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On the Total CR Twist of Transversal Curves in the 3-Sphere

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Abstract. We investigate the total CR twist functional on transversal curves in the standard CR 3-sphere $\mathrm{S}^3 \subset \mathbb{C}^2$. The question of the integration by quadratures of the critical curves and the problem of existence and properties of closed critical curves are addressed. A procedure for the explicit integration of general critical curves is provided and a characterization of closed curves within a specific class of general critical curves is given. Experimental evidence of the existence of infinite countably many closed critical curves is provided.

Key words: CR 3-sphere; transversal curves; CR invariants; total CR twist; Griffiths' formalism; Lax formulation of E-L equations; integration by quadratures; closed critical curves

2020 Mathematics Subject Classification: 53C50; 53C42; 53A10

Dedicated to Peter Olver
on the occasion of his 70th birthday

1 Introduction

The present paper finds its inspiration and theoretical framework in the subjects of moving frames, differential invariants, and invariant variational problems, three of the many research topics to which Peter Olver has made lasting contributions. Among the many publications of Peter Olver dedicated to these subjects, we like to mention [16, 17, 34, 35] as the ones that most influenced our research activity.

More specifically, in this paper we further develop some of the themes considered in [25, 31, 32] concerning the Cauchy–Riemann (CR) geometry of transversal and Legendrian curves in the 3-sphere. In three dimensions, a CR structure on a manifold is defined by an oriented contact distribution equipped with a complex structure. While the automorphism group of a contact manifold is infinite dimensional, that of a CR threefold is finite dimensional and of dimension less or equal than eight [5, 6, 7]. The maximally symmetric CR threefold is the 3-sphere S^3, realized as a real hyperquadric of $\mathbb{CP}^2$ acted upon transitively by the Lie group $G \cong \mathrm{SU}(2,1)$. This homogeneous model allows the application of differential-geometric techniques to the study of transversal and Legendrian curves in S^3. Since the seminal work of Bennequin [1], the study of the topological properties of transversal and Legendrian knots in 3-dimensional contact manifolds has been an important area of research (see, for instance, [10, 12, 13, 14, 15, 18] and the literature therein). Another reason of interest for 3-dimensional contact geometry comes from

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its applications to neuroscience. In fact, as shown by Hoffman [20], the visual cortex can be modeled as a bundle equipped with a contact structure. For more details, the interested reader is referred to the monograph [36, Section 5]. Recently, the CR geometry of Legendrian and transversal curves in S^3 has also found interesting applications in the framework of integrable system [4].

Let us begin by recalling some results from the CR geometry of transversal curves in S^3. According to [31], away from CR inflection points, a curve transversal to the contact distribution of S^3 can be parametrized by a natural *pseudoconformal* parameter s and in this parametrization it is uniquely determined, up to CR automorphisms, by two local CR invariants: the CR *bending* κ and the CR *twist* τ. This was achieved by developing the method of moving frames and by constructing a canonical frame field along *generic*[1] transversal curves. Moreover, for closed transversal curves, we defined three discrete global invariants, namely, the wave number, the CR spin, and the CR turning number. Next, we investigated the total strain functional, defined by integrating the strain element $\mathrm{d}s$. We proved that the corresponding critical curves have constant bending and twist, and hence arise as orbits of 1-parameter groups of CR automorphisms. Finally, closed critical curves are shown to be transversal positive torus knots with maximal Bennequin number.

In the present paper, we consider the CR invariant variational problem for generic transversal curves in S^3 defined by the *total CR twist functional*,

$$\mathcal{W}(\gamma) = \int_\gamma \tau \, \mathrm{d}s.$$

Our purpose is to address both the question of the explicit integration of critical curves and the problem of existence and properties of closed critical curves of $\mathcal{W}$.

We now give a brief outline of the content and results of this paper. In Section 2, we shortly describe the standard CR structure of the 3-sphere S^3, viewed as a homogeneous space of the group G, and collect some preliminary material. We then recall the basic facts about the CR geometry of transversal curves in S^3 as developed in [31] (see the description above). Moreover, besides the already mentioned discrete global invariants for a closed transversal curve, we introduce a fourth global invariant, the *trace* of the curve with respect to a spacelike line.

In Section 3, we apply the method of moving frames and the Griffiths approach to the calculus of variations [19, 21, 26] to compute the Euler–Lagrange equations of the total CR twist functional. We construct the momentum space of the corresponding variational problem and find a Lax pair formulation for the Euler–Lagrange equations satisfied by the critical curves. This is the content of Theorem A, the first main result of the paper, whose proof occupies the whole Section 3. As a consequence of Theorem A, to each critical curve we associate a *momentum* operator, which is a fixed element of the G-module $\mathfrak{h}$ of traceless selfadjoint endomorphisms of $\mathbb{C}^{2,1}$. From the conservation of the momentum along a critical curve, we derive two conservation laws, involving two real parameters c_1 and c_2. The pair $\mathbf{c} = (c_1, c_2)$ is referred to as the *modulus* of the critical curve.

In Section 4, we introduce the phase type of the modulus of a critical curve. We then define the phase curve of a given modulus and the associated notion of signature of a critical curve with that given modulus. For a generic modulus $\mathbf{c}$, the phase type of $\mathbf{c}$ refers to the properties of the roots of the quintic polynomial in principal form given by

$$\mathrm{P}_{\mathbf{c}}(x) = x^5 + \frac{3}{2}c_2 x^2 + 27 c_1 x - \frac{27}{2}c_1^2.$$

The phase curve of the modulus $\mathbf{c}$ is the real algebraic curve defined by the equation $y^2 = \mathrm{P}_{\mathbf{c}}(x)$. The signature of a critical curve γ with modulus $\mathbf{c}$ and nonconstant twist provides a parametrization of the connected components of the phase curve of $\mathbf{c}$ by the twist of γ. Importantly, the

[1] I.e., with no CR inflection points.

periodicity of the twist of γ amounts to the compactness of the image of the signature of γ. This will play a role in Sections 5 and 6, where the closedness question for critical curves is addressed. Using the Klein formulae for the icosahedral solutions of the quintic [23, 33, 38], the roots of $\mathrm{P}_{\mathbf{c}}$ can be evaluated in terms of hypergeometric functions. As a byproduct, we show that the twist and the bending of a critical curve can be obtained by inverting incomplete hyperelliptic integrals of the first kind. We further specialize our analysis by introducing the orbit type of the modulus $\mathbf{c}$ of a critical curve γ. The orbit type of $\mathbf{c}$ refers to the spectral properties of the momentum associated to γ. Depending on the phase type, the number of connected components of the phase curves, and the orbit type, the critical curves are then divided into twelve classes. The critical curves of only three of these classes have periodic twist.

In Section 5, we show that a *general* critical curve (cf. Definition 5.1) can be integrated by quadratures using the momentum of the curve. This is the content of Theorem B, the second main result of the paper. Theorem B is then specialized to one of the twelve classes of critical curves, the class characterized by the compactness of the connected component of the phase curve and by the existence of three distinct real eigenvalues of the momentum. Theorem C, the third main result, shows that the critical curves of this specific class can be explicitly written by inverting hyperelliptic integrals of the first and third kind. We then examine the closure conditions and prove that a critical curve in this class is closed if and only if certain complete hyperelliptic integrals depending on the modulus of the curve are rational. Finally, the relations between these rational numbers and the global CR invariants mentioned above are discussed.

In the last section, Section 6, we develop convincing heuristic and numerical arguments to support the claim that there exist infinite countably many distinct congruence classes of closed critical curves. These curves are uniquely determined by the four discrete geometric invariants: the wave number, the CR spin, the CR turning number, and the trace with respect to the spacelike λ_1-eigenspace of the momentum. Using numerical tools, we construct and illustrate explicit examples of approximately closed critical curves.

2 Preliminaries

2.1 The standard CR structure on the 3-sphere

Let $\mathbb{C}^{2,1}$ denote $\mathbb{C}^3$ with the indefinite Hermitian scalar product of signature $(2,1)$ given by

$$\langle \mathbf{z}, \mathbf{w} \rangle = {}^t\bar{\mathbf{z}}\mathbf{h}\mathbf{w}, \qquad \mathbf{h} = (h_{ij}) = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 1 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}. \tag{2.1}$$

Following common terminology in pseudo-Riemannian geometry, a nonzero vector $\mathbf{z} \in \mathbb{C}^{2,1}$ is *spacelike*, *timelike* or *lightlike*, depending on whether $\langle \mathbf{z}, \mathbf{z} \rangle$ is positive, negative or zero. By $\mathcal{N}$ we denote the *nullcone*, i.e., the set of all lightlike vectors.

Let $\mathcal{S} = \mathbb{P}(\mathcal{N})$ be the real hypersurface in $\mathbb{C}P^2$ defined by

$$\mathcal{S} = \big\{ [\mathbf{z}] \in \mathbb{C}P^2 \mid \langle \mathbf{z}, \mathbf{z} \rangle = \mathrm{i}(\overline{z}_1 z_3 - \overline{z}_3 z_1) + \overline{z}_2 z_2 = 0 \big\}.$$

The restriction of the affine chart

$$s\colon\ \mathbb{C}^2 \ni (z_1, z_2) \longmapsto \left[{}^t\!\left(\frac{1+z_1}{2}, \mathrm{i}\frac{z_2}{\sqrt{2}}, \mathrm{i}\frac{1-z_1}{2} \right) \right] \in \mathcal{S} \subset \mathbb{C}P^2$$

to the unit sphere S^3 of $\mathbb{C}^2$ defines a smooth diffeomorphism between S^3 and $\mathcal{S}$. For each $p = [\mathbf{z}] \in \mathcal{S}$, the differential $(1,0)$-form

$$\tilde{\zeta}\big|_p = -\frac{\mathrm{i}\langle \mathbf{z}, \mathbf{dz} \rangle}{\bar{\mathbf{z}}\,{}^t\mathbf{z}}\bigg|_p \in \Omega^{1,0}\big(\mathbb{C}P^2\big)\big|_p$$

is well defined. In addition, the null space of the imaginary part of $\tilde{\zeta}|_p$ is $T(\mathcal{S})|_p$, namely the tangent space of $\mathcal{S}$ at p. Thus, the restriction of $\tilde{\zeta}$ to $T(\mathcal{S})$ is a real-valued 1-form $\zeta \in \Omega^1(\mathcal{S})$. Since the pullback of ζ by the diffeomorphism $s\colon \mathrm{S}^3 \to \mathcal{S}$ is the standard contact form $\mathrm{i}\overline{\mathbf{z}} \cdot d\mathbf{z}|_{\mathrm{S}^3}$ of S^3, then ζ is a contact form whose contact distribution $\mathcal{D}$ is, by construction, a complex subbundle of $T\big(\mathbb{C}P^2\big)\big|_{\mathcal{S}}$. Therefore, $\mathcal{D}$ inherits from $T\big(\mathbb{C}P^2\big)\big|_{\mathcal{S}}$ a complex structure J. This defines a CR structure on $\mathcal{S}$.

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ denote the standard basis of $\mathbb{C}^3$. Consider $P_0 = [\mathbf{e}_1] \in \mathcal{S}$ and $P_\infty = [\mathbf{e}_3] \in \mathcal{S}$ as the origin and the point at infinity of $\mathcal{S}$. Then, $\dot{\mathcal{S}} := \mathcal{S} \setminus \{P_\infty\}$ can be identified with Euclidean 3-space with its standard contact structure $\mathrm{d}z - y\mathrm{d}x + x\mathrm{d}y$ by means of the *Heisenberg projection*[2]

$$\pi_H\colon\ \dot{\mathcal{S}} \ni [\mathbf{z}] \longmapsto {}^t(\mathrm{Re}(z_2/z_1), \mathrm{Im}(z_2/z_1), \mathrm{Re}(z_3/z_1)) \in \mathbb{R}^3.$$

The inverse of the Heisenberg projection is the *Heisenberg chart*

$$j_H\colon\ \mathbb{R}^3 \ni {}^t(x,y,z) \longmapsto \left[{}^t\big(1, x+\mathrm{i}y, z + \frac{\mathrm{i}}{2}(x^2+y^2)\big)\right] \in \dot{\mathcal{S}}.$$

The Heisenberg chart can be lifted to a map whose image is a 3-dimensional closed subgroup $\mathbb{H}^3$ of G, which is isomorphic to the 3-dimensional *Heisenberg group* [31].

Let G be the special pseudo-unitary group of (2.1), i.e., the 8-dimensional Lie group of unimodular complex 3×3 matrices preserving (2.1),

$$G = \big\{A \in \mathrm{SL}(3,\mathbb{C}) \mid {}^t\bar{A}\mathbf{h}A = \mathbf{h}\big\} \cong \mathrm{SU}(2,1),$$

and let $\mathfrak{g}$ denote the Lie algebra of G,

$$\mathfrak{g} = \big\{X \in \mathfrak{sl}(3,\mathbb{C}) \mid {}^t\bar{X}\mathbf{h} + \mathbf{h}X = 0\big\}.$$

The Maurer–Cartan form of the group G takes the form

$$\vartheta = A^{-1}\mathrm{d}A = \begin{pmatrix} \alpha_1^1 + \mathrm{i}\beta_1^1 & -\mathrm{i}\alpha_3^2 - \beta_3^2 & \alpha_3^1 \\ \alpha_1^2 + \mathrm{i}\beta_1^2 & -2\mathrm{i}\beta_1^1 & \alpha_3^2 + \mathrm{i}\beta_3^2 \\ \alpha_1^3 & \mathrm{i}\alpha_1^2 + \beta_1^2 & -\alpha_1^1 + \mathrm{i}\beta_1^1 \end{pmatrix},$$

where the 1-forms $\big(\alpha_1^1, \beta_1^1, \alpha_1^2, \beta_1^2, \alpha_1^3, \alpha_3^2, \beta_3^2, \alpha_3^1\big)$ form a basis of the dual Lie algebra $\mathfrak{g}^*$. The center of G is $Z = \big\{\varpi I_3 \mid \varpi \in \mathbb{C},\ \varpi^3 = 1\big\} \cong \mathbb{Z}_3$, where I_3 denotes the 3×3 identity matrix. Let $[G]$ denote the quotient Lie group G/Z and for $A \in G$ let $[A]$ denote its equivalence class in $[G]$. Thus $[A] = [B]$ if and only if $B = \varpi A$, for some cube root of unity ϖ. For any $A \in G$, the column vectors (A_1, A_2, A_3) of A form a basis of $\mathbb{C}^{2,1}$ satisfying $\langle A_i, A_j\rangle = h_{ij}$ and $\det(A_1, A_2, A_3) = 1$. Such a basis is referred to as a *lightcone basis*. On the other hand, a basis $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ of $\mathbb{C}^{2,1}$, such that $\det(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = 1$ and $\langle \mathbf{u}_i, \mathbf{u}_j\rangle = \delta_{ij}\epsilon_j$, where $\epsilon_1 = -1$, $\epsilon_2 = \epsilon_3 = 1$, is referred to as a unimodular *pseudo-unitary basis*.

The group G acts transitively and almost effectively on the left of $\mathcal{S}$ by

$$A[\mathbf{z}] = [A\mathbf{z}], \qquad \forall\, A \in G,\ \forall\, [\mathbf{z}] \in \mathcal{S}.$$

This action descends to an effective action of $[G] = G/Z$ on $\mathcal{S}$. It is a classical result of E. Cartan [5, 6, 7] that $[G]$ is the group of CR automorphisms of $\mathcal{S}$.

If we choose $[\mathbf{e}_1] = \big[{}^t(1,0,0)\big] \in \mathcal{S}$ as an origin of $\mathcal{S}$, the natural projection

$$\pi_{\mathcal{S}}\colon\ G \ni A \mapsto A[\mathbf{e}_1] = [A_1] \in \mathcal{S}$$

[2] This map is the analogue of the stereographic projection in Möbius (conformal) geometry.

makes G into a (trivial) principal fiber bundle with structure group

$$G_0 = \{A \in G \mid A[\mathbf{e}_1] = [\mathbf{e}_1]\}\,.$$

The elements of G_0 consist of all 3×3 unimodular matrices of the form

$$X(\rho, \theta, v, r) = \begin{pmatrix} \rho \mathrm{e}^{\mathrm{i}\theta} & -\mathrm{i}\rho \mathrm{e}^{-\mathrm{i}\theta}\bar{v} & \mathrm{e}^{\mathrm{i}\theta}(r - \frac{\mathrm{i}}{2}\rho|v|^2) \\ 0 & \mathrm{e}^{-2\mathrm{i}\theta} & v \\ 0 & 0 & \rho^{-1}\mathrm{e}^{\mathrm{i}\theta}, \end{pmatrix}, \tag{2.2}$$

where $v \in \mathbb{C}$, $r \in \mathbb{R}$, $0 \leq \theta < 2\pi$, and $\rho > 0$.

Remark 2.1. The left-invariant 1-forms α_1^2, β_1^2, α_1^3 are linearly independent and generate the semi-basic 1-forms for the projection $\pi_{\mathcal{S}}\colon G \to \mathcal{S}$. So, if $s\colon U \subseteq \mathcal{S} \to G$ is a local cross section of $\pi_{\mathcal{S}}$, then $\left(s^*\alpha_1^3, s^*\alpha_1^2, s^*\beta_1^2\right)$ defines a coframe on U and $s^*\alpha_1^3$ is a positive contact form.

2.2 Transversal curves

Definition 2.2. Let $\gamma\colon J \to \mathcal{S}$ be a smooth immersed curve. We say that γ is *transversal* (to the contact distribution $\mathcal{D}$) if the tangent vector $\gamma'(t) \notin \mathcal{D}|_{\gamma(t)}$, for every $t \in J$. The parametrization γ is said to be *positive* if $\zeta(\gamma'(t)) > 0$, for every t and for every positive contact form compatible with the CR structure. From now on, we assume that the parametrization of a transversal curve is positive.

Definition 2.3. Let $\gamma\colon J \to \mathcal{S}$ be a smooth curve. A *lift* of γ is a map $\Gamma\colon J \to \mathcal{N}$ into the nullcone $\mathcal{N} \subset \mathbb{C}^{2,1}$, such that $\gamma(t) = [\Gamma(t)]$, for every $t \in J$.

If Γ is a lift, any other lift is given by $r\Gamma$, where r is a smooth complex-valued function, such that $r(t) \neq 0$, for every $t \in J$. From the definition of the contact distribution, we have the following.

Proposition 2.4. *A parametrized curve $\gamma\colon J \to \mathcal{S}$ is transversal and positively oriented if and only if $-\mathrm{i}\langle\Gamma, \Gamma'\rangle|_t > 0$, for every $t \in J$ and for every lift Γ.*

Definition 2.5. A *frame field along* $\gamma\colon J \to \mathcal{S}$ is a smooth map $A\colon J \to G$ such that $\pi_{\mathcal{S}} \circ A = \gamma$. Since the fibration $\pi_{\mathcal{S}}$ is trivial, there exist frame fields along every transversal curve. If $A = (A_1, A_2, A_3)$ is a frame field along γ, A_1 is a lift of γ.

Let A be a frame field along γ. Then

$$A^{-1}A' = \begin{pmatrix} a_1^1 + \mathrm{i}b_1^1 & -\mathrm{i}a_3^2 - b_3^2 & a_3^1 \\ a_1^2 + \mathrm{i}b_1^2 & -2\mathrm{i}b_1^1 & a_3^2 + \mathrm{i}b_3^2 \\ a_1^3 & \mathrm{i}a_1^2 + b_1^2 & -a_1^1 + \mathrm{i}b_1^1 \end{pmatrix},$$

where a_1^3 is a strictly positive real-valued function. Any other frame field along γ is given by $\tilde{A} = AX(\rho, \theta, v, r)$, where ρ ($\rho > 0$), θ, $r\colon J \to \mathbb{R}$, $v = p + \mathrm{i}q\colon J \to \mathbb{C}$ are smooth functions and $X(\rho, \theta, v, r)\colon J \to G_0$ is as in (2.2). If we let

$$\tilde{A}^{-1}\tilde{A}' = \begin{pmatrix} \tilde{a}_1^1 + \mathrm{i}\tilde{b}_1^1 & -\mathrm{i}\tilde{a}_3^2 - \tilde{b}_3^2 & \tilde{a}_3^1 \\ \tilde{a}_1^2 + \mathrm{i}\tilde{b}_1^2 & -2\mathrm{i}\tilde{b}_1^1 & \tilde{a}_3^2 + \mathrm{i}\tilde{b}_3^2 \\ \tilde{a}_1^3 & \mathrm{i}\tilde{a}_1^2 + \tilde{b}_1^2 & -\tilde{a}_1^1 + \mathrm{i}\tilde{b}_1^1 \end{pmatrix},$$

then

$$\tilde{A}^{-1}\tilde{A}' = X^{-1}A^{-1}A'X + X^{-1}X',$$

which implies

$$\tilde{a}_1^3 = \rho^2 a_1^3, \qquad \tilde{a}_1^2 + \mathrm{i}\tilde{b}_1^2 = \rho \mathrm{e}^{3\mathrm{i}\theta}\big(a_1^2 + \mathrm{i}b_1^2\big) - \rho^2 \mathrm{e}^{2\mathrm{i}\theta}(p + \mathrm{i}q)a_1^3.$$

From this it follows that along any parametrized transversal curve there exists a frame field A for which $a_1^2 + \mathrm{i}b_1^2 = 0$. Such a frame field is said to be of *first order*.

