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*Original*

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# On the inverse of a Fibonacci number modulo a Fibonacci number being a Fibonacci number

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## Abstract

Let  $(F_n)_{n \geq 1}$  be the sequence of Fibonacci numbers. For all integers  $a$  and  $b \geq 1$  with  $\gcd(a, b) = 1$ , let  $[a^{-1} \bmod b]$  be the multiplicative inverse of  $a$  modulo  $b$ , which we pick in the usual set of representatives  $\{0, 1, \dots, b-1\}$ . Put also  $[a^{-1} \bmod b] := \infty$  when  $\gcd(a, b) > 1$ .

We determine all positive integers  $m$  and  $n$  such that  $[F_m^{-1} \bmod F_n]$  is a Fibonacci number. This extends a previous result of Premreesuk, Noppakaew, and Pongsriiam, who considered the special case  $m \in \{3, n-3, n-2, n-1\}$  and  $n \geq 7$ .

Let  $(L_n)_{n \geq 1}$  be the sequence of Lucas numbers. We also determine all positive integers  $m$  and  $n$  such that  $[L_m^{-1} \bmod L_n]$  is a Lucas number.

**Keywords:** congruences; Fibonacci numbers; Lucas numbers; modular arithmetic; modular multiplicative inverse.

**MSC2020:** Primary: 11B39, Secondary: 11A99.

## 1 Introduction

Let  $(F_n)_{n \geq 1}$  be the sequence of Fibonacci numbers, which is defined by  $F_1 := 1$ ,  $F_2 := 1$ , and  $F_n := F_{n-1} + F_{n-2}$  for every integer  $n \geq 3$ . Several authors studied modular multiplicative inverses related to Fibonacci numbers. For instance, Komatsu, Luca, and Tachiya [3] (see also [2]) studied the multiplicative order of  $F_{n+1}F_n^{-1}$  modulo  $F_m$ , where  $m$  and  $n$  are positive integers such that  $\gcd(F_m, F_{n+1}F_n) = 1$ . Luca, Stănică, and Yalçiner [5] studied the positive integers  $M$  such that the invertible residue classes modulo  $M$  represented by Fibonacci numbers form a subgroup. For all integers  $a$  and  $b \geq 1$  with  $\gcd(a, b) = 1$ , let  $[a^{-1} \bmod b]$  be the unique  $x \in \{0, 1, \dots, b-1\}$  such that  $ax \equiv 1 \pmod{b}$ . For the sake of convenience, put also  $[a^{-1} \bmod b] := \infty$  when  $\gcd(a, b) > 1$ . Alecci, Murru, and Sanna [1] determined the Zeckendorf representation of  $[a^{-1} \bmod F_n]$ , for every fixed  $a \geq 3$  and for every integer  $n \geq 1$ . (The case  $a = 2$  was previously solved by Premreesuk, Noppakaew, and Pongsriiam [6]). Motivated by some results in knot theory [4], Song [9] found four families of pairs  $(F_m, F_n)$  of Fibonacci numbers such that  $[F_m^{-1} \bmod F_n]$  and  $[F_n^{-1} \bmod F_m]$  are both Fibonacci numbers. Sanna [8] proved that these families, together with some isolated pairs, are indeed all the pairs of Fibonacci numbers with such a property. For integers  $n \geq 7$  and  $m \in \{3, n-3, n-2, n-1\}$ , Premreesuk, Noppakaew, and Pongsriiam [6] found necessary and sufficient conditions for  $[F_m^{-1} \bmod F_n]$  to be a Fibonacci number.

Our first result is the following.

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**Theorem 1.1.** *Let  $\ell, m, n$  be integers with  $\ell \geq 2$ ,  $m \geq 3$ ,  $n \geq 4$  and  $m < 4n$ . Then*

$$F_\ell = [F_m^{-1} \bmod F_n] \quad (1)$$

*if and only if:*

- (c1)  $\ell = n - \frac{1}{2}(3 + (-1)^n)$  and  $m = n - 2$ ; or
- (c2)  $\ell = n - \frac{1}{2}(3 - (-1)^n)$  and  $m \in \{n - 1, n + 1, n + 2\}$ ; or
- (c3)  $\ell = 2$ ,  $m = 2n - 2$ , and  $n$  is odd; or
- (c4)  $\ell = 2$ ,  $m \in \{2n - 1, 2n + 1, 2n + 2\}$ , and  $n$  is even; or
- (c5)  $\ell = n - 2$  and  $m = 3n - 2$ ; or
- (c6)  $\ell = n - 1$  and  $m \in \{3n - 1, 3n + 1, 3n + 2\}$ ; or
- (c7)  $\ell = 2$  and  $m = 4n - 1$ .

