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Nondimensional shape optimisation of non-prismatic beams with sinusoidal lateral profile

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ABSTRACT

The present paper deals with the optimal design of non-prismatic beams, i.e. beams with variable cross-section. To set the optimisation problem, Euler-Bernoulli unshearable beam theory is considered and the elastica equation expressing the transverse displacement as a function of the applied loads is reformulated into a system of four differential equations involving kinematic components and internal forces. The optimal solution (in terms of volume) must satisfy two constraints: the maximum Von Mises equivalent stress must not exceed an (ideal) strength and the maximum vertical displacement is limited to a fraction of beam length. To evaluate the maximum equivalent stress in the beam, normal and shear stresses have been considered. The former evaluated through Navier formula, the latter through a formula derived from Jourawsky and holding for straight and untwisted beams with bi-symmetric variable cross-sections. The optimal

24 solutions as function of material unit weight, maximum strength and applied load are presented
25 and discussed. It is shown that the binding constraint is usually represented by the maximum stress
26 in the beam, and that applied load and strength affect the solution more than material unit weight.
27 To maintain the generality of the solution, the nondimensionalisation according to Buckingham
28 Π -theorem is implemented and a design abacus is proposed.

29 INTRODUCTION

30 In the last decades, non-prismatic beams have been widely adopted in the structural engineering
31 field for civil, aerospace, and mechanical applications (El-Mezaini et al. 1991; Ascione et al. 2017;
32 Vilar et al. 2022; Cucuzza et al. 2021; Sardone et al. 2020; Marano and Quaranta 2010; De Biagi
33 et al. 2020; Magnucki et al. 2021). This type of beam is characterised by variable cross-section
34 along its centroidal axis (Gere and Timoshenko 1997), bestowing it a strong interconnection among
35 structural form, functionality, aesthetic and architectural requirements (Mercuri et al. 2020a). These
36 features determined their everlasting success over the centuries, referring e.g. to monumental and
37 historical architectures like Roman aqueducts and masonry arch structures. Non-prismatic elements
38 have been extensively adopted even for infrastructures, e.g. for bridges and viaducts (Kozy and
39 Tunstall 2007; Kaveh et al. 2022; Fiore et al. 2016; Muteb and Shaker 2017; Kaveh et al. 2020b;
40 Zhou et al. 2019; Balduzzi et al. 2017), and buildings, such as double-tapered roof beams for
41 industrial structures (Vilar et al. 2022; Bournas et al. 2014; McKinstry et al. 2016).

42 When dealing with prismatic beams, the classical Euler-Bernoulli beam theory holds, which
43 neglects the shear deformation contribution and assumes the Navier hypothesis (Carpinteri 2013).
44 Nonetheless, a more advanced theory is required to deal with non-prismatic beams, able to ac-
45 curately and reliably capture the actual structural response. Therefore, in this research work, the
46 Euler-Bernoulli unshearable beam theory was considered (Bertolini et al. 2019; Timoshenko and
47 Goodier 1934). Recently, in the scientific literature, various mechanical models were proposed
48 for non-prismatic beams. In (Medwadowski 1984), the authors proposed a solution of the differ-
49 ential equations for non-prismatic beams, denoted in that work as shear beams, considering the
50 effect of shear deformations. In (Bulte 1992) the differential equation formulation of the deflec-

51 tion curve was presented as a multi-point boundary value problem. In (Romano 1996) analytical
52 closed-form solutions were proposed for bending beams accounting for the shear deformation with
53 non-prismatic parabolic profiles with both varying width and depth. In (Katsikadelis and Tsiatas
54 2003), the nonlinear large deflection analysis was conducted on the Euler-Bernoulli beam with
55 variable stiffness with the analog equation method due to variable coefficients in the governing
56 differential equations. In (Balduzzi et al. 2016), the authors analyse the compatibility and equi-
57 librium of non-prismatic beams with a Timoshenko-like beam model, formulated as a system of
58 six coupled ordinary differential equations (ODE). Cazzani et al. (Cazzani et al. 2016) proposed a
59 Timoshenko beam model and a non-uniform rational B-splines (NURBS) interpolation to analyse
60 curved beams with the isogeometric analysis (Hughes et al. 2005). In (Bertolini et al. 2019),
61 the authors analysed the stress distribution in untwisted, straight, thin-walled beams with constant
62 taper with rectangular and circular cross-section shapes. Most of the analytical approaches for
63 non-prismatic beams proposed in the literature have been finally solved with the finite differences
64 methods, even considering non-homogeneous conditions (Al-Azzawi and Emad 2020; Tuominen
65 and Jaako 1992).

66 The non-prismatic geometry ensures great versatility for optimising specific structural aspects
67 of interest (Rath et al. 1999; Sarma and Adeli 1998; Colin and MacRae 1984; Mercuri et al. 2020a;
68 Kaveh et al. 2021), for instance, minimum material consumption, optimising structural perfor-
69 mances, etc. In addition, the material used, e.g. concrete, steel, or wood (Maki and Kuenzi 1965),
70 plays a crucial role in the shape and topology optimisation process due to distinct constitutive laws
71 and possible changing behaviour in tension and compression. Furthermore, nowadays new mate-
72 rials and technologies such as additive manufacturing are opening new possibilities and promising
73 research paths (Mercuri et al. 2020a). The problem of optimal design of non-prismatic beams has
74 been studied quite extensively, implementing both gradient-based (Rao 2019) and gradient-free
75 meta-heuristic algorithms (Resende et al. 2017; Plevris 2009), such as genetic algorithm (Cucuzza
76 et al. 2021; Cucuzza et al. 2022; Biswal et al. 2017) or particle swarm optimisation algorithm
77 (Rosso et al. 2022; Rosso et al. 2021). In (Luévanos-Rojas et al. 2020), the optimal design of

78 reinforced concrete rectangular cross-section beams with straight haunches was analysed with the
79 aim of reaching the minimum constitutive materials cost. In (Veenendaal et al. 2011), the optimal
80 form-finding problem has been studied for the design of non-prismatic fabric-formed beams. The
81 technical difficulties of traditional casting methods for these non-conventional variable curvature
82 structures are nowadays partially overcome by leveraging innovative production technologies such
83 as 3D printing and additive manufacturing (Asprone et al. 2018; Mercuri 2018; Costa et al. 2020).
84 This latter aspect further nourishes the current relevance and contemporary of the present study on
85 optimal variable-curvature non-prismatic solutions. In (Kaveh et al. 2020a; Kaveh et al. 2020b),
86 the optimal seismic design of three-dimensional steel frames were carried on with the response
87 spectrum analysis method. The same authors in a later study (Kaveh et al. 2021) analysed op-
88 timal performance-based reinforced concrete frames with objective function based on both cost
89 and sustainability, expressed in terms of carbon dioxide emissions. Similarly, (Yavari et al. 2017)
90 optimised environmental sustainability of non-prismatic slab frames bridge geometries. Recently,
91 (Wang et al. 2021) proposed an innovation from a computational point of view for sequentially
92 solving shape and topology optimization of beam structures, introducing the concept of 2.5D beam
93 model than traditional 3D modeling. Basically, standard 1D beam elements are interconnected
94 longitudinally, and, in every finite node, the section properties are retrieved from an additional bidi-
95 mensional section model. The shape of the beam has been parametrically defined by non-uniform
96 rational B-splines (NURBS) (Piegl and Tiller 1996).

97 In comparison to the literature previous studies, in the present work, the authors proposed
98 an optimal design criterion for homogeneous constant width non-prismatic beams based on the
99 *elastica* equation with a dimensionless perspective, eventually providing a design abacus. The
100 main findings of the present work are summarised below:

- 101 • the minimum weight, or proportionally the minimum volume, optimisation problem was
102 stated based on a dimensionless form of the *elastica* equation according to Buckingham
103 Π -theorem;
- 104 • the stress distributions considered Euler-Bernoulli unshearable beam theory (Bertolini et al.

105 2019);

- 106 • the constraints of the optimisation problem are expressed as Von Mises equivalent stress
107 limitation and maximum limit vertical deflection limited to a fraction of beam length;
- 108 • the optimal solutions as a function of material unit weight, maximum strength, and applied
109 load are presented in a design abacus graph form.

110 The current document is organised as follows. In Section 2, the analytical formulation of
111 the *elastica* governing ODEs is presented, even illustrating the dimensionless procedure and the
112 assumed stress distributions. The minimum volume (weight) optimisation problem statement is
113 described in Section 3, showing that the non-prismatic variable beam depth profile is defined
114 through an emptying sinusoidal function. Eventually, in Section 4 the optimal solutions as a
115 function of material unit weight, maximum strength, and applied load are presented and discussed,
116 finally delivering a useful design abacus encompassing the wide spectrum of design parameters
117 analysed.

