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Semiseparable functors ☆

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In this paper we introduce and investigate the notion of semiseparable functor. One of its first features is that it allows a novel description of separable and naturally full functors in terms of faithful and full functors, respectively. To any semiseparable functor we attach an invariant, given by an idempotent natural transformation, which controls when the functor is separable and yields a characterization of separable functors in terms of (dual) Maschke and conservative functors. We prove that any semiseparable functor admits a canonical factorization as a naturally full functor followed by a separable functor. Here the main tool is the construction of the coidentifier category attached to the associated idempotent natural transformation. Then we move our attention to the semiseparability of functors that have an adjoint. First we obtain a Rafael-type Theorem. Next we characterize the semiseparability of adjoint functors in terms of the (co)separability of the associated (co)monads and the natural fullness of the corresponding (co)comparison functor. We also focus on functors that are part of an adjoint triple. In particular, we describe bireflections as semiseparable (co)reflections, or equivalently, as either Frobenius or naturally full (co)reflections. As an application of our results, we study

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the semiseparability of functors traditionally attached to ring homomorphisms, coalgebra maps, corings and bimodules, introducing the notions of semicosplit coring and semiseparability relative to a bimodule which extend those of cosplit coring and Sugano's separability relative to a bimodule, respectively.

Contents

Introduction

The notion of separable ring extension occurs in Algebra, Number Theory and Algebraic Geometry. In [33] C. Năstăsescu et al. reinterpreted this notion at a categorical level by introducing the so-called separable functors. Explicitly, a functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *separable* if the associated natural transformation $\mathcal{F}_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(FX,FY)$, mapping f to Ff, has a left inverse, i.e. there is a natural transformation $\mathcal{P}_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(FX,FY) \to \mathrm{Hom}_{\mathcal{C}}(X,Y)$ such that $\mathcal{P}_{X,Y} \circ \mathcal{F}_{X,Y} = \mathrm{Id}_{\mathrm{Hom}_{\mathcal{C}}(X,Y)}$ for all X and Y in $\mathcal{C}$. A right version of this property yields to *naturally full* functors, as defined in [4]. In this paper, we introduce the notion of *semiseparable* functor, by requiring $\mathcal{F}_{X,Y}$ to be a regular natural transformation - an analogue of von Neumann regular element - i.e., by requiring $\mathcal{F}_{X,Y}$ to admit a natural transformation $\mathcal{P}_{X,Y}$ as above such that $\mathcal{F}_{X,Y} \circ \mathcal{P}_{X,Y} \circ \mathcal{F}_{X,Y} = \mathcal{F}_{X,Y}$. Semiseparability allows to treat separability and natural fullness in a unified way and from a new perspective that reveals further features of them. For instance, it is well-known that a separable functor is faithful and that a naturally full functor is full: In Proposition 1.3, we see how the reverse implications hold by adding the assumption of semiseparability. In Proposition 1.4, to any semiseparable functor $F : \mathcal{C} \to \mathcal{D}$ we attach, in a unique way, a suitable idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$, which is trivial only in case F is separable. Interestingly, as a particular case, in Corollary 1.6 we obtain that every naturally full functor admits such an idempotent natural transformation that is not trivial unless the functor is also separable whence fully faithful. By using the idempotent natural transformation e, in Corollary 1.9, we prove that a functor is separable if and only if it is semiseparable and either Maschke, dual Maschke or conservative. To such an e we can attach a suitable quotient category $\mathcal{C}_e$ of $\mathcal{C}$, the so-called coidentifier [23]. This is the main ingredient to prove the notable property, stated in Theorem 1.15, that any semiseparable functor $F : \mathcal{C} \to \mathcal{D}$ factors as the naturally full quotient functor $H : \mathcal{C} \to \mathcal{C}_e$ followed by a unique separable functor $F_e : \mathcal{C}_e \to \mathcal{D}$. As a consequence, in Corollary 1.16, a functor is shown to be semiseparable if and only if it factors as a naturally full functor followed by a separable functor.

Next we investigate semiseparable functors which have a right (resp. left) adjoint. In this setting a celebrated result for separable functors is the so-called Rafael Theorem [34], which provides a characterization of separability in terms of splitting properties of the (co)unit. It is natural to wonder whether such a result is also available for semiseparable functors, and in fact in Theorem 2.1 we prove that a functor which has a right (resp. left) adjoint is semiseparable if and only if the (co)unit of the adjunction is regular as a natural transformation. Then we study semiseparability in the context of Eilenberg-Moore categories. Our main result here is Theorem 2.9 stating that, given an adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$, the right adjoint G is semiseparable if and only if the monad GF is separable and the comparison functor $K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}$ is naturally full, where $\mathcal{C}_{GF}$ is the Eilenberg-Moore category of modules over GF. A similar result for F is given in Theorem 2.14. As a consequence, we recover similar characterizations for separable, naturally full and fully faithful functors.

Then, we focus on functors that have both a left and a right adjoint. It is well-known that in an adjoint triple $F \dashv G \dashv H$, the functor F is fully faithful if and only if so is H. Proposition 2.19 shows that a similar behaviour holds for semiseparable, separable and naturally full functors. As far as we know, this result is new even at the separable and naturally full cases. Next, as a consequence of Rafael-type Theorem, in Proposition 2.21 we obtain necessary and sufficient conditions for the semiseparability of a Frobenius functor.

We explore semiseparability in connection with functors admitting a fully faithful (left) right adjoint, which are known as *(co)reflections*, cf. [6], and with functors admitting a fully faithful left and right adjoint equal and satisfying a coherence condition relating the unit and counit of the two adjunctions, which are called *bireflections*, cf. [23]. Our main result in this direction is Theorem 2.24 where we prove that a (co)reflection is semiseparable if and only if it is naturally full if and only if it is Frobenius if and only if it is a bireflection. In Proposition 2.27 we see that, given a category $\mathcal{C}$ and an idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$, the quotient functor $H : \mathcal{C} \to \mathcal{C}_e$ is a bireflection if and only if e splits (e.g. $\mathcal{C}$ is idempotent complete). As a consequence, in Corollary 2.28 we show that a factorization of a semiseparable functor as a bireflection followed by a separable functor is available if and only if the associated idempotent natural transformation e splits, and that such a factorization amounts to the canonical one given by the coidentifier category.

Finally, the results we obtained so far are applied to several functors traditionally connected to the study of separability. The first functors we look at are the restriction of scalars functor $\varphi_* : S\text{-Mod} \to R\text{-Mod}$, whose semiseparability falls back to its separability, the extension of scalars functor $\varphi^* = S \otimes_R (-) : R\text{-Mod} \to S\text{-Mod}$ and the coinduction functor $\varphi^! = {}_R\mathrm{Hom}(S, -) : R\text{-Mod} \to S\text{-Mod}$ associated to a ring morphism $\varphi : R \to S$. These functors form an adjoint triple $\varphi^* \dashv \varphi_* \dashv \varphi^!$. In Proposition 3.1, we characterize the semiseparability of φ^*, equivalent to that of $\varphi^!$, in terms of the regularity of φ as a morphism of R-bimodules. Explicitly, the extension of scalars functor φ^* is semiseparable if and only if there exists an R-bimodule map $E : S \to R$ such that $\varphi \circ E \circ \varphi = \varphi$, i.e., such that $\varphi E(1_S) = 1_S$. In a similar fashion, we investigate the semiseparability of the corestriction of coscalars functor $\psi_* : \mathcal{M}^C \to \mathcal{M}^D$ and of the coinduction functor $\psi^* := (-)\square_D C : \mathcal{M}^D \to \mathcal{M}^C$ attached to a coalgebra morphism $\psi : C \to D$, obtaining in Proposition 3.8 that ψ^* is semiseparable if and only if ψ is a regular morphism of D-bicomodules if and only if there is a D-bicomodule morphism $\chi : D \to C$ such that $\varepsilon_C \circ \chi \circ \psi = \varepsilon_C$.

The subsequent functor we investigate is the induction functor $G := (-) \otimes_R \mathcal{C} : \text{Mod-}R \to \mathcal{M}^{\mathcal{C}}$ attached to an R-coring $\mathcal{C}$. Whereas an R-coring $\mathcal{C}$ is sometimes called cosplit in the literature whenever G is a separable functor, we say that $\mathcal{C}$ is *semicosplit* if G is semiseparable. In Theorem 3.10 we prove that $\mathcal{C}$ is semicosplit if and only if the coring counit $\varepsilon_{\mathcal{C}} : \mathcal{C} \to R$ is regular as a morphism of R-bimodules if and only if there is an invariant element $z \in \mathcal{C}^R = \{c \in \mathcal{C} \mid rc = cr,\ \forall r \in R\}$ such that $\varepsilon_{\mathcal{C}}(z)\varepsilon_{\mathcal{C}}(c) = \varepsilon_{\mathcal{C}}(c)$ (i.e., such that $\varepsilon_{\mathcal{C}}(z)c = c$), for every $c \in \mathcal{C}$.

Next we consider the coinduction functor $\sigma_* = \mathrm{Hom}_S(M, -) : \text{Mod-}S \to \text{Mod-}R$ associated to an (R, S)-bimodule M, together with its left adjoint $\sigma^* := (-) \otimes_R M : \text{Mod-}R \to \text{Mod-}S$. As we will prove in Theorem 3.18, the semiseparability of this functor, which results to be equivalent to the fact that the evaluation map $\mathrm{ev}_M : M^* \otimes_R M \to S$ is regular as a morphism of S-bimodules and $M \otimes_S \mathrm{ev}_M$ is surjective, can be also completely described in terms of a property of M that led us to introduce the M-*semiseparability over* R for the ring S, an extension of M-separability investigated by Sugano in [37]. In Corollary 3.20 we provide the following characterization: S is M-separable over R if and only if S is M-semiseparable over R and M is a generator in Mod-S. A different characterization of M-semiseparability of S over R is obtained in Proposition 3.22. This allows us to exhibit in Example 3.23 an instance where S is M-semiseparable but not M-separable over R. As in the separable case, if we add the assumption that M is finitely generated and projective as a right S-module, then the (co)monad associated to the adjunction (σ^*, σ_*) can be described in a easier way.

This allows to achieve further characterizations of the semiseparability of σ_* and σ^* in Proposition 3.26 and Proposition 3.27, respectively. Moreover, in Proposition 3.5, Corollary 3.12, and Proposition 3.24, an explicit factorization, as a bireflection followed by a separable functor, is provided for the above functors φ^*, G, and σ_*, respectively, when they are semiseparable. Finally, in Theorem 3.31 we study the semiseparability of the coinvariant functor $(-)^{\mathrm{co}B} : \mathfrak{M}_B^B \to \mathfrak{M}$ attached to a bialgebra B over a field $\Bbbk$, where $\mathfrak{M}_B^B$ and $\mathfrak{M}$ denote the category of right Hopf modules over B and the category of $\Bbbk$-vector spaces, respectively. Explicitly, $(-)^{\mathrm{co}B}$ is semiseparable if and only if B is a right Hopf algebra with anti-multiplicative and anti-comultiplicative right antipode.

