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Regular Article

Inclusions and noninclusions of Hardy type spaces on certain nondoubling manifolds ☆

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ABSTRACT

In this paper we establish inclusions and noninclusions between various Hardy type spaces on noncompact Riemannian manifolds M with Ricci curvature bounded from below, positive injectivity radius and spectral gap.

Our first main result states that, if $\mathscr{L}$ is the positive Laplace–Beltrami operator on M, then the Riesz–Hardy space $H^1_{\mathscr{R}}(M)$ is the isomorphic image of the Goldberg type space $\mathfrak{h}^1(M)$ via the map $\mathscr{L}^{1/2}(\mathscr{I}+\mathscr{L})^{-1/2}$, a fact that is false in $\mathbb{R}^n$. Specifically, $H^1_{\mathscr{R}}(M)$ agrees with the Hardy type space $\mathfrak{X}^{1/2}(M)$ recently introduced by the first three authors; as a consequence, we prove that $H^1_{\mathscr{R}}(M)$ does not admit an atomic characterisation.

Noninclusions are mostly proved in the special case where the manifold is a Damek–Ricci space S. Our second main result states that $H^1_{\mathscr{R}}(S)$, the heat Hardy space $H^1_{\mathscr{H}}(S)$ and the Poisson–Hardy space $H^1_{\mathscr{P}}(S)$ are mutually distinct spaces, a

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Riesz transform

fact which is in sharp contrast to the Euclidean case, where these three spaces agree.
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1. Introduction

The purpose of this paper is to prove a number of results concerning Hardy type spaces on a certain class of nondoubling Riemannian manifolds. Our results illustrate that the scenario of Hardy spaces on such class of manifolds may differ considerably from that we are familiar with on Euclidean spaces.

Hardy spaces have become a landmark in the panorama of Euclidean Harmonic Analysis (in several variables) after the 1972 seminal paper of C. Fefferman and E. M. Stein [14]. It is virtually impossible in a research paper to give appropriate credit to all the mathematicians who have contributed to develop the theory of Hardy spaces in a variety of settings besides $\mathbb{R}^n$. Some pointers to the existing literature may be found in the introduction of [22], to which we refer the interested reader.

One of the key features of $H^1(\mathbb{R}^n)$ is its "flexibility", which is a consequence of its many different characterisations: these include the atomic and the maximal characterisations and the characterisation via Riesz transforms. This makes it possible to choose the most useful characterisation of $H^1(\mathbb{R}^n)$ in connection with a specific application one has in mind.

Suppose that M is a complete connected noncompact Riemannian manifold. Denote by $\mathscr{L}$ the (nonnegative) Laplace–Beltrami operator on M and by ∇ the covariant derivative. Consider the *Riesz transform* $\mathscr{R} := \nabla \mathscr{L}^{-1/2}$ and the following Hardy type spaces, endowed with their natural norms:

(i) the *Riesz–Hardy space* $H^1_{\mathscr{R}}(M)$, defined by

$$H^1_{\mathscr{R}}(M) := \{f \in L^1(M) : |\mathscr{R} f| \in L^1(M)\}; \tag{1.1}$$

(ii) the *heat maximal Hardy space* $H^1_{\mathscr{H}}(M)$, associated to the heat semigroup $\{\mathscr{H}_t := \mathrm{e}^{-t\mathscr{L}} : t \geq 0\}$, defined by

$$H^1_{\mathscr{H}}(M) := \big\{f \in L^1(M) : \mathscr{H}_* f \in L^1(M)\big\}, \tag{1.2}$$

where $\mathscr{H}_* f := \sup_{t>0} |\mathscr{H}_t f|$;

(iii) the *Poisson maximal Hardy space* $H^1_{\mathscr{P}}(M)$, associated to the Poisson semigroup $\{\mathscr{P}_t := \mathrm{e}^{-t\mathscr{L}^{1/2}} : t \geq 0\}$, defined by

$$H^1_{\mathscr{P}}(M) := \big\{f \in L^1(M) : \mathscr{P}_* f \in L^1(M)\big\}, \tag{1.3}$$

where $\mathscr{P}_* f := \sup_{t>0} |\mathscr{P}_t f|$.

As mentioned above (see [14])

$$H^1_{\mathscr{R}}(\mathbb{R}^n) = H^1_{\mathscr{H}}(\mathbb{R}^n) = H^1_{\mathscr{P}}(\mathbb{R}^n). \tag{1.4}$$

It is natural to speculate whether a similar equality holds in wider generality. A discussion of this interesting problem may be found in the introduction of [22], to which the reader is referred for further information. Here we content ourselves to mention that, as a consequence of the efforts of various authors [4,12,18,33],

$$H^1_{\mathscr{R}}(M) \supseteq H^1_{\mathscr{H}}(M) = H^1_{\mathscr{P}}(M)$$

in the case where M is a **doubling** Riemannian manifold with Ricci curvature bounded from below and positive injectivity radius. The first inclusion is a trivial consequence of the boundedness of the Riesz transform $\mathscr{R}$ from, say, $H^1_{\mathscr{H}}(M)$ to $L^1(M)$. To the best of our knowledge, it is unknown whether, under the assumptions above, $H^1_{\mathscr{R}}(M) = H^1_{\mathscr{H}}(M)$.

We emphasize the fact that there are only a few (very specific) examples in the literature, besides $\mathbb{R}^n$, in which the Riesz–Hardy space $H^1_{\mathscr{R}}(M)$ is characterised in whatsoever form. For more on this, see the introduction of [22] and the references therein.

In this paper we are interested exclusively in analogues of the space $H^1(\mathbb{R}^n)$ on manifolds in the class $\mathscr{M}$ of all complete noncompact connected Riemannian manifolds M satisfying the following:

(i) the injectivity radius of M is positive;
(ii) the Ricci tensor of M is bounded from below;
(iii) M has spectral gap, i.e., the bottom b of the L^2-spectrum of the Laplace–Beltrami operator $\mathscr{L}$ is strictly positive.

Riemannian manifolds in $\mathscr{M}$ are nondoubling metric measure spaces. Notice that $\mathscr{M}$ includes all symmetric spaces of the noncompact type and Damek–Ricci spaces, as well as all the simply connected complete Riemannian manifolds with negative pinched sectional curvature such as, for instance, the universal coverings of compact manifolds of strictly negative curvature. Note that $\mathscr{M}$ is precisely the class of manifolds considered in [22]. We refer to [22, pp. 2064–2065] for some important analytic and geometric consequences of the assumptions above.

Many variants of Hardy type spaces have been considered on (subclasses of) the class $\mathscr{M}$. Each of them is taylored to obtain endpoint estimates for certain classes of operators. Without any pretence of exhaustiveness, we mention [1,6,7,20–30,36], and refer the reader to the introductive sections of these papers for more on Hardy type spaces and their role in obtaining sharp estimates for a variety of operators, and for pointers to the vast literature on the subject.

One of the main results of this paper, Corollary 4.9, is the perhaps surprising fact that in the class $\mathscr{M}$ the ideal chain of equalities (1.4) may fail. Specifically, it asserts that, if S is a Damek–Ricci space, then $H^1_{\mathscr{R}}(S)$, $H^1_{\mathscr{H}}(S)$ and $H^1_{\mathscr{P}}(S)$ are pairwise distinct spaces. Notice that this is in sharp contrast with the Euclidean case discussed above. It is likely that a similar phenomenon happens on more general manifolds than Damek–Ricci spaces. As a matter of fact, we can rule out that the spaces $H^1_{\mathscr{R}}(M)$ and $H^1_{\mathscr{P}}(M)$ coincide when M is an arbitrary manifold in the class $\mathscr{M}$ (see Corollary 3.5).

In order to effectively compare $H^1_{\mathscr{R}}(M)$, $H^1_{\mathscr{H}}(M)$ and $H^1_{\mathscr{P}}(M)$, we make use of two main ideas: the first is to consider certain variants of $H^1_{\mathscr{H}}(M)$ and $H^1_{\mathscr{P}}(M)$, the second is to characterise $H^1_{\mathscr{R}}(M)$ as an isomorphic image of the local Hardy space $\mathfrak{h}^1(M)$ of Goldberg type on M (we emphasize that an analogous characterisation fails in $\mathbb{R}^n$, see Proposition 2.2); see Section 2 for details on $\mathfrak{h}^1(M)$. We offer the following comments concerning the genesis of these two ideas.

The first is suggested by the reading of [1], where J.-Ph. Anker sharpened previous results of N. Lohoué [21] and proved that if M is a symmetric space of the noncompact type, then for every c in $[0,1)$ the following estimate holds:

$$\|\mathscr{P}^c_* f\|_{L^1(M)} \leq C \left(\|f\|_{L^1(M)} + \big\| |\mathscr{R} f| \big\|_{L^1(M)} \right).$$

Here

$$\mathscr{P}^c_* f := \sup_{t>0} \langle t \rangle^c \, |\mathscr{P}_t f|,$$

and $\langle t \rangle = \max\{1, t\}$. In other words, $H^1_{\mathscr{R}}(M) \subseteq H^1_{\mathscr{P}}(M)$ (and, more generally, $H^1_{\mathscr{R}}(M) \subseteq H^1_{\mathscr{P},c}(M)$ for all $c \in [0,1)$ in the notation of Section 3). This surprising result strongly suggests that it may be advantageous to consider a finer scale of spaces $H^1_{\mathscr{P},c}(M)$, including $H^1_{\mathscr{P}}(M)$, defined in terms of the modified maximal function $\mathscr{P}^c_*$ above. By analogy, it seems natural to consider a similar scale of spaces $H^1_{\mathscr{H},c}(M)$, including $H^1_{\mathscr{H}}(M)$, defined in terms of the modified maximal function $\mathscr{H}^c_*$, defined as follows:

$$\mathscr{H}^c_* f := \sup_{t>0} \langle t \rangle^c \, |\mathscr{H}_t f|.$$

The scales of spaces $\big\{H^1_{\mathscr{P},c}(M) : c \in \mathbb{R}\big\}$ and $\big\{H^1_{\mathscr{H},c}(M) : c \in \mathbb{R}\big\}$ will play a fundamental role in what follows (see Section 3).

The second idea was, in fact, one of the motivations behind the introduction and the study in [22] of the scale of Hardy type spaces $\{\mathfrak{X}^\gamma(M) : \gamma > 0\}$ (see Definition 2.6), which, by definition, are isometric images of the Goldberg type space $\mathfrak{h}^1(M)$ via the fractional powers $\mathscr{U}^\gamma$ of the operator $\mathscr{U} = \mathscr{L}(\mathscr{I} + \mathscr{L})^{-1}$. In [22] it was proved that for manifolds in the class $\mathscr{M}$ the space $\mathfrak{X}^{1/2}(M)$ agrees with $\{f \in \mathfrak{h}^1(M) : |\mathscr{R} f| \in L^1(M)\}$, which, in principle, may be strictly contained in $H^1_{\mathscr{R}}(M)$. In this paper, armed with the Riesz transform characterisation of $\mathfrak{h}^1(M)$ recently obtained in [30], we improve this result and show that the Riesz–Hardy space $H^1_{\mathscr{R}}(M)$ agrees with the Banach space

$\mathfrak{X}^{1/2}(M)$: see Theorem 2.10. We emphasize that this result justifies retrospectively the introduction of the space $\mathfrak{X}^{1/2}(M)$. Thus, for every M in the class $\mathscr{M}$, the Riesz–Hardy space $H^1_{\mathscr{R}}(M)$ turns out to be an isometric copy of $\mathfrak{h}^1(M)$ via the map $\mathscr{U}^{1/2}$, a fact that fails in the Euclidean setting (see Proposition 2.2).

The fact that if S is a Damek–Ricci space, then $H^1_{\mathscr{R}}(S)$, $H^1_{\mathscr{H}}(S)$ and $H^1_{\mathscr{P}}(S)$ are mutually distinct spaces is a consequence of fine estimates of the kernels of certain operators on Damek–Ricci spaces and of inclusions between the families of Hardy type spaces $H^1_{\mathscr{H},c}(M)$ and $H^1_{\mathscr{P},c}(M)$ for arbitrary manifolds M in the class $\mathscr{M}$. Our analysis shows that the containments amongst the spaces $H^1_{\mathscr{H},c}(S)$ and $H^1_{\mathscr{P},c}(S)$ and the abovementioned family of spaces $\mathfrak{X}^\gamma(S)$ are subtle and highly nontrivial. In particular, we show that $\mathfrak{X}^{1/2}(S)$ is very similar to but not quite the same as $H^1_{\mathscr{P},1}(S)$: see Corollary 3.5 and Theorem 4.8 below.

It is interesting to observe that $\mathfrak{X}^{1/2}(M)$ is the space of all functions of the form $\mathscr{L}^{1/2}(\mathscr{I}+\mathscr{L})^{-1/2}f$, where f belongs to the Goldberg type space $\mathfrak{h}^1(M)$. Amongst the direct consequences of this fact, we mention the following. Proving that the spectral operator $m(\mathscr{L})$ maps $H^1_{\mathscr{R}}(M)$ to $L^1(M)$ is equivalent to showing that the spectral operator $m(\mathscr{L})\mathscr{L}^{1/2}(\mathscr{I}+\mathscr{L})^{-1/2}$ maps $\mathfrak{h}^1(M)$ to $L^1(M)$. This does not involve estimates of gradients any more and can hopefully be done using spectral methods.

We mention that the results contained in this paper are versions in the continuous setting of similar results obtained in the discrete case of graphs and trees [8]. It is fair to observe that analysis at local scales, which turns out to be trivial on graphs, is highly nontrivial in our situation. Notice also that local and global analysis on manifolds cannot be neatly separated, and one influences the other. As a consequence, our methods turn out to be comparatively more involved than those in [8]. For instance, our first result (concerning the abovementioned characterisation of the Riesz–Hardy space), which hinges on the detailed analysis performed in [30] and on spectral methods, is rather trivial on graphs.

Our paper is organised as follows. Section 2 contains our first main result, Theorem 2.10, together with an important improvement (Proposition 2.5) of the main result in [30]. In Section 3, we introduce the spaces $H^1_{\mathscr{H},c}(M)$ and $H^1_{\mathscr{P},c}(M)$, defined in terms of certain weighted heat and Poisson maximal operators, respectively. We prove some inclusions amongst $H^1_{\mathscr{H},c}(M)$, $H^1_{\mathscr{P},c}(M)$ and $\mathfrak{X}^\gamma(M)$ on any manifold M in the class $\mathscr{M}$. Our methods, based on functional calculus, are comparatively simple and direct. Among other things (see Corollary 3.5), we show that

$$H^1_{\mathscr{P},2\gamma+\varepsilon}(M) \subset \mathfrak{X}^\gamma(M) \subset H^1_{\mathscr{P},2\gamma-\varepsilon}(M)$$

for every $\varepsilon > 0$ and $\gamma > 0$. A special case of interest of this chain of inclusions is

$$H^1_{\mathscr{P},1+\varepsilon}(M) \subset H^1_{\mathscr{R}}(M) \subset H^1_{\mathscr{P},1-\varepsilon}(M)$$

for every $\varepsilon > 0$, which implies in particular that $H^1_{\mathscr{R}}(M)$ is properly contained in $H^1_{\mathscr{P}}(M)$. Concerning the relationship between $H^1_{\mathscr{H},c}(M)$ and $\mathfrak{X}^\gamma(M)$ we are only able to show, for an arbitrary manifold M in the class $\mathscr{M}$, that

$$H^1_{\mathscr{H},\gamma+\varepsilon}(M) \subset \mathfrak{X}^\gamma(M)$$

for every $\varepsilon > 0$. We can prove an inclusion in the opposite direction in the special case where the manifold is a Damek–Ricci space, where spherical Fourier analysis allows us to obtain precise estimates of the heat kernel; this is discussed in Section 4, which is mainly devoted to proving noninclusions amongst $H^1_{\mathscr{R}}(S)$, $H^1_{\mathscr{H}}(S)$ and $H^1_{\mathscr{P}}(S)$, where S is a Damek–Ricci space. As a consequence, we show that $H^1_{\mathscr{R}}(S)$, $H^1_{\mathscr{H}}(S)$ and $H^1_{\mathscr{P}}(S)$ are mutually distinct spaces (see Corollary 4.9).

In the course of the paper, the letter C is used to denote a finite positive constant, whose value may change from place to place. We also write $A \asymp B$ to indicate that $A \leq CB$ and $B \leq CA$. The symbols $\subseteq$ and $\subset$ denote set inclusion and proper set inclusion respectively, while S^{c} denotes the complement of the set S. For a real number x, we write $\lceil x \rceil$ to denote the smallest integer greater than or equal to x.

