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# Lie pairs

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**Abstract.** Extending the theory of systems, we introduce a theory of Lie semialgebra "pairs" which parallels the classical theory of Lie algebras, but with a "null set" replacing 0. A selection of examples is given. These Lie pairs comprise two categories in addition to the universal algebraic definition, one with "weak Lie morphisms" preserving null sums, and the other with "$\preceq$-morphisms" preserving a surpassing relation $\preceq$ that replaces equality. We provide versions of the PBW (Poincaré-Birkhoff-Witt) Theorem in these categories.

## Contents



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## 1 Introduction

The purpose of this paper is to take a further step towards a general flexible framework for a unified treatment of classical algebraic structures together with those arising in a tropical context, where typically one cannot rely on the existence of an additive inverse (e.g., as in the celebrated max-plus algebra). The present research includes Lie algebras in this general picture, in the sense that we are about to explain. In other words, it may be considered as one more stage of a wider program initiated some years ago by the second author, through the theory of *triples* and *systems* (see e.g. [21, 22]), which has already proved successful in revisiting classical algebraic phenomena by embedding them in a tropical context. Among its applications, we recall the construction of an effective tropical substitute of the exterior algebra, along with a natural extension of the Cayley-Hamilton theorem [11] for endomorphisms of modules over semialgebras.

The simple but effective idea for remedying the lack of negation is to introduce an endomorphism $(-)$, whose square is the identity, to which is attached a *surpassing relation*. A further step was taken in [8], where a theory of Clifford semialgebra is proposed. In more traditional contexts, Clifford algebras are examples of Lie super-algebras, so [8] may be considered as the first relevant example of Lie semi-(super)algebras obtained within the already collocated framework of triples and systems. It was applied to extend to the tropical framework the polynomial representation of Lie algebras of endomorphisms of a vector space, in the same spirit of [10].

Meanwhile, a theory of semialgebra pairs has been developed in [2, 17], with the aim of exploiting, through their axiomatization, the formal properties enjoyed by the surpassing relation associated to a negation map, and eventually expunging the latter. For the reader's convenience, we recall here that many of the classical key properties of several algebraic structures in traditional frameworks are recovered by the formalism of *systems*, in which equality is replaced by the surpassing relation.

This premise should make clear that there is ample motivation to cope with the more tricky situation provided by the tropical version of Lie algebras; moreover, in view of [17], it is natural to investigate and to set the foundation of a theory of Lie pairs, generalizing Lie (semi)-algebras. These are pairs $(\mathcal{L}, \mathcal{L}_0)$ of modules over some commutative semiring $C$, endowed with a product $[\,] : \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}$, $(x, y) \mapsto [xy]$, satisfying suitable properties inspired by the classical Lie theory, and for which the skew-symmetric and "Jacobi identity" features of the theory are all subsumed in the submodule $\mathcal{L}_0$ of $\mathcal{L}$, which basically contains all of the relevant relations.

To show the reader quickly what we are talking about, the Lie bracket $[\,]$ satisfies the property $[xx] \in \mathcal{L}_0$ for all $x \in \mathcal{L}$. Moreover, one naturally requires $[xy] + [yx]$ to lie in $\mathcal{L}_0$, regardless of the choice of $(x, y) \in \mathcal{L} \times \mathcal{L}$. We thereby define, in case of free modules over a base semiring, the structure constants of a Lie pair. The attractiveness of the theory comes from the freedom in defining $\mathcal{L}_0$ as the basket containing all the undesired appurtenances (due to skew-symmetry or the Jacobi relation) occurring in the formal manipulations, which enhance the ability to construct families of examples of Lie pairs. It is also important to stress that the proposed axiomatization is natural, and one recovers the Lie semialgebras in the sense that when $\mathcal{L}$ is a module over a commutative ring and when $\mathcal{L}_0 = \{0\}$ one obtains the classical definition of Lie algebras, and all of our examples work in this case, and reproduce the classical ones, like, e.g., the cross product.

Although our take is more along traditional structural algebraic lines, following Jacobson [15] and Humphreys [14], but relying on the subset $\mathcal{L}_0$ taking the place of 0, it should be remarked that the literature has already seen research aimed to build theories of Lie semialgebras, for instance in the work by Hilgert and Hofmann [13], relying on the Campbell-Hausdorff formula.

As remarked, this theory of "pairs" is an outgrowth of "triples" and "systems," cf. [2, 11, 18, 21, 22], which have unified classical algebraic theory with tropical theory and other examples including hyperfields, as explained in [1]. Pairs are used in linear algebra in [2], and in generalizing commutative algebra theory in [2]. But whereas the set $\mathcal{T}$ of "tangible" elements (that is the elements of the ground set) played a crucial role in semiring and hyperring systems, in this study of "Lie pairs" we do not deal with tangible elements at all. In other words, $\mathcal{T} = \emptyset$.

We bring in a "surpassing relation" in §2.11, to be preserved by "$\preceq$-morphisms" in its appropriate category. There are three possible categories, corresponding to the three versions of morphisms given in Definition 2.3 and Definition 2.15. The "weak morphisms" and "$\preceq$-morphisms" are inspired by the theory of hyperfields, cf. [23].

Among the main thrusts of this paper is to lay out the categorical foundations of Lie pairs in Definition 3.3, paying attention to examples inspired by the classical theory, obtaining categories parallel to [1, 2]. At times negation can be replaced by a "pre-weak negation map" $\psi$ satisfying $x + \psi(x) \in \mathcal{L}_0$, cf. Theorem 4.3. We also introduce pre-Lie $\varepsilon$-pairs, the analog of pre-Lie algebras, in Definition 4.8, and show how to obtain a Lie pair from a pre-Lie $\varepsilon$-pairs in Theorem 4.9. The Lie versions of morphisms are given in Definition 3.20. It might seem strange that there are three different versions of Lie morphisms, but this also happens in other non-classical algebraic theories such as hyperfields [23]. Our main category uses "weak Lie morphisms," with many natural examples provided along the way.

We extend major examples from classical Lie theory, to be described shortly. On the other hand, there is a Lie version of Krasner's hyperfield construction of [20], given in §4.5.

To test the viability of these notions, we prove versions of the PBW (Poincare-Birkhoff-Witt) Theorem in these three categories (Theorems 6.13, 6.14, and 6.16).

### 1.1 Shape of the paper

To help the reader to get oriented in the exposition of so many new, though natural, notions, we now give a glimpse of how the paper is organized, also to share the feeling of what is in it.

To ease the reading, and to make the paper as self contained as possible, we collect in section 2 all the prerequisites and notation to be used in the article. The framework is very general, which explains why we put so much emphasis on very sparse algebraic structures like magmas and bimagmas. Pairs and negation maps are quickly recalled in Section 2.1 and 2.7. Weak Property N, introduced in Section 2.2.2, is necessary because we cannot expect, as easily seen by basic examples, for nontrivial negation maps to exist.

The theoretical core of the paper is Section 3, where we collect foundational material about the theory of Lie pairs in our sense, the basic morphisms used in the rest of the paper.

To show that our theory is not empty we devote Section 4 to major Lie constructions (such as a Lie pair from an associative pair in Theorem 4.3, and from an associative pair with involution in Theorems 4.10 and 4.12) and examples (the classical constructions of Theorem 4.14, and low dimensional examples in §4.4 including the cross product), as well as Filiform pairs in §4.4.2 and an example motivated by hyperfield theory in Theorem 4.26.

One standard technique for working with semialgebras, to cope with the unavailability of additive inverses, is that of doubling, which in a sense recalls the construction of the integers from the natural numbers, but where we avoid taking the quotient modulo a congruence.

Among the most natural examples of Lie pairs, is one where the Lie bracket is obtained as the Lie commutator in an associative semialgebra. The construction is straightforward. The standard model of any associative (semi)-algebra is that of a quotient of the tensor (semi)-algebra associated to a module. This is why in Section 6.1 we treat tensor semialgebras of free Lie pairs.

In §6 we address the natural question: given any Lie pair $(\mathcal{L}, \mathcal{L}_0)$, can we construct an associative pair $(\mathcal{A}, \mathcal{A}_0)$ in which $(\mathcal{L}, \mathcal{L}_0)$ is embedded, in such a way that the commutator restricts to the given Lie bracket? This would be the extension of the Poincaré–Birkhoff–Witt (PBW) theorem in our context. In our concluding subsection 6.3 we analyze the corresponding PBW situation in the various versions of Lie pairs. We shall see that the construction is unambiguous for each of the versions considered, although it must take into account the corresponding category.

## 2 Preliminaries and Notation

First we review some definitions from [17]. As usual we denote as $\mathbb{N}$ the semiring of natural numbers (including 0), and $\mathbb{N}^+ := \mathbb{N} \setminus \{0\}$.

**Definition 2.1.**

(i) A **magma** is a set with a binary operation denoted $(+)$ (addition) or $(\cdot)$ (multiplication). At times we also require a neutral element, written as 0 or 1 respectively.

A **semigroup** is a magma whose given binary operation satisfies the law of associativity.

A **bimagma** $\mathcal{A}$ is a multiplicative monoid $(\mathcal{A}, \cdot, 1)$ which also is an additive semigroup $(\mathcal{A}, +, 0)$, satisfying $0b = b0 = 0$ for all $b \in \mathcal{A}$. (Thus, for us, bimagmas are associative both for multiplication and addition.)

A **d-bimagma** is a bimagma which is **distributive**, by which we mean

$$\left(\sum_i x_i\right)\left(\sum_j y_j\right) = \sum_{i,j} x_i y_j, \quad \text{for all } x_i, y_j \in \mathcal{A}.$$

A **semiring** (cf. [9], [12]) $(\mathcal{A}, +, \cdot, 0, 1)$ is a (multiplicatively) associative d-bimagma also with a multiplicative identity 1. A **semifield** is a semiring in which every nonzero element is invertible.

(ii) $\mathcal{C}$ always will denote a commutative semiring, e.g. $\mathbb{N}$ or $\mathbb{Q}_+$ or the max-plus algebra. Often $\mathcal{C}$ will be a semifield.

(iii) A (left) $\mathcal{A}$-**module**[1] over a semigroup $C$ is a semigroup $(\mathcal{M}, +, 0)$ endowed with scalar multiplication $C \times \mathcal{M} \to \mathcal{M}$ satisfying the following axioms, for all $c, c_i \in C$ and $y, y_i \in \mathcal{M}$:

(a) $c0 = 0c = 0$, (i.e., 0 is absorbing).

(b) $c \sum y_i = \sum cy_i$, $(\sum c_i)y = \sum(c_i y)$.

(c) (when $C$ is a semiring) $(c_1)(c_2 y) = (c_1 c_2)y$.

A **basis** of an $C$-module $\mathcal{M}$ (if it exists) is a set $\{x_i : i \in I\} \subseteq \mathcal{M}$ such that any element of $A$ can be written uniquely as a sum $\sum c_i x_i$, $c_i \in C$, where almost all $c_i = 0$. In this case we call $\mathcal{M}$ a **free** $\mathcal{A}$-module of rank $|I|$.

(iv) A $\mathcal{C}$-**bimagma** is a $\mathcal{C}$-module which is also a bimagma and satisfies

$$(cy_1)y_2 = c(y_1 y_2) = y_1(cy_2)$$

for all $c \in \mathcal{C}$, $y_i \in \mathcal{A}$.. A $\mathcal{C}$-**bimagma ideal** of a $\mathcal{C}$-bimagma $\mathcal{A}$ is a sub-bimagma which is also an $\mathcal{A}$-module.

(v) A **multiplicative ideal** of a bimagma $\mathcal{A}$ is a subset $\mathcal{I} \subseteq \mathcal{A}$ satisfying $bd, db \in \mathcal{I}$, for each $b \in \mathcal{I}$ and $d \in \mathcal{A}$.

An **ideal** is a sub-semigroup $\mathcal{I} \subseteq \mathcal{A}$ which is also a multiplicative ideal.

(vi) An **involution** of a $\mathcal{C}$-bimagma $\mathcal{A}$ is an anti-automorphism $(*) : \mathcal{A} \to \mathcal{A}$ of order $\leq 2$, i.e., $(cb)^* = cb^*$, $(\sum b_i)^* = \sum b_i^*$, $(b^*)^* = b$, and $(b_1 b_2)^* = b_2^* b_1^*$ for $b, b_i \in \mathcal{A}$. (We have defined an involution of the first kind.)

(vii) A **semialgebra** over $\mathcal{C}$ is a $\mathcal{C}$-bimagma that is also a semiring.

(viii) A map $f : M \to N$ of $\mathcal{C}$-modules is **module multiplicative** if $f(cy) = cf(y)$, for all $c \in \mathcal{C}$, $y \in M$.

(ix) Module homomorphisms are defined as usual. For a $\mathcal{C}$-module $\mathcal{M}$, the semialgebra of module homomorphisms $\mathcal{M} \to \mathcal{M}$ is denoted as $\operatorname{End}_{\mathcal{C}} \mathcal{M}$. For notational convenience, we omit the subscript $\mathcal{C}$ when it is understood, and designate $0_{\mathcal{M}} \in \operatorname{End} \mathcal{M}$ for the 0 homomorphism, i.e., $0_{\mathcal{M}}(v) = 0$, for all $v \in \mathcal{M}$.

**Remark 2.2.** Any semiring $\mathcal{A}$ is a semialgebra over its **center** $C = Z(\mathcal{A})$.

[1] For convenience, we are defining modules with 0, but since we do not require negation, the 0 element could be dispensed with.

## 2.1 Pairs

**Definition 2.3.**

(i) A $\mathcal{C}$**-pair** $(\mathcal{A}, \mathcal{A}_0)$ is a $\mathcal{C}$-module $\mathcal{A}$ with a $\mathcal{C}$-subset $\mathcal{A}_0$. In particular, if $\mathcal{C}_0 \subset \mathcal{C}$ we have the pair $(\mathcal{C}, \mathcal{C}_0)$, which we call the **base pair**.

(ii) A map $f : (\mathcal{A}, \mathcal{A}_0) \to (\mathcal{A}', \mathcal{A}'_0)$ of pairs is:

  (a) a **homomorphism** if $f(b_1 + b_2) = f(b_1) + f(b_2)$, for all $b_1, b_2 \in \mathcal{A}$.

  (b) a **weak morphism** if $\sum f(b_i) \in \mathcal{A}'_0$ whenever $\sum b_i \in \mathcal{A}_0$; $f$ is $\mathcal{A}_0$**-injective** if $\sum f(b_i) \in \mathcal{A}'_0$ implies $\sum b_i \in \mathcal{A}_0$.

(iii) A $(\mathcal{C}, \mathcal{C}_0)$**-pair** is a $\mathcal{C}$-pair $(\mathcal{A}, \mathcal{A}_0)$ for which $\mathcal{C}_0\mathcal{A} \subseteq \mathcal{A}_0$.

**Important Note 2.4.** *Any $\mathcal{C}$-pair is automatically a $(\mathcal{C}, 0)$-pair. We will identify $\mathcal{C}$ with $(\mathcal{C}, 0)$ when appropriate.*

*$(\mathcal{C}, \mathcal{C}_0)$ is presumed given, and "pair" means $(\mathcal{C}, \mathcal{C}_0)$-pair. The justification for this approach is given in [22, Note 1.34] and [2].*

*Intuitively $\mathcal{A}_0$ replaces "zero." Often $\mathcal{A}_0$ is a bimagma ideal of $\mathcal{A}$.*

*The essential difference with [17] and [2] is that here we do not assume that $\mathcal{C} \subseteq \mathcal{A}$, and "tangible elements" do not play a role here.*

*Occasionally we will merely be given a semiring $\mathcal{A}$ and an ideal $\mathcal{A}_0$. Then $(\mathcal{A}, \mathcal{A}_0)$ becomes a pair when we define $C$ as in Remark 2.2, and $C_0 = C \cap \mathcal{A}_0$.*

**Definition 2.5.**

(i) A **bimagma pair** is a pair $(\mathcal{A}, \mathcal{A}_0)$ for which $\mathcal{A}$ is a bimagma, satisfying $\sum b_i \in \mathcal{A}_0$ implies $\sum bb_i \in \mathcal{A}_0$.