We remark that, in Theorem 1.1, the conditions on  $\ell, m, n$  are not restrictive. In fact, since  $F_1 = F_2 = 1$ , there is no loss in generality in assuming that  $\ell, m, n \geq 2$ . Moreover, the cases in which  $m = 2$  or  $n \in \{2, 3\}$  are easy to study. Hence, for the sake of brevity, we did not include them. Finally, in light of Lemma 2.3, the condition  $m < 4n$  is not a restriction.

Let  $(L_n)_{n \geq 1}$  be the sequence of Lucas numbers, which is defined by  $L_1 := 2$ ,  $L_2 := 1$ , and  $L_n = L_{n-1} + L_{n-2}$  for every integer  $n \geq 3$ .

Our second result is the following.

**Theorem 1.2.** *Let  $\ell, m, n$  be integers with  $\ell \geq 1$ ,  $m \geq 2$ ,  $n \geq 5$ , and  $m \leq 4n$ . Then*

$$L_\ell = [L_m^{-1} \bmod L_n] \quad (2)$$

*if and only if:*

- (d1)  $\ell = \frac{1}{2}(n \pm 1)$ ,  $m = \frac{1}{2}(n \mp 1)$ , and  $n \equiv 1 \pmod{4}$ ; or
- (d2)  $\ell = \frac{1}{2}(n + 1)$ ,  $m = \frac{1}{2}(3n + 1)$ , and  $n$  is odd; or
- (d3)  $\ell = 1$  and  $m = 2n - (-1)^n$ ; or
- (d4)  $\ell = \frac{1}{2}(n - 1)$ ,  $m = \frac{1}{2}(5n + 1)$ , and  $n \equiv 1 \pmod{4}$ ; or
- (d5)  $\ell = \frac{1}{2}(n + 1)$ ,  $m = \frac{1}{2}(5n - 1)$ , and  $n \equiv 1 \pmod{4}$ ; or
- (d6)  $\ell = \frac{1}{2}(n + 1)$ ,  $m = \frac{1}{2}(7n + 1)$ , and  $n$  is odd.

We remark that, in Theorem 1.2, the conditions on  $\ell, m, n$  are not restrictive. In fact, the cases in which  $m = 1$  or  $n \in \{2, 3, 4\}$  are easy to study. Hence, for the sake of brevity, we did not include them. Furthermore, in light of Lemma 2.3, the condition  $m \leq 4n$  is not a restriction.

The proof of Theorem 1.1 (resp. 1.2) is based on approximating  $F_\ell F_m / F_n$  (resp.  $L_\ell L_m / L_n$ ) with an appropriate linear combination of Fibonacci (resp. Lucas) numbers, which depends on the size of  $m$  relatively to  $\ell$  and  $n$ . This approximation is sufficiently accurate to imply certain identities for the integers  $\ell, m, n$  that satisfy (1) (resp. (2)). Then, these identities make possible to determine  $\ell, m, n$ .

## 2 Preliminaries

It is well known that the Binet formulas

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n \quad (3)$$

hold for every integer  $n \geq 1$ , where  $\alpha := (1 + \sqrt{5})/2$  and  $\beta := (1 - \sqrt{5})/2$  (see, e.g., [7, Ch. 1]).

In fact, it is useful to extend the sequences of Fibonacci and Lucas numbers to all integers by using (3). In particular, from (3) it follows that

$$F_{-n} = (-1)^{n+1}F_n \quad \text{and} \quad L_{-n} = (-1)^nL_n, \quad (4)$$

for every integer  $n$ .

We also need the following lemmas.

**Lemma 2.1.** *We have that*

$$(i) \quad L_aL_b = L_{a+b} + (-1)^bL_{a-b};$$

$$(ii) \quad 5F_aF_b = L_{a+b} - (-1)^bL_{a-b};$$

$$(iii) \quad F_aF_b = F_{a+b} - F_{a+b-2} - F_{a-1}F_{b-1};$$

for all integers  $a, b$ .