2. A characterisation of the Riesz–Hardy space

In this section we prove our first main result, Theorem 2.10, about the characterisation of the Riesz–Hardy space on a manifold M of class $\mathscr{M}$. As mentioned in the introduction, this result highlights a relationship between the local Hardy space $\mathfrak{h}^1(M)$ and the Riesz–Hardy space $H^1_{\mathscr{R}}(M)$ that has no counterpart in the Euclidean setting. In order to clearly illustrate this, we shall briefly discuss the Euclidean case (see Proposition 2.2 below) before moving to the case of manifolds. First of all, however, as the methods in this section are based on functional calculus techniques, some preliminary considerations on the matter are in order.

2.1. The extended Dunford class

If $\theta \in (0, \pi)$, by $\mathbf{S}_\theta$ we denote the open sector in the complex plane $\mathbb{C}$ with aperture 2θ symmetric about the positive real axis. We recall that the *extended Dunford class* $\mathscr{E}(\mathbf{S}_\theta)$ is the space of all bounded holomorphic functions $F : \mathbf{S}_\theta \to \mathbb{C}$ for which there exist $w_0, w_\infty \in \mathbb{C}$ and $\varepsilon > 0$ such that

$$|F(z) - w_0| + |F(1/z) - w_\infty| = O(|z|^\varepsilon) \quad \text{as } z \to 0 \tag{2.1}$$

(see [16, Lemma 2.2.3]). We record here some elementary properties of $\mathscr{E}(\mathbf{S}_\theta)$ that will be of use in the sequel (cf. [16, Examples 2.2.4 and 2.2.5]).

Lemma 2.1. *Let $\theta \in (0, \pi)$.*

(i) $\mathscr{E}(\mathbf{S}_\theta)$ *is a unital algebra under pointwise addition and multiplication.*
(ii) *If* F *is in* $\mathscr{E}(\mathbf{S}_\theta)$ *and* $1/F$ *is a bounded holomorphic function on* $\mathbf{S}_\theta$*, then* $1/F$ *is in* $\mathscr{E}(\mathbf{S}_\theta)$*.*
(iii) *Any function of the form*

$$z \mapsto \left(\frac{a+bz}{c+dz}\right)^\gamma, \tag{2.2}$$

with $\gamma, a, b \in [0,\infty)$ *and* $c, d \in (0,\infty)$*, is in* $\mathscr{E}(\mathbf{S}_\theta)$*.*

Proof. Part (i) is discussed in [16, Section 2.2].

As for part (ii), if $F \in \mathscr{E}(\mathbf{S}_\theta)$ and $1/F$ is bounded, then the limits $w_0, w_\infty \in \mathbb{C}$ of F at 0 and ∞ must be nonzero. Consequently

$$\left|\frac{1}{F(z)} - \frac{1}{w_0}\right| + \left|\frac{1}{F(1/z)} - \frac{1}{w_\infty}\right| = \frac{|F(z)-w_0|}{|F(z)||w_0|} + \frac{|F(1/z)-w_0|}{|F(1/z)||w_\infty|} = O(|z|^\varepsilon)$$

as $z \to 0$, for some $\varepsilon > 0$, where we used (2.1) and the boundedness of $1/F$.

Finally, for part (iii), any function F of the form (2.2) is bounded and holomorphic on $\mathbf{S}_\theta$, and has finite limits $w_0 = (a/c)^\gamma$ and $w_\infty = (b/d)^\gamma$ as $z \to 0$ and $z \to \infty$ respectively. If $w_0 \neq 0$, then F extends to a holomorphic function in a neighbourhood of 0, thus clearly $|F(z) - w_0| = O(|z|)$ as $z \to 0$. If instead $w_0 = 0$, then $a = 0$ and $|F(z)| = O(|z|^\gamma)$ as $z \to 0$. Similar considerations apply to $F(1/z)$ and w_∞ in place of $F(z)$ and w_0, thus showing that F is in $\mathscr{E}(\mathbf{S}_\theta)$. □

The relevance for our discussion of the extended Dunford class lies in its role in the functional calculus for sectorial operators, as described, e.g., in [16, Chapter 2]. Specifically:

- if $\mathscr{T}$ is a sectorial operator of angle $\omega \in [0,\pi)$ on a Banach space $\mathscr{X}$ (see, e.g., [16, Section 2.1]), then, for all $\theta \in (\omega,\pi)$ and $F \in \mathscr{E}(\mathbf{S}_\theta)$, the operator $F(\mathscr{T})$ is bounded on $\mathscr{X}$ [16, Theorem 2.3.3];
- if $\mathscr{T}$ is the generator of a uniformly bounded semigroup of class C_0 on $\mathscr{X}$, then $\mathscr{T}$ is sectorial of angle $\pi/2$ (see [16, Section 2.1.1, p. 24]).

These facts will be repeatedly used in what follows.

2.2. Hardy spaces in the Euclidean setting

The local Hardy space $\mathfrak{h}^1(\mathbb{R}^n)$ was introduced in D. Goldberg's paper [15]. Since then, $\mathfrak{h}^1(\mathbb{R}^n)$ is commonly referred to as the Goldberg space. We start by establishing a result that relates the classical Hardy space $H^1(\mathbb{R}^n) = H^1_{\mathscr{R}}(\mathbb{R}^n)$ and the Goldberg space $\mathfrak{h}^1(\mathbb{R}^n)$. Consider the operator $\mathscr{L}^{1/2}(\mathscr{I}+\mathscr{L})^{-1/2}$, also denoted by $\mathscr{U}^{1/2}$, where $-\mathscr{L} = \Delta$ denotes the standard Laplacian.

Proposition 2.2. *The operator $\mathscr{U}^{1/2}$ is bounded and injective, but not surjective, from $\mathfrak{h}^1(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$. Thus, the space $H^1(\mathbb{R}^n)$ contains $\mathscr{U}^{1/2}[\mathfrak{h}^1(\mathbb{R}^n)]$ properly.*

Proof. First we show that if g is in $\mathfrak{h}^1(\mathbb{R}^n)$, then $\mathscr{U}^{1/2}g$ is in $H^1(\mathbb{R}^n)$. Since $\mathscr{L}$ generates a contractive semigroup of class C_0 on $L^1(\mathbb{R}^n)$ and the function

$$\varphi(z) := \left(\frac{z}{1+z}\right)^{1/2}$$

is in the extended Dunford class $\mathscr{E}(\mathbf{S}_\theta)$ for every θ in $(\pi/2, \pi)$ (see Lemma 2.1), the operator $\mathscr{U}^{1/2} = \varphi(\mathscr{L})$ is bounded on $L^1(\mathbb{R}^n)$ (see [16, Theorem 2.3.3]). Hence $\mathscr{U}^{1/2}g$ is in $L^1(\mathbb{R}^n)$. In order to show that $\mathscr{U}^{1/2}g$ is in $H^1(\mathbb{R}^n)$, it suffices to prove that $|\mathscr{R}\mathscr{U}^{1/2}g|$ is in $L^1(\mathbb{R}^n)$. Notice that

$$|\mathscr{R}\mathscr{U}^{1/2}g| = |\nabla \mathscr{L}^{-1/2}\mathscr{U}^{1/2}g| = |\nabla(\mathscr{I}+\mathscr{L})^{-1/2}g|,$$

which belongs to $L^1(\mathbb{R}^n)$, because g belongs to $\mathfrak{h}^1(\mathbb{R}^n)$, by assumption. A close examination of the argument above shows that there exists a constant C such that

$$\|\mathscr{U}^{1/2}g\|_{H^1(\mathbb{R}^n)} \le C\left(1 + \left\|\mathscr{U}^{1/2}\right\|_{L^1(\mathbb{R}^n)}\right) \|g\|_{\mathfrak{h}^1(\mathbb{R}^n)} \qquad \forall g \in \mathfrak{h}^1(\mathbb{R}^n).$$

Note that $\mathscr{U}^{1/2}$ is injective on $\mathfrak{h}^1(\mathbb{R}^n)$. Indeed, if $\mathscr{U}^{1/2}g = 0$, then

$$\frac{|\xi|}{\sqrt{1+|\xi|^2}}\,\widehat{g}(\xi) = 0 \qquad \forall \xi \in \mathbb{R}^n.$$

Therefore $\widehat{g}(\xi)$ vanishes on $\mathbb{R}^n \setminus \{0\}$, hence everywhere, for $\widehat{g}$ is continuous. Thus, $g = 0$, as required.

Finally, we show that $\mathscr{U}^{1/2}\left[\mathfrak{h}^1(\mathbb{R}^n)\right]$ is properly contained in $H^1(\mathbb{R}^n)$. We argue by contradiction. If $\mathscr{U}^{1/2}$ were surjective, then $\mathscr{U}^{-1/2}$ would be bounded from $H^1(\mathbb{R}^n)$ to $\mathfrak{h}^1(\mathbb{R}^n)$, and in particular from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. We prove that this fails. Indeed, $(\mathscr{I}+\mathscr{L})^{-1/2}\mathscr{U}^{-1/2} = \mathscr{L}^{-1/2}$, and $(\mathscr{I}+\mathscr{L})^{-1/2}$ is bounded on $L^1(\mathbb{R}^n)$. Thus, if $\mathscr{U}^{-1/2}$ were bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$, the same would be true of $\mathscr{L}^{-1/2}$. However, the boundedness of $\mathscr{L}^{-1/2}$ from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ is easily disproved by homogeneity considerations.

Indeed, for any $R > 0$, the formula $T_R f(x) = R^{-n} f(x/R)$ defines an isometric automorphism T_R of both $L^1(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$. On the other hand, by homogeneity of $\mathscr{L} = -\Delta$ on $\mathbb{R}^n$, we deduce that $\mathscr{L}^{-1/2}T_R f = R\,T_R\mathscr{L}^{-1/2}f$; thus, for any nonzero $f \in H^1(\mathbb{R}^n)$,

$$\|\mathscr{L}^{-1/2}T_R f\|_{L^1(\mathbb{R}^n)} = R\|\mathscr{L}^{-1/2}f\|_{L^1(\mathbb{R}^n)} \to \infty \qquad \text{as } R \to \infty,$$

while

$$\|T_R f\|_{H^1(\mathbb{R}^n)} = \|f\|_{H^1(\mathbb{R}^n)} \qquad \forall R > 0,$$

whence $\mathscr{L}^{-1/2}$ cannot be bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. □

2.3. Local Hardy spaces on manifolds with bounded geometry

We now move to the case of manifolds. The construction of a local Hardy space by Goldberg was extended to a certain class of Riemannian manifolds with strongly bounded geometry by M. Taylor [36], further generalised to the setting of metric measure spaces [31], and recently investigated in [23]. In the case where M is a complete Riemannian manifold with Ricci curvature bounded from below, the space $\mathfrak{h}^1(M)$ can be characterised in several ways. In particular, it admits an atomic and an ionic decomposition [31]. If we assume further that M has positive injectivity radius, then $\mathfrak{h}^1(M)$ can be equivalently defined in terms of either the local heat maximal operator or the local Poisson maximal operator [23].

For the reader's convenience, we report below one of the several equivalent definitions of $\mathfrak{h}^1(M)$ given in [31] in terms of atomic decompositions. As in [23], here we specialise to the case where the manifold M belongs to the class $\mathscr{M}_0$ of *Riemannian manifolds with bounded geometry*, that is, complete connected Riemannian manifolds with positive injectivity radius and Ricci curvature bounded from below. We point out that the class $\mathscr{M}_0$ is strictly larger than the class $\mathscr{M}$ discussed in the Introduction, for $\mathscr{M}_0$ contains also doubling manifolds, such as $\mathbb{R}^n$ with the standard Euclidean structure, as well as compact manifolds. Here and in the sequel, we denote by μ the Riemannian measure on the manifold M.

Definition 2.3. Let M be a manifold in the class $\mathscr{M}_0$. A *standard* $\mathfrak{h}^1(M)$*-atom* is a function a in $L^1(M)$ supported in a ball B of radius at most one, satisfying the following conditions:

(i) *size condition*: $\|a\|_2 \leq \mu(B)^{-1/2}$;
(ii) *cancellation condition*: $\int_B a \, \mathrm{d}\mu = 0$.

A *global* $\mathfrak{h}^1(M)$*-atom* is a function a in $L^1(M)$ supported in a ball B of radius exactly one, satisfying the size condition above (but possibly not the cancellation condition). Standard and global $\mathfrak{h}^1(M)$-atoms will be referred to simply as $\mathfrak{h}^1(M)$*-atoms*.

Definition 2.4. Let M be a manifold in the class $\mathscr{M}_0$. The *local atomic Hardy space* $\mathfrak{h}^1(M)$ is the space of all functions f in $L^1(M)$ that admit a decomposition of the form

$$f = \sum_{j=1}^{\infty} \lambda_j a_j, \tag{2.3}$$

where the a_j's are $\mathfrak{h}^1(M)$-atoms and $\sum_{j=1}^{\infty} |\lambda_j| < \infty$. The norm $\|f\|_{\mathfrak{h}^1}$ of f is the infimum of $\sum_{j=1}^{\infty} |\lambda_j|$ over all decompositions (2.3) of f.

We refer the reader to [31] and [23] for further details on $\mathfrak{h}^1(M)$, including other equivalent characterisations.

Given a Riemannian manifold M in the class $\mathscr{M}_0$ and a positive number τ, we consider the translated Laplacian $\tau\mathscr{I} + \mathscr{L}$ and define the associated *translated Riesz transform* $\mathscr{R}_\tau := \nabla(\tau\mathscr{I} + \mathscr{L})^{-1/2}$, and the *local Riesz–Hardy* space

$$\mathfrak{h}^1_{\mathscr{R}_\tau}(M) := \big\{ f \in L^1(M) : |\mathscr{R}_\tau f| \in L^1(M) \big\}. \tag{2.4}$$

We equip $\mathfrak{h}^1_{\mathscr{R}_\tau}(M)$ with the norm

$$\|f\|_{\mathfrak{h}^1_{\mathscr{R}_\tau}(M)} := \|f\|_1 + \big\| |\mathscr{R}_\tau f| \big\|_1 .$$

It is a classical result that $\mathfrak{h}^1_{\mathscr{R}_\tau}(\mathbb{R}^n) = \mathfrak{h}^1(\mathbb{R}^n)$ for all $\tau > 0$. It was recently proved in [30, Theorem 7.9] that, if M is in the class $\mathscr{M}_0$ and τ is a large positive number, then $\mathfrak{h}^1_{\mathscr{R}_\tau}(M) = \mathfrak{h}^1(M)$ and their norms are equivalent. In fact, the requirement that τ be a large positive number is not needed for any manifold M with bounded geometry, as shown in the next proposition that improves [30, Theorem 7.9].

Proposition 2.5. *Suppose that M is Riemannian manifold in the class $\mathscr{M}_0$, and let τ be a positive number. Then $\mathfrak{h}^1_{\mathscr{R}_\tau}(M) = \mathfrak{h}^1(M)$ and their norms are equivalent.*

Proof. Consider positive numbers τ and τ', and the function

$$\varphi_\tau^{\tau'}(z) = \left(\frac{\tau' + z}{\tau + z} \right)^{1/2}.$$

By Lemma 2.1, the function $\varphi_\tau^{\tau'}$ belongs to the extended Dunford class $\mathscr{E}(\mathbf{S}_\theta)$ for any θ in $(\pi/2, \pi)$. Since $\mathscr{L}$ is the generator of a uniformly bounded semigroup of class C_0 on $\mathfrak{h}^1(M)$ (see [22, Theorem 3.1]), the operator $\varphi_\tau^{\tau'}(\mathscr{L})$ is bounded on $\mathfrak{h}^1(M)$ by [16, Theorem 2.3.3] for every pair of positive numbers τ and τ'. The same also applies with $L^1(M)$ in place of $\mathfrak{h}^1(M)$. As $\varphi_{\tau'}^{\tau}(\mathscr{L})$ is the inverse of $\varphi_\tau^{\tau'}(\mathscr{L})$, we conclude that $\varphi_\tau^{\tau'}(\mathscr{L})$ is a topological vector space automorphism of $L^1(M)$ and also of $\mathfrak{h}^1(M)$, for any $\tau, \tau' > 0$. Additionally,

$$\mathscr{R}_{\tau'} = \mathscr{R}_\tau \, \varphi_\tau^{\tau'}(\mathscr{L}),$$

so from (2.4) we readily deduce that $\varphi_\tau^{\tau'}(\mathscr{L}) : \mathfrak{h}^1_{\mathscr{R}_{\tau'}}(M) \to \mathfrak{h}^1_{\mathscr{R}_\tau}(M)$ is bounded, and actually an isomorphism with inverse $\varphi_{\tau'}^{\tau}(\mathscr{L})$.

