}\int_\Omega |\nabla\widehat{\theta}_\varepsilon(t,x)|^2\,\mathrm{d}x + \frac{\varepsilon}{4}\|\theta_\varepsilon(t)\|^2_H + \frac{\varepsilon}{2}\int_0^t\int_\Omega |\nabla\theta_\varepsilon(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\leq K + K\left(\int_0^t\int_\Omega |\nabla\widehat{\theta}_\varepsilon(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \|f_{\varepsilon 1}(s)\|_H \varepsilon^{1/2}\|\theta_\varepsilon(s)\|_H\,\mathrm{d}s + \int_0^t \varepsilon\|\theta_\varepsilon(s)\|^2_H\,\mathrm{d}s\right),\end{aligned}$$

which, together with a generalized version of the Gronwall Lemma (cf. [2, Theorem 2.1]), allows us to conclude. □

Now we establish the estimate for the phase χ.

Lemma 4.5. *Under the assumptions of Theorem 2.2, there exists a constant C_2 independent of ε, but depending on T, Ω, α, ψ, f, u, θ_0, χ_0, such that if $(\theta_\varepsilon, \chi_\varepsilon)$ is the only solution of Problem $(\mathbf{P}_\varepsilon)$, then*

$$\|\chi_\varepsilon\|_{L^\infty(Q)} + \varepsilon\|\chi_\varepsilon'\|_{L^2(Q)} \leq C_2. \tag{4.21}$$

Proof. We already know that the sequence χ_ε is bounded in $L^\infty(Q)$ by virtue of Lemma 4.2. From equation (2.24) for the phase relaxation and from the Lipschitz continuity (2.5) of ψ, we get that

$$\begin{aligned}&\varepsilon^2\int_0^t\int_\Omega |\chi_\varepsilon'(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s = \int_0^t\int_\Omega |\psi(\theta_\varepsilon(s,x) + u_\varepsilon(s,x), \chi_\varepsilon(s,x))|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\leq 2\int_0^t\int_\Omega |\psi(\theta_\varepsilon(s,x) + u_\varepsilon(s,x), \chi_\varepsilon(s,x)) - \psi(0,0)|^2\,\mathrm{d}x\,\mathrm{d}s + 2\int_0^t\int_\Omega |\psi(0,0)|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\leq 2L^2\int_0^t\int_\Omega (|\theta_\varepsilon(s,x) + u_\varepsilon(s,x)|^2 + |\chi_\varepsilon(s,x)|^2)\,\mathrm{d}x\,\mathrm{d}s + 2T|\Omega||\psi(0,0)|^2 \\ &\leq 4L^2\|\theta\|^2_{L^2(0,t;H)} + 4L^2\|u\|^2_{L^2(Q)} + 2L^2 t|\Omega|M_1^2 + 2T\Omega|\psi(0,0)|^2,\end{aligned}$$

where M_1 is the constant found in (4.18) thanks to (2.35) and to Lemma 4.2. We conclude by invoking Lemma 4.4. □

We are now ready to prove the main result of this paper.

Proof of Theorem 2.2. From Lemma 4.4 and Lemma 4.5 we deduce that there exist two functions

$$\theta \in L^2(Q), \qquad \chi \in L^\infty(Q) \tag{4.22}$$

such that, at least for a subsequence which we do not relabel,

$$\theta_\varepsilon \rightharpoonup \theta \qquad \text{in } L^2(Q), \tag{4.23}$$

$$\widehat{\theta}_\varepsilon \overset{*}{\rightharpoonup} \widehat{\theta} \qquad \text{in } L^\infty(0,T;V) \cap H^1(0,T;H), \tag{4.24}$$

$$\chi_\varepsilon \overset{*}{\rightharpoonup} \chi \qquad \text{in } L^\infty(Q). \tag{4.25}$$