118 **BEAM MODEL**

119 A beam of length L , straight centerline and a variable cross-section is considered (Figure 1).
120 A Cartesian coordinate system ($Oxyz$) is introduced, setting: the origin O in the centroid of one
121 of the end cross-sections; the x - and y -axes as the principal central axes of inertia of the cross-
122 section; the z -axis along the beam centerline. We assume plane bending in the yz -plane, where the
123 beam is subjected to distributed transverse load $q(z)$ and its deflection is described by transverse
124 displacement $v(z)$. The constituting material is assumed to be homogeneous, isotropic and linear
125 elastic with Young's modulus E .

126 ***Elastica* equation**

127 The beam is supposed of solid doubly-symmetric cross-section with variable depth $h(z)$. Based
128 on the Euler-Bernoulli theory, beam deflection is governed by a fourth-order ODE, the *elastica*

129 equation, which reads, for a variable cross-section beam,

$$130 \quad \frac{d}{dz^2} \left[EJ(z) \frac{d^2 v(z)}{dz^2} \right] = q(z), \quad (1)$$

131 where $J(z) = J_x(Z)$ is the area moment of inertia of the cross-section. By solving for differentiation
132 and dividing both members by $EJ(z)$, the equation of the deflection curve reads

$$133 \quad \frac{d^4 v(z)}{dz^4} + 2 \frac{d^3 v(z)}{dz^3} \frac{dJ(z)}{dz} \frac{1}{J(z)} + \frac{d^2 v(z)}{dz^2} \frac{d^2 J(z)}{dz^2} \frac{1}{J(z)} = \frac{q(z)}{EJ(z)}. \quad (2)$$

134 To give a more general description of the beam model, Buckingham Π -theorem is adopted (**Baren-**
135 **blatt 1987**) and a suitable nondimensionalisation is introduced by rescaling lengths by the beam
136 span L and forces by EL^2 . Nondimensional variables $\tilde{z} = z/L$ (with $\tilde{z} \in [0, 1]$), $\tilde{v} = v/L$ and
137 functions $\tilde{J} = J/L^4$ and $\tilde{q} = q/EL$ are thus defined, while the derivative with respect to dimensional
138 variable z is expressed as

$$139 \quad \frac{d}{dz} = \frac{d}{d\tilde{z}} \frac{d\tilde{z}}{dz} = \frac{1}{L} \frac{d}{d\tilde{z}}. \quad (3)$$

140 Accordingly, Equation (2) can be rewritten as

$$141 \quad \tilde{v}^{IV}(\tilde{z}) + 2\tilde{v}'''(\tilde{z}) \frac{\tilde{J}'(\tilde{z})}{\tilde{J}(\tilde{z})} + \tilde{v}''(\tilde{z}) \frac{\tilde{J}''(\tilde{z})}{\tilde{J}(\tilde{z})} = \frac{\tilde{q}(\tilde{z})}{\tilde{J}(\tilde{z})} \quad (4)$$

142 where the notation $(\cdot)'$ denotes the derivative with respect to nondimensional variable \tilde{z} .

143 **First-order ODEs**

144 Alternative to the *elastica* equation, Eqn. (1), the shear-bending problem of the variable cross-
145 section beam can be formulated as a system of four first-order ODEs (Bulte 1992)

$$146 \left\{ \begin{array}{l} \frac{dv}{dz} = -\phi(z), \\ \frac{d\phi}{dz} = \frac{M(z)}{EJ(z)}, \\ \frac{dM}{dz} = V(z), \\ \frac{dV}{dz} = -q(z), \end{array} \right. \quad (5)$$

147 Thanks to the functional \mathbf{f} , Eqn. (5) can be rewritten in vectorial notation as

$$148 \mathbf{w}'(z) = \mathbf{f}(z, \mathbf{w}), \quad (6)$$

149 where vector \mathbf{w} has components $w_1 = v(z)$, $w_2 = \phi(z)$, $w_3 = M(z)$ and $w_4 = V(z)$, with ϕ the
150 rotation of the cross-section (Figure 1). In this way, the variability of the cross-section is taken
151 implicitly into account only by $J(z)$ and, due to the fact that all the equation are coupled, this is taken
152 into account in the entire system avoiding to explicitly solve the fourth order equation depending
153 by the derivative of the inertia. Moreover, in this way the solutions of the system directly represent
154 shear, moment, rotation and deflection curves.

155 The distributed load $q(z)$ includes two contributions: (i) the beam self weight per unit length,
156 equal to the product of the material unit weight γ by the cross-sectional area $A(z)$; (ii) the applied
157 force per unit length q_0 , assumed to be constant along the beam. It can thus be expressed as

$$158 q(z) = q_0(z) + \gamma A(z). \quad (7)$$

159 According to Buckingham Π -theorem, it is possible to rewrite the system of Eqn. (5) as

$$160 \left\{ \begin{array}{l} \frac{d\tilde{v}}{d\tilde{z}} = -\phi(\tilde{z}), \\ \frac{d\phi}{d\tilde{z}} = \frac{\tilde{M}(\tilde{z})}{\tilde{J}(\tilde{z})}, \\ \frac{d\tilde{M}}{d\tilde{z}} = \tilde{V}(\tilde{z}), \\ \frac{d\tilde{V}}{d\tilde{z}} = -\tilde{q}(\tilde{z}), \end{array} \right. \quad (8)$$

where

$$\tilde{M}(\tilde{z}) = \frac{M(\tilde{z})}{EL^2}, \quad (9)$$

$$\tilde{V}(\tilde{z}) = \frac{V(\tilde{z})}{EL^3}. \quad (10)$$

161 Accordingly, Eqn. (6), turns into

$$162 \tilde{\mathbf{w}}'(z) = \mathbf{f}(\tilde{z}, \tilde{\mathbf{w}}). \quad (11)$$

163 As previously illustrated, the normalized distributed load $\tilde{q}(\tilde{z})$ can be divided in two components

164 as

$$165 \tilde{q}(\tilde{z}) = \tilde{\psi}_q(\tilde{z}) + \tilde{\psi}_\gamma \tilde{A}(\tilde{z}), \quad (12)$$

where

$$\tilde{\psi}_q(\tilde{z}) = \frac{q_0(\tilde{z})}{EL}, \quad (13)$$

$$\tilde{\psi}_\gamma = \frac{\gamma L}{E}, \quad (14)$$

166 and $\tilde{A}(\tilde{z}) = A(\tilde{z})/L^2$. Considering a constant distributed applied force, Eqn. (12), turns into

$$167 \tilde{q}(\tilde{z}) = \tilde{\psi}_q + \tilde{\psi}_\gamma \tilde{A}(\tilde{z}). \quad (15)$$

168 Stress distributions

169 Beams with variable cross-section exhibit non-trivial stress distributions which differ from
170 those predicted by the classical formulae of prismatic beam theory, in particular regarding shear
171 stresses (Timoshenko 1956b; Oden 1981; Bruhns 2003). Under the assumption of plane bending,
172 the beam is in a plane state of stress with $\sigma_x = \tau_{xy} = \tau_{zx} = 0$. Transverse normal stress σ_y , although
173 non vanishing by equilibrium in non-prismatic beams, is generally small and can be neglected
174 without appreciable error (Balduzzi et al. 2016). Distributions of normal stresses $\sigma := \sigma_z$ and
175 shear stresses $\tau := \tau_{zy}$ acting on the cross-section are given as follows.

176 The distribution of normal stresses σ can be recovered by using the Navier flexure formula

$$177 \sigma(y, z) = \frac{M(z)}{J_x(z)} y \quad (16)$$

178 which holds with a good approximation for non-prismatic beams, provided the variation of the
179 cross-section is not too rapid (Timoshenko 1956b; Boley 1963).