Now, by [30, Theorem 7.9], we can find τ' large enough that $\mathfrak{h}^1(M) = \mathfrak{h}^1_{\mathscr{R}_{\tau'}}(M)$. Then, for any positive number τ,

$$\mathfrak{h}^1_{\mathscr{R}_\tau}(M) = \varphi_\tau^{\tau'}(\mathscr{L})[\mathfrak{h}^1_{\mathscr{R}_{\tau'}}(M)] = \varphi_\tau^{\tau'}(\mathscr{L})[\mathfrak{h}^1(M)] = \mathfrak{h}^1(M),$$

as required. □

2.4. A family of Hardy type spaces

We recall the definition of the Hardy type spaces $\mathfrak{X}^\gamma(M)$, $\gamma > 0$, recently introduced in [22], where M is a manifold in the class $\mathscr{M}$. Consider the family $\{\mathscr{U}_\tau := \mathscr{L}\,(\tau\mathscr{I} + \mathscr{L})^{-1} : \tau > 0\}$ of (spectrally defined) operators. We also write $\mathscr{U}$ instead of $\mathscr{U}_1$ for brevity. For any $\tau, \gamma > 0$, the mapping $\mathscr{U}_\tau^\gamma$ is bounded on $\mathfrak{h}^1(M)$ [22, Section 3]. We emphasize that $\mathscr{U}_\tau^\gamma\big[\mathfrak{h}^1(M)\big]$ is independent of τ and there exists a positive constant C such that

$$C^{-1}\,\|\mathscr{U}^{-\gamma} f\|_{\mathfrak{h}^1(M)} \le \|\mathscr{U}_\tau^{-\gamma} f\|_{\mathfrak{h}^1(M)} \le C\,\|\mathscr{U}^{-\gamma} f\|_{\mathfrak{h}^1(M)}$$

for every f in $\mathscr{U}^\gamma\big[\mathfrak{h}^1(M)\big]$ (see [22, Proposition 3.5]).

Definition 2.6. We denote by $\mathfrak{X}^\gamma(M)$ the space $\mathscr{U}^\gamma\big[\mathfrak{h}^1(M)\big]$, endowed with the norm that makes $\mathscr{U}^\gamma$ an isometry, i.e.,

$$\|f\|_{\mathfrak{X}^\gamma(M)} := \|\mathscr{U}^{-\gamma} f\|_{\mathfrak{h}^1(M)} \qquad \forall f \in \mathscr{U}^\gamma\big[\mathfrak{h}^1(M)\big].$$

See [22] for further information on $\mathfrak{X}^\gamma(M)$, $\gamma > 0$. We record here a few useful properties of the spaces $\mathfrak{X}^\gamma(M)$ that will be important in the sequel.

Proposition 2.7. *Let M be a manifold in the class $\mathscr{M}$ and let $\gamma > 0$.*

(i) *$\mathfrak{X}^\gamma(M)$ properly contains $\mathfrak{X}^{\gamma+\varepsilon}(M)$ for all $\varepsilon > 0$.*
(ii) *$f \in \mathfrak{X}^\gamma(M)$ if and only if $f \in \mathfrak{h}^1(M)$ and $\mathscr{L}^{-\gamma} f \in \mathfrak{h}^1(M)$.*
(iii) *If $\psi \in C_c^\infty(M)$, then $\mathscr{L}^\gamma \psi \in \mathfrak{X}^\gamma(M)$.*

Proof. Parts (i) and (ii) are proved in [22, Proposition 4.14 and Proposition 4.2]. As for part (iii), note that

$$\mathscr{U}^{-\gamma}\mathscr{L}^\gamma\psi = (1+\mathscr{L})^\gamma\psi = (1+\mathscr{L})^{\gamma-N}(1+\mathscr{L})^N\psi$$

for all $N \in \mathbb{N}$. Since $\psi \in C_c^\infty(M)$, we deduce that $(1+\mathscr{L})^N\psi \in C_c^\infty(M)$, and therefore $(1+\mathscr{L})^N\psi \in \mathfrak{h}^1(M)$; the conclusion follows by the $\mathfrak{h}^1$ boundedness of $(1+\mathscr{L})^{\gamma-N}$ for $N > \gamma$ [22, Theorem 3.1]. □

2.5. Negative powers of the Laplacian and Riesz transform

As a consequence of the spectral gap assumption for manifolds M of the class $\mathscr{M}$, the powers $\mathscr{L}^{-\gamma}$, $\gamma > 0$, are bounded on $L^p(M)$ for any $p \in (1, \infty)$, though this fails for $p = 1$. Nevertheless we can show that $\mathscr{L}^{-\gamma}$ maps $L^1(M)$ into $L^1_{\mathrm{loc}}(M)$.

Proposition 2.8. *Let M be a manifold in the class $\mathscr{M}$ and $\gamma \in (0,\infty)$. Then, there exists a positive constant C such that, for all $f \in L^1(M)$ and $o \in M$,*

$$\|\mathscr{L}^{-\gamma} f\|_{L^1(B_R(o))} \le C R^\gamma \|f\|_{L^1} \qquad \forall R \ge 1.$$

Proof. Recall that $\mathscr{H}_t = e^{-t\mathscr{L}}$ denotes the heat semigroup. By functional calculus, we can write, at least formally,

$$\mathscr{L}^{-\gamma} = c_\gamma \int_0^\infty t^\gamma \mathscr{H}_t \, \frac{\mathrm{d}t}{t},$$

where $c_\gamma := 1/\Gamma(\gamma)$. As a consequence, for all $T > 0$,

$$\mathscr{L}^{-\gamma} f = c_\gamma \int_0^T t^\gamma \mathscr{H}_t f \, \frac{\mathrm{d}t}{t} + c_\gamma \int_T^\infty t^\gamma \mathscr{H}_t f \, \frac{\mathrm{d}t}{t} =: f_T^0 + f_T^\infty.$$

Now, as $\mathscr{H}_t$ is a contractive semigroup on $L^1(M)$,

$$\|f_T^0\|_{L^1} \le c_\gamma \|f\|_{L^1} \int_0^T t^\gamma \, \frac{\mathrm{d}t}{t} \le C T^\gamma \|f\|_{L^1},$$

since $\gamma > 0$. On the other hand, in light of the ultracontractive estimate

$$\|\mathscr{H}_t\|_{1\to 2} \le C \mathrm{e}^{-bt} \qquad \forall t \ge 1,$$

where $b > 0$ is the bottom of the L^2-spectrum of $\mathscr{L}$ (see, e.g., [22, eq. (2.4)]), we deduce that

$$\|f_T^\infty\|_{L^2} \le C \|f\|_{L^1} \int_T^\infty t^{\gamma-1} \mathrm{e}^{-bt} \, \mathrm{d}t \le C T^{\gamma-1} \mathrm{e}^{-bT} \|f\|_{L^1}$$

for all $T \ge 1$. As M has exponential volume growth (see, e.g., [23, eq. (2.3)]), there exists $\beta > 0$ such that

$$\mu(B_R(o)) \le C \mathrm{e}^{2\beta R} \qquad \forall R \ge 1$$

and consequently, by the Cauchy–Schwarz inequality,

$$\|f_T^\infty\|_{L^1(B_R(o))} \le C \mathrm{e}^{\beta R} \|f_T^\infty\|_{L^2} \le C T^{\gamma-1} \mathrm{e}^{\beta R - bT}.$$

In conclusion,

$$\|\mathscr{L}^{-\gamma} f\|_{L^1(B_R(o))} \le C T^\gamma \left[1 + T^{-1} \mathrm{e}^{\beta R - bT}\right]$$

for all $R, T \ge 1$, and the desired estimate follows by taking $T = (1 + \beta/b)R$. □

On the basis of the previous estimate, the Riesz transform $\mathscr{R}f = \nabla \mathscr{L}^{-1/2} f$ of any function $f \in L^1(M)$ is well-defined at least in the sense of distributions. We can then define the Riesz–Hardy space $H^1_{\mathscr{R}}(M)$ as described in (1.1) in the Introduction, and equip it with the norm $\|f\|_{L^1(M)} + \big\||\mathscr{R}f|\big\|_{L^1(M)}$.

2.6. The Federer–Fleming inequality

As we shall see, $H^1_{\mathscr{R}}(M)$ can be characterised as one of the spaces $\mathfrak{X}^\gamma(M)$ introduced above. An important tool in this characterisation is the validity of the following L^1 Sobolev inequality, also known as the Federer–Fleming inequality [5,9], which is again a consequence of the spectral gap assumption.

Proposition 2.9. *Let M be a manifold in the class $\mathscr{M}$. There exists a positive constant κ such that, if $h \in L^1_{\mathrm{loc}}(M)$ satisfies the estimate*

$$\|h\|_{L^1(B_R(o))} \le CR \qquad \forall R \ge 1 \tag{2.5}$$

for some $o \in M$ and $C > 0$, and moreover $|\nabla h| \in L^1(M)$, then $h \in L^1(M)$ and

$$\|h\|_{L^1} \le \kappa \big\||\nabla h|\big\|_{L^1}. \tag{2.6}$$

Proof. As the manifold M is in the class $\mathscr{M}$, the Cheeger isoperimetric constant of M is strictly positive (see [5] and [6, Theorem 9.5]), whence the Federer–Fleming inequality (2.6) is known to hold for all $h \in C^\infty_c(M)$ (see [9, Theorem V.2.1, p. 131], and also [22, eq. (2.6)]). On the other hand, as M is complete, the space $C^\infty_c(M)$ is dense in the Sobolev space

$$W^{1,1}(M) = \{f \in L^1(M) : |\nabla f| \in L^1(M)\}$$

(see, e.g., [17, Theorem 2.7] or [10, Proposition 5.5]), thus the inequality (2.6) holds for all $h \in W^{1,1}(M)$. So it just remains to show that, if $h \in L^1_{\mathrm{loc}}(M)$ is only assumed to satisfy the a-priori estimate (2.5) and $|\nabla h| \in L^1(M)$, then actually $h \in L^1(M)$.

To this purpose, let us consider, for all $R \ge 1$, a compactly supported Lipschitz cutoff $\psi_R : M \to [0,1]$ which is identically 1 on $B_R(o)$, vanishes outside $B_{2R}(o)$, and satisfies

$$\big\||\nabla \psi_R|\big\|_\infty \le CR^{-1}; \tag{2.7}$$

the existence of such cutoffs is a consequence of the completeness of M (see, e.g., [35, Lemma 2.2]).

Let now h satisfy the assumptions of Proposition 2.9. As $h \in L^1_{\mathrm{loc}}(M)$, we deduce that $\psi_R h$ is in $L^1(M)$. Moreover, as $|\nabla h| \in L^1(M)$, by the Leibniz rule,

$$|\nabla(\psi_R h)| \le \psi_R |\nabla h| + |\nabla \psi_R||h|,$$

so $|\nabla(\psi_R h)|$ is in $L^1(M)$ too, with

$$\begin{aligned} \big\||\nabla(\psi_R h)|\big\|_1 &\le \big\||\nabla h|\big\|_1 + CR^{-1}\|h\|_{L^1(B_{2R}(o))} \\ &\le \big\||\nabla h|\big\|_1 + C; \end{aligned} \tag{2.8}$$

in the first inequality we used (2.7) and information on the support of ψ_R, while the second inequality follows from (2.5).

As $\psi_R h \in W^{1,1}(M)$, we can then apply (2.6) to $\psi_R h$ and obtain that

$$\|\psi_R h\|_1 \le \kappa\big\||\nabla(\psi_R h)|\big\|_1 \le \kappa\big\||\nabla h|\big\|_1 + C$$

for all $R \ge 1$. If we take the limit as $R \to +\infty$, then by monotone convergence we deduce that $h \in L^1(M)$, as required. □

2.7. Characterisation of the Riesz–Hardy space

In view of Proposition 2.2, the following characterisation of $H^1_{\mathscr{R}}(M)$ highlights an interesting noneuclidean phenomenon.

Theorem 2.10. *Let M be a manifold in the class $\mathscr{M}$. Then $H^1_{\mathscr{R}}(M) = \mathfrak{X}^{1/2}(M)$, with equivalence of norms.*

This characterisation substantially refines the result in [22, Theorem 5.3], where $\mathfrak{X}^{1/2}(M)$ is identified with the space $\{f \in \mathfrak{h}^1(M) : |\mathscr{R} f| \in L^1(M)\}$. While a number of ideas from the proof of [22, Theorem 5.3] also apply here, for the reader's convenience we opt to provide a self-contained proof of Theorem 2.10 below. Beside favouring the readability of the argument, this also gives us the opportunity to clarify why the Federer–Fleming inequality is applicable in this context, a point that was somewhat skimmed over in the proof presented in [22].

Proof. The proof of the inclusion $\mathfrak{X}^{1/2}(M) \subseteq H^1_{\mathscr{R}}(M)$ is analogous to that of the inclusion $\mathscr{U}^{1/2}[\mathfrak{h}^1(\mathbb{R}^n)] \subseteq H^1_{\mathscr{R}}(\mathbb{R}^n)$ given in Proposition 2.2. Indeed, if f is in $\mathfrak{X}^{1/2}(M)$, then f is in $\mathfrak{h}^1(M)$, thus in $L^1(M)$, and moreover $f = \mathscr{U}^{1/2} g$ for some $g \in \mathfrak{h}^1(M)$. In addition,

$$\mathscr{R} f = \nabla \mathscr{L}^{-1/2} f = \nabla(\mathscr{I} + \mathscr{L})^{-1/2} \mathscr{U}^{-1/2} f = \mathscr{R}_1 g. \tag{2.9}$$

So from Proposition 2.5 and the fact that $g \in \mathfrak{h}^1(M)$ we deduce that $|\mathscr{R} f| \in L^1(M)$, and in conclusion $f \in H^1_{\mathscr{R}}(M)$.

As for the opposite inclusion, if f is in $H^1_{\mathscr{R}}(M)$, then f and $|\mathscr{R} f|$ are in $L^1(M)$. If $h := \mathscr{L}^{-1/2} f$, then, from Proposition 2.8, we know that $h \in L^1_{\mathrm{loc}}(M)$ and $\|h\|_{L^1(B_R(o))} \le CR^{1/2}$ for all $R \ge 1$. Moreover $\nabla h = \mathscr{R} f$, thus $|\nabla h| \in L^1(M)$. We can then apply Proposition 2.9 to h and deduce that $h = \mathscr{L}^{-1/2} f$ is in $L^1(M)$.

Notice now that

$$\mathscr{U}^{-1/2} = (\mathscr{I} + \mathscr{L})^{1/2}\mathscr{L}^{-1/2} = \varphi(\mathscr{L})(\mathscr{I} + \mathscr{L}^{-1/2}), \tag{2.10}$$

where

$$\varphi(z) := \frac{(1+z)^{1/2}}{1+z^{1/2}}.$$

It is easily seen that φ is in $\mathscr{E}(\mathbf{S}_\theta)$ for all $\theta \in (\pi/2, \pi)$: indeed, by parts (i) and (iii) of Lemma 2.1, the reciprocal function

$$\frac{1}{\varphi(z)} = \frac{1}{(1+z)^{1/2}} + \frac{z^{1/2}}{(1+z)^{1/2}}$$

is in $\mathscr{E}(\mathbf{S}_\theta)$, thus φ is too by part (ii) of the same Lemma. So $\varphi(\mathscr{L})$ is bounded on $L^1(M)$ by [16, Theorem 2.3.3]. From (2.10) we then deduce that the function

$$g := \mathscr{U}^{-1/2} f = \varphi(\mathscr{L})(f + \mathscr{L}^{-1/2} f)$$

is in $L^1(M)$, as f and $\mathscr{L}^{-1/2} f$ are. Moreover, as in (2.9), $\mathscr{R} f = \mathscr{R}_1 g$. Therefore both g and $|\mathscr{R}_1 g|$ are in $L^1(M)$, that is, $g \in \mathfrak{h}^1_{\mathscr{R}_1}(M)$. However, by Proposition 2.5, this is equivalent to saying that $g \in \mathfrak{h}^1(M)$, and consequently $f = \mathscr{U}^{1/2} g \in \mathfrak{X}^{1/2}(M)$, by definition. □

As a consequence of the above characterisation, in stark contrast to the Euclidean case, we can rule out that $H^1_{\mathscr{R}}(M)$ can be characterised in terms of atomic decompositions, at least for a subclass of the manifolds of class $\mathscr{M}$.

