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# Nondimensional shape optimisation of non-prismatic beams with sinusoidal lateral profile

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## ABSTRACT

The present paper deals with the optimal design of non-prismatic beams, i.e. beams with variable cross-section. To set the optimisation problem, Euler-Bernoulli unshearable beam theory is considered and the elastica equation expressing the transverse displacement as a function of the applied loads is reformulated into a system of four differential equations involving kinematic components and internal forces. The optimal solution (in terms of volume) must satisfy two constraints: the maximum Von Mises equivalent stress must not exceed an (ideal) strength and the maximum vertical displacement is limited to a fraction of beam length. To evaluate the maximum equivalent stress in the beam, normal and shear stresses have been considered. The former evaluated through Navier formula, the latter through a formula derived from Jourawsky and holding for straight and untwisted beams with bi-symmetric variable cross-sections. The optimal

solutions as function of material unit weight, maximum strength and applied load are presented and discussed. It is shown that the binding constraint is usually represented by the maximum stress in the beam, and that applied load and strength affect the solution more than material unit weight. To maintain the generality of the solution, the nondimensionalisation according to Buckingham  $\Pi$ -theorem is implemented and a design abacus is proposed.

## INTRODUCTION

In the last decades, non-prismatic beams have been widely adopted in the structural engineering field for civil, aerospace, and mechanical applications (El-Mezaini et al. 1991; Ascione et al. 2017; Vilar et al. 2022; Cucuzza et al. 2021; Sardone et al. 2020; Marano and Quaranta 2010; De Biagi et al. 2020; Magnucki et al. 2021). This type of beam is characterised by variable cross-section along its centroidal axis (Gere and Timoshenko 1997), bestowing it a strong interconnection among structural form, functionality, aesthetic and architectural requirements (Mercuri et al. 2020a). These features determined their everlasting success over the centuries, referring e.g. to monumental and historical architectures like Roman aqueducts and masonry arch structures. Non-prismatic elements have been extensively adopted even for infrastructures, e.g. for bridges and viaducts (Kozy and Tunstall 2007; Kaveh et al. 2022; Fiore et al. 2016; Muteb and Shaker 2017; Kaveh et al. 2020b; Zhou et al. 2019; Balduzzi et al. 2017), and buildings, such as double-tapered roof beams for industrial structures (Vilar et al. 2022; Bournas et al. 2014; McKinstry et al. 2016).

When dealing with prismatic beams, the classical Euler-Bernoulli beam theory holds, which neglects the shear deformation contribution and assumes the Navier hypothesis (Carpinteri 2013). Nonetheless, a more advanced theory is required to deal with non-prismatic beams, able to accurately and reliably capture the actual structural response. Therefore, in this research work, the Euler-Bernoulli unshearable beam theory was considered (Bertolini et al. 2019; Timoshenko and Goodier 1934). Recently, in the scientific literature, various mechanical models were proposed for non-prismatic beams. In (Medwadowski 1984), the authors proposed a solution of the differential equations for non-prismatic beams, denoted in that work as shear beams, considering the effect of shear deformations. In (Bulte 1992) the differential equation formulation of the deflec-

51 tion curve was presented as a multi-point boundary value problem. In (Romano 1996) analytical  
52 closed-form solutions were proposed for bending beams accounting for the shear deformation with  
53 non-prismatic parabolic profiles with both varying width and depth. In (Katsikadelis and Tsiatas  
54 2003), the nonlinear large deflection analysis was conducted on the Euler-Bernoulli beam with  
55 variable stiffness with the analog equation method due to variable coefficients in the governing  
56 differential equations. In (Balduzzi et al. 2016), the authors analyse the compatibility and equi-  
57 librium of non-prismatic beams with a Timoshenko-like beam model, formulated as a system of  
58 six coupled ordinary differential equations (ODE). Cazzani et al. (Cazzani et al. 2016) proposed a  
59 Timoshenko beam model and a non-uniform rational B-splines (NURBS) interpolation to analyse  
60 curved beams with the isogeometric analysis (Hughes et al. 2005). In (Bertolini et al. 2019),  
61 the authors analysed the stress distribution in untwisted, straight, thin-walled beams with constant  
62 taper with rectangular and circular cross-section shapes. Most of the analytical approaches for  
63 non-prismatic beams proposed in the literature have been finally solved with the finite differences  
64 methods, even considering non-homogeneous conditions (Al-Azzawi and Emad 2020; Tuominen  
65 and Jaako 1992).

66 The non-prismatic geometry ensures great versatility for optimising specific structural aspects  
67 of interest (Rath et al. 1999; Sarma and Adeli 1998; Colin and MacRae 1984; Mercuri et al. 2020a;  
68 Kaveh et al. 2021), for instance, minimum material consumption, optimising structural perfor-  
69 mances, etc. In addition, the material used, e.g. concrete, steel, or wood (Maki and Kuenzi 1965),  
70 plays a crucial role in the shape and topology optimisation process due to distinct constitutive laws  
71 and possible changing behaviour in tension and compression. Furthermore, nowadays new mate-  
72 rials and technologies such as additive manufacturing are opening new possibilities and promising  
73 research paths (Mercuri et al. 2020a). The problem of optimal design of non-prismatic beams has  
74 been studied quite extensively, implementing both gradient-based (Rao 2019) and gradient-free  
75 meta-heuristic algorithms (Resende et al. 2017; Plevris 2009), such as genetic algorithm (Cucuzza  
76 et al. 2021; Cucuzza et al. 2022; Biswal et al. 2017) or particle swarm optimisation algorithm  
77 (Rosso et al. 2022; Rosso et al. 2021). In (Luévanos-Rojas et al. 2020), the optimal design of

78 reinforced concrete rectangular cross-section beams with straight haunches was analysed with the  
79 aim of reaching the minimum constitutive materials cost. In (Veenendaal et al. 2011), the optimal  
80 form-finding problem has been studied for the design of non-prismatic fabric-formed beams. The  
81 technical difficulties of traditional casting methods for these non-conventional variable curvature  
82 structures are nowadays partially overcome by leveraging innovative production technologies such  
83 as 3D printing and additive manufacturing (Asprone et al. 2018; Mercuri 2018; Costa et al. 2020).  
84 This latter aspect further nourishes the current relevance and contemporary of the present study on  
85 optimal variable-curvature non-prismatic solutions. In (Kaveh et al. 2020a; Kaveh et al. 2020b),  
86 the optimal seismic design of three-dimensional steel frames were carried on with the response  
87 spectrum analysis method. The same authors in a later study (Kaveh et al. 2021) analysed op-  
88 timal performance-based reinforced concrete frames with objective function based on both cost  
89 and sustainability, expressed in terms of carbon dioxide emissions. Similarly, (Yavari et al. 2017)  
90 optimised environmental sustainability of non-prismatic slab frames bridge geometries. Recently,  
91 (Wang et al. 2021) proposed an innovation from a computational point of view for sequentially  
92 solving shape and topology optimization of beam structures, introducing the concept of 2.5D beam  
93 model than traditional 3D modeling. Basically, standard 1D beam elements are interconnected  
94 longitudinally, and, in every finite node, the section properties are retrieved from an additional bidi-  
95 mensional section model. The shape of the beam has been parametrically defined by non-uniform  
96 rational B-splines (NURBS) (Piegl and Tiller 1996).

97 In comparison to the literature previous studies, in the present work, the authors proposed  
98 an optimal design criterion for homogeneous constant width non-prismatic beams based on the  
99 *elastica* equation with a dimensionless perspective, eventually providing a design abacus. The  
100 main findings of the present work are summarised below:

- 101 • the minimum weight, or proportionally the minimum volume, optimisation problem was  
102 stated based on a dimensionless form of the *elastica* equation according to Buckingham  
103  $\Pi$ -theorem;
- 104 • the stress distributions considered Euler-Bernoulli unshearable beam theory (Bertolini et al.

105 2019);

- 106 • the constraints of the optimisation problem are expressed as Von Mises equivalent stress  
107 limitation and maximum limit vertical deflection limited to a fraction of beam length;
- 108 • the optimal solutions as a function of material unit weight, maximum strength, and applied  
109 load are presented in a design abacus graph form.

110 The current document is organised as follows. In Section 2, the analytical formulation of  
111 the *elastica* governing ODEs is presented, even illustrating the dimensionless procedure and the  
112 assumed stress distributions. The minimum volume (weight) optimisation problem statement is  
113 described in Section 3, showing that the non-prismatic variable beam depth profile is defined  
114 through an emptying sinusoidal function. Eventually, in Section 4 the optimal solutions as a  
115 function of material unit weight, maximum strength, and applied load are presented and discussed,  
116 finally delivering a useful design abacus encompassing the wide spectrum of design parameters  
117 analysed.

## 118 **BEAM MODEL**

119 A beam of length  $L$ , straight centerline and a variable cross-section is considered (Figure 1).  
120 A Cartesian coordinate system ( $Oxyz$ ) is introduced, setting: the origin  $O$  in the centroid of one  
121 of the end cross-sections; the  $x$ - and  $y$ -axes as the principal central axes of inertia of the cross-  
122 section; the  $z$ -axis along the beam centerline. We assume plane bending in the  $yz$ -plane, where the  
123 beam is subjected to distributed transverse load  $q(z)$  and its deflection is described by transverse  
124 displacement  $v(z)$ . The constituting material is assumed to be homogeneous, isotropic and linear  
125 elastic with Young's modulus  $E$ .