180 Conversely, the distribution of shear stresses τ is considerably altered compared to prismatic
181 beams. In non-prismatic beams, shear stresses τ are dependent not only upon the internal shear force
182 V , but also upon the internal axial force N and bending moment M , as well as on the changing rate
183 of height and width of the beam (Bruhns 2003). This result follows from the equilibrium boundary
184 condition on the beam's lateral surface, which requires the shear stress τ to be proportional to the
185 normal stress σ due to the taper angle (Auricchio et al. 2015). Jourawsky's theory (Timoshenko
186 1956a) is consequently ineffective in predicting the actual shear stress distribution because (i)
187 it violates the boundary equilibrium, (ii) cannot reproduce the correct distribution shape and
188 magnitude and (iii) fails to identify the position and value of the maximum shear stress (Bruhns 2003;
189 Paglietti and Carta 2009; Beltempo et al. 2015; Balduzzi et al. 2017; Mercuri et al. 2020b). In view
190 of these considerations, we calculate the distribution of shear stresses by using the shear formula
191 derived by Bertolini *et al.* (Bertolini et al. 2019, Equation 5), an extension of the Jourawsky formula
192 holding for straight and untwisted beams with bi-symmetric variable cross-sections. Assuming null

193 distributed couples applied to the beam and null internal axial force, the extended shear formula
 194 simplifies to

$$195 \quad \tau(y, z) = \frac{1}{c(y, z)} \left[V(z) \frac{S^*(y, z)}{J(z)} + M(z) \frac{d}{dz} \left(\frac{S^*(y, z)}{J(z)} \right) \right], \quad (17)$$

196 where $c(y, z)$ is the cross-sectional width at the arbitrary level y where the shear stress τ is evaluated;
 197 $S^*(y, z) := S_x^*(y, z)$ is the first moment of area, with respect to the bending neutral axis x , of the
 198 cross-sectional region below the arbitrary level y . Specifically, for the rectangular cross-section,
 199 with constant width b and variable height $h(z)$, it holds

$$200 \quad c(y, z) = b, \quad S^*(y, z) = \frac{b}{2} \left(\frac{h^2(z)}{4} - y^2 \right), \quad J(z) = \frac{1}{12} b h^3(z), \quad (18)$$

201 and Eqn. (17) reads

$$202 \quad \tau(y, z) = \frac{3}{2} \frac{1}{bh} \left[V(z) \left(1 - 4 \frac{y^2}{h^2} \right) + M(z) \frac{dh}{dz} \left(-\frac{1}{h} + 12 \frac{y^2}{h^3} \right) \right]. \quad (19)$$

203 OPTIMISATION PROBLEM

204 The optimisation problem tries to define the minimum volume which determines the minimum
 205 weight directly linked to the minimum usage of material respecting stress and deflection constraints
 206 (Cucuzza et al. 2021), which evaluations derive from structural analysis conducted with the sys-
 207 tem, Eqn. (8). Despite the minimization of the self-weight may not comprehensively cover all
 208 the numerous aspects for a general minimum cost design problem (Adeli and Sarma 2006), as a
 209 first approximation, and in the absence of precise requirements and prescription, it may be suc-
 210 cessfully employed as an indirect indicator of the cost, directly related to the minimum material
 211 consumption (Rao 2019; Cucuzza et al. 2021; Spillers and MacBain 2009). The minimization of
 212 self-weight also provides benefits for earthquake design situations (Plevris 2012; Rao 2019), for
 213 shells design loading (Adriaenssens et al. 2014), and also accounting for transportation and installa-
 214 tion aspects especially involving precast elements solutions (Veenendaal 2008). In the dimensional

215 problem, the stress constraints are treated in a simplified way adopting Von Mises criterion,

$$216 \quad \sigma^2(z) + 3\tau^2(z) \leq \sigma_{id}^2. \quad (20)$$

217 where the ideal stress σ_{id} is assumed to be the yielding stress for an ideal material (same behaviour
218 both in tension and in compression). Considering the above Von Mises stress constraint, Eqn. (20),
219 and the specific forms for normal and shear stresses, Eqns. (16) and (19), respectively, it is possible
220 to look for a dimensionless form to make consistency with the dimensionless system of Eqn. (8).
221 In order to obtain a dimensionless stress it is sufficient to divide it by the elastic modulus E , and
222 after some mathematical elaborations, it is possible to prove that

$$223 \quad \tilde{\sigma}^2(\tilde{z}) + 3\tilde{\tau}^2(\tilde{z}) \leq \tilde{\psi}_\sigma^2, \quad (21)$$

224 in which a new dimensionless parameter is introduced, $\tilde{\psi}_\sigma = \sigma_{id}/E$. It is also possible to express
225 the deflection constraint in a dimensionless form. Assuming a limit value of $v_{lim} = L/250$, the
226 dimensionless deflection constraint is defined as

$$227 \quad \tilde{v}(\tilde{z}) \leq \frac{1}{250}. \quad (22)$$

228 The above deflection limit value may be retrieved by general deformability requirements un-
229 der service conditions contained in current structural codes regulations, e.g. the Eurocodes
230 (EN1990 2002).

231 For evaluating the above-mentioned constraints of the optimisation problem herein investigated,
232 various structural analyses have been conducted in order to account for possible multiple load cases
233 conditions (Spillers and MacBain 2009; Cucuzza et al. 2022; Rao 2019). According to the basic
234 principles of structural design (EN1990 2002), every structure has to be designed and assessed for
235 the toughest loading conditions likely occurring in its lifespan. Therefore, it implies considering
236 the envelope of the maximum actions' effects coming from different load combinations. For the

237 sake of simplicity, in the current study, two different load conditions have been considered. The first
 238 load configuration accounts for the uniformly distributed load, as described in Eqn. (15), applied
 239 over the entire span length. The second load condition accounts for an asymmetric live load
 240 applied over the half-span length only. This latter configuration is usually more burdensome than
 241 the first load case for non-prismatic geometries, especially due to potential instability phenomena
 242 (Bazzucchi et al. 2017; Virgin et al. 2014). Since we are dealing with beam structures that may be
 243 employed, at different scales, both for buildings or bridges under uncertain locations of live loads
 244 (EN1990 2002), the asymmetric load condition must be applied on both the half-spans alternatively
 245 for accounting all the possible loading cases. In this sense, it should be expected that the optimal
 246 beam solution will still present a symmetric shape along the longitudinal axis. This optimal
 247 solution is expected stiffer profile than the one loaded with the first load case only, thus with a
 248 greater cross-section in general, but able to withstand both symmetric and asymmetric loading
 249 conditions.

250 **Beam geometry definition**

251 The previous constraints are applied to a doubly end-fixed beam having cross-section height that
 252 varies along the z -coordinate by way of an *emptying* function $\eta(z)$, given as a linear combination
 253 of sines

$$254 \quad h(z) = h_0 - \eta(z) = h_0 - \sum_{i=1,3,5\dots}^N \Delta h_i \sin\left(i\frac{\pi}{L}z\right), \quad (23)$$

255 where h_0 is the height of the end cross-sections, N is the number of harmonics combined in the
 256 emptying function and Δh_i is the amplitude of the i -th harmonic. A sketch of the beam is reported
 257 in Figure 2. The structural design principles and the load cases remarks mentioned in the previous
 258 section justify the authors' choice to focus only on the even sinusoidal harmonics in Eq.(23),
 259 thus delivering symmetrical beam profiles solutions. In this work, depending on the number of
 260 harmonics considered, we denote the beam with $N = 1$ as *one-lobe solution*, the one with $N = 3$
 261 as *three-lobes solution*, and so forth. As an example, the height profile of the solution with three

lobes is

$$h(z) = h_0 - \left[\Delta h_1 \sin\left(\frac{\pi}{L}z\right) + \Delta h_3 \sin\left(3\frac{\pi}{L}z\right) \right]. \quad (24)$$

In general, the volume of the emptied beam with sine emptying functions is equal to

$$V = \int_0^L A(z) dz \quad \text{with} \quad A(z) = f(h(z)), \quad (25)$$

and therefore, the dimensionless volume definition may be expressed as

$$\tilde{V} = \frac{V}{L^3} \quad (26)$$

For instance, detailing the above-mentioned Eqn. (25) for a rectangular bisymmetrical cross-section it holds:

$$V = b \left[\int_0^L h(z) dz \right] = bL \left[h_0 - \sum_{i=1}^N \Delta h_i \frac{2}{i\pi} \right]. \quad (27)$$

According to the nondimensionalisation introduced in Section 2, it results

$$\tilde{h}(\tilde{z}) = \tilde{h}_0 - \tilde{\eta}(\tilde{z}) = \tilde{h}_0 - \sum_{i=1}^N \Delta \tilde{h}_i \sin(i\pi\tilde{z}) \quad (28)$$

and

$$\tilde{V} = \frac{V}{L^3} = \tilde{b} \left[\tilde{h}_0 - \sum_{i=1}^N \Delta \tilde{h}_i \frac{2}{i\pi} \right]. \quad (29)$$