The organization of the paper is the following. In Section 1 we introduce the notion of semiseparable functor and we investigate its interaction with separable and naturally full functors, relative separable functors and composition. We show that any semiseparable functor admits the associated idempotent natural transformation and that it factors as a naturally full functor followed by a separable one. We also see how the existence of a suitable type of generator within its source category implies that a functor is semiseparable if and only if it is separable. Section 2 collects results on semiseparable functors that have an adjoint. Explicitly, we obtain a Rafael-type theorem for semiseparable functors, we study the behaviour of semiseparable adjoint functors in terms of (co)monads and the associated (co)comparison functor, we investigate functors that have both a (possibly equal or fully faithful) left and right adjoint. In Section 3 we test the notion of semiseparability on relevant functors attached to ring homomorphisms, coalgebra maps, corings, bimodules and Hopf modules. This section closes with a discussion on (co)reflections that provides an overview of the notions considered in this work highlighting their mutual interaction.

0.1. Preliminaries and notations

Given an object X in a category $\mathcal{C}$, the identity morphism on X will be denoted either by Id_X or X for short. For categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F : \mathcal{C} \to \mathcal{D}$ just means a covariant functor. By $\mathrm{Id}_{\mathcal{C}}$ we denote the identity functor on $\mathcal{C}$. For any functor $F : \mathcal{C} \to \mathcal{D}$, we denote $\mathrm{Id}_F : F \to F$ (or just F, if no confusion may arise) the natural transformation defined by $(\mathrm{Id}_F)_X := \mathrm{Id}_{FX}$.

Let $\mathcal{C}$ be a category. Denote by $\mathcal{C}^{\mathrm{op}}$ the opposite category of $\mathcal{C}$. An object X and a morphism $f : X \to Y$ in $\mathcal{C}$ will be denoted by X^{op} and $f^{\mathrm{op}} : Y^{\mathrm{op}} \to X^{\mathrm{op}}$ respectively when regarded as an object and a morphism in $\mathcal{C}^{\mathrm{op}}$. Given a functor $F : \mathcal{C} \to \mathcal{D}$, one defines its opposite functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ by setting $F^{\mathrm{op}}X^{\mathrm{op}} = (FX)^{\mathrm{op}}$ and $F^{\mathrm{op}}f^{\mathrm{op}} = (Ff)^{\mathrm{op}}$.

A morphism (natural transformation) f is called *regular*[1] provided there is a morphism (resp. natural transformation) g with $f \circ g \circ f = f$. Note that the asymmetry of this

[1] This terminology is derived from von Neumann regularity of rings. See e.g. [28, page 21].

definition is apparent as g could be replaced by $g' = g \circ f \circ g$ in such a way that $f \circ g' \circ f = f$ and $g' \circ f \circ g' = g'$.

By a ring we mean a unital associative ring.

1. The notion of semiseparable functor

In this section we introduce and investigate the notion of semiseparable functor. Subsection 1.1 presents its definition and characterizes the known notions of separable and naturally full functors in terms of it. In Subsection 1.2 we attach an invariant to any semiseparable functor, that we call the associated idempotent natural transformation, which controls the separability of the functor and allows a characterization of separable functors in terms of (dual) Maschke and conservative functors. In Subsection 1.3 the connection between semiseparable and relative separable functors is explored. Subsection 1.4 concerns the behaviour of semiseparable functors with respect to composition. Subsection 1.5 shows how semiseparable functors admit a canonical factorization as a naturally full functor followed by a separable one. Here the main tool is the construction of the coidentifier category attached to the associated idempotent natural transformation. In Subsection 1.6 we investigate under which conditions the existence of a suitable type of generator within its source category implies that a functor is semiseparable if and only if it is separable.

1.1. Semiseparable functors

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and consider the associated natural transformation

$$\mathcal{F}^F : \mathrm{Hom}_{\mathcal{C}}(-,-) \to \mathrm{Hom}_{\mathcal{D}}(F-, F-),$$

defined by setting $\mathcal{F}^F_{C,C'}(f) = F(f)$, for any $f : C \to C'$ in $\mathcal{C}$. We recall that F is said to be *separable* [33] if there is a natural transformation $\mathcal{P}^F : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$ such that $\mathcal{P}^F \circ \mathcal{F}^F = \mathrm{Id}$. Similarly, a functor $F : \mathcal{C} \to \mathcal{D}$ is called *naturally full* [4] if there exists a natural transformation $\mathcal{P}^F : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$ such that $\mathcal{F}^F \circ \mathcal{P}^F = \mathrm{Id}$.

Clearly, a functor is fully faithful if and only if it is both separable and naturally full.

Definition 1.1. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is **semiseparable** if the natural transformation $\mathcal{F}^F$ is regular, i.e. if there exists a natural transformation $\mathcal{P}^F : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$ such that $\mathcal{F}^F \circ \mathcal{P}^F \circ \mathcal{F}^F = \mathcal{F}^F$.

Remark 1.2. Since $\mathcal{F}^F_{X,Y} = \mathcal{F}^{F^{\mathrm{op}}}_{Y^{\mathrm{op}},X^{\mathrm{op}}}$ it is clear that a functor $F : \mathcal{C} \to \mathcal{D}$ is semiseparable (resp. separable, naturally full, full, faithful, fully faithful) if and only if so is $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$.

When the functor F is obvious from the context, we will simply write $\mathcal{F}, \mathcal{P}$ instead of $\mathcal{F}^F, \mathcal{P}^F$.

It is well-known that a separable functor is faithful and that a naturally full functor is full. Let us see how adding the notion of semiseparable functor to the picture allows us to turn these implications into equivalences.

Proposition 1.3. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then,*

(i) *F is separable if and only if F is semiseparable and faithful;*
(ii) *F is naturally full if and only if F is semiseparable and full.*

Proof. We only prove (i), the proof of (ii) being similar. Assume that F is separable. From $\mathcal{P} \circ \mathcal{F} = \mathrm{Id}$ it follows that $\mathcal{F} \circ \mathcal{P} \circ \mathcal{F} = \mathcal{F} \circ \mathrm{Id} = \mathcal{F}$, i.e. F is semiseparable, and that for all $C, C' \in \mathcal{C}$, the map $\mathcal{F}_{C,C'}$ is injective, i.e. F is faithful. Conversely, if F is semiseparable, we have that there exists a natural transformation $\mathcal{P}$ such that $\mathcal{F} \circ \mathcal{P} \circ \mathcal{F} = \mathcal{F}$, hence, if F is faithful, $\mathcal{P} \circ \mathcal{F} = \mathrm{Id}$, as $\mathcal{F}$ is injective on components. □

In view of Proposition 1.3, both separable and naturally full functors are instances of semiseparable functors. Next aim is to endow any semiseparable functor with an invariant that will play a central role in our treatment.

1.2. The associated idempotent

Here we attach, in a canonical way, a suitable idempotent natural transformation to any semiseparable functor.

Proposition 1.4. *Let $F : \mathcal{C} \to \mathcal{D}$ be a semiseparable functor. Then there is a unique idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ such that $Fe = \mathrm{Id}_F$ with the following universal property: if $f, g : A \to B$ are morphisms, then $Ff = Fg$ if and only if $e_B \circ f = e_B \circ g$.*

Proof. Since F is semiseparable, there is a natural transformation $\mathcal{P}$ such that $\mathcal{F} \circ \mathcal{P} \circ \mathcal{F} = \mathcal{F}$. Set $e_X := \mathcal{P}_{X,X}(\mathrm{Id}_{FX})$. Note that $Fe_X = F\mathcal{P}_{X,X}(\mathrm{Id}_{FX}) = \mathcal{F}_{X,X}\mathcal{P}_{X,X}\mathcal{F}_{X,X}(\mathrm{Id}_X) = \mathcal{F}_{X,X}(\mathrm{Id}_X) = \mathrm{Id}_{FX}$. Thus $e_X \circ e_X = \mathcal{P}_{X,X}(\mathrm{Id}_{FX}) \circ e_X = \mathcal{P}_{X,X}(\mathrm{Id}_{FX} \circ Fe_X) = \mathcal{P}_{X,X}(\mathrm{Id}_{FX}) = e_X$ and hence e_X is idempotent. Moreover, for every morphism $f : X \to Y$ we have $f \circ e_X = f \circ \mathcal{P}_{X,X}(\mathrm{Id}_{FX}) = \mathcal{P}_{X,Y}(Ff \circ \mathrm{Id}_{FX}) = \mathcal{P}_{X,Y}(\mathrm{Id}_{FY} \circ Ff) = \mathcal{P}_{Y,Y}(\mathrm{Id}_{FY}) \circ f = e_Y \circ f$ so that $f \circ e_X = e_Y \circ f$, i.e. $e = (e_X)_{X \in \mathcal{C}} : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ is an idempotent natural transformation such that $Fe = \mathrm{Id}_F$. Now, consider morphisms $f, g : A \to B$. If $Ff = Fg$, then $\mathcal{P}_{A,B}(Ff) = \mathcal{P}_{A,B}(Fg)$ i.e. $\mathcal{P}_{B,B}(\mathrm{Id}_{FB}) \circ f = \mathcal{P}_{B,B}(\mathrm{Id}_{FB}) \circ g$, i.e. $e_B \circ f = e_B \circ g$. Conversely, from $e_B \circ f = e_B \circ g$ we get $Fe_B \circ Ff = Fe_B \circ Fg$ and hence $Ff = Fg$ as $Fe = \mathrm{Id}_F$. Finally, let $e' : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ be an idempotent natural transformation such that, if $f, g : A \to B$ are morphisms, then $Ff = Fg$ if and only if

$e'_B \circ f = e'_B \circ g$. From $e'_X \circ e'_X = e'_X \circ \mathrm{Id}_X$ we get $Fe'_X = F\mathrm{Id}_X$ (whence $Fe' = \mathrm{Id}_F$). From the universal property of e we get $e_X \circ e'_X = e_X \circ \mathrm{Id}_X$ i.e. $e_X \circ e'_X = e_X$. If we interchange the roles of e and e', in a similar way we get $e'_X \circ e_X = e'_X$. By naturality we have $e_X \circ e'_X = e'_X \circ e_X$ whence $e_X = e'_X$, i.e. $e = e'$. □

Definition 1.5. The idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ we have attached to a semiseparable functor $F : \mathcal{C} \to \mathcal{D}$ in Proposition 1.4 will be called the **associated idempotent natural transformation**. Thus e is defined on components by $e_X := \mathcal{P}_{X,X}(\mathrm{Id}_{FX})$ where $\mathcal{P}$ is any natural transformation such that $\mathcal{F} \circ \mathcal{P} \circ \mathcal{F} = \mathcal{F}$.