Definition 2.6. Let Γ be a lift of a transversal curve $\gamma\colon J \to \mathcal{S}$. If

$$\det(\Gamma, \Gamma', \Gamma'')|_{t_0} = 0,$$

for some $t_0 \in J$, then $\gamma(t_0)$ is called a *CR inflection point*. The notion of CR inflection point is independent of the lift Γ. A transversal curve with no CR inflection points is said to be *generic*. The notion of a CR inflection point is invariant under reparametrizations and under the action of the group of CR automorphisms.

Remark 2.7. If $A\colon J \to G$ is a frame field along a transversal curve γ, then $\gamma(t_0)$ is a CR inflection point if $\det(A_1, A_1', A_1'')\big|_{t_0} = 0$. A transversal curve all of whose points are CR inflection points is called a *chain*. The notion of chain on a CR manifold goes back to Cartan [5, 6] (see also [22] and the literature therein).

If γ is transversal and Γ is one of its lifts, then the complex plane $[\Gamma \wedge \Gamma']_t$ is of type $(1,1)$ and the set of null complex lines contained in $[\Gamma \wedge \Gamma']_t$ is a chain which is independent of the choice of the lift Γ. This chain, denoted by $\mathcal{C}_\gamma|_t$, is called the *osculating chain* of γ at $\gamma(t)$. By construction, $\mathcal{C}_\gamma|_t$ is the unique chain passing through $\gamma(t)$ and tangent to γ at the contact point $\gamma(t)$. For more details on the CR-geometry of transversal curves in the 3-sphere, we refer to [31]. As a basic reference for transversal knots and their topological invariants in the framework of 3-dimensional contact geometry, we refer to [14] and the literature therein.

2.3 The canonical frame and the local CR invariants

In the following, we will consider generic transversal curves.

Definition 2.8. Let γ be a generic transversal curve. A lift Γ of γ, such that

$$\det(\Gamma, \Gamma', \Gamma'') = -1,$$

is said to be a *Wilczynski lift* (W-lift) of γ. If Γ is a Wilczynski lift, any other is given by $\varpi\Gamma$, where $\varpi \in \mathbb{C}$ is a cube root of unity. The function

$$a_\gamma = \mathrm{i}\langle \Gamma, \Gamma' \rangle^{-1}$$

is smooth, real-valued, and independent of the choice of Γ. We call a_γ the *strain density* of the parametrized transversal curve γ. The linear differential form $\mathrm{d}s = a_\gamma \mathrm{d}t$ is called the *infinitesimal strain*.

Proposition 2.9 ([31])**.** *The strain density and the infinitesimal strain are invariant under the action of the CR transformation group. In addition, if $h\colon I \to J$ is a change of parameter, then the infinitesimal strains $\mathrm{d}s$ and $\mathrm{d}\tilde{s}$ of γ and $\tilde{\gamma} = \gamma \circ h$, respectively, are related by $\mathrm{d}\tilde{s} = h^*(\mathrm{d}s)$.*

Proof. This proof corrects a few misprints contained in the original one. If $A \in G$ and if Γ is a Wilczynski lift of γ, then $\hat{\Gamma} = A\Gamma$ is a Wilczynski lift of $\hat{\gamma} = A\gamma$. This implies that $a_\gamma = a_{\hat{\gamma}}$. Next, consider a reparametrization $\tilde{\gamma} = \gamma \circ h$ of γ. Then, $\Gamma^* = \Gamma \circ h$ is a lift of $\tilde{\gamma}$, such that

$$\det\big(\Gamma^*, (\Gamma^*)', (\Gamma^*)''\big) = -(h')^3.$$

This implies that $\tilde{\Gamma} = (h')^{-1}\Gamma^*$ is a Wilczynski lift of $\tilde{\gamma}$. Hence

$$\langle \tilde{\Gamma}, (\tilde{\Gamma})' \rangle = (h')^{-1} \langle \Gamma, \Gamma' \rangle \circ h.$$

Therefore, the strain densities of γ and $\tilde{\gamma}$ are related by $a_{\tilde{\gamma}} = h' a_\gamma \circ h$. Consequently, we have $h^*(\mathrm{d}s) = h' a_\gamma \circ h \, \mathrm{d}t = a_{\tilde{\gamma}} \mathrm{d}t = \mathrm{d}\tilde{s}$. ■

As a straightforward consequence of Proposition 2.9, we have the following.

Corollary 2.10. *A generic transversal curve γ can be parametrized so that $a_\gamma = 1$.*

Definition 2.11. If $a_\gamma = 1$, we say that $\gamma\colon J \to \mathcal{S}$ is a *natural parametrization*, or a parametrization by the *pseudoconformal strain* or *pseudoconformal parameter*. In the following, the *natural parameter* will be denoted by s.

We can state the following.

Proposition 2.12 ([31])**.** *Let $\gamma\colon J \to \mathcal{S}$ be a generic transversal curve, parametrized by the natural parameter. There exists a (first order) frame field $\mathcal{F} = (F_1, F_2, F_3)\colon J \to G$ along γ, such that F_1 is a W-lift and*

$$\mathcal{F}^{-1}\mathcal{F}' = \begin{pmatrix} \mathrm{i}\kappa & -\mathrm{i} & \tau \\ 0 & -2\mathrm{i}\kappa & 1 \\ 1 & 0 & \mathrm{i}\kappa \end{pmatrix} =: K_{\kappa,\tau}(s), \tag{2.3}$$

where $\kappa, \tau\colon J \to \mathbb{R}$ are smooth functions, called the CR bending *and the* CR twist*, respectively. The frame field $\mathcal{F}$ is called a* Wilczynski frame*. If $\mathcal{F}$ is a Wilczynski frame, any other is given by $\varpi\mathcal{F}$, where ϖ is a cube root of unity. Thus, there exists a unique frame field $[\mathcal{F}]\colon J \to [G]$ along γ, called the* canonical frame *of γ.*

Remark 2.13. Given two smooth functions $\kappa, \tau\colon J \to \mathbb{R}$, there exists a generic transversal curve $\gamma\colon J \to \mathcal{S}$, parametrized by the natural parameter, whose bending is κ and whose twist is τ. The curve γ is unique up to CR automorphisms of $\mathcal{S}$.

Remark 2.14 (cf. [31])**.**

(1) Let $\gamma\colon J \to \mathcal{S}$ be as above and $\mathcal{F} = (F_1, F_2, F_3)\colon J \to G$ be a Wilczynski frame along γ. Then,

$$\gamma^{\#}\colon \ J \ni s \mapsto [F_3(s)]_{\mathbb{C}} \in \mathcal{S}$$

is an immersed curve, called the *dual* of γ. The dual curve is Legendrian (i.e., tangent to the contact distribution) if and only if $\tau = 0$. Thus, the twist can be viewed as a measure of how the dual curve differs from being a Legendrian curve.

(2) Generic transversal curves with constant bending and twist have been studied by the authors in [31]. In the following we will consider generic transversal curves with nonconstant CR invariant functions.

Remark 2.15. Regarding the CR 3-sphere $\mathrm{S}^3 \cong \mathcal{S}$ with its standard pseudo-hermitian (PSH) structure (J, ζ), Chiu and Ho (cf. [8]) obtained a complete set of local PSH invariants for *horizontally regular* curves in S^3 parametrized by the *horizontal arc length* w, namely the *p-curvature* $k_{\mathrm{PSH}}(\mathrm{w})$ and the *T-variation* $\tau_{\mathrm{PSH}}(\mathrm{w})$. The canonical PSH frame field producing the PSH invariants originates a CR frame field which can be further adapted to a canonical CR frame following the reduction procedure developed in [31]. From this one can read the CR invariants. Thus, in principle, the CR bending $\kappa(s)$ and the CR twist $\tau(s)$ can be expressed in terms of the PSH invariants $k_{\mathrm{PSH}}(\mathrm{w}(s))$, $\tau_{\mathrm{PSH}}(\mathrm{w}(s))$ and their derivatives with respect to s.

2.4 Discrete CR invariants of a closed transversal curve

Referring to [31, 32], we briefly recall some CR invariants for closed transversal curves, namely the notions of wave number, CR spin, and CR turning number (or Maslov index). These invariants will be used in Sections 5 and 6. The *wave number* is the ratio between the least period ω_γ of γ and the least period ω of the functions (κ, τ). The *CR spin* is the ratio between ω_γ and the least period of a Wilczynski lift of γ. The *CR turning number* is the degree (winding number) of the map $F_1 - \mathrm{i}F_3 \colon \mathbb{R}/\omega_\gamma\mathbb{Z} \cong \mathrm{S}^1 \to \dot{\mathbb{C}} = \mathbb{C}\backslash\{0\}$, where $\mathcal{F} = (F_1, F_2, F_3)$ is a Wilczynski frame along γ.

We will also make use of another invariant.

Definition 2.16. Let $[\mathbf{z}] \in \mathbb{CP}^2$ be a spacelike line. Denote by $\mathtt{C}_{[\mathbf{z}]}$ the chain of all null lines orthogonal to $[\mathbf{z}]$, equipped with its positive orientation. Consider a closed generic transversal curve γ with its positive orientation. Since γ is closed and generic, the intersection of γ with $\mathtt{C}_{[\mathbf{z}]}$ is either a finite set of points, or the empty set. The *trace of γ with respect to* $[\mathbf{z}]$, denoted by $\mathrm{tr}_{[\mathbf{z}]}(\gamma)$, is the integer defined as follows: (1) if $\gamma \cap \mathtt{C}_{[\mathbf{z}]} \neq \varnothing$, then $\mathrm{tr}_{[\mathbf{z}]}(\gamma)$ counts the number of intersection points of γ with $\mathtt{C}_{[\mathbf{z}]}$ (since γ is not necessarily a simple curve, the intersection points are counted with their multiplicities); (2) otherwise, $\mathrm{tr}_{[\mathbf{z}]}(\gamma) = \mathrm{Lk}(\gamma, \mathtt{C}_{[\mathbf{z}]})$, the linking number of γ with $\mathtt{C}_{[\mathbf{z}]}$. The trace of γ is a G-equivariant map, that is, $\mathrm{tr}_{[\mathbf{z}]}(\gamma) = \mathrm{tr}_{A[\mathbf{z}]}(A\gamma)$, for every $A \in G$.

3 The total CR twist functional

Let $\mathfrak{T}$ be the space of generic transversal curves in $\mathcal{S}$, parametrized by the natural parameter. We consider the *total CR twist functional* $\mathcal{W} \colon \mathfrak{T} \to \mathbb{R}$, defined by

$$\mathcal{W}[\gamma] = \int_{J_\gamma} \tau_\gamma \eta_\gamma,$$

where J_γ is the domain of definition of the transversal curve γ, τ_γ is its twist, and $\eta_\gamma = \mathrm{d}s_\gamma$ is the infinitesimal strain of γ (cf. Section 2.3).

A curve $\gamma \in \mathfrak{T}$ is said to be a *critical curve* in $\mathcal{S}$ if it is a critical point of $\mathcal{W}$ when one considers compactly supported variations through generic transversal curves.

The main result of this section is the following.

Theorem A. *Let $\gamma \colon J \to \mathcal{S}$ be a generic transversal curve parametrized by the natural parameter. Then, γ is a critical curve if and only if* [3]

$$L'(s) = [L(s), K_{\kappa,\tau}(s)], \tag{3.1}$$

where

$$L = \begin{pmatrix} 0 & \mathrm{i}\tau' + 3(1 - \tau\kappa) & 2\mathrm{i}\tau \\ \tau & 0 & \tau' + 3\mathrm{i}(1 - \tau\kappa) \\ 3\mathrm{i} & -\mathrm{i}\tau & 0 \end{pmatrix} \tag{3.2}$$

and $K_{\kappa,\tau}$ is defined as in (2.3).

Proof. The proof of Theorem A is organized in four steps and three lemmas.

Step 1. We show that generic transversal curves are in 1-1 correspondence with the integral curves of a suitable Pfaffian differential system.

[3] As usual, we write $[L, K] = LK - KL$ for the commutator of L and K.

Let $\gamma\colon J \to \mathcal{S}$ be a generic transversal curve parametrized by the natural parameter. According to Proposition 2.12, the canonical frame of γ defines a unique lift $[\mathcal{F}]\colon J \to [G]$. The map

$$\mathfrak{f}\colon\ J \ni s \longmapsto ([\mathcal{F}(s)], \kappa(s), \tau(s)) \in [G] \times \mathbb{R}^2$$

is referred to as the *extended frame* of γ. The product space $M := [G] \times \mathbb{R}^2$ is called the *configuration space*. The coordinates on $\mathbb{R}^2$ will be denoted by (κ, τ).

With some abuse of notation, we use α_1^1, β_1^1, α_1^2, β_1^2, α_1^3, α_3^2, β_3^2, α_3^1 to denote the entries of the Maurer–Cartan form of $[G]$ as well as their pull-backs on the configuration space M. By Proposition 2.12, the extended frames of γ are the integral curves of the Pfaffian differential system $(\mathcal{A}, \eta)$ on M generated by the linearly independent 1-forms

$$\begin{aligned} &\mu^1 = \alpha_1^2, \qquad \mu^2 = \beta_1^2, \qquad \mu^3 = \alpha_3^2 - \alpha_1^3, \qquad \mu^4 = \beta_3^2,\\ &\mu^5 = \alpha_1^1, \qquad \mu^6 = \beta_1^1 - \kappa\alpha_1^3, \qquad \mu^7 = \alpha_3^1 - \tau\alpha_1^3, \end{aligned}$$

with the independence condition $\eta := \alpha_1^3$.

If $\mathfrak{f} = ([\mathcal{F}], \kappa, \tau)\colon J \to M$ is an integral curve of $(\mathcal{A}, \eta)$, then $\gamma = [F_1]\colon J \to \mathcal{S}$ defines a generic transversal curve, such that $[\mathcal{F}]$ is its canonical frame, κ its bending and τ its twist. Accordingly, the integral curves of $(\mathcal{A}, \eta)$ are the extended frames of generic transversal curves in $\mathcal{S}$.

Thus, generic transversal curves are in 1-1 correspondence with the integral curves of the Pfaffian system $(\mathcal{A}, \eta)$ on the configuration space M.

If we put

$$\pi^1 = \mathrm{d}\kappa, \qquad \pi^2 = \mathrm{d}\tau,$$

the 1-forms $\big(\eta, \mu^1, \dots, \mu^7, \pi^1, \pi^2\big)$ define an absolute parallelism on M. Exterior differentiation and use of the Maurer–Cartan equations of G yield the following structure equations for the coframe $\big(\eta, \mu^1, \dots, \mu^7, \pi^1, \pi^2\big)$:

$$\begin{cases} \mathrm{d}\eta = 2\mu^1 \wedge \mu^2 + 2\mu^5 \wedge \eta, \\ \mathrm{d}\pi^1 = d\pi^2 = 0, \end{cases} \tag{3.3}$$

$$\begin{cases} \mathrm{d}\mu^1 = -\mu^1 \wedge \mu^5 + 3\mu^2 \wedge \mu^6 + \big(3\kappa\mu^2 - \mu^3\big) \wedge \eta, \\ \mathrm{d}\mu^2 = -3\mu^1 \wedge \mu^6 - \mu^2 \wedge \mu^3 - \big(3\kappa\mu^1 + \mu^4\big) \wedge \eta, \\ \mathrm{d}\mu^3 = -2\mu^1 \wedge \mu^2 - \mu^1 \wedge \mu^7 + \mu^3 \wedge \mu^5 + 3\mu^4 \wedge \mu^6 - \big(\tau\mu^1 - 3\kappa\mu^4 + 3\mu^5\big) \wedge \eta, \\ \mathrm{d}\mu^4 = -\mu^2 \wedge \mu^7 - 3\mu^3 \wedge \mu^6 + \mu^4 \wedge \mu^5 - \big(\tau\mu^2 + 3\kappa\mu^3 - 3\mu^6\big) \wedge \eta, \\ \mathrm{d}\mu^5 = -\mu^1 \wedge \mu^4 + \mu^2 \wedge \mu^3 + \big(\mu^2 - \mu^7\big) \wedge \eta, \\ \mathrm{d}\mu^6 = -2\kappa\mu^1 \wedge \mu^2 - \mu^1 \wedge \mu^3 - \mu^2 \wedge \mu^4 - \big(\mu^1 + 2\kappa\mu^5\big) \wedge \eta - \pi^1 \wedge \eta, \\ \mathrm{d}\mu^7 = -2\tau\mu^1 \wedge \mu^2 - 2\mu^3 \wedge \mu^4 - 2\mu^5 \wedge \mu^7 + \big(2\mu^4 - 2\tau\mu^5\big) \wedge \eta - \pi^2 \wedge \eta. \end{cases} \tag{3.4}$$

Remark 3.1. From the structure equations it follows that the derived flag of $(\mathcal{A}, \eta)$ is given by $\mathcal{A}_{(4)} \subset \mathcal{A}_{(3)} \subset \mathcal{A}_{(2)} \subset \mathcal{A}_{(1)}$, where $\mathcal{A}_{(4)} = \{0\}$, $\mathcal{A}_{(3)} = \operatorname{span}\big\{\mu^1\big\}$, $\mathcal{A}_{(2)} = \operatorname{span}\big\{\mu^1, \mu^2, \mu^3\big\}$, $\mathcal{A}_{(1)} = \operatorname{span}\big\{\mu^1, \mu^2, \mu^3, \mu^4, \mu^5\big\}$. Thus, all the derived systems of $(\mathcal{A}, \eta)$ have constant rank. For the notion of derived flag, see [19].

Step 2. We develop a construction due to Griffiths [19] on an affine subbundle of $T^*(M)$ (cf. also [2, 21, 26]) in order to derive the Euler–Lagrange equations.

Let $\mathcal{Z} \subset T^*(M)$ be the affine subbundle defined by the 1-forms $\mu^1, \dots, \mu^7$ and $\lambda := \tau\eta$, namely

$$\mathcal{Z} = \lambda + \operatorname{span}\big\{\mu^1, \dots, \mu^7\big\} \subset T^*(M).$$

We call $\mathcal{Z}$ the *phase space* of the Pfaffian system $(\mathcal{A},\eta)$. The 1-forms $\big(\mu^1,\dots,\mu^7,\lambda\big)$ induce a global affine trivialization of $\mathcal{Z}$, which may be identified with $M\times\mathbb{R}^7$ by the map

$$M\times\mathbb{R}^7\ni(([\mathcal{F}],\kappa,\tau),p_1,\dots,p_7)\longmapsto\lambda_{|_{([\mathcal{F}],\kappa,\tau)}}+\sum_{j=1}^7 p_j\mu^j{}_{|_{([\mathcal{F}],\kappa,\tau)}}\in\mathcal{Z},$$

where $p_1,\dots,p_7$ are the fiber coordinates of the bundle map $\mathcal{Z}\to M$ with respect to the trivialization. Under this identification, the restriction to $\mathcal{Z}$ of the Liouville (canonical) 1-form of $T^*(M)$ takes the form

$$\xi=\tau\eta+\sum_{j=1}^7 p_j\mu^j.$$

Exterior differentiation and use of the quadratic equations (3.3) and (3.4) yield

$$\begin{aligned}\mathrm{d}\xi\equiv{}&\pi^2\wedge\eta+2\tau\mu^5\wedge\eta+\sum_{j=1}^7\mathrm{d}p_j\wedge\mu^j+p_1\big(3\kappa\mu^2-\mu^3\big)\wedge\eta\\&-p_2\big(3\kappa\mu^1+\mu^4\big)\wedge\eta-p_3\big(\tau\mu^1-3\kappa\mu^4+3\mu^5\big)\wedge\eta\\&-p_4\big(\tau\mu^2+3\kappa\mu^3-3\mu^6\big)\wedge\eta+p_5\big(\mu^2-\mu^7\big)\wedge\eta\\&-p_6\big(\pi^1+\mu^1+2\kappa\mu^5\big)\wedge\eta-p_7\big(\pi^2-2\mu^4+4\tau\mu^5\big)\wedge\eta,\end{aligned}$$

where the sign '$\equiv$' denotes equality modulo the span of $\{\mu^i\wedge\mu^j\}_{i,j=1,\dots,7}$.

The Cartan system $(\mathcal{C}(\mathrm{d}\xi),\eta)$ of the 2-form $\mathrm{d}\xi$ is the Pfaffian system on $\mathcal{Z}$ generated by the 1-forms

$$\{X\lrcorner\,\mathrm{d}\xi\mid X\in\mathfrak{X}(\mathcal{Z})\}\subset\Omega^1(\mathcal{Z}),$$

with independence condition $\eta\neq0$.

By Step 1, generic transversal curves are in 1-1 correspondence with the integral curves of the Pfaffian system $(\mathcal{A},\eta)$.

Let $\mathfrak{f}\colon J\to M$ be the extended frame corresponding to the generic transversal curve $\gamma\colon J\to\mathcal{S}$ parametrized by the natural parameter. According to Griffiths approach to the calculus of variations (cf. [2, 19, 21, 26]), if the extended frame $\mathfrak{f}$ admits a lift $y\colon J\to\mathcal{Z}$ to the phase space $\mathcal{Z}$ which is an integral curve of the Cartan system $(\mathcal{C}(\mathrm{d}\xi),\eta)$, then γ is a critical curve of the total twist functional with respect to compactly supported variations.