(ii) A bimagma pair $(\mathcal{A}, \mathcal{A}_0)$ is $\mathcal{A}_0$**-additive** if $\mathcal{A}_0$ is an ideal of $\mathcal{A}$.

(iii) A **semiring pair** is a bimagma pair, for which $\mathcal{A}$ is a semiring.

(iv) An $\varepsilon$**-pair**, for $\varepsilon \in \mathcal{C}$, is an $\mathcal{A}_0$-additive bimagma pair $(\mathcal{A}, \mathcal{A}_0)$, for which

$$xy + \varepsilon yx \in \mathcal{A}_0, \text{ for all } x, y \in \mathcal{A}.$$

(v) Given a bimagma pair $(\mathcal{A}, \mathcal{A}_0)$, an **involution** of $(\mathcal{A}, \mathcal{A}_0)$ is an involution $(*)$ of $\mathcal{A}$ such that $\mathcal{A}_0^* = \mathcal{A}_0$.

(vi) A pair $(\mathcal{A}, \mathcal{A}_0)$ is $\mathcal{A}_0$**-cancellative** if, for $y \in \mathcal{A}$, $c \in \mathcal{C}$, $cy \in \mathcal{A}_0$ implies $c \in \mathcal{A}_0$ or $y \in \mathcal{A}_0$.

(vii) A pair $(\mathcal{A}, \mathcal{A}_0)$ satisfies $\mathcal{A}_0$**-elimination** if $y_0 + y_1 \in \mathcal{A}_0$ for $y_0 \in \mathcal{A}_0$, $y_1 \in \mathcal{A}$, implies $y_1 \in \mathcal{A}_0$.

(viii) A **homomorphism** of bimagma pairs $\varphi : (\mathcal{A}, \mathcal{A}_0) \to (\mathcal{A}', \mathcal{A}'_0)$ is a bimagma homomorphism $\varphi : \mathcal{A} \to \mathcal{A}'$ (i.e., which preserves addition and multiplication), with $\varphi(\mathcal{A}_0) \subseteq \mathcal{A}'_0$.

(ix) A (left) **module pair** $(\mathcal{M}, \mathcal{M}_0)$ over a pair $(\mathcal{A}, \mathcal{A}_0)$ is an $\mathcal{A}$-module $\mathcal{M}$ together with a subset $\mathcal{M}_0$ and a bilinear product $\mathcal{A} \times \mathcal{M} \to \mathcal{M}$ satisfying the following properties for all $b \in \mathcal{A},\ y \in \mathcal{M}$:

(a) $0y = 0_{\mathcal{M}} = y0$,

(b) $b0_{\mathcal{M}} = 0_{\mathcal{M}}$,

(c) $b\mathcal{M}_0 \subseteq \mathcal{M}_0$,

(d) $\mathcal{A}_0 y \subseteq \mathcal{M}_0$.

(x) If $(\mathcal{M}, \mathcal{M}_0)$ is a pair, then define

$$\operatorname{End} \mathcal{M}_0 = \{f \in \operatorname{End} \mathcal{M} : f(\mathcal{M}) \subseteq \mathcal{M}_0\},$$

and take $\operatorname{End}(\mathcal{M}, \mathcal{M}_0)$ to be the pair $(\operatorname{End} \mathcal{M}, \operatorname{End} \mathcal{M}_0)$.

(xi) A **sub-pair** of a pair $(\mathcal{A}, \mathcal{A}_0)$ is a pair $(S, S_0)$ where $S \subseteq \mathcal{A}$ and $S_0 \subseteq S \cap \mathcal{A}_0$.

## 2.2 Substitutes for negation

Although we have bypassed negation, we need some versions to carry out the theory.

### 2.2.1 Pre-negation and negation maps

**Definition 2.6** ([17]). A **pre-negation map** on a bimagma pair $(\mathcal{A}, \mathcal{A}_0)$ is a semigroup endomorphism $b \mapsto \psi(b)$ of $\mathcal{A}$, satisfying the following conditions, for all $b, b_1 \in \mathcal{A}$:

(i) $\psi(bb') = b\psi(b') = \psi(b)b'$,

(ii) $b + \psi(b) \in \mathcal{A}_0$ for all $b \in \mathcal{A}$,

(iii) $\psi(\mathcal{A}_0) \subseteq \mathcal{A}_0$.

**Definition 2.7** ([17]).

1. A **negation map** on a bimagma pair is a pre-negation map, denoted $(-)$, on $(\mathcal{A}, \mathcal{A}_0)$ of order $\leq 2$, i.e., satisfying $(-)((-)b) = b$ for all $b \in \mathcal{A}$.

2. We write $b(-)b'$ for $b + (-)b'$.

3. A $\psi$-**pair** is a bimagma pair with a pre-negation map $\psi$.

### 2.2.2 Weak Property N

We avoid negation maps, and instead use the following generalization.

**Definition 2.8.** A pair $(\mathcal{A}, \mathcal{A}_0)$ satisfies **Weak Property N** if for each $b \in \mathcal{A}$ there is an element $b'$ such that $b + b' = b' + b \in \mathcal{A}_0$.

(We used "Weak" here to be consistent with the terminology of [2]. It has nothing to do with "weak morphisms," to be defined below.) The following very easy example is illustrative.

**Example 2.9.** For any $\mathcal{C}$-bimagma $\mathcal{A}$, picking any element $\varepsilon$ in $\mathcal{C}$ for which $1+\varepsilon \in C_0$, the map $b \mapsto \varepsilon b$ is a pre-negation map $\psi$ of $(\mathcal{A}, (1+\varepsilon)\mathcal{A})$, which is a $\psi$-pair satisfying Weak Property $N$, since $b + \varepsilon b = (1+\varepsilon)b \in \mathcal{A}_0$.

**Important Note 2.10.** *When $\mathcal{C}$ has a negation map, we can take $\varepsilon = (-)1$ in Example 2.9. In general, the element $\varepsilon$ is a more general version of $(-)1$, since we do not require $\varepsilon^2 = 1$, but nevertheless $1+\varepsilon$ replaces $0$.*

## 2.3 Surpassing relations

**Definition 2.11.** A **surpassing relation** $\preceq$ on a bimagma pair $(\mathcal{A}, \mathcal{A}_0)$ is a pre-order satisfying the following conditions for all $b_i, b_i' \in \mathcal{A}$, $c \in \mathcal{C}$:

(i) If $b_1 \preceq b_2$ and $b_1' \preceq b_2'$ then $b_1 + b_1' \preceq b_2 + b_2'$ and $b_1 b_1' \preceq b_2 b_2'$.

(ii) If $b_1 + b_0 = b_1'$ for some $b_0 \in \mathcal{A}_0$, then $b_1 \preceq b_1'$.

(iii) If $c \in \mathcal{C}$ and $b_1 \preceq b_1'$ then $cb_1 \preceq cb_1'$.

(iv) When $\mathcal{A}$ has a given negation map $(-)$, if $b_1 \preceq b_1'$ then $(-)b_1 \preceq (-)b_1'$.

We also write $b_1 \succeq b_2$ to denote that $b_2 \preceq b_1$.

**Example 2.12.**

- Our main example in this paper of a surpassing relation on a pair $(\mathcal{A}, \mathcal{A}_0)$, denoted $\preceq_0$, is given by $b_1 \preceq_0 b_2$ iff $b_2 = b_1 + z$ for some $z \in \mathcal{A}_0$. Then $\mathcal{A}_0 = \{b \in \mathcal{A} : 0 \preceq_0 b\}$. When $\mathcal{A}_0 = 0$, the relation $\preceq$ becomes equality.
- For $\psi$-pairs, we write $\preceq_\psi$ for $\preceq_0$; i.e., $b_1 \preceq_0 b_2$ iff $b_2 = b_1 + (b + \psi(b))$ for some $b \in \mathcal{A}$.
- Another example, motivated by hypergroup theory, to be used in Theorem 4.26: Let $\mathcal{P}(\mathcal{H})$ denote the power set of a set $\mathcal{H}$. We say that $S_1 \preceq_\subseteq S_2$ if $S_1 \subseteq S_2$.

**Remark 2.13.**

1. Any surpassing relation $\preceq$ on a pair $(\mathcal{M}, \mathcal{M}_0)$ induces a surpassing relation elementwise on $\mathrm{End}(\mathcal{M}, \mathcal{M}_0)$, by $f \preceq g$ if $f(y) \preceq g(y)$, for all $y \in \mathcal{M}$.

2. The surpassing relation $\preceq_0$ restricts to a surpassing relation on sub-pairs.

**Important Note 2.14.** *A surpassing relation $\preceq$ can be useful, since we may generalize classical formulas by replacing equality by $\preceq$.*

### 2.3.1 $\preceq$-morphisms

**Definition 2.15.** A map $f : (\mathcal{A}, \mathcal{A}_0) \to (\mathcal{A}', \mathcal{A}'_0)$ of pairs is a **$\preceq$-morphism** if satisfies $f(b_1 + b_2) \preceq f(b_1) + f(b_2)$ for $b_1, b_2$ in $\mathcal{A}$, and $f(b_1) \preceq f(b_2)$ whenever $b_1 \preceq b_2$; $f$ is **$\preceq$-injective** if $f(b_1) \preceq f(b_2)$ implies $b_1 \preceq b_2$.

**Lemma 2.16.** *Every $\preceq$-morphism is a weak morphism.*

*Proof.* If $b_1 + b_2 \in \mathcal{A}_0$ then $b_1 + b_2 \succeq 0$, so $f(b_1) + f(b_2) \succeq 0$, i.e., $f(b_1) + f(b_2) \in \mathcal{A}_0$. □

## 2.4 Identities and varieties

We appeal to some of the notions from universal algebra, without going into the technicalities. Jacobson's book [16] is a good resource.

In brief, an **$\Omega$-algebra** is a set which has $n$-ary operations which we assume here include addition and multiplication and their bimagma laws, and when appropriate, the negation map. The 0-ary operations are just distinguished elements. We also admit identities, i.e., equality of universal atomic formulas (in terms of the operations). A **homomorphism** of $\Omega$-algebras is a function which preserves all the given operations.

We generalize this notion to an **$\Omega$-algebra pair** to be a pair $(\mathcal{A}, \mathcal{A}_0)$ of $\Omega$-algebras such that $\mathcal{A}_0$ is a multiplicative ideal of $\mathcal{A}$ invariant under the given unary operations.

**Definition 2.17.** A **free object** for $\mathcal{V}$ is some $U \in \mathcal{V}$ together with an index set $I$ and a set $X = \{x_i : i \in I\}$, such that for any $\mathcal{A} \in \mathcal{V}$ and $\{b_i : i \in I\} \subseteq \mathcal{A}$ there is a unique homomorphism $\Phi : U \to \mathcal{A}$ for which $\Phi(x_i) = b_i$, for all $i \in I$.

**Example 2.18.** Let $I$ be an index set, and $I_0 \subset I$.

- The free $\mathcal{C}$-module of rank $|I|$ was defined in Definition 2.1, which we denote as $\mathcal{C}^{(I)}$.
- The free $\mathcal{C}$-pair $(\mathcal{C}^{(I)}, \mathcal{C}^{(I_0)})$ of rank $(|I|, |I_0|)$; one can notice that $\mathcal{C}^{(I_0)} = \sum_{i \in I_0} \mathcal{C} x_i$ has a basis $\{x^i : i \in I_0\}$ which is expanded to a basis $\{x^i : i \in I\}$ of $\mathcal{C}^{(I)} = \sum_{i \in I} \mathcal{C} x_i$.
- One can take the free $(\mathcal{C}, \mathcal{C}_0)$-module $(\mathcal{C}, \mathcal{C}_0)^{(I)} = (\mathcal{C}^{(I)}, \mathcal{C}_0^{(I)})$ over $(\mathcal{C}, \mathcal{C}_0)$.
- The free $\mathcal{C}$-module with a formal negation map, of rank $|2I|$, has a formal basis

  $$\{x_i : i \in I\} \cup \{y_i : i \in I\},$$

  where we define $(-)x_i = y_i$ and $(-)y_i = x_i$. (This idea will be pursued in Example 5.1.)

- The free multiplicative magma $\mathcal{M}(I)$ is constructed as the set of words in the indeterminates $x_i,\ i \in I$, without associativity. Multiplication is juxtaposition, but with putting in parentheses at each stage. To wit, the $x_i$ are words, and if $w_1$ and $w_2$ are words, then $(w_1w_2)$ is a word. For example, $(x_1(x_2x_3))$ and $((x_1x_2)x_3)$ are different words.

  We get a pair by taking $\mathcal{M}(I)_0$ to be the submagma consisting of words, at least one of whose indeterminates is $x_i$ for $i \in I_0$.

- The free d-bimagma $\mathcal{F}(I)$ is the magma semialgebra of the free multiplicative magma, i.e., is built from the free module having as basis the free multiplicative magma $\mathcal{M}(I)$, whose multiplication is extended via distributivity, as elaborated below in Example 2.23.

- The free d-bimagma pair is $(\mathcal{F}(I), \mathcal{F}(I)_0)$.

- The free semigroup is constructed as the set of words in indeterminates $x_i$, with multiplication being juxtaposition, but without parentheses.

- The free semiring is the semigroup semiring of the free semigroup.

Recall that a **variety** $\mathcal{V}$ in universal algebra is closed under direct products, substructures, and homomorphic images.

**Lemma 2.19.**

(i) *If $(\mathcal{A}, \mathcal{A}_0)$ is a pair and $\varphi : \mathcal{A} \to \overline{\mathcal{A}}$ is a homomorphism, then $(\overline{\mathcal{A}},\ \overline{\mathcal{A}}_0 := \varphi(\mathcal{A}_0))$ is a pair. Furthermore, a surpassing relation $\preceq$ on $(\mathcal{A}, \mathcal{A}_0)$ induces a surpassing relation $(\overline{\mathcal{A}}, \overline{\mathcal{A}}_0)$, by putting $\bar{b}_1 \preceq \bar{b}_2$ if $b_1 \preceq b_2$.*

(ii) *If $(\mathcal{A}_i, \mathcal{A}_{i0})$ are pairs for each $i \in I$, then the direct product $(\prod_{i\in I} \mathcal{A}_i, \prod_{i\in I} \mathcal{A}_{i0})$ is a pair, and surpassing relations on each pair $(\mathcal{A}_i, \mathcal{A}_{i0})$ induce a surpassing relation on $(\prod_{i\in I} \mathcal{A}_i, \prod_{i\in I} \mathcal{A}_{i0})$, componentwise.*

(iii) *Generalizing (ii), for any filter $\mathcal{F}$ on $I$, one can define the **reduced product** (cf. [7])*

$$\prod_{i\in I} \mathcal{A}_i/\mathcal{F}$$

*by saying $(b_i) \cong (c_i)$ if $\{i \in I : b_i = c_i\} \in \mathcal{F}$, and then get the **reduced pair** $(\prod_{i\in I} \mathcal{A}_i/\mathcal{F}, \prod_{i\in I} \mathcal{A}_{i0}/\mathcal{F})$, which inherits the surpassing relation; i.e., $(b_i) \preceq (c_i)$ if and only if $\{i : b_i \preceq c_i\} \subseteq \mathcal{F}$.*

(iv) *Negation maps are preserved under reduced products.*

(v) *Weak Property N is preserved under reduced products.*

*Proof.* The verifications are routine, noting that a filter is closed under finite intersections. □

For the $\preceq$-theory, we introduce the relation $\preceq$ into the language, even though it introduces difficulties.