Since naturally full functors are in particular semiseparable, we get the following result which was unknown before to the best of our knowledge.

Corollary 1.6. *Any naturally full functor admits the associated idempotent natural transformation.*

We now see how the idempotent natural transformation associated to a semiseparable functor controls its separability.

Corollary 1.7. *Let $F : \mathcal{C} \to \mathcal{D}$ be a semiseparable functor and let $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ be the associated idempotent natural transformation. Then, F is separable if and only if $e = \mathrm{Id}$.*

Proof. By construction $e_X = \mathcal{P}_{X,X}(\mathrm{Id}_{FX})$ where $\mathcal{P}$ is a natural transformation such that $\mathcal{F} \circ \mathcal{P} \circ \mathcal{F} = \mathcal{F}$. If F is separable then $\mathcal{P} \circ \mathcal{F} = \mathrm{Id}$ and hence $e_X = \mathcal{P}_{X,X}(\mathrm{Id}_{FX}) = \mathcal{P}_{X,X}\mathcal{F}_{X,X}(\mathrm{Id}_X) = \mathrm{Id}_X$. Conversely, suppose $e = \mathrm{Id}$. Then, for every $f : X \to Y$, we have $\mathcal{P}_{X,Y}(Ff) = \mathcal{P}_{X,Y}(Ff \circ \mathrm{Id}_{FX}) = f \circ \mathcal{P}_{X,X}(\mathrm{Id}_{FX}) = f \circ e_X = f$ so that $\mathcal{P} \circ \mathcal{F} = \mathrm{Id}$ and F is separable. □

The existence of the associated idempotent natural transformation leads us to a further characterization of separable functors in terms of Maschke, dual Maschke and conservative functors. Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is called a *Maschke functor* if it reflects split-monomorphisms, i.e. for every morphism i in $\mathcal{C}$ such that Fi is split-mono, then i is split-mono.[2] Similarly, F is a *dual Maschke functor* if it reflects split-epimorphisms. A functor is called *conservative* if it reflects isomorphisms.

Remark 1.8. By [33, Proposition 1.2] a separable functor is both Maschke and dual Maschke. Moreover a functor which is both Maschke and dual Maschke is conservative.

Corollary 1.9. *The following assertions are equivalent for a functor $F : \mathcal{C} \to \mathcal{D}$.*

[2] This is equivalent to [15, Remark 6], where F is called a Maschke functor if every object in $\mathcal{C}$ is relative injective. Recall that an object M is called relative injective if, for every morphism $i : C \to C'$ such that Fi is split-mono, then the map $\mathrm{Hom}_{\mathcal{C}}(i, M) : \mathrm{Hom}_{\mathcal{C}}(C', M) \to \mathrm{Hom}_{\mathcal{C}}(C, M)$, $f \mapsto f \circ i$, is surjective.

(1) *F is separable;*
(2) *F is semiseparable and Maschke;*
(3) *F is semiseparable and dual Maschke;*
(4) *F is semiseparable and conservative.*

Proof. $(1) \Rightarrow (2),(3),(4)$. By Proposition 1.3 (i), a separable functor is semiseparable. Moreover, by Remark 1.8, a separable functor is both Maschke and dual Maschke whence conservative.

$(2),(3),(4) \Rightarrow (1)$. Since F is semiseparable, we can consider its associated idempotent natural transformation e such that $Fe_X = \mathrm{Id}_{FX}$, for every object X in $\mathcal{C}$. Thus Fe_X is split-mono, split-epi and iso. Depending on whether F is either Maschke, dual Maschke or conservative, we get that e_X is either split-mono, split-epi or iso. Since e_X is idempotent, we get $e_X = \mathrm{Id}_X$ so that F is separable by Corollary 1.7. $\square$

1.3. Relative separability

Let $F : \mathcal{C} \to \mathcal{D}$ and $H : \mathcal{C} \to \mathcal{E}$ be functors. We recall from [15, Definition 4, page 97] that F is called H-*separable* if there exists a natural transformation

$$\mathcal{P}^{F,H} : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{E}}(H-, H-)$$

such that $\mathcal{P}^{F,H} \circ \mathcal{F}^F = \mathcal{F}^H$. In particular, a $\mathrm{Id}_{\mathcal{C}}$-separable functor coincides with a separable functor. The following result represents a connection between semiseparable functors and H-separable ones, and it will be used in Lemma 1.13 to study what happens if $G \circ F$ is semiseparable and G is faithful.

Proposition 1.10. *Let $H : \mathcal{C} \to \mathcal{E}$ be a semiseparable functor with associated idempotent natural transformation e and let $F : \mathcal{C} \to \mathcal{D}$ be a H-separable functor. If $Fe = \mathrm{Id}_F$ (e.g. $\mathcal{P}^{F,H}$ is injective on components), then F is semiseparable.*

Proof. By definition $\mathcal{P}^{F,H} \circ \mathcal{F}^F = \mathcal{F}^H$. Since H is semiseparable, there exists a natural transformation $\mathcal{P}^H : \mathrm{Hom}_{\mathcal{E}}(H-, H-) \to \mathrm{Hom}_{\mathcal{C}}(-, -)$ such that $\mathcal{F}^H \circ \mathcal{P}^H \circ \mathcal{F}^H = \mathcal{F}^H$. Set $\mathcal{P}^F := \mathcal{P}^H \circ \mathcal{P}^{F,H}$, for every X, Y in $\mathcal{C}$. Then $\mathcal{P}^F \circ \mathcal{F}^F = \mathcal{P}^H \circ \mathcal{P}^{F,H} \circ \mathcal{F}^F = \mathcal{P}^H \circ \mathcal{F}^H$. Thus, for every $f : FX \to FY$, we have $\mathcal{P}^F_{X,Y}\mathcal{F}^F_{X,Y}(f) = \mathcal{P}^H_{X,Y}\mathcal{F}^H_{X,Y}(f) = \mathcal{P}^H_{X,Y}(Hf) = f \circ \mathcal{P}^H_{X,X}(\mathrm{Id}_{HX}) = f \circ e_X$ and hence $\mathcal{F}^F_{X,Y}\mathcal{P}^F_{X,Y}\mathcal{F}^F_{X,Y}(f) = F(f \circ e_X) = Ff \circ Fe_X = Ff = \mathcal{F}^F_{X,Y}(f)$ so that $\mathcal{F}^F_{X,Y}\mathcal{P}^F_{X,Y}\mathcal{F}^F_{X,Y} = \mathcal{F}^F_{X,Y}$ i.e. F is semiseparable. If $\mathcal{P}^{F,H}$ is injective on components, then from $\mathcal{P}^{F,H}_{X,X}(Fe_X) = \mathcal{P}^{F,H}_{X,X}\mathcal{F}^F_{X,X}(e_X) = \mathcal{F}^H_{X,X}(e_X) = He_X = H\mathrm{Id}_X = \mathcal{F}^H_{X,X}(\mathrm{Id}_X) = \mathcal{P}^{F,H}_{X,X}(F\mathrm{Id}_X)$ we infer $Fe_X = \mathrm{Id}_{FX}$. $\square$

Corollary 1.11. *Let $H : \mathcal{C} \to \mathcal{E}$ be a semiseparable functor with associated idempotent natural transformation e and assume H is a retract of a functor $F : \mathcal{C} \to \mathcal{D}$. If $Fe = \mathrm{Id}_F$ then F is semiseparable. As a consequence, semiseparable functors are closed under isomorphisms.*

Proof. Since H is a retract of F, there are natural transformations $\varphi : F \to H$ and $\psi : H \to F$ such that $\varphi \circ \psi = \mathrm{Id}_H$. Define $\mathcal{P}^{F,H}$ by setting $\mathcal{P}^{F,H}_{X,Y}(g) := \varphi_Y \circ g \circ \psi_X$, for every $g : FX \to FY$, and note that $\mathcal{P}^{F,H} \circ \mathcal{F}^F = \mathcal{F}^H$ so that $F : \mathcal{C} \to \mathcal{D}$ is H-separable. Thus, by Proposition 1.10, if $Fe = \mathrm{Id}_F$, the functor F is semiseparable. Let us prove the last part of the statement. Let $\varphi : F \to H$ be an isomorphism of the functors $F, H : \mathcal{C} \to \mathcal{D}$, where H is semiseparable with associated idempotent natural transformation e. Clearly H is a retract of F via $\psi := \varphi^{-1}$. Thus F is semiseparable, as $Fe = Fe \circ \psi \circ \varphi = \psi \circ He \circ \varphi = \psi \circ \mathrm{Id}_H \circ \varphi = \mathrm{Id}_F$. □

1.4. Behaviour with respect to composition

It is known that if $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ are separable functors so is their composition $G \circ F$ and, the other way around, if the composition $G \circ F$ is separable so is F, see [33, Lemma 1.1]. A similar result with some difference, holds for naturally full functors, see [4, Proposition 2.3]. Here we study the behaviour of semiseparable functors with respect to composition. The first difference, with respect to the separable and naturally full cases, is that semiseparable functors are not closed under composition as we will see later in Example 3.3. However the closeness is available in some cases, as the following result shows.

Lemma 1.12. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors and consider the composite $G \circ F : \mathcal{C} \to \mathcal{E}$.*

(i) *If F is semiseparable and G is separable, then $G \circ F$ is semiseparable.*
(ii) *If F is naturally full and G is semiseparable, then $G \circ F$ is semiseparable.*

Proof. If F is semiseparable with respect to $\mathcal{P}^F$ and G is separable with respect to $\mathcal{P}^G$, then for every X, Y in $\mathcal{C}$ we have

$$\mathcal{F}^{GF}_{X,Y}\mathcal{P}^F_{X,Y}\mathcal{P}^G_{FX,FY}\mathcal{F}^{GF}_{X,Y} = \mathcal{F}^{GF}_{X,Y}\mathcal{P}^F_{X,Y}\mathcal{P}^G_{FX,FY}\mathcal{F}^G_{FX,FY}\mathcal{F}^F_{X,Y}$$
$$= \mathcal{F}^{GF}_{X,Y}\mathcal{P}^F_{X,Y}\mathcal{F}^F_{X,Y} = \mathcal{F}^G_{FX,FY}\mathcal{F}^F_{X,Y}\mathcal{P}^F_{X,Y}\mathcal{F}^F_{X,Y} = \mathcal{F}^G_{FX,FY}\mathcal{F}^F_{X,Y} = \mathcal{F}^{GF}_{X,Y},$$

hence $G \circ F$ is semiseparable through $\mathcal{P}^{GF}_{X,Y} := \mathcal{P}^F_{X,Y}\mathcal{P}^G_{FX,FY}$. The proof of (ii) is similar. □

We now provide a variant of the property that if $G \circ F$ is separable so is F.