As observed by Bryant [2], if all the derived systems of $(\mathcal{A},\eta)$ are of constant rank, as in the case under discussion (cf. Remark 3.1), then the converse is also true. Hence all extremal trajectories arise as projections of integral curves of the Cartan system $(\mathcal{C}(\mathrm{d}\xi),\eta)$.

Next, we compute the Cartan system $(\mathcal{C}(\mathrm{d}\xi),\eta)$. Contracting the 2-form $\mathrm{d}\xi$ with the vector fields of the tangent frame

$$(\partial_\eta,\partial_{\mu^1},\dots,\partial_{\mu^7},\partial_{\pi^1},\partial_{\pi^2},\partial_{p_1},\dots,\partial_{p_7})$$

on $\mathcal{Z}$, dual to the coframe $\big(\eta,\mu^1,\dots,\mu^7,\pi^1,\pi^2,\mathrm{d}p_1,\dots,\mathrm{d}p_7\big)$, yields the 1-forms

$$\begin{aligned}&\partial_{p_j}\lrcorner\,\mathrm{d}\xi\equiv\mu^j,\qquad j=1,\dots,7,\\&\begin{cases}-\partial_{\pi^1}\lrcorner\,\mathrm{d}\xi\equiv p_6\eta=:\dot\pi_1,\\-\partial_{\pi^2}\lrcorner\,\mathrm{d}\xi\equiv(p_7-1)\eta=:\dot\pi_2,\\-\partial_\eta\lrcorner\,\mathrm{d}\xi\equiv(1-p_7)\pi^2=:\dot\eta,\end{cases}\end{aligned}\tag{3.5}$$

$$\begin{cases} -\partial_{\mu^1} \lrcorner\, \mathrm{d}\xi \equiv \mathrm{d}p_1 + (3\kappa p_2 + \tau p_3 + p_6)\eta =: \dot{\mu}^1, \\ -\partial_{\mu^2} \lrcorner\, \mathrm{d}\xi \equiv \mathrm{d}p_2 - (3\kappa p_1 - \tau p_4 + p_5)\eta =: \dot{\mu}^2, \\ -\partial_{\mu^3} \lrcorner\, \mathrm{d}\xi \equiv \mathrm{d}p_3 + (p_1 + 3\kappa p_4)\eta =: \dot{\mu}^3, \\ -\partial_{\mu^4} \lrcorner\, \mathrm{d}\xi \equiv \mathrm{d}p_4 + (p_2 - 3\kappa p_3 - 2p_7)\eta =: \dot{\mu}^4, \\ -\partial_{\mu^5} \lrcorner\, \mathrm{d}\xi \equiv \mathrm{d}p_5 - (2\tau - 3p_3 - 2\kappa p_6 - 4\tau p_7)\eta =: \dot{\mu}^5, \\ -\partial_{\mu^6} \lrcorner\, \mathrm{d}\xi \equiv \mathrm{d}p_6 - 3p_4\eta =: \dot{\mu}^6, \\ -\partial_{\mu^7} \lrcorner\, \mathrm{d}\xi \equiv \mathrm{d}p_7 + p_5\eta =: \dot{\mu}^7. \end{cases}$$

We have proved the following.

Lemma A1. *The Cartan system $(\mathcal{C}(\mathrm{d}\xi), \eta)$ is the Pfaffian system on $\mathcal{Z} \cong M \times \mathbb{R}^7$ generated by the 1-forms*

$$\{\mu^1, \dots, \mu^7, \dot{\pi}_1, \dot{\pi}_2, \dot{\eta}, \dot{\mu}^1, \dots, \dot{\mu}^7\}$$

and with independence condition $\eta \neq 0$.

Now, the Cartan system $(\mathcal{C}(\mathrm{d}\xi), \eta)$ is reducible, i.e., there exists a nonempty submanifold $\mathcal{Y} \subseteq \mathcal{Z}$, called the reduced space, such that: (1) at each point of $\mathcal{Y}$ there exists an integral element of $(\mathcal{C}(\mathrm{d}\xi), \eta)$ tangent to $\mathcal{Y}$; (2) if $\mathcal{X} \subseteq \mathcal{Z}$ is any other submanifold with the same property of $\mathcal{Y}$, then $\mathcal{X} \subseteq \mathcal{Y}$. The reduced space $\mathcal{Y}$ is called the *momentum space* of the variational problem. Moreover, the restriction of the Cartan system $(\mathcal{C}(\mathrm{d}\xi), \eta)$ to $\mathcal{Y}$ is called the *Euler–Lagrange system* of the variational problem, and will be denoted by $(\mathcal{J}, \eta)$.

A basic result states that the Pfaffian systems $(\mathcal{C}(\mathrm{d}\xi), \eta)$ and $(\mathcal{J}, \eta)$ have the same integral curves (cf. [19, 26]).

The system $(\mathcal{J}, \eta)$ can be constructed by an algorithmic procedure (cf. [19]).

Lemma A2. *The momentum space $\mathcal{Y}$ is the 11-dimensional submanifold of $\mathcal{Z}$ defined by the equations*

$$p_7 = 1, \qquad p_6 = p_5 = p_4 = 0, \qquad p_3 = -\frac{2}{3}\tau, \qquad p_2 = 2(1 - \tau\kappa).$$

The Euler–Lagrange system $(\mathcal{J}, \eta)$ is the Pfaffian system on $\mathcal{Y} \cong M \times \mathbb{R}$, with independence condition $\eta \neq 0$, generated by the 1-forms

$$\begin{cases} {\mu^1}_{|\mathcal{Y}}, \dots, {\mu^7}_{|\mathcal{Y}}, \\ \sigma_1 = \mathrm{d}p_1 + 6\kappa(1 - \tau\kappa)\eta - \frac{2}{3}\tau^2\eta, \\ \sigma_2 = -2\tau\mathrm{d}\kappa - 2\kappa\mathrm{d}\tau - 3kp_1\eta, \\ \sigma_3 = -2\mathrm{d}\tau + 3p_1\eta. \end{cases} \tag{3.6}$$

Proof of Lemma A2. Let $V_1(\mathrm{d}\xi) \hookrightarrow \mathbb{P}(T(\mathcal{Z})) \to \mathcal{Z}$ be the totality of 1-dimensional integral elements of $(\mathcal{C}(\mathrm{d}\xi), \eta)$. In view of (3.5), we find that

$$V_1(\mathrm{d}\xi)_{|_{(([\mathcal{F}],\kappa,\tau);p_1,\dots,p_7)}} \neq \varnothing \iff p_6 = 0,\ p_7 = 1.$$

Thus, the image $\mathcal{Z}_1 \subset \mathcal{Z}$ of $V_1(\mathrm{d}\xi)$ with respect to the natural projection $V_1(\mathrm{d}\xi) \to \mathcal{Z}$, is given by

$$\mathcal{Z}_1 = \{(([\mathcal{F}], \kappa, \tau); p_1, \dots, p_7) \in \mathcal{Z} \mid p_6 = 0,\ p_7 = 1\}.$$

Next, the restriction of $\dot{\mu}^6$ and $\dot{\mu}^7$ to $\mathcal{Z}_1$ take the form $\dot{\mu}^6 = -3p_4\eta$ and $\dot{\mu}^7 = p_5\eta$. Thus, the second reduction $\mathcal{Z}_2$ is given by

$$\mathcal{Z}_2 = \{(([\mathcal{F}], \kappa, \tau); p_1, \dots, p_7) \in \mathcal{Z}_1 \mid p_4 = p_5 = 0\}.$$

Considering the restriction of $\dot\mu^4$ and $\dot\mu^5$ to $\mathcal{Z}_2$ yields the equations

$$p_2 = 2(1-\tau\kappa), \qquad p_3 = -\frac{2}{3}\tau,$$

which define the third reduction $\mathcal{Z}_3$. Now, the restriction $\mathcal{C}_3(\mathrm{d}\xi)$ to $\mathcal{Z}_3$ of the Cartan system $\mathcal{C}(\mathrm{d}\xi)$ is generated by the 1-forms $\mu^1,\dots,\mu^7$ and

$$\begin{cases} \sigma_1 = \mathrm{d}p_1 + 6\kappa(1-\tau\kappa)\eta - \frac{2}{3}\tau^2\eta, \\ \sigma_2 = \mathrm{d}p_2 - 3kp_1\eta = -2\tau\mathrm{d}\kappa - 2\kappa\mathrm{d}\tau - 3kp_1\eta, \\ \sigma_3 = -2\mathrm{d}\tau + 3p_1\eta. \end{cases}$$

This implies that there exists an integral element of $V_1(\mathrm{d}\xi)$ over each point of $\mathcal{Z}_3$, i.e., for each $p\in\mathcal{Z}_3$, $V_1(\mathrm{d}\xi)_{|p}\neq\varnothing$. Hence, $\mathcal{Y}:=\mathcal{Z}_3$ is the momentum space and $(\mathcal{J},\eta):=(\mathcal{C}_3(\mathrm{d}\xi),\eta)$ is the reduced system of $(\mathcal{C}(\mathrm{d}\xi),\eta)$. ■

Step 3. We derive the Euler–Lagrange equations.

By the previous discussion, all the extremal trajectories of $\mathcal{S}$ arise as projections of the integral curves of the Euler–Lagrange system. If $y\colon J\to\mathcal{Y}$ is an integral curve of the Euler–Lagrange system $(\mathcal{J},\eta)$ and $\mathtt{pr}\colon\mathcal{Y}\to\mathcal{S}$ is the natural projection of $\mathcal{Y}$ onto $\mathcal{S}$, then $\gamma=\mathtt{pr}\circ y\colon J\to\mathcal{S}$ is a critical curve of the total twist functional with respect to compactly supported variations.

We can prove the following.

Lemma A3. *A curve $y\colon J\to\mathcal{Y}$ is an integral curve of the Euler–Lagrange system $(\mathcal{J},\eta)$ if and only if the bending κ and the twist τ of the transversal curve $\gamma=\mathtt{pr}\circ y\colon J\to\mathcal{S}$ satisfy the equations*

$$2\kappa\tau' + \tau\kappa' = 0, \tag{3.7}$$

$$\tau'' + 9\kappa(1-\tau\kappa) - \tau^2 = 0. \tag{3.8}$$

Proof of Lemma A3. If $y=(([\mathcal{F}],\kappa,\tau);p_1)\colon J\to\mathcal{Y}$ is an integral curve of the Euler–Lagrange system $(\mathcal{J},\eta)$, the projection $\gamma=\mathtt{pr}\circ y$ is the smooth curve $\gamma(s)=[F_1(s)]$, where F_1 is the first column of $\mathcal{F}$. The equations

$$\mu^1=\dots=\mu^7=0$$

together with the independence condition $\eta\neq0$ tell us that $([\mathcal{F}],\kappa,\tau)$ is an integral curve of the Pfaffian system $(\mathcal{A},\eta)$ on the configuration space M. Hence γ is a generic transversal curve with bending κ, twist τ and $\mathcal{F}$ is a Wilczynski frame along γ. Next, for the smooth function $\kappa,\tau\colon J\to\mathbb{R}$, let κ', κ'' and τ', τ'', etc., be defined by

$$\mathrm{d}\kappa=\kappa'\eta, \qquad \mathrm{d}\kappa'=\kappa''\eta, \qquad \mathrm{d}\tau=\tau'\eta, \qquad \mathrm{d}\tau'=\tau''\eta.$$

With reference to (3.6), equation $\sigma_3=0$ implies

$$p_1=\frac{2}{3}\tau'.$$

Further, $\sigma_2=0$ gives

$$2\kappa\tau'+\tau\kappa'=0.$$

Finally, equation $\sigma_1=0$ yields

$$\tau''+9\kappa(1-\tau\kappa)-\tau^2=0.$$

Conversely, let $\gamma\colon J \to \mathcal{S}$ be a generic transversal curve, parametrized by the natural parameter, satisfying (3.7) and (3.8) and let $[\mathcal{F}]$ its canonical frame. Then,

$$y(s) = \left(([\mathcal{F}], \kappa, \tau); \frac{2}{3}\tau' \right)$$

is, by construction, an integral curve of the Euler–Lagrange system $(\mathcal{J}, \eta)$. ■

Step 4. We eventually provide a Lax formulation for the Euler–Lagrange equations (cf. (3.7) and (3.8)) of a critical curve $\gamma\colon J \to \mathcal{S}$.

Using the Killing form of $\mathfrak{g}$, the dual Lie algebra $\mathfrak{g}^*$ can be identified with $\mathfrak{h} = \mathrm{i}\mathfrak{g}$, the G-module of traceless selfadjoint endomorphisms of $\mathbb{C}^{2,1}$. Under this identification, the restriction to $\mathcal{Y}$ of the tautological 1-form ξ goes over to an element of $\mathfrak{h}$ which originates the $\mathfrak{h}$-valued function $L\colon J \to \mathfrak{h}$ given by

$$L(s) = \begin{pmatrix} 0 & \mathrm{i}\tau' + 3(1-\tau\kappa) & 2\mathrm{i}\tau \\ \tau & 0 & \tau' + 3\mathrm{i}(1-\tau\kappa) \\ 3\mathrm{i} & -\mathrm{i}\tau & 0 \end{pmatrix}. \tag{3.9}$$

A direct computation shows that the Euler–Lagrange equations (3.7) and (3.8) of the critical curve γ are satisfied if and only if

$$L'(s) = [L(s), K_{\kappa,\tau}(s)],$$

where $K_{\kappa,\tau}$ is given by (2.3). This concludes the proof of Theorem A. ■

As a consequence of Theorem A, we have the following.

Corollary 3.2. *Let $\gamma\colon J \to \mathcal{S}$ be a generic transversal curve parametrized by the natural parameter. Let $[\mathcal{F}]\colon J \to [G]$ be the canonical frame of γ and let $L\colon J \to \mathfrak{h}$ be as in (3.9). If γ is a critical curve, the Lax equation (3.1) implies that*

$$\mathcal{F}(s)L(s)\mathcal{F}^{-1}(s) = \mathfrak{M}, \qquad \forall\, s \in J,$$

where $\mathfrak{M}$ is a fixed element of $\mathfrak{h}$ corresponding to a chosen value $L(s_0)$ of $L(s)$.

Definition 3.3. The element $\mathfrak{M} \in \mathfrak{h}$ is called the *momentum* of the critical curve γ.

The characteristic polynomial of the momentum $\mathfrak{M}$ is

$$-x^3 - 6\kappa\tau^2 x + 54\kappa\tau - 27\kappa^2\tau^2 + 2\tau^3 - 3\tau'^2 - 27.$$

The conservation of the momentum along γ yields the two conservation laws

$$\kappa\tau^2 = c_1,$$
$$-18\kappa\tau + 9\kappa^2\tau^2 - \frac{2}{3}\tau^3 + \tau'^2 = C_2 - 9,$$

for real constants c_1 and C_2. We let $c_2 := C_2 - 9$. Using this notation, the (opposite of the) characteristic polynomial of the momentum is

$$\mathrm{Q}(x) = x^3 + 6c_1 x + (27 + 3c_2).$$

If $c_1 \neq 0$, the twist and the bending are never zero and the conservation laws can be rewritten as

$$\kappa = c_1\tau^{-2}, \qquad \frac{3}{2}\tau^2\tau'^2 = \tau^5 + \frac{3}{2}c_2\tau^2 + 27c_1\tau - \frac{27}{2}c_1^2. \tag{3.10}$$

If $c_1 = 0$, it can be easily proved that $\kappa = 0$ and the second conservation law takes the form

$$\tau'^2 = \frac{2}{3}\tau^3 + c_2.$$

Definition 3.4. The pair of real constants $\mathbf{c} = (c_1, c_2)$ is called the *modulus* of the critical curve γ.

Remark 3.5. For the application of Griffiths' approach to other geometric variational problems, the reader is referred to [9, 11, 19, 26, 27, 28, 30].

4 The CR twist of a critical curve

4.1 Phase types

For $\mathbf{c} = (c_1, c_2) \in \mathbb{R}^2$, we denote by $\mathrm{P}_{\mathbf{c}}$ the quintic polynomial in principal form given by

$$\mathrm{P}_{\mathbf{c}}(x) = x^5 + \frac{3}{2}c_2x^2 + 27c_1x - \frac{27}{2}c_1^2$$

and by $\mathrm{Q}_{\mathbf{c}}$ the cubic polynomial given by

$$\mathrm{Q}_{\mathbf{c}}(x) = x^3 + 6c_1x + (27 + 3c_2). \tag{4.1}$$

Excluding the case $\mathbf{c} = 0$, $\mathrm{P}_{\mathbf{c}}$ possesses at least a pair of complex conjugate roots.

Definition 4.1. We adopt the following terminology.

- $\mathbf{c} \in \mathbb{R}^2$ is of *phase type* $\mathcal{A}$ if $\mathrm{P}_{\mathbf{c}}$ has four complex roots $a_j \pm \mathrm{i}b_j$, $j = 1, 2$, $0 < b_1 < b_2$, and a simple real root e_1;
- $\mathbf{c} \in \mathbb{R}^2$ is of *phase type* $\mathcal{B}$ if $\mathrm{P}_{\mathbf{c}}$ has two complex roots $a \pm \mathrm{i}b$, $b > 0$, and three simple real roots $e_1 < e_2 < e_3$;
- $\mathbf{c} \in \mathbb{R}^2$ is of *phase type* $\mathcal{C}$ if $\mathrm{P}_{\mathbf{c}}$ has a multiple real root.

In the latter case, two possibilities may occur: (1) $\mathrm{P}_{\mathbf{c}}$ has a double real root and a simple real root; or (2) $\mathrm{P}_{\mathbf{c}}$ has a real root of multiplicity 5.

By the same letters, we also denote the corresponding sets of moduli of phase types $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, respectively.

Next, we give a more detailed description of the sets $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$. To this end, we start by defining the *separatrix curve*. Let (m, n) be the homogeneous coordinates of $\mathbb{RP}^1$ and let $[(m_*, n_*)]$ be the point of $\mathbb{RP}^1$ such that $3m_*^3 + 6m_*^2n_* + 4m_*n_*^2 + 2n_*^3 = 0$ (i.e., $m_* = 1$ and $n_* \approx -0.72212$).

Definition 4.2. The *separatrix curve* $\Xi \subset \mathbb{R}^2$ is the image of the parametrized curve $\xi = (\xi_1, \xi_2)\colon \mathbb{RP}^1 \setminus \{[(m_*, n_*)]\} \to \mathbb{R}^2$, defined by

$$\xi_1([(m,n)]) = \frac{6\sqrt[3]{2}mn^{4/3}\big(3m^2 + 2mn + n^2\big)^{4/3}}{\big(3m^3 + 6m^2n + 4mn^2 + 2n^3\big)^{5/3}},$$
$$\xi_2([(m,n)]) = -\frac{36n\big(3m^2 + 2mn + n^2\big)\big(4m^3 + 3m^2n + 2mn^2 + n^3\big)}{\big(3m^3 + 6m^2n + 4mn^2 + 2n^3\big)^2}.$$

Remark 4.3. The map ξ is injective and Ξ has a cusp at $\xi([(1,1)]) = \big(\frac{4}{5}\big(\frac{6}{5}\big)^{2/3}, -\frac{48}{5}\big)$. It is regular elsewhere. In addition, Ξ has a horizontal inflection point at $\xi([(0,1)]) = (0, -9)$. Let J_ξ be the interval $(\arctan(n_*), \arctan(n_*) + \pi) \approx (-0.625418, 2.51617)$. Then, $\widetilde{\xi}\colon t \in \mathrm{J}_\xi \to \xi(\cos(t), \sin(t)) \in \Xi$ is another parametrization of Ξ. The inflection point is $\widetilde{\xi}(\pi/2)$. The "negative part" $\Xi_- = \Xi \cap \big\{\mathbf{c} \in \mathbb{R}^2 \mid c_1 < 0\big\}$ of Ξ is parametrized by the restriction of $\widetilde{\xi}$ to $\hat{\mathrm{J}}_\xi = (\pi/2, \pi + \arctan(n_*))$. The left picture of Figure 1 reproduces the separatrix curve (in black); the negative part of the separatrix curve is highlighted in dashed-yellow. The cusp is the red point and the horizontal inflection point is coloured in green.

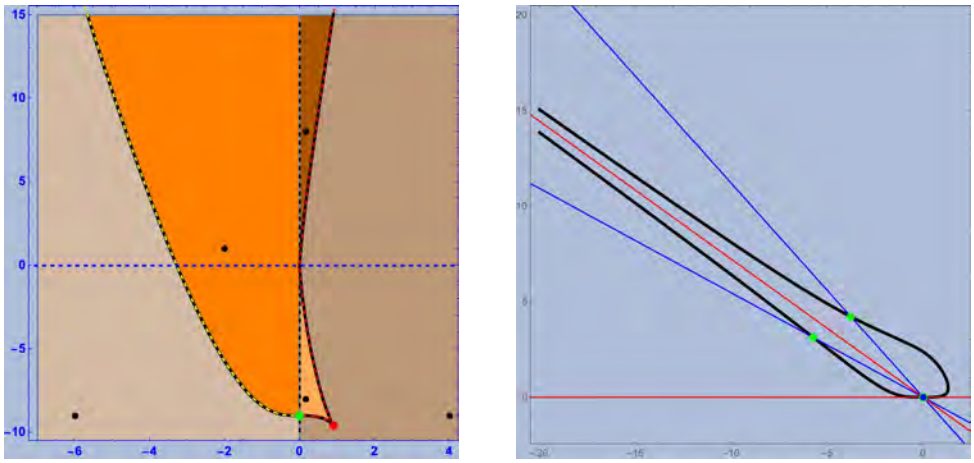

Figure 1. On the left: the separatrix curve (in black), the upper domain $\mathcal{M}_+$ (coloured in three orange tones) and the lower domain $\mathcal{M}_-$ (coloured in two brown tones). On the right: the curve $\mathtt{C}$ and its parametrization obtained by intersecting $\mathtt{C}$ with lines through the origin.