Corollary 2.11. *Suppose that M is a rank one symmetric space of the noncompact type. Then the Riesz–Hardy space $H^1_{\mathscr{R}}(M)$ does not admit an atomic characterisation. More precisely, the set of the compactly supported elements of $H^1_{\mathscr{R}}(M)$ is not dense in $H^1_{\mathscr{R}}(M)$.*

Proof. According to [22, Theorem 4.15], the space $\mathfrak{X}^{1/2}(M)$ does not admit an atomic decomposition. Since, by Theorem 2.10, $H^1_{\mathscr{R}}(M)$ agrees with $\mathfrak{X}^{1/2}(M)$, the conclusion follows. □

3. Maximal operators and function spaces for the class $\mathscr{M}$

In this section we consider various maximal operators and function spaces associated thereto on any manifold M in the class $\mathscr{M}$. We collect here their definitions and some of their elementary properties.

3.1. Maximal operators and associated Hardy type spaces

Recall that $\mathscr{H}_t = \mathrm{e}^{-t\mathscr{L}}$ and that $\mathscr{P}_t = \mathrm{e}^{-t\mathscr{L}^{1/2}}$: thus, $\{\mathscr{H}_t\}$ and $\{\mathscr{P}_t\}$ denote the heat and the Poisson semigroups on M, respectively. For every $c \in \mathbb{R}$ we denote by $\mathscr{H}_*^c$ the *weighted heat maximal operator*, defined by

$$\mathscr{H}_*^c f = \sup_{t>0} \langle t \rangle^c \, |\mathscr{H}_t f| \,,$$

where $\langle t \rangle = \max\{1, t\}$. When $c = 0$ we simply denote by $\mathscr{H}_*$ the maximal operator $\mathscr{H}_*^0$. We denote by $H^1_{\mathscr{H},c}$ the space

$$H^1_{\mathscr{H},c} = \{f \in L^1(M) : \mathscr{H}_*^c f \in L^1(M)\} \,,$$

endowed with the norm

$$\|f\|_{H^1_{\mathscr{H},c}} = \|f\|_{L^1} + \|\mathscr{H}_*^c f\|_{L^1} \,.$$

Clearly $\{H^1_{\mathscr{H},c}(M) : c \in \mathbb{R}\}$ is a decreasing family of spaces. We denote by $\mathscr{H}_*^{\mathrm{loc}}$ the *local heat maximal operator*, defined by

$$\mathscr{H}_*^{\mathrm{loc}} f = \sup_{0<t\leq 1} |\mathscr{H}_t f| \,.$$

We define the *local heat Hardy space* as

$$\mathfrak{h}^1_{\mathscr{H}}(M) = \{f \in L^1(M) : \mathscr{H}_*^{\mathrm{loc}} f \in L^1(M)\} \,,$$

endowed with the norm

$$\|f\|_{\mathfrak{h}^1_{\mathscr{H}}} = \|f\|_{L^1} + \|\mathscr{H}_*^{\mathrm{loc}} f\|_{L^1} \,.$$

Much in the same vein, for any real number c in $\mathbb{R}$ we consider the *Poisson maximal operator $\mathscr{P}_*^c$ with parameter c*, which acts on a function f on M by

$$\mathscr{P}_*^c f = \sup_{t>0} \langle t \rangle^c \, |\mathscr{P}_t f| .$$

We shall write $\mathscr{P}_*$ instead of $\mathscr{P}_*^0$. We then define $H^1_{\mathscr{P},c}(M)$ by

$$H^1_{\mathscr{P},c}(M) = \{f \in L^1(M) : \mathscr{P}_*^c f \in L^1(M)\} .$$

We equip $H^1_{\mathscr{P},c}(S)$ with the norm

$$\|f\|_{H^1_{\mathscr{P},c}} = \|f\|_{L^1} + \|\mathscr{P}_*^c f\|_{L^1} \,.$$

We denote by $\mathscr{P}_*^{\mathrm{loc}}$ the *local Poisson maximal operator*, defined by

$$\mathscr{P}_*^{\mathrm{loc}} f = \sup_{0<t\leq 1} |\mathscr{P}_t f| \,.$$

We define the *local Poisson Hardy space* as

$$\mathfrak{h}^1_{\mathscr{P}}(M) = \{f \in L^1(M) : \mathscr{P}_*^{\mathrm{loc}} f \in L^1(M)\} \,,$$

endowed with the norm

$$\|f\|_{\mathfrak{h}^1_{\mathscr{P}}} = \|f\|_{L^1} + \|\mathscr{P}_*^{\mathrm{loc}} f\|_{L^1} \,.$$

We shall need the following result, which relates $\mathfrak{h}^1_{\mathscr{H}}(M)$ and $\mathfrak{h}^1_{\mathscr{P}}(M)$ with the Goldberg type space $\mathfrak{h}^1(M)$, and is valid more generally for any manifold M in the class $\mathscr{M}_0$.

Proposition 3.1. *The spaces $\mathfrak{h}^1(M)$, $\mathfrak{h}^1_{\mathscr{H}}(M)$ and $\mathfrak{h}^1_{\mathscr{P}}(M)$ coincide and their norms are mutually equivalent.*

Proof. See [23, Corollary 5.4]. □

3.2. Relationships between $H^1_{\mathscr{H},c}(M)$, $H^1_{\mathscr{P},c}(M)$ and $\mathfrak{X}^\gamma(M)$

In this subsection we establish some inclusions between $H^1_{\mathscr{H},c}(M)$, $H^1_{\mathscr{P},c}(M)$ and $\mathfrak{X}^\gamma(M)$, whenever M belongs to the class $\mathscr{M}$ defined in the Introduction.

It is well known that the Poisson semigroup $\mathscr{P}_t$ can be subordinated to the heat semigroup $\mathscr{H}_t$ via the formula

$$\mathscr{P}_t = t \int_0^\infty \frac{\mathrm{e}^{-t^2/(4s)}}{\sqrt{4\pi s}} \, \mathscr{H}_s \, \frac{\mathrm{d}s}{s} \tag{3.1}$$

(see, for instance, [38, formula (2), p. 260] or [34, formula (*), p. 47]).

One of the results we want to prove in this generality is that $\mathfrak{X}^{1/2}(M) \subseteq H^1_{\mathscr{P},c}(M)$ for all $c < 1$. This will generalise a result of Anker [1], who proved, in the case where M is a Riemannian symmetric space of the noncompact type, that for every $c < 1$ there exists a constant C such that

$$\|\mathscr{P}_*^c f\|_{L^1} \leq C \, \left(\|f\|_{L^1} + \big\| |\mathscr{R} f| \big\|_{L^1} \right) , \tag{3.2}$$

where $\mathscr{R}$ denotes the Riemannian Riesz transform.

From Proposition 2.7 (ii) we know that a function f belongs to $\mathfrak{X}^\gamma(M)$ if and only if both f and $\mathscr{L}^{-\gamma} f$ belong to $\mathfrak{h}^1(M)$. In order to prove the inclusion $\mathfrak{X}^{1/2}(M) \subseteq H^1_{\mathscr{P},c}(M)$ for $c < 1$, and more generally that $\mathfrak{X}^\gamma(M) \subseteq H^1_{\mathscr{P},c}(M)$ for $c < 2\gamma$ it is natural to write $\mathscr{P}_t f$ as $\mathscr{L}^\gamma \mathscr{P}_t \mathscr{L}^{-\gamma} f$, and to look for estimates for the operator $\mathscr{L}^\gamma \mathscr{P}_t$. We can obtain

subordination formulae for the operators $\mathscr{L}^\gamma \mathscr{P}_t$ in terms of the heat semigroup similar to formula (3.1) for $\mathscr{P}_t$, as shown by taking $\alpha = 1/2$ in the following technical lemma.

Lemma 3.2. *For $\alpha \in (0,1)$ and $\gamma \geq 0$, let $F_{\alpha,\gamma} : (0,\infty) \to \mathbb{C}$ be defined by contour integration as follows:*

$$F_{\alpha,\gamma}(s) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} s\, \mathrm{e}^{sz - z^\alpha} z^\gamma \, \mathrm{d}z \tag{3.3}$$

for any $\sigma > 0$.

(i) *There exist constants $C, \rho > 0$ (possibly depending on α and γ) such that*

$$|F_{\alpha,\gamma}(s)| \leq C s^{-\gamma}\, \mathrm{e}^{-2\rho s^{-\alpha/(1-\alpha)}}. \tag{3.4}$$

Moreover

$$z^\gamma \mathrm{e}^{-z^\alpha} = \int_0^\infty F_{\alpha,\gamma}(s)\, \mathrm{e}^{-sz}\, \frac{\mathrm{d}s}{s} \tag{3.5}$$

for every complex number z with $\operatorname{Re} z > 0$.

(ii) *For all real numbers $c < \gamma/\alpha$,*

$$\int_0^\infty \sup_{t \geq 1} t^c \left| t^{-\gamma/\alpha} F_{\alpha,\gamma}(s/t^{1/\alpha}) \right| \frac{\mathrm{d}s}{s} < \infty.$$

We point out that, in the case $\gamma = 0$, a more precise estimate of $F_{\alpha,\gamma}(s)$ for $s \to \infty$ can be proved than the one in (3.4), as shown in [23, Lemma 5.2]. However we will not need this here.

Proof. We first prove part (i). The construction of the function $F_{\alpha,\gamma}$ satisfying (3.5) through the contour integration (3.3) is a small modification of that in [38, Section IX.11], and can be justified through the theory of the Laplace transform.

Let $a := \cos(\alpha\pi/2)/3$. Since $\operatorname{Re}(z^\alpha) \geq 3a|z|^\alpha$ whenever $\operatorname{Re} z > 0$, from the representation (3.3) we obtain that, for any $\sigma \geq (a\alpha)^{1/(1-\alpha)}$,

$$\begin{aligned} |F_{\alpha,\gamma}(s)| &\leq \frac{1}{2\pi} s \mathrm{e}^{s\sigma} \sigma^{\gamma+1} \int_{\mathbb{R}} (1+t^2)^{\gamma/2} \mathrm{e}^{-3a\sigma^\alpha (1+t^2)^{\alpha/2}}\, \mathrm{d}t \\ &\leq C s \mathrm{e}^{2s\sigma} \mathrm{e}^{-2a\sigma^\alpha}, \end{aligned}$$

where the constant C may depend on α and γ but not on σ; by choosing $\sigma = (a\alpha/s)^{1/(1-\alpha)}$ and observing that

$$s(a\alpha/s)^{1/(1-\alpha)} - a(a\alpha/s)^{\alpha/(1-\alpha)} = -(1-\alpha)\alpha^{\alpha/(1-\alpha)} a^{1/(1-\alpha)} s^{-\alpha/(1-\alpha)},$$

we obtain the estimate in (3.4) in the case $s \leq 1$. As for the remaining case, by changing the contour of integration in (3.3) to the concatenation of the two half-lines $z = re^{-i\theta}$ $(-\infty < -r < 0)$ and $z = re^{i\theta}$ $(0 < r < \infty)$ for any $\theta \in (\pi/2, \pi)$, we obtain, much as in [38, p. 263], the representation

$$F_{\alpha,\gamma}(s) = \frac{1}{\pi} \int_0^\infty sr^\gamma e^{sr\cos\theta - r^\alpha \cos(\alpha\theta)} \sin\left(sr\sin\theta - r^\alpha \sin(\alpha\theta) + \theta(1+\gamma)\right) \, dr;$$

if we take $\theta = \pi/(1+\alpha)$, then $\alpha\theta = \pi - \theta$, thus $k := \cos(\alpha\theta) = -\cos(\theta) > 0$ and

$$|F_{\alpha,\gamma}(s)| \leq C \int_0^\infty sr^\gamma e^{-k(sr+r^\alpha)} \, dr \leq Cs^{-\gamma}$$

for $s \geq 1$. This completes the proof of part (i).

Next we prove part (ii). Note that, by (3.4),

$$\begin{aligned} \sup_{t\geq 1} t^c \left| t^{-\gamma/\alpha} F_{\alpha,\gamma}(s/t^{1/\alpha}) \right| &\leq Cs^{\alpha c - \gamma} \sup_{t\geq 1} (t/s^\alpha)^c e^{-2\rho(t/s^\alpha)^{1/(1-\alpha)}} \\ &\leq Cs^{\alpha c - \gamma} e^{-\rho s^{-\alpha/(1-\alpha)}}, \end{aligned}$$

and the latter is integrable with respect to ds/s provided $c < \gamma/\alpha$. □

In view of the equality $\mathfrak{X}^{1/2}(M) = H^1_{\mathscr{R}}(M)$ proved in Section 2, the estimate (3.2) corresponds to the case $\gamma = 1/2$ of Theorem 3.3 (iii) below.

Theorem 3.3. *Suppose that γ is a positive number. The following hold:*

(i) $H^1_{\mathscr{H},c}(M) \subseteq \mathfrak{h}^1(M)$ *for all* $c \in \mathbb{R}$ *and* $H^1_{\mathscr{H},c}(M) \subseteq \mathfrak{X}^\gamma(M)$ *for all* $c > \gamma$;
(ii) $H^1_{\mathscr{P},c}(M) \subseteq \mathfrak{h}^1(M)$ *for all* $c \in \mathbb{R}$ *and* $H^1_{\mathscr{P},c}(M) \subseteq \mathfrak{X}^\gamma(M)$ *for all* $c > 2\gamma$;
(iii) $H^1_{\mathscr{P},c}(M) = \mathfrak{h}^1(M)$ *for all* $c < 0$ *and* $\mathfrak{X}^\gamma(M) \subseteq H^1_{\mathscr{P},c}(M)$ *for all* $c < 2\gamma$;
(iv) $H^1_{\mathscr{H},c}(M) \subseteq H^1_{\mathscr{P},2c}(M)$ *for all* $c \in \mathbb{R}$.

Proof. First we prove part (i). Suppose that f is in $H^1_{\mathscr{H},c}(M)$, i.e. $\mathscr{H}^c_* f$ is in $L^1(M)$. Then $\mathscr{H}^{\mathrm{loc}}_* f$ is in $L^1(M)$, whence f is in $\mathfrak{h}^1(M)$, by Proposition 3.1, and

$$\|f\|_{\mathfrak{h}^1} \leq C \, \|\mathscr{H}^c_* f\|_{L^1}.$$

Now suppose that $c > \gamma$ and assume that f is in $H^1_{\mathscr{H},c}(M)$. We have already proved that f is in $\mathfrak{h}^1(M)$. By Proposition 2.7 (ii), in order to conclude f is in $\mathfrak{X}^\gamma(M)$, it suffices to prove that $g := \mathscr{L}^{-\gamma} f$ is in $\mathfrak{h}^1(M)$, and there exists a constant C such that

$$\|\mathscr{H}^{\mathrm{loc}}_*(\mathscr{L}^{-\gamma} f)\|_{L^1} \leq C \, \|\mathscr{H}^c_* f\|_{L^1} \qquad \forall f \in H^1_{\mathscr{H},c}(M).$$

By spectral theory,

$$\mathscr{I} = c_\gamma \int_0^\infty (t\mathscr{L})^\gamma \, \mathrm{e}^{-t\mathscr{L}} \, \frac{\mathrm{d}t}{t}, \tag{3.6}$$

where $c_\gamma := 1/\Gamma(\gamma)$. Thus, for each s in $(0,1]$,

$$\mathscr{H}_s g = c_\gamma \int_0^\infty t^\gamma \, \mathscr{H}_{t+s} f \, \frac{\mathrm{d}t}{t} = c_\gamma \int_0^\infty \frac{t^\gamma}{\langle t+s\rangle^c} \, \langle t+s\rangle^c \, \mathscr{H}_{t+s} f \, \frac{\mathrm{d}t}{t}.$$

By taking the supremum with respect to s in $(0,1]$, we see that

$$\|\mathscr{H}_*^{\mathrm{loc}} g\|_{L^1} \le C \, \|\mathscr{H}_*^c f\|_{L^1} \int_0^\infty t^{\gamma-1} \langle t\rangle^{-c} \, \mathrm{d}t.$$

The integral on the right hand side is convergent, because $c > \gamma > 0$, and therefore g is in $\mathfrak{h}^1(M)$, by Proposition 3.1, as required.