Lemma 1.13. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors. If $G \circ F$ is semiseparable and G is faithful, then F is semiseparable.*

Proof. It follows from Proposition 1.10, by setting for every X, Y in $\mathcal{C}$, $\mathcal{P}^{F,GF}_{X,Y} := \mathcal{F}^G_{FX,FY}$, which is injective, as G is faithful. □

Afterwards, in Proposition 2.23 we will see how, under stronger assumptions on F, the functor G comes out to be semiseparable whenever $G \circ F$ is.

In Theorem 1.15, we will give a criterion to factorize any semiseparable functor as the composition of a naturally full functor followed by a separable functor. The main ingredient will be the coidentifier category which is the object of the following subsection.

1.5. The coidentifier

Given a category $\mathcal{C}$ and an idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$, consider the coidentifier $\mathcal{C}_e$ defined as in [23, Example 17]. This is the quotient category $\mathcal{C}/\sim$ of $\mathcal{C}$ where $\sim$ is the congruence relation on the hom-sets defined, for all $f, g : A \to B$, by setting $f \sim g$ if and only if $e_B \circ f = e_B \circ g$. Thus, $\mathrm{Ob}\,(\mathcal{C}_e) = \mathrm{Ob}\,(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}_e}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)/\sim$. We denote by $\overline{f}$ the class of $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ in $\mathrm{Hom}_{\mathcal{C}_e}(A, B)$. We have the quotient functor $H : \mathcal{C} \to \mathcal{C}_e$ acting as the identity on objects and as the canonical projection on morphisms. Note that H is naturally full with respect to $\mathcal{P}_{A,B} : \mathrm{Hom}_{\mathcal{C}_e}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, B)$ defined by $\mathcal{P}_{A,B}(\overline{f}) = e_B \circ f$ and that the idempotent natural transformation associated to H is exactly e.

Lemma 1.14. *Let $\mathcal{C}$ be a category, let $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ be an idempotent natural transformation and let $H : \mathcal{C} \to \mathcal{C}_e$ be the quotient functor.*

(1) *A functor $F : \mathcal{C} \to \mathcal{D}$ satisfies $Fe = \mathrm{Id}_F$ if and only if there is a functor $F_e : \mathcal{C}_e \to \mathcal{D}$ (necessarily unique) such that $F = F_e \circ H$. Given $F, F' : \mathcal{C} \to \mathcal{D}$ such that $Fe = \mathrm{Id}_F$ and $F'e = \mathrm{Id}_{F'}$, and a natural transformation $\beta : F \to F'$, there is a unique natural transformation $\beta_e : F_e \to F'_e$ such that $\beta = \beta_e H$.*
(2) *The functor $H : \mathcal{C} \to \mathcal{C}_e$ is orthogonal to any faithful functor $S : \mathcal{D} \to \mathcal{E}$ i.e., given functors F and G such that $S \circ F = G \circ H$, then there is a unique functor $F_e : \mathcal{C}_e \to \mathcal{D}$ such that $F_e \circ H = F$ and $S \circ F_e = G$.*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "H"] \arrow[d, "F"'] & \mathcal{C}_e \arrow[dl, dotted, "F_e"] \\ \mathcal{D} & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "H"] \arrow[d, "F"'] & \mathcal{C}_e \arrow[d, "G"] \arrow[dl, dotted, "F_e"'] \\ \mathcal{D} \arrow[r, "S"] & \mathcal{E} \end{tikzcd}$$

Proof. First note that $He = \mathrm{Id}_H$ as e is the idempotent natural transformation associated to H.

(1). This property is the universal property of the coidentifier that can be deduced from the dual version of [23, Definition 14(1)]. We just point out that the functor $F_e : \mathcal{C}_e \to \mathcal{D}$ acts as F on objects and maps the class $\overline{f}$ into Ff and that, for every object X in $\mathcal{C}$, we have $(\beta_e)_X = \beta_X$.

(2). We compute $SFe_X = GHe_X = G\mathrm{Id}_{HX} = \mathrm{Id}_{GHX} = \mathrm{Id}_{SFX} = S\mathrm{Id}_{FX}$ so that, since S is faithful, we get that $Fe_X = \mathrm{Id}_{FX}$ and hence $Fe = \mathrm{Id}_F$. Thus, by (1) there is a unique functor $F_e : \mathcal{C}_e \to \mathcal{D}$, such that $F_e \circ H = F$, which acts as F on objects and maps the class $\overline{f}$ into Ff. Moreover $SF_eX = SF_eHX = SFX = GHX = GX$ and $SF_e\overline{f} = SF_eHf = SFf = GHf = G\overline{f}$ so that $S \circ F_e = G$. □

Another way to prove that $H : \mathcal{C} \to \mathcal{C}_e$ is orthogonal to any faithful functor $S : \mathcal{D} \to \mathcal{E}$ is to observe it is eso (essentially surjective on objects, i.e. for every $D \in \mathcal{C}_e$ there is $C \in \mathcal{C}$ such that $D \cong H(C)$) and full and that there is an (eso and full, faithful) factorization system, see e.g. [20, Example 7.9]. In the following result, we show that any semiseparable functor admits a special type of (eso and full, faithful) factorization. In fact $F = F_e \circ H$, where H is eso and (naturally) full while F_e is separable whence faithful.

Theorem 1.15. *Let $F : \mathcal{C} \to \mathcal{D}$ be a semiseparable functor and let $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ be the associated idempotent natural transformation. Then, there is a unique functor $F_e : \mathcal{C}_e \to \mathcal{D}$ (necessarily separable) such that $F = F_e \circ H$ where $H : \mathcal{C} \to \mathcal{C}_e$ is the quotient functor. Furthermore, if F also factors as $S \circ N$ where $S : \mathcal{E} \to \mathcal{D}$ is a separable functor and $N : \mathcal{C} \to \mathcal{E}$ is a naturally full functor, then there is a unique functor $N_e : \mathcal{C}_e \to \mathcal{E}$ (necessarily fully faithful) such that $N_e \circ H = N$ and $S \circ N_e = F_e$, and e is also the idempotent natural transformation associated to N.*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "H"] \arrow[d, "N"'] & \mathcal{C}_e \arrow[d, "F_e"] \arrow[dl, dotted, "N_e"'] \\ \mathcal{E} \arrow[r, "S"] & \mathcal{D} \end{tikzcd}$$

Proof. By Lemma 1.14, there is a unique functor $F_e : \mathcal{C}_e \to \mathcal{D}$ such that $F = F_e \circ H$ where $H : \mathcal{C} \to \mathcal{C}_e$ is the quotient functor. If $F_e\overline{f} = F_e\overline{g}$, then $Ff = Fg$ so that, by Proposition 1.4, we get $e_B \circ f = e_B \circ g$ which means $\overline{f} = \overline{g}$. Thus F_e is faithful. Moreover $\mathcal{F}^F_{X,Y} \circ \mathcal{P}^F_{X,Y} \circ \mathcal{F}^F_{X,Y} = \mathcal{F}^F_{X,Y}$ rewrites as $\mathcal{F}^{F_e}_{X,Y} \circ \mathcal{F}^H_{X,Y} \circ \mathcal{P}^F_{X,Y} \circ \mathcal{F}^{F_e}_{X,Y} \circ \mathcal{F}^H_{X,Y} = \mathcal{F}^{F_e}_{X,Y} \circ \mathcal{F}^H_{X,Y}$. Since $\mathcal{F}^{F_e}_{X,Y}$ is injective and $\mathcal{F}^H_{X,Y}$ is surjective, we get $\mathcal{F}^H_{X,Y} \circ \mathcal{P}^F_{X,Y} \circ \mathcal{F}^{F_e}_{X,Y} = \mathrm{Id}$ which implies that F_e is separable (and also that H is naturally full, fact that we already know).

Concerning the last sentence, since S is separable, then it is faithful. By Lemma 1.14 H is orthogonal to S so that there is a unique functor $N_e : \mathcal{C}_e \to \mathcal{E}$ such that $N_e \circ H = N$ and $S \circ N_e = F_e$. Since $N_e \circ H = N$ and N is full, we deduce that N_e is full (this is not true in general, but here H acts as the identity on objects) and since $S \circ N_e = F_e$ and F_e is faithful, we deduce that N_e is faithful. Thus N_e is fully faithful.

It remains to prove that F and N share the same associated idempotent natural transformation[3] $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$. Indeed, by Corollary 1.6, N has an associated idempotent natural transformation $e' : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ and by definition we have $e'_X := \mathcal{P}^N_{X,X}(\mathrm{Id}_{NX})$, for any $X \in \mathcal{C}$. Since $F = S \circ N$, by the proof of Lemma 1.12 (i), we can choose

[3] thus the functor $N_e : \mathcal{C}_e \to \mathcal{E}$ such that $N_e \circ H = N$ and $S \circ N_e = F_e$, is exactly the separable functor achieved from the first part of this theorem applied to the naturally full functor N.

$\mathcal{P}^F_{X,Y} := \mathcal{P}^N_{X,Y} \circ \mathcal{P}^S_{NX,NY}$, so that $e_X = \mathcal{P}^F_{X,X}(\mathrm{Id}_{FX}) = \mathcal{P}^N_{X,X}(\mathcal{P}^S_{NX,NX}(\mathrm{Id}_{SNX})) = \mathcal{P}^N_{X,X}(\mathrm{Id}_{NX}) = e'_X$ whence $e = e'$. $\square$

We are now ready to prove the desired characterization of semiseparable functors in terms of separable and naturally full functors.

Corollary 1.16. *A functor is semiseparable if and only if it factors as $S \circ N$ where S is a separable functor and N is a naturally full functor.*

Proof. If a functor is semiseparable, it factors as a naturally full functor followed by a separable one by Theorem 1.15. Conversely, by Lemma 1.12 (i), the composition $S \circ N$, of a separable functor S by a naturally full (whence semiseparable) functor N, is semiseparable. $\square$

1.6. Generators

We want to investigate how the existence of a suitable type of generator within a category $\mathcal{C}$ could imply that a functor $F : \mathcal{C} \to \mathcal{D}$ is semiseparable if and only if it is separable.

Recall, from [26, Definition 7], that a morphism $k : X \to Y$ in a category $\mathcal{C}$ is called *constant* provided that for each object Z in $\mathcal{C}$ and for each pair of morphisms $g, h : Z \to X$, it follows $k \circ g = k \circ h$. A category $\mathcal{C}$ is said to be *constant-generated* provided that, for any pair of morphisms $f, g : X \to Y$ in $\mathcal{C}$ such that $f \neq g$, then there exist an object $\mathfrak{G}$ and a constant morphism $k : \mathfrak{G} \to X$ such that $f \circ k \neq g \circ k$. We point out that the definition of constant-generated category we are giving here differs from the original one of [26, Definition 8] in the fact that we do not require that $\mathrm{Hom}_{\mathcal{C}}(X, Y) \neq \emptyset$, condition which is superfluous for our purposes.