Design vector and problem statement

Considering that the height of the beam must always be a positive number, i.e. $\tilde{h}(\tilde{z}) > 0$, we defined the dimensionless height of the end cross-section \tilde{h}_0 as the sum of a minimum height \tilde{h}_{min} , to be strictly positive, and the maximum emptying function, resulting in

$$\tilde{h}_0 = \tilde{h}_{min} + \max_{\tilde{z} \in [0,1]} \tilde{\eta}(\tilde{z}) \quad (30)$$

The design vector \mathbf{D} collects the dimensionless values of the minimum height and the amplitudes

281 coefficients of the sine function $\Delta\tilde{h}_i$. The optimization problem can be formulated in the following
 282 way

$$\begin{aligned}
 &\text{Find } \mathbf{D} = \{\tilde{h}_{min}; \Delta\tilde{h}_i\}_{i=1,3,5\dots} \text{ such that} \\
 &\min \tilde{V}(\mathbf{D}) \\
 &\text{s.t. } \tilde{\sigma}^2(\tilde{z}) + 3\tilde{\tau}^2(\tilde{z}) \leq \tilde{\psi}_\sigma^2, \\
 &\quad \tilde{v}(\tilde{z}) \leq \frac{1}{250}
 \end{aligned}
 \tag{31}$$

284 Thanks to the procedure previously described and implemented in Matlab, the optimal geom-
 285 etry of beams with different combinations of parameters ψ_q , ψ_γ and ψ_σ was investigated. The
 286 dimensionless form allows covering all the possible situations for the specific problem parameters
 287 values such as the span length, geometric and material properties included in the aforementioned
 288 parameters.

289 For the sake of better controlling the optimization process, limiting the mathematical topology
 290 complexity of the search space, and in order to avoid an excessive over-parametrization of the
 291 beam's shape longitudinal profile, the authors studied the optimization process using the number
 292 of sine-emptying lobes as a fixed parameter rather than a design variable. Specifically, the authors
 293 provided a detailed comparison and discussion of four different structural configurations, i.e. from
 294 one-lobe to seven-lobes. For a number of lobes greater than seven-lobes, the authors observed that
 295 the influence of higher lobes was practically negligible compared to the increase of the beam's
 296 shape profile complexity.

297 **RESULTS AND DISCUSSION**

298 In this section, the results of optimization analyses for a beam with rectangular cross-section
 299 are presented. The structural analyses have been conducted under two different load conditions.
 300 The first load configuration accounts for the uniformly distributed load, as described in Eqn. (15),
 301 applied over the entire span length. The second load condition accounts for an asymmetric live load
 302 applied over the half-span length only. This latter configuration is usually more burdensome than
 303 the first load case for non-prismatic geometries, especially due to potential instability phenomena

304 (Bazzucchi et al. 2017; Virgin et al. 2014). Furthermore, in the current study, point loads have not
305 been explicitly considered since they are properly representative of specific design situations, see
306 e.g. (Yang et al. 2022). Nonetheless, the current methodology may account for point loads as well
307 by implementing equivalent distributed loads over a short finite length, simulating its actual load
308 footprint.

309 Several analyses were carried to determine the influence of the number of lobes on the optimal
310 beam solution and to highlight the effects of the maximum allowable stress level and material
311 unit weight. A design abacus is proposed to summarise the results. The optimisation problem
312 was implemented in a Matlab code and solved with the *fmincon* function provided within the
313 *Optimization Toolbox* package (MATLAB Optimization Toolbox). The input parameters of the
314 *fmincon* function are the objective function defined in Eqn. (29) and the non-linear constraints
315 defined in Eqs. (21)-(22), both summarized in the optimisation problem statement in Eqn. (31). The
316 solver algorithm option has been set to the well-acknowledged and efficient nonlinear programming
317 method named sequential quadratic programming (SQP) (Schittkowski 1986). This gradient-
318 based iterative method is based on quasi-Newton approximation of the Hessian of the Lagrangian
319 function for constrained optimization problems (Rao 2019), which translates in the resolution of
320 quadratic programming subproblems forming an active set strategy for a line search procedure
321 (Biggs 1975; Han 1977; Powell 2006; Powell 1978). Since the current implementation requires
322 strict feasibility with respect to constraints, it implements an automatic adaptation of the finite
323 difference gradient step along the line search, and due to the quasi-Newton approximation of the
324 Hessian, any second-order eigenvalue sensitivity is not strictly necessary (Li et al. 2016).

325 **Influence of the number of lobes**

326 Figure 3 shows the optimal solutions (in grey) considering the material and geometric properties
327 reported in Table 1. Four different configurations were compared, from one-lobe to seven-lobes,
328 i.e. considering $N = 1, 3, 5, 7$. For each case, the components of vector $\tilde{\mathbf{w}}$, i.e. nondimensional
329 displacement \tilde{v} , rotation ϕ , bending moment \tilde{M} and shear \tilde{V} are reported in the top subplots. The
330 displacement plot (top left) includes a horizontal red line denoting the limit value, i.e. $1/250$. The

331 bottom plot refers to the maximum Von Mises stress along the beam and includes (in red) the
332 threshold ψ_σ .

333 For all the examined cases, the maximum stress in the beam represents the most strict (binding)
334 constraint. Table 2 reports the values of the components of the design vector \mathbf{D} . Comparing the
335 various solutions, it results that increasing the number of lobes reduces the volume of the optimal
336 beam. The presence of two parts with limited height, which emerges for $N = 3$ and is further
337 highlighted for $N = 5, 7$, implies larger rotations and, by consequence, increased displacements.
338 The number of lobes in the solution affects Von Mises equivalent stress. For $N = 1$, the maximum
339 stresses are observed at beam ends and at midspan, where heights h_0 and h_{min} can be optimised.
340 A different trend is noted for $N = 3$, where the maximum stress occurs at 1/6 and 5/6 of beam
341 length, roughly. Considering the area below the stress curve as an ideal measure of the material
342 exploitation rate, it results that the best use is obtained when the stress level tends to the threshold
343 value in any section of the beam. Comparing the solutions with different number of lobes, it clearly
344 emerges that the larger the number of lobes, the better the exploitation rate. Five- and seven-lobes
345 solutions produce similar maximum vertical displacements, but different material exploitation, in
346 particular in the first and last sixth of the beam. In detail, $N = 7$ solution exhibits a stress plateau
347 in the first and last part of the beam. The similarity in five- and seven-lobes solutions emerges in
348 analysing the components of the design vector reported in Table 2. Checking the \tilde{V} column, i.e.
349 the values of the objective function, it results that the reduction in the optimal volume (target of the
350 optimisation) is more evident up to $N = 5$, while for $N = 7$, the resulting \tilde{V} is close to the five-lobes
351 solution. As a conclusion, three-lobes and five-lobes represent feasible solutions for fixed-fixed
352 beams with uniformly distributed load.

353 For the sake of completeness, other boundary conditions should be analysed in future studies
354 since the herein-presented double fixed condition is mainly representative of concrete structures.
355 Indeed, the authors preliminary tested the current optimization procedure considering other beam
356 boundary conditions, in particular the double-hinged one. However, the obtained optimal results ap-
357 pear not relevant for the scope of the current study, and they have not been herein reported. Nonethe-

358 less, it is worth reminding that for other structural materials, such as steel or timber, the semi-rigid
359 restraints condition is the actual one. A proper embedding of these aspects is out of the scope of the
360 current manuscript and may require future deeper investigations. Indeed, special attention should
361 be paid to the specific technical choice adopted for the restrain joints, which affects their rotational
362 stiffness capacity on the moment rotation plane (Daniūnas and Urbonas 2008; Du et al. 2022).

363 **Influence of the maximum stress**

364 Three-lobes solution was adopted for assessing the effect of maximum stress on the optimal beam
365 height profile. The optimisation problem of Eqn. (31) was solved considering geometric and load
366 parameters reported in Table 3. To highlight the dependency of the optimal solution on the value of
367 the stress parameter ψ_σ , three different values were considered, i.e. $\psi_\sigma = 3.33 \times 10^{-4}$, 6.66×10^{-4} ,
368 and 1×10^{-3} . These correspond to ideal stresses σ_{id} of 10, 20 and 30 MPa. Figure 4 shows the
369 optimal solutions for the three stress levels and Table 4 details the amplitudes of the sine functions
370 and the value of the objective function. Comparing stress and displacement curves of the three
371 solutions it emerges that different trends emerge. For low stress levels, say $\psi_\sigma = 3.33 \times 10^{-4}$, the
372 relevant constraint for the optimal solution is represented by the maximum allowable stress itself.
373 For high stresses, $\psi_\sigma = 1 \times 10^{-3}$, the maximum displacement is the binding term. For medium
374 stresses, $\psi_\sigma = 6.66 \times 10^{-4}$, both constraints are relevant for the optimal solution.