### 126 ***Elastica* equation**

127 The beam is supposed of solid doubly-symmetric cross-section with variable depth  $h(z)$ . Based  
128 on the Euler-Bernoulli theory, beam deflection is governed by a fourth-order ODE, the *elastica*

129 equation, which reads, for a variable cross-section beam,

$$130 \quad \frac{d}{dz^2} \left[ EJ(z) \frac{d^2 v(z)}{dz^2} \right] = q(z), \quad (1)$$

131 where  $J(z) = J_x(Z)$  is the area moment of inertia of the cross-section. By solving for differentiation  
132 and dividing both members by  $EJ(z)$ , the equation of the deflection curve reads

$$133 \quad \frac{d^4 v(z)}{dz^4} + 2 \frac{d^3 v(z)}{dz^3} \frac{dJ(z)}{dz} \frac{1}{J(z)} + \frac{d^2 v(z)}{dz^2} \frac{d^2 J(z)}{dz^2} \frac{1}{J(z)} = \frac{q(z)}{EJ(z)}. \quad (2)$$

134 To give a more general description of the beam model, Buckingham  $\Pi$ -theorem is adopted (**Baren-**  
135 **blatt 1987**) and a suitable nondimensionalisation is introduced by rescaling lengths by the beam  
136 span  $L$  and forces by  $EL^2$ . Nondimensional variables  $\tilde{z} = z/L$  (with  $\tilde{z} \in [0, 1]$ ),  $\tilde{v} = v/L$  and  
137 functions  $\tilde{J} = J/L^4$  and  $\tilde{q} = q/EL$  are thus defined, while the derivative with respect to dimensional  
138 variable  $z$  is expressed as

$$139 \quad \frac{d}{dz} = \frac{d}{d\tilde{z}} \frac{d\tilde{z}}{dz} = \frac{1}{L} \frac{d}{d\tilde{z}}. \quad (3)$$

140 Accordingly, Equation (2) can be rewritten as

$$141 \quad \tilde{v}^{IV}(\tilde{z}) + 2\tilde{v}'''(\tilde{z}) \frac{\tilde{J}'(\tilde{z})}{\tilde{J}(\tilde{z})} + \tilde{v}''(\tilde{z}) \frac{\tilde{J}''(\tilde{z})}{\tilde{J}(\tilde{z})} = \frac{\tilde{q}(\tilde{z})}{\tilde{J}(\tilde{z})} \quad (4)$$

142 where the notation  $(\cdot)'$  denotes the derivative with respect to nondimensional variable  $\tilde{z}$ .

143 **First-order ODEs**

144 Alternative to the *elastica* equation, Eqn. (1), the shear-bending problem of the variable cross-  
145 section beam can be formulated as a system of four first-order ODEs (Bulte 1992)

$$146 \left\{ \begin{array}{l} \frac{dv}{dz} = -\phi(z), \\ \frac{d\phi}{dz} = \frac{M(z)}{EJ(z)}, \\ \frac{dM}{dz} = V(z), \\ \frac{dV}{dz} = -q(z), \end{array} \right. \quad (5)$$

147 Thanks to the functional  $\mathbf{f}$ , Eqn. (5) can be rewritten in vectorial notation as

$$148 \mathbf{w}'(z) = \mathbf{f}(z, \mathbf{w}), \quad (6)$$

149 where vector  $\mathbf{w}$  has components  $w_1 = v(z)$ ,  $w_2 = \phi(z)$ ,  $w_3 = M(z)$  and  $w_4 = V(z)$ , with  $\phi$  the  
150 rotation of the cross-section (Figure 1). In this way, the variability of the cross-section is taken  
151 implicitly into account only by  $J(z)$  and, due to the fact that all the equation are coupled, this is taken  
152 into account in the entire system avoiding to explicitly solve the fourth order equation depending  
153 by the derivative of the inertia. Moreover, in this way the solutions of the system directly represent  
154 shear, moment, rotation and deflection curves.

155 The distributed load  $q(z)$  includes two contributions: (i) the beam self weight per unit length,  
156 equal to the product of the material unit weight  $\gamma$  by the cross-sectional area  $A(z)$ ; (ii) the applied  
157 force per unit length  $q_0$ , assumed to be constant along the beam. It can thus be expressed as

$$158 q(z) = q_0(z) + \gamma A(z). \quad (7)$$

159 According to Buckingham  $\Pi$ -theorem, it is possible to rewrite the system of Eqn. (5) as

$$160 \left\{ \begin{array}{l} \frac{d\tilde{v}}{d\tilde{z}} = -\phi(\tilde{z}), \\ \frac{d\phi}{d\tilde{z}} = \frac{\tilde{M}(\tilde{z})}{\tilde{J}(\tilde{z})}, \\ \frac{d\tilde{M}}{d\tilde{z}} = \tilde{V}(\tilde{z}), \\ \frac{d\tilde{V}}{d\tilde{z}} = -\tilde{q}(\tilde{z}), \end{array} \right. \quad (8)$$

where

$$\tilde{M}(\tilde{z}) = \frac{M(\tilde{z})}{EL^2}, \quad (9)$$

$$\tilde{V}(\tilde{z}) = \frac{V(\tilde{z})}{EL^3}. \quad (10)$$

161 Accordingly, Eqn. (6), turns into

$$162 \tilde{\mathbf{w}}'(z) = \mathbf{f}(\tilde{z}, \tilde{\mathbf{w}}). \quad (11)$$

163 As previously illustrated, the normalized distributed load  $\tilde{q}(\tilde{z})$  can be divided in two components

164 as

$$165 \tilde{q}(\tilde{z}) = \tilde{\psi}_q(\tilde{z}) + \tilde{\psi}_\gamma \tilde{A}(\tilde{z}), \quad (12)$$

where

$$\tilde{\psi}_q(\tilde{z}) = \frac{q_0(\tilde{z})}{EL}, \quad (13)$$

$$\tilde{\psi}_\gamma = \frac{\gamma L}{E}, \quad (14)$$

166 and  $\tilde{A}(\tilde{z}) = A(\tilde{z})/L^2$ . Considering a constant distributed applied force, Eqn. (12), turns into

$$167 \tilde{q}(\tilde{z}) = \tilde{\psi}_q + \tilde{\psi}_\gamma \tilde{A}(\tilde{z}). \quad (15)$$

## 168 Stress distributions

169 Beams with variable cross-section exhibit non-trivial stress distributions which differ from  
170 those predicted by the classical formulae of prismatic beam theory, in particular regarding shear  
171 stresses (Timoshenko 1956b; Oden 1981; Bruhns 2003). Under the assumption of plane bending,  
172 the beam is in a plane state of stress with  $\sigma_x = \tau_{xy} = \tau_{zx} = 0$ . Transverse normal stress  $\sigma_y$ , although  
173 non vanishing by equilibrium in non-prismatic beams, is generally small and can be neglected  
174 without appreciable error (Balduzzi et al. 2016). Distributions of normal stresses  $\sigma := \sigma_z$  and  
175 shear stresses  $\tau := \tau_{zy}$  acting on the cross-section are given as follows.

176 The distribution of normal stresses  $\sigma$  can be recovered by using the Navier flexure formula

$$177 \sigma(y, z) = \frac{M(z)}{J_x(z)} y \quad (16)$$

178 which holds with a good approximation for non-prismatic beams, provided the variation of the  
179 cross-section is not too rapid (Timoshenko 1956b; Boley 1963).

180 Conversely, the distribution of shear stresses  $\tau$  is considerably altered compared to prismatic  
181 beams. In non-prismatic beams, shear stresses  $\tau$  are dependent not only upon the internal shear force  
182  $V$ , but also upon the internal axial force  $N$  and bending moment  $M$ , as well as on the changing rate  
183 of height and width of the beam (Bruhns 2003). This result follows from the equilibrium boundary  
184 condition on the beam's lateral surface, which requires the shear stress  $\tau$  to be proportional to the  
185 normal stress  $\sigma$  due to the taper angle (Auricchio et al. 2015). Jourawsky's theory (Timoshenko  
186 1956a) is consequently ineffective in predicting the actual shear stress distribution because (i)  
187 it violates the boundary equilibrium, (ii) cannot reproduce the correct distribution shape and  
188 magnitude and (iii) fails to identify the position and value of the maximum shear stress (Bruhns 2003;  
189 Paglietti and Carta 2009; Beltempo et al. 2015; Balduzzi et al. 2017; Mercuri et al. 2020b). In view  
190 of these considerations, we calculate the distribution of shear stresses by using the shear formula  
191 derived by Bertolini *et al.* (Bertolini et al. 2019, Equation 5), an extension of the Jourawsky formula  
192 holding for straight and untwisted beams with bi-symmetric variable cross-sections. Assuming null