An integration in time of the energy balance equation (2.23) yields

$$\theta_\varepsilon + \chi_\varepsilon + \mathsf{A}\widehat{\theta}_\varepsilon = \theta_{0\varepsilon} + \chi_{0\varepsilon} + \widehat{f}_\varepsilon, \qquad \text{in } L^2(0,T;V') \tag{4.26}$$

therefore taking the limit as $\varepsilon \to 0$ along the subsequence established above we get

$$\theta + \chi + \mathsf{A}\widehat{\theta} = \theta_0 + \chi_0 + \widehat{f} \qquad \text{in } L^2(0,T;V') \tag{4.27}$$

which turns out to be equivalent to (2.30). From the Lipschitz continuity (2.5) of ψ we have that

$$\begin{aligned} &\int_Q |\varepsilon\chi_\varepsilon'(t,x) - \psi(\theta(t,x)+u(t,x),\chi(t,x))||v(t,x)|\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_Q |\psi(\theta_\varepsilon(t,x)+u_\varepsilon(t,x),\chi_\varepsilon(t,x)) - \psi(\theta(t,x)+u(t,x),\chi(t,x))||v(t,x)|\,\mathrm{d}x\,\mathrm{d}t \\ &\le L\int_Q (|\theta_\varepsilon(t,x)-\theta(t,x)| + |u_\varepsilon(t,x)-u(t,x)| + |\chi(t,x)-\chi_\varepsilon(t,x)|)|v(t,x)|\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{4.28}$$

for every $v \in L^2(Q)$, therefore if $\xi \in L^2(Q)$ is defined by

$$\xi(t,x) := \psi(\theta(t,x)+u(t,x),\chi(t,x)), \qquad (t,x)\in Q,. \tag{4.29}$$

we have that

$$\varepsilon\chi_\varepsilon' \rightharpoonup \xi \qquad \text{in } L^2(Q). \tag{4.30}$$

On the other hand from (4.25) we have that $\chi_\varepsilon' \to \chi'$ in Q in the sense of distributions, therefore $\varepsilon\chi_\varepsilon' \to 0$ in Q in the sense of distributions and we infer that

$$\varepsilon\chi_\varepsilon' \;\rightharpoonup\; 0 \quad \text{in } L^2(Q). \tag{4.31}$$

Thus from (4.29), (4.30) and (4.31) we infer that

$$\psi(\theta(t,x)+u(t,x),\chi(t,x)) = 0 \qquad \text{for a.e. } (t,x)\in Q, \tag{4.32}$$

so that by (2.11) we get that

$$\chi(t,x) \in \alpha(\theta(t,x)+u(t,x)) \qquad \text{for a.e. } (t,x)\in Q \tag{4.33}$$

and also (2.31) is proved. It remains to prove uniqueness, which also allows us to deduce that the whole sequences (θ_ε) and (χ_ε) converge. Let (θ_i,χ_i), $i=1,2$, be two solutions, and set

$$\Theta := \theta_1 - \theta_2, \qquad \mathcal{X} := \chi_1 - \chi_2. \tag{4.34}$$

Taking the difference of the equations (2.30) written for (θ_1,χ_1) and (θ_2,χ_2), we find

$$\widehat{\Theta} \in L^\infty(0,T;V)\cap H^1(0,T;H), \tag{4.35}$$

$$\mathcal{X} \in L^\infty(0,T;H), \tag{4.36}$$

$$\Theta + \mathcal{X} + A\widehat{\Theta} = 0 \quad \text{in } V', \quad \text{in } \left]0,T\right[\,. \tag{4.37}$$

By a comparison in the last equation, we see that $A\widehat{\Theta} \in L^2(0,T;H)$, therefore multiplying (4.37) by Θ and integrating over $\Omega\times(0,t)$, we get

$$\|\Theta\|^2_{L^2(0,t;H)} + \int_0^t\!\!\int_\Omega \mathcal{X}(s,x)\Theta(s,x)\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\int_\Omega |\nabla\widehat{\Theta}(t,x)|\,\mathrm{d}x = 0. \tag{4.38}$$

Therefore, since $\mathcal{X}\Theta \ge 0$ a.e. in Q by the monotonicity of α and (2.31), from (4.38) we infer that $\Theta = 0$ a.e. in Q and, by a comparison in (4.37), that $\mathcal{X} = 0$ a.e. in Q, and the uniqueness of Problem **(P)** is proved. This uniqueness property, together with the fact that the formulation (2.15)–(2.20) is stronger than the formulation of Problem **(P)**, let us infer that (θ,χ) is indeed the solution of (2.15)–(2.20). □

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