Definition 4.4. The (open) *upper and lower domains* bounded by the separatrix curve Ξ are denoted by $\mathcal{M}_\pm$. In Figure 1, the upper domain $\mathcal{M}_+$ is coloured in three orange tones: orange, dark-orange and light-orange; the lower domain $\mathcal{M}_-$ is coloured in two brown tones: light-brown and brown.

Proposition 4.5. *The polynomial* $\mathrm{P}_{\mathbf{c}}$ *has multiple roots if and only if* $\mathbf{c} \in \Xi \cup Oy$*, has four complex roots if and only if* $\mathbf{c} \in \mathcal{M}_- \setminus (Oy \cap \mathcal{M}_-)$*, and has three distinct real roots if and only if* $\mathbf{c} \in \mathcal{M}_+ \setminus (Oy \cap \mathcal{M}_+)$*. Equivalently,*

$$\mathcal{A} = \mathcal{M}_- \setminus (Oy \cap \mathcal{M}_-), \qquad \mathcal{B} = \mathcal{M}_+ \setminus (Oy \cap \mathcal{M}_+), \qquad \mathcal{C} = \Xi \cup Oy.$$

Proof. First, we prove the following claim.

Claim. $\mathrm{P}_{\mathbf{c}}$ *has a double root* $a_3 \neq 0$ *if and only if* $\mathbf{c}$ *belongs to the separatrix curve minus the cusp.*

Note that $c_1 \neq 0$ (otherwise the double root would be 0). Let a_4 be the other simple real root and $b_1 + \mathrm{i}b_2$, $b_1 - \mathrm{i}b_2$, $b_2 > 0$, be the two complex conjugate roots. Since the sum of the roots of $\mathrm{P}_{\mathbf{c}}$ is zero, we have $b_1 = -\frac{1}{2}(2a_3 + a_4)$. Since the coefficient of x^3 is zero and $b_2 > 0$, we get $b_2 = \sqrt{2a_3^2 + a_3a_4 + 3a_4^2/4}$. Expanding $(x - a_3)^2(x - a_4)(x - b_1 - \mathrm{i}b_2)(x - b_1 + \mathrm{i}b_2)$ and comparing the coefficients of the monomials x^n, $n = 1, \dots, 4$, with the coefficients of $\mathrm{P}_{\mathbf{c}}$ we may write c_1 and c_2 as functions of a_3 and a_4,

$$\begin{aligned} c_1 &= \frac{1}{27}\big(3a_3^4 + 6a_3^3a_4 + 4a_3^2a_4^2 + 2a_3a_4^3\big), \\ c_2 &= -\frac{2}{3}\big(4a_3^3 + 3a_3^2a_4 + 2a_3a_4^2 + a_4^3\big). \end{aligned} \tag{4.2}$$

In addition,

$$c_1^2 = \frac{2}{27}\big(3a_3^4a_4 + 2a_3^3a_4^2 + a_3^2a_4^3\big).$$

Taking into account that $a_3 \neq 0$, it follows that (a_3, a_4) belongs to the algebraic curve $\mathtt{C}$ (the black curve on the right picture in Figure 1) defined by the equation

$$54y\big(3x^2 + 2xy + y^2\big) - \big(3x^3 + 6x^2y + 4xy^2 + 2y^3\big)^2 = 0.$$

Now, consider the line $\ell_{m,n}$ through the origin, with homogeneous coordinates (m,n), i.e., the line with parametric equations $p_{m,n}(t) = (mt, nt)$. If $(m,n) \neq (1,0)$ and $3m^3+6m^2n+4mn^2+2n^3 \neq 0$ (we are excluding the two red lines on the right picture in Figure 1), $\ell_{m,n}$ intersects $\mathtt{C}$ when $t = 0$ and $t = t_{m,n}$, where

$$t_{m,n} = \frac{3\sqrt[3]{2}\sqrt[3]{n\left(3m^2+2mn+n^2\right)}}{\sqrt[3]{\left(3m^3+6m^2n+4mn^2+2n^3\right)^2}}.$$

If $(m,n) = (1,0)$ or $3m^3+6m^2n+4mn^2+2n^3 = 0$, $\ell_{m,n}$ intersects $\mathtt{C}$ only at the origin (see the right picture in Figure 1). Hence $\beta\colon [(m,n)] \to t_{m,n}\cdot(m,n)$, $[(m,n)] \neq [(1,0)]$, $3m^3+6m^2n+4mn^2+2n^3 \neq 0$, is a parametrization of $\mathtt{C}\setminus\{(0,0)\}$. Thus, using (4.2), the map

$$[(m,n)] \to (c_1(\beta([(m,n])), c_2(\beta([(m,n])) \in \mathbb{R}^2$$

is a parametrization of the set of all $\mathbf{c}$, $c_1 \neq 0$, such that $\mathrm{P}_{\mathbf{c}}$ has multiple roots. It is now a computational matter to check that $(c_1(\beta([(m,n])), c_2(\beta([(m,n])) = \xi([(m,n)])$. This proves the claim. It also shows that $\mathrm{P}_{\mathbf{c}}$ has multiple roots if and only if $\mathbf{c} \in \Xi \cup Oy$.

To prove the other assertions, we begin by observing that the discriminant of the derived polynomial $\mathrm{P}'_{\mathbf{c}}$ is negative. Hence $\mathrm{P}'_{\mathbf{c}}$ has two distinct real roots and a pair of complex conjugate roots. Denote by $x'_{\mathbf{c}}$ and $x''_{\mathbf{c}}$ the real roots of $\mathrm{P}'_{\mathbf{c}}$, ordered so that $x'_{\mathbf{c}} < x''_{\mathbf{c}}$. Observe that $x'_{\mathbf{c}}$ and $x''_{\mathbf{c}}$ are differentiable functions of $\mathbf{c}$. Then, $\mathrm{P}_{\mathbf{c}}$ possesses three distinct real roots if and only if $x'_{\mathbf{c}}\cdot x''_{\mathbf{c}} < 0$, one simple real root if and only if $x'_{\mathbf{c}}\cdot x''_{\mathbf{c}} > 0$, and a multiple root if and only if $x'_{\mathbf{c}}\cdot x''_{\mathbf{c}} = 0$.

From the first part of the proof, the set of all $\mathbf{c} \in \mathbb{R}^2$, such that $\mathrm{P}_{\mathbf{c}}$ has only simple roots is the complement of $\Xi \cup Oy$. This set has five connected components:

$$\begin{aligned}
\mathcal{M}'_+ &= \{\mathbf{c} \in \mathcal{M}_+ \setminus (Oy \cap \mathcal{M}_+) \mid c_1 < 0\},\\
\mathcal{M}''_+ &= \{\mathbf{c} \in \mathcal{M}_+ \setminus (Oy \cap \mathcal{M}_+) \mid c_1 > 0 \text{ and } c_2 > 0\},\\
\mathcal{M}'''_+ &= \{\mathbf{c} \in \mathcal{M}_+ \setminus (Oy \cap \mathcal{M}_+) \mid c_1 > 0 \text{ and } c_2 < 0\},\\
\mathcal{M}'_- &= \{\mathbf{c} \in \mathcal{M}_- \setminus (Oy \cap \mathcal{M}_-) \mid c_1 < 0\},\\
\mathcal{M}''_- &= \{\mathbf{c} \in \mathcal{M}_- \setminus (Oy \cap \mathcal{M}_-) \mid c_1 > 0\}.
\end{aligned} \tag{4.3}$$

Referring to the left picture in Figure 1, $\mathcal{M}'_+$ is the orange domain, $\mathcal{M}''_+$ is the dark-orange domain, $\mathcal{M}'''_+$ is the light-orange domain, $\mathcal{M}'_-$ is the light-brown domain, and $\mathcal{M}''_-$ is the brown domain.

Consider the following points (the black points in Figure 1):

$$\begin{aligned}
&\mathbf{c}_1 = (-2,1) \in \mathcal{M}'_+, \qquad \mathbf{c}_2 = (1/6,8) \in \mathcal{M}''_+, \qquad \mathbf{c}_3 = (1/6,-8) \in \mathcal{M}'''_+,\\
&\mathbf{c}_4 = (-6,-9) \in \mathcal{M}'_-, \qquad \mathbf{c}_5 = (4,-9) \in \mathcal{M}''_-.
\end{aligned}$$

Using Klein's formulas for the icosahedral solution of a quintic polynomial in principal form (cf. [23, 33, 38]),[4] we find that the polynomials $\mathrm{P}_{\mathbf{c}_j}$, $j = 1,2,3$, have three distinct real roots and that $\mathrm{P}_{\mathbf{c}_j}$, $j = 4,5$, have one real root. The domain $\mathcal{M}'_+$ is connected and the function $\mathcal{M}'_+ \ni \mathbf{c} \mapsto x'_{\mathbf{c}}\cdot x''_{\mathbf{c}}$ is differentiable and nowhere zero. Since $x'_{\mathbf{c}_1}\cdot x''_{\mathbf{c}_1} < 0$, it follows that $\mathbf{c} \mapsto x'_{\mathbf{c}}\cdot x''_{\mathbf{c}}$ is strictly negative. Then, $\mathrm{P}_{\mathbf{c}}$ has three distinct real roots, for every $\mathbf{c} \in \mathcal{M}'_+$. Similarly, $\mathrm{P}_{\mathbf{c}}$ has three distinct real roots, for every $\mathbf{c} \in \mathcal{M}''_+ \cup \mathcal{M}'''_+$ and a unique real root for every $\mathbf{c} \in \mathcal{M}'_- \cup \mathcal{M}''_-$. This concludes the proof. ∎

[4] We used the Trott and Adamchik code (cf. [38]) implementing Klein's formulas in the software MATHEMATICA.

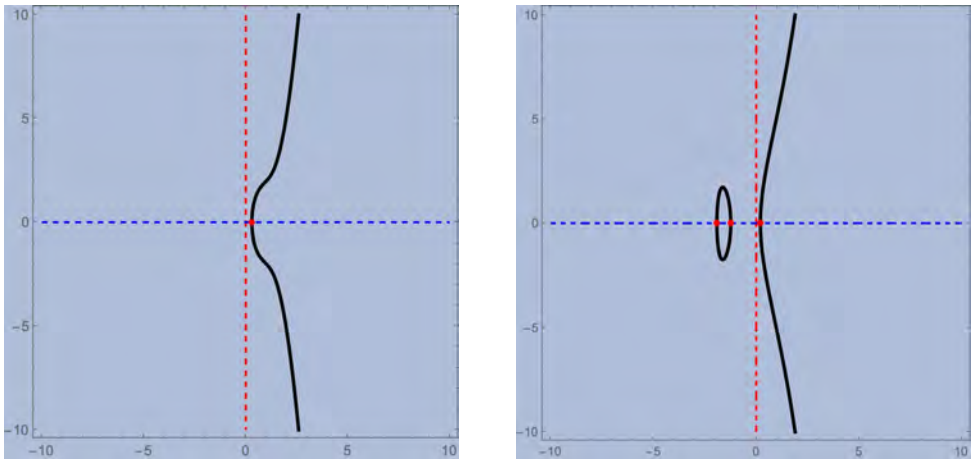

Figure 2. On the left: the phase curve of $\mathbf{c} \in \mathcal{A}$. On the right: the phase curve of $\mathbf{c} \in \mathcal{B}$.

Remark 4.6. The real roots of $\mathrm{P}_{\mathbf{c}_1}$ are $e_1 = -2.44175 < e_2 = -0.9904 < 0 < e_3 = 2.87645$ and those of $\mathrm{P}_{\mathbf{c}_2}$ are $e_1 = -2.14118 < e_2 = -0.448099 < 0 < e_3 = 0.0701938$. Instead, the roots of $\mathrm{P}_{\mathbf{c}_3}$ are $0 < e_1 = 0.12498 < e_2 = 0.250656 < 2.15383$. Since the product $e_2(\mathbf{c})e_3(\mathbf{c})$ is a continuous function on the connected components $\mathcal{M}'_+$, $\mathcal{M}''_+$, and $\mathcal{M}'''_+$, we deduce that the lowest roots of $\mathrm{P}_{\mathbf{c}}$ are negative if $\mathbf{c} \in \mathcal{M}'_+ \cup \mathcal{M}''_+$ and positive if $\mathbf{c} \in \mathcal{M}'''_+$.

4.2 Phase curves and signatures

Definition 4.7. Let $\Sigma_{\mathbf{c}}$ be the real algebraic curve defined by $y^2 = \mathrm{P}_{\mathbf{c}}(x)$. We call $\Sigma_{\mathbf{c}}$ the *phase curve* of $\mathbf{c}$.

If $\mathbf{c} \in \mathcal{A} \cup \mathcal{B}$, $\Sigma_{\mathbf{c}}$ is a smooth real cycle of a hyperelliptic curve of genus 2. If $\mathbf{c} \in \mathcal{C}$, and $\mathbf{c} \neq 0$, $\Sigma_{\mathbf{c}}$ is a singular real cycle of an elliptic curve. If $\mathbf{c} = 0$, $\Sigma_{\mathbf{c}}$ is a singular rational curve. The following facts can be easily verified:

- if $\mathbf{c} \in \mathcal{A}$, $\Sigma_{\mathbf{c}}$ is connected, unbounded, and intersects the Ox-axis at $(e_1, 0)$ (see Figure 2);
- if $\mathbf{c} \in \mathcal{B}$, $\Sigma_{\mathbf{c}}$ has two smooth connected components, one is compact and the other is unbounded. Let $\Sigma'_{\mathbf{c}}$ be the compact connected component and $\Sigma''_{\mathbf{c}}$ be the noncompact one. $\Sigma'_{\mathbf{c}}$ intersects the Ox-axis at $(e_1, 0)$ and $(e_2, 0)$, while $\Sigma''_{\mathbf{c}}$ intersects the Ox-axis at $(e_3, 0)$ (see Figure 2);
- if $\mathbf{c} \in \mathcal{C}$ and $c_1 \neq 0$, $\Sigma_{\mathbf{c}}$ has a smooth, unbounded connected component $\Sigma''_{\mathbf{c}}$ and an isolated singular point $(e_1, 0)$, where $e_1 = e_2$ is the double real root of $\mathrm{P}_{\mathbf{c}}(x)$. The unbounded connected component intersects the Ox-axis at $(e_3, 0)$, where e_3 is the simple real root of $\mathrm{P}_{\mathbf{c}}(x)$ (see Figure 3). If $c_1 = 0$ and $c_2 \neq 0$, $\Sigma_{\mathbf{c}}$ is connected, with an ordinary double point (see Figure 3). If $\mathbf{c} = 0$, $\Sigma_{\mathbf{c}}$ is connected with a cusp at the origin (see Figure 3).

Definition 4.8. Let γ be a critical curve with nonconstant twist and modulus $\mathbf{c}$. Let $\mathrm{J}_\gamma \subset \mathbb{R}$ be the maximal interval of definition of γ. With reference to (3.10), we adapt to our context the terminology used in [3, 24, 29] and call

$$\sigma_\gamma \colon \ \mathrm{J}_\gamma \to \mathbb{R}^2, \qquad s \mapsto \big(\tau(s), \sqrt{3/2}\tau(s)\tau'(s)\big)$$

the *signature* of γ.

Remark 4.9. From the Poincaré–Bendixson theorem, it follows that the twist of γ is periodic if and only if $\sigma_\gamma(\mathrm{J}_\gamma)$ is compact. Observing that $\sigma_\gamma(\mathrm{J}_\gamma)$ is one of the 1-dimensional connected

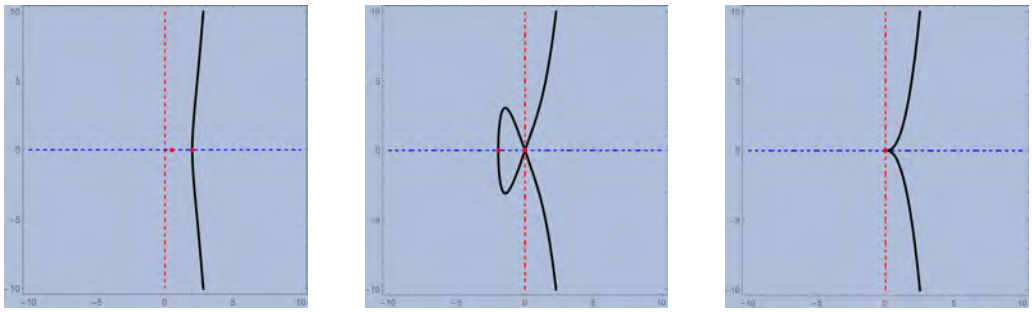

Figure 3. On the left: the phase curve of $\mathbf{c} \in \mathcal{C}$, $c_1 \neq 0$. On the center: the phase curve of $\mathbf{c} \in \mathcal{C}$, $c_1 = 0$ and $c_2 \neq 0$. On the right: the phase curve of $\mathbf{c} = (0, 0)$.

components of $\Sigma_{\mathbf{c}}$, we can conclude that the twist is a periodic function if and only if $\mathbf{c} \in \mathcal{B}$ and $\sigma_\gamma(\mathrm{J}_\gamma) = \Sigma'_{\mathbf{c}}$.

Definition 4.10. A critical curve γ with modulus $\mathbf{c}$ is said to be of type $\mathcal{B}'$ if $\mathbf{c} \in \mathcal{B}$ and $\sigma_\gamma(\mathrm{J}_\gamma) = \Sigma'_{\mathbf{c}}$; it is said to be of type $\mathcal{B}''$ if $\mathbf{c} \in \mathcal{B}$ and $\sigma_\gamma(\mathrm{J}_\gamma) = \Sigma''_{\mathbf{c}}$.

4.3 The twist of a critical curve

4.3.1 The twist of a critical curve of type $\mathcal{A}$

Let γ be a critical curve of type $\mathcal{A}$, i.e., with modulus $\mathbf{c} \in \mathcal{A}$. Then $\mathrm{P}_{\mathbf{c}}$ has a unique real root e_1. The polynomial $\mathrm{P}_{\mathbf{c}}(x)$ is positive if $x > e_1$ and is negative if $x < e_1$. Since $\mathrm{P}_{\mathbf{c}}(0) = -27c_1^2/2 < 0$, the root is positive. Let $\omega_{\mathbf{c}} > 0$ be the improper hyperelliptic integral of the first kind defined by

$$\omega_{\mathbf{c}} = \sqrt{\frac{3}{2}} \int_{e_1}^{+\infty} \frac{\tau \mathrm{d}\tau}{\sqrt{\mathrm{P}_{\mathbf{c}}(\tau)}} > 0.$$

The incomplete hyperelliptic integral

$$h_{\mathbf{c}}(\tau) = \sqrt{\frac{3}{2}} \int_{e_1}^{\tau} \frac{u \mathrm{d}u}{\sqrt{\mathrm{P}_{\mathbf{c}}(u)}}, \qquad u \geq e_1$$

is a strictly increasing diffeomorphism of $[e_1, +\infty)$ onto $[0, \omega_{\mathbf{c}})$ (see Figure 4). The twist is the unique even function $\tau_{\mathbf{c}} \colon (-\omega_{\mathbf{c}}, \omega_{\mathbf{c}}) \to \mathbb{R}$, such that $\tau_{\mathbf{c}} = h_{\mathbf{c}}^{-1}$ on $[0, \omega_{\mathbf{c}})$. The maximal domain of definition is $\mathrm{J}_{\mathbf{c}} = (-\omega_{\mathbf{c}}, \omega_{\mathbf{c}})$. $\tau_{\mathbf{c}}$ is strictly positive, with vertical asymptotes as $s \to \mp\omega_{\mathbf{c}}^{\pm}$ (see Figure 4). Note that $\tau_{\mathbf{c}}$ is the solution of the Cauchy problem

$$\tau'' = \tau^2 - 9c_1\tau^{-2}\big(1 - c_1\tau^{-1}\big), \qquad \tau(0) = e_1, \qquad \tau'(0) = 0.$$

4.3.2 The twist of a critical curve of type $\mathcal{B}'$

Let $e_1 < e_2 < e_3$ be the simple real roots of $\mathrm{P}_{\mathbf{c}}$. The highest root e_3 is positive. The lower roots e_1 and e_2 are either both negative or both positive and $\mathrm{P}_{\mathbf{c}}$ is positive on (e_1, e_2). Let $\omega_{\mathbf{c}} > 0$ be the complete hyperelliptic integral of the first kind

$$\omega_{\mathbf{c}} = \mathrm{sign}(e_1) \sqrt{\frac{3}{2}} \int_{e_2}^{e_1} \frac{\tau \mathrm{d}\tau}{\sqrt{\mathrm{P}_{\mathbf{c}}(\tau)}} > 0. \tag{4.4}$$

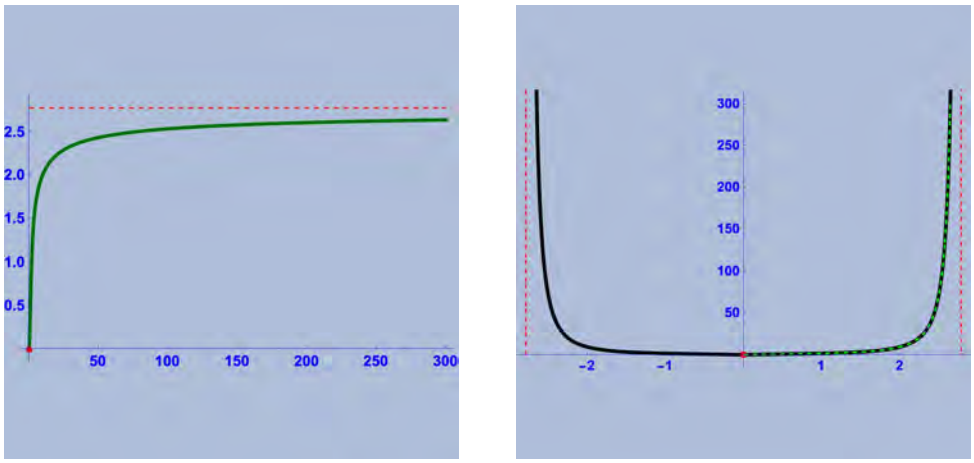

Figure 4. On the left: the graph of the function $h_{\mathbf{c}}$, $\mathbf{c} = (1/2, -4.8) \in \mathcal{A}$. The red line is the horizontal asymptote $y = e_1$. On the right: the graph of the twist. The red lines are the vertical asymptotes $x = \pm\omega_{\mathbf{c}}$.