*Proof.* These identities follow easily from (3). □

**Lemma 2.2.** *We have that*

$$(i) \quad F_aF_b - F_cF_d = (-1)^{a+1}F_{c-a}F_{d-a};$$

$$(ii) \quad L_aL_b - L_cL_d = (-1)^a5F_{c-a}F_{d-a};$$

$$(iii) \quad 5F_aF_b - L_cL_d = (-1)^{a+1}L_{c-a}L_{d-a};$$

for all integers  $a, b, c, d$  with  $a + b = c + d$ .

*Proof.* These identities can be proved either directly from (3), or by taking appropriate differences of pairs of the identities of Lemma 2.1 and, eventually, employing (4). □

**Lemma 2.3.** *We have that*

$$(i) \quad F_{a+4n} \equiv F_a \pmod{F_n};$$

$$(ii) \quad L_{a+4n} \equiv L_a \pmod{L_n};$$

for all integers  $a$  and  $n \geq 1$ .

*Proof.* Using (3) one can verify that

$$F_{a+4n} - F_a = F_nL_nL_{a+2n} \quad \text{and} \quad L_{a+4n} - L_a = 5F_nF_{a+2n}L_n,$$

from which the claim follows. □

**Lemma 2.4.** *Let  $a, b, n$  be integers. We have that:*

$$(i) \quad |F_aF_b| < F_n - 1 \text{ if } |a| + |b| \leq n \text{ and } n \geq 5;$$

$$(ii) \quad |5F_aF_b| < L_n - 1 \text{ if } |a| + |b| \leq n - 1 \text{ and } n \geq 6;$$

$$(iii) \quad |L_a| < L_n - 1 \text{ if } |a| \leq n - 1 \text{ and } n \geq 4;$$

$$(iv) \quad |L_aL_b| < L_n - 1 \text{ if } |a| + |b| \leq n - 1, \{|a|, |b|\} \notin \{0, n - 1\}, \text{ and } n \geq 6.$$

*Proof.* By (4), we have that  $|F_k| = F_{|k|}$  and  $|L_k| = L_{|k|}$ , for every integer  $k$ . Hence, throughout the proof, we can assume that  $a, b \geq 0$ . Moreover, by symmetry, we can assume that  $a \geq b$ .

Let us prove (i) and (ii). If  $b = 0$  or  $a + b \leq 4$ , then (i) and (ii) follow easily. Hence, assume that  $b \geq 1$  and  $a + b \geq 5$ . Then, by Lemma 2.1(iii), we have that

$$F_a F_b = F_{a+b} - F_{a+b-2} - F_{a-1} F_{b-1} \leq F_n - F_3 < F_n - 1,$$

whenever  $a + b \leq n$ , which proves (i). Furthermore, by Lemma 2.1(ii), we have that

$$5F_a F_b = L_{a+b} - (-1)^b L_{a-b} \leq L_{a+b} + L_{a-b} \leq L_{n-1} + L_{n-3} = L_n - L_{n-4} < L_n - 1,$$

whenever  $a + b \leq n - 1$  and  $n \geq 6$ , which proves (ii).

If  $a \leq n - 1$  and  $n \geq 4$ , then

$$L_a \leq \max\{2, L_{n-1}\} = L_{n-1} = L_n - L_{n-2} < L_n - 1,$$

which proves (iii).

By Lemma 2.1(i), we have that

$$L_{a+b} = L_{a+b} + (-1)^b L_{a-b} \leq L_{a+b} + L_{a-b}.$$

Therefore, if  $a + b \leq n - 2$  and  $n \geq 5$ , then

$$L_{a+b} \leq 2L_{n-2} = L_n - L_{n-3} < L_n - 1.$$

Furthermore, if  $a + b = n - 1$ ,  $b \geq 1$ , and  $n \geq 6$ , then

$$L_{a+b} \leq L_{n-1} + L_{n-3} = L_n - L_{n-4} < L_n - 1.$$

Thus (iv) is proved.  $\square$

### 3 Proof of Theorem 1.1

With a bit of patience, one can check that Theorem 1.1 holds for  $n = 4$ . Hence, hereafter, assume that  $n \geq 5$ .