193 distributed couples applied to the beam and null internal axial force, the extended shear formula  
 194 simplifies to

$$195 \quad \tau(y, z) = \frac{1}{c(y, z)} \left[ V(z) \frac{S^*(y, z)}{J(z)} + M(z) \frac{d}{dz} \left( \frac{S^*(y, z)}{J(z)} \right) \right], \quad (17)$$

196 where  $c(y, z)$  is the cross-sectional width at the arbitrary level  $y$  where the shear stress  $\tau$  is evaluated;  
 197  $S^*(y, z) := S_x^*(y, z)$  is the first moment of area, with respect to the bending neutral axis  $x$ , of the  
 198 cross-sectional region below the arbitrary level  $y$ . Specifically, for the rectangular cross-section,  
 199 with constant width  $b$  and variable height  $h(z)$ , it holds

$$200 \quad c(y, z) = b, \quad S^*(y, z) = \frac{b}{2} \left( \frac{h^2(z)}{4} - y^2 \right), \quad J(z) = \frac{1}{12} b h^3(z), \quad (18)$$

201 and Eqn. (17) reads

$$202 \quad \tau(y, z) = \frac{3}{2} \frac{1}{bh} \left[ V(z) \left( 1 - 4 \frac{y^2}{h^2} \right) + M(z) \frac{dh}{dz} \left( -\frac{1}{h} + 12 \frac{y^2}{h^3} \right) \right]. \quad (19)$$

## 203 OPTIMISATION PROBLEM

204 The optimisation problem tries to define the minimum volume which determines the minimum  
 205 weight directly linked to the minimum usage of material respecting stress and deflection constraints  
 206 (Cucuzza et al. 2021), which evaluations derive from structural analysis conducted with the sys-  
 207 tem, Eqn. (8). Despite the minimization of the self-weight may not comprehensively cover all  
 208 the numerous aspects for a general minimum cost design problem (Adeli and Sarma 2006), as a  
 209 first approximation, and in the absence of precise requirements and prescription, it may be suc-  
 210 cessfully employed as an indirect indicator of the cost, directly related to the minimum material  
 211 consumption (Rao 2019; Cucuzza et al. 2021; Spillers and MacBain 2009). The minimization of  
 212 self-weight also provides benefits for earthquake design situations (Plevris 2012; Rao 2019), for  
 213 shells design loading (Adriaenssens et al. 2014), and also accounting for transportation and installa-  
 214 tion aspects especially involving precast elements solutions (Veenendaal 2008). In the dimensional

215 problem, the stress constraints are treated in a simplified way adopting Von Mises criterion,

$$216 \quad \sigma^2(z) + 3\tau^2(z) \leq \sigma_{id}^2. \quad (20)$$

217 where the ideal stress  $\sigma_{id}$  is assumed to be the yielding stress for an ideal material (same behaviour  
218 both in tension and in compression). Considering the above Von Mises stress constraint, Eqn. (20),  
219 and the specific forms for normal and shear stresses, Eqns. (16) and (19), respectively, it is possible  
220 to look for a dimensionless form to make consistency with the dimensionless system of Eqn. (8).  
221 In order to obtain a dimensionless stress it is sufficient to divide it by the elastic modulus  $E$ , and  
222 after some mathematical elaborations, it is possible to prove that

$$223 \quad \tilde{\sigma}^2(\tilde{z}) + 3\tilde{\tau}^2(\tilde{z}) \leq \tilde{\psi}_\sigma^2, \quad (21)$$

224 in which a new dimensionless parameter is introduced,  $\tilde{\psi}_\sigma = \sigma_{id}/E$ . It is also possible to express  
225 the deflection constraint in a dimensionless form. Assuming a limit value of  $v_{lim} = L/250$ , the  
226 dimensionless deflection constraint is defined as

$$227 \quad \tilde{v}(\tilde{z}) \leq \frac{1}{250}. \quad (22)$$

228 The above deflection limit value may be retrieved by general deformability requirements un-  
229 der service conditions contained in current structural codes regulations, e.g. the Eurocodes  
230 (EN1990 2002).

231 For evaluating the above-mentioned constraints of the optimisation problem herein investigated,  
232 various structural analyses have been conducted in order to account for possible multiple load cases  
233 conditions (Spillers and MacBain 2009; Cucuzza et al. 2022; Rao 2019). According to the basic  
234 principles of structural design (EN1990 2002), every structure has to be designed and assessed for  
235 the toughest loading conditions likely occurring in its lifespan. Therefore, it implies considering  
236 the envelope of the maximum actions' effects coming from different load combinations. For the

237 sake of simplicity, in the current study, two different load conditions have been considered. The first  
 238 load configuration accounts for the uniformly distributed load, as described in Eqn. (15), applied  
 239 over the entire span length. The second load condition accounts for an asymmetric live load  
 240 applied over the half-span length only. This latter configuration is usually more burdensome than  
 241 the first load case for non-prismatic geometries, especially due to potential instability phenomena  
 242 (Bazzucchi et al. 2017; Virgin et al. 2014). Since we are dealing with beam structures that may be  
 243 employed, at different scales, both for buildings or bridges under uncertain locations of live loads  
 244 (EN1990 2002), the asymmetric load condition must be applied on both the half-spans alternatively  
 245 for accounting all the possible loading cases. In this sense, it should be expected that the optimal  
 246 beam solution will still present a symmetric shape along the longitudinal axis. This optimal  
 247 solution is expected stiffer profile than the one loaded with the first load case only, thus with a  
 248 greater cross-section in general, but able to withstand both symmetric and asymmetric loading  
 249 conditions.

## 250 **Beam geometry definition**

251 The previous constraints are applied to a doubly end-fixed beam having cross-section height that  
 252 varies along the  $z$ -coordinate by way of an *emptying* function  $\eta(z)$ , given as a linear combination  
 253 of sines

$$254 \quad h(z) = h_0 - \eta(z) = h_0 - \sum_{i=1,3,5\dots}^N \Delta h_i \sin\left(i\frac{\pi}{L}z\right), \quad (23)$$

255 where  $h_0$  is the height of the end cross-sections,  $N$  is the number of harmonics combined in the  
 256 emptying function and  $\Delta h_i$  is the amplitude of the  $i$ -th harmonic. A sketch of the beam is reported  
 257 in Figure 2. The structural design principles and the load cases remarks mentioned in the previous  
 258 section justify the authors' choice to focus only on the even sinusoidal harmonics in Eq.(23),  
 259 thus delivering symmetrical beam profiles solutions. In this work, depending on the number of  
 260 harmonics considered, we denote the beam with  $N = 1$  as *one-lobe solution*, the one with  $N = 3$   
 261 as *three-lobes solution*, and so forth. As an example, the height profile of the solution with three

lobes is

$$h(z) = h_0 - \left[ \Delta h_1 \sin\left(\frac{\pi}{L}z\right) + \Delta h_3 \sin\left(3\frac{\pi}{L}z\right) \right]. \quad (24)$$

In general, the volume of the emptied beam with sine emptying functions is equal to

$$V = \int_0^L A(z) dz \quad \text{with} \quad A(z) = f(h(z)), \quad (25)$$

and therefore, the dimensionless volume definition may be expressed as

$$\tilde{V} = \frac{V}{L^3} \quad (26)$$

For instance, detailing the above-mentioned Eqn. (25) for a rectangular bisymmetrical cross-section it holds:

$$V = b \left[ \int_0^L h(z) dz \right] = bL \left[ h_0 - \sum_{i=1}^N \Delta h_i \frac{2}{i\pi} \right]. \quad (27)$$

According to the nondimensionalisation introduced in Section 2, it results

$$\tilde{h}(\tilde{z}) = \tilde{h}_0 - \tilde{\eta}(\tilde{z}) = \tilde{h}_0 - \sum_{i=1}^N \Delta \tilde{h}_i \sin(i\pi\tilde{z}) \quad (28)$$

and

$$\tilde{V} = \frac{V}{L^3} = \tilde{b} \left[ \tilde{h}_0 - \sum_{i=1}^N \Delta \tilde{h}_i \frac{2}{i\pi} \right]. \quad (29)$$

### Design vector and problem statement

Considering that the height of the beam must always be a positive number, i.e.  $\tilde{h}(\tilde{z}) > 0$ , we defined the dimensionless height of the end cross-section  $\tilde{h}_0$  as the sum of a minimum height  $\tilde{h}_{min}$ , to be strictly positive, and the maximum emptying function, resulting in

$$\tilde{h}_0 = \tilde{h}_{min} + \max_{\tilde{z} \in [0,1]} \tilde{\eta}(\tilde{z}) \quad (30)$$

The design vector  $\mathbf{D}$  collects the dimensionless values of the minimum height and the amplitudes

281 coefficients of the sine function  $\Delta\tilde{h}_i$ . The optimization problem can be formulated in the following  
 282 way

$$\begin{aligned}
 &\text{Find } \mathbf{D} = \{\tilde{h}_{min}; \Delta\tilde{h}_i\}_{i=1,3,5\dots} \text{ such that} \\
 &\min \tilde{V}(\mathbf{D}) \\
 &\text{s.t. } \tilde{\sigma}^2(\tilde{z}) + 3\tilde{\tau}^2(\tilde{z}) \leq \tilde{\psi}_\sigma^2, \\
 &\quad \tilde{v}(\tilde{z}) \leq \frac{1}{250}
 \end{aligned}
 \tag{31}$$

284 Thanks to the procedure previously described and implemented in Matlab, the optimal geom-  
 285 etry of beams with different combinations of parameters  $\psi_q$ ,  $\psi_\gamma$  and  $\psi_\sigma$  was investigated. The  
 286 dimensionless form allows covering all the possible situations for the specific problem parameters  
 287 values such as the span length, geometric and material properties included in the aforementioned  
 288 parameters.