**Remark 2.20.**
1. In general a surpassing relation $\preceq$ need not pass to homomorphic images, because one could conceivably have $b_1 \preceq b_2$ and $b_3 \preceq b_4$ with $\bar{b}_2 = \bar{b}_3$, but $\bar{b}_1 \not\preceq \bar{b}_4$.
2. The surpassing relation $\preceq_0$ does remain a surpassing relation under homomorphic images. Namely, if $b_2 = b_1 + c_0$ and $b_4 = b_3 + d_0$ for $c_0, d_0 \in \mathcal{A}_0$, with $\bar{b}_2 = \bar{b}_3$, then
$$\bar{b}_4 = \bar{b}_3 + \bar{d}_0 = \bar{b}_2 + \bar{d}_0 = \bar{b}_1 + \bar{c}_0 + \bar{d}_0,$$
i.e., $\bar{b}_1 \preceq_0 \bar{b}_4$.
3. The surpassing relation $\preceq_0$ need not pass to sub-pairs, because we might lose null elements.

**Definition 2.21.**

(i) An **identity** is a universal atomic formula $f(x_1, \dots, x_m) = g(x_1, \dots, x_{m'})$.

(ii) A $\preceq$-**identity** is a universal atomic formula $f(x_1, \dots, x_m) \preceq g(x_1, \dots, x_{m'})$.

**Proposition 2.22.** *Any class of $\Omega$-algebras defined by identities and $\preceq_0$-identities on pairs has free objects.*

*Proof.* For identities we simply impose the relations on the elements of $U$ and $U_0$ by means of congruences, as is customary in universal algebra. For $\preceq_0$-identities $f \preceq_0 g$ we adjoin fresh distinct indeterminates $y_{f,g}$ to $U_0$ and impose the relations $f + y_{f,g} = g$. □

**Example 2.23.** The elements of the free d-bimagma are obtained by repeated addition and multiplication. Distributivity permits us to rewrite any element $f(x_1, \dots, x_m)$ as a sum $\sum_j h_j$ of **monomials**, i.e., products of the $x_i$, together with some coefficient. We say that the monomial $h_j$ is **multilinear of degree** $m$ if each $x_i$, $1 \le i \le m$, appears exactly once in $h_j$; moreover $f$ is **multilinear of degree** $m$ if each of its monomials $h_j$ is multilinear of degree $m$.

In each case, an identity or $\preceq$-identity is **multilinear of degree** $m$ if in the notation of Definition 2.21, $m = m'$ and both $f$ and $g$ are multilinear of degree $m$.

**Lemma 2.24.** *To verify a multilinear identity or $\preceq$-identity in an $\mathcal{A}_0$-additive bimagma pair, it is enough to check it on a spanning set $S$ over $\mathcal{C}$.*

*Proof.* Just write each element $b_i$ as $\sum_j c_{ij} s_j$ for $s_j \in S$, and open up the expression. □

## 3 The basic theory of Lie pairs

We introduce adjoints as a preparation for the Lie theory.

**Definition 3.1.** For a bimagma pair $(\mathcal{A}, \mathcal{A}_0)$, and $x \in \mathcal{A}$, we define the **adjoint maps** $\mathrm{ad}_x : \mathcal{A} \to \mathcal{A}$ and $\mathrm{ad}^\dagger_x : \mathcal{A} \to \mathcal{A}$, by $\mathrm{ad}_x(y) = xy$ and $\mathrm{ad}^\dagger_x(y) = yx$ for $y \in \mathcal{A}$. We also define $\mathrm{Ad}_{\mathcal{A}} = \{\mathrm{ad}_x : x \in \mathcal{A}\}$, $\mathrm{Ad}^\dagger_{\mathcal{A}} = \{\mathrm{ad}^\dagger_x : x \in \mathcal{A}\}$, $\mathrm{AD}(\mathcal{A}) = \mathrm{Ad}_{\mathcal{A}} + \mathrm{Ad}^\dagger_{\mathcal{A}}$, and $\mathrm{AD}(\mathcal{A})_0 = \{f \in \mathrm{AD}(\mathcal{A}) : f(\mathcal{A}) \subseteq \mathcal{A}_0\}$.

**Remark 3.2.**

(i) Suppose $(\mathcal{A}, \mathcal{A}_0)$ is a pair $(\mathcal{A}, \mathcal{A}_0)$, and $x \in \mathcal{A}$. Then $\mathrm{ad}_x, \mathrm{ad}^\dagger_x \in \mathrm{End}(\mathcal{A}, \mathcal{A}_0)$.

(ii) $(\mathrm{AD}(\mathcal{A}), \mathrm{AD}(\mathcal{A})_0)$ is a sub-pair of $(\mathrm{End}\,\mathcal{A}, \mathrm{End}\,\mathcal{A}_0)$.

### 3.1 Lie brackets and Lie pairs

We are ready to bring in the Lie bracket, the focus of this paper.

**Definition 3.3.**

(i) A $\mathcal{L}_0$-**Lie bracket** on a pair $(\mathcal{L}, \mathcal{L}_0)$ is a map, written $[\quad] : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$, satisfying the following **Lie bracket axioms**, for all $x, y \in \mathcal{L}$, with $\mathrm{ad}_x$ and $\mathrm{ad}^\dagger_x$ as in Definition 3.1:

  (a) $\mathrm{ad}_x(x) \in \mathcal{L}_0$, i.e., $[xx] \in \mathcal{L}_0$,

  (b) $\mathrm{ad}_x + \mathrm{ad}^\dagger_x \in \mathrm{End}(\mathcal{L}, \mathcal{L}_0)_0$, i.e., $[xy] + [yx] \in \mathcal{L}_0$ (the intuition being that right multiplication acts like the negation of left multiplication),

  (c) $\mathrm{ad}_{[xy]} + \mathrm{ad}_x \mathrm{ad}^\dagger_y + \mathrm{ad}^\dagger_y \mathrm{ad}_x \in \mathrm{End}(\mathcal{L}, \mathcal{L}_0)_0$, called the **Jacobi $\mathcal{L}_0$- identity**.

  (c′) $\mathrm{ad}^\dagger_{[xy]} + \mathrm{ad}^\dagger_y \mathrm{ad}_x + \mathrm{ad}_y \mathrm{ad}^\dagger_x \in \mathrm{End}(\mathcal{L}, \mathcal{L}_0)_0$, called the **reflected Jacobi $\mathcal{L}_0$- identity**.

  (d) $\mathrm{ad}_{cx} = c\,\mathrm{ad}_x$ for all $c \in C$;

  (e) If $\sum_i x_i \in \mathcal{L}_0$, then $\sum_i \mathrm{ad}^\dagger_y(x_i) \in \mathcal{L}_0$, and $\sum_i \mathrm{ad}_y(x_i) \in \mathcal{L}_0$ for all $y \in \mathcal{L}$.

**Remark 3.4.**

(i) For $z \in \mathcal{L}$, Axiom (c) of (i) translates to

$$[[xy]z] + [x[yz]] + [y[zx]] \in \mathcal{L}_0. \tag{1}$$

(ii) Interchanging $y$ and $z$ in (2) yields $\mathrm{ad}_x \mathrm{ad}_y = \mathrm{ad}^\dagger_y \mathrm{ad}^\dagger_x$. If this holds then axiom $(c')$ is superfluous.

**Lemma 3.5.** *Axiom (b) is implied by (a) in any Lie pair satisfying $\mathcal{L}_0$-elimination.*

*Proof.* $[xx] + [yy] + [xy] + [yx] = [(x + y)(x + y)] \in \mathcal{L}_0$. But $[xx] + [yy] \in \mathcal{L}_0$ by (a), so $[xy] + [yx] \in \mathcal{L}_0$. □

**Definition 3.6.** A **quasi Lie pair** is a pair $(\mathcal{L}, \mathcal{L}_0)$ endowed with a $\mathcal{L}_0$-Lie bracket. A **Lie pair** is a quasi Lie pair $(\mathcal{L}, \mathcal{L}_0)$ whose $\mathcal{L}_0$-Lie bracket is $C$-bilinear, also satisfying the condition that if $\sum_i x_i \in \mathcal{L}_0$, then $\sum_i \mathrm{ad}_{x_i} \in \mathrm{End}(\mathcal{L}, \mathcal{L}_0)_0$, $\sum_i \mathrm{ad}^\dagger_{x_i} \in \mathrm{End}(\mathcal{L}, \mathcal{L}_0)_0$. The $\mathcal{L}_0$-Lie bracket is **†-reversible** if $\mathrm{ad}^\dagger_x \mathrm{ad}_y = \mathrm{ad}_x \mathrm{ad}^\dagger_y$, for all $x, y \in \mathcal{L}$.

The †-reversibility translates to

$$[[yz]x] = [x[zy]]. \tag{2}$$

Here are other desirable properties that hold in the classical situation.

**Definition 3.7.**

(i) The **reflected** $\mathcal{L}_0$-Lie bracket is defined as $[xy]^\dagger = [yx]$. $(\mathcal{L}, \mathcal{L}_0)^\dagger$ is $(\mathcal{L}, \mathcal{L}_0)$ with the reflected $\mathcal{L}_0$-Lie bracket.

(ii) The $\mathcal{L}_0$-Lie bracket is $\mathcal{L}_0$**-reversible** if $[xy] \in \mathcal{L}_0$ implies $[yx] \in \mathcal{L}_0$,

(iii) The $\mathcal{L}_0$-Lie bracket is $\mathcal{L}_0$**-symmetric** if $\mathrm{ad}_x = \mathrm{ad}^\dagger_x$ for $x \in \mathcal{L}_0$.

(iv) The **reflected quasi Lie pair** is the quasi Lie pair $(\mathcal{L}, \mathcal{L}_0)$ with reflected $\mathcal{L}_0$-Lie bracket.

(v) A **reversible Lie pair** is a Lie pair $(\mathcal{L}, \mathcal{L}_0)$ whose $\mathcal{L}_0$-Lie bracket is †-reversible.

**Remark 3.8.** The reflection of a Lie pair is a Lie pair.

**Important Note 3.9.**

*(i) We always assume that $\mathcal{L} \neq \mathcal{L}_0$, since otherwise the axioms are vacuous.*

*(ii) If $\mathcal{L}_0 = \{0\}$, the axioms revert to classical Lie theory.*

*(iii) We often view $\mathcal{L}$ as a bimagma, whose multiplication is the Lie bracket. Usually $\mathcal{L}_0$ is a sub-bimagma.*

*(iv) We need bilinearity to determine a Lie bracket in terms of products of basis elements. But there is an interesting example (Theorem 4.26) arising from hyperrings, which fails bilinearity yet satisfies part of distributivity.*

*(v) In general neither †-reversibility nor $\mathcal{L}_0$-reversibility holds, cf. the cross product examples of §4.4.3. But if $\mathcal{L}_0$-elimination holds, then $\mathcal{L}_0$ reversibility holds. Indeed, if $[xy] \in \mathcal{L}_0$, then $[xy] + [yx] \in \mathcal{L}_0$, implying $[yx] \in \mathcal{L}_0$.*

### 3.1.1 Lie brackets on a free module over a base pair $(\mathcal{C}, \mathcal{C}_0)$

The following observation provides a method of constructing Lie brackets on a free module over a base pair $(\mathcal{C}, \mathcal{C}_0)$, especially in the finite dimensional case.

**Lemma 3.10.** *If $\mathcal{L}$ is a free module over a base pair $(\mathcal{C}, \mathcal{C}_0)$ with basis $\{b_i : i \in I\}$, then the Lie bracket can be defined in terms of the products*

$$[b_i b_j] = \sum_k c_{ij}^k b_k, \quad c_{ij}^k \in \mathcal{C},$$

*and $(\mathcal{L}, \mathcal{L}_0)$ is a Lie pair if and only if these coefficients satisfy the following axioms for each $i, j, k, m \in I$:*

1. $c_{ii}^m \in \mathcal{C}_0$,
2. $c_{i,j}^m + c_{j,i}^m \in \mathcal{C}_0$,
3. $\sum_l (c_{ij}^l c_{lk}^m + c_{kj}^l c_{li}^m + c_{ki}^l c_{lj}^m) \in \mathcal{C}_0$,
4. $\sum_l (c_{ij}^l c_{lk}^m + c_{kj}^l c_{li}^m + c_{ki}^l c_{lj}^m) \in \mathcal{C}_0$.

*Moreover †-reversibility holds if and only if $\sum_l c_{ij}^l c_{lk}^m = \sum_l c_{kj}^l c_{li}^m$, for all $i, j, k, m$.*

*Proof.* The only axiom in Definition 3.3(i) which is not multilinear is (a), which says for all $c_i \in C$ that

$$\left(\sum c_i b_i\right)^2 = \sum_i c_i^2 b_i^2 + \sum c_i c_j (b_i b_j + b_j b_i) \in \mathcal{L}_0,$$

implied by $b_i^2 \in \mathcal{L}_0$ and (b).

So we need $\sum_m c_{ii}^m b_m \in \mathcal{C}_0$, which means $c_{ii}^m \in \mathcal{C}_0$ for each $m \in I$.

We check all the other axioms on basis elements $b_i, b_j, b_k$. Axiom (b) requires that $\sum_m (c_{i,j}^m + c_{j,i}^m) b_m = b_i b_j + b_j b_i \in \mathcal{C}_0$, so $c_{i,j}^m + c_{j,i}^m \in \mathcal{C}_0$ for each $m \in I$.

The Jacobi $\mathcal{L}_0$-identity reads as

$$[[b_i b_j] b_k] + [[b_j b_k] b_i] + [[b_k, b_i] b_j] = \sum_{l,m} (c_{ij}^l c_{lk}^m + c_{jk}^l c_{li}^m + c_{ki}^l c_{lj}^m) b_m,$$

so we need $\sum_l (c_{ij}^l c_{lk}^m + c_{jk}^l c_{li}^m + c_{ki}^l c_{lj}^m) \in \mathcal{C}_0$, for each $i, j, k, m \in I$.

Similarly, †-reversibility means $\sum_l (c_{ij}^l c_{lk}^m + c_{kj}^l c_{li}^m + c_{ki}^l c_{lj}^m) \in \mathcal{C}_0$ for each $i, j, k, m \in I$. $\square$

In this way, any Lie pair is determined via the $\mathcal{L}$ valued matrix

$$([b_i b_j])_{1 \le i,j \le n}$$

such that all the diagonal elements $[b_i b_i] \in \mathcal{L}_0$ and $[b_i b_j] + [b_j b_i] \in \mathcal{L}_0$ ($\mathcal{L}_0$-skew symmetry).

**Definition 3.11.**

(i) For $V, W$ subsets of a bimagma $\mathcal{L}$, we define $[VW]$ to be the $\mathcal{C}$-subspace of $\mathcal{L}$ generated by $\{[vw] : v \in V,\ w \in W\}$.

(ii) An ideal of a Lie pair (with respect to the Lie bracket) $(\mathcal{L}, \mathcal{L}_0)$ is also called a **Lie ideal**, for emphasis.

**Lemma 3.12.**

*(i) Any bimagma sub-pair $(W, W_0)$ of a Lie pair $(\mathcal{L}, \mathcal{L}_0)$ is itself a Lie pair, which is $W_0$-reversible (resp. †-reversible) if $(\mathcal{L}, \mathcal{L}_0)$ is $\mathcal{L}_0$-reversible (resp. †-reversible).*

*(ii) $([W\mathcal{L}], [W\mathcal{L}] \cap \mathcal{L}_0)$ is a Lie ideal of $(\mathcal{L}, \mathcal{L}_0)$ for any ideal $W$ of $\mathcal{L}$.*

*(iii) Let $\mathcal{L}' := [\mathcal{L}\mathcal{L}]$. $(\mathcal{L}', \mathcal{L} \cap \mathcal{L}_0)$ is a Lie ideal of $(\mathcal{L}, \mathcal{L}_0)$ that satisfies Weak Property N.*

*Proof.* (i) We show that $(W, W_0)$ is a Lie pair, by verifying the conditions of Definition 3.3(i), for $x, y, z \in W$.