Suppose that (1) is satisfied. Hence, we have that  $F_\ell < F_n$  and

$$F_\ell F_m - A F_n = 1, \tag{5}$$

for some integer  $A \geq 0$ . In particular, from  $F_\ell < F_n$  it follows that  $\ell < n$ . Moreover, since  $m \geq 3$ , we get that  $A \geq 1$ . Consequently, we have that  $F_\ell F_m \geq F_n + 1$ . Hence, by Lemma 2.4(i), we get that  $m > n - \ell$ .

Define the four disjoint intervals

$$I_1 := (n - \ell, n + \ell], \quad I_2 := (n + \ell, 3n - \ell], \quad I_3 := (3n - \ell, 3n + \ell], \quad I_4 := (3n + \ell, 4n).$$

By the previous considerations, we have that  $m$  belongs to exactly one of such intervals.

For every integer  $k$ , put  $F_k^+ := F_k$  if  $k \geq 0$ , and  $F_k^+ := 0$  if  $k < 0$ . Then, define

$$A_{\ell, m, n} := F_{\ell+m-n} - (-1)^\ell F_{-\ell+m-n}^+ + (-1)^n F_{\ell+m-3n}^+ - (-1)^{\ell+n} F_{-\ell+m-3n}^+$$

and

$$B_{\ell, m, n} := F_\ell F_m - A_{\ell, m, n} F_n. \tag{6}$$

Let us prove that

$$B_{\ell, m, n} = \begin{cases} (-1)^{\ell+1} F_{m-n} F_{n-\ell} & \text{if } m \in I_1; \\ (-1)^n F_\ell F_{m-2n} & \text{if } m \in I_2; \\ (-1)^{\ell+n+1} F_{m-3n} F_{n-\ell} & \text{if } m \in I_3; \\ F_\ell F_{m-4n} & \text{if } m \in I_4. \end{cases} \tag{7}$$

By Lemma 2.2(i), we have that

$$F_\ell F_m - F_{\ell+m-n} F_n = (-1)^{\ell+1} F_{m-n} F_{n-\ell}. \quad (8)$$

Hence, we get that  $B_{\ell,m,n} = (-1)^{\ell+1} F_{m-n} F_{n-\ell}$  for each  $m \in I_1$ .

By (8), Lemma 2.2(i), and (4), we have that

$$\begin{aligned} F_\ell F_m - (F_{\ell+m-n} - (-1)^\ell F_{-\ell+m-n}) F_n &= (-1)^{\ell+1} (F_{m-n} F_{n-\ell} - F_{-\ell+m-n} F_n) \\ &= (-1)^{\ell+m+n} F_{-\ell} F_{2n-m} = (-1)^n F_\ell F_{m-2n} \end{aligned} \quad (9)$$

Hence, we get that  $B_{\ell,m,n} = (-1)^n F_\ell F_{m-2n}$  for each  $m \in I_2$ .

By (9) and Lemma 2.2(i), we have that

$$\begin{aligned} F_\ell F_m - (F_{\ell+m-n} - (-1)^\ell F_{-\ell+m-n} + (-1)^n F_{\ell+m-3n}) F_n \\ = (-1)^n (F_\ell F_{m-2n} - F_{\ell+m-3n} F_n) = (-1)^{\ell+n+1} F_{m-3n} F_{n-\ell}. \end{aligned} \quad (10)$$

Hence, we get that  $B_{\ell,m,n} = (-1)^{\ell+n+1} F_{m-3n} F_{n-\ell}$  for each  $m \in I_3$ .

Finally, by (10), (4), and Lemma 2.2(i), we have that

$$\begin{aligned} F_\ell F_m - (F_{\ell+m-n} - (-1)^\ell F_{-\ell+m-n} + (-1)^n F_{\ell+m-3n} - (-1)^{\ell+n} F_{-\ell+m-3n}) F_n \\ = (-1)^{\ell+n+1} (F_{m-3n} F_{n-\ell} - F_{-\ell+m-3n} F_n) = (-1)^{\ell+m} F_{-\ell} F_{4n-m} = F_\ell F_{m-4n}. \end{aligned}$$

Hence, we get that  $B_{\ell,m,n} = F_\ell F_{m-4n}$  for each  $m \in I_4$ . The proof of (7) is complete.

At this point, considering the four cases in (7), one can easily check that  $B_{\ell,m,n}$  is equal to  $\pm F_a F_b$ , where  $a$  and  $b$  are integers (depending on  $\ell, m, n$ ) such that  $|a| + |b| \leq n$ .