289 For the sake of better controlling the optimization process, limiting the mathematical topology  
 290 complexity of the search space, and in order to avoid an excessive over-parametrization of the  
 291 beam's shape longitudinal profile, the authors studied the optimization process using the number  
 292 of sine-emptying lobes as a fixed parameter rather than a design variable. Specifically, the authors  
 293 provided a detailed comparison and discussion of four different structural configurations, i.e. from  
 294 one-lobe to seven-lobes. For a number of lobes greater than seven-lobes, the authors observed that  
 295 the influence of higher lobes was practically negligible compared to the increase of the beam's  
 296 shape profile complexity.

## 297 **RESULTS AND DISCUSSION**

298 In this section, the results of optimization analyses for a beam with rectangular cross-section  
 299 are presented. The structural analyses have been conducted under two different load conditions.  
 300 The first load configuration accounts for the uniformly distributed load, as described in Eqn. (15),  
 301 applied over the entire span length. The second load condition accounts for an asymmetric live load  
 302 applied over the half-span length only. This latter configuration is usually more burdensome than  
 303 the first load case for non-prismatic geometries, especially due to potential instability phenomena

(Bazzucchi et al. 2017; Virgin et al. 2014). Furthermore, in the current study, point loads have not been explicitly considered since they are properly representative of specific design situations, see e.g. (Yang et al. 2022). Nonetheless, the current methodology may account for point loads as well by implementing equivalent distributed loads over a short finite length, simulating its actual load footprint.

Several analyses were carried to determine the influence of the number of lobes on the optimal beam solution and to highlight the effects of the maximum allowable stress level and material unit weight. A design abacus is proposed to summarise the results. The optimisation problem was implemented in a Matlab code and solved with the *fmincon* function provided within the *Optimization Toolbox* package (MATLAB Optimization Toolbox). The input parameters of the *fmincon* function are the objective function defined in Eqn. (29) and the non-linear constraints defined in Eqs. (21)-(22), both summarized in the optimisation problem statement in Eqn. (31). The solver algorithm option has been set to the well-acknowledged and efficient nonlinear programming method named sequential quadratic programming (SQP) (Schittkowski 1986). This gradient-based iterative method is based on quasi-Newton approximation of the Hessian of the Lagrangian function for constrained optimization problems (Rao 2019), which translates in the resolution of quadratic programming subproblems forming an active set strategy for a line search procedure (Biggs 1975; Han 1977; Powell 2006; Powell 1978). Since the current implementation requires strict feasibility with respect to constraints, it implements an automatic adaptation of the finite difference gradient step along the line search, and due to the quasi-Newton approximation of the Hessian, any second-order eigenvalue sensitivity is not strictly necessary (Li et al. 2016).

### **Influence of the number of lobes**

Figure 3 shows the optimal solutions (in grey) considering the material and geometric properties reported in Table 1. Four different configurations were compared, from one-lobe to seven-lobes, i.e. considering  $N = 1, 3, 5, 7$ . For each case, the components of vector  $\tilde{\mathbf{w}}$ , i.e. nondimensional displacement  $\tilde{v}$ , rotation  $\phi$ , bending moment  $\tilde{M}$  and shear  $\tilde{V}$  are reported in the top subplots. The displacement plot (top left) includes a horizontal red line denoting the limit value, i.e.  $1/250$ . The

331 bottom plot refers to the maximum Von Mises stress along the beam and includes (in red) the  
332 threshold  $\psi_\sigma$ .

333 For all the examined cases, the maximum stress in the beam represents the most strict (binding)  
334 constraint. Table 2 reports the values of the components of the design vector  $\mathbf{D}$ . Comparing the  
335 various solutions, it results that increasing the number of lobes reduces the volume of the optimal  
336 beam. The presence of two parts with limited height, which emerges for  $N = 3$  and is further  
337 highlighted for  $N = 5, 7$ , implies larger rotations and, by consequence, increased displacements.  
338 The number of lobes in the solution affects Von Mises equivalent stress. For  $N = 1$ , the maximum  
339 stresses are observed at beam ends and at midspan, where heights  $h_0$  and  $h_{min}$  can be optimised.  
340 A different trend is noted for  $N = 3$ , where the maximum stress occurs at 1/6 and 5/6 of beam  
341 length, roughly. Considering the area below the stress curve as an ideal measure of the material  
342 exploitation rate, it results that the best use is obtained when the stress level tends to the threshold  
343 value in any section of the beam. Comparing the solutions with different number of lobes, it clearly  
344 emerges that the larger the number of lobes, the better the exploitation rate. Five- and seven-lobes  
345 solutions produce similar maximum vertical displacements, but different material exploitation, in  
346 particular in the first and last sixth of the beam. In detail,  $N = 7$  solution exhibits a stress plateau  
347 in the first and last part of the beam. The similarity in five- and seven-lobes solutions emerges in  
348 analysing the components of the design vector reported in Table 2. Checking the  $\tilde{V}$  column, i.e.  
349 the values of the objective function, it results that the reduction in the optimal volume (target of the  
350 optimisation) is more evident up to  $N = 5$ , while for  $N = 7$ , the resulting  $\tilde{V}$  is close to the five-lobes  
351 solution. As a conclusion, three-lobes and five-lobes represent feasible solutions for fixed-fixed  
352 beams with uniformly distributed load.

353 For the sake of completeness, other boundary conditions should be analysed in future studies  
354 since the herein-presented double fixed condition is mainly representative of concrete structures.  
355 Indeed, the authors preliminary tested the current optimization procedure considering other beam  
356 boundary conditions, in particular the double-hinged one. However, the obtained optimal results ap-  
357 pear not relevant for the scope of the current study, and they have not been herein reported. Nonethe-

358 less, it is worth reminding that for other structural materials, such as steel or timber, the semi-rigid  
359 restraints condition is the actual one. A proper embedding of these aspects is out of the scope of the  
360 current manuscript and may require future deeper investigations. Indeed, special attention should  
361 be paid to the specific technical choice adopted for the restrain joints, which affects their rotational  
362 stiffness capacity on the moment rotation plane (Daniūnas and Urbonas 2008; Du et al. 2022).

### 363 **Influence of the maximum stress**

364 Three-lobes solution was adopted for assessing the effect of maximum stress on the optimal beam  
365 height profile. The optimisation problem of Eqn. (31) was solved considering geometric and load  
366 parameters reported in Table 3. To highlight the dependency of the optimal solution on the value of  
367 the stress parameter  $\psi_\sigma$ , three different values were considered, i.e.  $\psi_\sigma = 3.33 \times 10^{-4}$ ,  $6.66 \times 10^{-4}$ ,  
368 and  $1 \times 10^{-3}$ . These correspond to ideal stresses  $\sigma_{id}$  of 10, 20 and 30 MPa. Figure 4 shows the  
369 optimal solutions for the three stress levels and Table 4 details the amplitudes of the sine functions  
370 and the value of the objective function. Comparing stress and displacement curves of the three  
371 solutions it emerges that different trends emerge. For low stress levels, say  $\psi_\sigma = 3.33 \times 10^{-4}$ , the  
372 relevant constraint for the optimal solution is represented by the maximum allowable stress itself.  
373 For high stresses,  $\psi_\sigma = 1 \times 10^{-3}$ , the maximum displacement is the binding term. For medium  
374 stresses,  $\psi_\sigma = 6.66 \times 10^{-4}$ , both constraints are relevant for the optimal solution.

375 As a matter of evidence, the optimal solution would benefit in terms of volume of material if the  
376 maximum allowable stress level increases. To measure such benefit, Table 4 reports in the values  
377 of the nondimensional volume  $\tilde{V}$ . The change of  $\psi_\sigma$  affects the value of the objective function in  
378 a nonlinear manner, with no direct relationship between the value of  $\psi_\sigma$  and  $\tilde{V}$ . To address such  
379 issue, a parametric analysis was performed to highlight the specific binding constraint and study  
380 the value of the objective function. Figure 5 details the results in term of  $\tilde{V}$  (contour lines) and  
381 relevant constraint in the optimisation (coloured bullets) for  $N = 3$ ,  $\psi_\gamma = 8.33 \times 10^{-6}$  and  $\tilde{b} = 0.05$ .  
382 The load parameter  $\psi_q$  varies in the range from  $3.33 \times 10^{-8}$  to  $1.67 \times 10^{-7}$  that corresponds to a  
383 distributed load between 10 and 100 kN/m (the remaining variables are those reported in Table 3).  
384 The stress parameter  $\psi_\sigma$  varies in the range from  $3.33 \times 10^{-4}$  to  $1 \times 10^{-3}$ . It is shown that, for the

385 larges part of the investigated cases, the binding constraint is represented by the maximum stress  
 386 in the beam (similarly to what shown in Figure 4.a). For large maximum stresses, the relevant  
 387 condition is the maximum displacement, highlighted with blue bullets. The transition between the  
 388 two limit conditions depends on the value of the distributed load, in particular for  $\psi_\sigma > 6 \times 10^{-4}$ .  
 389 Observing the trends of  $\tilde{V}$  in the black contour plot, it is seen that for high  $\psi_\sigma$ , the optimal volume  
 390 depends on the external load, only, as the beam shape is constrained by the maximum displacement.  
 391 For high values of  $\psi_q$ , the maximum stress controls the optimal volume.