375 As a matter of evidence, the optimal solution would benefit in terms of volume of material if the
376 maximum allowable stress level increases. To measure such benefit, Table 4 reports in the values
377 of the nondimensional volume \tilde{V} . The change of ψ_σ affects the value of the objective function in
378 a nonlinear manner, with no direct relationship between the value of ψ_σ and \tilde{V} . To address such
379 issue, a parametric analysis was performed to highlight the specific binding constraint and study
380 the value of the objective function. Figure 5 details the results in term of \tilde{V} (contour lines) and
381 relevant constraint in the optimisation (coloured bullets) for $N = 3$, $\psi_\gamma = 8.33 \times 10^{-6}$ and $\tilde{b} = 0.05$.
382 The load parameter ψ_q varies in the range from 3.33×10^{-8} to 1.67×10^{-7} that corresponds to a
383 distributed load between 10 and 100 kN/m (the remaining variables are those reported in Table 3).
384 The stress parameter ψ_σ varies in the range from 3.33×10^{-4} to 1×10^{-3} . It is shown that, for the

385 larges part of the investigated cases, the binding constraint is represented by the maximum stress
 386 in the beam (similarly to what shown in Figure 4.a). For large maximum stresses, the relevant
 387 condition is the maximum displacement, highlighted with blue bullets. The transition between the
 388 two limit conditions depends on the value of the distributed load, in particular for $\psi_\sigma > 6 \times 10^{-4}$.
 389 Observing the trends of \tilde{V} in the black contour plot, it is seen that for high ψ_σ , the optimal volume
 390 depends on the external load, only, as the beam shape is constrained by the maximum displacement.
 391 For high values of ψ_q , the maximum stress controls the optimal volume.

392 **Influence of material unit weight**

393 To study the influence of the unit weight of the material constituting the beam, three scenarios
 394 were considered and compared. The solution reported in Figure 4.c obtained for $\psi_\gamma = 8.33 \times$
 395 10^{-6} , $\psi_q = 3.33 \times 10^{-8}$ and $\psi_\sigma = 1 \times 10^{-3}$ is considered as reference for the analysis. Two
 396 additional cases were considered, keeping fixed all the parameters except ψ_γ which is halved and
 397 doubled. The results of the optimisation are reported in Table 5. It is found that the modification of
 398 ψ_γ affects in a limited way the values of the amplitudes of the optimal solution, nor the volume of the
 399 beam, that is, its weight. To understand the reason of such trend, it is necessary to consider the total
 400 weight of the beam, namely G , computed as $G = \gamma V$, which can be expressed in nondimensional
 401 form as

$$402 \quad \tilde{G} = \psi_\gamma \tilde{V}. \quad (32)$$

403 The total applied load Q is computed as $Q = q_0 L$, which can be further expressed as

$$404 \quad \tilde{Q} = \psi_q. \quad (33)$$

405 For all the analysed cases, the result of Eqn. (32) (reported in the seventh column of Table 5), is
 406 smaller than ψ_q (3.33×10^{-8}), showing that the dead load is not relevant in the solution.

407 **Design abacus**

408 The analyses performed highlighted that three- and five-lobes solutions provide good results for
 409 the minization of the objective function. Considering the parameters describing material weight,

410 load and maximum allowable stress, it was shown that ψ_q and ψ_σ play a relevant role in the optimal
 411 solution, while the parameter associated to the unit weight ψ_γ has a secondary importance since
 412 it slightly affects the optimal design. To let the solution the more general as possible, various
 413 combination of real construction materials, geometries and loads were considered. A summary of
 414 these is reported in Table 6; among all the cases, ψ_γ varies between 3.71×10^{-7} and 8.33×10^{-6} , ψ_σ
 415 varies between 1.00×10^{-3} and 3.75×10^{-3} , and ψ_q varies between 9.52×10^{-10} and 6.25×10^{-6} .
 416 The design abacus, which would serve for defining the optimal height profile of the beam, was
 417 formulated for a fixed value of $\psi_\gamma = 1 \times 10^{-6}$, ψ_σ in the range 0.5×10^{-3} to 1.5×10^{-3} (3 values) and
 418 ψ_q in the range 4×10^{-8} to 4×10^{-6} (in 7 logarithmically equally spaced values), trying to represent
 419 the possible materials, beam lengths and loads configurations. The nondimensional beam width is
 420 $\tilde{b} = 0.05$.

421 Table 7 reports the optimal values of the design vector and the corresponding \tilde{V} . It results that
 422 \tilde{V} is in the range 0.001 to 0.0085, roughly. In general, the values of h_{min} and the absolute values
 423 of the amplitudes of the sine function $\Delta\tilde{h}_1$, $\Delta\tilde{h}_3$ and $\Delta\tilde{h}_5$ increase for increasing ψ_q . Besides, the
 424 increase of ψ_σ causes a reduction of the terms. These trends reflect the findings of the specific
 425 studies reported in the previous sections.

426 Figure 6 shows the height profiles of the optimal beams. The scale in Y-axes are kept constant
 427 in all the plots for a better understanding of the effects of the parameters on the optimal solution.
 428 The results of Table 7 can be used for the design of real beams: the design values, i.e., the minimum
 429 height and the sine amplitudes can be determined by interpolation for a given ψ_σ and ψ_q .

430 CONCLUSIONS

431 The present paper deals with the optimal design of beams with variable cross-section. To this
 432 aim, Euler-Bernoulli beam theory has been adopted. The fourth order elastica equation has been
 433 rewritten according to the formulation proposed by Bulte as a system of four differential equations.
 434 According to Buckingham II-theorem, a nondimensionalisation has been done to let the solution
 435 as general as possible. The loads acting of the beam are the self weight and a distributed line
 436 load. The minimum volume (weight) solution must satisfy two constraints: the maximum Von

437 Mises equivalent stress must not exceed an (ideal) strength and the maximum vertical displacement
438 is limited to a fraction (1/250) of beam length. To evaluate the maximum equivalent stress
439 in the beam, normal and shear stresses have been considered. The former evaluated through
440 Navier formula, the latter through a formula derived from Jourawsky and holding for straight and
441 untwisted beams with bi-symmetric variable cross-sections. The optimisation problem has focused
442 on a beam with fixed-fixed ends subjected to a uniformly distributed load. To create the variable
443 height profile, an emptying function resulting as a combination of sine functions with different
444 amplitudes has been introduced. For the sake of completeness, other boundary conditions should
445 be analysed in future studies. The double fixed condition is mainly representative of concrete
446 structures, whereas for e.g. steel or timber structures, the semi-rigid condition is the actual one.
447 However, considering these aspects may require future investigations accounting for the specific
448 technical choice adopted for the restrain joints, thus affecting their rotational stiffness capacity
449 (Daniūnas and Urbonas 2008; Du et al. 2022).

450 The parametric analyses showed that:

- 451 • the choice of the number of sines in the emptying function that describes the shape of the
452 beam, is relevant up to $N = 5$, i.e., a five-lobes solution. For finer solution, for examples,
453 seven-lobes solution, there is not an improvement in the optimal solution in terms of
454 minimum weight;
- 455 • the maximum stress in the material influences the binding constraint. In general, it has been
456 noted that the stress constraint is relevant for the optimal solution for the large majority
457 of cases. The displacement constraint affects the solution for low external loads and high
458 strength;
- 459 • material unit weight does not affect the optimal solution as the total weight of the beam is
460 smaller than the total applied load. For this reason, the variability of the material can be
461 avoided in a preliminary design of a beam.

462 A design abacus with a profiles plot encompassing the wide spectrum of design parameters has

463 been proposed to help in the design of an optimal five-lobes solution. The findings of the present
464 paper would serve for the design of beams optimised with respect to weight. It should be em-
465 phasized that the current optimization problem statement in Eqn. (31) may be further refined,
466 e.g. peculiarly referring to more detailed constraints derived from actual structural codes based
467 on the specific constitutive materials adopted (NTC 2018; EN1990 2002). Furthermore, tradi-
468 tional casting methods for concrete non-prismatic beams with variable curvature profiles are still
469 challenging (Veenendaal et al. 2011), especially for rebar placing operations, and often lead to
470 more expensive solutions than classical alternatives. Nevertheless, in the novel panorama of ad-
471 ditive manufacturing and 3D printing (Costa et al. 2020; Mercuri 2018), the herein-studied struc-
472 tural solution is already becoming more feasible, revolutionizing the current construction indus-
473 try. Therefore, future research efforts will concern the numerous aspects related to promising
474 3D printing casting solutions, e.g. involving innovative and printing-technologically compat-
475 ible materials, life cycle assessment, and non-prismatic beams industrialization among others
476 (Costa et al. 2020; Gregori et al. 2019; Fiore et al. 2014; Asprone et al. 2018).