The proof of part (ii) is entirely analogous to that of part (i). One simply needs to replace γ with 2γ, $\mathscr{H}$ with $\mathscr{P}$, as well as $\mathscr{L}$ with $\mathscr{L}^{1/2}$ in the reproducing formula (3.6). We omit the details.

Next, we prove part (iii). Due to the inclusion already established in part (ii), we only need to prove that $\mathfrak{h}^1(M) \subseteq H^1_{\mathscr{P},c}(M)$ for all $c < 0$, and $\mathfrak{X}^\gamma(M) \subseteq H^1_{\mathscr{P},c}(M)$ for all $\gamma > 0$ and $c < 2\gamma$. In light of the characterisation of $\mathfrak{X}^\gamma(M)$ in Proposition 2.7 (ii), we are effectively reduced to proving that, for all $\gamma \ge 0$ and $c < 2\gamma$,

$$\text{if } f \in \mathfrak{h}^1(M) \text{ and } \mathscr{L}^{-\gamma} f \in \mathfrak{h}^1(M), \text{ then } f \in H^1_{\mathscr{P},c}(M).$$

Let then $\gamma \ge 0$, and assume that both f and $\mathscr{L}^{-\gamma} f$ are in $\mathfrak{h}^1(M)$. As f is in $\mathfrak{h}^1(M)$, we deduce that $\mathscr{P}_*^{\mathrm{loc}} f$ is in $L^1(M)$, by Proposition 3.1, and

$$\|\mathscr{P}_*^{\mathrm{loc}} f\|_{L^1} \le C \, \|f\|_{\mathfrak{h}^1} \, . \tag{3.7}$$

Let $c < 2\gamma$; in order to prove that f is in $H^1_{\mathscr{P},c}(M)$, it remains to show that $\sup_{t>1} t^c |\mathscr{P}_t f|$ is in $L^1(M)$. We write $\mathscr{P}_t f = \mathscr{L}^\gamma \mathscr{P}_t \mathscr{L}^{-\gamma} f$, and set $g := \mathscr{L}^{-\gamma} f$. Formula (3.5), applied with $\alpha = 1/2$, and spectral theory yield the subordination formula

$$\mathscr{L}^\gamma \mathscr{P}_t g = t^{-2\gamma} \int_0^\infty F_{1/2,\gamma}(s) \mathscr{H}_{st^2} g \, \frac{\mathrm{d}s}{s} = \int_0^\infty \left[t^{-2\gamma} F_{1/2,\gamma}(s/t^2)\right] \mathscr{H}_s g \, \frac{\mathrm{d}s}{s}. \tag{3.8}$$

Then

$$\left\| \sup_{t>1} t^c |\mathscr{P}_t f| \right\|_{L^1} \le \int_0^\infty \sup_{t>1} t^c \left| t^{-2\gamma} F_{1/2,\gamma}(s/t^2) \right| \, \|\mathscr{H}_s g\|_{L^1} \, \frac{\mathrm{d}s}{s} \le C \, \|g\|_{L^1} \, ; \tag{3.9}$$

the last inequality follows from the contractivity of $\mathscr{H}_s$ on $L^1(M)$ and Lemma 3.2 (ii) (which applies for $c < 2\gamma$). Combining this inequality with (3.7) finally yields

$$\|\mathscr{P}_*^c f\|_{L^1} \le C \left[\|f\|_{\mathfrak{h}^1} + \|\mathscr{L}^{-\gamma} f\|_{\mathfrak{h}^1}\right],$$

as required.

Finally, we prove part (iv). If $c < 0$, then $H^1_{\mathscr{H},c}(M) \subseteq \mathfrak{h}^1(M) = H^1_{\mathscr{P},2c}(M)$ by parts (i) and (iii). Assume instead that $c \geq 0$. By changing variables in the subordination formula (3.1), we see that

$$\mathscr{P}_t = \frac{1}{\sqrt{\pi}} \int_0^\infty v^{-1/2}\, \mathrm{e}^{-v}\, \mathscr{H}_{t^2/4v}\, \mathrm{d}v,$$

and consequently, since $c \geq 0$,

$$t^{2c}|\mathscr{P}_t f| \leq \frac{4^c}{\sqrt{\pi}} \int_0^\infty v^{c-1/2}\, \mathrm{e}^{-v}\, \langle t^2/4v\rangle^c |\mathscr{H}_{t^2/4v} f|\, \mathrm{d}v.$$

Since $\int_0^\infty v^{c-1/2}\, \mathrm{e}^{-v}\, \mathrm{d}v < \infty$, this implies the pointwise estimate

$$\sup_{t>0} t^{2c}|\mathscr{P}_t f| \leq C \mathscr{H}_*^c f,$$

and therefore

$$\mathscr{P}_*^{2c} f \leq \sup_{0<t\leq 1} |\mathscr{P}_t f| + \sup_{t\geq 1} t^{2c}|\mathscr{P}_t f| \leq C(\mathscr{H}_* f + \mathscr{H}_*^c f) \leq C \mathscr{H}_*^c f,$$

as desired. □

Remark 3.4. The estimate (3.9), namely, the fact that

$$\left\| \sup_{t>1} t^c |\mathscr{P}_t f| \right\|_{L^1} \leq C\, \|\mathscr{L}^{-\gamma} f\|_{L^1} \tag{3.10}$$

for $c < 2\gamma$, could also be proved without using the subordination formula (3.8) and the estimate (3.4), but as a consequence of a more general estimate for generators of analytic semigroups. Specifically, if $\mathscr{T}$ is a sectorial operator of angle $\omega \in [0, \pi/2)$ on a Banach space $\mathscr{X}$ (i.e., $-\mathscr{T}$ generates an analytic semigroup of angle $\pi/2 - \omega$ on $\mathscr{X}$, cf. [16, Section 3.4]), then, for all $\beta > 0$ and $c < \beta$, the estimate

$$\left\| \sup_{t>1} t^c |e^{-t\mathscr{T}} f| \right\|_{\mathscr{X}} \leq C\, \|\mathscr{T}^{-\beta} f\|_{\mathscr{X}} \tag{3.11}$$

holds for all f in the domain of $\mathscr{T}^{-\beta}$. As $\mathscr{L}$ is sectorial on $L^1(M)$ of angle $\pi/2$ [22, Theorem 3.1], the fractional power $\mathscr{L}^{1/2}$ is sectorial on $L^1(M)$ of angle $\pi/4$ [16, Proposition 3.1.2], thus (3.10) follows by applying (3.11) with $\mathscr{T} = \mathscr{L}^{1/2}$, $\mathscr{X} = L^1(M)$ and $\beta = 2\gamma$. More generally, (3.11) could be applied with $\mathscr{T} = \mathscr{L}^\alpha$ for $\alpha \in (0, 1)$ (cf. Remark 4.10 below).

In order to prove (3.11), one can write $\mathscr{T}^\beta e^{-t\mathscr{T}}$ as a contour integral, namely,

$$\mathscr{T}^\beta e^{-t\mathscr{T}} = \frac{1}{2\pi i}\int_{\Gamma_\theta} z^\beta e^{-tz}(z-\mathscr{T})^{-1}\,\mathrm{d}z,$$

where $\Gamma_\theta = \{re^{i\theta} : r>0\} \cup \{re^{-i\theta} : r>0\}$ and $\theta \in (\omega, \pi/2)$ [16, eq. (2.5)] and observe that, as $\mathscr{T}$ is sectorial of angle ω,

$$\left\|\!\left|(z-\mathscr{T})^{-1}\right|\!\right\|_{\mathscr{X}} \le C|z|^{-1} \qquad \forall z \in \Gamma_\theta$$

[16, Section 2.1]; thus, if f is in the domain of $\mathscr{T}^{-\beta}$ and $g = \mathscr{T}^{-\beta} f$, then

$$\begin{aligned}\left\|\sup_{t>1} t^c |e^{-t\mathscr{T}} f|\right\|_{\mathscr{X}} = \left\|\sup_{t>1} t^c |\mathscr{T}^\beta e^{-t\mathscr{T}} g|\right\|_{\mathscr{X}} &\le C\|g\|_{\mathscr{X}} \int_0^\infty \left[\sup_{t>1} t^c e^{-tr\cos\theta}\right] r^\beta \,\frac{\mathrm{d}r}{r} \\ &\le C\|\mathscr{T}^{-\beta} f\|_{\mathscr{X}} \int_0^\infty e^{-(r\cos\theta)/2} r^{\beta-c}\,\frac{\mathrm{d}r}{r}\end{aligned}$$

and the estimate (3.11) follows as $\cos\theta > 0$ and $c < \beta$, so the last integral is finite.

The above approach does not work when $\mathscr{T}$ is sectorial of angle $\omega \ge \pi/2$; as a matter of fact, Proposition 4.3 and Theorem 4.5 below show that, if M is a Damek–Ricci space, then the estimate (3.11) with $\mathscr{T} = \mathscr{L}$ and $\mathscr{X} = L^1(M)$ does not hold in the whole range $c < \beta$, but only for $c < (\beta-1)/2$.

We thank the referee for suggesting that the validity of (3.10) may be related to the analyticity of the Poisson semigroup.

From Theorem 3.3 we deduce the following strict inclusions, which imply among other things that the Riesz Hardy space and the Poisson–Hardy space on M cannot coincide.

Corollary 3.5. *For each $\gamma, \varepsilon > 0$ the following proper inclusions hold:*

$$H^1_{\mathscr{P},2\gamma+\varepsilon}(M) \subset \mathfrak{X}^\gamma(M) \subset H^1_{\mathscr{P},2\gamma-\varepsilon}(M).$$

In particular,

$$H^1_{\mathscr{P},1+\varepsilon}(M) \subset H^1_{\mathscr{R}}(M) \subset H^1_{\mathscr{P},1-\varepsilon}(M)$$

and therefore

$$H^1_{\mathscr{R}}(M) \subset H^1_{\mathscr{P}}(M).$$

Proof. Up to the fact that the inclusions are strict, the first chain of inclusions follows by combining parts (ii) and (iii) of Theorem 3.3. Moreover, in light of the equality $\mathfrak{X}^{1/2}(M) = H^1_{\mathscr{R}}(M)$ proved in Theorem 2.10, the second chain of inclusions is a restatement of the first one in the case where $\gamma = 1/2$, and the last inclusion follows by

further specialising to $\varepsilon = 1$. So we only need to check the strictness of the first chain of inclusions.

To prove that $H^1_{\mathscr{P},2\gamma+\varepsilon}(M)$ is properly contained in $H^1_{\mathscr{R}}(M)$, we argue as follows. Choose δ so that $2\gamma + \varepsilon > \delta > 2\gamma$. Note that $H^1_{\mathscr{P},2\gamma+\varepsilon}(M) \subseteq \mathfrak{X}^{\delta/2}(M)$, by Theorem 3.3 (ii). Since the family of spaces $\{\mathfrak{X}^\gamma(M) : \gamma > 0\}$ is strictly decreasing (see Proposition 2.7 (i)) and $\delta > 2\gamma$, $\mathfrak{X}^\gamma(M)$ contains $\mathfrak{X}^{\delta/2}(M)$ properly, whence $\mathfrak{X}^\gamma(M)$ contains $H^1_{\mathscr{P},2\gamma+\varepsilon}(M)$ properly, as required.

The proof that $\mathfrak{X}^\gamma(M)$ is properly contained in $H^1_{\mathscr{P},2\gamma-\varepsilon}(M)$ is similar, and is omitted. □

In light of Corollary 3.5, a natural question is whether $H^1_{\mathscr{P},1}(M)$ agrees with $H^1_{\mathscr{R}}(M)$. We shall see in Section 4 that the answer is in the negative when M is a Damek–Ricci space. In fact, a similar remark applies to $H^1_{\mathscr{P},2\gamma}(M)$ and $\mathfrak{X}^\gamma(M)$ for any $\gamma > 0$, as shown in Theorem 4.8 below.

4. Noninclusions on Damek–Ricci spaces

The results in this section state that certain inclusions between various type of Hardy spaces introduced in the previous sections fail in the case where the Riemannian manifold M is a Damek–Ricci space, which we shall henceforth denote by S. The main reason for restricting to Damek–Ricci spaces is that we need rather precise estimates for the heat semigroup that we obtain via spherical Fourier analysis. The noninclusions we prove on Damek–Ricci spaces are likely to hold on all symmetric spaces of the noncompact type. However, this would add little knowledge at the expense of a presumably long analysis based on quite intricated and technical proofs. Altogether, we believe this is not worth doing here.

4.1. Damek–Ricci spaces

We refer to [2,3,11,13] for more details on the analysis on Damek–Ricci spaces. We just recall here that a Damek–Ricci space S is the one-dimensional harmonic extension of an Heisenberg-type group N obtained by making $\mathbb{R}^+$ act on N by automorphic dilations. In particular, S is a connected Lie group, with a Riemannian structure which is invariant by left translations, and the Riemannian measure μ is a left Haar measure. We denote by Q the homogeneous dimension of N and by n the dimension of S. It is known that in this case the bottom of the L^2-spectrum of the Laplace–Beltrami operator $\mathscr{L}$ is $b = Q^2/4$.

A radial function on S is a function that depends only on the distance from the base point o. If f is radial, then we may write $f(r)$ in place of $f(x)$ when $d(x, o) = r$; moreover

$$\int_S f(x)\,\mathrm{d}\mu(x) = \int_0^\infty f(r)A(r)\,\mathrm{d}r\,, \quad \text{where } A(r) \asymp \begin{cases} r^{n-1} & r < 1 \\ \mathrm{e}^{Qr} & r \geq 1\,. \end{cases} \tag{4.1}$$

A radial function is called *spherical* if it is an eigenfunction of $\mathscr{L}$ and if it takes value 1 at o. For $\lambda \in \mathbb{C}$ we denote by ϕ_λ the spherical function such that

$$\mathscr{L}\phi_\lambda = (\lambda^2 + Q^2/4)\,\phi_\lambda. \tag{4.2}$$

The *spherical transform* of an integrable radial function is defined as

$$\widetilde{f}(\lambda) = \int_S f(x)\,\phi_\lambda(x)\,\mathrm{d}\mu(x)\,.$$

Note that $\widetilde{f}$ is an even function, as $\phi_\lambda = \phi_{-\lambda}$; moreover, for "nice" radial functions f the following inversion formula holds:

$$f(x) = C\int_0^\infty \widetilde{f}(\lambda)\,\phi_\lambda(x)\,|\mathbf{c}(\lambda)|^{-2}\,\mathrm{d}\lambda\,,$$

where the density $|\mathbf{c}(\lambda)|^{-2}$ is expressed in terms of the Harish-Chandra function

$$\mathbf{c}(\lambda) = \frac{2^{Q-2i\lambda}\Gamma(2i\lambda)}{\Gamma(Q/2+i\lambda)}\,\frac{\Gamma(n/2)}{\Gamma((n-Q)/2+i\lambda)} = \overline{\mathbf{c}(-\overline{\lambda})} \tag{4.3}$$

(see [3, p. 151]).

In the following statement we collect a few well-known properties of the function $\mathbf{c}$ and the spherical functions ϕ_λ, that will be of use in the sequel.