### 392 **Influence of material unit weight**

393 To study the influence of the unit weight of the material constituting the beam, three scenarios  
 394 were considered and compared. The solution reported in Figure 4.c obtained for  $\psi_\gamma = 8.33 \times$   
 395  $10^{-6}$ ,  $\psi_q = 3.33 \times 10^{-8}$  and  $\psi_\sigma = 1 \times 10^{-3}$  is considered as reference for the analysis. Two  
 396 additional cases were considered, keeping fixed all the parameters except  $\psi_\gamma$  which is halved and  
 397 doubled. The results of the optimisation are reported in Table 5. It is found that the modification of  
 398  $\psi_\gamma$  affects in a limited way the values of the amplitudes of the optimal solution, nor the volume of the  
 399 beam, that is, its weight. To understand the reason of such trend, it is necessary to consider the total  
 400 weight of the beam, namely  $G$ , computed as  $G = \gamma V$ , which can be expressed in nondimensional  
 401 form as

$$402 \quad \tilde{G} = \psi_\gamma \tilde{V}. \quad (32)$$

403 The total applied load  $Q$  is computed as  $Q = q_0 L$ , which can be further expressed as

$$404 \quad \tilde{Q} = \psi_q. \quad (33)$$

405 For all the analysed cases, the result of Eqn. (32) (reported in the seventh column of Table 5), is  
 406 smaller than  $\psi_q$  ( $3.33 \times 10^{-8}$ ), showing that the dead load is not relevant in the solution.

### 407 **Design abacus**

408 The analyses performed highlighted that three- and five-lobes solutions provide good results for  
 409 the minization of the objective function. Considering the parameters describing material weight,

410 load and maximum allowable stress, it was shown that  $\psi_q$  and  $\psi_\sigma$  play a relevant role in the optimal  
411 solution, while the parameter associated to the unit weight  $\psi_\gamma$  has a secondary importance since  
412 it slightly affects the optimal design. To let the solution the more general as possible, various  
413 combination of real construction materials, geometries and loads were considered. A summary of  
414 these is reported in Table 6; among all the cases,  $\psi_\gamma$  varies between  $3.71 \times 10^{-7}$  and  $8.33 \times 10^{-6}$ ,  $\psi_\sigma$   
415 varies between  $1.00 \times 10^{-3}$  and  $3.75 \times 10^{-3}$ , and  $\psi_q$  varies between  $9.52 \times 10^{-10}$  and  $6.25 \times 10^{-6}$ .  
416 The design abacus, which would serve for defining the optimal height profile of the beam, was  
417 formulated for a fixed value of  $\psi_\gamma = 1 \times 10^{-6}$ ,  $\psi_\sigma$  in the range  $0.5 \times 10^{-3}$  to  $1.5 \times 10^{-3}$  (3 values) and  
418  $\psi_q$  in the range  $4 \times 10^{-8}$  to  $4 \times 10^{-6}$  (in 7 logarithmically equally spaced values), trying to represent  
419 the possible materials, beam lengths and loads configurations. The nondimensional beam width is  
420  $\tilde{b} = 0.05$ .

421 Table 7 reports the optimal values of the design vector and the corresponding  $\tilde{V}$ . It results that  
422  $\tilde{V}$  is in the range 0.001 to 0.0085, roughly. In general, the values of  $h_{min}$  and the absolute values  
423 of the amplitudes of the sine function  $\Delta\tilde{h}_1$ ,  $\Delta\tilde{h}_3$  and  $\Delta\tilde{h}_5$  increase for increasing  $\psi_q$ . Besides, the  
424 increase of  $\psi_\sigma$  causes a reduction of the terms. These trends reflect the findings of the specific  
425 studies reported in the previous sections.

426 Figure 6 shows the height profiles of the optimal beams. The scale in Y-axes are kept constant  
427 in all the plots for a better understanding of the effects of the parameters on the optimal solution.  
428 The results of Table 7 can be used for the design of real beams: the design values, i.e., the minimum  
429 height and the sine amplitudes can be determined by interpolation for a given  $\psi_\sigma$  and  $\psi_q$ .

## 430 CONCLUSIONS

431 The present paper deals with the optimal design of beams with variable cross-section. To this  
432 aim, Euler-Bernoulli beam theory has been adopted. The fourth order elastica equation has been  
433 rewritten according to the formulation proposed by Bulte as a system of four differential equations.  
434 According to Buckingham II-theorem, a nondimensionalisation has been done to let the solution  
435 as general as possible. The loads acting of the beam are the self weight and a distributed line  
436 load. The minimum volume (weight) solution must satisfy two constraints: the maximum Von

437 Mises equivalent stress must not exceed an (ideal) strength and the maximum vertical displacement  
438 is limited to a fraction (1/250) of beam length. To evaluate the maximum equivalent stress  
439 in the beam, normal and shear stresses have been considered. The former evaluated through  
440 Navier formula, the latter through a formula derived from Jourawsky and holding for straight and  
441 untwisted beams with bi-symmetric variable cross-sections. The optimisation problem has focused  
442 on a beam with fixed-fixed ends subjected to a uniformly distributed load. To create the variable  
443 height profile, an emptying function resulting as a combination of sine functions with different  
444 amplitudes has been introduced. For the sake of completeness, other boundary conditions should  
445 be analysed in future studies. The double fixed condition is mainly representative of concrete  
446 structures, whereas for e.g. steel or timber structures, the semi-rigid condition is the actual one.  
447 However, considering these aspects may require future investigations accounting for the specific  
448 technical choice adopted for the restrain joints, thus affecting their rotational stiffness capacity  
449 (Daniūnas and Urbonas 2008; Du et al. 2022).

450 The parametric analyses showed that:

- 451 • the choice of the number of sines in the emptying function that describes the shape of the  
452 beam, is relevant up to  $N = 5$ , i.e., a five-lobes solution. For finer solution, for examples,  
453 seven-lobes solution, there is not an improvement in the optimal solution in terms of  
454 minimum weight;
- 455 • the maximum stress in the material influences the binding constraint. In general, it has been  
456 noted that the stress constraint is relevant for the optimal solution for the large majority  
457 of cases. The displacement constraint affects the solution for low external loads and high  
458 strength;
- 459 • material unit weight does not affect the optimal solution as the total weight of the beam is  
460 smaller than the total applied load. For this reason, the variability of the material can be  
461 avoided in a preliminary design of a beam.

462 A design abacus with a profiles plot encompassing the wide spectrum of design parameters has

463 been proposed to help in the design of an optimal five-lobes solution. The findings of the present  
464 paper would serve for the design of beams optimised with respect to weight. It should be em-  
465 phasized that the current optimization problem statement in Eqn. (31) may be further refined,  
466 e.g. peculiarly referring to more detailed constraints derived from actual structural codes based  
467 on the specific constitutive materials adopted (NTC 2018; EN1990 2002). Furthermore, tradi-  
468 tional casting methods for concrete non-prismatic beams with variable curvature profiles are still  
469 challenging (Veenendaal et al. 2011), especially for rebar placing operations, and often lead to  
470 more expensive solutions than classical alternatives. Nevertheless, in the novel panorama of ad-  
471 ditive manufacturing and 3D printing (Costa et al. 2020; Mercuri 2018), the herein-studied struc-  
472 tural solution is already becoming more feasible, revolutionizing the current construction indus-  
473 try. Therefore, future research efforts will concern the numerous aspects related to promising  
474 3D printing casting solutions, e.g. involving innovative and printing-technologically compat-  
475 ible materials, life cycle assessment, and non-prismatic beams industrialization among others  
476 (Costa et al. 2020; Gregori et al. 2019; Fiore et al. 2014; Asprone et al. 2018).

#### 477 **Data Availability Statement**

478 All data, models, or code that support the findings of this study are available from the corre-  
479 sponding author upon reasonable request.