477 **Data Availability Statement**

478 All data, models, or code that support the findings of this study are available from the corre-
479 sponding author upon reasonable request.

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TABLE 1. Material and geometric properties for the analysis related to the number of lobes.

γ	25 kN/m ³
E	30 GPa
L	10 m
q_0	20 kN/m
σ_{id}	20 MPa
\tilde{b}	0.05
ψ_γ	8.33×10^{-6}
ψ_q	6.66×10^{-8}
ψ_σ	6.66×10^{-4}

TABLE 2. Optimal design values related to the cases of Figure 3.

N	\tilde{h}_{min}	$\Delta\tilde{h}_1$	$\Delta\tilde{h}_3$	$\Delta\tilde{h}_5$	$\Delta\tilde{h}_7$	\tilde{V}
1	0.0151	0.0231				1.17×10^{-3}
3	0.0157	0.0301	0.0055			1.07×10^{-3}
5	0.0119	0.0262	0.0040	-0.0026		9.93×10^{-4}
7	0.0122	0.0262	0.0040	-0.0026	0.00028	9.95×10^{-4}

TABLE 3. Material and geometric properties for the analysis related to the effect of maximum material stress.

γ	25 kN/m ³
E	30 GPa
L	10 m
q_0	10 kN/m
\tilde{b}	0.05
ψ_γ	8.33×10^{-6}
ψ_q	3.33×10^{-8}

TABLE 4. Optimal design values related to the cases of Figure 4.

N	ψ_σ	\tilde{h}_{min}	$\Delta\tilde{h}_1$	$\Delta\tilde{h}_3$	\tilde{V}
3	3.33×10^{-4}	0.0165	0.0317	0.0058	1.13×10^{-3}
3	6.66×10^{-4}	0.0112	0.0233	0.0062	8.01×10^{-4}
3	1×10^{-3}	0.0099	0.0229	0.0087	7.95×10^{-4}

TABLE 5. Optimal design values related to the cases of Figure 4.

N	ψ_γ	\tilde{h}_{min}	$\Delta\tilde{h}_1$	$\Delta\tilde{h}_3$	\tilde{V}	$\psi_\gamma\tilde{V}$
3	4.17×10^{-6}	0.0096	0.0217	0.0086	7.73×10^{-4}	3.22×10^{-9}
3	8.33×10^{-6}	0.0099	0.0229	0.0087	7.95×10^{-4}	6.62×10^{-9}
3	1.66×10^{-5}	0.0106	0.0257	0.0088	8.37×10^{-4}	1.39×10^{-8}

TABLE 6. Combination of structural and load configurations to be considered for determining the range of parameters of the design abaci.

E GPa	γ kN/m ³	σ_{id} MPa	L m	q_0 kN/m	ψ_γ	ψ_σ	ψ_q
Concrete							
30	25	30	1	2	8.33×10^{-7}	1.00×10^{-3}	6.67×10^{-8}
30	25	30	10	2	8.33×10^{-6}	1.00×10^{-3}	6.67×10^{-9}
30	25	30	1	50	8.33×10^{-7}	1.00×10^{-3}	1.67×10^{-6}
30	25	30	10	50	8.33×10^{-6}	1.00×10^{-3}	1.67×10^{-7}
Timber							
8	5	30	1	2	5.63×10^{-7}	3.75×10^{-3}	2.50×10^{-7}
8	5	30	10	2	5.63×10^{-6}	3.75×10^{-3}	2.50×10^{-8}
8	5	30	1	50	5.63×10^{-7}	3.75×10^{-3}	6.25×10^{-6}
8	5	30	10	50	5.63×10^{-6}	3.75×10^{-3}	6.25×10^{-7}
Alluminium							
69	27	100	1	2	3.91×10^{-7}	1.46×10^{-3}	2.92×10^{-8}
69	27	100	10	2	3.91×10^{-6}	1.46×10^{-3}	2.92×10^{-9}
69	27	100	1	50	3.91×10^{-7}	1.46×10^{-3}	7.30×10^{-7}
69	27	100	10	50	3.91×10^{-6}	1.46×10^{-3}	7.30×10^{-8}
Steel							
210	78	250	1	2	3.71×10^{-7}	1.19×10^{-3}	9.52×10^{-9}
210	78	250	10	2	3.71×10^{-6}	1.19×10^{-3}	9.52×10^{-10}
210	78	250	1	50	3.71×10^{-7}	1.19×10^{-3}	2.38×10^{-7}
210	78	250	10	50	3.71×10^{-6}	1.19×10^{-3}	2.38×10^{-8}

TABLE 7. Combination of structural and load configurations to be considered for determining the range of parameters of the design abacus.

ψ_σ	ψ_q	h_{min}	$\Delta\tilde{h}_1$	$\Delta\tilde{h}_3$	$\Delta\tilde{h}_5$	\tilde{V}
0.5×10^{-3}	4.0×10^{-8}	0.0101	0.0223	0.0037	-0.0025	0.000858
	8.6×10^{-8}	0.0129	0.0341	0.0022	-0.0056	0.001290
	1.8×10^{-7}	0.0221	0.0482	0.0073	-0.0048	0.001838
	4.0×10^{-7}	0.0325	0.0705	0.0107	-0.0071	0.002695
	8.6×10^{-7}	0.0478	0.1032	0.0158	-0.0103	0.003955
	1.8×10^{-6}	0.0708	0.1508	0.0234	-0.0148	0.005810
	4.0×10^{-6}	0.1057	0.2192	0.0351	-0.0209	0.008551
1.0×10^{-3}	4.0×10^{-8}	0.0077	0.0163	0.0044	-0.0041	0.000781
	8.6×10^{-8}	0.0099	0.0209	0.0057	-0.0053	0.001006
	1.8×10^{-7}	0.0152	0.0321	0.0070	-0.0042	0.001311
	4.0×10^{-7}	0.0229	0.0499	0.0075	-0.0050	0.001904
	8.6×10^{-7}	0.0337	0.0731	0.0111	-0.0073	0.002794
	1.8×10^{-6}	0.0496	0.1070	0.0164	-0.0107	0.004102
	4.0×10^{-6}	0.0750	0.1567	0.0233	-0.0140	0.006033
1.5×10^{-3}	4.0×10^{-8}	0.0077	0.0163	0.0044	-0.0041	0.000781
	8.6×10^{-8}	0.0099	0.0209	0.0057	-0.0053	0.001006
	1.8×10^{-7}	0.0128	0.0269	0.0074	-0.0069	0.001298
	4.0×10^{-7}	0.0167	0.0350	0.0095	-0.0086	0.001675
	8.6×10^{-7}	0.0275	0.0597	0.0090	-0.0060	0.002280
	1.8×10^{-6}	0.0404	0.0875	0.0133	-0.0088	0.003347
	4.0×10^{-6}	0.0597	0.1279	0.0197	-0.0127	0.004917

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- 1 Beam with variable cross-section: coordinate system, load, displacements, internal forces. The displacement field is denoted with components v, w, ϕ . The internal forces are N (axial), V (shear) and M (bending moment). 39
- 2 Beam with variable cross-section generated with the emptying function of Eqn. (23). 40
- 3 Comparison of optimal beam solutions with variable cross-section considering different number of lobes. The parameters related to the weight per unit mass, the load and the maximum allowable stress are reported in Table 2. 41
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- 6 Beam height profiles of the beams for various ψ_σ and ψ_q . The values of the design vector are reported in Table 7. 44