1. $[xx] \in \mathcal{L}_0 \cap W = W_0$.
2. $[xy] + [yx] \in \mathcal{L}_0 \cap W = W_0$.
3. Likewise the Jacobi $W_0$-identity (and its reflection) and $\mathrm{ad}_x = \mathrm{ad}_x^\dagger$ for $x \in W_0$ are a fortiori, as well as †-reversibility.
4. The other axioms are clear.

(ii) Clearly $[x[yw]], [[yw]x] \in [W\mathcal{L}]$ for $w \in W$.

(iii) Using (ii), we only need to verify the Weak Property N, which is clear since $[xy] + [yx] \in \mathcal{L}_0$. $\square$

### 3.1.2 Negated Lie pairs

**Definition 3.13.** A **negated Lie pair** is a Lie pair with a negation map (cf. Definition 2.7), such that $[yx] = (-)[xy]$ for all $x, y$.

**Remark 3.14.** Negated Lie pairs are rather restrictive. For example any negated Lie pair is reflexive and $\mathcal{L}_0$-symmetric.

## 3.2 Lie pairs with a surpassing relation

One could introduce a surpassing relation $\preceq$.

**Definition 3.15.**

1. A $\preceq$-**Lie bracket** with regard to a surpassing relation $\preceq$ is a $\mathcal{L}_0$-Lie bracket also satisfying, for all $x, x_i, y \in \mathcal{L}$:

   (a) (The Jacobi $\preceq$-identity) $\mathrm{ad}_{[xy]} \preceq \mathrm{ad}_x \mathrm{ad}_y + \mathrm{ad}_y \mathrm{ad}_x^\dagger$.

(b) $\mathrm{ad}_{\sum_i x_i} \preceq \sum_i \mathrm{ad}_{x_i}$,

(c) $\mathrm{ad}_x(\sum y_i) \preceq \sum \mathrm{ad}_x(y_i)$ and $\mathrm{ad}_x^\dagger(\sum y_i) \preceq \sum \mathrm{ad}_x^\dagger(y_i)$ for all $x, y_i \in \mathcal{L}$.

(d) $\mathrm{ad}^\dagger_{\sum_i x_i} \preceq \sum_i \mathrm{ad}^\dagger_{x_i}$.

(e) If $x \preceq y$, then $\mathrm{ad}_x \preceq \mathrm{ad}_y$ and $\mathrm{ad}_x^\dagger \preceq \mathrm{ad}_y^\dagger$.

2. A $\preceq$-**weak Lie pair** is a weak Lie pair $(\mathcal{L}, \mathcal{L}_0)$ with a surpassing relation $\preceq$.

3. A $\preceq$-**Lie pair** is a Lie pair $(\mathcal{L}, \mathcal{L}_0)$ with a surpassing relation $\preceq$.

**Important Note 3.16.** *The classical Lie theory has equality holding in (1), but we find this too restrictive to obtain a workable algebraic theory for semialgebras.*

**Remark 3.17.** Assume that $(\mathcal{L}, \mathcal{L}_0)$ is a reversible Lie pair. One can rewrite Axiom (1)(a) as

$$[[xy]z] \preceq [x[yz]] + [y[zx]]. \tag{3}$$

**Lemma 3.18.** *Assume that $(\mathcal{L}, \mathcal{L}_0)$ is $\mathcal{L}_0$-reversible (Definition 3.7).*

*(i) The Jacobi $\preceq$-identity also is equivalent to each of:*

*(d′)* $\mathrm{ad}_y \mathrm{ad}_x^\dagger \preceq \mathrm{ad}_{[xy]} + \mathrm{ad}_x \mathrm{ad}_y^\dagger$.

*(d″)* $\mathrm{ad}^\dagger_{[xy]} \preceq \mathrm{ad}_x^\dagger \mathrm{ad}_y^\dagger + \mathrm{ad}_y^\dagger \mathrm{ad}_x$.

*(ii) If $[[xy]z] + w = [x[yz]] + [y[zx]]$ for $w \in \mathcal{L}_0$, then*

$$[z[yx]] + w = [[zy]x] + [[xz]y].$$

*Proof.* Use Equation (3) throughout.

(i) To obtain Axiom (d′) switch $y$ and $z$. The reverse argument gives us (3) from (d′).

To obtain Axiom (d″), plug Definition 3.7(ii) into each term of (3), and then exchange $x$ and $y$. The reverse argument gives us (3) from (d″).

(ii) When $[[xy]z] + w = [x[yz]] + [y[zx]]$ for $w \in \mathcal{L}_0$,

$$[z[yx]] + w = [[xy]z] + w = [x[yz]] + [y[zx]] = [[zy]x] + [[xz]y].$$

∎

**Remark 3.19.** Axioms (d′) and (d″) remain consistent when $\mathcal{A}_0$-symmetry holds.

### 3.3 Categories involving Lie pairs

There are three natural kinds of morphisms of Lie pairs, each of which defines its category.

**Definition 3.20.** A $\mathcal{C}$-module homomorphism $f : (\mathcal{L}, \mathcal{L}_0) \to (\mathcal{N}, \mathcal{N}_0)$ of Lie pairs is a **Lie bracket map** if $f(\mathcal{L}_0) \subseteq \mathcal{N}_0$ and $f([b_1 b_2]) = [f(b_1)f(b_2)]$, for all $b_1, b_2 \in \mathcal{L}$. A Lie bracket map $f$ is a **weak Lie morphism**, **$\preceq$-Lie morphism**, resp. **Lie homomorphism**, if $f$ is a weak morphism, resp. $\preceq$-morphism, resp. homomorphism.

**Lemma 3.21.**

*(i) The Lie pairs and their weak Lie morphisms comprise a category.*

*(ii) The $\preceq$-Lie pairs and their Lie $\preceq$-morphisms comprise a subcategory of (i).*

*(iii) The Lie pairs and their Lie homomorphisms comprise a subcategory of (ii).*

*Proof.* One checks easily that the composition of two Lie homomorphisms is a Lie homomorphism, and likewise for weak Lie morphisms and Lie $\preceq$-morphisms. The other assertions are by Lemma 2.16. $\square$

**Important Note 3.22.** *Although Lie homomorphisms are the definition from universal algebra, the first category, using weak Lie morphisms, fits best into the general theory of pairs.*

**Example 3.23.** Suppose $(\mathcal{L}, \mathcal{L}_0)$ is any Lie pair, and $\mathcal{L}_0 \subset \mathcal{L}_1 \subset \mathcal{L}$. Then $(\mathcal{L}, \mathcal{L}_1)$ also is a Lie pair, and the identity map can be viewed as a Lie homomorphism from $(\mathcal{L}, \mathcal{L}_0)$ to $(\mathcal{L}, \mathcal{L}_1)$. Likewise for weak Lie pairs and $\preceq$-Lie pairs.

## 4 Lie pair constructions

In this section we show how celebrated examples of Lie theory can be generalized to Lie pairs.

### 4.1 $\psi$-Lie pairs from associative and pre-Lie $\varepsilon$-pairs

In the classical theory of Lie algebras one knows that for each associative algebra (and more generally for pre-Lie algebras [4]), the additive commutator makes it into a Lie algebra. In our situation, this cannot work since we do not have negation. Nevertheless, there is an analogous procedure for pairs.

**Definition 4.1.**

(i) A $\psi$-**Lie pair** is a Lie pair having a pre-negation map $\psi$.

(ii) A **strong** $\psi$-**Lie pair** is a $\psi$-Lie pair satisfying $[yx] = \psi([xy])$ for each $x, y \in \mathcal{L}$.

(iii) For $\varepsilon \in C$, an $\varepsilon$-**Lie pair** is a Lie pair satisfying $x + \varepsilon x \in \mathcal{L}_0$ for all $x \in \mathcal{L}$ and $[xy] + [yx] \in (1+\varepsilon)\mathcal{L} \subseteq \mathcal{L}_0$.

(iv) For $\varepsilon \in C$, a **strong** $\varepsilon$-**Lie pair** is a $\varepsilon$-**Lie pair** satisfying $[yx] = \varepsilon([xy])$ for each $x, y \in \mathcal{L}$.

When $\mathcal{L}_0 = (1+\varepsilon)\mathcal{L}$ for $\varepsilon \in C$, (iii), (iv) are special cases of (i),(ii) respectively, taking $\psi$ to be the map $x \mapsto \varepsilon x$.

**Lemma 4.2.** *Any strong $\psi$-Lie pair $(\mathcal{L}, \mathcal{L}_0)$ is †-reversible, and is $\mathcal{L}_0$-reversible.*

*Proof.*

$$[[xy]z] = \psi(\psi[[xy]z]) = [[\psi(x)y]\psi(z)] = [z[yx]],$$

so †-reversibility holds.

If $[xy] \in \mathcal{L}_0$, then $[yx] = \psi([xy]) \in \mathcal{L}_0$. □ □

**Theorem 4.3.** *Given a semiring $\psi$-pair $(\mathcal{R}, \mathcal{R}_0)$, define $(\mathcal{R}, \mathcal{R}_0)_\psi := (\mathcal{R}, \mathcal{R}_0)$, endowed with the Lie bracket defined by $[xy]_\psi = xy + \psi(y)x$. Then $(\mathcal{R}, \mathcal{R}_0)_\psi$ is a $\psi$-Lie pair. Moreover, $(\mathcal{R}, \mathcal{R}_0)_\psi$ is a strong $\psi$-Lie pair when $\psi^2 = 1_\mathcal{R}$.*

*Proof.* We verify the axioms in Definition 3.3(i).

(a) $[xx] = x^2 + \psi(x)x = x^2 + \psi(x^2) \in \mathcal{R}_0$.

(b) $[xy] + [yx] = xy + \psi(y)x + yx + \psi(x)y = xy + yx + \psi(xy + yx) \in \mathcal{R}_0$.

(c) $[[xy]z] + [[yz]x] + [[zx]y]$

$$\begin{aligned}
&= (xy + \psi(y)x)z + \psi(z)(xy + \psi(y)x) + (yz + \psi(z)y)x \\
&\quad + \psi(x)(yz + \psi(z)y) + (zx + \psi(y)x)y + \psi(y)(zx + \psi(x)z) \\
&= \psi(xyz) + yxz + \psi(zxy) + \psi^2(zyx) + yzx + \psi(zyx) \\
&\quad + \psi(xyz) + \psi^2(xzy) + zxy + \psi(xzy) + \psi(yzx) + \psi^2(yxz) \\
&= \psi\big((xyz + yzx + zyx) + \psi(zyx + xzy + yxz)\big) \\
&= \psi\big([xy]z + [yz]x + [zx]y + \psi([xy]z + [yz]x + [zx]y)\big) \in \mathcal{R}_0.
\end{aligned}$$

(c′) is analogous, and (d),(e) are easy.

When $\psi^2 = 1_\mathcal{R}$, we have $[yx] = yx + \psi(x)y = \psi(xy + \psi(y)x) = \psi([xy])$. □

**Remark 4.4.**

(i) For instance, for a $\mathcal{C}$-semialgebra $\mathcal{R}$, pick any element $\varepsilon \in \mathcal{C}$, and define $\mathcal{C}_0 = \mathcal{C}(1+\varepsilon)$ and $\mathcal{R}_0 = (1+\varepsilon)\mathcal{R}$. Then $(\mathcal{R}, \mathcal{R}_0)$ is a semiring pair satisfying the hypothesis of the theorem, taking the pre-negation $\psi$ to be $b \mapsto \varepsilon b$.

(ii) As an example for when $\mathcal{A}_0$-reversibility holds, in any bimagma with a negation map $(-)$, we write $[b, b']$ for the **Lie commutator** $bb'(-)b'b$.

**Corollary 4.5.** *Any semiring pair $(\mathcal{A}, \mathcal{A}_0)$ with a negation map $(-)$ becomes a negated Lie pair, denoted by $(\mathcal{A}, \mathcal{A}_0)^{(-)}$, under the Lie product $[bb'] = [b, b']$, which also satisfies $\dagger$-reversibility and $\mathcal{A}_0$-reversibility.*

*Proof.* Take $\varepsilon = (-)1$. ☐

We also can obtain a $\preceq$-version, by extending the **Leibniz $\preceq$-identities** given in [22, Lemma 2.35] to a pre-negation map $\psi$ cf. Definition 2.6:

**Remark 4.6.** If $\psi$ is a pre-negation map on a semiring, then

$$\psi(x_1 \dots x_n) = x_1 \dots x_{i-1}\psi(x_i)x_{i+1} \dots x_n$$

for all $i$, by induction on $n$.

**Lemma 4.7 (Leibniz $\psi$-identities).** *In any semiring $\mathcal{A}$, defining $[x, y]_\psi = xy + \psi(yx)$.*

*(i)* $[x, y]_\psi z + y[x, z]_\psi = [x, yz]_\psi + yxz + \psi(yxz)$.

*(ii)* $z[x, y]_\psi + [x, z]_\psi y = [x, zy]_\psi + zxy + \psi(zxy)$.

*(iii)* $[x, [yz]_\psi]_\psi + yxz + zxy + \psi(yxz + zxy) = [[x, y]_\psi, z]_\psi + [y, [x, z]_\psi]_\psi$. *In particular,* $[x, [y, z]_\psi]_\psi \preceq_\psi [[x, y]_\psi, z]_\psi + [y, [x, z]_\psi]_\psi$.

*Proof.* (i) As in [22, Lemma 2.35], we compute:

$$[x, y]_\psi z + y[x, z]_\psi = (xy + \psi(yx))z + y(xz + \psi(z)x) = xyz + \psi(yzx) + yxz + \psi(yxz).$$

(ii) By symmetry.
(iii) Add (i) to (ii).

☐

#### 4.1.1 Pre-Lie $\psi$-pairs

The construction of Lie algebras from pre-Lie algebras also can be extended to Lie pairs. Recall that $\psi$ is a pre-negation map.

**Definition 4.8.** The **$\psi$-associator** in a $C$-bimagma is given by

$$(x, y, z)_\psi := (xy)z + \psi(x(yz)).$$

An $\mathcal{A}_0$-additive bimagma pair $(\mathcal{A}, \mathcal{A}_0)$ is a **pre-Lie $\psi$-pair** if $(x, y, z)_\psi + \psi((x, z, y)_\psi) \in \mathcal{A}_0$ for all $x, y, z \in \mathcal{A}$.

**Theorem 4.9.** *Any $\mathcal{A}_0$-additive pre-Lie $\psi$-pair $(\mathcal{A}, \mathcal{A}_0)$ becomes an $\psi$-Lie pair under the Lie bracket $[xy]_\psi := xy + \psi(yx)$.*

*If $\psi$ is invertible, the reverse bracket of $[xy]_\psi$ is $\psi([xy]_{\psi^{-1}})$.*

*Proof.* We modify the proof of Theorem 4.3. (a) and (b) are the same, and to get (c) we have

$$\begin{aligned}
[[xy]z] + [[yz]x] + [[zx]y] &= (xy)z + \psi(yx)z + \psi(z)(xy) + \psi(z)\psi(yx) + (yz)x \\
&+ \psi((zy))x + \psi(x)(yz) + \psi(x)\psi(zy) + (zx)y \\
&+ \psi(xz)y + \psi(y)(zx) + \psi(y)\psi(xz) \\
&= (x,y,z)_\psi + (y,z,x)_\psi + (z,x,y)_\psi + \psi((z,y,x)_\psi \\
&+ (x,z,y)_\psi + (y,x,z)_\psi) \\
&= ((x,y,z)_\psi + \psi((x,z,y)_\psi) + ((y,z,x) + (y,x,z)_\psi) \\
&+ ((z,x,y)_\psi + \psi((z,y,x)_\psi))
\end{aligned} \tag{4}$$

which is in $\mathcal{R}_0$.