Proposition 1.17. *If $\mathcal{C}$ is a constant-generated category, then* $\mathrm{Nat}(\mathrm{Id}_{\mathcal{C}}, \mathrm{Id}_{\mathcal{C}}) = \{\mathrm{Id}\}$. *As a consequence, a functor $F : \mathcal{C} \to \mathcal{D}$ is semiseparable if and only if it is separable.*

Proof. Let $e \in \mathrm{Nat}(\mathrm{Id}_{\mathcal{C}}, \mathrm{Id}_{\mathcal{C}})$ and suppose that $e_X \neq \mathrm{Id}_X$, for some object X in $\mathcal{C}$. Since $\mathcal{C}$ is constant-generated, there are an object $\mathfrak{G}$ and a constant morphism $k : \mathfrak{G} \to X$ such that $e_X \circ k \neq \mathrm{Id}_X \circ k$. By naturality of e and since k is constant, we have $e_X \circ k = k \circ e_{\mathfrak{G}} = k \circ \mathrm{Id}_{\mathfrak{G}} = \mathrm{Id}_X \circ k$, a contradiction. Therefore $e_X = \mathrm{Id}_X$ and hence $e = \mathrm{Id}$. We conclude by Corollary 1.7. $\square$

Recall that an object $\mathfrak{G}$ in a category $\mathcal{C}$ is called a *generator* if, for every pair of morphisms $f, g : X \to Y$ in $\mathcal{C}$ such that $f \neq g$, there is a morphism $p : \mathfrak{G} \to X$ such that $f \circ p \neq g \circ p$. If the domain of a functor F is a category with a generator, instead of a constant-generated category, it is not obvious that F is semiseparable if and only if it is separable. However we are able to retrieve the same conclusion by adding suitable assumptions.

A first example in this direction is given by taking a *well-pointed* category, i.e. a category that has a generator which is at the same time a terminal object.

Corollary 1.18. *If $\mathcal{C}$ is a well-pointed category, then it is constant-generated. As a consequence, a functor $F : \mathcal{C} \to \mathcal{D}$ is semiseparable if and only if it is separable.*

Proof. Let $\mathfrak{G}$ be a generator which is a terminal object. Given morphisms $f, g : X \to Y$ in $\mathcal{C}$ such that $f \neq g$, since $\mathfrak{G}$ is a generator there is a morphism $k : \mathfrak{G} \to X$ such that $f \circ k \neq g \circ k$. On the other hand, since $\mathfrak{G}$ is a terminal object, then k is constant. We conclude by Proposition 1.17. $\square$

Example 1.19. Corollary 1.18 applies in case $\mathcal{C}$ is either the category Set of sets or the category Top of topological spaces or the category Comp of compact Hausdorff spaces which are well-pointed. In fact the singleton $\{*\}$ is both a terminal object and a generator in all of these categories, see [9, 2.3.2.a, 2.1.7g, 4.5.17.a, 4.5.17.f and 4.5.17.g].

Remark 1.20. In view of Proposition 1.17 and Corollary 1.18, we get that in a well-pointed category $\mathcal{C}$ one has $\mathrm{Nat}(\mathrm{Id}_{\mathcal{C}}, \mathrm{Id}_{\mathcal{C}}) = \{\mathrm{Id}\}$. This result already appeared in [23, Corollary 21].

Looking for other additional conditions guaranteeing the equivalence between the semiseparability and the separability of a functor, we need the notion of *central idempotent endomorphism* of an object $\mathfrak{G}$ in a category $\mathcal{C}$. By this, we mean a central idempotent in the monoid $(\mathrm{End}(\mathfrak{G}), \circ, \mathrm{Id}_{\mathfrak{G}})$, i.e. a morphism $g : \mathfrak{G} \to \mathfrak{G}$ such that $g \circ g = g$ and $g \circ f = f \circ g$ for every morphism $f : \mathfrak{G} \to \mathfrak{G}$.

Proposition 1.21. *Let $\mathcal{C}$ be a category with a generator $\mathfrak{G}$ and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Assume there is no central idempotent endomorphism $g \neq \mathrm{Id}_{\mathfrak{G}} : \mathfrak{G} \to \mathfrak{G}$ such that $Fg = \mathrm{Id}_{F\mathfrak{G}}$.*

Then, F is semiseparable if and only if it is separable.

Proof. Consider an idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ such that $Fe = \mathrm{Id}_F$. Then $e_{\mathfrak{G}}$ is a central idempotent endomorphism of $\mathfrak{G}$ such that $Fe_{\mathfrak{G}} = \mathrm{Id}_{F\mathfrak{G}}$ and hence $e_{\mathfrak{G}} = \mathrm{Id}_{\mathfrak{G}}$ by hypothesis. Let X be an object in $\mathcal{C}$ and suppose that $e_X \neq \mathrm{Id}_X$. Since $\mathfrak{G}$ is a generator, there is a morphism $p : \mathfrak{G} \to X$ such that $e_X \circ p \neq \mathrm{Id}_X \circ p$ but, by naturality of e, we have $e_X \circ p = p \circ e_{\mathfrak{G}} = p \circ \mathrm{Id}_{\mathfrak{G}} = p$ so that we are led to a contradiction. Therefore $e_X = \mathrm{Id}_X$ and hence $e = \mathrm{Id}$. We conclude by Corollary 1.7. $\square$

We are now going to apply Proposition 1.21 to the category R-Mod of left R-modules. First we need the following easy lemma.

Lemma 1.22. *Let R be a ring. Then $g : R \to R$ is a central idempotent endomorphism of left R-modules if and only if $g = z\mathrm{Id}_R$ for a central idempotent $z \in R$, namely $z = g(1)$.*

Proof. For every $r \in R$, consider the morphism of left R-modules $f_r : R \to R, f_r(x) := xr$. Assume that g is a central idempotent endomorphism of left R-modules. Then $g(r) = rg(1) = rz$. From $g \circ f_r = f_r \circ g$ we get $zr = f_r(z) = f_r(g(1)) = g(f_r(1)) = g(r) = rz$ so that z is in the center of R. Moreover, since g is left R-linear and idempotent, we get $zz = g(z) = g(g(1)) = g(1) = z$. Conversely, it is clear that $z\mathrm{Id}_R : R \to R$ is a central idempotent endomorphism of left R-modules in case z is a central idempotent in R. □

Corollary 1.23. *Let R be a ring with no non-trivial central idempotent (e.g. R is a domain). A functor $F : R\text{-Mod} \to \mathcal{D}$ such that $F0 \neq \mathrm{Id}_{FR}$ is semiseparable if and only if it is separable.*

Proof. Let $g : R \to R$ be a central idempotent endomorphism of left R-modules such that $Fg = \mathrm{Id}_{FR}$. By Lemma 1.22, we have that $g = z\mathrm{Id}_R$ for a central idempotent $z \in R$. By hypothesis z is trivial i.e. $z = 1$ or $z = 0$ and hence we get either $g = \mathrm{Id}_R$ or $g = 0$. Since $Fg = \mathrm{Id}_{FR}$, we must have $g = \mathrm{Id}_R$. Since R is a generator in R-Mod, by Proposition 1.21, we conclude. □

2. Semiseparability and adjunctions

This section collects results on semiseparable functors which have an adjoint. Explicitly in Subsection 2.1, we investigate a Rafael-type theorem for semiseparable functors. In Subsection 2.2, we study the behaviour of semiseparable adjoint functors in terms of (co)monads and the associated (co)comparison functor. Subsection 2.3 contains results on semiseparability of functors that have both a (possibly equal) left and right adjoint.

2.1. Rafael-type Theorem

Rafael Theorem [34] provides a characterization of separable functors which have an adjoint: explicitly, given an adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$ with unit η and counit ϵ, then F is separable if and only if there exists a natural transformation $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $\nu \circ \eta = \mathrm{Id}_{\mathrm{Id}_{\mathcal{C}}}$ while G is separable if and only if there exists a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $\epsilon \circ \gamma = \mathrm{Id}_{\mathrm{Id}_{\mathcal{D}}}$. Next result extends Rafael Theorem to semiseparable functors.

Theorem 2.1. *(Rafael-type Theorem) Let $(F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C})$ be an adjoint pair of functors, with unit $\eta : \mathrm{Id} \to GF$ and counit $\epsilon : FG \to \mathrm{Id}$. Then:*

(i) *F is semiseparable if and only if η is regular.*
(ii) *G is semiseparable if and only if ϵ is regular.*

Proof. One can prove this result in a similar way as in [28, Theorem 1, page 90], by means of Yoneda Lemma. We include here an alternative proof following the lines of Rafael Theorem for separable functors. (i) Assume that F is semiseparable and let $\mathcal{P}$ be a natural transformation such that $\mathcal{F} \circ \mathcal{P} \circ \mathcal{F} = \mathcal{F}$. We define $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ on components by setting $\nu_X := \mathcal{P}_{GFX,X}(\epsilon_{FX}) : GFX \to X$, for any object X in $\mathcal{C}$. The naturality of ν_X in X follows from the one of $\mathcal{P}$. Moreover, by naturality of $\mathcal{P}$, for any X, Y in $\mathcal{C}$ and $g : FX \to FY$ we also have

$$\begin{aligned}\nu_Y \circ Gg \circ \eta_X &= \mathcal{P}_{GFY,Y}(\epsilon_{FY}) \circ Gg \circ \eta_X = \mathcal{P}_{X,Y}(\epsilon_{FY} \circ FGg \circ F\eta_X) \\ &= \mathcal{P}_{X,Y}(g \circ \epsilon_{FX} \circ F\eta_X) = \mathcal{P}_{X,Y}(g \circ \mathrm{Id}_{FX}) = \mathcal{P}_{X,Y}(g).\end{aligned}$$

The associated natural transformation is defined by $e_X := \mathcal{P}_{X,X}(\mathrm{Id}_{FX}) = \nu_X \circ G\mathrm{Id}_{FX} \circ \eta_X = \nu_X \circ \eta_X$ so that $e = \nu \circ \eta$. We compute $\eta \circ \nu \circ \eta = \eta \circ e = GFe \circ \eta = G\mathrm{Id}_F \circ \eta = \eta$. Thus, η is regular.