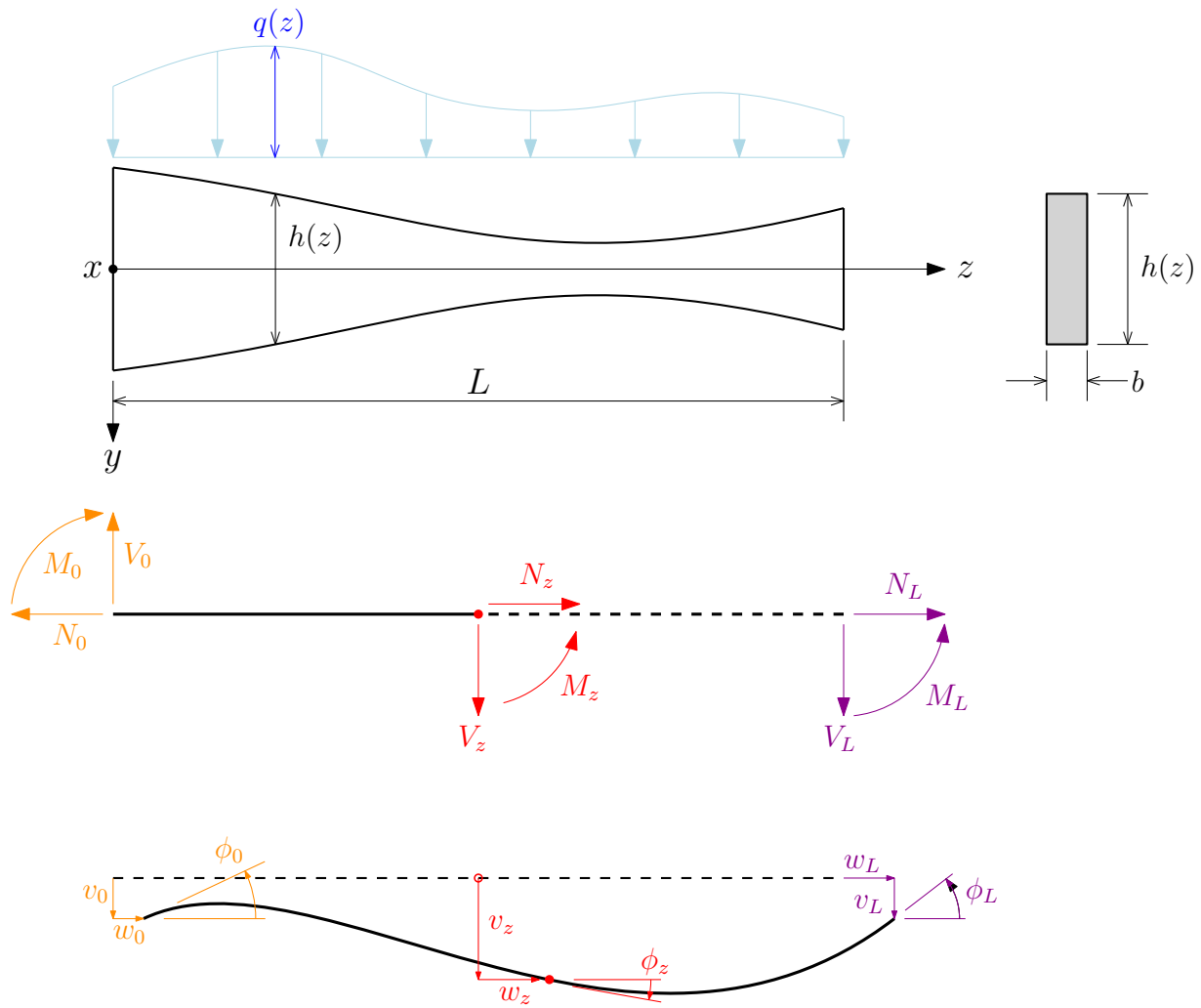


Fig. 1. Beam with variable cross-section: coordinate system, load, displacements, internal forces. The displacement field is denoted with components v , w , ϕ . The internal forces are N (axial), V (shear) and M (bending moment).

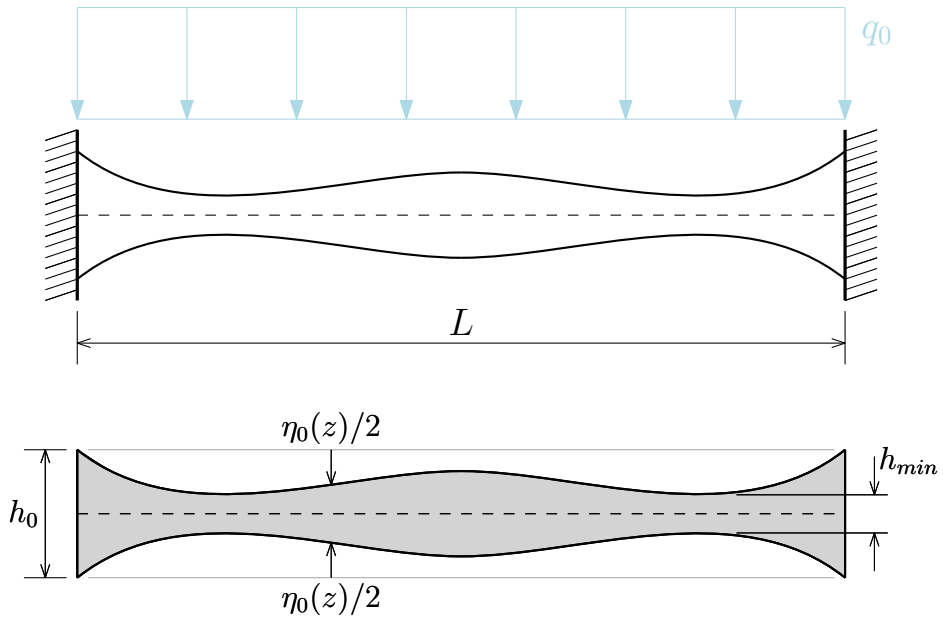
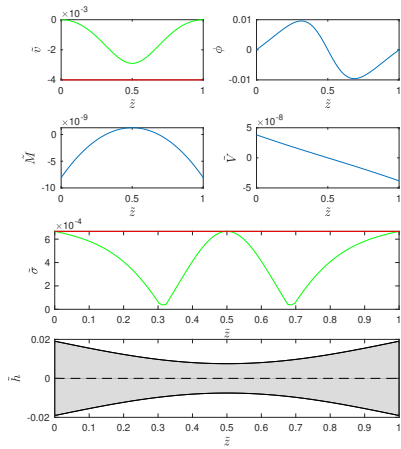
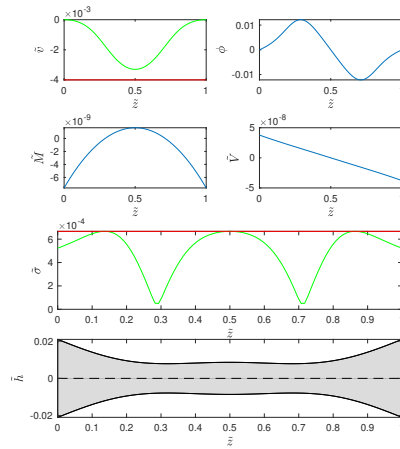


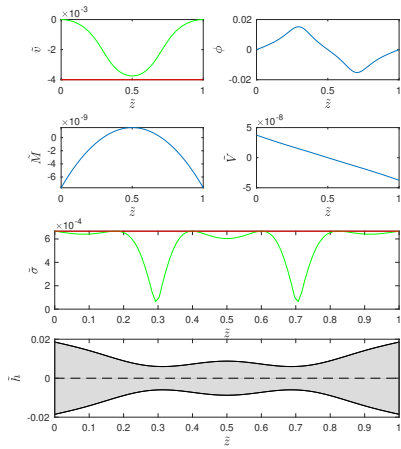
Fig. 2. Beam with variable cross-section generated with the emptying function of Eqn. (23).



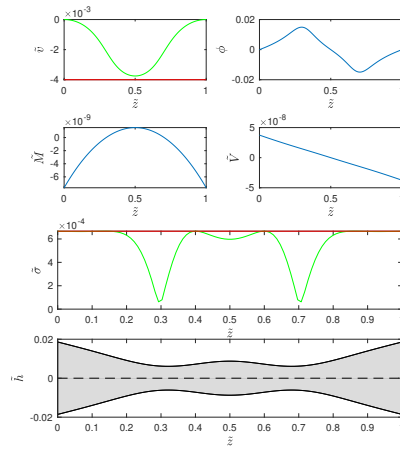
(a) One-lobe solution, $N = 1$



(b) Three-lobes solution, $N = 3$



(c) Five-lobes solution, $N = 5$



(d) Seven-lobes solution, $N = 7$

Fig. 3. Comparison of optimal beam solutions with variable cross-section considering different number of lobes. The parameters related to the weight per unit mass, the load and the maximum allowable stress are reported in Table 2.