Let $h_{\mathbf{c}}$ be the incomplete hyperelliptic integrals of the first kind

$$h_{\mathbf{c}}(\tau) = \begin{cases} \sqrt{\frac{3}{2}} \displaystyle\int_{e_2}^{\tau} \frac{u\mathrm{d}u}{\sqrt{\mathrm{P}_{\mathbf{c}}(u)}}, & \tau \in [e_1, e_2],\ e_1 < e_2 < 0, \\ \sqrt{\frac{3}{2}} \displaystyle\int_{e_1}^{\tau} \frac{u\mathrm{d}u}{\sqrt{\mathrm{P}_{\mathbf{c}}(u)}}, & \tau \in [e_1, e_2],\ 0 < e_1 < e_2. \end{cases}$$

The function $h_{\mathbf{c}}$ is a diffeomorphism of $[e_1, e_2]$ onto $[0, \omega_{\mathbf{c}}]$, strictly decreasing if $e_1 < e_2 < 0$ and strictly increasing if $0 < e_1 < e_2$ (see Figure 5). The twist $\tau_{\mathbf{c}}$ is the even periodic function with least period $2\omega_{\mathbf{c}}$, obtained by extending periodically the function $\tau(s) = h_{\mathbf{c}}^{-1}(s)$ defined on $[0, \omega_{\mathbf{c}}]$ and on $[-\omega_{\mathbf{c}}, 0]$, respectively.

- If $e_1 < e_2 < 0$, then $\tau_{\mathbf{c}}$ is strictly negative with minimum value e_1 and maximum value e_2, attained, respectively, at $s \equiv \omega_{\mathbf{c}} \mod 2\omega_{\mathbf{c}}$ and at $s \equiv 0 \mod 2\omega_{\mathbf{c}}$ (see Figure 5).
- If $0 < e_1 < e_2$, then $\tau_{\mathbf{c}}$ is strictly positive, with minimum value e_1 and maximum value e_2, attained, respectively, at $s \equiv 0 \mod 2\omega_{\mathbf{c}}$ and at $s \equiv \omega_{\mathbf{c}} \mod 2\omega_{\mathbf{c}}$.

Observe that $\tau_{\mathbf{c}}$ is the solution of the Cauchy problem

$$\begin{array}{lllll} (i) & \tau'' = \tau^2 - 9c_1\tau^{-2}\big(1 - c_1\tau^{-1}\big), & \tau(0) = e_2, & \tau'(0) = 0 & \text{if } e_1 < e_2 < 0, \\ (ii) & \tau'' = \tau^2 - 9c_1\tau^{-2}\big(1 - c_1\tau^{-1}\big), & \tau(0) = e_1, & \tau'(0) = 0 & \text{if } 0 < e_1 < e_2. \end{array} \tag{4.5}$$

4.3.3 The twist of a critical curve of type $\mathcal{B}''$

The twist of a critical curve of type $\mathcal{B}''$ can be constructed as in the case of a critical curve of type $\mathcal{A}$. More precisely, let $e_3 > 0$ be the highest real root of $\mathrm{P}_{\mathbf{c}}$ and $\omega_{\mathbf{c}}$ be the improper hyperelliptic integral of the first kind given by

$$\omega_{\mathbf{c}} = \sqrt{\frac{3}{2}} \int_{e_3}^{+\infty} \frac{\tau\mathrm{d}\tau}{\sqrt{\mathrm{P}_{\mathbf{c}}(\tau)}} > 0.$$

Let $h_{\mathbf{c}}(\tau)$ be the incomplete hyperelliptic integral

$$h_{\mathbf{c}}(\tau) = \sqrt{\frac{3}{2}} \int_{e_3}^{\tau} \frac{u\mathrm{d}u}{\sqrt{\mathrm{P}_{\mathbf{c}}(u)}}, \qquad \tau \geq e_3.$$

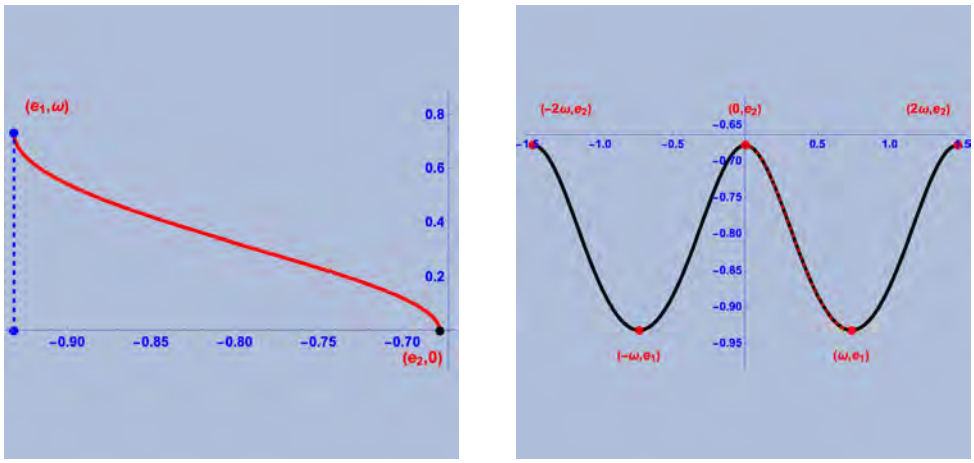

Figure 5. On the left: the graph of the function $h_{\mathbf{c}}$ for a critical curve of type $\mathcal{B}'$, with modulus $\mathbf{c} \approx (-0.828424, -8.349417) \in \mathcal{B}$. On the right: the graph of the twist, an even periodic function with least period $2\omega_{\mathbf{c}}$. The lowest roots e_1 and e_2 are negative.

Then, $h_{\mathbf{c}}$ is a strictly increasing diffeomorphism of $[e_3, +\infty)$ onto $[0, \omega_{\mathbf{c}})$. The twist is the unique even function $\tau_{\mathbf{c}} \colon (-\omega_{\mathbf{c}}, \omega_{\mathbf{c}}) \to \mathbb{R}$, such that $\tau_{\mathbf{c}} = h_{\mathbf{c}}^{-1}$ on $[0, \omega_{\mathbf{c}})$. The maximal interval of definition of $\tau_{\mathbf{c}}$ is $\mathrm{J}_{\mathbf{c}} = (-\omega_{\mathbf{c}}, \omega_{\mathbf{c}})$. The function $\tau_{\mathbf{c}}$ is positive, with vertical asymptotes as $s \to \mp\omega_{\mathbf{c}}^{\pm}$, and is the solution of the Cauchy problem

$$\tau'' = \tau^2 - 9c_1\tau^{-2}\big(1 - c_1\tau^{-1}\big), \qquad \tau(0) = e_3, \qquad \tau'(0) = 0.$$

4.3.4 The twist of a critical curve of type $\mathcal{C}$ with $c_1 \neq 0$

The twist of a critical curve of type $\mathcal{C}$, with $c_1 \neq 0$, can be constructed as for curves of types $\mathcal{A}$ or $\mathcal{B}''$. Let $e_3 > 0$ be simple real root of $\mathrm{P}_{\mathbf{c}}$ and $\omega_{\mathbf{c}}$ be the improper elliptic integral of the first kind

$$\omega_{\mathbf{c}} = \sqrt{\frac{3}{2}} \int_{e_3}^{+\infty} \frac{\tau \mathrm{d}\tau}{\sqrt{\mathrm{P}_{\mathbf{c}}(\tau)}} > 0.$$

Let $h_{\mathbf{c}}(\tau)$ be the incomplete elliptic integral

$$h_{\mathbf{c}}(\tau) = \sqrt{\frac{3}{2}} \int_{e_3}^{\tau} \frac{u \mathrm{d}u}{\sqrt{\mathrm{P}_{\mathbf{c}}(u)}}, \qquad \tau \geq e_3.$$

Then, $h_{\mathbf{c}}$ is a strictly increasing diffeomorphism of $[e_3, +\infty)$ onto $[0, \omega_{\mathbf{c}})$. The twist $\tau_{\mathbf{c}}$ is the unique even function $\tau_{\mathbf{c}} \colon (-\omega_{\mathbf{c}}, \omega_{\mathbf{c}}) \to \mathbb{R}$, such that $\tau_{\mathbf{c}} = h_{\mathbf{c}}^{-1}$ on $[0, \omega_{\mathbf{c}})$. The maximal interval of definition of $\tau_{\mathbf{c}}$ is $\mathrm{J}_{\mathbf{c}} = (-\omega_{\mathbf{c}}, \omega_{\mathbf{c}})$. The twist is positive, with vertical asymptotes as $s \to \mp\omega_{\mathbf{c}}^{\pm}$. Note that $\tau_{\mathbf{c}}$ is the solution of the Cauchy problem

$$\tau'' = \tau^2 - 9c_1\tau^{-2}\big(1 - c_1\tau^{-1}\big), \qquad \tau(0) = e_3, \qquad \tau'(0) = 0.$$

4.3.5 The twist of a critical curve with $c_1 = 0$

If $c_1 = 0$, the bending vanishes identically and the twist is a solution of the second order ODE $\tau'' - \tau^2 = 0$. Then,

$$\tau(s) = \sqrt[3]{6}\wp\left(\frac{s+a}{\sqrt[3]{6}}\middle| 0, g_3\right), \qquad g_3 = -\sqrt[3]{2/243}c_2, \qquad \kappa(s) = 0,$$

where a is an unessential constant and $\wp(-, g_2, g_3)$ is the Weierstrass function with invariants g_2, g_3.

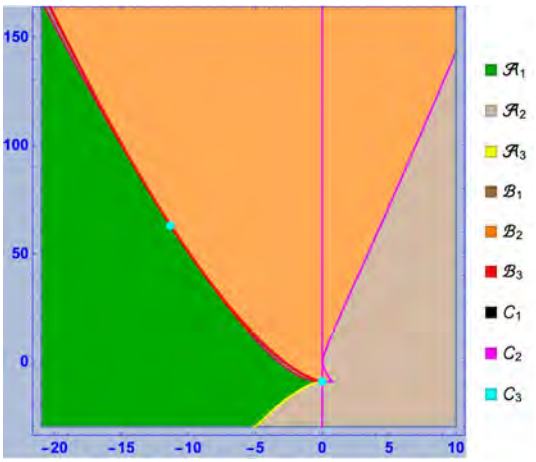

Figure 6. The nine regions $\mathcal{A}_j$, $\mathcal{B}_j$ and $\mathcal{C}_j$, $j = 1, 2, 3$.

4.4 Orbit types and the twelve classes of critical curves with nonconstant twist

The moduli of the critical curves can be classified depending on the properties of the eigenvalues of the momenta.

Definition 4.11. For $\mathbf{c} = (c_1, c_2) \in \mathbb{R}^2$, let $\Delta_1(\mathbf{c}) = -27\big(32c_1^3 + 9(9 + c_2)^2\big)$ be the discriminant of the cubic polynomial $\mathrm{Q}_{\mathbf{c}}$ (cf. (4.1)). We say that $\mathbf{c} \in \mathbb{R}^2$ is

1. Of *orbit type* 1 (in symbols, $\mathbf{c} \in \mathrm{OT}_1$) if $\Delta_1(\mathbf{c}) > 0$; the momentum of a critical curve with modulus $\mathbf{c} \in \mathrm{OT}_1$ has three distinct real eigenvalues: $\lambda_1 = -(\lambda_2 + \lambda_3) < 0 < \lambda_2 < \lambda_3$.
2. Of *orbit type* 2 (in symbols, $\mathbf{c} \in \mathrm{OT}_2$) if $\Delta_1(\mathbf{c}) < 0$; the momentum of a critical curve with modulus $\mathbf{c} \in \mathrm{OT}_2$ has a real eigenvalue λ_1 and two complex conjugate roots: λ_2, with positive imaginary part, and $\lambda_3 = \overline{\lambda}_2$.
3. Of *orbit type* 3 (in symbols, $\mathbf{c} \in \mathrm{OT}_3$) if $\Delta_1(\mathbf{c}) = 0$; the momentum of a critical curve with modulus $\mathbf{c} \in \mathrm{OT}_3$ has an eigenvalue with algebraic multiplicity greater than one (> 1).

Correspondingly, $\mathbb{R}^2$ is partitioned into nine regions (see Figure 6):

$$\mathcal{A}_j = \mathcal{A} \cap \mathrm{OT}_j, \qquad \mathcal{B}_j = \mathcal{B} \cap \mathrm{OT}_j, \qquad \mathcal{C}_j = \mathcal{C} \cap \mathrm{OT}_j, \qquad j = 1, 2, 3.$$

Definition 4.12. Let γ be a critical curve with modulus $\mathbf{c}$ and $j \in \{1, 2, 3\}$. We say that γ is of *type* $\mathcal{A}_j$ if $\mathbf{c} \in \mathcal{A}_j$; of *type* $\mathcal{B}'_j$ if $\mathbf{c} \in \mathcal{B}_j$ and the image of its signature σ_γ is compact; of *type* $\mathcal{B}''_j$ if $\mathbf{c} \in \mathcal{B}_j$ and the image of σ_γ is unbounded; and of *type* $\mathcal{C}_j$ if $\mathbf{c} \in \mathcal{C}_j$, $j = 1, 2, 3$.

Remark 4.13. The only critical curves with periodic twist are those of the types $\mathcal{B}'_j$, $j = 1, 2, 3$. Consequently, critical curves of the other types cannot be closed.

Remark 4.14. $\mathcal{B}_1$ lies in the half-plane $\{(c_1, c_2) \mid c_1 < 0\}$; it is bounded below by $\Xi' = \{\mathbf{c} \in \Xi \mid c_1 < 0\}$ and above by $\Delta' = \big\{\mathbf{c} \in \mathbb{R}^2 \mid \Delta_1(\mathbf{c}) = 0,\ c_2 > -9\big\}$. The curves Ξ' and Δ' intersect each other tangentially at $\mathbf{c}' = (c'_1, c'_2) \approx (-11.339754, 63.004420)$ (see Figure 7). Thus, $\mathcal{B}_1$ has two connected components:

$$\mathcal{B}_1^- = \{\mathbf{c} \in \mathcal{B}_1 \mid c_1 \in (c'_1, -9)\}, \qquad \mathcal{B}_1^+ = \{\mathbf{c} \in \mathcal{B}_1 \mid c_1 > c'_1\}.$$

Referring to Remark 4.3, Ξ' is parametrized by the restriction of $\widetilde{\xi}$ to the interval $\hat{\mathrm{J}}_\xi = (\pi/2, \pi + \arctan(n_*))$. Let t' be the point of $\hat{\mathrm{J}}_\xi$ such that $\widetilde{\xi}(t') = \mathbf{c}'$, ($t' \approx 2.3008$). Put $\hat{\mathrm{J}}_\xi^- = (\pi/2, t')$ and

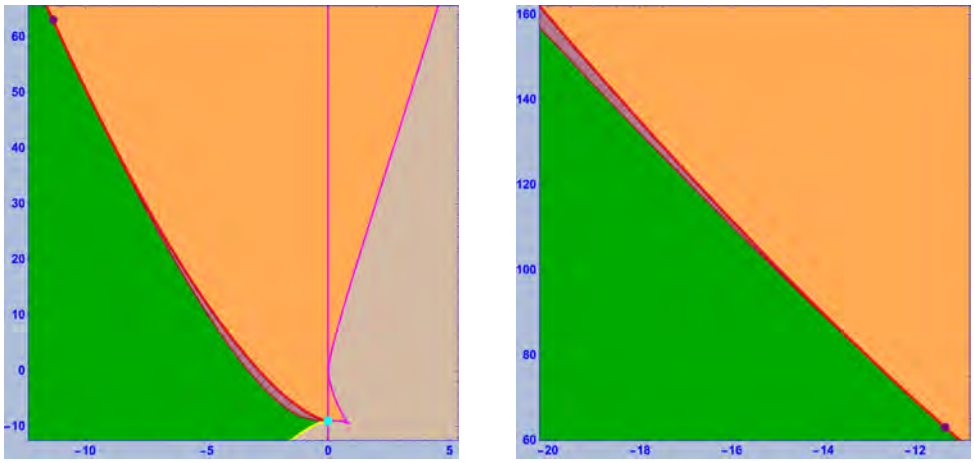

Figure 7. On the left: the connected component $\mathcal{B}_1^-$ (dark brown) of $\mathcal{B}_1$. The point coloured in cyan is the inflection point of Ξ (the union of the arcs coloured in black and magenta and of the two points) and the cusp of $\Delta_1 = 0$ (the union of the yellow and red arcs and of the two points). The point coloured in purple is the point of tangential contact. On the right: the connected component $\mathcal{B}_1^+$ (dark brow) of $\mathcal{B}_1$.

$\hat{\mathrm{J}}_\xi^+ = (t', \pi + \arctan(n_*))$. The restriction of $\widetilde{\xi}$ to $\hat{\mathrm{J}}_\xi^-$ is a parametrization of $\Xi_- = \{\mathbf{c} \in \Xi' \mid c_1 \in (c_1', 0)\}$ and the restriction to $\hat{\mathrm{J}}_\xi^+$ is a parametrization of $\Xi_+ = \{\mathbf{c} \in \Xi' \mid c_1 < c_1'\}$. Consequently, $\mathcal{B}_1^\pm$ are parametrized by

$$\psi_\pm \colon\ K_\pm \ni (t, s) \longmapsto \big(\widetilde{\xi}(t) - p(t)\big)s + p(t), \tag{4.6}$$

where $K_\pm$ are the rectangles $\hat{\mathrm{J}}_\xi^\pm \times (0, 1)$ and

$$p(t) = \left(\widetilde{\xi}_1(t), \frac{1}{3}\left(4\sqrt{-2\widetilde{\xi}_1(t)^3} - 27\right)\right).$$

5 Integrability by quadratures

5.1 Integrability by quadratures of general critical curves

Definition 5.1. Let Δ_2 be the polynomial

$$\Delta_2(\mathbf{c}) = 9c_1^3\big(c_1^3 + 216\big) + 6c_1^3c_2(c_2 + 36) + (c_2 + 9)(c_2 + 18)^3.$$

A critical curve γ with modulus $\mathbf{c}$ is said to be *general* if $\Delta_1(\mathbf{c})\Delta_2(\mathbf{c}) \neq 0$. Since $\Delta_1(\mathbf{c}) \neq 0$, the momentum $\mathfrak{M}_\gamma$ of a general critical curve γ has three distinct eigenvalues λ_1, λ_2, λ_3, sorted as in Definition 4.11.

Let J be the maximal interval of definition of the twist (it can be computed in terms of the modulus). Define $\mathbf{y}_j \colon \mathrm{J} \to \mathbb{C}^{1,2}$, $j = 1, 2, 3$, by

$$\mathbf{y}_j = {}^t\left(\tau(3 - \mathrm{i}\tau') - \lambda_j^2 - 3c_1, 9 - \frac{9c_1}{\tau} - \lambda_j\tau - 3\mathrm{i}\tau', \mathrm{i}\big(\tau^2 - 3\lambda_j\big)\right). \tag{5.1}$$

Let $\mathrm{V} \colon \mathrm{J} \to \mathfrak{gl}(3, \mathbb{C})$ be the matrix-valued map with column vectors $\mathbf{y}_1$, $\mathbf{y}_2$ and $\mathbf{y}_3$. Let $\mathrm{D}(z_1, z_2, z_3)$ denote the diagonal matrix with z_j as the jth element on the diagonal. Recall that, if $c_1 \neq 0$, then τ is nowhere zero.