Conversely, suppose η is regular, i.e. there exists a natural transformation $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $\eta \circ \nu \circ \eta = \eta$, and for any $f \in \mathrm{Hom}_{\mathcal{D}}(FX, FY)$ define $\mathcal{P}_{X,Y}(f) := \nu_Y \circ Gf \circ \eta_X$. From the naturality of η and ν, for any $h : X \to Y$, $k : FY \to FZ$, and $l : Z \to T$ we have $\mathcal{P}_{X,T}(Fl \circ k \circ Fh) = \nu_T \circ G(Fl \circ k \circ Fh) \circ \eta_X = (\nu_T \circ GFl) \circ Gk \circ (GFh \circ \eta_X) = l \circ (\nu_Z \circ Gk \circ \eta_Y) \circ h = l \circ \mathcal{P}_{Y,Z}(k) \circ h$, thus $\mathcal{P} : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$ is a natural transformation. Since $\mathcal{P}_{GFX,X}(\epsilon_{FX}) = \nu_X \circ G\epsilon_{FX} \circ \eta_{GFX} = \nu_X \circ \mathrm{Id}_{GFX} = \nu_X$, the correspondence between $\mathcal{P}$ and ν is bijective. Set $e := \nu \circ \eta$. Then $Fe = F(\nu \circ \eta) = \mathrm{Id}_F \circ F(\nu\circ\eta) = \epsilon F \circ F\eta \circ F(\nu\circ\eta) = \epsilon F \circ F(\eta\circ\nu\circ\eta) = \epsilon F \circ F\eta = \mathrm{Id}_F$ i.e. $Fe = \mathrm{Id}_F$. Therefore F is semiseparable by the following computation, that holds for every $f : X \to Y$

$$\begin{aligned}(\mathcal{F}_{X,Y} \circ \mathcal{P}_{X,Y} \circ \mathcal{F}_{X,Y})(f) &= F(\mathcal{P}_{X,Y}(F(f))) = F(\nu_Y \circ GF(f) \circ \eta_X) \\ &= F(\nu_Y \circ \eta_Y \circ f) = F(e_Y) \circ Ff = \mathrm{Id}_{FY} \circ \mathcal{F}_{X,Y}(f) = \mathcal{F}_{X,Y}(f).\end{aligned}$$

(ii) It follows by duality. □

We include here a useful lemma, which characterizes the regularity of unit and counit.

Lemma 2.2. *Let $(F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C})$ be an adjoint pair of functors, with unit η and counit ϵ.*

(i) *The following equalities are equivalent for a natural transformation $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$:*

  (1) *$\eta \circ \nu \circ \eta = \eta$ (i.e. η is regular);*
  (2) *$F\nu \circ F\eta = \mathrm{Id}_F$;*
  (3) *$\nu G \circ \eta G = \mathrm{Id}_G$.*

(ii) *The following equalities are equivalent for a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$:*

(1) $\epsilon \circ \gamma \circ \epsilon = \epsilon$ *(i.e. ϵ is regular);*
(2) $G\epsilon \circ G\gamma = \mathrm{Id}_G$*;*
(3) $\epsilon F \circ \gamma F = \mathrm{Id}_F$.

Proof. We just prove (i) as (ii) follows dually.
(1) ⇒ (2). $F\nu \circ F\eta = \mathrm{Id}_F \circ F\nu \circ F\eta = \epsilon F \circ F\eta \circ F\nu \circ F\eta = \epsilon F \circ F\eta = \mathrm{Id}_F$.
(2) ⇒ (1). By naturality of η, we have $\eta \circ \nu \circ \eta = \eta \circ (\nu \circ \eta) = GF(\nu \circ \eta) \circ \eta = G(F\nu \circ F\eta) \circ \eta = \eta$.
(1) ⇒ (3). $\nu G \circ \eta G = \mathrm{Id}_G \circ \nu G \circ \eta G = G\epsilon \circ \eta G \circ \nu G \circ \eta G = G\epsilon \circ \eta G = \mathrm{Id}_G$.
(3) ⇒ (1). By naturality of $\nu \circ \eta$, we have $\eta \circ \nu \circ \eta = \eta \circ (\nu \circ \eta) = (\nu \circ \eta)GF \circ \eta = (\nu G \circ \eta G)F \circ \eta = \eta$. □

Remark 2.3. Let $(F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C})$ be an adjunction with unit η and counit ϵ.

1) Assume that there is a natural transformation $\nu : GF \to \mathrm{Id}_\mathcal{C}$ such that $\eta \circ \nu \circ \eta = \eta$. By Theorem 2.1, we know that F is semiseparable so that we can take the associated idempotent natural transformation $e : \mathrm{Id}_\mathcal{C} \to \mathrm{Id}_\mathcal{C}$. We can write it explicitly in terms of ν. Indeed, by the proof of Theorem 2.1, we can define $\mathcal{P}_{X,Y} : \mathrm{Hom}_\mathcal{D}(FX, FY) \to \mathrm{Hom}_\mathcal{C}(X, Y)$ by setting $\mathcal{P}_{X,Y}(f) := \nu_Y \circ Gf \circ \eta_X$ for every morphism $f : FX \to FY$. By definition, $e_X := \mathcal{P}_{X,X}(\mathrm{Id}_{FX}) = \nu_X \circ \eta_X$ so that $e = \nu \circ \eta$.

2) Dually, if there is a natural transformation $\gamma : \mathrm{Id}_\mathcal{D} \to FG$ such that $\epsilon \circ \gamma \circ \epsilon = \epsilon$, then G is semiseparable and the associated idempotent natural transformation is $e = \epsilon \circ \gamma : \mathrm{Id}_\mathcal{D} \to \mathrm{Id}_\mathcal{D}$.

2.2. Eilenberg-Moore category

In order to study the behaviour of semiseparable adjoint functors in terms of separable (co)monads and associated (co)comparison functor, we remind some basic facts concerning Eilenberg-Moore categories [21].

Given a monad $(\top, m : \top\top \to \top, \eta : \mathrm{Id}_\mathcal{C} \to \top)$ on a category $\mathcal{C}$ we denote by $\mathcal{C}_\top$ the Eilenberg-Moore category of modules (or algebras) over it. The forgetful functor $U_\top : \mathcal{C}_\top \to \mathcal{C}$ has a left adjoint, namely the *free functor*

$$V_\top : \mathcal{C} \to \mathcal{C}_\top, \qquad C \mapsto (\top C, m_C), \qquad f \mapsto \top(f).$$

The unit $\mathrm{Id}_\mathcal{C} \to U_\top V_\top = \top$ is exactly η while the counit $\beta : V_\top U_\top \to \mathrm{Id}_{\mathcal{C}_\top}$ is completely determined by the equality by $U_\top \beta_{(X,\mu)} = \mu$ for every object (X, μ) in $\mathcal{C}_\top$ (see [10, Proposition 4.1.4]). Dually, given a comonad $(\bot, \Delta : \bot \to \bot\bot, \epsilon : \bot \to \mathrm{Id}_\mathcal{C})$ on a category $\mathcal{C}$ we denote by $\mathcal{C}^\bot$ the Eilenberg-Moore category of comodules (or coalgebras) over it. The forgetful functor $U^\bot : \mathcal{C}^\bot \to \mathcal{C}$ has a right adjoint, namely the *cofree functor*

$$V^\bot : \mathcal{C} \to \mathcal{C}^\bot, \qquad C \mapsto (\bot C, \Delta_C), \qquad f \mapsto \bot(f).$$

The unit $\alpha : \mathrm{Id}_{\mathcal{C}^\perp} \to V^\perp U^\perp$ is completely determined by the equality by $U^\perp \alpha_{(X,\rho)} = \rho$ for every object (X, ρ) in $\mathcal{C}^\perp$ while the counit $U^\perp V^\perp = \perp \to \mathrm{Id}_{\mathcal{C}}$ is exactly ϵ.

Given an adjunction $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C})$, with unit η and counit ϵ, we can consider the monad $(GF, G\epsilon F, \eta)$ and the comonad $(FG, F\eta G, \epsilon)$. We have the *comparison functor*

$$K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}, \qquad D \mapsto (GD, G\epsilon_D), \qquad f \mapsto G(f),$$

and the *cocomparison functor*

$$K^{FG} : \mathcal{C} \to \mathcal{D}^{FG}, \qquad C \mapsto (FC, F\eta_C), \qquad f \mapsto F(f),$$

that fit into the diagram

$$\begin{tikzcd}
\mathcal{D}^{FG} \arrow[r, shift left=1.5, "U^{FG}"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left=1.5, "V^{FG}"] \arrow[d, shift left=1.5, "G"] \arrow[dr, bend left, "K_{GF}"] & \\
& \mathcal{C} \arrow[u, shift left=1.5, "F"] \arrow[u, phantom, "\dashv"] \arrow[ul, bend left, "K^{FG}"] \arrow[r, shift left=1.5, "V_{GF}"] \arrow[r, phantom, "\perp"] & \mathcal{C}_{GF}, \arrow[l, shift left=1.5, "U_{GF}"]
\end{tikzcd} \tag{1}$$

where $U_{GF} \circ K_{GF} = G$, $K_{GF} \circ F = V_{GF}$, $U^{FG} \circ K^{FG} = F$ and $K^{FG} \circ G = V^{FG}$.

We recall that a monad $(\top, m : \top\top \to \top, \eta : \mathrm{Id}_{\mathcal{C}} \to \top)$ on a category $\mathcal{C}$ is said to be *separable* [11] if there exists a natural transformation $\sigma : \top \to \top\top$ such that $m \circ \sigma = \mathrm{Id}_\top$ and $\top m \circ \sigma \top = \sigma \circ m = m\top \circ \top\sigma$; in particular, a separable monad is a monad satisfying the equivalent conditions of [11, Proposition 6.3].

Dually, a comonad $(\perp, \Delta : \perp \to \perp\perp, \epsilon : \perp \to \mathrm{Id}_{\mathcal{C}})$ on a category $\mathcal{C}$ is said to be *coseparable* if there exists a natural transformation $\tau : \perp\perp \to \perp$ satisfying $\tau \circ \Delta = \mathrm{Id}_\perp$ and $\perp\tau \circ \Delta\perp = \Delta \circ \tau = \tau\perp \circ \perp\Delta$.

Furthermore, an *idempotent monad* is a monad $(\top, m, \eta)$ on a category $\mathcal{C}$ whose multiplication m is an isomorphism or, equivalently, such that the forgetful functor $U_\top : \mathcal{C}_\top \to \mathcal{C}$ is fully faithful, see [10, Proposition 4.2.3]. Dually, an *idempotent comonad* is a comonad $(\perp, \Delta, \epsilon)$ on a category $\mathcal{C}$ whose comultiplication Δ is an isomorphism or, equivalently, such that the forgetful functor $U^\perp : \mathcal{C}^\perp \to \mathcal{C}$ is fully faithful, see [2, Section 6].

An adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$ with unit $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ and counit $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$ is said to be an *idempotent adjunction*[4] if the monad $(GF, G\epsilon F, \eta)$ is idempotent, or equivalently if the comonad $(FG, F\eta G, \epsilon)$ is idempotent, see e.g. [19, Subsection 3.4]. Indeed by [27, Proposition 2.8] this is equivalent to ask that anyone of the natural transformations ϵF, $G\epsilon$, $F\eta$ and ηG is an isomorphism.

[4] As underlined in [19], the first hint of *idempotent adjunctions* can be found in [29] under the name of *idempotent constructions*.