Lemma 4.1. *The following hold.*

(i) *The function* $\mathbf{c}^{-1}$ *is holomorphic in the half-plane* $\{\lambda : \operatorname{Im}\lambda < Q/2\}$.
(ii) *For all* $k \in \mathbb{N}$ *and all* $\sigma > 0$, *there exists a constant* C *such that*

$$|\partial_\lambda^k\big(\lambda^{-1}\mathbf{c}(-\lambda)^{-1}\big)| \le C\,(1+|\operatorname{Re}\lambda|)^{(n-3)/2-k}$$

for all $\lambda \in \mathbb{C}$ *with* $0 \le \operatorname{Im}\lambda \le \sigma$.
(iii) ϕ_λ *is bounded if and only if* $|\operatorname{Im}\lambda| \le Q/2$, *and in that case* $\|\phi_\lambda\|_\infty \le 1$.
(iv) $\phi_{\pm iQ/2} \equiv 1$.
(v) *For every* $\lambda \in \mathbb{C} \setminus \frac{1}{2}i\mathbb{Z}$ *and* $r > 0$,

$$\phi_\lambda(r) = \mathbf{c}(\lambda)\Phi_\lambda(r) + \mathbf{c}(-\lambda)\Phi_{-\lambda}(r),$$

where

$$\Phi_\lambda(r) = \sum_{\ell=0}^\infty \Gamma_\ell(\lambda)\,\mathrm{e}^{(i\lambda-Q/2-\ell)r};$$

here $\Gamma_0 \equiv 1$, *and moreover there exist* $d, C > 0$ *such that*

$$|\Gamma_\ell(\lambda)| \le C(1+\ell)^d$$

for every $\ell \in \mathbb{N}$ *and every* λ *in*

$$\{\lambda \in \mathbb{C} \,:\, \operatorname{Im}\lambda \ge 0\} \cup \{\lambda \in \mathbb{C} \,:\, |\operatorname{Im}\lambda| \le |\operatorname{Re}\lambda|\}.$$

Proof. Parts (i) and (ii) follow from (4.3); in particular, for part (ii), see the proof of [19, Proposition A1(b)]. Part (iii) is in [11, Theorem 5.12], while part (iv) is clear from (4.2). As for part (v), see [3, formula (5) and Theorem 3.2]. □

It is useful to recall that on the group S the convolution of two functions is defined by

$$f * g(x) = \int_S f(y)\, g(y^{-1}x)\, \mathrm{d}\mu(y)\,.$$

Since the Laplace–Beltrami operator $\mathscr{L}$ on S is left-invariant, its heat and Poisson semigroups $\mathscr{H}_t$ and $\mathscr{P}_t$ can be realised by the right convolution with the heat and Poisson kernels h_t and p_t, i.e.

$$\mathscr{H}_t f = f * h_t, \qquad \mathscr{P}_t f = f * p_t.$$

As we shall see, spherical analysis on S allows us to obtain precise estimates for the heat and Poisson kernels, which will provide a sharper picture with respect to inclusions and noninclusions of various Hardy type spaces.

4.2. Estimates for the heat maximal operator on S

We start with a technical lemma, which will be useful in our estimates for the heat kernel on S.

Lemma 4.2. *Let* $a \in (0,\infty)$.

(i) *If* $\varepsilon, k \in [0,\infty)$, *then*

$$\sup_{t\ge 1} t^{-\varepsilon}(1+t^{-1/2}|r-at|)^{-k} \asymp r^{-\min\{\varepsilon,k\}},$$

for all $r \in [1,\infty)$. *The implicit constants may depend on* ε, a *and* k, *but not on* r.

(ii) *If* $k > 1$, *then*

$$\int_1^\infty (1+t^{-1/2}|r-at|)^{-k}\, \mathrm{d}r \le C\sqrt{t}.$$

Proof. We first prove part (i). Note that, if $r \le a+1$, then, trivially,

$$\sup_{t\ge 1} t^{-\varepsilon}(1+t^{-1/2}|r-at|)^{-k} \asymp 1 \asymp r^{-\min\{\varepsilon,k\}}.$$

So we only need to consider the range where $r \ge a+1$.

Assuming that $r \ge a+1$, we first observe that

$$\sup_{t\ge r/(a+1)} t^{-\varepsilon}(1+t^{-1/2}|r-at|)^{-k} \asymp r^{-\varepsilon}.$$

Indeed, the lower bound is immediately obtained by taking $t = r/a$, while the upper bound simply follows by the majorisation $(1+t^{-1/2}|r-at|)^{-k} \le 1$.

Let us now consider the range $1 \le t \le r/(a+1)$. Here $|r-at| \asymp r$, and $r/t^{1/2} \ge r/t \ge 1$, so

$$t^{-\varepsilon}(1+t^{-1/2}|r-at|)^{-k} \asymp t^{-\varepsilon+k/2} r^{-k}$$

and therefore

$$\sup_{1\le t\le r/(a+1)} t^{-\varepsilon}(1+t^{-1/2}|r-at|)^{-k} \asymp \begin{cases} r^{-k/2-\varepsilon} & \text{if } k \ge 2\varepsilon, \\ r^{-k} & \text{if } k \le 2\varepsilon. \end{cases}$$

The desired estimate follows by combining the ones above, and the proof of part (i) is complete.

As for part (ii), we simply have

$$\begin{aligned} \int_1^\infty (1+t^{-1/2}|r-at|)^{-k}\,\mathrm{d}r &= \int_{|r-at|<\sqrt{t}} + \int_{|r-at|\ge\sqrt{t}} \\ &\le 2\sqrt{t} + t^{k/2}\int_{|r|\ge\sqrt{t}} |r|^{-k}\,\mathrm{d}r \\ &\le C\sqrt{t}, \end{aligned}$$

provided $k > 1$. □

Using spherical Fourier analysis, we can now deduce a number of precise estimates for the heat kernel h_t. Related results on any symmetric space of the noncompact type may be found in [1].

Proposition 4.3. *Suppose that γ and c are real numbers and $\gamma \ge 0$.*

(i) *For each radial compactly supported distribution ψ on S such that $\widetilde{\psi}(iQ/2) \ne 0$ there exists $\bar{r} \ge 1$ such that, for all $c \in \mathbb{R}$,*

$$\sup_{t\geq 1} t^c |\mathscr{L}^\gamma(\mathscr{H}_t\psi)(x)| \geq C\,\frac{\mathrm{e}^{-Q|x|}}{|x|^{(\gamma+1)/2-c}} \qquad \forall x \in B_{\bar{r}}(o)^{\complement}. \tag{4.4}$$

In particular, $\sup_{t\geq 1} t^c |\mathscr{L}^\gamma(\mathscr{H}_t\psi)| \notin L^1(S)$ *whenever* $c \geq (\gamma-1)/2$.

(ii) *If* $c \leq (\gamma+1)/2$, *then the bound*

$$\sup_{t\geq 1} t^c |\mathscr{L}^\gamma h_t(x)| \leq C\,\frac{\mathrm{e}^{-Q|x|}}{(1+|x|)^{(\gamma+1)/2-c}} \qquad \forall x \in S \tag{4.5}$$

holds whenever $\gamma \in \mathbb{N}$ *or* $c > -(\gamma+1)/2$.

(iii) *If* $c < (\gamma-1)/2$, *then* $\sup_{t\geq 1} t^c |\mathscr{L}^\gamma h_t| \in L^1(S)$.

(iv) *For all* $t > 0$,

$$\|\mathscr{L}^\gamma h_t\|_{L^1} \asymp \begin{cases} t^{-\gamma} & \text{if } t \leq 1, \\ t^{-\gamma/2} & \text{if } t > 1. \end{cases}$$

Remark 4.4. If $\psi \in L^1(S)$ is radial, then $\widetilde{\psi}(iQ/2) = \int_S \psi\, d\mu$ (see Lemma 4.1 (iv)), so the assumption $\widetilde{\psi}(iQ/2) \neq 0$ in part (i) above has a clear "geometric" meaning. On the other hand, the pointwise lower bound in part (i) applies not only to (compactly supported) radial functions $\psi \in L^1(S)$, but also more generally to radial distributions, such as the Dirac delta δ_o. Indeed, in the case where $\psi = \delta_o$, the upper bound in part (ii) shows (for certain ranges of the parameters γ and c) the optimality of the lower bound in part (i).

Proof. We first prove part (i). Let us immediately observe that the nonintegrability of $\sup_{t\geq 1} t^c |\mathscr{L}^\gamma(\mathscr{H}_t\psi)|$ when $c \geq (\gamma-1)/2$ follows from the pointwise estimate (4.4) and the integration formula (4.1) for radial functions. Thus, we are reduced to proving (4.4) for any $c \in \mathbb{R}$.

Up to replacing ψ with $\psi/\widetilde{\psi}(iQ/2)$, we may and shall assume that $\widetilde{\psi}(iQ/2) = 1$. Observe that, by spherical Fourier analysis, for all $r \geq 0$,

$$\mathscr{L}^\gamma(\mathscr{H}_t\psi)(r) = \int_{\mathbb{R}} \widetilde{\psi}(\lambda)\,(\lambda^2+Q^2/4)^\gamma\,\mathrm{e}^{-t(\lambda^2+Q^2/4)}\,\phi_\lambda(r)\,|\mathbf{c}(\lambda)|^{-2}\,\mathrm{d}\lambda \tag{4.6}$$

and, if $r > 0$, by Lemma 4.1 (v),

$$\mathscr{L}^\gamma(\mathscr{H}_t\psi)(r) = 2\int_{\mathbb{R}} \widetilde{\psi}(\lambda)\,(\lambda^2+Q^2/4)^\gamma\,\mathrm{e}^{-t(\lambda^2+Q^2/4)}\,\Phi_\lambda(r)\,\mathbf{c}(-\lambda)^{-1}\,\mathrm{d}\lambda\,, \tag{4.7}$$

where we have used the fact that $\widetilde{\psi}$ is even.

Assume that $r \geq 1$ and $t \geq 1$. By the Paley–Wiener theorem for the spherical Fourier transform [32, Theorem 3.6], since ψ is a compactly supported distribution, $\widetilde{\psi}$ extends to an entire function of exponential type. In particular, there exist $m, R > 0$ such that

$$\big|\widetilde{\psi}(\lambda)\big| \leq C\,(1+|\lambda|)^m\,\mathrm{e}^{R\,|\mathrm{Im}\,\lambda|} \qquad \forall \lambda \in \mathbb{C}\,. \tag{4.8}$$

Additionally, we can write

$$(\lambda^2 + Q^2/4)^\gamma = (Q/2 + i\lambda)^\gamma (Q/2 - i\lambda)^\gamma$$

for all $\lambda \in \mathbb{C}$ with $|\operatorname{Im}\lambda| \le Q/2$ and $\lambda \neq \pm iQ/2$; here we choose the branch of $w \mapsto w^\gamma$ on the complex plane slit along the half-line $(-\infty, 0]$ that agrees with the arithmetic γ power on $(0, \infty)$. By Cauchy's theorem, we may shift the contour of integration in (4.7) from the real line to $\mathbb{R} + iQ/2$, and obtain that

$$\mathscr{L}^\gamma(\mathscr{H}_t\psi)(r) = 2\int_{\mathbb{R}} \widetilde{\psi}(\lambda + iQ/2)\,(i\lambda)^\gamma\, p(\lambda)\,\mathrm{e}^{-t(\lambda^2 + i\lambda Q)}\,\Phi_{\lambda+iQ/2}(r)\,\mathrm{d}\lambda\,, \tag{4.9}$$

where $p(\lambda) = (Q - i\lambda)^\gamma\,\mathbf{c}(-\lambda - iQ/2)^{-1}$. In particular, from Lemma 4.1 (ii) we deduce that the function p, thought of as a function of a real variable, belongs to a Hörmander symbol class, namely,

$$p \in S^M, \quad \text{where } M = \gamma + (n-1)/2. \tag{4.10}$$

By Lemma 4.1 (v), we can write

$$\mathrm{e}^{Qr}\Phi_{\lambda+iQ/2}(r) = \mathrm{e}^{i\lambda r} + R_\lambda(r)\,,$$

where

$$|R_\lambda(r)| \le C_\delta\,\mathrm{e}^{-\delta r} \tag{4.11}$$

for all $r \ge 1$, $\delta \in (0,1)$ and $\lambda \in \mathbb{R}$. Hence

$$\begin{aligned}\frac{1}{2}\mathrm{e}^{Qr}\mathscr{L}^\gamma(\mathscr{H}_t\psi)(r) &= \int_{\mathbb{R}} \widetilde{\psi}(\lambda + iQ/2)\,(i\lambda)^\gamma\, p(\lambda)\,\mathrm{e}^{-t\lambda^2}\,\mathrm{e}^{i\lambda(r-tQ)}\,\mathrm{d}\lambda \\ &\quad + \int_{\mathbb{R}} \widetilde{\psi}(\lambda + iQ/2)\,(i\lambda)^\gamma\, p(\lambda)\,\mathrm{e}^{-t(\lambda^2+i\lambda Q)}\,R_\lambda(r)\,\mathrm{d}\lambda \\ &= I(t,r) + J(t,r)\,.\end{aligned} \tag{4.12}$$

Now, from (4.8), (4.10) and (4.11) we deduce that

$$|J(t,r)| \le C_\delta\,\mathrm{e}^{-\delta r}\int_{\mathbb{R}} |\lambda|^\gamma (1+|\lambda|)^{M+m}\,\mathrm{e}^{-t\,\lambda^2}\,\mathrm{d}\lambda \le C_\delta\,t^{-(1+\gamma)/2}\mathrm{e}^{-\delta r}\,, \tag{4.13}$$

for every $t \ge 1$ and $\delta \in (0,1)$.

Next, we show that the main contribution in $I(t,r)$ comes from integration in a neighbourhood of the origin. Indeed, denote by η a smooth even function on $\mathbb{R}$ with support contained in $[-2,2]$ such that $0 \le \eta \le 1$ and η is equal to 1 in $[-1,1]$. Then, we may write $I(t,r)$ as $I_\eta(t,r) + I_{1-\eta}(t,r)$, where

$$I_\eta(t,r) = \int_{\mathbb{R}} \eta(\lambda)\, \widetilde{\psi}(\lambda + iQ/2)\, (i\lambda)^\gamma\, p(\lambda)\, \mathrm{e}^{-t\lambda^2}\, \mathrm{e}^{i\lambda(r-tQ)}\, \mathrm{d}\lambda\,,$$

$$I_{1-\eta}(t,r) = \int_{\mathbb{R}} (1-\eta(\lambda))\, \widetilde{\psi}(\lambda + iQ/2)\, (i\lambda)^\gamma\, p(\lambda)\, \mathrm{e}^{-t\lambda^2}\, \mathrm{e}^{i\lambda(r-tQ)}\, \mathrm{d}\lambda\,.$$

Now, by (4.8) and (4.10),

$$\begin{aligned} |I_{1-\eta}(t,r)| &\le C \int_{|\lambda|\ge 1} \mathrm{e}^{-t\lambda^2} (1+|\lambda|)^{M+m+\gamma}\, \mathrm{d}\lambda \\ &\le C\, \mathrm{e}^{-t/2}\,. \end{aligned} \tag{4.14}$$

Next, we estimate $I_\eta(t,r)$. Since $\lambda \mapsto \widetilde{\psi}(\lambda + iQ/2)p(\lambda)$ is smooth near $\lambda = 0$ and $\widetilde{\psi}(iQ/2) = 1$, we deduce that

$$\widetilde{\psi}(\lambda + iQ/2)p(\lambda) = p(0) + O(|\lambda|)$$

for $|\lambda| \le 2$. Therefore

$$\begin{aligned} I_\eta(t,r) &= p(0) \int_{\mathbb{R}} \eta(\lambda)\, (i\lambda)^\gamma\, \mathrm{e}^{-t\lambda^2}\, \mathrm{e}^{i\lambda(r-tQ)}\, \mathrm{d}\lambda + \int_{-2}^{2} \mathrm{e}^{-t\lambda^2} O(|\lambda|^{1+\gamma})\, \mathrm{d}\lambda \\ &= p(0)\, I_\eta^0(t,r) + I_\eta^1(t,r), \end{aligned}$$

and clearly

$$|I_\eta^1(t,r)| \le C \int_{-2}^{2} |\lambda|^{1+\gamma}\, \mathrm{e}^{-t\lambda^2}\, \mathrm{d}\lambda \le C\, t^{-(2+\gamma)/2}\,. \tag{4.15}$$

Since $p(0) = Q^\gamma\, \mathbf{c}(-iQ/2)^{-1} \neq 0$ by (4.3), it remains to consider $I_\eta^0(t,r)$.