We can prove the following.

Theorem B. *Let $\gamma\colon \mathrm{J} \to \mathcal{S}$ be a general critical curve. The functions $\det(\mathrm{V})$ and $\tau^2 - 3\lambda_j$, $j = 1, 2, 3$, are nowhere zero. Let r_j be continuous determinations of $\sqrt{\tau^2 - 3\lambda_j}$ and let ϕ_j be the functions defined by*[5]

$$\phi_j(s) = \int_0^s \frac{3c_1\lambda_j - \big(4c_1 + \lambda_j^2\big)\tau^2(u) + 3\tau^3(u)}{\tau^2(u)\big(3\lambda_j - \tau^2(u)\big)}\mathrm{d}u. \tag{5.2}$$

Then, γ is congruent to

$$\mathrm{J} \ni s \longmapsto \big[\mathrm{MD}\big(r_1\mathrm{e}^{\mathrm{i}\phi_1}, r_2\mathrm{e}^{\mathrm{i}\phi_2}, r_3\mathrm{e}^{\mathrm{i}\phi_3}\big)\mathrm{V}^{-1}\mathbf{e}_1\big] \in \mathcal{S},$$

where $\mathrm{M} = \mathrm{V}(0)\mathrm{D}(r_1(0), r_2(0), r_3(0))^{-1}$.

Proof. The proof of Theorem B is organized into three lemmas.

Lemma B1. *The following statements hold true:*

(1) *if the momentum has three distinct real eigenvalues, then $\pm\sqrt{3\lambda_2}$ and $\pm\sqrt{3\lambda_3}$ cannot be roots of $\mathrm{P}_{\mathbf{c}}$;*

(2) *if the momentum has two complex conjugate eigenvalues and a positive real eigenvalue λ_1, then $\pm\sqrt{3\lambda_1}$ cannot be roots of $\mathrm{P}_{\mathbf{c}}$.*

Proof of Lemma B1. First, note that the image of the parametrized curve

$$\alpha(t) = \left(-t\left(\frac{t^3}{3} + \sqrt{3}\right), t^3\left(\frac{t^3}{3} + 2\sqrt{3}\right) - 9\right)$$

is contained in the zero locus of Δ_2. This can be proved by a direct computation. Secondly, from the expression of $\mathrm{Q}_{\mathbf{c}}$, it follows that

$$\begin{aligned} c_1 &= -\frac{1}{6}\big(\lambda_2^2 + \lambda_2\lambda_3 + \lambda_3^2\big) = -\frac{1}{6}\big(\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2\big), \\ c_2 &= \frac{1}{3}\big(\lambda_2^2\lambda_3 + \lambda_2\lambda_3^2 - 27\big) = \frac{1}{3}\big(\lambda_1^2\lambda_2 + \lambda_1\lambda_2^2 - 27\big). \end{aligned} \tag{5.3}$$

1. Suppose that the momentum has three distinct real eigenvalues. By contradiction, suppose that $\sqrt{3\lambda_2}$ is a root of $\mathrm{P}_{\mathbf{c}}$. Then

$$\begin{aligned} 0 = -\frac{8}{3}\mathrm{P}_{\mathbf{c}}\big(\sqrt{3\lambda_2}\big) = \lambda_3^4 + 2\lambda_2\lambda_3^3 - \big(\lambda_2^2 - 12\sqrt{3}\lambda_2^{1/2}\big)\lambda_3^2 - 2\big(\lambda_2^3 - 6\sqrt{3}\lambda_2^{3/2}\big)\lambda_3 \\ + \big(\lambda_2^4 - 12\sqrt{3}\lambda_2^{5/2} + 108\lambda_2\big). \end{aligned}$$

Solving this equation with respect to λ_3, taking into account that $\lambda_3 > 0$, we obtain

$$\lambda_3 = \frac{1}{2}\left(-\lambda_2 + \sqrt{5\lambda_2^2 - 24\sqrt{3\lambda_2}}\right).$$

Substituting into (5.3), we find

$$c_1 = \sqrt{3\lambda_2} - \frac{1}{3}\lambda_2^2, \qquad c_2 = -9 - 2\sqrt{3}\lambda_2^{3/2} + \frac{1}{3}\lambda_2^3.$$

Then, $\mathbf{c} = \alpha\big(-\sqrt{\lambda_2}\big)$. This implies that $\mathbf{c}$ belongs to the zero locus of Δ_2, which is a contradiction. By an analogous argument, we prove that also $-\sqrt{3\lambda_2}$ cannot be a root of $\mathrm{P}_{\mathbf{c}}$. By

[5] If $c_1 \neq 0$, the denominator of the integrand in nowhere zero and the ϕ_j are real-analytic. If $c_1 = 0$, the integrand reduces to $\big(3\tau - \lambda_j^2\big)\big(3\lambda_j - \tau^2\big)^{-1}$. Thus, also in this case the functions ϕ_j are real-analytic.

interchanging the role of λ_2 and λ_3 and arguing as above, it follows that also $\pm\sqrt{3\lambda_3}$ cannot be roots of $\mathrm{P}_{\mathbf{c}}$.

2. Next, suppose that the momentum has two complex conjugate eigenvalues and a nonnegative real eigenvalue λ_1. Recall that the eigenvalues are sorted so that the imaginary part of λ_2 is positive. By contradiction, suppose that $\sqrt{3\lambda_1}$ is a root of $\mathrm{P}_{\mathbf{c}}$. Then,

$$0 = -\frac{8}{3}\mathrm{P}_{\mathbf{c}}\big(\sqrt{3\lambda_1}\big) = \lambda_2^4 + 2\lambda_1\lambda_2^3 - \big(\lambda_1^2 - 12\sqrt{3}\lambda_1^{1/2}\big)\lambda_2^2 - 2\big(\lambda_1^3 - 6\sqrt{3}\lambda_1^{3/2}\big)\lambda_2 + \big(\lambda_1^4 - 12\sqrt{3}\lambda_1^{5/2} + 108\lambda_1\big).$$

Solving this equation with respect to λ_2, taking into account that the imaginary part of λ_2 is positive, we find

$$\lambda_2 = \frac{1}{2}\left(-\lambda_1 + \sqrt{5\lambda_1^2 - 24\sqrt{3\lambda_1}}\right).$$

Substituting into (5.3) yields $\mathbf{c} = \alpha\big(-\sqrt{\lambda_1}\big)$. Thus, $\mathbf{c}$ is a root of Δ_2, which is a contradiction. An analogous argument shows that $-\sqrt{3\lambda_1}$ cannot be a root of $\mathrm{P}_{\mathbf{c}}$. This concludes the proof of the lemma. ∎

Lemma B2. $\det(\mathrm{V})(s) \neq 0$, *for every* $s \in \mathrm{J}_\gamma$.

Proof of Lemma B2. Let $\mathbb{L}_j$ be the 1-dimensional eigenspaces of the momentum $\mathfrak{M}_\gamma$ relative to the eigenvalues λ_j. Let L be as in (3.2). By Corollary 3.2 of Theorem A, we have $\mathcal{F}L\mathcal{F}^{-1} = \mathfrak{M}$, where $\mathcal{F}$ is a Wilczynski frame field along γ. Then, $L(s)$ and $\mathfrak{M}$ have the same eigenvalues. Next, consider the line bundles

$$\Lambda_j = \big\{(s, \mathbf{y}) \in \mathrm{J}_\gamma \times \mathbb{C}^{1,2} \mid L(s)\mathbf{y} = \lambda_j \mathbf{y}\big\}, \qquad j = 1, 2, 3.$$

Note that $(s, \mathbf{y}) \in \Lambda_j$ if and only if $\mathcal{F}(s)\mathbf{y} \in \mathbb{L}_j$. Let $\mathbf{y}_j$, $j = 1, 2, 3$, be as in (5.1). A direct computation shows that $L\mathbf{y}_j = \lambda_j \mathbf{y}_j$. Thus, $\mathbf{y}_j$ is a cross section of the eigenbundle Λ_j. Hence, $\det(\mathrm{V})(s) \neq 0$ if and only if $\mathbf{y}_j(s) \neq \vec{0}$, for every s.

Case I. The eigenvalues of the momentum are real and distinct. Let y_j^i, $i = 1, 2, 3$, denote the components of $\mathbf{y}_j$. Since λ_1 is negative, it follows from (5.1) that $\mathrm{y}_1^3(s) \neq 0$, for every s, and hence $\mathbf{y}_1(s) \neq \vec{0}$. We prove that $\mathbf{y}_2(s) \neq \vec{0}$. Suppose, by contradiction, that $\mathbf{y}_2(s_*) = \vec{0}$, for some $s_* \in \mathrm{J}_\gamma$. From $\mathrm{y}_2^1(s_*) = \mathrm{y}_2^2(s_*) = 0$, it follows that $\tau'(s_*) = 0$. Hence $e := \tau(s_*)$ is a root of $\mathrm{P}_{\mathbf{c}}$. From $\mathrm{y}_2^3(s_*) = 0$, it follows that $e = \pm\sqrt{3\lambda_2}$, which contradicts Lemma B1. An analogous argument leads to the conclusion that $\mathbf{y}_3(s) \neq \vec{0}$, for every $s \in \mathrm{J}_\gamma$.

Case II. The momentum has a real eigenvalue λ_1 and two complex conjugate eigenvalues λ_2, λ_3 (λ_2 with positive imaginary part). Since λ_2 and λ_3 have nonzero imaginary parts and τ is real valued, $\mathrm{y}_2^3(s) \neq 0$ and $\mathrm{y}_3^3(s) \neq 0$, for every s. If $\lambda_1 < 0$, then $\mathrm{y}_1^3(s) \neq 0$, for every s. If $\lambda_1 \geq 0$, suppose, by contradiction, that $\mathbf{y}_1(s_*) = \vec{0}$. From $\mathrm{y}_1^1(s_*) = \mathrm{y}_1^2(s_*) = 0$, we infer that $\tau'(s_*) = 0$. Hence $e = \tau(s_*)$ is a root of $\mathrm{P}_{\mathbf{c}}$. From $\mathrm{y}_1^3(s_*) = 0$, we have $e = \pm\sqrt{3\lambda_1}$, which contradicts Lemma B1. ∎

We are now in a position to conclude the proof. For $j = 1, 2, 3$, let $\mathbf{w}_j$ be defined by

$$\mathbf{w}_j = \mathcal{F}\mathbf{y}_j \colon \ \mathrm{J}_\gamma \to \mathbb{C}^{1,2}.$$

Then, $\mathbf{w}_j(s) \in \mathbb{L}_j$ and $\mathbf{w}_j(s) \neq \vec{0}$, for every s. Thus, there exist smooth functions $\Phi_j \colon \mathrm{J}_\gamma \to \mathbb{C}$, such that $\mathbf{w}_j' = \Phi_j \mathbf{w}_j$. From (2.3), we have

$$\Phi_j \mathbf{y}_j = \mathbf{y}_j' + K\mathbf{y}_j, \qquad j = 1, 2, 3, \tag{5.4}$$

where

$$K = \begin{pmatrix} \mathrm{i}c_1\tau^{-2} & -\mathrm{i} & \tau \\ 0 & -2\mathrm{i}c_1\tau^{-2} & 1 \\ 1 & 0 & \mathrm{i}c_1\tau^{-2} \end{pmatrix}.$$

Then, the third component of $\mathbf{y}_j' + K\mathbf{y}_j$ is equal to

$$3\tau + 3c_1\lambda_j\tau^{-2} - \big(\lambda_j^2 + 4c_1\big) + \mathrm{i}\tau\tau'.$$

Hence, using (5.4) we obtain

$$\Phi_j = -\frac{\tau\tau'}{3\lambda_j - \tau^2} + \mathrm{i}\frac{3c_1\lambda_j - \big(4c_1 + \lambda_j^2\big)\tau^2 + 3\tau^3}{\tau^2\big(3\lambda_j - \tau^2\big)}. \tag{5.5}$$

Lemma B3. *The functions $3\lambda_j - \tau^2$, $j = 1, 2, 3$, are nowhere zero.*

Proof of Lemma B3. The statement is obvious if λ_j is real and negative or complex, with nonzero imaginary part. If λ_j is real non-negative, the smoothness of Φ_j implies that $\tau\tau'\big(3\lambda_j - \tau^2\big)^{-1}$ is differentiable. Then $\big(3\lambda_j - \tau^2\big)(s) \neq 0$, for every s, such that $\tau(s)\tau'(s) \neq 0$. If $\tau(s)\tau'(s) = 0$, it follows that $\tau(s)$ is a root of the polynomial $\mathrm{P}_{\mathbf{c}}$. Therefore, by Lemma B1, we have that $\big(3\lambda_j - \tau^2\big)(s) \neq 0$. ■

From (5.5), we have

$$\int_0^s \Phi_j \mathrm{d}u = \log\Big(\sqrt{\tau^2 - 3\lambda_j}\Big) + \mathrm{i}\phi_j + b_j, \qquad j = 1, 2, 3,$$

where b_j is a constant of integration, $\sqrt{\tau^2 - 3\lambda_j}$ is a continuous determination of the square root of $\tau^2 - 3\lambda_j$ and $\log\big(\sqrt{\tau^2 - 3\lambda_j}\big)$ is a continuous determination of the logarithm of $\sqrt{\tau^2 - 3\lambda_j}$. Since $\mathbf{w}_j' = \Phi_j\mathbf{w}_j$, we obtain

$$\mathcal{F}\mathbf{y}_j r_j^{-1}\mathrm{e}^{-\mathrm{i}\phi_j} = \mathbf{m}_j, \qquad j = 1, 2, 3,$$

where $\mathbf{m}_j$ is a constant vector belonging to the eigenspace $\mathbb{L}_j$ of $\mathfrak{M}$. This implies

$$\mathcal{F} = \mathrm{M}\mathrm{D}\big(r_1\mathrm{e}^{\mathrm{i}\phi_1}, r_2\mathrm{e}^{\mathrm{i}\phi_2}, r_2\mathrm{e}^{\mathrm{i}\phi_2}\big)\mathrm{V}^{-1},$$

where M is an invertible matrix such that $\mathrm{M}^{-1}\mathfrak{M}\mathrm{M} = \mathrm{D}(\lambda_1, \lambda_2, \lambda_3)$. By possibly replacing γ with a congruent curve, we may suppose that $\mathcal{F}(0) = I_3$. Then, since $\phi_j(0) = 0$, we have $\mathrm{M} = \mathrm{V}(0)\mathrm{D}(r_1(0), r_2(0), r_3(0))^{-1}$. This concludes the proof of Theorem B. ■

5.2 Integrability by quadratures of general critical curves of type $\mathcal{B}_1'$

We now specialize the above procedure to the case of general critical curves of type $\mathcal{B}_1'$ (i.e., general critical curves with modulus $\mathbf{c} \in \mathcal{B}_1$ and with periodic twist). Let $\mathcal{M}_+'$ be as in (4.3). Since $\mathcal{B}_1$ is contained in $\mathcal{M}_+'$, the lowest roots e_1 and e_2 of $\mathrm{P}_{\mathbf{c}}$ are negative, for every $\mathbf{c} \in \mathcal{B}_1$ (cf. Remark 4.6).

Lemma 5.2. *Let γ be a general critical curve of type $\mathcal{B}_1'$. The λ_1-eigenspace of the momentum is spacelike.*

Proof. Let $\mathbf{y}_j$ be as in (5.1). Then $\mathcal{F}(s)\mathbf{y}_j(s)$ belongs to the λ_j-eigenspace of $\mathfrak{M}$, for every $s \in \mathbb{R}$. Using the conservation law $\frac{3}{2}\tau^2(\tau')^2 = \mathrm{P}_{\mathbf{c}}(\tau)$ (cf. (3.10)) and taking into account that $\lambda_j^3 + 6c_1\lambda_j + 3(9 + c_2) = 0$, we compute

$$\langle \mathcal{F}\mathbf{y}_j, \mathcal{F}\mathbf{y}_j\rangle = \langle \mathbf{y}_j, \mathbf{y}_j\rangle = 3\big(\tau^2 - 3\lambda_j\big)\big(2c_1 + \lambda_j^2\big).$$

Moreover, since $\lambda_1 = -(\lambda_2 + \lambda_3)$ and $c_1 = -\big(\lambda_2^2 + \lambda_2\lambda_3 + \lambda_3^2\big)/6$, we have

$$\begin{aligned} 2c_1 + \lambda_1^2 &= \frac{1}{3}(2\lambda_2 + \lambda_3)(\lambda_2 + 2\lambda_3) > 0,\\ 2c_1 + \lambda_3^2 &= \frac{1}{3}(\lambda_3 - \lambda_2)(\lambda_2 + 2\lambda_3) > 0,\\ 2c_1 + \lambda_2^2 &= -\frac{1}{3}(\lambda_3 - \lambda_2)(2\lambda_2 + \lambda_3) < 0. \end{aligned}$$

From the fact that $\lambda_1 < 0$, it follows that $\langle \mathbf{y}_1, \mathbf{y}_1\rangle > 0$. This proves that the λ_1-eigenspace of the momentum is spacelike. ∎

Definition 5.3. There are two possible cases: either the λ_3-eigenspace of $\mathfrak{M}$ is spacelike, or else is timelike. In the first case, we say that γ is *positively polarized*, while in the second case, we say that γ is *negatively polarized*.

Remark 5.4. In view of the above lemma, γ is positively polarized if and only if $e_1^2 - 3\lambda_3 > 0$ and is negatively polarized if and only if $e_2^2 - 3\lambda_3 < 0$. It is a linear algebra exercise to prove the existence of $A \in G$, such that $A^{-1}\mathfrak{M}A = \mathfrak{M}_{\lambda_1,\lambda_2,\lambda_3}$, where

$$\mathfrak{M}_{\lambda_1,\lambda_2,\lambda_3} = \begin{pmatrix} \frac{1}{2}(\lambda_2 + \lambda_3) & 0 & \frac{\varepsilon \mathrm{i}}{2}(\lambda_2 - \lambda_3)\\ 0 & \lambda_1 & 0\\ -\frac{\varepsilon \mathrm{i}}{2}(\lambda_2 - \lambda_3) & 0 & \frac{1}{2}(\lambda_2 + \lambda_3) \end{pmatrix}, \tag{5.6}$$

where $\varepsilon = \pm 1$ accounts for the polarization of γ (see below). It is clear that any critical curve of type $\mathcal{B}_1'$ is congruent to a critical curve whose momentum is in the canonical form $\mathfrak{M}_{\lambda_1,\lambda_2,\lambda_3}$.

Definition 5.5. A critical curve of type $\mathcal{B}_1'$ is said to be in a *standard configuration* if its momentum is in the canonical form (5.6). Two standard configurations with the same twist are congruent with respect to the left action of the maximal compact abelian subgroup $\mathbb{T}^2 = \{\mathrm{A} \in \mathrm{G} \mid \mathrm{A}\mathbf{e}_2 \wedge \mathbf{e}_2 = 0\}$.

Let $\mathbf{c} \in \mathcal{B}_1$, such that $\Delta_1(\mathbf{c})\Delta_2(\mathbf{c}) \neq 0$. Let $e_1 < e_2 < e_3$ be the real roots of $\mathrm{P}_{\mathbf{c}}$ and let $\lambda_1 = -(\lambda_2 + \lambda_3) < 0 < \lambda_2 < \lambda_3$ be the roots of $\mathrm{Q}_{\mathbf{c}}$. Let τ be the periodic function defined as in the first of the (4.5) and ϕ_j, $j = 1, 2, 3$, be as in (5.2). Let ρ_j be the constants

$$\rho_1 = \frac{1}{\sqrt{(2\lambda_2 + \lambda_3)(\lambda_2 + 2\lambda_3)}}, \qquad \rho_2 = \frac{1}{\sqrt{2(\lambda_3 - \lambda_2)(2\lambda_2 + \lambda_3)}},$$
$$\rho_3 = \frac{1}{\sqrt{2(\lambda_3 - \lambda_2)(\lambda_2 + 2\lambda_3)}}$$

and z_j be the functions

$$z_1 = \rho_1\sqrt{3(\lambda_2 + \lambda_3) + \tau^2}\mathrm{e}^{\mathrm{i}\phi_1}, \qquad z_2 = \rho_2\sqrt{3\lambda_2 - \tau^2}\mathrm{e}^{\mathrm{i}\phi_2}, \qquad z_3 = \rho_3\sqrt{3\lambda_3 - \tau^2}\mathrm{e}^{\mathrm{i}\phi_3}. \tag{5.7}$$

Let $\varepsilon = -\mathrm{sign}\big(e_2^2 - 3\lambda_3\big)$. We can state the following.

Theorem C. *A general critical curve of type $\mathcal{B}_1'$ with modulus* $\mathbf{c}$ *is congruent to*

$$\gamma\colon\ \mathbb{R}\ni s\longmapsto \big[{}^t(z_2+z_3,\varepsilon \mathrm{i} z_1,-\varepsilon \mathrm{i}(z_2-z_3))\big]\in\mathcal{S}. \tag{5.8}$$

In addition, γ is in a standard configuration.