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688  
689  
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**List of Tables**

1 Material and geometric properties for the analysis related to the number of lobes. . . 31

2 Optimal design values related to the cases of Figure 3. . . . . 32

3 Material and geometric properties for the analysis related to the effect of maximum  
material stress. . . . . 33

4 Optimal design values related to the cases of Figure 4. . . . . 34

5 Optimal design values related to the cases of Figure 4. . . . . 35

6 Combination of structural and load configurations to be considered for determining  
the range of parameters of the design abaci. . . . . 36

7 Combination of structural and load configurations to be considered for determining  
the range of parameters of the design abacus. . . . . 37

**TABLE 1.** Material and geometric properties for the analysis related to the number of lobes.

$\gamma$	25 kN/m <sup>3</sup>
$E$	30 GPa
$L$	10 m
$q_0$	20 kN/m
$\sigma_{id}$	20 MPa
$\tilde{b}$	0.05
$\psi_\gamma$	$8.33 \times 10^{-6}$
$\psi_q$	$6.66 \times 10^{-8}$
$\psi_\sigma$	$6.66 \times 10^{-4}$

**TABLE 2.** Optimal design values related to the cases of Figure 3.

N	$\tilde{h}_{min}$	$\Delta\tilde{h}_1$	$\Delta\tilde{h}_3$	$\Delta\tilde{h}_5$	$\Delta\tilde{h}_7$	$\tilde{V}$
1	0.0151	0.0231				$1.17 \times 10^{-3}$
3	0.0157	0.0301	0.0055			$1.07 \times 10^{-3}$
5	0.0119	0.0262	0.0040	-0.0026		$9.93 \times 10^{-4}$
7	0.0122	0.0262	0.0040	-0.0026	0.00028	$9.95 \times 10^{-4}$

**TABLE 3.** Material and geometric properties for the analysis related to the effect of maximum material stress.

$\gamma$	25 kN/m <sup>3</sup>
$E$	30 GPa
$L$	10 m
$q_0$	10 kN/m
$\tilde{b}$	0.05
$\psi_\gamma$	$8.33 \times 10^{-6}$
$\psi_q$	$3.33 \times 10^{-8}$

**TABLE 4.** Optimal design values related to the cases of Figure 4.

N	$\psi_\sigma$	$\tilde{h}_{min}$	$\Delta\tilde{h}_1$	$\Delta\tilde{h}_3$	$\tilde{V}$
3	$3.33 \times 10^{-4}$	0.0165	0.0317	0.0058	$1.13 \times 10^{-3}$
3	$6.66 \times 10^{-4}$	0.0112	0.0233	0.0062	$8.01 \times 10^{-4}$
3	$1 \times 10^{-3}$	0.0099	0.0229	0.0087	$7.95 \times 10^{-4}$

**TABLE 5.** Optimal design values related to the cases of Figure 4.

N	$\psi_\gamma$	$\tilde{h}_{min}$	$\Delta\tilde{h}_1$	$\Delta\tilde{h}_3$	$\tilde{V}$	$\psi_\gamma\tilde{V}$
3	$4.17 \times 10^{-6}$	0.0096	0.0217	0.0086	$7.73 \times 10^{-4}$	$3.22 \times 10^{-9}$
3	$8.33 \times 10^{-6}$	0.0099	0.0229	0.0087	$7.95 \times 10^{-4}$	$6.62 \times 10^{-9}$
3	$1.66 \times 10^{-5}$	0.0106	0.0257	0.0088	$8.37 \times 10^{-4}$	$1.39 \times 10^{-8}$

**TABLE 6.** Combination of structural and load configurations to be considered for determining the range of parameters of the design abaci.

$E$ GPa	$\gamma$ kN/m <sup>3</sup>	$\sigma_{id}$ MPa	$L$ m	$q_0$ kN/m	$\psi_\gamma$	$\psi_\sigma$	$\psi_q$
<b>Concrete</b>							
30	25	30	1	2	$8.33 \times 10^{-7}$	$1.00 \times 10^{-3}$	$6.67 \times 10^{-8}$
30	25	30	10	2	$8.33 \times 10^{-6}$	$1.00 \times 10^{-3}$	$6.67 \times 10^{-9}$
30	25	30	1	50	$8.33 \times 10^{-7}$	$1.00 \times 10^{-3}$	$1.67 \times 10^{-6}$
30	25	30	10	50	$8.33 \times 10^{-6}$	$1.00 \times 10^{-3}$	$1.67 \times 10^{-7}$
<b>Timber</b>							
8	5	30	1	2	$5.63 \times 10^{-7}$	$3.75 \times 10^{-3}$	$2.50 \times 10^{-7}$
8	5	30	10	2	$5.63 \times 10^{-6}$	$3.75 \times 10^{-3}$	$2.50 \times 10^{-8}$
8	5	30	1	50	$5.63 \times 10^{-7}$	$3.75 \times 10^{-3}$	$6.25 \times 10^{-6}$
8	5	30	10	50	$5.63 \times 10^{-6}$	$3.75 \times 10^{-3}$	$6.25 \times 10^{-7}$
<b>Alluminium</b>							
69	27	100	1	2	$3.91 \times 10^{-7}$	$1.46 \times 10^{-3}$	$2.92 \times 10^{-8}$
69	27	100	10	2	$3.91 \times 10^{-6}$	$1.46 \times 10^{-3}$	$2.92 \times 10^{-9}$
69	27	100	1	50	$3.91 \times 10^{-7}$	$1.46 \times 10^{-3}$	$7.30 \times 10^{-7}$
69	27	100	10	50	$3.91 \times 10^{-6}$	$1.46 \times 10^{-3}$	$7.30 \times 10^{-8}$
<b>Steel</b>							
210	78	250	1	2	$3.71 \times 10^{-7}$	$1.19 \times 10^{-3}$	$9.52 \times 10^{-9}$
210	78	250	10	2	$3.71 \times 10^{-6}$	$1.19 \times 10^{-3}$	$9.52 \times 10^{-10}$
210	78	250	1	50	$3.71 \times 10^{-7}$	$1.19 \times 10^{-3}$	$2.38 \times 10^{-7}$
210	78	250	10	50	$3.71 \times 10^{-6}$	$1.19 \times 10^{-3}$	$2.38 \times 10^{-8}$

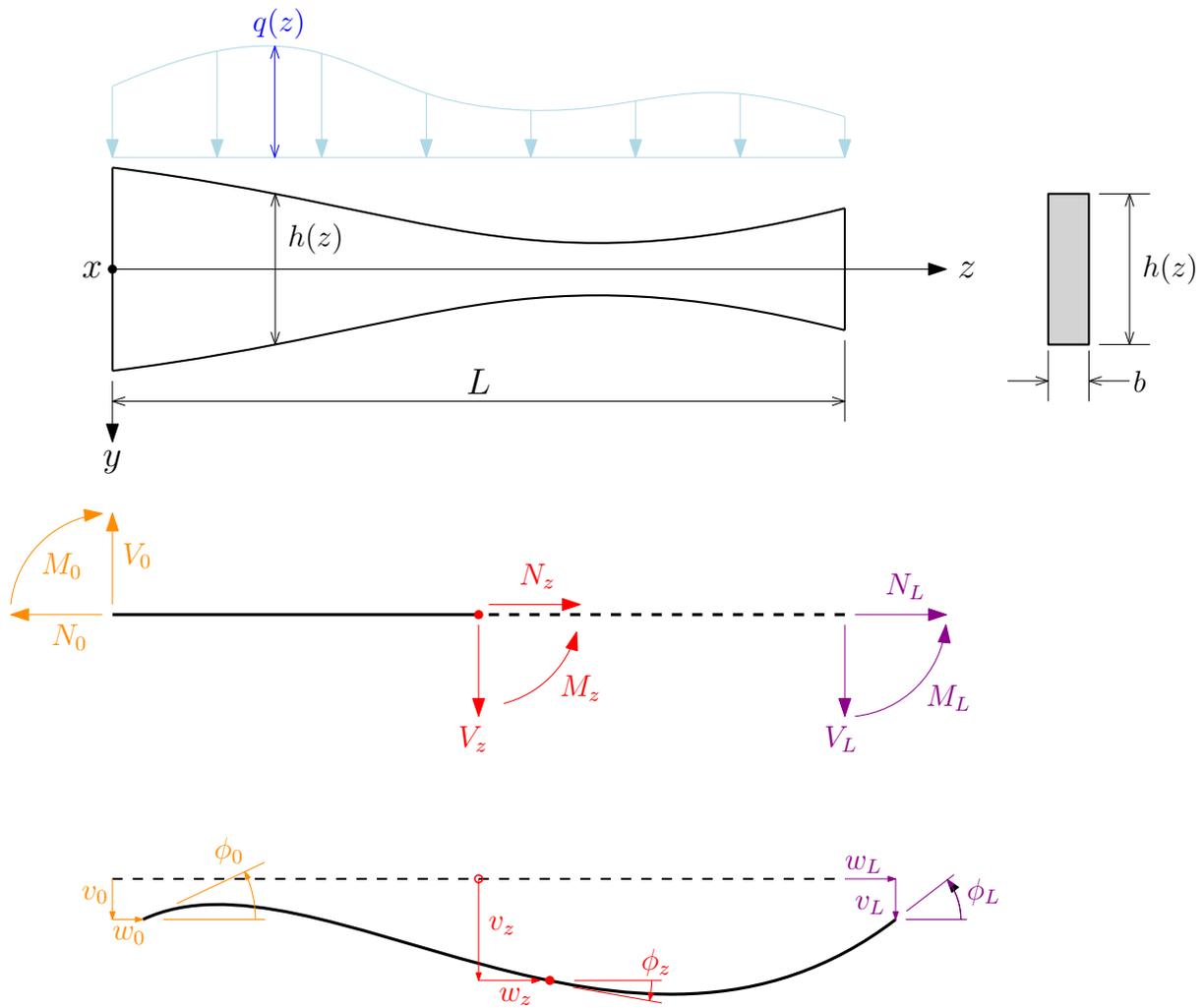
**TABLE 7.** Combination of structural and load configurations to be considered for determining the range of parameters of the design abacus.