Therefore, from (6) and Lemma 2.4(i), we get that

$$\left| \frac{F_\ell F_m}{F_n} - A_{\ell,m,n} \right| = \frac{|B_{\ell,m,n}|}{F_n} = \frac{|F_a F_b|}{F_n} < 1 - \frac{1}{F_n}.$$

Consequently, recalling (5), we have that

$$|A - A_{\ell,m,n}| \leq \left| A - \frac{F_\ell F_m}{F_n} \right| + \left| \frac{F_\ell F_m}{F_n} - A_{\ell,m,n} \right| < \frac{1}{F_n} + \left( 1 - \frac{1}{F_n} \right) = 1,$$

which implies that  $A = A_{\ell,m,n}$ , since  $A$  and  $A_{\ell,m,n}$  are both integers. Then, from (5) and (6), we get that  $B_{\ell,m,n} = 1$ .

Note that, for every integer  $k$ , we have that  $|F_k| = 1$  if and only if  $k \in \{-2, -1, 1, 2\}$ . In particular, we have that  $F_{-2} = -1$  and  $F_{-1} = F_1 = F_2 = 1$ . Therefore, from  $B_{\ell,m,n} = 1$  we can determine  $\ell$  and  $m$  in terms of  $n$  in each of the four cases in (7).

If  $m \in I_1$ , then  $(-1)^{\ell+1} F_{m-n} F_{n-\ell} = 1$ . Hence, we have that  $m \in \{n-2, n-1, n+1, n+2\}$  and  $\ell \in \{n-2, n-1\}$  (recall that  $\ell < n$ ). If  $m = n-2$ , then either  $\ell = n-2$  and  $n$  is even, or  $\ell = n-1$  and  $n$  is odd. This is case (c1). If  $m \in \{n-1, n+1, n+2\}$ , then either  $\ell = n-1$  and  $n$  is even, or  $\ell = n-2$  and  $n$  is odd. This is case (c2).

If  $m \in I_2$ , then  $(-1)^n F_\ell F_{m-2n} = 1$ . Hence,  $\ell = 2$  and  $m \in \{2n-2, 2n-1, 2n+1, 2n+2\}$ . If  $m = 2n-2$ , then  $F_{m-2n} = -1$  and consequently  $n$  is odd, which is case (c3). If  $m \in \{2n-1, 2n+1, 2n+2\}$ , then  $F_{m-2n} = 1$  and consequently  $n$  is even, which is case (c4).

If  $m \in I_3$ , then  $(-1)^{\ell+n+1} F_{m-3n} F_{n-\ell} = 1$ . Hence, it follows that  $\ell \in \{n-2, n-1\}$  and  $m \in \{3n-2, 3n-1, 3n+1, 3n+2\}$ . If  $\ell = n-2$ , then  $(-1)^{\ell+n+1} F_{n-\ell} = -1$  and consequently  $m = 3n-2$ , which is case (c5). If  $\ell = n-1$ , then  $(-1)^{\ell+n+1} F_{n-\ell} = 1$  and consequently  $m \in \{3n-1, 3n+1, 3n+2\}$ , which is case (c6).

If  $m \in I_4$ , then  $F_\ell F_{m-4n} = 1$ . Hence, we have that  $\ell = 2$  and  $m = 4n-1$  (recall that  $m < 4n$ ), which is case (c7).

At this point, we have proved that if (1) is true then the integers  $\ell, m, n$  are of the form given by (c1)–(c7).

Vice versa, using (7), one can easily verify that if  $\ell, m, n$  are of the form given by (c1)–(c7) then  $B_{\ell,m,n} = 1$ . In turn, by (6), this implies that (1) is true.

The proof of Theorem 1.1 is complete.

## 4 Proof of Theorem 1.2

With a bit of patience, one can check that Theorem 1.2 holds for  $n = 5$ . Hence, hereafter, assume that  $n \geq 6$ .

Suppose that (2) is satisfied. Hence, we have that  $L_\ell < L_n$  and

$$L_\ell L_m - C L_n = 1, \quad (11)$$

for some integer  $C \geq 0$ . In particular, from  $L_\ell < L_n$  it follows that  $\ell < n$ . Moreover, since  $m \geq 2$ , we get that  $C \geq 1$ . Consequently, we have that  $L_\ell L_m \geq L_n + 1$ . Hence, by Lemma 2.4(iv), we get that  $m \geq n - \ell$ .