Proof. Let $\widetilde{\gamma}$ be a critical curve of type $\mathcal{B}_1'$ with modulus $\mathbf{c}$. Let $\mathcal{F}$ be a Wilczynski frame along $\widetilde{\gamma}$. Suppose $\varepsilon=1$ (i.e., $\langle \mathbf{y}_3,\mathbf{y}_3\rangle<0$). Let $\mathbf{u}_j$ be the maps defined by

$$\mathbf{u}_1=\frac{1}{\sqrt{3}\sqrt{2c_1+\lambda_1^2}\sqrt{\tau^2-3\lambda_1}}\mathbf{y}_1,\qquad \mathbf{u}_2=\frac{1}{\sqrt{3}\sqrt{-\big(2c_1+\lambda_2^2\big)}\sqrt{\tau^2-3\lambda_2}}\mathbf{y}_2,$$

$$\mathbf{u}_3=\frac{1}{\sqrt{3}\sqrt{2c_1+\lambda_3^2}\sqrt{\tau^2-3\lambda_3}}\mathbf{y}_3.$$

Consider the map $U=(\mathbf{u}_3,\mathbf{u}_2,\mathbf{u}_1)\colon\mathbb{R}\to \mathrm{GL}(3,\mathbb{C})$. From Theorem B and Lemma 5.2, we have

- $\langle \mathbf{u}_1,\mathbf{u}_1\rangle=\langle \mathbf{u}_2,\mathbf{u}_2\rangle=-\langle \mathbf{u}_3,\mathbf{u}_3\rangle=1$, and $\langle \mathbf{u}_i,\mathbf{u}_j\rangle=0$, for $i\neq j$, that is, $U(s)$ is a pseudo-unitary basis of $\mathbb{C}^{1,2}$, for every $s\in\mathbb{R}$;
- $U^{-1}LU=D(\lambda_3,\lambda_2,\lambda_1)$.

Using again Theorem B, we obtain

$$\mathcal{F}UD\big(\mathrm{e}^{-\mathrm{i}\phi_3},\mathrm{e}^{-\mathrm{i}\phi_2},\mathrm{e}^{-\mathrm{i}\phi_1}\big)=\mathrm{M}D\Big(\sqrt{3\big(2c_1+\lambda_3^2\big)},\sqrt{-3\big(2c_1+\lambda_2^2\big)},\sqrt{3\big(2c_1+\lambda_1^2\big)}\Big)^{-1}, \tag{5.9}$$

where the matrix $\mathrm{M}\in\mathrm{GL}(3,\mathbb{C})$ diagonalizes the momentum of $\widetilde{\gamma}$, i.e., $\mathrm{M}^{-1}\widetilde{\mathfrak{M}}\mathrm{M}=D(\lambda_3,\lambda_2,\lambda_1)$. In particular, the column vectors of the right hand side of (5.9), denoted by $\widetilde{\mathrm{B}}$, constitutes a pseudo-unitary basis. Let ϵ be the inverse of a cubic root of $\widetilde{\mathrm{B}}$. Then, the column vectors of $\epsilon\widetilde{\mathrm{B}}$ constitute a unimodular pseudo-unitary basis. Therefore, there exists a unique $A\in G$, such that $\epsilon A\widetilde{\mathrm{B}}=\mathrm{B}$, where

$$\mathrm{B}=\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ 0 & 0 & \mathrm{i}\\ \frac{\mathrm{i}}{\sqrt{2}} & \frac{\mathrm{i}}{\sqrt{2}} & 0\end{pmatrix}. \tag{5.10}$$

Then

$$A\mathcal{F}=\epsilon^{-1}\mathrm{B}D\big(\mathrm{e}^{\mathrm{i}\phi_3},\mathrm{e}^{\mathrm{i}\phi_2},\mathrm{e}^{\mathrm{i}\phi_1}\big)U^{-1}=\epsilon^{-1}\mathrm{B}D\big(\mathrm{e}^{\mathrm{i}\phi_3},\mathrm{e}^{\mathrm{i}\phi_2},\mathrm{e}^{\mathrm{i}\phi_1}\big)D(-1,1,1)\bar{{}^tU}\mathbf{h}. \tag{5.11}$$

It is now a computational matter to check that the first column vector of the right hand side of (5.11) is $\epsilon^{-1}\,{}^t(z_2+z_3,\mathrm{i}z_1,-\mathrm{i}(z_2-z_3))$. This implies $\widetilde{\gamma}=A^{-1}\gamma$ (i.e., $\widetilde{\gamma}$ and γ are congruent to each other).

Taking into account that $U^{-1}LU=D(\lambda_3,\lambda_2,\lambda_1)$ and using (5.9), the momentum of $\widetilde{\gamma}$ is $\widetilde{\mathfrak{M}}=\widetilde{\mathrm{B}}D(\lambda_3,\lambda_2,\lambda_1)\widetilde{\mathrm{B}}^{-1}$. Therefore, the momentum of γ is

$$\mathfrak{M}=A\widetilde{\mathrm{B}}D(\lambda_3,\lambda_2,\lambda_1)\widetilde{\mathrm{B}}^{-1}A^{-1}=\mathrm{B}D(\lambda_3,\lambda_2,\lambda_1)\mathrm{B}^{-1}=\mathfrak{M}_{\lambda_1,\lambda_2,\lambda_3}.$$

This proves that γ is in standard configuration.

If $\varepsilon=-1$ (i.e., $\langle \mathbf{y}_3,\mathbf{y}_3\rangle>0$), considering $U=(\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_1)$ and arguing as above, we get the same conclusion. ∎

Remark 5.6. Theorem C implies that a standard configuration γ does not pass through the pole $[\mathbf{e}_3]$ of the Heisenberg projection π_H. Thus $\check{\gamma}:=\pi_H\circ\gamma$ is a transversal curve of $\mathbb{R}^3$, which does not intersect the Oz-axis.

Remark 5.7. Breaking the integrands into partial fractions, the integrals

$$f_j(\tau) = \sqrt{\frac{3}{2}} \int_{e_2}^{\tau} \frac{3c_1\lambda_j - \big(4c_1 + \lambda_j^2\big) + 3\tau^3}{\tau\big(3\lambda_j - \tau^2\big)\sqrt{\mathrm{P}_{\mathbf{c}}(\tau)}} \mathrm{d}\tau, \qquad j = 1, 2, 3,$$

can be written as linear combinations of standard hyperelliptic integrals of the first and third kind. Then ϕ_j is the odd quasi-periodic function with quasi-period 2ω such that $\phi_j(s) = f_j[\tau(s)]$. In practice, we compute τ and ϕ_j, $j = 1, 2, 3$, by numerically solving the following system of ODE,

$$\begin{aligned} \tau'' &= \tau^2 - 9c_1\tau^{-2}\big(1 - c_1\tau^{-1}\big), \\ \phi_j' &= \frac{3c_1\lambda_j - \big(4c_1 + \lambda_j^2\big)\tau^2 + 3\tau^3}{\tau^2\big(3\lambda_j - \tau^2\big)}, \qquad j = 1, 2, 3, \end{aligned} \tag{5.12}$$

with initial conditions

$$\tau(0) = e_2, \qquad \tau'(0) = 0, \qquad \phi_j(0) = 0, \qquad j = 1, 2, 3. \tag{5.13}$$

5.3 Closing conditions

From Theorem C, it follows that a critical curve of type $\mathcal{B}_1'$ is closed if and only if

$$\mathfrak{P}_j = \frac{1}{2\pi}\phi_j(2\omega) \in \mathbb{Q}, \qquad j = 1, 2, 3.$$

On the other hand,

$$\frac{1}{2\pi}\phi_j(2\omega) = \frac{1}{\pi} \int_{e_2}^{e_1} \frac{\sqrt{3}\big(3c_1\lambda_j - \big(4c_1 + \lambda_j^2\big)\tau^2 + 3\tau^3\big)}{\sqrt{2}\tau\big(3\lambda_j - \tau^2\big)\sqrt{\mathrm{P}_{\mathbf{c}}(\tau)}} \mathrm{d}\tau. \tag{5.14}$$

Thus, γ is closed if and only if the complete hyperelliptic integrals on the right hand side of (5.14) are rational. For a closed critical curve γ, we put $\mathfrak{P}_j = q_j = m_j/n_j$, where $n_j > 0$ and $\gcd(m_j, n_j) = 1$. We call q_j, the *quantum numbers* of γ. By construction, $\mathrm{e}^{\mathrm{i}2\pi\mathfrak{P}_1}, \mathrm{e}^{\mathrm{i}2\pi\mathfrak{P}_2}$ and $\mathrm{e}^{\mathrm{i}2\pi\mathfrak{P}_3}$ are the eigenvalues of the *monodromy* $\mathtt{M}_\gamma = \mathcal{F}(2\omega)\mathcal{F}(0)^{-1}$ of γ. Since $\det(\mathtt{M}_\gamma) = 1$, we have

$$\sum_{j=1}^{3} \mathfrak{P}_j \equiv 0, \qquad \mod \mathbb{Z}.$$

Then, γ is closed if and only if two among the integrals $\mathfrak{P}_j$, $j = 1, 2, 3$, are rational.

Remark 5.8. The closing conditions can be rephrased as follows. Consider the even quasi-periodic functions ϕ_1, ϕ_3. Then, the critical curve is closed if and only if the jumps $\phi_j|_0^{2\omega}$, $j = 1, 3$, are rational.

Example 5.9. We now consider an example, which will be taken up again in the last section. Choose $\mathbf{c} \approx (-0.8284243304411575, -8.349417691746162) \in \mathcal{B}_1^-$. The real roots of the quintic polynomial are

$$e_1 \approx -0.931924 < e_2 \approx -0.678034 < 0 < e_3 \approx 2.79051$$

and the eigenvalues of the momentum are

$$\lambda_1 \approx -2.40462 < 0 < \lambda_2 \approx 0.40614 < \lambda_3 \approx 1.99848.$$

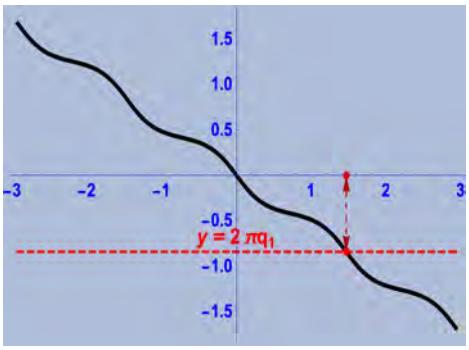

Figure 8. The graph of ϕ_1 for $\mathbf{c} \approx (-0.8284243304411575, -8.349417691746162) \in \mathcal{B}_1^-$.

The half-period of the twist is computed by numerically evaluating the hyperelliptic integral (4.4). We evaluate τ, ϕ_1, ϕ_2, ϕ_3 by solving numerically the system (5.12), with initial conditions (5.13) on the interval $[-4\omega, 4\omega]$. Figure 8 reproduces the graph of the quasi-periodic function ϕ_1 on the interval $[-4\omega, 4\omega]$ (the graph of the twist was depicted in Figure 5). The red point on the Ox-axis is 2ω and the length of the arrows is the jump $\phi_1|_0^{2\omega}$. In this example,

$$\left| -\frac{1}{2\pi}\frac{2}{15} - \phi_1|_0^{2\omega} \right| = 1.6151 \cdot 10^{-8}, \qquad \left| -\frac{1}{2\pi}\frac{10}{21} - \phi_3|_0^{2\omega} \right| = 4.46887 \cdot 10^{-8}.$$

So, modulo negligible numerical errors, the corresponding critical curves are closed, with quantum number $q_1 = -2/15$ and $q_3 = -10/21$. In the last section, we will explain how we computed the modulus. A standard configuration of a curve with modulus $\mathbf{c}$ is represented in Figure 13.

5.4 Discrete global invariants of a closed critical curve

Consider a closed general critical curve γ of type $\mathcal{B}_1'$, with modulus $\mathbf{c}$ and quantum numbers $q_1 = m_1/n_1$, $q_2 = m_2/n_2$, $q_3 = m_3/n_3$, $q_1 + q_2 + q_3 \equiv 0 \mod \mathbb{Z}$. The half-period ω of the twist is given by the complete hyperelliptic integral (4.4). Let $\mathtt{M}_\gamma = \mathcal{F}(\omega)\mathcal{F}(0)^{-1}$ be the monodromy of γ. The monodromy does not depend on the choice of the canonical lift. It is a diagonalizable element of G with eigenvalues $\mathrm{e}^{2\pi i q_1}$, $\mathrm{e}^{2\pi i q_2}$, and $\mathrm{e}^{2\pi i q_3}$. Thus, $\mathtt{M}_\gamma$ has finite order $n = \mathrm{lcm}(n_1, n_3)$. The momentum $\mathfrak{M}_\gamma$ has three distinct real eigenvalues, so its stabilizer is a maximal compact abelian subgroup $\mathbb{T}^2_\gamma \cong \mathrm{S}^1 \times \mathrm{S}^1$ of G (if γ is a standard configuration, $\mathbb{T}^2_\gamma = \mathbb{T}^2$). Since $[\mathtt{M}_\gamma, \mathfrak{M}_\gamma] = 0$, $\mathtt{M}_\gamma \in \mathbb{T}^2_\gamma$. Let s_1, s_3 be the integers defined by $n = s_1 n_1 = s_3 n_3$. The *CR spin* of γ is $1/3$ if and only if $n \equiv 0 \mod 3$ and $m_1 s_1 \equiv m_3 s_3 \not\equiv 0 \mod 3$. The *wave number* $\mathbf{n}_\gamma$ of γ is n if the spin is 1 and $n/3$ if the spin is $1/3$. Let $|[\gamma]|$ denote the trajectory of γ. The stabilizer $\hat{G}_\gamma = \{[A] \in [G] \mid [A] \cdot |[\gamma]| = |[\gamma]|\}$ is spanned by $[\mathfrak{M}_\gamma]$ and is a cyclic group of order $\mathbf{n}_\gamma$. Geometrically, $\hat{G}_\gamma$ is the *symmetry group* of the critical curve γ. The *CR turning number* w_γ is the degree of the map $\mathbb{R}/2n\omega\mathbb{Z} \ni s \mapsto F_1 - \mathrm{i}F_3 \in \dot{\mathbb{C}} := \mathbb{C}\setminus\{0\}$, where the F_j's are the components of a Wilczynski frame along γ. Without loss of generality, we may suppose that γ is in a standard configuration. From (5.8), it follows that w_γ is the degree of $\mathbb{R}/2n\omega\mathbb{Z} \ni s \mapsto z_3 \in \dot{\mathbb{C}}$, if $\varepsilon_\gamma = 1$, and is the degree of $\mathbb{R}/2n\omega\mathbb{Z} \ni s \mapsto z_2 \in \dot{\mathbb{C}}$, if $\varepsilon_\gamma = -1$. Therefore,

$$w_\gamma = \begin{cases} s_3 m_3 & \text{if } \varepsilon_\gamma = 1, \\ s_2 m_2 & \text{if } \varepsilon_\gamma = -1. \end{cases}$$

A closed critical curve γ has an additional discrete CR invariant, denoted by $\mathrm{tr}_*(\gamma)$, the trace of γ with respect to the spacelike λ_1-eigenspace of the momentum. To clarify the geometrical meaning of the trace, it is convenient to consider a standard configuration. In this case, $\mathbb{L}_1$ is

spanned by $\mathbf{e}_2 \in \mathbb{C}^{1,2}$ and the corresponding chain is the intersection of $\mathcal{S}$ with the projective line $z_2 = 0$. The Heisenberg projection of this chain is the upward oriented Oz-axis. Thus, $\mathrm{tr}_*(\gamma)$ is the linking number $\mathrm{Lk}\big(\check{\gamma}, Oz^{\uparrow}\big)$ of the Heisenberg projection of γ with the upward oriented Oz-axis.

Proposition 5.10. *Let γ be as above. Then*

$$\mathrm{tr}_*(\gamma) = \begin{cases} (q_1 - q_3)\mathbf{n}_\gamma & \text{if } \varepsilon_\gamma = 1, \\ (q_1 - q_2)\mathbf{n}_\gamma & \text{if } \varepsilon_\gamma = -1. \end{cases}$$

Proof. Without loss of generality, we may assume that γ is in standard configuration. The Heisenberg projection of γ is

$$\check{\gamma} = {}^t\!\left(\mathrm{Re}\left(\frac{\varepsilon i z_1,}{z_2 + z_3}\right), \mathrm{Im}\left(\frac{\varepsilon \mathrm{i} z_1,}{z_2 + z_3}\right), \mathrm{Re}\left(\frac{-\varepsilon \mathrm{i}(z_2 - z_3)}{z_2 + z_3}\right)\right).$$

Since $\check{\gamma}$ does not intersect the Oz-axis, the linking number $\mathrm{Lk}\big(\check{\gamma}, Oz^{\uparrow}\big)$ is the degree of

$$\mathbb{R}/2\mathbf{n}_\gamma\omega\mathbb{Z} \ni s \longmapsto \frac{z_1}{z_2 + z_3} \in \dot{\mathbb{C}}.$$

From (5.7), it follows that this degree is the degree of

$$f\colon\ \mathbb{R}/2\mathbf{n}_\gamma\omega\mathbb{Z} \ni s \longmapsto \frac{\rho_1\sqrt{3(\lambda_2+\lambda_3)+\tau^2(s)}\mathrm{e}^{\mathrm{i}\phi_1}}{\rho_2\sqrt{3\lambda_2-\tau^2(s)}\mathrm{e}^{\mathrm{i}\phi_2} + \rho_3\sqrt{3\lambda_3-\tau^2(s)}\mathrm{e}^{\mathrm{i}\phi_3}}.$$

Suppose that γ is negatively polarized. Then, $\tau^2 - 3\lambda_3 < \tau^2 - 3\lambda_2 < 0$ and $0 < \tau^2 < 3\lambda_2$. Therefore,

$$0 < \frac{\rho_2\sqrt{3\lambda_2 - \tau^2}}{\rho_3\sqrt{3\lambda_3 - \tau^2}} = \sqrt{\frac{\big(3\lambda_2 - \tau^2\big)(\lambda_2 + 2\lambda_3)}{\big(3\lambda_3 - \tau^2\big)(2\lambda_2 + \lambda_3)}} \leq \sqrt{\frac{\lambda_2(\lambda_2 + 2\lambda_3)}{\lambda_3(2\lambda_2 + \lambda_3)}} < 1.$$

Thus

$$f = \frac{\rho_1\sqrt{3(\lambda_2+\lambda_3)+\tau^2}}{\rho_3\sqrt{3\lambda_3-\tau^2}} \frac{\mathrm{e}^{\mathrm{i}(\phi_1-\phi_3)}}{1 + h\mathrm{e}^{\mathrm{i}(\phi_2-\phi_3)}},$$

where

$$h = \frac{\rho_2\sqrt{3\lambda_2 - \tau^2}}{\rho_3\sqrt{3\lambda_3 - \tau^2}}.$$

Since $0 < h < 1$, the image of $1 + h\mathrm{e}^{\mathrm{i}(\phi_2-\phi_3)}$ is a curve contained in a disk of radius < 1 centered at $(1, 0)$. Hence $1 + h\mathrm{e}^{\mathrm{i}(\phi_2-\phi_3)}$ is null-homotopic in $\dot{\mathbb{C}}$. This implies

$$\deg(f) = \frac{1}{2\pi}(\phi_1 - \phi_3)\Big|_0^{2\mathbf{n}_\gamma\omega} = \mathbf{n}_\gamma(q_1 - q_3).$$

Suppose that γ is positively polarized. Then, $\tau^2 - 3\lambda_2 > \tau^2 - 3\lambda_3 > 0$. In particular, $\tau^2 > 3\lambda_3 > 0$ and

$$0 < \frac{\rho_3\sqrt{\tau^2 - 3\lambda_3}}{\rho_2\sqrt{\tau^2 - 3\lambda_2}} = \sqrt{\frac{(\tau^2 - 3\lambda_3)(2\lambda_2 + \lambda_3)}{(\tau^2 - 3\lambda_2)(\lambda_2 + 2\lambda_3)}} < \sqrt{\frac{2\lambda_2 + \lambda_3}{\lambda_2 + 2\lambda_3}} < 1.$$

Then

$$f = -\mathrm{i}\frac{\rho_1\sqrt{3(\lambda_2+\lambda_3)+\tau^2}}{\rho_2\sqrt{\tau^2-3\lambda_2}}\frac{\mathrm{e}^{\mathrm{i}(\phi_1-\phi_2)}}{1+\tilde{h}\mathrm{e}^{\mathrm{i}(\phi_3-\phi_2)}},$$

where

$$\tilde{h} = \frac{\rho_3\sqrt{\tau^2-3\lambda_3}}{\rho_2\sqrt{\tau^2-3\lambda_2}}.$$

Since $0<\tilde{h}<1$, the image of $1+\tilde{h}\mathrm{e}^{\mathrm{i}(\phi_3-\phi_2)}$ is a curve contained in a disk of radius <1 centered at $(1,0)$. Hence $1+\tilde{h}\mathrm{e}^{\mathrm{i}(\phi_3-\phi_2)}$ is null-homotopic in $\dot{\mathbb{C}}$. This implies

$$\deg(f) = \frac{1}{2\pi}(\phi_1-\phi_2)\Big|_0^{2\mathbf{n}_\gamma\omega} = \mathbf{n}_\gamma(q_1-q_2). \qquad \blacksquare$$

Summarizing: *the quantum numbers of a closed critical curve are determined by the wave number, the CR spin, the CR turning number, and the trace.*

6 Experimental evidence of the existence of infinite countably many closed critical curves of type $\mathcal{B}_1'$ and examples

This section is of an experimental nature. We use numerical tools, implemented in the software MATHEMATICA 13.3, to support the claim that there exist countably many closed critical curves of type $\mathcal{B}_1'$, with moduli belonging to the connected component $\mathcal{B}_1^-$ of $\mathcal{B}_1$ (cf. Remark 4.14). The same reasoning applies, as well, if the modulus belongs to the other connected component $\mathcal{B}_1^+$ of $\mathcal{B}_1$. We parametrize $\mathcal{B}_1^-$ by the map $\psi_-\colon K_-\to\mathcal{B}_1^-$, defined in (4.6), where K_- is the rectangle $\hat{\mathrm{J}}_\xi^-\times(0,1)$, $\hat{\mathrm{J}}_\xi^- = (\pi/2, 2.3008)$. We take $\mathbf{p}=(p_1,p_2)\in K_-$ as the fundamental parameters. The modulus $\mathbf{c}=(c_1,c_2)$, the roots $e_1<e_2<0<e_3$ of the quintic polynomial, and the eigenvalues $\lambda_1=-(\lambda_2+\lambda_3)<0<\lambda_2<\lambda_3$ of the momentum are explicit functions of the parameters (p_1,p_2). Let K_-^* be the open set of the general parameters, that is,

$$K_-^* = \{\mathbf{p}\in K_- \mid \Delta_1(\psi_-(\mathbf{p}))\Delta_2(\psi_-(\mathbf{p}))\neq 0\}.$$

The complete hyperelliptic integrals $\mathfrak{P}_j$ can be evaluated numerically as functions of $\mathbf{p}\in K_-^*$. Consider the real analytic map $\mathfrak{P}=(\mathfrak{P}_1,\mathfrak{P}_3)\colon K_-^*\to\mathbb{R}^2$.[6] Choose $\mathbf{p}_*=(2,1/2)\in K_-^*$ and plot the graphs of the functions $f_{11}(p_1)=\mathfrak{P}_1(p_1,1/2)$, $f_{12}(p_2)=\mathfrak{P}_1(2,p_2)$, $f_{31}(p_1)=\mathfrak{P}_3(p_1,1/2)$, and $f_{32}(p_2)=\mathfrak{P}_3(2,p_2)$ (see Figures 9 and 10).