$(c')$ is analogous, and (d),(e) are easy.

Finally, $[xy]_\psi^\dagger = [yx]_\psi = yx + \psi(xy) = \psi((xy + \psi^{-1}(y)x)$. ☐

### 4.2 Lie pairs from semiring pairs with involution

We can also get examples from involutions. Let $\mathcal{R}$ be an associative $\mathcal{C}$-semialgebra equipped with an involution $* : \mathcal{R} \to \mathcal{R}$, cf. Definition 2.5. Define $\mathcal{L}$ to be $\mathcal{R}$, endowed with the bracket:

$$[xy] = xy + y^* x \tag{5}$$

and $\mathcal{L}_0$ be the $\mathcal{C}$-module

$$\sum_{x\in\mathcal{R}} \mathcal{R}(x + x^*) + \sum_{x\in\mathcal{R}} (x + x^*)\mathcal{R} + \sum_{x\in\mathcal{R}} \mathcal{R}(x + x^*)\mathcal{R}.$$

In particular $x(y + y^*)$, $(x + x^*)y$, and $x(y + y^*)z$ belong to $\mathcal{L}_0$ for any arbitrary choice of $x, y, z \in \mathcal{R}$.

**Theorem 4.10.** $(\mathcal{L}, \mathcal{L}_0)$ *as defined above is a Lie pair.*

*Proof.* Let us check the axioms 3.3(i).

(a) $[xx] = x^2 + x^* x = (x + x^*)x \in \mathcal{L}_0$.

(b) $[xy] + [yx] = xy + y^* x + yx + x^* y = (x + x^*)y + (y + y^*)x \in \mathcal{L}_0$.

(c) The Jacobi $\mathcal{L}_0$- identity holds, since

$$\begin{aligned}
[[xy]z] + [[yz]x] + [[zx]y] &= (xy + y^* x)z + z^*(xy + y^* x) + (yz + z^* y)x + x^*(yz + z^* y) \\
&+ (zx + x^* z)y + y^* zx + x^* z) \\
&= \underbrace{(x + x^*)}\, yz + \underbrace{(y^* + y)}\, zx + \underbrace{(z^* + z)}\, xy \\
&+ y^* \underbrace{(x + x^*)}\, z + z^* \underbrace{(y + y^*)}\, x + x^* \underbrace{(z + z^*)}\, y.
\end{aligned} \tag{6}$$

Since each bracketed term belongs to $\mathcal{L}_0$, the left side of (6) belongs to $\mathcal{L}_0$.

(d) Follows from the fact that $\mathcal{L}_0$ is an ideal. $\square$

**Important Note 4.11.** *Theorem 4.10 is the $\mathcal{L}_0$-version of skew symmetric elements (defined by $x + x^* = 0$), since here we have stipulated $x + x^* \in \mathcal{L}_0$.*

### 4.2.1 $\varepsilon$-Skew symmetric pairs[2]

More generally, let us now fix $\varepsilon \in C$ and define the bracket on $\mathcal{R}$ by

$$[xy] = xy + \varepsilon y^* x. \tag{7}$$

Take $\mathcal{L} = R$ and stipulate that $\mathcal{L}_0$ contains $\{x + \varepsilon x^* : x \in \mathcal{R}\}$.

**Theorem 4.12.** *$(\mathcal{L}, \mathcal{L}_0)$ is a Lie pair under the bracket of (7).*

*Proof.* Again let us check the axioms of Definition 3.3(i). First of all

$$[xx] = x^2 + \varepsilon x^* x = x(x + \varepsilon x^*) \in \mathcal{L}_0.$$

This proves that axiom 3.3(i)(a) holds.

To check (b), $[xy] + [yx] = xy + \varepsilon y^* x + yx + \varepsilon x^* y = (x + \varepsilon x^*)y + (y + \varepsilon y^*)x$ which belongs to $\mathcal{L}_0$ because $(x + \varepsilon x^*)$ and $(y + \varepsilon y^*)$ do.

For (c), the Jacobi $\mathcal{L}_0$-identity, we use again expression (6) with $\varepsilon$ inserted in the appropriate places.

(d) and (e) are obvious. $\square$

## 4.3 The "classical" Lie pairs

We can now describe the paired version of the classical Lie algebras $A_n, B_n, C_n, D_n$. We need the *trace* $\mathrm{tr}\,(A) := \sum a_{ii}$ of a matrix $A = (a_{ij})$.

**Lemma 4.13.** $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ *for matrices $A, B$ over $\mathcal{C}$.*

*Proof.* $\mathrm{tr}(AB) = \sum a_{ij} b_{ji} = \sum b_{ji} a_{ij} = \mathrm{tr}(BA)$, since $\mathcal{C}$ is commutative. $\square$

**Theorem 4.14.** *Fix $\varepsilon \in \mathcal{C}$ such that $\varepsilon + 1 \in \mathcal{C}_0$. (See footnote 2.)*

*(i) The paired version of the classical Lie algebra $A_n$ is given by the* ***special linear pair*** *$\mathrm{sl}_n := (\mathcal{L}, \mathcal{L}_0)$, where $\mathcal{L} = \{x \in M_n(\mathcal{C}) : \mathrm{tr}(x) \in \mathcal{C}_0\}$, and $\mathcal{L}_0$ is obtained as in Theorem 4.9.*

*(ii) The $\varepsilon$-paired version of the classical Lie algebra $B_n$ is given by $\mathfrak{so}^{(\varepsilon)}_{2n+1} := (\mathcal{L}, \mathcal{L}_0)$, where $\mathcal{L} = x \in M_{2n+1}(\mathcal{C}) : x + \varepsilon x^T \in M_{2n+1}(\mathcal{C}_0)\}$, and $\mathcal{L}_0$ is obtained as in Theorem 4.10.*

[2] We could do this for a general pre-negation map $\boldsymbol{\varepsilon}$ if we stipulate that $\boldsymbol{\psi}$ preserves the involution, i.e., $\boldsymbol{\psi(x^*) = \psi(x)^*}$.

*(iii)* *The $\varepsilon$-paired version of the classical Lie algebra $C_n$ is given by* $\mathfrak{sp}_{2n}^{(\varepsilon)} := (\mathcal{L}, \mathcal{L}_0)$, *with* $\mathcal{L} = \{x \in M_{2n}(\mathcal{C}) : Jx + \varepsilon x^T J \in M_{2n+1}(\mathcal{C}_0)\}$, *where $J$ is the matrix* $\begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}$, *and $\mathcal{L}_0$ is obtained as in Theorem 4.10.*

*(iv)* *The $\varepsilon$-paired version of the classical Lie algebra $D_n$ is given by* $\mathfrak{so}_{2n}^{(\varepsilon)} := (\mathcal{L}, \mathcal{L}_0)$, *where* $\mathcal{L} = \{x \in M_{2n}(\mathcal{C}) : Jx + \varepsilon x^T J \in M_{2n}(\mathcal{C}_0)\}$, *and $\mathcal{L}_0$ is obtained as in Theorem 4.10.*

*Proof.* (i) By Theorem 4.9, noting that $\mathrm{sl}_n^{(\varepsilon)}$ is closed under $[\ \ ]_\psi$ since $\operatorname{tr} AB + \varepsilon BA \in \mathcal{C}_0$.

(ii), (iv) By Theorem 4.10.

(iii) Also by Theorem 4.10, using the involution $x \mapsto J^{-1}x^T J$ (formally adjoining $\varepsilon^{-1}$ if necessary). □

The exceptional Lie pairs could also be defined, but this is effected most easily via the Jordan version.

## 4.4 Non-classical examples

Other Lie pairs cannot be obtained by means of Lie commutators.

**Example 4.15.** The $\mathcal{C}_0$-skew $3 \times 3$ matrices deserve further analysis. In $\mathcal{L} := M_3(\mathcal{C})$ we consider matrices of type $J_0, J_1, J_2$, which depend on two $\mathcal{C}_0$-constrained parameters of $\mathcal{C}$, namely

$$J_0 := \left\{ J_0(a, a') := \begin{pmatrix} 0 & a & 0 \\ a' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \ a + a' \in \mathcal{C}_0 \right\} \in \mathcal{C}^{3\times 3},$$

$$J_1 := \left\{ J_1(b, b') = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ b' & 0 & 0 \end{pmatrix} \middle| \ b + b' \in \mathcal{C}_0 \right\} \in \mathcal{C}^{3\times 3},$$

$$J_2 := \left\{ J_2(c, c') = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & c' & 0 \end{pmatrix} \middle| \ c + c' \in \mathcal{C}_0 \right\} \in \mathcal{C}^{3\times 3}.$$

Clearly each element of $\mathcal{L}$ can be written (not uniquely) as a $\mathcal{C}$-linear combination of a matrix of type $J_0$, one of type $J_1$, and one of type $J_2$. We claim that

(a) $[J_i J_i] \in \mathcal{L}_0$;

(b) $[J_i, J_j] \subseteq J_{(i+j) \mod 3}$.

We know already that $(\mathcal{L}, \mathcal{L}_0)$ is a Lie pair but nevertheless it is instructive to perform explicit computations to obtain Property (b).

$$[J_0(a, a'), J_1(b, b')] \ = \ J_0(a, a')J_1(b, b') + J_1(b, b')^T J_0(a, a')$$

$$= \begin{pmatrix} 0 & a & 0 \\ a' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ b' & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & b' \\ 0 & 0 & 0 \\ b & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a & 0 \\ a' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & ab \\ 0 & ab' & 0 \end{pmatrix} = aJ_2(b, b') \in J_2.$$

Clearly the same argument holds for the other possible choices of indices, and b) is proven.

To go further in producing examples, we start with the most fundamental ones. Given a pair $(\mathcal{L}, \mathcal{L}_0)$ we will write its product as the bracket $[xy]$, with the hope of showing that it is a Lie bracket. Write $\mathcal{L}' := [\mathcal{L}\mathcal{L}]$.

**Definition 4.16.** A pair $(\mathcal{L}, \mathcal{L}_0)$ is $\mathcal{L}_0$-**Lie abelian** if $\mathcal{L}' \subseteq \mathcal{L}_0$. More generally, $(\mathcal{L}, \mathcal{L}_0)$ is $\mathcal{L}_0$-**Lie nilpotent** of **index** 2 if $[\mathcal{L}\mathcal{L}'] \subseteq \mathcal{L}_0$.

**Lemma 4.17.**

*(i) All $\mathcal{L}_0$-Lie nilpotent pairs of index 2 satisfy the Jacobi $\mathcal{L}_0$- identity.*

*(ii) If $(\mathcal{L}, \mathcal{L}_0)$ is $\mathcal{L}_0$-Lie abelian, then the Jacobi $\preceq$-identity holds.*

*Proof.* (i) All the terms are in $\mathcal{L}_0$.

(ii) $[xy], [yz] \in \mathcal{L}_0$ imply $[[xy]z] = [[zy]x]] = [x[yz]]$, and $[y[zx]] \in \mathcal{L}_0$, which in turn implies that $[[xy]z] \preceq [x[yz] + [y[zx]]$. □

### 4.4.1 Low dimensional examples

If $\mathcal{L} = \bigoplus_{i=1}^n \mathcal{C}x_i$ we say that $\mathcal{L}$ has dimension $n$. Many of the lowest dimensional examples lack negation maps. Blachar [6, §2.3] provided the 3-dimensional examples of Lie pairs over a semifield $\mathcal{C}$ having a negation map, so we shall only consider Lie pairs over a semifield pair $(\mathcal{C}, \mathcal{C}_0)$.

**Example 4.18.** The only 1-dimensional example is supplied by the trivial algebra $\mathcal{L} = \mathcal{C}x$, with $[xx] = 0$.

**Example 4.19.** The 2-dimensional examples where the Lie pair is $\mathcal{L}_0$-Lie abelian, so the Jacobi $\mathcal{L}_0$-identity holds. Let $\mathcal{L} = \mathcal{C}x + \mathcal{C}y$.

1. $\mathcal{L}_0 = \{0\}$; then we get the classical examples in [15].

2. $\mathcal{L}_0 = \mathcal{C}y$, where $[xx] = y$ and one of the following holds:

   - $[xy] = [yx] = [yy] = y$.
   - $[xy] = [yx] = y$ and $[yy] = 0$.

- $[xy] = [yx] = [yy] = 0$.

3. Now $\mathcal{L}_0 = \mathcal{C}(\mu x + \nu y)$, with $\mu, \nu \neq 0$. One example is $\mu + \nu = 1$, $[xx] = [yy] = 0$, and $[xy] = [yx] = \mu x + \nu y$.

   $[x[xy]] = \nu[xy]$ and $[y[xy]] = \mu[xy]$.

   $[x[yx]] = \mu[xy]$ and $[y[yx]] = [[xy]x] = \nu[yx]$.

A relevant 3-dimensional example, whose aim is to recover the situation of the cross product, will be studied separately in Section 4.4.3 below.

**Example 4.20.** Some 4-dimensional examples. Let $\mathcal{L} = \mathcal{C}x \oplus \mathcal{C}y \oplus \mathcal{C}z_1 \oplus \mathcal{C}z_2$, and let $\mathcal{L}_0 = \mathcal{C}z_1 \oplus \mathcal{C}z_2$, where $[xx] = [yy] = 0$, $[xy] = z_1$, $[yx] = z_2$, and

1. $[xz_i] = [yz_i] = [z_iz_j] = 0$ for all $i, j$.

2. (The Heisenberg pair) $[xz_1] = [z_1y] = [z_2x] = [yz_2] = z_1$, $[z_iz_j] = 0$ and additionally $[xz_2] = [z_2y] = [z_1x] = [yz_1] = z_2$.

3. $[xz_1] = [z_1y] = [z_2x] = [yz_2] = [z_1z_2] = z_1$, $[xz_2] = [z_2y] = [z_1x] = [yz_1] = [z_2z_1] = z_2$.

Example 4.19 (1) satisfies the Jacobi $\mathcal{L}_0$-identity, by computation. The other Lie pairs are $\mathcal{L}_0$-Lie abelian, so satisfy the Jacobi $\mathcal{L}_0$-identity.

### 4.4.2 Filiform pairs

Another large class of examples is provided by the *filiform algebras* [3], an important class of nilpotent Lie algebras which has a *Verne basis* $\{x_1, \dots, x_n\}$ satisfying

(i) $[x_1x_i] = x_{i+1}$, $1 \leq i \leq n-1$,

(ii) $[x_1, x_n] = 0$,

(iii) $[x_ix_j] = \sum_{k \geq i+j} c_{i,j}x_k$, $c_{i,j} \in C$.

**Definition 4.21.** A **filiform pair** is a Lie pair $(\mathcal{L}, \mathcal{L}_0)$ such that $\mathcal{L}$ is a free $(C, C_0)$-module with basis $x_1, \dots, x_n$ satisfying the conditions

1. $[x_1x_i] = x_{i+1}$, $1 \leq i \leq n-1$,

2. $[x_1, x_n] \in \mathcal{L}_0$,

3. $[x_ix_j] \in \sum_{k \geq i+j} c_{i,j}x_k + \mathcal{L}_0$, where $c_{i,j} + c_{j,i} \in \mathcal{L}_0$.

   (In particular $[xx], [xy] + [yx] \in \mathcal{L}_0$, for all $x, y \in \mathcal{L}$.)