Define the four disjoint intervals

$$J_1 := [n - \ell, n + \ell), \quad J_2 := [n + \ell, 3n - \ell), \quad J_3 := [3n - \ell, 3n + \ell), \quad J_4 := [3n + \ell, 4n].$$

By the previous considerations, we have that  $m$  belongs to exactly one of such intervals.

For every integer  $k$ , put  $L_k^+ := L_k$  if  $k \geq 0$ , and  $L_k^+ := 0$  if  $k < 0$ . Then, define

$$C_{\ell, m, n} := L_{\ell+m-n} + (-1)^\ell L_{-\ell+m-n}^+ - (-1)^n L_{\ell+m-3n}^+ - (-1)^{\ell+n} L_{-\ell+m-3n}^+$$

and

$$D_{\ell, m, n} := L_\ell L_m - C_{\ell, m, n} L_n. \quad (12)$$

Let us prove that

$$D_{\ell, m, n} = \begin{cases} (-1)^\ell 5F_{m-n} F_{n-\ell} & \text{if } m \in J_1; \\ (-1)^{n+1} L_\ell L_{m-2n} & \text{if } m \in J_2; \\ (-1)^{\ell+n+1} 5F_{m-3n} F_{n-\ell} & \text{if } m \in J_3; \\ L_\ell L_{m-4n} & \text{if } m \in J_4. \end{cases} \quad (13)$$

From Lemma 2.2(ii), it follows that

$$L_\ell L_m - L_{\ell+m-n} L_n = (-1)^\ell 5F_{m-n} F_{n-\ell}. \quad (14)$$

Hence, we have that  $D_{\ell, m, n} = (-1)^\ell 5F_{m-n} F_{n-\ell}$  for every  $m \in J_1$ .

From (14), Lemma 2.2(iii), and (4), it follows that

$$\begin{aligned} L_\ell L_m - (L_{\ell+m-n} + (-1)^\ell L_{-\ell+m-n}) L_n &= (-1)^\ell (5F_{m-n} F_{n-\ell} - L_{-\ell+m-n} L_n) \\ &= (-1)^{\ell+m+n+1} L_{-\ell} L_{2n-m} = (-1)^{n+1} L_\ell L_{m-2n}. \end{aligned} \quad (15)$$

Hence, we have that  $D_{\ell, m, n} = (-1)^{n+1} L_\ell L_{m-2n}$  for every  $m \in J_2$ .

From (15) and Lemma 2.2(ii), it follows that

$$\begin{aligned} L_\ell L_m - (L_{\ell+m-n} + (-1)^\ell L_{-\ell+m-n} - (-1)^n L_{\ell+m-3n}) L_n \\ = (-1)^{n+1} (L_\ell L_{m-2n} - L_{\ell+m-3n} L_n) = (-1)^{\ell+n+1} 5F_{m-3n} F_{n-\ell}. \end{aligned} \quad (16)$$

Hence, we have that  $D_{\ell, m, n} = (-1)^{\ell+n+1} 5F_{m-3n} F_{n-\ell}$  for every  $m \in J_3$ .

Finally, from (16), Lemma 2.2(iii), and (4), it follows that

$$\begin{aligned} L_\ell L_m - (L_{\ell+m-n} + (-1)^\ell L_{-\ell+m-n} - (-1)^n L_{\ell+m-3n} - (-1)^{\ell+n} L_{-\ell+m-3n}) L_n \\ = (-1)^{\ell+n+1} (5F_{m-3n} F_{n-\ell} - L_{-\ell+m-3n} L_n) = (-1)^{\ell+m} L_{-\ell} L_{4n-m} = L_\ell L_{m-4n}. \end{aligned}$$

Hence, we have that  $D_{\ell, m, n} = L_\ell L_{m-4n}$  for every  $m \in J_4$ . The proof of (13) is complete.

Suppose that  $C', D'$  are integers such that

$$\left| \frac{L_\ell L_m}{L_n} - C' \right| < 1 - \frac{1}{L_n} \quad \text{and} \quad D' = L_\ell L_m - C' L_n.$$

Then, by (11), we have that

$$|C - C'| \leq \left| C - \frac{L_\ell L_m}{L_n} \right| + \left| \frac{L_\ell L_m}{L_n} - C' \right| < \frac{1}{L_n} + \left( 1 - \frac{1}{L_n} \right) = 1.$$

Consequently, we have that  $C' = C$  and, by (11) again, that  $D' = 1$ .