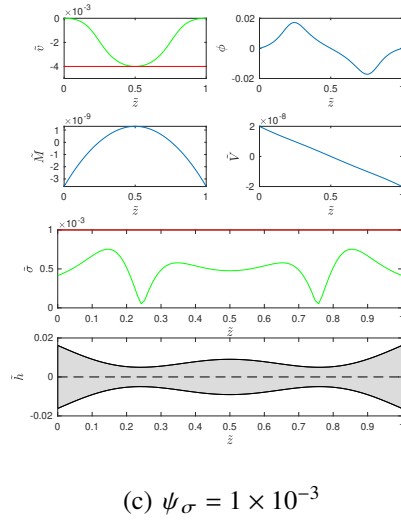
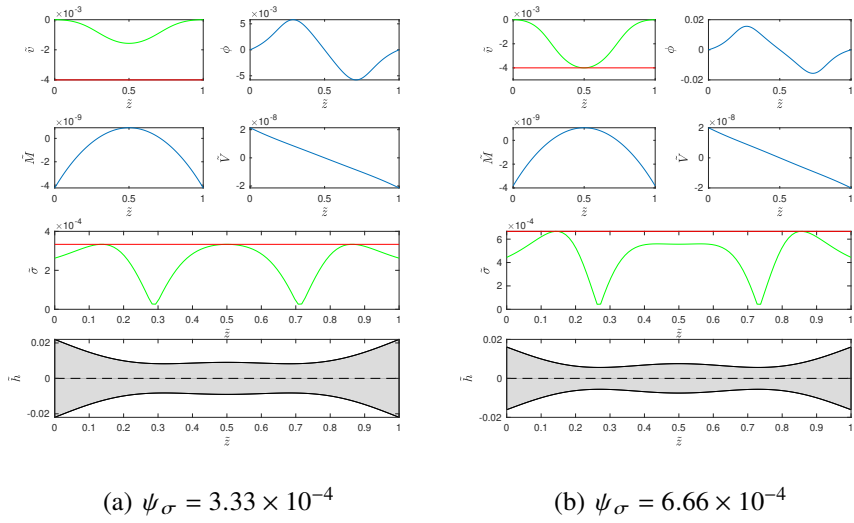


Fig. 4. Comparison of optimal beam solutions of three-lobes variable cross-section considering different maximum stress levels, i.e. the value of parameter ψ_σ . The parameters related to the weight per unit mass, the load and the maximum allowable stress, as well as the optimal solution are reported in Table 4.

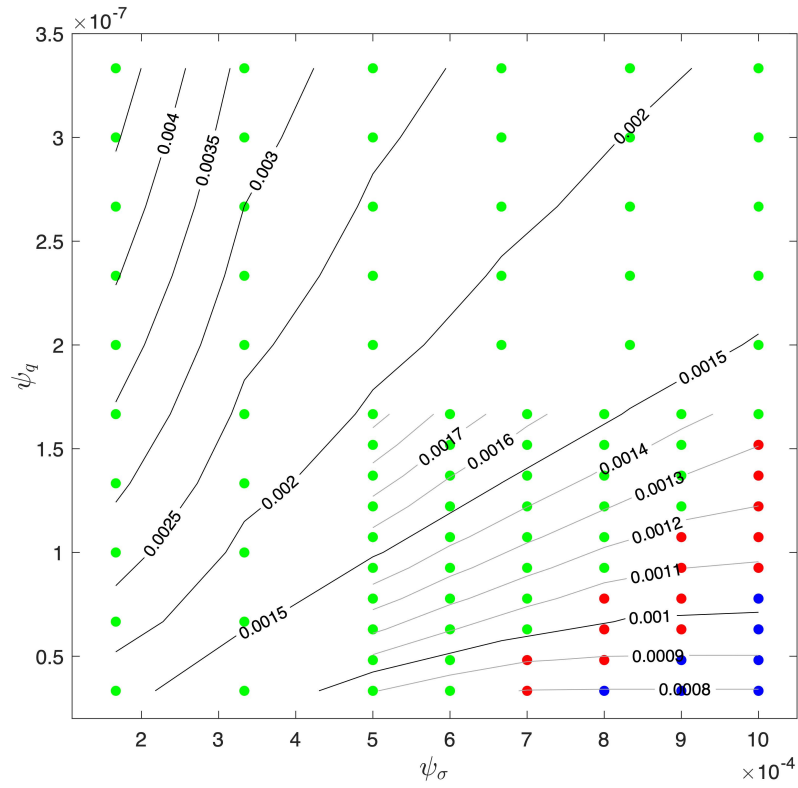


Fig. 5. Value of the dimensionless volume of the beam \tilde{V} and binding solution constraints for different ψ_σ and ψ_q . The bullets indicate whether the relevant solution constraint is the maximum stress (green), Eqn. (21), the maximum displacement (blue), Eqn. (22), or both (red).

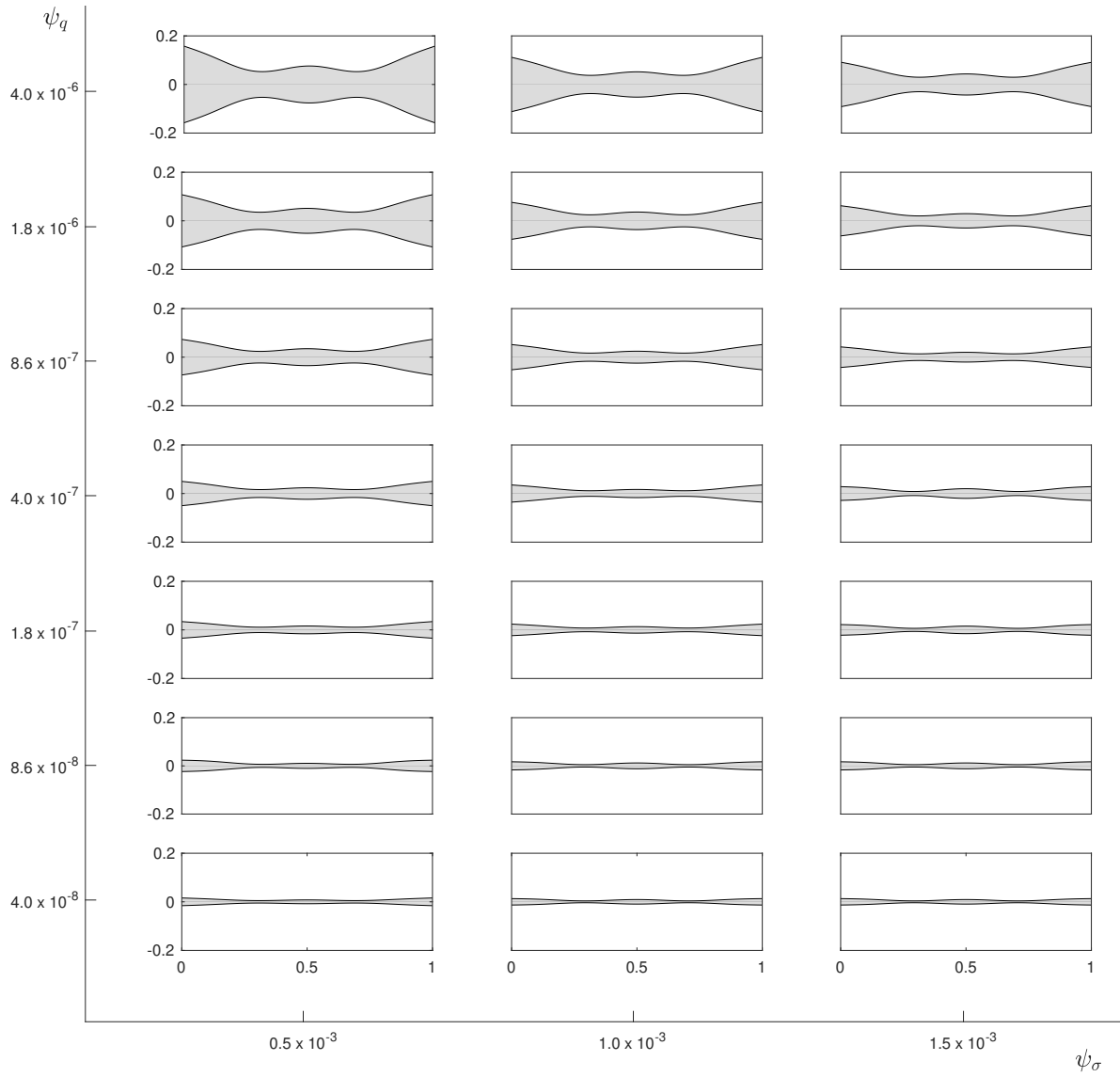


Fig. 6. Beam height profiles of the beams for various ψ_σ and ψ_q . The values of the design vector are reported in Table 7.