$\psi_\sigma$	$\psi_q$	$h_{min}$	$\Delta\tilde{h}_1$	$\Delta\tilde{h}_3$	$\Delta\tilde{h}_5$	$\tilde{V}$
$0.5 \times 10^{-3}$	$4.0 \times 10^{-8}$	0.0101	0.0223	0.0037	-0.0025	0.000858
	$8.6 \times 10^{-8}$	0.0129	0.0341	0.0022	-0.0056	0.001290
	$1.8 \times 10^{-7}$	0.0221	0.0482	0.0073	-0.0048	0.001838
	$4.0 \times 10^{-7}$	0.0325	0.0705	0.0107	-0.0071	0.002695
	$8.6 \times 10^{-7}$	0.0478	0.1032	0.0158	-0.0103	0.003955
	$1.8 \times 10^{-6}$	0.0708	0.1508	0.0234	-0.0148	0.005810
	$4.0 \times 10^{-6}$	0.1057	0.2192	0.0351	-0.0209	0.008551
$1.0 \times 10^{-3}$	$4.0 \times 10^{-8}$	0.0077	0.0163	0.0044	-0.0041	0.000781
	$8.6 \times 10^{-8}$	0.0099	0.0209	0.0057	-0.0053	0.001006
	$1.8 \times 10^{-7}$	0.0152	0.0321	0.0070	-0.0042	0.001311
	$4.0 \times 10^{-7}$	0.0229	0.0499	0.0075	-0.0050	0.001904
	$8.6 \times 10^{-7}$	0.0337	0.0731	0.0111	-0.0073	0.002794
	$1.8 \times 10^{-6}$	0.0496	0.1070	0.0164	-0.0107	0.004102
	$4.0 \times 10^{-6}$	0.0750	0.1567	0.0233	-0.0140	0.006033
$1.5 \times 10^{-3}$	$4.0 \times 10^{-8}$	0.0077	0.0163	0.0044	-0.0041	0.000781
	$8.6 \times 10^{-8}$	0.0099	0.0209	0.0057	-0.0053	0.001006
	$1.8 \times 10^{-7}$	0.0128	0.0269	0.0074	-0.0069	0.001298
	$4.0 \times 10^{-7}$	0.0167	0.0350	0.0095	-0.0086	0.001675
	$8.6 \times 10^{-7}$	0.0275	0.0597	0.0090	-0.0060	0.002280
	$1.8 \times 10^{-6}$	0.0404	0.0875	0.0133	-0.0088	0.003347
	$4.0 \times 10^{-6}$	0.0597	0.1279	0.0197	-0.0127	0.004917

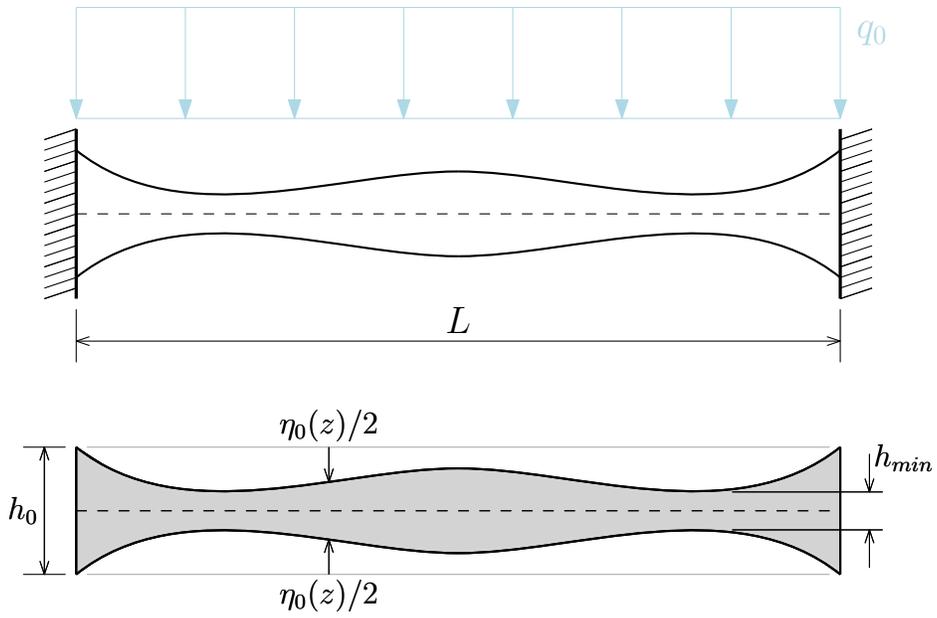
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## List of Figures

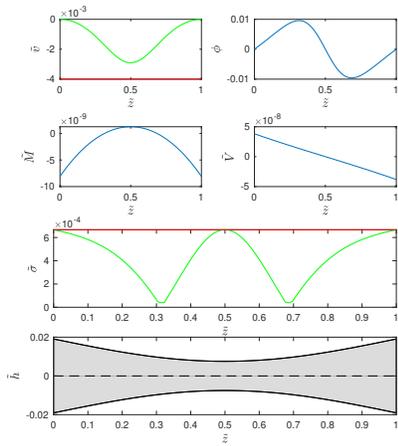
- 1 Beam with variable cross-section: coordinate system, load, displacements, internal forces. The displacement field is denoted with components  $v, w, \phi$ . The internal forces are  $N$  (axial),  $V$  (shear) and  $M$  (bending moment). . . . . 39
- 2 Beam with variable cross-section generated with the emptying function of Eqn. (23). 40
- 3 Comparison of optimal beam solutions with variable cross-section considering different number of lobes. The parameters related to the weight per unit mass, the load and the maximum allowable stress are reported in Table 2. . . . . 41
- 4 Comparison of optimal beam solutions of three-lobes variable cross-section considering different maximum stress levels, i.e. the value of parameter  $\psi_\sigma$ . The parameters related to the weight per unit mass, the load and the maximum allowable stress, as well as the optimal solution are reported in Table 4. . . . . 42
- 5 Value of the dimensionless volume of the beam  $\tilde{V}$  and binding solution constraints for different  $\psi_\sigma$  and  $\psi_q$ . The bullets indicate whether the relevant solution constraint is the maximum stress (green), Eqn. (21), the maximum displacement (blue), Eqn. (22), or both (red). . . . . 43
- 6 Beam height profiles of the beams for various  $\psi_\sigma$  and  $\psi_q$ . The values of the design vector are reported in Table 7. . . . . 44



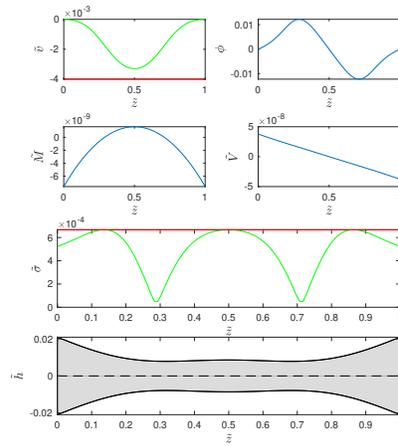
**Fig. 1.** Beam with variable cross-section: coordinate system, load, displacements, internal forces. The displacement field is denoted with components  $v, w, \phi$ . The internal forces are  $N$  (axial),  $V$  (shear) and  $M$  (bending moment).



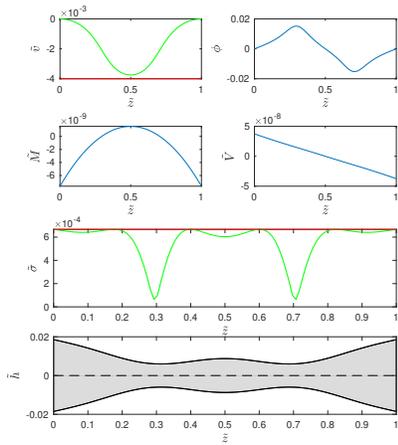
**Fig. 2.** Beam with variable cross-section generated with the emptying function of Eqn. (23).