Hereafter, we will make use of such a fact several times, by taking  $C' = C_{\ell, m, n} + s$  and  $D' = D_{\ell, m, n} - sL_n$  for some  $s \in \{-1, 0, 1\}$ .

We will also use the fact that, for every integer  $k$ , we have that  $|L_k| = 1$  if and only if  $k \in \{-1, 1\}$ . Precisely, we have that  $L_{-1} = -1$  and  $L_1 = 1$ .

If  $m = n - \ell$ , then from (13), (4), and Lemma 2.1(ii) it follows that

$$D_{\ell, m, n} = (-1)^{\ell} 5F_{-\ell} F_{n-\ell} = -5F_\ell F_{n-\ell} = -L_n + (-1)^{n-\ell} L_{2\ell-n}.$$

Hence, by Lemma 2.4(iii), we have that

$$\left| \frac{L_\ell L_m}{L_n} - (C_{\ell, m, n} - 1) \right| = \frac{|L_{2\ell-n}|}{L_n} < 1 - \frac{1}{L_n},$$

which implies that  $(-1)^{n-\ell} L_{2\ell-n} = 1$ . Therefore, either  $n - \ell$  is even and  $2\ell - n = 1$ , or  $n - \ell$  is odd and  $2\ell - n = -1$ . Recalling that  $m = n - \ell$ , it follows that  $\ell = \frac{1}{2}(n \pm 1)$ ,  $m = \frac{1}{2}(n \mp 1)$ , and  $n \equiv 1 \pmod{4}$ , which is case (d1).

If  $m \in J_1 \setminus \{n - \ell\}$ , then (13) and Lemma 2.4(ii) yield that

$$\left| \frac{L_\ell L_m}{L_n} - C_{\ell, m, n} \right| = \frac{5|F_{m-n} F_{n-\ell}|}{L_n} < 1 - \frac{1}{L_n},$$

which implies that  $(-1)^{\ell} 5F_{m-n} F_{n-\ell} = 1$ . However, this last equality is clearly impossible.

If  $m = n + \ell$ , then (13) and Lemma 2.1(i) yield that

$$D_{\ell, m, n} = (-1)^{n+1} L_\ell L_{\ell-n} = (-1)^{n+1} L_{2\ell-n} - (-1)^\ell L_n.$$

Hence, by Lemma 2.4(iii), we have that

$$\left| \frac{L_\ell L_m}{L_n} - (C_{\ell, m, n} - (-1)^\ell) \right| = \frac{|L_{2\ell-n}|}{L_n} < 1 - \frac{1}{L_n},$$

which implies that  $(-1)^{n+1} L_{2\ell-n} = 1$ . Therefore, either  $n$  is odd and  $2\ell - n = 1$ , or  $n$  is even and  $2\ell - n = -1$ . However, the latter case is impossible. Hence, recalling that  $m = n + \ell$ , we get that  $\ell = \frac{1}{2}(n + 1)$ ,  $m = \frac{1}{2}(3n + 1)$ , and  $n$  is odd, which is case (d2).

If  $m \in J_2 \setminus \{n + \ell\}$  and  $(\ell, m) \neq (n - 1, 2n)$ , then (13) and Lemma 2.4(iv) yield that

$$\left| \frac{L_\ell L_m}{L_n} - C_{\ell, m, n} \right| = \frac{|L_\ell L_{m-2n}|}{L_n} < 1 - \frac{1}{L_n},$$

which implies that  $(-1)^{n+1} L_\ell L_{m-2n} = 1$ . Thus  $\ell = 1$  and either  $m = 2n + 1$  and  $n$  is odd, or  $m = 2n - 1$  and  $n$  is even. This is case (d3).

If  $\ell = n - 1$  and  $m = 2n$  then, by (13), we get that

$$D_{\ell, m, n} = (-1)^{n+1} 2L_{n-1} = (-1)^{n+1} L_n + (-1)^{n+1} L_{n-3}.$$

Hence, by Lemma 2.4(iii), we have that

$$\left| \frac{L_\ell L_m}{L_n} - (C_{\ell, m, n} + (-1)^{n+1}) \right| = \frac{|L_{n-3}|}{L_n} < 1 - \frac{1}{L_n},$$

which implies that  $(-1)^{n+1} L_{n-3} = 1$ , but this last equality is impossible.