**Example 4.22.** Let us see a few instances.

1. The **standard 3-dimensional filiform pair** has generators $x_1, x_2, x_3$. Choose $\ell_{21}$ as $c_1x_1 + c_2x_2 + c_3x_3 \in \mathcal{L}$ arbitrarily ($c_i \in \mathcal{C}$), and take $\mathcal{L}_0 = \mathcal{C}(x_3 + \ell_{21})$, together with the relations

$$[x_1x_2] = x_3, \qquad [x_2x_1] = \ell_{21}, \qquad [x_ix_j] = 0 \quad \text{otherwise.}$$

The Jacobi $\mathcal{L}_0$- identity then holds trivially, checked on generators.

2. More generally choose $\ell_{13}, \ell_{31}, \ell_{23}, \ell_{32}$, and define $\mathcal{L}_0$ as being the $\mathcal{C}$-submodule generated by $(\ell_{ij} + \ell_{ji}, \ell_{21} + x_3)$ $(i + j \geq 4)$. Then the commutators

$$[x_1x_2] = x_3, \qquad [x_2x_1] = \ell_{21}, \quad [x_1x_3] = \ell_{13},$$
$$[x_3x_1] = \ell_{31}, \quad [x_2x_3] = \ell_{23}, \quad [x_3x_2] = \ell_{32},$$

define a Lie pair $(\mathcal{L}, \mathcal{L}_0)$.

### 4.4.3 The cross product Lie pair

In any reasonable theory of Lie pairs one should be able to recover the classical example of the Lie algebra $(\mathbb{R}^3, \times)$, the cross product in the three-dimensional real vector space. We will do it via the procedure described in Section 3.1.1.

**Example 4.23** (The cross product)**.** Let

$$\mathcal{L} = \mathcal{C}b_0 \oplus \mathcal{C}b_1 \oplus \mathcal{C}b_2.$$

Take arbitrarily two arbitrary 3-tuples $(c_0, c_1, c_2)$ and $(d_0, d_1, d_2)$ in $\mathcal{L}^3$, not necessarily $\mathcal{C}$-linearly independent. We define a Lie bracket on $\mathcal{L}$, depending on the choice of $(d_i)$ and $(c_i)$ (i.e. a 6- parameter family) by encoding it into a $\mathcal{L}$-valued $3\times3$ matrix $A : \mathcal{L}\times\mathcal{L} \longrightarrow \mathcal{L}$ given by:

$$A := \begin{pmatrix} d_0 & b_2 & c_1 \\ c_2 & d_1 & b_0 \\ b_1 & c_0 & d_2 \end{pmatrix} \in \mathcal{L}^{3\times3} \cong \mathcal{L}^* \otimes \mathcal{L}^* \otimes \mathcal{L}, \tag{8}$$

stipulating that

$$[b_ib_j] = A(i, j), \qquad 0 \leq i, j \leq 2.$$

We obtain a Lie pair generically, imitating the natural structure of the cross product. For this reason we define

$$\mathcal{L}_0(A) = \mathcal{C}\langle d_i, b_i + c_i, b_ic_i\rangle.$$

The notation reflects the fact that the submodule $\mathcal{L}_0(A)$ of $\mathcal{L}$ depends on the matrix $A$. Let us check that $(\mathcal{L}, \mathcal{L}_0(A))$ satisfies the axioms of $(a)$, $(b)$ $(c)$, $(c')$, $(d)$ and $(e)$ of Definition 3.3(i). To this purpose, we first compute the product of two generic elements

$$x = x_0b_0 + x_1b_1 + x_2b_2 \qquad \textit{and} \qquad y = y_0b_0 + y_1b_1 + y_2b_2$$

of $\mathcal{L}$, using the multiplication matrix. A simple computation yields:

$$\begin{aligned} [xy] &= x_0y_0d_0 + x_1y_1d_1 + x_2y_2d_2 \\ &= x_1y_2b_0 + x_2y_1c_0 + x_2y_0b_1 + x_0y_2c_1 + x_2y_0b_1 + x_0y_1b_2 + x_1y_0c_2 \end{aligned} \tag{9}$$

(a) Let us check that $[xx] \in \mathcal{L}_0(A)$. Indeed,

$$[xx] = x_0^2 d_0 + x_1^2 d_1 + x_2^2 d_2 + x_0 x_1 (b_2 + c_2) + x_0 x_2 (b_1 + c_1) + x_1 x_2 (b_0 + c_0) \in \mathcal{L}_0(A);$$

(b) Let us check that $ad_x(y) + ad_x^\dagger(y) = [xy] + [yx] \in \mathcal{L}_0(A)$.

$$\begin{aligned}[xy] + [yx] = \quad & 2x_0 y_0 d_0 + 2x_1 y_1 d_1 + 2x_2 y_2 d_2 + (x_0 y_1 + x_1 y_0)(b_2 + c_2) \\ & + (x_0 y_2 + x_2 y_0)(b_1 + c_1) + (x_1 y_2 + x_2 y_1)(b_0 + c_0) \in \mathcal{L}_0(A)\end{aligned} \tag{10}$$

(c) We now come to the Jacobi identity. Besides the generic elements $x$ and $y$ mentioned before, let $z = z_0 b_0 + z_1 b_1 + z_2 b_2$ and consider the Jacobi sum

$$(xy)z + (yz)x + (zx)y. \tag{11}$$

Expanding (11) in terms of the component of $x$, $y$ and $z$, one easily get

$$\begin{aligned}&[[xy]z] + [[yz]x] + [[zx]y] \\ = & \sum_{0 \le i,j,k \le 2} x_i y_j z_k \big([[b_i b_j] b_k] + [[b_j b_k] b_i] + [[b_k b_i] b_j]\big).\end{aligned}$$

For each choice of $(i, j, k) \in \{0, 1, 2\}^3$ we have basically two cases.

i) $(i, j, k)$ is either an even or odd permutation of $(0, 1, 2)$. In the even case we have

$$\begin{aligned}&[[b_i b_j] b_k] + [[b_j b_k] b_i] + [[b_k b_i] b_j] \\ = & [b_k b_k] + [b_i b_i] + [b_j b_j] = d_0 + d_1 + d_2 \in \mathcal{L}_0(A).\end{aligned}$$

In the odd case:

$$[[b_i b_j] b_k] + [[b_j b_k] b_i] + [[b_k b_i] b_j = c_k b_k + c_i b_i + c_j b_j \in \mathcal{L}_0(A).$$

ii) If $i = j$ then

$$[[b_i b_i] b_k] + [[b_i b_k] b_i] + [[b_k b_i] b_i] = [d_i b_k] + [(b_j + c_j) b_i] \in \mathcal{L}_0(A).$$

iii) The case $i = k$ works the same as in (ii).

The $\dagger$-reversibility does not hold in general. We compute $[[b_0 b_1] b_2] = [b_2 b_2] = d_2$, whereas $[[b_2 b_1] b_0] = [c_0 b_0]$, so in general we need $[c_i b_i] = d_i$, which normally fails.

**Example 4.24.** Generalizing Example 4.23, let $V$ be a free module over $\mathcal{C}$ and also let $A : V \otimes V \to V$ be a $V$-valued bilinear form over $\mathcal{C}$. If $V = \bigoplus_{1 \le i \le n} \mathcal{C} b_i$, let us denote $A = (a_{ij})$ for $a_{ij} \in V$, where $b_i b_j = a_{ij}$, and let

$$\mathcal{L} = \bigoplus_{i<j} \mathcal{C} \cdot a_{ij}$$

Define

$$\mathcal{L}_0(A) := \mathcal{C} \cdot \langle a_{ij} + a_{ji},\, a_{ii},\, a_{ij}a_{ji} \rangle \tag{12}$$

i.e., $\mathcal{L}_0(A)$ is the $\mathcal{C}$-submodule spanned by the expressions listed in (12). Then $(\mathcal{L}, \mathcal{L}_0(A))$ is a Lie pair. The verification works the same as in the case of $n = 3$ (Example 4.23), so we omit it.

**Remark 4.25.** If $\mathcal{A}$ is an algebra (i.e., with additive inverses) and $c_i = -b_i$ and $d_i = 0$, the matrix $C$ as in (8) defines the usual cross product (for $\mathcal{A} = \mathbb{R}$).

### 4.5 Krasner type

One can also insert some hypergroup theory into the theory of Lie pairs.

**Theorem 4.26.**

*(i) (Inspired by [20]) Let $R$ be a semiring, and $G$ a normal multiplicative subgroup of $R$. Pick $\varepsilon \in R$. Then $H = R/G$ is a hyper-semiring, and let $\mathcal{A} = \mathcal{P}(R/G)$, i.e., the elements $S \in \mathcal{A}$ are unions $\cup a_i G$ of cosets of $R$. In other words, if $a \in S$ then $a_i g \in S$ for each $g \in G$. Addition is defined by*

$$\boxplus a_i G = \{\sum a_i g_i : g_i \in G\}.$$

*$(\mathcal{A}, \mathcal{A}_0)^{(\varepsilon)}$ of Theorem 4.3, and $\mathcal{A}_0 = \{S \in \mathcal{P}(R/G) : 0 \in S\}$. $H$ satisfies all of the axioms of a reversible weak $\preceq_\subseteq$-Lie pair, under the Lie bracket $[aG\, bG] = [a, b]G$.*

*(ii) In (i), we could take a Lie multiplicative ideal $M$ of $R$ and instead take*

$$\mathcal{A}_0 = \{S \in \mathcal{P}(R/G) : S \cap M \neq \emptyset\}.$$

*(iii) Let $R$ be a semiring with an involution $(*)$, and $G$ a normal multiplicative symmetric subgroup of $R$. Then the analog of $H$ of (i), in Theorem 4.10, is a weak Lie pair, with surpassing relation $\subseteq$.*

*(iv) In (iii), we could take a symmetric Lie multiplicative ideal $M$ of $R$ and instead take $\mathcal{A}_0 = \{S \in \mathcal{P}(R/G) : S \cap M \neq \emptyset\}$.*

*Proof.* (i) We get the Lie product in $R$ as in [22, Proposition 10.7]. [2, Proposition 5.18] yields associativity of addition.

(ii) Analogous to (i).

(iii) $\mathcal{A}/G$ is a hyper-semiring, as in [2]. Then we apply (i), defining

$$[S_1, S_2] = \{[a_{i1}, a_{j2}] : a_{i1} \in S_1,\ a_{j2} \in S_2\},$$

and have a weak Lie pair with surpassing relation $\subseteq$.

(iv) Analogous to (iii). ∎

**Remark 4.27.** $(\boxplus_i a_i G)(\boxplus_j b_j G) \subseteq \boxplus_{i,j} a_i b_j G$ in each of the Krasner-type constructions, in view of (4.7), which shows that there formally are more terms in the right side of Definition 2.11(ii) than the left, and the extra ones are paired off.

# 5 Doubling

**Example 5.1.** (Abstract doubling of a $C$-module, also see [2, Example 1.7(iii)]). This is a way to create a pair with a negation map, from any $C$-module $\mathcal{A}$.

1. Define $\widehat{\mathcal{A}} := \mathcal{A} \times \mathcal{A}$ with pointwise addition. We think of the first component as a positive copy of $\mathcal{A}$, and the second component as a negative copy of $\mathcal{A}$.

2. Define multiplication in $\widehat{\mathcal{A}}$ by the **twist action**
$$(b_1, b_2)(b_1', b_2') = (b_1 b_1' + b_2 b_2',\ b_1 b_2' + b_2 b_1'). \tag{13}$$

3. $\widehat{\mathcal{A}}$ has the "switch" negation map given by $(-)(b_1, b_2) = (b_2, b_1)$.

4. If $\mathcal{A}$ is a $C$-module, then $\widehat{\mathcal{A}}$ is a $\widehat{C}$-module with the respect to the twist action
$$(c_1, c_2)(b_1, b_2) = c_1 b_2 + c_2 b_2, c_1 b_2 + c_2 b_1).$$
Note that $(-)(1, 0) = (0, 1)$.

5. If $f, g : (\mathcal{A}, \mathcal{T}) \to (\mathcal{A}', \mathcal{T}')$ are homomorphisms, then define $(f, g) : \widehat{\mathcal{A}} \to \widehat{\mathcal{A}'}$ given by $(f, g)(b_1, b_2) = (f(b_1) + g(b_2), f(b_2) + g(b_1))$.

**Lemma 5.2.** *Any doubled d-bimagma $\widehat{\mathcal{A}}$ is $\mathbb{Z}_2$-graded as $(\mathcal{A} \times \{0\}) \oplus (\{0\} \times \mathcal{A})$.*

*Proof.* $\mathcal{A} \times \{0\}$ is the "+" part, and $\{0\} \times \mathcal{A}$ is the "−" part. We need d-bimagmas to decompose multiplication according to the grading. □

**Remark 5.3.** As in [22] one could obtain a pair by defining
$$\widehat{\mathcal{A}}_0 = \mathrm{Diag} := \{(b, b) : b \in \mathcal{A}\},$$
noting that $(b_1, b_2)(-)(b_1, b_2) = (b_1, b_2) + (b_2, b_1) = (b_1 + b_2, b_1 + b_2)$.

## 5.1 Doubling a pair

As in [2], we rather modify the doubling construction when working in the category of pairs, as follows:

**Example 5.4** (Doubling a pair)**.**

(i) Given a pair $(\mathcal{A}, \mathcal{A}_0)$, we obtain a pair $\widehat{(\mathcal{A}, \mathcal{A}_0)} := (\widehat{\mathcal{A}}, \widehat{\mathcal{A}}_0)$ by defining
$$\widehat{\mathcal{A}}_0 = \mathrm{Diag} + \{(b_1, b_2) : b_1 + b_2 \in \mathcal{A}_0\}.$$

(ii) If $(\mathcal{A}, \mathcal{A}_0)$ is a $C$-pair, then $(\widehat{\mathcal{A}}, \widehat{\mathcal{A}}_0)$ is a $\widehat{C}$-pair under the twist action.

**Lemma 5.5.** *If $\mathcal{A}_0$ is an ideal of $\mathcal{A}$, then $\widehat{\mathcal{A}_0}$ as defined in Example 5.4 is an ideal of $\widehat{\mathcal{A}}$.*

*Proof.* By (13), noting that if $b_1 + b_2 \in \mathcal{A}_0$ then

$$b_1 b_1' + b_2 b_2' + b_1 b_2' + b_2 b_1' = (b_1 + b_2)(b_1' + b_2') \in \mathcal{A}_0.$$

□

**Proposition 5.6.** *Any multilinear identity or $\preceq$-identity of an $\mathcal{A}_0$-additive bimagma pair (See Definition 2.3) $(\mathcal{A}, \mathcal{A}_0)$ also holds in the doubled pair.*

*Proof.* By Lemma 2.24 we need only check homogeneous elements, and they are preserved via the grading. □

### 5.2 Negated Lie pairs from a semiring

Motivated by Theorem 4.3, we construct a Lie pair from any semiring, with the switch a negation map.