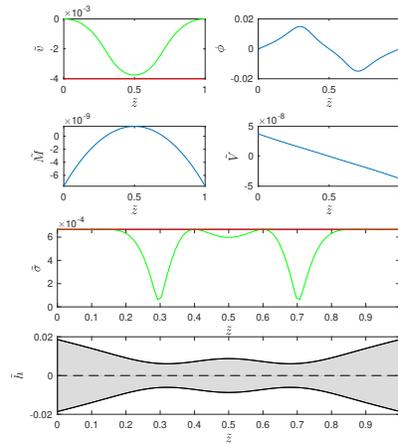
(a) One-lobe solution,  $N = 1$



(b) Three-lobes solution,  $N = 3$

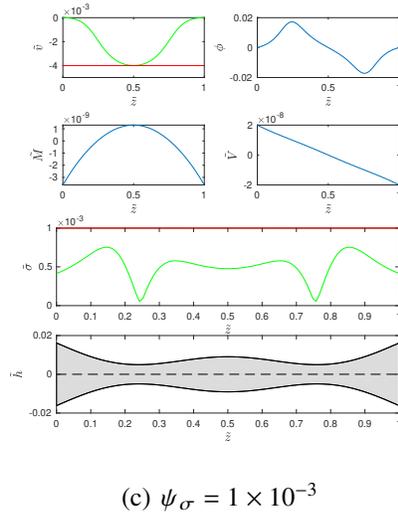
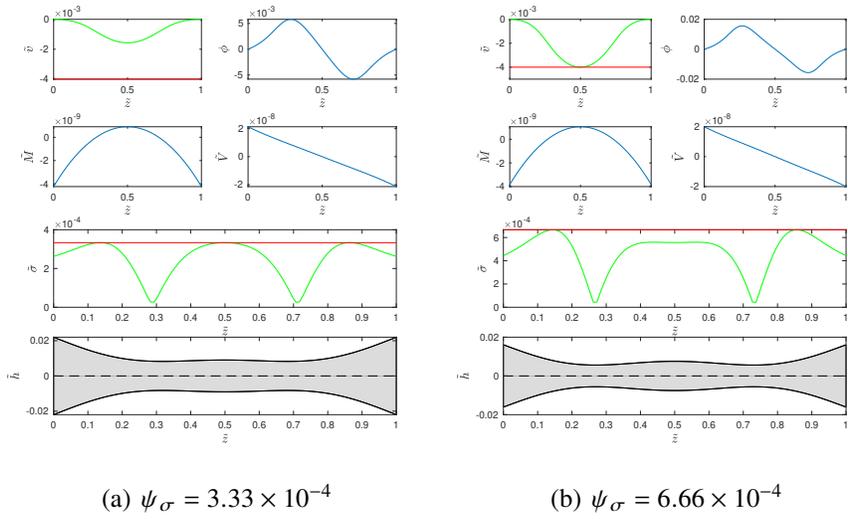


(c) Five-lobes solution,  $N = 5$

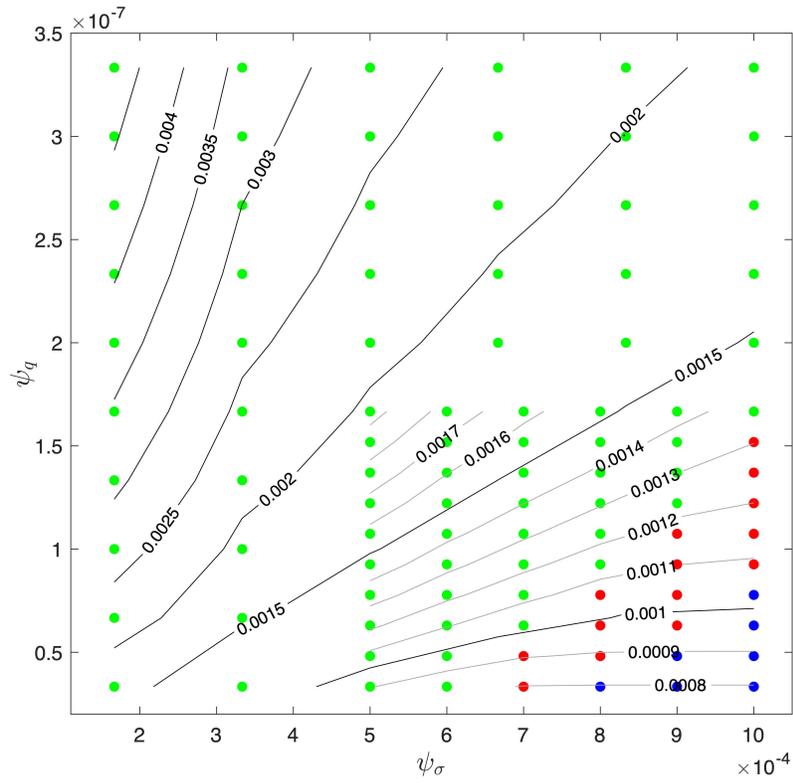


(d) Seven-lobes solution,  $N = 7$

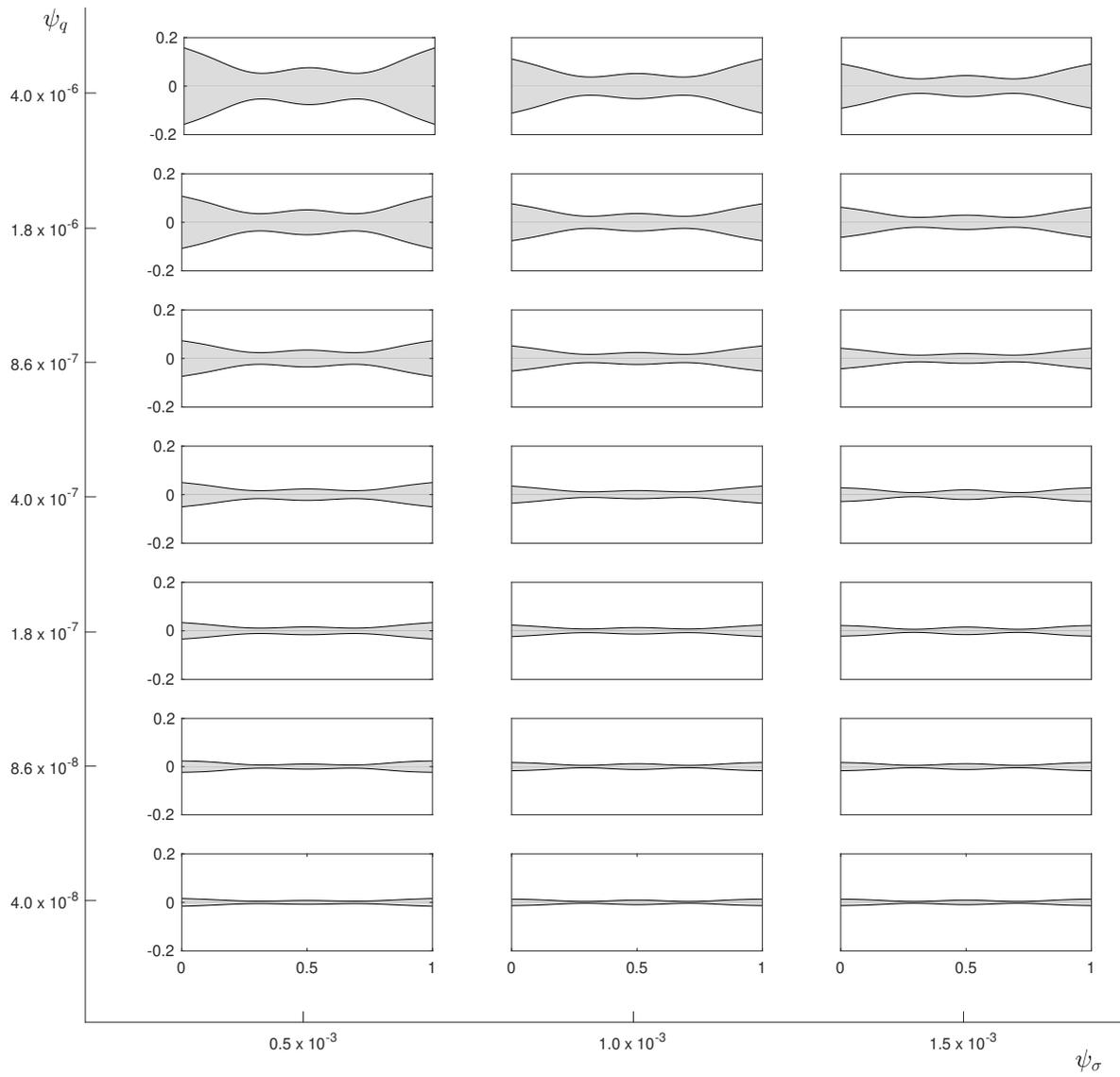
**Fig. 3.** Comparison of optimal beam solutions with variable cross-section considering different number of lobes. The parameters related to the weight per unit mass, the load and the maximum allowable stress are reported in Table 2.



**Fig. 4.** Comparison of optimal beam solutions of three-lobes variable cross-section considering different maximum stress levels, i.e. the value of parameter  $\psi_\sigma$ . The parameters related to the weight per unit mass, the load and the maximum allowable stress, as well as the optimal solution are reported in Table 4.



**Fig. 5.** Value of the dimensionless volume of the beam  $\tilde{V}$  and binding solution constraints for different  $\psi_\sigma$  and  $\psi_q$ . The bullets indicate whether the relevant solution constraint is the maximum stress (green), Eqn. (21), the maximum displacement (blue), Eqn. (22), or both (red).



**Fig. 6.** Beam height profiles of the beams for various  $\psi_\sigma$  and  $\psi_q$ . The values of the design vector are reported in Table 7.