If  $m = 3n - \ell$ , then (13), (4), and Lemma 2.1(ii) yield that

$$\begin{aligned} D_{\ell,m,n} &= (-1)^{\ell+n+1} 5F_{-\ell}F_{n-\ell} = (-1)^n 5F_{\ell}F_{n-\ell} = (-1)^n (L_n - (-1)^{n-\ell} L_{2\ell-n}) \\ &= (-1)^n L_n - (-1)^{\ell} L_{2\ell-n}. \end{aligned}$$

Hence, by Lemma 2.4(iii), we have that

$$\left| \frac{L_{\ell}L_m}{L_n} - (C_{\ell,m,n} + (-1)^n) \right| = \frac{|L_{2\ell-n}|}{L_n} < 1 - \frac{1}{L_n},$$

which implies that  $(-1)^{\ell+1} L_{2\ell-n} = 1$ . Therefore, either  $\ell$  is even and  $2\ell - n = -1$ , or  $\ell$  is odd and  $2\ell - n = 1$ . That is, we have that  $n \equiv 1 \pmod{4}$  and either  $\ell = \frac{1}{2}(n-1)$  and  $m = \frac{1}{2}(5n+1)$ , or  $\ell = \frac{1}{2}(n+1)$  and  $m = \frac{1}{2}(5n-1)$ . These are cases (d4) and (d5).

If  $m \in J_3 \setminus \{3n - \ell\}$ , then (13) and Lemma 2.4(ii) yield that

$$\left| \frac{L_{\ell}L_m}{L_n} - C_{\ell,m,n} \right| = \frac{|5F_{m-3n}F_{n-\ell}|}{L_n} < 1 - \frac{1}{L_n},$$

which implies that  $(-1)^{\ell+n+1} 5F_{m-3n}F_{n-\ell} = 1$ , but this last equality is impossible.

If  $m = 3n + \ell$ , then (13) and Lemma 2.1(i) yield that

$$D_{\ell,m,n} = L_{\ell}L_{\ell-n} = L_{2\ell-n} + (-1)^{\ell+n} L_n.$$

Hence, by Lemma 2.4(iii), we get that

$$\left| \frac{L_{\ell}L_m}{L_n} - (C_{\ell,m,n} + (-1)^{\ell+n}) \right| = \frac{|L_{2\ell-n}|}{L_n} < 1 - \frac{1}{L_n},$$

which implies that  $L_{2\ell-n} = 1$ . Hence, we get that  $2\ell - n = 1$ . Recalling that  $m = 3n + \ell$ , it follows that  $\ell = \frac{1}{2}(n+1)$ ,  $m = \frac{1}{2}(7n+1)$ , and  $n$  is odd, which is case (d6).

If  $m \in J_4 \setminus \{3n + \ell, 4n\}$  and  $(\ell, m) \neq (n-1, 4n)$ , then (13) and Lemma 2.4(iv) yield that

$$\left| \frac{L_{\ell}L_m}{L_n} - C_{\ell,m,n} \right| = \frac{|L_{\ell}L_{m-4n}|}{L_n} < 1 - \frac{1}{L_n},$$

which implies that  $L_{\ell}L_{m-4n} = 1$ . Hence, we have that  $\ell = 1$  and  $m = 4n + 1$ , which is impossible, since  $m \leq 4n$ .

If  $\ell = n - 1$  and  $m = 4n$  then, by (13), we get that

$$D_{\ell,m,n} = 2L_{n-1} = L_n + L_{n-3}.$$

Hence, by Lemma 2.4(iii), we have that

$$\left| \frac{L_{\ell}L_m}{L_n} - (C_{\ell,m,n} + 1) \right| = \frac{|L_{n-3}|}{L_n} < 1 - \frac{1}{L_n},$$

which implies that  $L_{n-3} = 1$ , but this last equality is impossible.

At this point, we have proved that if (2) is true then the integers  $\ell, m, n$  are of the form given by (d1)–(d6).

Vice versa, using (13), one can easily verify that if  $\ell, m, n$  are of the form given by (d1)–(d6) then  $D_{\ell,m,n} = 1$ . In turn, by (12), this implies that (2) is true.

The proof of Theorem 1.2 is complete.

## Statements and declarations

### Competing interests

The author declare that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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