The function f_{11} is strictly increasing, while the other three functions are strictly decreasing. This implies that $\mathfrak{P}$ has maximal rank at $\mathbf{p}_*$. Thus $\mathcal{P}_-=\mathfrak{P}(K_-)$ is a set with non empty interior. In particular $\mathcal{P}_-^r := \mathcal{P}_-\cap\mathbb{Q}$ is an infinite countable set and, for every $\mathbf{q}=(q_1,q_2)\in\mathcal{P}_-^r$, there exists a closed critical curve of type $\mathcal{B}'^-_1$ with quantum numbers q_1 and q_2. Figure 11 reproduces the plot of the map $\mathfrak{P}$, an open convex set. The mesh supports a stronger conclusion: the map $\mathfrak{P}$ is 1-1. Therefore, one can assume that, for every rational point $(q_1,q_3)\in\mathcal{P}_-$, there exists a unique congruence class of closed critical curves with quantum numbers q_1 and q_3. The construction of a standard configuration of a critical curve associated to a rational point $\mathbf{q}\in\mathcal{P}_-$ can be done in three steps.

Step 1. Choose a rational point $\mathbf{q}=(q_1,q_3)=(m_1/n_1,m_3/n_3)\in\mathcal{P}_-$. To find the parameter $\mathbf{p}\in K_-$, such that $\mathfrak{P}(\mathbf{p})=\mathbf{q}$, we may proceed as follows: plot the level curves $\mathtt{X}_{q_1}=\mathfrak{P}_1^{-1}(q_1)$

[6] Actually, $\mathfrak{P}$ is real-analytic on all K_-. Instead, $\mathfrak{P}_2$ has a jump discontinuity at the exceptional locus.

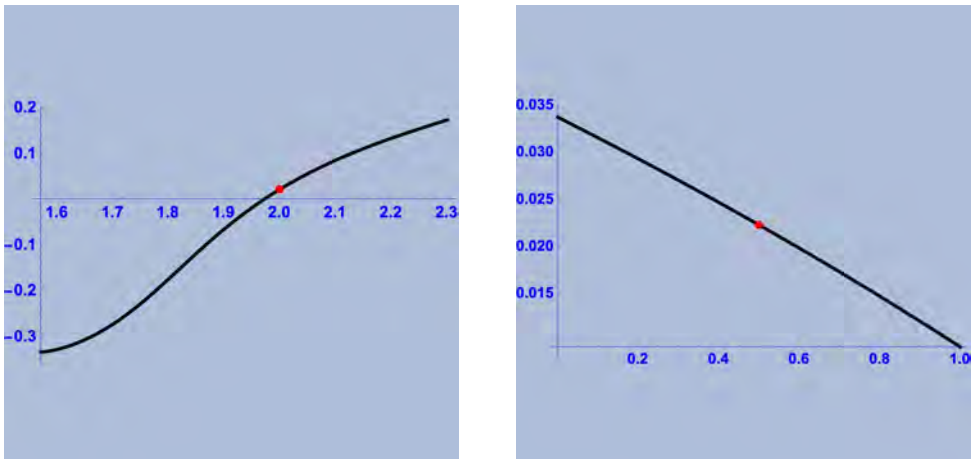

Figure 9. On the left: the graph of the function f_{11}. On the right: the graph of the function f_{12}.

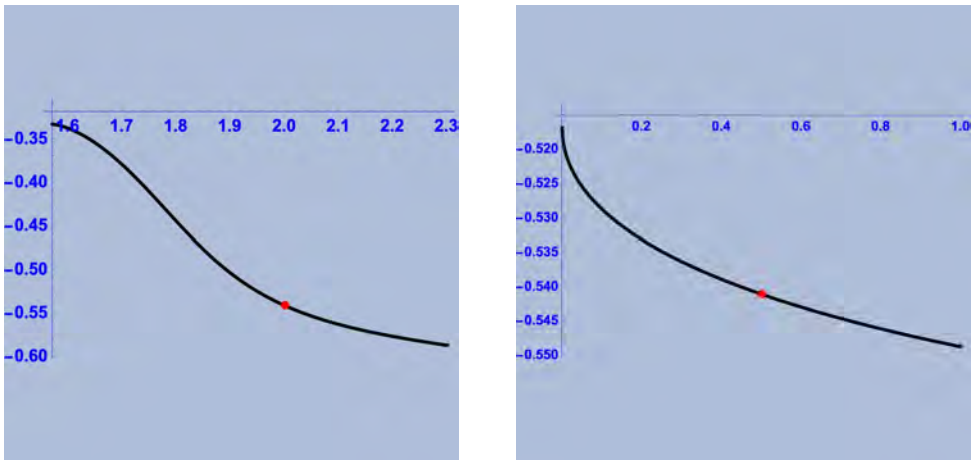

Figure 10. On the left: the graph of the function f_{31}. On the right: the graph of the function f_{32}.

and $\mathtt{Y}_{q_3} = \mathfrak{P}_3^{-1}(q_3)$ and choose a small rectangle $\mathrm{R} \subset K_-$ containing $\mathtt{X}_{q_1} \cap \mathtt{Y}_{q_2}$ (see Figure 12). Then we minimize numerically the function

$$\delta_{\mathbf{q}}\colon\ \mathrm{R} \ni \mathbf{p} \longmapsto \sqrt{(\mathfrak{P}_1(\mathbf{p}) - q_1)^2 + (\mathfrak{P}_3(\mathbf{p}) - q_3)^2}.$$

We use the stochastic minimization method named "differential evolution" [37] implemented in MATHEMATICA.

Example 6.1. Let us revisit Example 5.9. Choose $\mathbf{q} = (-2/15, -10/21) \in \mathcal{P}_-$. The plot of the level curves $\mathtt{X}_{q_1}$ and $\mathtt{Y}_{q_3}$ is depicted in Figure 12. Minimizing $\delta_{\mathbf{q}}$ on the rectangle $\mathrm{R} = [1.83, 1.86] \times [0.65, 0.75]$ (depicted on the right picture in Figure 12) we obtain $\mathbf{p} = (1.84438, 0.719473)$ and $\delta_{\mathbf{q}}(\mathbf{p}) = 3.26867 \cdot 10^{-9}$. So, up to negligible numerical errors, we may assume $\mathbf{p} = \mathfrak{P}^{-1}(\mathbf{q})$. Computing $\psi_-(\mathbf{p})$, we find the modulus $\mathbf{c} = (c_1, c_2)$ of the curve, where $c_1 = -0.828424$ and $c_2 = -8.349418$. With the modulus at hand, we compute the lowest real roots of the quintic polynomial, $e_1 \approx -0.931924 < e_2 \approx -0.678034$, and the roots of the momentum, namely $\lambda_1 \approx -2.40462 < 0 < \lambda_2 \approx 0.40614 < \lambda_3 \approx 1.99848$.

Step 2. We evaluate numerically the integral (4.4) and we get the half-period ω of the twist of the critical curve. In our example $\omega \approx 0.732307$. The next step is to evaluate the twist τ. This can be done by solving numerically the Cauchy problem (4.5) on the interval $[0, 2n\omega]$,

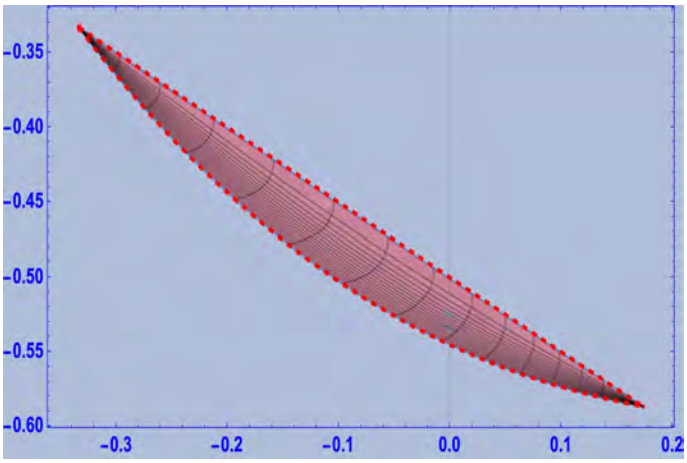

Figure 11. The plot of the map $\mathfrak{P}$.

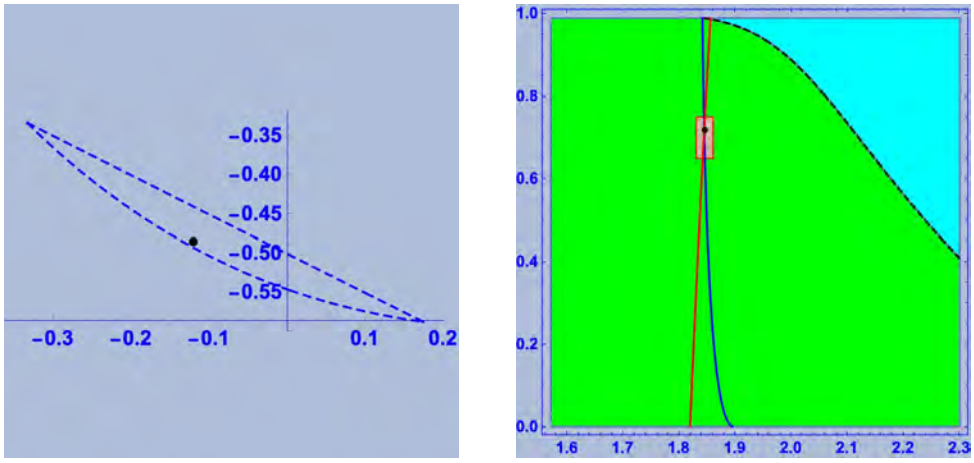

Figure 12. On the left: the point $\mathbf{q} = (-2/15, -10/21) \in \mathcal{P}_-$. On the right: the level curves $\mathtt{X}_{-2/15}$ and $\mathtt{Y}_{-10/21}$. The dotted curve is the exceptional locus. The green and the cyan domains are the two connected components of K^*_-. The brow rectangle is the one chosen for the numerical minimization of the function $\delta_{\mathbf{q}}$.

$n = \operatorname{lcm}(n_1, n_2)$. The bending is given by $\kappa = c_1/\tau^2$. Next, we solve the Frenet type linear system (2.3), with initial condition $\mathcal{F}(0) = I_3$. Then, $\widetilde{\gamma}\colon [0, 2n\omega] \ni s \longmapsto [F_1(s)] \in \mathcal{S}$ is a critical curve with quantum numbers q_1 and q_3 and $\mathcal{F}$ is a Wilczynski frame field along $\widetilde{\gamma}$. However, $\widetilde{\gamma}$ is not in a standard configuration.

Step 3. The last step consists in building the standard configuration. The momentum $\mathfrak{M}$ of $\widetilde{\gamma}$ is $L(0)$, where L is as in (3.2). Taking into account that $\tau(0) = e_2$, $\tau'(0) = 0$, and that $\kappa(0) = c_1/e_2^2$, we get

$$\mathfrak{M} = \begin{pmatrix} 0 & 3\big(1 - \frac{c_1}{e_2}\big) & 2\mathrm{i}e_2 \\ e_2 & 0 & 3\mathrm{i}\big(1 - \frac{c_1}{e_2}\big) \\ 3\mathrm{i} & -\mathrm{i}e_2 & 0 \end{pmatrix}.$$

The eigenspace of the highest eigenvalue is timelike (i.e., these critical curves are negatively polarized). We compute the eigenvectors and we build a unimodular pseudo-unitary basis $A = (A_1, A_2, A_3)$, such that A_1 is an eigenvector of λ_3, A_2 is an eigenvector of λ_2, and A_3 is an eigenvector of λ_1. Let B be as in (5.10). Consider $\mathrm{M} = \mathrm{B}A^{-1} \in G$. Then, $\gamma = \mathrm{M}\widetilde{\gamma}$ is a standard configuration of a critical curve with quantum numbers q_1 and q_3.

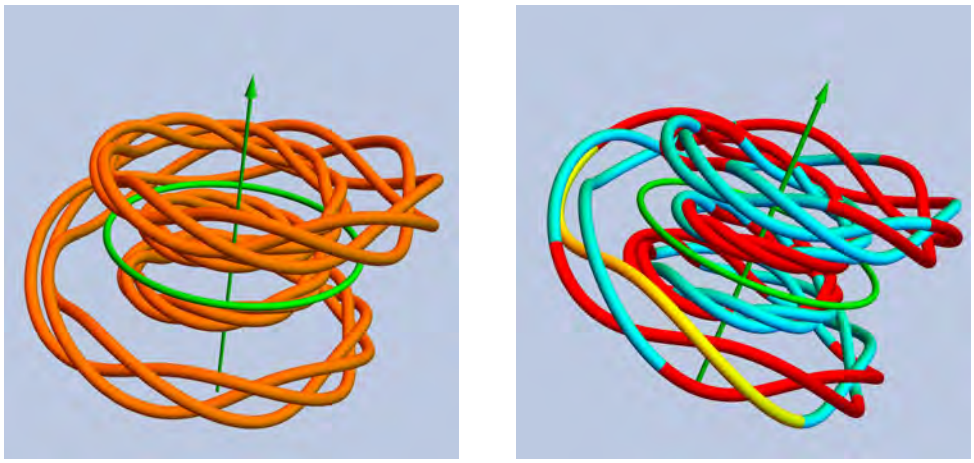

Figure 13. The Heisenberg projection of a standard configuration of a critical curve with quantum numbers $q_1 = -2/15$ and $q_2 = -10/21$. The figure on the left reproduces the fundamental arc $\gamma([0, 2\omega))$ (coloured in yellow). The curve can be constructed by acting with the monodromy on the fundamental arc.

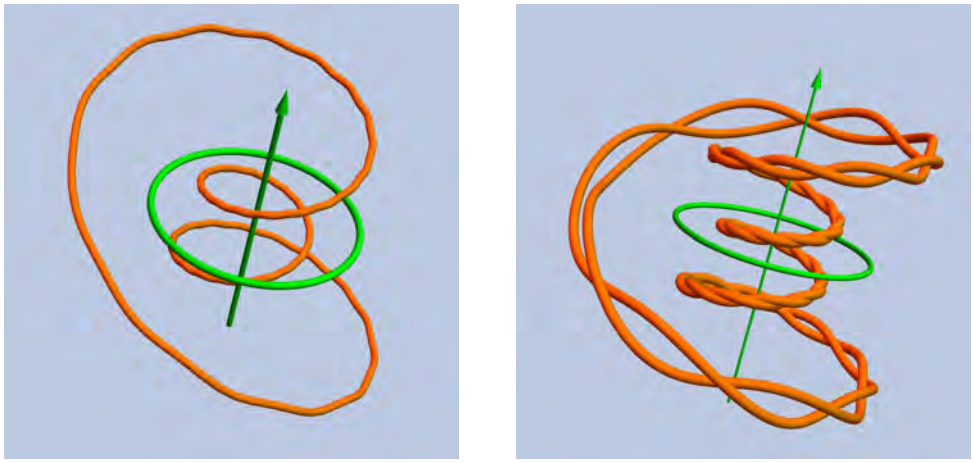

Figure 14. On the left: the Heisenberg projection of a standard configuration of a critical curve of type $\mathcal{B}'^{-}_{1}$ with quantum numbers $q_1 = -3/10$, $q_2 = -9/25$. On the right: the Heisenberg projection of a standard configuration of a critical curve of type $\mathcal{B}'^{-}_{1}$ with quantum numbers $q_1 = -1/5$, $q_2 = -3/7$.

Remark 6.2. The curve γ does not pass through the pole of the Heisenberg projection π_H. So, $\widehat{\gamma} = \pi_H \circ \gamma$ is a closed transversal curve of $\mathbb{R}^3$ which does not intersect the Oz-axis and $\mathrm{tr}_*(\gamma) = \mathrm{Lk}(\widehat{\gamma}, Oz)$.

Example 6.3. Applying Steps 2 and 3 to Example 6.1 and computing the Heisenberg projection, we obtain the transversal curve depicted in Figure 13, a non-trivial transversal knot. The quantum numbers are $q_1 = -2/15$ and $q_2 = -10/21$. Recalling what has been said about the discrete invariants of a critical curve (cf. Section 5.4), the spin is $1/3$, the wave number is $n = 35$, the CR turning number is -50, and the trace is 12.

Example 6.4. Figures 14 and 15 reproduce the Heisenberg projections of the standard configurations of critical curves of type $\mathcal{B}'^{-}_{1}$ with quantum numbers $(-3/10, -9/25)$, $(-1/5, -3/7)$, $(5/49, -4/7)$, and $(-7/36, -23/54)$, respectively. All of them have spin 1. The first is a trivial torus knot with wave number $n = 50$, $\mathrm{tr}_* = 3$, and CR turning number $w = -18$; the second example is a nontrivial transversal knot with $n = 35$, $\mathrm{tr}_* = 8$, and $w = -15$. The third example

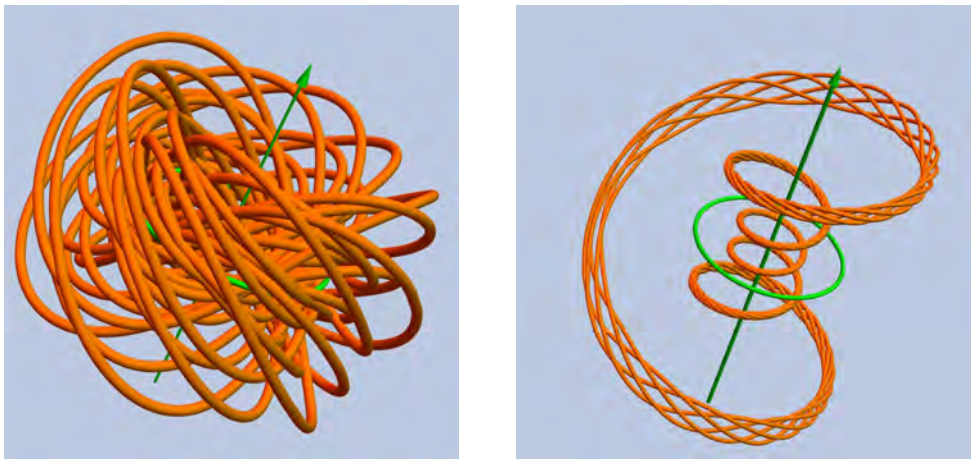

Figure 15. On the left: the Heisenberg projection of a standard configuration of a critical curve of type $\mathcal{B}'^{-}_{1}$, with quantum numbers $q_1 = 5/49$, $q_2 = -4/7$. On the right: the Heisenberg projection of a standard configuration of a critical curve of type $\mathcal{B}'^{-}_{1}$, with quantum numbers $q_1 = -7/36$, $q_2 = -23/54$.

is a "tangled" transversal curve with $n = 49$, $\mathrm{tr}_* = 33$, and $w = -28$. The last example is a nontrivial transversal torus knot with wave number 108, $\mathrm{tr}_* = 25$, and $w = -46$.

Remark 6.5. It is clear that, being numerical approximations, the parametrizations obtained with this procedure are only approximately periodic.

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