**Example 5.7.** (Lie bracket on a doubled pair)
Building on Example 5.4, we can define the Lie bracket

$$[(x_1, y_1)(x_2, y_2)] = (x_1x_2 + y_1y_2 + x_2y_1 + y_2x_1,\ x_1y_2 + y_1x_2 + x_2x_1 + y_2y_1).$$

**Theorem 5.8.** *If $\mathcal{A}$ is a semiring then the Lie bracket of Example 5.7 makes $(\widehat{\mathcal{A}}, \widehat{\mathcal{A}}_0)$ a $\preceq_0$-Lie pair, where $\widehat{\mathcal{A}}_0 = \{(a, a) : a \in \mathcal{A}\}$.*

*If $(\mathcal{A}, \mathcal{A}_0)$ is a semiring pair over $(C, C_0)$, then $(\widehat{\mathcal{A}, \mathcal{A}_0})$ of Example 5.4 is a reversible Lie pair.*

*Proof.* We get the axioms of Definition 3.3(i) by passing to $\hat{A}$ and applying Theorem 4.5.
We could also verify them directly:

$$[(x, y), (x, y)] = (xx + yy + xy + yx, xy + yx + xx + yy) \in \hat{\mathcal{A}}_0.$$

$$\begin{aligned}
[(x_1, y_1),&(x_2, y_2)] + [(x_2, y_2), (x_1, y_1)] \\
&= (x_1x_2 + y_1y_2 + x_2y_1 + y_2x_1,\ x_1y_2 + y_1x_2 + x_2x_1 + y_2y_1) \\
&\quad + (x_2x_1 + y_2y_1 + x_1y_2 + y_1x_2,\ x_2y_1 + y_2x_1 + x_1x_2 + y_1y_2) \in \hat{\mathcal{A}}_0,
\end{aligned} \tag{14}$$

and

$$[[(x_1, y_1), (x_2, y_2)](x_3, y_3)] = [[(x_3, y_3), (x_2, y_2)](x_1, y_1)]$$

holds by symmetry of the definition.
To prove the Jacobi $\preceq_0$-identity, we need to show that

$$[(x_1, y_1), (x_2, y_2)], (x_3, y_3)] \preceq_0 [(x_1, y_1)[(x_2, y_2)(x_3, y_3)]] + [[(x_2, y_2), (x_3, y_3)(x_1, y_1)]],$$

which is straightforward but lengthy. □

$\widehat{\mathcal{A}}_0$-reversibility may fail since $[(x_1, y_1)(x_2, x_2)] = (x_1x_2 + y_1x_2, x_1x_2 + y_1x_2)$ whereas $[(x_2, x_2)(x_1, y_1)] = (x_2x_1 + x_2y_1, x_2x_1 + x_2y_1)$.

### 5.3 Doubling of Lie pairs

We use the method of doubling to construct a negation map for a Lie pair.

**Theorem 5.9.** *If $(\mathcal{L}, \mathcal{L}_0)$ is a Lie pair, then $(\widehat{\mathcal{L}}, \widehat{\mathcal{L}}_0)$ is a Lie pair, and there is a Lie homomorphism $(\mathcal{L}, \mathcal{L}_0) \to (\widehat{\mathcal{L}}, \widehat{\mathcal{L}}_0)$ given by $y \mapsto (y, 0)$.*

*If $[\ \ ]$ is a $\preceq$-Lie bracket on $(\mathcal{L}, \mathcal{L}_0)$, then $[\ \ ]$ naturally induces a $\preceq$-Lie bracket on $(\widehat{\mathcal{L}}, \widehat{\mathcal{L}}_0)$.*

*If $(\mathcal{L}, \mathcal{L}_0)$ satisfies $\dagger$-reversibility (resp. $\mathcal{L}_0$-reversibility), then so does $(\widehat{\mathcal{L}}, \widehat{\mathcal{L}}_0)$.*

*Proof.* We verify the axioms of Definition 3.3(i).

$$[(x, y)(x, y)] = ([xx] + [yy], [xy] + [yx]) \in \mathcal{A}_0 \times \mathcal{A}_0 \subseteq \hat{\mathcal{A}}_0.$$

$$[(x_1, x_2)(y_1, y_2)] + [(y_1, y_2)(x_1, x_2)] =$$
$$([x_1y_1] + [x_2y_2], [x_1y_2] + [x_2y_1]) + ([y_1x_1] + [y_2x_2], [y_2x_1] + [y_1x_2]) \in \hat{\mathcal{A}}_0.$$

If $y_1 + y_2 \in \mathcal{A}_0$ then $[(x_1, x_2)(y_1, y_2)] \in \hat{\mathcal{A}}_0$ since

$$[x_1y_1] + [x_2y_2] + [x_1y_2] + [x_2y_1] = [(x_1 + x_2)(y_1 + y_2)] \in \mathcal{A}_0.$$

The other defining identities (as in Remark 3.4) and $\preceq$-identities (as in Remark 3.17) are multilinear and thus pass to $(\widehat{\mathcal{L}}, \widehat{\mathcal{L}}_0)$ by Lemma 5.6. $\square$

**Important Note 5.10.** *The doubled Lie pair need not be a negated Lie pair even though it has a negation map. Indeed,*

$$(-)[(x_1, x_2)(y_1, y_2)] = (-)([x_1y_1] + [x_2y_2], [x_1y_2] + [x_2y_1]) = ([x_1y_2] + [x_2y_1], [x_1y_1] + [x_2y_2])$$

*whereas*

$$[(y_1, y_2)(x_1, x_2)] = ([y_1x_1] + [y_2x_2], [y_2x_1] + [y_1x_2]).$$

## 6 Representing Lie pairs inside semiring pairs

As always, we work with pairs over $(C, C_0)$. Our goal in this section is to embed a Lie pair in an appropriate associative pair. In order to obtain such a pair, we need to consider tensor pairs.

### 6.1 The tensor d-bimagma and free Lie pairs

We would like to add free Lie pairs to the list of Example 2.18. We need a preliminary. Tensor products of modules over semialgebras can be defined in the usual universal way, cf. [19]. But we do not require this generality.

**Example 6.1.**

1. When constructing the tensor d-bimagma $T(V)$, rather than having it associative, we take the free d-bimagma given by tensor multiplication of monomials, cf. Example 2.18. In other words, let $V^{\otimes m}$ denote all tensor powers of $V$ over $C$, distinguished by parentheses, in the sense that $(V \otimes V) \otimes V$ and $V \otimes (V \otimes V)$ are distinct; for example,
$$V^{\otimes 3} := (V \otimes V) \otimes V \ \oplus \ V \otimes (V \otimes V).$$
To emphasize nonassociativity, we put parentheses around each monomial. We set $T(V) := \bigoplus_{m \geq 1} V^{\otimes m}$, with multiplication defined by juxtaposition, i.e., define $((h_1)(h_2)) = (h_1)\otimes(h_2)$, for monomials $(h_1)$ and $(h_2)$. For example if $(h_1), (h_2) \in V^{\otimes 2}$ then writing $(h_i) = (v_i \otimes w_i)$ we get
$$(h_1)(h_2) = (v_1 \otimes w_1) \otimes (v_2 \otimes w_2).$$
Thus $V^{\otimes m}$ is spanned over tensor products of the $x_i$; these are customarily called **pure simple tensors**. A **simple tensor** is a pure simple tensor with a coefficient from $C$.

   We form a d-bimagma pair $(T(V), T(V)_0)$ over a pair $(C, C_0)$ by putting $T(V)_0$ to be the subspace of $T(V)$ spanned by:

   (a) all simple tensors containing a factor in $V_0$, and

   (b) all simple tensors with coefficients from $C_0$,

   clearly an ideal of $T(V)$. Note that (b) is 0 when $C_0 = 0$.

2. For the associative case, let $\bar{V}^{\otimes m}$ denote all associative tensor powers of $V$ over $C$, written without parentheses, and $\bar{T}(V) := \bigoplus_{m \geq 1} \bar{V}^{\otimes m}$, with multiplication defined by $h_1 h_2 = h_1 \otimes h_2$, for monomials $h_1$ and $h_2$. $\bar{T}(V)$ is isomorphic to the **free associative algebra** over a basis of $V$.

We also want to make such a construction with vector space pairs. Let us consider $(V, V_0) = \bigoplus_{i \in I} (C, C_0) \cdot x_i$ be the free $(C, C_0)$-module, with basis $B = \{x_i : i \in I\}$, cf. Example 2.18.

**Remark 6.2.**

(i) We could take $C_0 = 0$ if we want.

(ii) In the other direction given a free $C$-module $V$, we could pass to $\tilde{V}$ and $\tilde{C}$, to reduce to the case that $(-)1 \in C$.

**Example 6.3.** Let $(C, C_0)$ be a pair, and take $(T(V), T(V)_0)$ to be the tensor d-bimagma of Example 6.1.

1. (The free $\mathcal{L}_0$-additive Lie pair) We take $\mathcal{L}$ to be $T(V)$, and $\mathcal{L}(V)_0$ to be the $C$-module generated by $T(V)_0$ and all expressions

   (a) $(\mathbf{xx})$,

   (b) $(\mathbf{xy} + \mathbf{yx})$,

   (c) $(\mathbf{xy})\mathbf{z} + (\mathbf{yz})\mathbf{x} + (\mathbf{zx})\mathbf{y}$,

   where the $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are simple tensors. In view of Lemma 2.24, the axioms of Definition 3.3(i) are satisfied by $(\mathcal{L}\mathcal{L}_0)$.

2. If one is willing to modify $C$, we can define the free Lie pair with basis indexed by any set $I$. Namely, we take commuting associative indeterminates $c_{i,j}^k$ over $C$, and use Lemma 3.10 to define $\mathcal{L}$ over $C[c_{i,j}^k]$, and formally defining $C_0$ to be the ideal defined by conditions (1)-(4) of Lemma 3.10.

3. When $(-)1 \in C$ (which can be attained using Remark 6.2), $(T(V), T(V)_0)^-$ is a Lie pair by means of Corollary 4.5.

But we need a surpassing relation $\preceq$ to work with the $\preceq$-adjoint algebra, so we also take a more intricate construction modeled on Proposition 2.22.

**Example 6.4** (The free bilinear $\preceq$-Lie pair)**.** Re-indexing the subscripts of the $y_i$, we adjoin a formal indeterminate $y_{h_1,h_2,h_3}$ for each 3-tuple of simple tensors. We take $\mathcal{C}$ to be the congruence generated by all pairs

$$(h_1 \otimes (h_2 \otimes h_3) + y_{h_1,h_2,h_3}, h_2 \otimes (h_3 \otimes h_1) + \ h_3 \otimes (h_1 \otimes h_2))$$

and let $\mathcal{U} = T(V)/\mathcal{C}$; i.e., we declare that

$$h_1 \otimes (h_2 \otimes h_3) + y_{h_1,h_2,h_3} = h_2 \otimes (h_3 \otimes h_1) + \ h_3 \otimes (h_1 \otimes h_2)$$

Let $\mathcal{U}_0$ be the multiplicative ideal of $\mathcal{U}$ generated by all terms

$$h_i \otimes h_i, \quad h_i \otimes h_j + h_j \otimes h_i, \quad y_{h_1,h_2,h_3}, \quad i,j,k\{1,2,3\}$$

where $h_i$ are monomials.

**Theorem 6.5.** *$(\mathcal{U}, \mathcal{U}_0))$ is a $\preceq_0$-Lie pair. Furthermore if $(\mathcal{L}, \mathcal{L}_0)$ is a $\preceq_0$-Lie pair then for any $a_i$ in $\mathcal{L}$, $i \in I$, there is a Lie homomorphism $(\mathcal{L}, \mathcal{L}_0) \to (\mathcal{L}, \mathcal{L}_0)$ sending $x_i \to \bar{x}_i := a_i$ and $y_{h_1,h_2,h_3}$ to an element $\bar{y}_{h_1,h_2,h_3}$ of $\mathcal{L}_0$ for which*

$$[\bar{h}_1[\bar{h}_2\bar{h}_3]] + \bar{y}_{h_1,h_2,h_3} = [[\bar{h}_2\bar{h}_3]\bar{h}_1] \ + [[\bar{h}_3[\bar{h}_1\bar{h}_2].$$

*Proof.* All the relations except the Jacobi $\preceq$-identity can be written as identities just in terms of $\mathcal{L}$ and $\mathcal{L}_0$, so are preserved under substitution. The only difficulty is the Jacobi $\preceq$-identity, which as in Proposition 2.22 we rewrite as an identity by inserting the extra term from $\mathcal{L}_0$. (We did not claim uniqueness, since several terms of $\mathcal{L}_0$ might provide equality.)

□

**Remark 6.6.** There is a natural map from the degree 2 part of the exterior semialgebra as in [11] to the Lie pair of Example 4.24. In fact we can construct a congruence of $\mathcal{L} \otimes \mathcal{L}$ which provides the map to $\mathcal{L}$.

### 6.2 Lie sub-pairs

**Definition 6.7.**

(i) A **weak $\psi$-Lie sub-pair** of a bimagma pair $(\mathcal{A}, \mathcal{A}_0)$ with a pre-negation map $\psi$ is a sub-pair $(\mathcal{L}, \mathcal{L}_0)$, together with and a map $[\ \ ]_\psi : \mathcal{L} \times \mathcal{L} \to \mathcal{A}$ satisfying the Lie bracket axioms of Definition 3.3(i), as well as the condition

$$b_1 b_2 + \psi(b_2 b_1) + [b_2 b_1]_\psi \in \mathcal{L}_0, \quad \text{for all } b_1, b_2 \in \mathcal{L}.$$

(ii) A **$\preceq$-Lie sub-pair** of a bimagma pair $(\mathcal{A}, \mathcal{A}_0)$ with a surpassing map is a sub-pair $(\mathcal{L}, \mathcal{L}_0)$, together with a bilinear map $[\ \ ] : \mathcal{L} \times \mathcal{L} \to \mathcal{A}$ satisfying the Lie bracket axioms of Definition 3.3(i), as well as the condition

$$b_1 b_2 \preceq_0 b_2 b_1 + [b_1 b_2], \quad \text{for all } b_1, b_2 \in \mathcal{L}.$$

(iii) An **$\psi$-Lie sub-pair** of a bimagma pair $(\mathcal{A}, \mathcal{A}_0)$ is a sub-pair $(\mathcal{L}, \mathcal{L}_0)$, together with a map $[\ \ ] : \mathcal{L} \times \mathcal{L} \to \mathcal{A}$ satisfying the Lie bracket axioms of Definition 3.3(i), as well as the condition

$$[b_1 b_2] = b_1 b_2 + \psi(b_2) b_1, \quad \text{for all } b_1, b_2 \in \mathcal{L}.$$

*We shall call $[\ \ ]$ a bracket, even though we do not require $\mathcal{L}$ to be closed under $[\ \ ]$.*

**Lemma 6.8.**

*(i) Any $\psi$-Lie sub-pair is a quasi Lie pair.*

*(ii) For any pre-negation map $\psi$, the bimagma pair $(\mathcal{L}, \mathcal{L}_0)$ is an $[\ \ ]_\psi$ $\preceq$-sub-pair of itself.*

*Proof.* (i)

$$\begin{aligned} b_1 b_2 + \psi(b_2 b_1) + [b_2 b_1] &= b_1 b_2 + \psi(b_2 b_1) + b_2 b_1 + \psi(b_1 b_2) \\ &= b_1 b_2 + b_2 b_1 + \psi(b_1 b_2 + b_2 b_1) \in \mathcal{L}_0. \end{aligned} \tag{15}$$

(ii) $b_1 b_2 \preceq_0 b_1 b_2 + b_2 b_1 + \psi(b_2 b_1) = b_2 b_1 + [b_1 b_2]_\psi$. □

**Important Note 6.9.** *Lemma 6.8 is applicable quite generally, since one can pass to the doubled bimagma pair and even take $\psi$ to be multiplication by $(0,1)$. An instance where one needs to take quasi Lie sub-pairs: We want to view the free Lie pair inside the free negated associative pair $(T(\mathcal{L}), T(\mathcal{L})_0)$, which we obtain by doubling in Remark 6.2. If we send $x \mapsto (x,0)$, then $[xy] \mapsto ([xy],0)$ whereas $[(x,0),(y,0)] = (xy,yx)$, which is different. By adjoining all elements of the form $(xy+[yx],yx)$ and $(xy,[xy]+yx)$ to $T(\mathcal{L})_0$ we have a quasi Lie sub-pair.*

### 6.2.1 The weak adjoint morphism

Following classical Lie theory, we want to represent Lie pairs inside semiring pairs. The following observation is easy.