Remark 2.4. An idempotent (co)monad on a category $\mathcal{C}$ is always (co)separable with splitting given by the inverse of the (co)multiplication. Another way to arrive at the same conclusion is to observe that the forgetful functor $U_\top : \mathcal{C}_\top \to \mathcal{C}$ (resp. $U^\perp : \mathcal{C}^\perp \to \mathcal{C}$) is both separable and naturally full whenever it is fully faithful.

Remark 2.5. Let $(F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C})$ be an adjunction with unit $\eta : \mathrm{Id}_\mathcal{C} \to GF$ and counit $\epsilon : FG \to \mathrm{Id}_\mathcal{D}$.

1) The adjunctions (F, G) and (V_{GF}, U_{GF}) have the same associated monad $(GF, G\epsilon F, \eta)$, whereas the adjunctions (F, G) and (U^{FG}, V^{FG}) have the same associated comonad $(FG, F\eta G, \epsilon)$.
2) By 1), (F, G) is idempotent if and only if (V_{GF}, U_{GF}) is idempotent, if and only if (U^{FG}, V^{FG}) is idempotent.
3) The counit of an adjunction coincides with the counit of the associated comonad. Thus, by 2), the adjunctions (F, G) and (U^{FG}, V^{FG}) have the same counit. As a consequence, G is semiseparable (resp. separable, naturally full, fully faithful) if and only if so is V^{FG} in view of the corresponding Rafael-type Theorems (i.e. Theorem 2.1, [34, Theorem 1.2] and [4, Theorem 2.6]) and their combination for fully faithfulness.
4) Similarly, the adjunctions (F, G) and (V_{GF}, U_{GF}) have the same unit and hence F is semiseparable (resp. separable, naturally full, fully faithful) if and only if so is V_{GF}.

Now, let (F, G, η, ϵ) be an adjunction. In [18, Lemma 3.1] it is proved that if the right adjoint G is separable then the associated monad $(GF, G\epsilon F, \eta)$ is separable. We show that the semiseparability of G is enough to gain the separability of the associated monad. The proof is similar to the separable case but uses Lemma 2.2. We also prove the analogous result involving the left adjoint and the associated comonad.

Lemma 2.6. *Let (F, G, η, ϵ) be an adjunction.*

(i) *If G is semiseparable, then the associated monad $(GF, G\epsilon F, \eta)$ is separable.*
(ii) *If F is semiseparable, then the associated comonad $(FG, F\eta G, \epsilon)$ is coseparable.*

Proof. (i) Assume G is semiseparable. Then, by Theorem 2.1 and Lemma 2.2 (ii), there is a natural transformation $\gamma : \mathrm{Id}_\mathcal{D} \to FG$ such that $G\epsilon \circ G\gamma = \mathrm{Id}_G$. Set $\sigma := G\gamma F : G\mathrm{Id}_\mathcal{D}F \to GFGF$. It follows that $G\epsilon F \circ \sigma = G\epsilon F \circ G\gamma F = \mathrm{Id}_{GF}$. Moreover, from the naturality of ϵ and γ, we have $\gamma\circ\epsilon = \epsilon FG \circ FG\gamma$ and $\gamma\circ\epsilon = FG\epsilon\circ\gamma FG$, respectively, hence $GFG\epsilon F\circ\sigma GF = GFG\epsilon F\circ G\gamma FGF = G\gamma F\circ G\epsilon F = G\epsilon FGF\circ GFG\gamma F = G\epsilon FGF\circ GF\sigma$. Therefore, the monad $(GF, G\epsilon F, \eta)$ is separable.

(ii) The proof is dual by using Lemma 2.2 (i). $\square$

Remark 2.7. We have recalled that if the right adjoint G of an adjunction (F, G) is separable then the associated monad $(GF, G\epsilon F, \eta)$ is separable. It is known that the converse is not true as [18, Example 3.7(2)] shows. Explicitly, let $\mathcal{C}$ and $\mathcal{C}'$ be two nontrivial additive categories, and consider the product category $\mathcal{D} = \mathcal{C} \times \mathcal{C}'$. Let $F : \mathcal{C} \to \mathcal{D}$ be the canonical functor sending an object C to $(C, 0)$ and a morphism f to $(f, 0)$. Its right adjoint is the projection functor $G : \mathcal{D} \to \mathcal{C}$. Then, the associated monad GF equals the identity monad on $\mathcal{C}$, which is separable, but G is not separable, as it is not faithful. Nevertheless, G results to be semiseparable. Indeed, let $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$ be the counit of the adjunction given for any $D = (C, C')$ in $\mathcal{D}$ by $\epsilon_D = (\mathrm{Id}_C, \varphi^I_{C'}) : FGD \to D$, where $\varphi^I_{C'}$ is the unique map from the zero object 0 of $\mathcal{C}$ to C'. Consider the natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$, given for any $D = (C, C')$ in $\mathcal{D}$ by $\gamma_D = (\mathrm{Id}_C, \varphi^T_{C'}) : D \to FGD$, where $\varphi^T_{C'}$ is the unique map from C' to 0. Then, from $\gamma_D \circ \epsilon_D = (\mathrm{Id}_C, \varphi^T_{C'}) \circ (\mathrm{Id}_C, \varphi^I_{C'}) = (\mathrm{Id}_C, \mathrm{Id}_0) = \mathrm{Id}_{FGD}$ it follows that G is naturally full by [4, Theorem 2.6], hence in particular semiseparable.

Remark 2.8. As it happens for the separable case, the fact that the associated (co)monad is (co)separable does not imply that the right (left) adjoint is semiseparable, i.e. the converse of Lemma 2.6 is not necessarily true. To see this, note that if (F, G) is an adjunction with G (resp. F) fully faithful, then the associated monad (resp. comonad) is always idempotent (this will be proved in Corollary 2.13, resp. Corollary 2.17) whence separable (resp. coseparable). However F (resp. G) needs not to be semiseparable in this case. For instance, we consider the usual adjunction (φ^*, φ_*) attached to a ring homomorphism $\varphi : R \to S$ (we will be back on it in Subsection 3.1). In [4, Example 3.3] it is shown an example of a ring epimorphism $\varphi : R \to S$ (in this case φ_* is fully faithful) such that the extension of scalars functor φ^* is full, but not naturally full, thus φ^* is not semiseparable by Proposition 1.3.

The following result characterizes the semiseparability of a right adjoint functor in terms of properties of the comparison functor and of the forgetful functor from the Eilenberg-Moore category of modules over the associated monad. We remark that by Proposition 1.3 the separability of the forgetful functor coincides with its semiseparability as it is faithful.

Theorem 2.9. *Let $(F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C})$ be an adjunction. Then, G is semiseparable if and only if the forgetful functor $U_{GF} : \mathcal{C}_{GF} \to \mathcal{C}$ is separable (equivalently, the monad $(GF, G\epsilon F, \eta)$ is separable) and the comparison functor $K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}$ is naturally full.*

Proof. Set $U := U_{GF}$ and $K := K_{GF}$. Let η and ϵ be the unit and counit of (F, G) respectively. Assume G is semiseparable. By Theorem 2.1 and Lemma 2.2 (ii), there is a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $G\epsilon \circ G\gamma = \mathrm{Id}_G$. By Lemma 2.6 (i), $(GF, G\epsilon F, \eta)$ is a separable monad, and by [7, 2.9 (1)], the separability of this monad is equivalent to the separability of U. We now prove that $K : \mathcal{D} \to \mathcal{C}_{GF}$ is naturally full. Let $h : KX \to KY$ be a morphism in $\mathcal{C}_{GF}$. Note that this means the equality $G\epsilon_Y \circ GFUh = Uh \circ G\epsilon_X$ holds true. Set $h' := \epsilon_Y \circ FUh \circ \gamma_X$. Then, since $U \circ K = G$, which is semiseparable by assumption, we obtain

$$UKh' = G(\epsilon_Y \circ FUh \circ \gamma_X) = (G\epsilon_Y \circ GFUh) \circ G\gamma_X = Uh \circ G\epsilon_X \circ G\gamma_X = Uh.$$

So K is full, as $Kh' = h$. Moreover, since U is faithful and UK is semiseparable, by Lemma 1.13 (i) K is semiseparable. By Proposition 1.3 (ii) this means that K is naturally full. Conversely, if U is separable and K is naturally full, then by Corollary 1.16 $G = U \circ K$ is semiseparable. □

Remark 2.10. Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction. If G is semiseparable, let $e : \mathrm{Id}_\mathcal{D} \to \mathrm{Id}_\mathcal{D}$ be the associated idempotent natural transformation. Then, by Theorem 1.15 there is a unique functor $G_e : \mathcal{D}_e \to \mathcal{C}$ (necessarily separable) such that $G = G_e \circ H$, where $H : \mathcal{D} \to \mathcal{D}_e$ is the quotient functor onto the coidentifier category $\mathcal{D}_e$, which in turn is naturally full. By Theorem 2.9 G also factors as $U_{GF} \circ K_{GF}$ where U_{GF} is separable and K_{GF} is naturally full. These two factorizations of G as a naturally full functor followed by a separable one are related, in view of Theorem 1.15, by a unique functor $(K_{GF})_e : \mathcal{D}_e \to \mathcal{C}_{GF}$ (necessarily fully faithful) such that $(K_{GF})_e \circ H = K_{GF}$ and $U_{GF} \circ (K_{GF})_e = G_e$. The same result also establishes that the idempotent natural transformation associated to K_{GF} is still e.

$$\begin{tikzcd} \mathcal{D} \arrow[r, "H"] \arrow[d, "K_{GF}"'] & \mathcal{D}_e \arrow[d, "G_e"] \arrow[dl, dotted, "(K_{GF})_e"'] \\ \mathcal{C}_{GF} \arrow[r, "U_{GF}"'] & \mathcal{C} \end{tikzcd}$$

As a consequence of Theorem 2.9 we can now recover similar characterizations for separable, naturally full and fully faithful right adjoints. Let us start with the separable case.

Corollary 2.11 *(cf. [18, proof of Proposition 3.5] and [5, Proposition 2.16]). Let $(F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C})$ be an adjunction. Then, G is separable if and only if the forgetful functor $U_{GF} : \mathcal{C}_{GF} \to \mathcal{C}$ is separable (equivalently, the monad $(GF, G\epsilon F, \eta)$ is separable) and the comparison functor $K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}$ is fully faithful (i.e. G is premonadic).*

Proof. Set $U := U_{GF}$ and $K := K_{GF}$. By Proposition 1.3 (i), G is separable if and only if it is semiseparable and faithful. By Theorem 2.9, G is semiseparable if and only if U is separable and K is naturally full. Since $G = U \circ K$ and U is faithful, we get that G is faithful if and only if K is faithful. Putting all together we get that G is separable if and only if U is separable and K is both naturally full and faithful. The latter means that K is fully faithful, i.e. G is premonadic. □

We now provide a new characterization of natural fullness of a right adjoint functor in terms of idempotence of its adjunction/monad and natural fullness of the comparison functor.