Set $\bar{t}(r) = \frac{r - A\sqrt{r}}{Q}$, where $A \in \mathbb{R}$ is a constant to be determined. Clearly, for all r sufficiently large, we have that $\bar{t}(r) \ge 1$, and moreover

$$\begin{aligned} I_\eta^0(\bar{t}(r), r) &= \int_{\mathbb{R}} \eta(\lambda)\, (i\lambda)^\gamma\, \mathrm{e}^{-\bar{t}(r)\lambda^2}\, \mathrm{e}^{iA\lambda\sqrt{r}}\, \mathrm{d}\lambda \\ &= r^{-(1+\gamma)/2} \int_{\mathbb{R}} \eta(v/\sqrt{r})\, \mathrm{e}^{-\bar{t}(r)v^2/r}\, \mathrm{e}^{iAv} (iv)^\gamma\, \mathrm{d}v\,, \end{aligned}$$

where we used the change of variables $\lambda\sqrt{r} = v$. Notice that

$$\frac{\bar{t}(r)}{r} = \frac{1}{Q} - \frac{A}{Q\sqrt{r}},$$

which tends to $1/Q$ as r tends to infinity. An application of the dominated convergence theorem shows that

$$\lim_{r\to\infty}\int_{\mathbb{R}}\eta(v/\sqrt{r})\,\mathrm{e}^{-\bar{t}(r)v^2/r}\,\mathrm{e}^{iAv}(iv)^\gamma\,\mathrm{d}v=\int_{\mathbb{R}}\mathrm{e}^{iAv}\,\mathrm{e}^{-v^2/Q}\,(iv)^\gamma\,\mathrm{d}v,$$

as $\eta(0)=1$; the last integral is the value at $-A$ of the Fourier transform of $v\mapsto \mathrm{e}^{-v^2/Q}(iv)^\gamma$, so we can choose A such that the integral does not vanish. With this choice of A, there exist $C, r_0\geq 1$ such that, for all $r\geq r_0$, we have $\bar{t}(r)\geq 1$ and

$$|I^0_\eta(\bar{t}(r),r)|\geq Cr^{-(1+\gamma)/2}.$$

Note, on the other hand, that, by the previous estimates (4.13), (4.14) and (4.15),

$$|I^1_\eta(\bar{t}(r),r)|,|J(\bar{t}(r),r)|,|I_{1-\eta}(\bar{t}(r),r)|\leq Cr^{-(2+\gamma)/2},$$

since $\bar{t}(r)\asymp r$ for r large. By combining the above estimates, we conclude that, for r sufficiently large,

$$\sup_{t>1}t^c|\mathscr{L}^\gamma(\mathscr{H}_t\psi)(r)|\geq\bar{t}(r)^c|\mathscr{L}^\gamma(\mathscr{H}_{\bar{t}(r)}\psi)(r)|\geq Cr^{c-(\gamma+1)/2}\,\mathrm{e}^{-Qr},$$

which proves (4.4) and thus part (i).

We now prove part (ii), that is, the pointwise upper bound (4.5). Note that $\mathscr{L}^\gamma h_t=\mathscr{L}^\gamma(\mathscr{H}_t\delta_o)$, hence all the above computations can be applied with $\psi=\delta_o$. In particular, by (4.6) and Lemma 4.1 (ii)-(iii), for every $t\geq 1$,

$$|\mathscr{L}^\gamma h_t(r)|\leq C\,\mathrm{e}^{-tQ^2/4}\,, \tag{4.16}$$

where C does not depend on t. This proves (4.5) for $x\in B_1(o)$, irrespective of the values of γ and c.

Take now $r\geq 1$. Then, by (4.12),

$$\begin{aligned}\frac{1}{2}\mathrm{e}^{Qr}\mathscr{L}^\gamma h_t(r)&=\int_{\mathbb{R}}(i\lambda)^\gamma\,p(\lambda)\,\mathrm{e}^{-t\lambda^2}\mathrm{e}^{i\lambda(r-tQ)}\,\mathrm{d}\lambda\\&\quad+\int_{\mathbb{R}}(i\lambda)^\gamma\,p(\lambda)\,\mathrm{e}^{-t(\lambda^2+i\lambda Q)}\,R_\lambda(r)\,\mathrm{d}\lambda\\&=I(t,r)+J(t,r)\,,\end{aligned} \tag{4.17}$$

where, as before, by (4.13),

$$|J(t,r)|\leq C_\delta\,t^{-(1+\gamma)/2}\mathrm{e}^{-\delta r}, \tag{4.18}$$

for every $t\geq 1$ and $\delta\in(0,1)$. Moreover, as $p\in S^M$ (see (4.10)), an estimate for $I(t,r)$ is immediately obtained by taking absolute values inside the integral:

$$|I(t,r)|\leq C\int_{\mathbb{R}}|\lambda|^\gamma(1+|\lambda|)^M\mathrm{e}^{-t\lambda^2}\,\mathrm{d}\lambda\leq C\,t^{-(1+\gamma)/2} \tag{4.19}$$

for every $r, t \geq 1$. The above estimates are enough to conclude the proof of (4.5) when $c = (\gamma+1)/2$. However, when $c < (\gamma+1)/2$, the estimate for $I(t,r)$ in (4.19) is not enough, as it does not give any decay in r. To improve on this, we will use integration by parts in order to take advantage of the oscillatory term $\mathrm{e}^{i\lambda(r-tQ)}$.

Let us first assume that $\gamma \in \mathbb{N}$. If we integrate by parts k times in the expression for $I(t,r)$, we obtain

$$I(t,r) = \frac{i^k}{(r-Qt)^k} \int_{\mathbb{R}} \partial_\lambda^k [(i\lambda)^\gamma p(\lambda)\, \mathrm{e}^{-t\lambda^2}]\, \mathrm{e}^{i\lambda(r-tQ)}\, \mathrm{d}\lambda. \tag{4.20}$$

Recall from (4.10) that $p \in S^M$. Hence, by arguing inductively, one readily shows that, for every nonnegative integer k, the kth derivative $\partial_\lambda^k [(i\lambda)^\gamma\, p(\lambda)\, \mathrm{e}^{-t\lambda^2}]$ is a finite sum of terms of the form $t^a\, (i\lambda)^b\, q(\lambda)\, \mathrm{e}^{-t\lambda^2}$, where

$$a, b, \ell \in \mathbb{N}, \quad q \in S^{M-\ell} \quad \text{and} \quad b - 2a - \ell = \gamma - k;$$

in particular $a - b/2 \leq (k-\gamma)/2$ and

$$\begin{aligned} \int_{\mathbb{R}} \left| t^a\, (i\lambda)^b\, q(\lambda)\, \mathrm{e}^{-t\lambda^2} \right| \mathrm{d}\lambda &\leq C t^{a-b/2} \int_{\mathbb{R}} (t\lambda^2)^{b/2} (1+|\lambda|)^{M-\ell} \mathrm{e}^{-t\lambda^2}\, \mathrm{d}\lambda \\ &\leq C t^{(k-\gamma-1)/2} \end{aligned} \tag{4.21}$$

for all $t \geq 1$. Therefore, from (4.20), we deduce that

$$|I(t,r)| \leq C_k\, |r - tQ|^{-k}\, t^{(k-\gamma-1)/2}. \tag{4.22}$$

By combining the previous estimates (4.18), (4.19) and (4.22), we conclude that, for all $k \in \mathbb{N}$ and $\delta \in (0,1)$ there exists a positive constant C such that

$$|\mathscr{L}^\gamma h_t(r)| \leq C\, \mathrm{e}^{-Qr}\, t^{-(\gamma+1)/2} \Big[\mathrm{e}^{-\delta r} + \big(1 + t^{-1/2} |r - Qt|\big)^{-k} \Big] \tag{4.23}$$

for all $t, r \geq 1$. Consequently, from Lemma 4.2 (i) we deduce that, for all $c \leq (\gamma+1)/2$ and $r \geq 1$,

$$\sup_{t \geq 1} t^c\, |\mathscr{L}^\gamma h_t(r)| \leq C e^{-Qr} r^{-\min\{k, (1+\gamma)/2 - c\}}. \tag{4.24}$$

As in this case we can pick any $k \in \mathbb{N}$, the above estimate completes the proof of the upper bound (4.5) and of part (ii) in the case where $\gamma \in \mathbb{N}$.

Suppose instead that γ is not an integer. In this case, we cannot integrate by parts in (4.20) arbitrarily many times, as repeated differentiation of the term $(i\lambda)^\gamma$ may produce a non-integrable singularity at 0. Nevertheless, we can integrate by parts at least $\lceil \gamma \rceil$ times (note that $|\lambda|^{\gamma - \lceil \gamma \rceil}$ is locally integrable at 0, as $\gamma - \lceil \gamma \rceil > -1$), and the previous argument can still be run (indeed, the estimate (4.21) remains true even for noninteger

$b > -1$), thus (4.23) and (4.24) are valid for $k = \lceil \gamma \rceil$. As a consequence, we may still deduce the upper bound (4.5) whenever $\lceil \gamma \rceil \geq (\gamma+1)/2 - c$, that is, $c \geq (\gamma+1)/2 - \lceil \gamma \rceil$.

To further extend the range of validity of (4.5) and complete the proof of part (ii), we make use of complex interpolation. Namely, the above argument leading to (4.23) can be applied, *mutatis mutandis*, when the real exponent γ is replaced by the complex exponent $z = \gamma + i\theta$, with $\gamma \in (-1, \infty)$ and $\theta \in \mathbb{R}$. In this case, we can integrate by parts $\lceil \gamma \rceil$ times and, instead of (4.23), we obtain, for any $\delta \in (0,1)$, the estimate

$$|\mathscr{L}^z h_t(r)| \leq C(1+|\theta|)^{\lceil \gamma \rceil} \, \mathrm{e}^{-Qr} \, t^{-(\gamma+1)/2} \Big[\mathrm{e}^{-\delta r} + \big(1 + t^{-1/2} |r - Qt|\big)^{-\lceil \gamma \rceil} \Big] \tag{4.25}$$

for all $t, r \geq 1$, where the constant C does not depend on θ. Consequently, for any $\varepsilon, \delta \in (0,1)$, we deduce the estimate

$$\Big| \mathrm{e}^{z^2} \mathscr{L}^z h_t(r) \Big| \leq C \, \mathrm{e}^{-Qr} \, t^{-(\gamma+1)/2} \Big[\mathrm{e}^{-\delta r} + \big(1 + t^{-1/2} |r - Qt|\big)^{\varepsilon - \gamma - 1} \Big] \tag{4.26}$$

whenever $\gamma > -1$ and $\lceil \gamma \rceil - \gamma \geq 1 - \varepsilon$; again, the constant C may depend on $\gamma, \varepsilon, \delta$ but not on θ. As $\mathscr{L}^z h_t(x)$ is, for fixed x and t, a holomorphic function of z on the half-plane where $\operatorname{Re} z = \gamma > -1$ (cf. (4.6)), an application of Hadamard's three-line theorem allows us to dispense with the constraint $\lceil \gamma \rceil - \gamma \geq 1 - \varepsilon$ and extend the validity of (4.26) to all $\gamma > -1$: more precisely, for all $\gamma > -1$ and $\varepsilon \in (0,1)$, there exists $\delta > 0$ such that (4.26) holds with $z = \gamma + i\theta$ for all $\theta \in \mathbb{R}$ and $r, t \geq 1$.

If we now apply (4.26) with $\theta = 0$ and $\gamma \geq 0$, then Lemma 4.2 (i) yields, for all $\varepsilon \in (0,1)$ and $c \leq (\gamma+1)/2$,

$$\sup_{t \geq 1} t^c \, |\mathscr{L}^\gamma h_t(r)| \leq C e^{-Qr} r^{-\min\{1+\gamma-\varepsilon, (1+\gamma)/2 - c\}}. \tag{4.27}$$

On the other hand, if $c > -(1+\gamma)/2$, then $(1+\gamma)/2 - c < 1 + \gamma$, so we can pick $\varepsilon > 0$ sufficiently small that $1 + \gamma - \varepsilon \geq (1+\gamma)/2 - c$, and (4.27) gives the desired upper bound (4.5) in full generality. This completes the proof of part (ii).

We now prove part (iii). Let us first note that the integrability of $\sup_{t \geq 1} t^c |\mathscr{L}^\gamma h_t|$ for $c < (\gamma - 1)/2$ immediately follows from (4.1) and the pointwise bound (4.5), provided the latter applies. Thus, in light of part (ii), the aforementioned integrability does hold true whenever γ is an integer or $c \in (-(\gamma+1)/2, (\gamma-1)/2)$. On the other hand, when γ is not an integer, necessarily $\gamma > 0$, thus the interval $(-(\gamma+1)/2, (\gamma-1)/2)$ is not empty; as $\sup_{t \geq 1} t^c |\mathscr{L}^\gamma h_t|$ is increasing in c, we conclude that the latter is integrable whenever $c < (\gamma - 1)/2$, as required.

Finally, let us prove part (iv). We first prove upper estimates. Since $\mathscr{L}$ is a sectorial operator on L^1, by the moment inequality [16, Proposition 6.6.4] it is enough to consider the case where $\gamma \in \mathbb{N}$. In this case, the estimate for $t \leq 1$ is known in greater generality (see, e.g., [37, Theorem IX.1.3 (ii)], and note that $\mathscr{L}^\gamma h_t = (-\partial_t)^\gamma h_t$). If instead $t \geq 1$, then we can use the estimates from the proof of part (ii). Namely, (4.16) shows that

$$\int_{B_1(o)} |\mathscr{L}^\gamma h_t(x)| \, \mathrm{d}\mu(x) \le C \, \mathrm{e}^{-tQ^2/4},$$

thus the main contribution comes from integration over $B_1(o)^\complement$. From (4.23) and (4.1), instead, we deduce that, for all $k \in \mathbb{N}$ with $k > 1$,

$$\int_{B_1(o)^\complement} |\mathscr{L}^\gamma h_t(x)| \, \mathrm{d}\mu(x) \le \frac{C}{t^{(1+\gamma)/2}} \Big[1 + \int_1^\infty \Big(1 + t^{-1/2} |r - Qt| \Big)^{-k} \, \mathrm{d}r \Big] \le C t^{-\gamma/2},$$

where we applied Lemma 4.2 (ii) in the last inequality, and the required upper estimate follows.

It remains to prove matching lower estimates. From Lemma 4.1 (iii) it follows that, if f is in $L^1(S)$, then

$$\sup_{\lambda \in \mathbf{T}_1} \big| \widetilde{f}(\lambda) \big| \le \|f\|_{L^1(S)} \,,$$

where $\mathbf{T}_1 := \big\{ \lambda \in \mathbb{C} : \big| \operatorname{Im} \lambda \big| \le Q/2 \big\}$. As

$$\big(\mathscr{L}^\gamma h_t \big)\widetilde{\ }(\lambda) = (\lambda^2 + Q^2/4)^\gamma \, \mathrm{e}^{-t(\lambda^2 + Q^2/4)},$$

we easily deduce that

$$\|\mathscr{L}^\gamma h_t\|_{L^1} \ge \big| \big(\mathscr{L}^\gamma h_t \big)\widetilde{\ }(t^{-1/2} + iQ/2) \big| \asymp t^{-\gamma/2}$$

for $t \ge 1$, and

$$\|\mathscr{L}^\gamma h_t\|_{L^1} \ge \big| \big(\mathscr{L}^\gamma h_t \big)\widetilde{\ }(t^{-1/2}) \big| \asymp t^{-\gamma}$$

for $0 < t \le 1$. This concludes the proof of part (iv) and of the proposition. □

The above estimates imply a number of inclusions and noninclusions between the spaces $\mathfrak{X}^\gamma(S)$ and $H^1_{\mathscr{H},c}(S)$, complementing those in Theorem 3.3.

Theorem 4.5. *Let $\gamma \in (0, \infty)$ and $c \in \mathbb{R}$.*

(i) *The space $\mathfrak{X}^{2c+1}(S)$ is not included in $H^1_{\mathscr{H},c}(S)$ whenever $2c + 1 > 0$.*
(ii) *The space $\mathfrak{X}^\gamma(S)$ is included in $H^1_{\mathscr{H},c}(S)$ whenever $\gamma > 2c + 1$.*
(iii) *$\mathfrak{h}^1(S) = H^1_{\mathscr{H},c}(S)$ for all $c < -1/2$, but $\mathfrak{h}^1(S) \nsubseteq H^1_{\mathscr{H},-1/2}(S)$.*

Proof. Let us prove part (i). Let $\delta = 2c+1 > 0$. Take any radial nonnegative $\psi \in C_c^\infty(S)$, which is not identically zero; in particular $\widetilde{\psi}(iQ/2) = \int_S \psi \, \mathrm{d}\mu > 0$ (see Lemma 4.1 (iv)), and moreover $\mathscr{L}^\delta \psi \in \mathfrak{X}^\delta(S)$ by Proposition 2.7 (iii). On the other hand, by Proposition 4.3 (i), $\mathscr{H}^c_*(\mathscr{L}^\delta \psi)$ is not in $L^1(S)$, whence $\mathscr{L}^\delta \psi \notin H^1_{\mathscr{H},c}(S)$.