**Proposition 6.10** ( [22, Proposition 10.6]). *For any $\psi$-Lie pair $(\mathcal{L}, \mathcal{L}_0)$, there are weak Lie morphisms $\mathrm{ad} : (\mathcal{L}, \mathcal{L}_0) \to \mathrm{End}(\mathcal{L}, \mathcal{L}_0)_\psi$, given by $b \mapsto \mathrm{ad}_b$, and, for $\psi$ invertible, $\mathrm{ad}^\dagger : (\mathcal{L}, \mathcal{L}_0) \to (\mathrm{End}\,\mathcal{L}, \mathrm{End}\,\mathcal{L}_0)_{\psi^{-1}}$, given by $b \mapsto \psi\,\mathrm{ad}_b^\dagger$.*

*Proof.* We verify the conditions of Definition 3.20. (i) and (ii) are immediate, and (iii) follows from the Jacobi $\mathcal{L}_0$-identity. □

Clearly $(\mathrm{AD}_{\mathcal{L}}, \mathrm{AD}_{\mathcal{L}_0})$ is a pair. We would like to say that it is a Lie pair under the obvious candidate for Lie bracket, namely $[\mathrm{ad}_x\,\mathrm{ad}_y] := \mathrm{ad}_x\,\mathrm{ad}_y + \mathrm{ad}_y^\dagger\,\mathrm{ad}_x$, but unfortunately this need not be closed.

## 6.3 PBW Theorems for Lie pairs

Throughout this section suppose that $(\mathcal{L}, \mathcal{L}_0)$ is a Lie pair, where $\mathcal{L}$ is also a free $C$-module with basis $\{x^i : i \in J\}$, and $\mathcal{L}_0$ is the submodule with basis $\{x^i : i \in J_0 \subset J\}$. Reversing the direction of Theorem 4.3, we want a universal enveloping construction of a semiring pair from the Lie pair $(\mathcal{L}, \mathcal{L}_0)$. In classical theory this is the celebrated PBW (Poincare-Birkhoff-Witt) Theorem.

For Lie pairs there are three possible versions $\iota : (\mathcal{L}, \mathcal{L}_0) \to U$ where $U$ is respectively $U_\psi(\mathcal{L}, \mathcal{L}_0)$, $U_\preceq(\mathcal{L}, \mathcal{L}_0)$, $U_\varepsilon(\mathcal{L}, \mathcal{L}_0)$, depending on which type of Lie pair and which type of morphism $\iota$ we use (resp. weak $\varepsilon$-Lie morphism, $\preceq$-Lie morphism, $\varepsilon$-Lie homomorphism), which we fix in the next definition.

**Definition 6.11. Universal Property.** If $(\mathcal{A}, \mathcal{A}_0)$ is any associative pair given together with a morphism $f : (\mathcal{L}, \mathcal{L}_0) \to (\mathcal{A}, \mathcal{A}_0)$ such that $f$ satisfies resp. (i), (ii), (iii) of Definition 6.7, then there is a unique respective morphism $\phi_f : \mathcal{U}(\mathcal{L}, \mathcal{L}_0) \to (\mathcal{A}, \mathcal{A}_0)$ such that $f = \phi_f \circ \iota$.

Note that we did not require $\iota$ to be injective; this will be examined each time. The reduction techniques used in classical Lie theory become unusable without cancellation, but in the semialgebra case we can often apply a degree argument to the elements of the tensor algebra in the following situation, since we only adjoin monomials of degree $\geq 2$ in the $x^i$ to $\mathcal{A}_0$.

**Definition 6.12.** A semigroup $(\mathcal{A}, 0)$ satisfies the **lacks zero sums** (LZS) property if the sum of nonzero elements of $\mathcal{A}$ cannot equal 0.

The LZS property will be the key to obtaining an injection in Theorems 6.13 and 6.14.

### 6.3.1 The weak $\psi$-version of PBW

**Theorem 6.13.** *Suppose $(\mathcal{L}, \mathcal{L}_0)$ is a $\psi$-Lie pair. Let $V = \mathcal{L}$, extended by a formal set of indeterminates $Y = \{y_{i,j} : i, j \in J\}$. Define $U_{\mathrm{weak};\psi}(\mathcal{L}) = T(V)$ using the construction of Example 6.3, and, identifying $x$ with $\iota(x)$ for $x$ in $\mathcal{L}$, let $U_{\mathrm{weak};\psi}(\mathcal{L})_0$ be the $C$-submodule generated by $\mathcal{L}(V)_0$ and*

$$\{x^i x^j + \psi(x^j)x^i + [x^j x^i] : i, j \in I\}.$$

*Define $U_{\mathrm{weak};\psi}(\mathcal{L}, \mathcal{L}_0) := (U_{\mathrm{weak};\psi}(\mathcal{L}), U_{\mathrm{weak};\psi}(\mathcal{L})_0)$. It is worth noticing that only the null part depends on $\psi$.*

1. *$U_{\mathrm{weak};\psi}(\mathcal{L}, \mathcal{L}_0)$ is a $\psi$-Lie pair, as in Theorem 4.3.*
2. *There is a universal weak $\psi$-Lie morphism $\iota_\psi\ (\mathcal{L}, \mathcal{L}_0) \to U_{\mathrm{weak};\psi}(\mathcal{L}, \mathcal{L}_0)$ given by $x^i \mapsto x^i$, satisfying the Universal Property in this setting.*
3. *The universal $\iota_\psi$ is $\mathcal{L}_0$-injective when $\mathcal{L}$ satisfies LZS.*

*Proof.* By definition $\iota(\mathcal{L}_0) \subseteq U_{\mathrm{weak};\psi}(\mathcal{L}) = T(V)$. Also

$$[x^i, x^j]_\psi + [x^j x^i] = x^i x^j + \psi(x^j)x^i + [x^j x^i] \in \mathcal{A}_0$$

by definition, so $\iota$ is a weak Lie morphism. Uniqueness is clear since $\varphi$ must satisfy $\varphi(\iota(x^i)) = f(x_i)$.

It remains to prove that $\iota$ is $\mathcal{L}_0$-injective when $\mathcal{L}$ satisfies LZS. This is seen seen by checking degrees in the tensor semialgebra. Namely, the degree 1 cannot be in $\mathcal{L}_0$ because of the LZS Property. (Here the lack of negation makes life easier, because there is no ambiguity!)

□

### 6.3.2 The $\preceq$ version of PBW

Now, given a Lie pair $(\mathcal{L}, \mathcal{L}_0)$ endowed with a surpassing map $\preceq$, we want to construct an associative negated pair $(\mathcal{U}_\preceq(\mathcal{L}, \mathcal{L}_0))$ such that there is a universal $\preceq$-embedding $\iota$ of $(\mathcal{L}, \mathcal{L}_0)$ such that

$$xy \preceq \iota([xy]) + yx, \tag{16}$$

satisfying the $\preceq$-universal Property.

There exists a map $\phi : \mathcal{U}(\mathcal{L}, \mathcal{L}_0) \to (\mathcal{A}, \mathcal{A}_0)$ such that $f = \phi \circ \iota$.

This is a bit subtler than before. Bergman [5] found a beautiful method of proving the PBW Theorem, related to Gröbner bases, to determine bases of algebras, but lacking negation is both a hindrance and an asset, as we shall see.

**Theorem 6.14.** *Suppose $(\mathcal{L}, \mathcal{L}_0)$ is a $\preceq$-Lie pair satisfying LZS, where $\mathcal{L}$ is also a free $C$-module with basis $\{x^i : i \in I\}$, where we order the index set $I$, and $\mathcal{L}_0$ is the submodule with basis $\{x^i : i \in J_0 \subset J\}$. We refine Example 6.3. Define $T(V)_>$ to be the subspace of $T(V)$ spanned by monomials $x_{\mathbf{i}} := x^{i_1} \otimes \cdots \otimes x^{i_m}$, where $i_1 > \cdots > i_m$.*

*Let $V = \mathcal{L}$. We take $W_0 = \{y_{j,i} :=: i < j \in I\}$, and $U_\preceq(\mathcal{L})$ the semialgebra freely generated by $T(V)_>$ and $W_0$, modulo the relations in the congruence generated by the relations $x^j x^i + y_{j,i} = x^i x^j + [x^j x^i]$, for all $j > i$, and $U_\preceq(\mathcal{L})_0$ the multiplicative ideal of $U_\preceq(\mathcal{L}, \mathcal{L}_0)$ generated by $W_0$ and $\mathcal{L}_0$. Then*

1. *$U_\preceq(\mathcal{L}; \mathcal{L}_0) := (U_\preceq(\mathcal{L}), U_\preceq(\mathcal{L}))_0$ defines a $\preceq$-Lie pair, and there is a universal $\preceq$-morphism $\varphi : U_\preceq(\mathcal{L}, \mathcal{L}_0) \to (\tilde{\mathcal{L}}, \tilde{\mathcal{L}}_0)$ given by $x^i \mapsto \bar{x}^i$.*

2. *Furthermore, $\iota\,(\mathcal{L}, \mathcal{L}_0) \to (U_\preceq(\mathcal{L}, \mathcal{L}_0))$ is $\mathcal{L}_0$-injective when $\mathcal{L}$ satisfies LZS.*

*Proof.* First we note that $\iota$ is a $\preceq$-homomorphism. By definition

$$x^i x^j + y_{j,i} = x^j x^i + y_{j,i} \succeq x^j x^i.$$

This extends to the congruence.

To prove that $\iota$ is $\mathcal{L}_0$-injective when $\mathcal{L}$ satisfies LZS, we simply note that all the relations have degree $\geq 2$ in the $x^i$, so they intersect trivially with $\mathcal{L}_0$. ∎

**Remark 6.15.** What can be said when $\mathcal{L}$ does not satisfy LZS? Since the Jordan algebraic version of the PBW fails, we must deal with the ambiguities using the Lie product. Any ambiguity involves rearranging sequences of $x^i$ into ascending sequences. But the parts of $T(V)_>$ match and these are stipulated to be canceled, so we are left with relations in $\mathcal{A}_0$; for example for $i < j < k$ one considers $x^k(x^j x^i)$ versus $(x^k x^j)x^i$, which is resolved by rearranging them and canceling $x^i x^j x^k$ :

$$\begin{aligned} x^k(x^j x^i) &= x^k(x^i x^j + [x^j x^i] + y_{j,i}) = x^i x^k x^j + y_{k,i} x^j + x^k y_{j,i} + x^k [x^j x^i] \\ &= x^i x^j x^k + (x^i y_{k,j} + y_{k,i} x_j + x^k y_{j,i} + x^k [x^j x^i]). \end{aligned} \tag{17}$$

Canceling out $x^j x^i x^k$ yields a relation holding in any classical Lie algebra, so we need some further cancellative property to be in a position to apply the classical PBW theorem.

### 6.3.3 The $\varepsilon$-version of PBW

Suppose that $(\mathcal{L}, \mathcal{L}_0)$ is a $\varepsilon$-Lie pair, with $\varepsilon \in C$.[3] Recall that $[x, y]_\epsilon = xy + \epsilon yx$. Then

$$[x, y]_\epsilon + [y, x]_\epsilon \in T(\mathcal{L}_0) := (1 + \epsilon) T(\mathcal{L}) \tag{18}$$

$$xy + yx(1 + \epsilon) = xy + \epsilon yx + yx = yx + [x, y]_\epsilon \quad \text{for all } x, y \in T(\mathcal{L})$$

[3] We could work more generally with a pre-negation map $\psi$ on $\mathcal{L}$ if we mod $T(\mathcal{L})$ by the congruence generated by $(\psi(x) \otimes y, x \otimes \psi(y))$ for all $x, y \in \mathcal{L}$.

i.e.

$$xy \preceq_0 yx + [x,y]_\epsilon \quad \text{for all } x,y \in T(\mathcal{L}) \tag{19}$$

There is a natural injection $\iota : (\mathcal{L};\mathcal{L}_0) \longrightarrow (T(\mathcal{L}), T(\mathcal{L}_0))$.

Define now $\mathcal{U}_\varepsilon(\mathcal{L})$ to be $T(\mathcal{L})$ modulo the congruence Cong generated by

$$([x_i x_j]_\varepsilon, x_i x_j + \varepsilon x_j x_i)$$

for elements in $\mathcal{L} = T^1(\mathcal{L})$. In other words, if $x_i, x_j \in \mathcal{L}$ then $\iota(x_i)\iota(x_j) + \varepsilon\iota(x_j)\iota(x_i)$ is identified with the $\iota$ image of $\epsilon[x_i x_j] \in \mathcal{L}$ in $T^1(\mathcal{L}) \subseteq T(\mathcal{L})$. Similarly let $\mathcal{U}(\mathcal{L}_0) = (1+\epsilon)\mathcal{U}$.

**Theorem 6.16.**

1. *$\mathcal{U}_\varepsilon(\mathcal{L};\mathcal{L}_0) := (\mathcal{U}_\varepsilon(\mathcal{L}), \mathcal{U}_\varepsilon(\mathcal{L})_0)$ defines a $\varepsilon$-Lie pair, which is strong when $(\mathcal{L};\mathcal{L}_0)$ is a strong $\varepsilon$-Lie pair.*
2. *There is a universal $\varepsilon$-Lie homorphism $\varphi : \mathcal{U}_\varepsilon(\mathcal{L},\mathcal{L}_0) \to (\tilde{\mathcal{L}}, \tilde{\mathcal{L}}_0)$ given by $x^i \mapsto \bar{x}^i$.*
3. *Let $(\mathcal{A},\mathcal{A}_0)$ be any associative $(\mathcal{C};\mathcal{C}_0)$-semiring pair, and let $f : (\mathcal{L},\mathcal{L}_0) \to (\mathcal{A},\mathcal{A}_0)$ be any map such that*

$$f(x)f(y) + \varepsilon f(y)f(x) = f([x,y]). \tag{20}$$

*Then there is a unique homomorphism $\psi_f : (\mathcal{U}_\varepsilon(\mathcal{L}), \mathcal{U}_\varepsilon(\mathcal{L}_0)) \to (\mathcal{A},\mathcal{A}_0)$ such that $f = \psi_f \circ \iota$.*

*Proof.* (1) and (2) are as in the proofs of Theorems 6.13 and 6.14, using Theorem 4.3.

(3) We define the map $T(\mathcal{L}) \to \mathcal{A}$ given by $x_i \mapsto f(x_i)$. By (20) this map factors through Cong, yielding the desired homomorphism $\psi_f : U(\mathcal{L}) \to \mathcal{A}$. Moreover we also have $\psi_f(U(\mathcal{L})) \subseteq (1+\varepsilon)\mathcal{A} = \mathcal{A}_0$. $\square$

**Description of $\mathcal{U}_\varepsilon(\mathcal{L})$.** As a $\mathcal{C}$-module, $\mathcal{U}_\varepsilon(\mathcal{L})$ is spanned by finite linear combinations of monomials $x_1^{i_1} \cdots x_k^{i_k}$, using (20) to reduce whenever possible.

We can obtain surpassing reductions, as in the following example:

**Example 6.17.**

1. Let $x, y, z, \in \mathcal{L}$. Then

$$\begin{aligned} zxy &\preceq (xz + \iota([zx]))y = xzy + \iota([zx])y \\ &\preceq x(yz + (\iota[z,y]) + \iota([zx])y = xyz + x\iota([zx]) + \iota([zx])y \end{aligned}$$

In other words

$$zxy \preceq xyz + x\iota([zx]) + \iota([zx])y.$$

Notice that the right hand side only involves product of brackets in $\mathcal{L}$ and of product of $x$, $y$, $z$ in alphabetical order. We can re-arrange the factors, paying the price of adding elements of lower degree. However, we may have extra terms of degree 1, so we may not have an injection.

**Remark 6.18.** The identity map on $\mathcal{L}$ induces a weak Lie morphism from $U_{\mathrm{weak}}(\mathcal{L}) = T(V)$ to $\mathcal{U}_\varepsilon(\mathcal{L})$, extending the identity on $\mathcal{L}$.

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