Let us now prove part (ii). If $f \in \mathfrak{X}^\gamma(S)$, then, by Proposition 2.7 (ii), $f \in \mathfrak{h}^1(S)$ and $\mathscr{L}^{-\gamma} f \in \mathfrak{h}^1(S)$, with

$$\|f\|_{\mathfrak{h}^1} + \|\mathscr{L}^{-\gamma} f\|_{\mathfrak{h}^1} \le C\|f\|_{\mathfrak{X}^\gamma}.$$

In particular, by Proposition 3.1,

$$\|f\|_{L^1} + \|\mathscr{H}_*^{\mathrm{loc}} f\|_{L^1} \le C\|f\|_{\mathfrak{h}^1} \le C\|f\|_{\mathfrak{X}^\gamma}.$$

On the other hand,

$$\mathscr{H}_t f = (\mathscr{L}^{-\gamma} f) * (\mathscr{L}^\gamma h_t),$$

whence

$$\sup_{t\ge 1} t^c |\mathscr{H}_t f| \le |\mathscr{L}^{-\gamma} f| * \sup_{t\ge 1} t^c |\mathscr{L}^\gamma h_t|.$$

By Proposition 4.3 (iii), the latter supremum is integrable whenever $c < (\gamma - 1)/2$; so, by Young's inequality,

$$\left\| \sup_{t\ge 1} t^c |\mathscr{H}_t f| \right\|_{L^1} \le C\|\mathscr{L}^{-\gamma} f\|_{L^1} \le C\|\mathscr{L}^{-\gamma} f\|_{\mathfrak{h}^1} \le C\|f\|_{\mathfrak{X}^\gamma},$$

as required.

As for part (iii), the inclusion $H^1_{\mathscr{H},c}(S) \subseteq \mathfrak{h}^1(S)$ follows from Theorem 3.3 (i); the remaining inclusion and noninclusion results are proved as in parts (i) and (ii), by taking $\delta = \gamma = 0$. □

Remark 4.6. Theorems 3.3 (i) and 4.5 (ii) imply that, for all $\gamma, \varepsilon > 0$,

$$H^1_{\mathscr{H},\gamma+\varepsilon}(S) \subset \mathfrak{X}^\gamma(S) \subset H^1_{\mathscr{H},(\gamma-1)/2-\varepsilon}(S).$$

It would be interesting to know whether these inclusion ranges are sharp (this is the case for the second inclusion in the series, but we do not know about the first). If this were the case, we would have a substantially different situation compared to the case of the Poisson maximal operator discussed in Corollary 3.5 above (note that the gap between $(\gamma-1)/2$ and γ increases with γ). On the other hand, in the "limit case" $\gamma = 0$ (with $\mathfrak{X}^\gamma(S)$ replaced by $\mathfrak{h}^1(S)$), the first inclusion is certainly not sharp, given that $\mathfrak{h}^1(S) = H^1_{\mathscr{H},-1/2-\varepsilon}(S)$ for all $\varepsilon > 0$.

4.3. Estimates of the Poisson maximal operator on S

The following result is the analogue for the semigroups $\{\mathrm{e}^{-t\mathscr{L}^\alpha} : t \ge 0\}$ of Proposition 4.3 (i): the needed estimates for the Poisson semigroups correspond to the case where $\alpha = 1/2$.

Proposition 4.7. *Let α, c and γ be real numbers such that $0 < \alpha < 1$ and $\gamma \geq 0$. Let ψ be a radial distribution with compact support such that $\widetilde{\psi}(iQ/2) \neq 0$. Then there exists a positive constant C and $\bar{r} \geq 1$ such that*

$$\sup_{t\geq 1} t^c \left|\mathrm{e}^{-t\mathscr{L}^\alpha}\big(\mathscr{L}^\gamma\psi\big)(x)\right| \geq C\,\frac{\mathrm{e}^{-Q|x|}}{|x|^{1+\gamma-c\alpha}} \qquad \forall x \in B_{\bar{r}}(o)^{\complement}. \tag{4.28}$$

In particular, $\sup_{t\geq 1} t^c \left|\mathrm{e}^{-t\mathscr{L}^\alpha}\big(\mathscr{L}^\gamma\psi\big)\right| \notin L^1(S)$ *whenever* $c \geq \gamma/\alpha$.

Proof. We first observe that the nonintegrability of $\sup_{t\geq 1} t^c|\mathscr{L}^\gamma(\mathrm{e}^{-t\mathscr{L}^\alpha}\psi)|$ when $c \geq \gamma/\alpha$ follows from the pointwise estimate (4.28) and the integration formula (4.1). Thus, we are reduced to proving the lower bound (4.28).

The proof of the latter closely follows that of part (i) of Proposition 4.3. Indeed, much as in that proof, we may assume that $\widetilde{\psi}(iQ/2) = 1$ and write, for all $r > 0$,

$$\begin{aligned}\frac{1}{2}\mathscr{L}^\gamma\big(\mathrm{e}^{-t\mathscr{L}^\alpha}\psi\big)(r) &= \int_{\mathbb{R}} \widetilde{\psi}(\lambda)\,(\lambda^2+Q^2/4)^\gamma\,\mathrm{e}^{-t(\lambda^2+Q^2/4)^\alpha}\,\Phi_\lambda(r)\,\mathbf{c}(-\lambda)^{-1}\,\mathrm{d}\lambda\\ &= \int_{\mathbb{R}} \widetilde{\psi}(\lambda+iQ/2)\,(i\lambda)^\gamma p(\lambda)\,\mathrm{e}^{-t(\lambda^2+iQ\lambda)^\alpha}\,\Phi_{\lambda+iQ/2}(r)\,\mathrm{d}\lambda\end{aligned}$$

(the only difference being the exponent α in place of 1); as before, we are using the branches of $w \mapsto w^\alpha$ and $w \mapsto w^\gamma$ on the complex plane slit along the half-line $(-\infty, 0]$ that coincide with the arithmetic α and γ powers on $(0,\infty)$. Still following the proof of Proposition 4.3 (i), we then proceed to split

$$\begin{aligned}\frac{1}{2}e^{Qr}\mathscr{L}^\gamma\big(\mathrm{e}^{-t\mathscr{L}^\alpha}\psi\big)(r) &= I(r,t) + J(r,t),\\ I(r,t) &= I_\eta(r,t) + I_{1-\eta}(r,t),\\ I_\eta(r,t) &= p(0)\,I^0_\eta(r,t) + I^1_\eta(r,t),\end{aligned}$$

where again $p(0) = Q^\gamma\,\mathbf{c}(-iQ/2)^{-1} \neq 0$.

A substantial difference between the case $\alpha \in (0,1)$ considered here and the case $\alpha = 1$ discussed in Proposition 4.3 is that, when $\alpha \in (0,1)$,

$$\mathrm{Re}\left[(\lambda^2+i\lambda Q)^\alpha\right] = |\lambda|^\alpha(\lambda^2+Q^2)^{\alpha/2}\cos(\alpha\arctan(Q/\lambda)) \asymp |\lambda|^\alpha(1+|\lambda|)^\alpha$$

for all $\lambda \in \mathbb{R}$ (the implicit constants may depend on α), and in particular

$$\left|\mathrm{e}^{-t(\lambda^2+iQ\lambda)^\alpha}\right| \leq \mathrm{e}^{-\kappa t|\lambda|^\alpha} \tag{4.29}$$

for some $\kappa > 0$. (In the case $\alpha = 1$, we have $\left|\mathrm{e}^{-t(\lambda^2+iQ\lambda)}\right| = \mathrm{e}^{-t|\lambda|^2}$ instead, with an exponent 2 in place of 1.) Armed with the estimate (4.29), we can follow the arguments in the proof of Proposition 4.3 (i) to obtain that, for all $\delta \in (0,1)$,

$$\begin{aligned} |J(r,t)| &\le C_\delta\, t^{-(1+\gamma)/\alpha} \mathrm{e}^{-\delta r}, \\ |I_{1-\eta}(r,t)| &\le C\, \mathrm{e}^{-\kappa t/2}, \\ |I^1_\eta(r,t)| &\le C\, t^{-(2+\gamma)/\alpha} \end{aligned} \tag{4.30}$$

for all $r, t \ge 1$ (cf. (4.13), (4.14) and (4.15)).

We are then left with estimating the main term

$$I^0_\eta(r,t) = \int_{\mathbb{R}} \eta(\lambda)\, (i\lambda)^\gamma\, \mathrm{e}^{-t(\lambda^2+iQ\lambda)^\alpha}\, \mathrm{e}^{i\lambda r}\, \mathrm{d}\lambda\,,$$

where $\eta \in C^\infty_c(\mathbb{R})$ is even and supported in $[-2,2]$, with $0 \le \eta \le 1$ and $\eta|_{[-1,1]} \equiv 1$. Let $\sigma > 0$ be a parameter to be fixed later, and notice that

$$\begin{aligned} I^0_\eta(r,(r/\sigma)^\alpha) &= \int_{\mathbb{R}} \eta(\lambda)\, (i\lambda)^\gamma\, \mathrm{e}^{-[r(\lambda^2+iQ\lambda)/\sigma]^\alpha}\, \mathrm{e}^{i\lambda r}\, \mathrm{d}\lambda \\ &= (\sigma/r)^{\gamma+1} \int_{\mathbb{R}} \eta(\sigma v/r)\, (iv)^\gamma\, \mathrm{e}^{-(\sigma v^2/r+iQv)^\alpha}\, \mathrm{e}^{i\sigma v}\, \mathrm{d}v. \end{aligned} \tag{4.31}$$

By dominated convergence, and since $\eta(0) = 1$, one can show that, as $r \to +\infty$, the latter integral tends to

$$\int_{\mathbb{R}} (iv)^\gamma\, \mathrm{e}^{-(iQv)^\alpha}\, \mathrm{e}^{i\sigma v}\, \mathrm{d}v, \tag{4.32}$$

that is, the value $\hat{F}(-\sigma)$ of the Fourier transform of $F : v \mapsto (iv)^\gamma\, \mathrm{e}^{-(iQv)^\alpha}$.

We now observe that, due to our choices of the branches of $w \mapsto w^\gamma$ and $w \mapsto w^\alpha$, F admits a holomorphic extension to the lower half-plane $\Omega_- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$; moreover, if $z = v + is$, $s < 0$, then

$$|F(z)| = |iv - s|^\gamma\, \mathrm{e}^{-Q^\alpha \operatorname{Re}[(iv-s)^\alpha]} \le |v|^\gamma\, \mathrm{e}^{-\kappa Q^\alpha |v|^\alpha},$$

and the latter expression is in $L^2(\mathbb{R})$ and independent of s. Consequently, F is in the holomorphic Hardy space $H^2(\Omega_-)$ and, by the Paley–Wiener theorem, $\hat{F}|_{(0,\infty)} \equiv 0$. It follows that there must exist a $\sigma > 0$ such that $\hat{F}(-\sigma) \neq 0$, for otherwise $\hat{F}$ and F would vanish identically.

Thus, with this choice of σ, the integral (4.32) does not vanish, and therefore from (4.31) we deduce that there exists $r_0 \ge \max\{1, \sigma\}$ sufficiently large that

$$|I^0_\eta(r,(r/\sigma)^\alpha)| \ge C\, r^{-(\gamma+1)},$$

for all $r \ge r_0$. On the other hand, the previous estimates (4.30) give

$$|J(r,(r/\sigma)^\alpha)|, |I_{1-\eta}(r,(r/\sigma)^\alpha)|, |I^1_\eta(r,(r/\sigma)^\alpha)| \le C\, r^{-(2+\gamma)}.$$

So we can conclude that, for $r \geq r_0$ sufficiently large,

$$\sup_{t\geq 1} t^c |\mathscr{L}^\gamma(\mathrm{e}^{-t\mathscr{L}^\alpha}\psi)(r)| \geq (r/\sigma)^{\alpha c} |\mathscr{L}^\gamma(\mathrm{e}^{-(r/\sigma)^\alpha \mathscr{L}^\alpha}\psi)(r)| \geq C\mathrm{e}^{-Qr} r^{c\alpha-\gamma-1},$$

as required. □

Thanks to the above estimate, we can now complement the inclusions of Corollary 3.5 with the following result, ruling out that the spaces $\mathfrak{X}^\gamma(S)$ and $H^1_{\mathscr{R}}(S)$ can be characterised in terms of the weighted Poisson maximal function.

Theorem 4.8. *Let $\gamma > 0$ and $c \in \mathbb{R}$.*

(i) *$\mathfrak{X}^\gamma(S)$ is not included in $H^1_{\mathscr{P},2\gamma}(S)$.*
(ii) *$\mathfrak{X}^\gamma(S)$ and $H^1_{\mathscr{P},c}(S)$ are different spaces.*
(iii) *$H^1_{\mathscr{R}}(S)$ and $H^1_{\mathscr{P},c}(S)$ are different spaces.*

Proof. Part (i) is proved in the same way as Theorem 4.5 (i). Indeed, take a nontrivial, compactly supported, nonnegative smooth radial function ψ on S. Then, by Proposition 2.7 (iii), $\mathscr{L}^\gamma\psi \in \mathfrak{X}^\gamma(S)$. On the other hand, $\widetilde{\psi}(iQ/2) = \int_S \psi \,\mathrm{d}\mu \neq 0$. Therefore, by Proposition 4.7, we conclude that $\mathscr{P}_*^{2\gamma}(\mathscr{L}^\gamma\psi) \notin L^1(S)$, that is, $\mathscr{L}^\gamma\psi \notin H^1_{\mathscr{P},2\gamma}(S)$.

We now prove part (ii). By Corollary 3.5, we already know that $\mathfrak{X}^\gamma(S) \neq H^1_{\mathscr{P},c}(S)$ whenever $c \neq 2\gamma$. On the other hand, when $c = 2\gamma$, part (i) rules out that $\mathfrak{X}^\gamma(S)$ can coincide with $H^1_{\mathscr{P},2\gamma}(S)$.

Finally, as $H^1_{\mathscr{R}}(S) = \mathfrak{X}^{1/2}(S)$ by Theorem 2.10, part (iii) follows from part (ii). □

Drawing on the theory developed above, we can finally conclude that the Riesz, Poisson and heat Hardy spaces on a Damek–Ricci space S are pairwise different.

Corollary 4.9. *The following hold.*

(i) *$H^1_{\mathscr{R}}(S)$ is properly contained in $H^1_{\mathscr{P}}(S)$.*
(ii) *$H^1_{\mathscr{H}}(S)$ is properly contained in $H^1_{\mathscr{P}}(S)$.*
(iii) *$H^1_{\mathscr{R}}(S)$ is not contained in $H^1_{\mathscr{H}}(S)$.*

Proof. The proper containment in part (i) is among those proved in Corollary 3.5 for any manifold of class $\mathscr{M}$.

We now prove part (ii). Observe that the containment is the case $c = 0$ of Theorem 3.3 (iv). Now $\mathfrak{X}^1(S) \subset H^1_{\mathscr{P}}(S)$, by Corollary 3.5, and $\mathfrak{X}^1(S) \nsubseteq H^1_{\mathscr{H}}(S)$, by Theorem 4.5 (i). Therefore $H^1_{\mathscr{H}}(S)$ cannot possibly coincide with $H^1_{\mathscr{P}}(S)$. This proves part (ii).

Finally we prove part (iii). We already know that $\mathfrak{X}^1(S) \subset \mathfrak{X}^{1/2}(S) = H^1_{\mathscr{R}}(S)$ by Theorem 2.10, and that $\mathfrak{X}^1(S) \nsubseteq H^1_{\mathscr{H}}(S)$. Thus, $H^1_{\mathscr{R}}(S)$ and $H^1_{\mathscr{H}}(S)$ cannot possibly agree. □

Remark 4.10. Entirely analogous results to those in Theorem 3.3, Corollary 3.5 and Theorem 4.8 hold when the weighted Poisson–Hardy spaces $H^1_{\mathscr{P},c}(M)$ are replaced by the spaces $H^1_{\mathscr{P}^\alpha,c}(M)$ similarly defined in terms of the subordinated semigroup $\{\mathrm{e}^{-t\mathscr{L}^\alpha}\}$ for any $\alpha \in (0,1)$; the case of the Poisson semigroup corresponds to the choice $\alpha = 1/2$. For example, for any manifold M in the class $\mathscr{M}$, one can prove the proper inclusions

$$H^1_{\mathscr{P}^\alpha,\gamma/\alpha+\varepsilon}(M) \subset \mathfrak{X}^\gamma(M) \subset H^1_{\mathscr{P}^\alpha,\gamma/\alpha-\varepsilon}(M)$$

for all $\gamma, \varepsilon > 0$, and also show that $H^1_{\mathscr{P}^\alpha,\gamma/\alpha}(S) \neq \mathfrak{X}^\gamma(S)$ when S is a Damek–Ricci space. Indeed, the analogue of Proposition 3.1 holds for an arbitrary subordinated semigroup (see [23, Corollary 5.4]), while the estimates in Lemma 3.2 and Proposition 4.7 apply to any $\alpha \in (0,1)$. We leave the remaining easy details to the interested reader.

Data availability

No data was used for the research described in the article.

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