Corollary 2.12. *The following are equivalent for an adjunction* $(F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C})$.

(1) G is naturally full.
(2) The adjunction (F, G) is idempotent and G is semiseparable.
(3) The forgetful functor $U_{GF} : \mathcal{C}_{GF} \to \mathcal{C}$ is fully faithful (i.e. the monad $(GF, G\epsilon F, \eta)$ is idempotent) and the comparison functor $K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}$ is naturally full.

Proof. Let $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ be the unit and let $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$ be the counit of the adjunction (F, G).

(1) ⇒ (2). If G is naturally full, by Rafael-type Theorem for naturally full functors [4, Theorem 2.6 (2)], there is a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $\gamma \circ \epsilon = \mathrm{Id}_{FG}$. Thus $G\gamma \circ G\epsilon = \mathrm{Id}_{GFG}$. On the other hand we have the triangular identity $G\epsilon \circ \eta G = \mathrm{Id}_G$ and hence $G\epsilon$ is invertible and (F, G) is idempotent. Moreover G is semiseparable by Proposition 1.3 (ii).

(2) ⇒ (3). It follows from the definition of an idempotent adjunction and from Theorem 2.9.

(3) ⇒ (1). Since $G = U_{GF} \circ K_{GF}$ we get that G is naturally full as a composition of naturally full functors, see [4, Proposition 2.3]. □

Now, putting together the above corollaries, we recover the characterization for a fully faithful right adjoint. Although it is well-known that $(F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C})$ is an idempotent adjunction, that is, the forgetful functor $U_{GF} : \mathcal{C}_{GF} \to \mathcal{C}$ (resp. $U^{FG} : \mathcal{D}^{FG} \to \mathcal{D}$) is fully faithful, provided the functor G (resp. F) is fully faithful, see e.g. [5, Proposition 2.5], the equivalence (1) ⇔ (2) in Corollary 2.13 (resp. Corollary 2.17) is new as far as we know.

Corollary 2.13. *The following are equivalent for an adjunction* $(F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C})$.

(1) G is fully faithful.
(2) The forgetful functor $U_{GF} : \mathcal{C}_{GF} \to \mathcal{C}$ is fully faithful (i.e. the monad $(GF, G\epsilon F, \eta)$ is idempotent) and the comparison functor $K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}$ is fully faithful (i.e. G is premonadic).
(3) The adjunction (F, G) is idempotent and the comparison functor $K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}$ is an equivalence (i.e. G is monadic).

Proof. (1) ⇔ (2). Put together Corollary 2.11 and Corollary 2.12.

(1) ⇔ (3). This follows by [5, Proposition 2.5]. □

Let us consider the dual context of Theorem 2.9. We characterize the semiseparability of a left adjoint functor in terms of the natural fullness of the cocomparison functor and of the separability of the forgetful functor from the Eilenberg-Moore category of comodules over the associated comonad.

Theorem 2.14. *Let $(F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C})$ be an adjunction. Then, F is semiseparable if and only if the forgetful functor $U^{FG} : \mathcal{D}^{FG} \to \mathcal{D}$ is separable (equivalently, the comonad $(FG, F\eta G, \epsilon)$ is coseparable) and the cocomparison functor $K^{FG} : \mathcal{C} \to \mathcal{D}^{FG}$ is naturally full.*

Dually to Remark 2.10, if $F \dashv G : \mathcal{D} \to \mathcal{C}$ is an adjunction with F semiseparable and $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ is the idempotent natural transformation associated to F, then Theorem 1.15 and Theorem 2.14 yield two factorizations $F_e \circ H = F = U^{FG} \circ K^{FG}$ of F as a naturally full functor followed by a separable one, and they are related by a unique functor $(K^{FG})_e : \mathcal{C}_e \to \mathcal{D}^{FG}$ (necessarily fully faithful) such that $(K^{FG})_e \circ H = K^{FG}$ and $U^{FG} \circ (K^{FG})_e = F_e$.

For future reference, we now state the dual of Corollaries 2.11, 2.12 and 2.13.

Corollary 2.15. *Let $(F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C})$ be an adjunction. Then, F is separable if and only if the forgetful functor $U^{FG} : \mathcal{D}^{FG} \to \mathcal{D}$ is separable (equivalently, the comonad $(FG, F\eta G, \epsilon)$ is coseparable) and the cocomparison functor $K^{FG} : \mathcal{C} \to \mathcal{D}^{FG}$ is fully faithful (i.e. F is precomonadic).*

Corollary 2.16. *The following are equivalent for an adjunction $(F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C})$.*

(1) F is naturally full.
(2) The adjunction (F, G) is idempotent and F is semiseparable.
(3) The forgetful functor $U^{FG} : \mathcal{D}^{FG} \to \mathcal{D}$ is fully faithful (i.e. the comonad $(FG, F\eta G, \epsilon)$ is idempotent) and the cocomparison functor $K^{FG} : \mathcal{C} \to \mathcal{D}^{FG}$ is naturally full.

Corollary 2.17. *The following are equivalent for an adjunction $(F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C})$.*

(1) F is fully faithful.
(2) The comonad $(FG, F\eta G, \epsilon)$ is idempotent and the cocomparison functor $K^{FG} : \mathcal{C} \to \mathcal{D}^{FG}$ is fully faithful (i.e. F is precomonadic).
(3) The adjunction (F, G) is idempotent and the cocomparison functor $K^{FG} : \mathcal{C} \to \mathcal{D}^{FG}$ is an equivalence (i.e. F is comonadic).

We include here a consequence of Corollaries 2.12 and 2.16 that will be used later on.

Corollary 2.18. *Let (F, G) be an idempotent adjunction. Then, F (resp. G) is semiseparable if and only if it is naturally full.*

2.3. Adjoint triples and bireflections

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Recall that an *adjoint triple* $F \dashv G \dashv H : \mathcal{C} \to \mathcal{D}$ of functors is a triple of functors $F, H : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $F \dashv G$ and $G \dashv H$. The following result, which is new to the best of our knowledge, shows how semiseparable, separable and naturally full functors behave with respect to adjoint triples.

Proposition 2.19. *Let $F \dashv G \dashv H : \mathcal{C} \to \mathcal{D}$ be an adjoint triple. Then, F is semiseparable (resp. separable, naturally full) if and only if so is H.*

Proof. We denote by η^l, ϵ^l and η^r, ϵ^r the unit and the counit of the adjunction $F \dashv G$ and of the adjunction $G \dashv H$, respectively. We just prove the "only if" part of the statement. For the other direction consider the adjoint triple $H^{\mathrm{op}} \dashv G^{\mathrm{op}} \dashv F^{\mathrm{op}}$ together with Remark 1.2. To a natural transformation $\nu^l : GF \to \mathrm{Id}_{\mathcal{C}}$ we can attach the natural transformation $\gamma^r := GH\nu^l \circ G\eta^r F \circ \eta^l : \mathrm{Id}_{\mathcal{C}} \to GH$ such that

$$\epsilon^r \circ \gamma^r = \epsilon^r \circ GH\nu^l \circ G\eta^r F \circ \eta^l = \nu^l \circ \epsilon^r GF \circ G\eta^r F \circ \eta^l = \nu^l \circ \eta^l. \tag{2}$$

Assume F is semiseparable. By Theorem 2.1 (i), there exists a natural transformation $\nu^l : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $\eta^l \circ \nu^l \circ \eta^l = \eta^l$. Define $\gamma^r : \mathrm{Id}_{\mathcal{C}} \to GH$ that fulfils (2) as above. We show that it is the required natural transformation of Theorem 2.1 (ii) such that $\epsilon^r \circ \gamma^r \circ \epsilon^r = \epsilon^r$. Indeed, by naturality of ϵ^r, we have $\epsilon^r \circ \gamma^r \circ \epsilon^r = \nu^l \circ \eta^l \circ \epsilon^r = \epsilon^r \circ \nu^l GH \circ \eta^l GH = \epsilon^r$, where the last equality follows from (1) $\Leftrightarrow$ (3) of Lemma 2.2 (i).

If F is separable, by Rafael Theorem, there exists a natural transformation $\nu^l : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $\nu^l \circ \eta^l = \mathrm{Id}$. Then, for γ^r defined as above and (2), we have $\epsilon^r \circ \gamma^r = \nu^l \circ \eta^l = \mathrm{Id}$ so that H is separable again by Rafael Theorem.

Assume F is naturally full. By [4, Theorem 2.6 (1)], there exists a natural transformation $\nu^l : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $\eta^l \circ \nu^l = \mathrm{Id}_{GF}$. Define $\gamma^r : \mathrm{Id}_{\mathcal{C}} \to GH$ as above. Observe that, from $\eta^l G \circ \nu^l G = \mathrm{Id}_{GFG}$ and $G\epsilon^l \circ \eta^l G = \mathrm{Id}_G$, it follows that $(\eta^l G)^{-1} = \nu^l G = G\epsilon^l$. Then, by naturality of γ^r and η^r we have $\gamma^r \circ \epsilon^r = GH\epsilon^r \circ \gamma^r GH = GH\epsilon^r \circ GH\nu^l GH \circ G\eta^r FGH \circ \eta^l GH = GH\epsilon^r \circ GHG\epsilon^l H \circ G\eta^r FGH \circ \eta^l GH = GH\epsilon^r \circ G(HG\epsilon^l \circ \eta^r FG)H \circ \eta^l GH = GH\epsilon^r \circ G\eta^r H \circ G\epsilon^l H \circ \eta^l GH = \mathrm{Id}_{GH} \circ \mathrm{Id}_{GH} = \mathrm{Id}_{GH}$. $\square$

Remark 2.20. We already observed that a functor is fully faithful if and only if it is at the same time separable and naturally full. Thus, by Proposition 2.19, we recover the well-known result that in an adjoint triple $F \dashv G \dashv H$, the functor F is fully faithful if and only if so is H, see e.g. [9, Proposition 3.4.2]. Adjoint triples $F \dashv G \dashv H$ where F and H are fully faithful are called *fully faithful adjoint triples*.

We will apply Proposition 2.19 in Subsection 3.1 to an adjoint triple associated to a ring morphism and in Subsection 3.5 in the study of a particular adjoint triple attached to a bialgebra.

Now, recall that a functor $G : \mathcal{D} \to \mathcal{C}$ is called *Frobenius* if there exists a functor $F : \mathcal{C} \to \mathcal{D}$ which is both a left and a right adjoint to G. Thus, a Frobenius functor $G : \mathcal{D} \to \mathcal{C}$ fits into an adjoint triple $F \dashv G \dashv F : \mathcal{C} \to \mathcal{D}$ where the left and right adjoint F are equal. As a consequence of Theorem 2.1 and Lemma 2.2, in the following result we obtain necessary and sufficient conditions for the semiseparability of a Frobenius functor. This is a semiseparable version of [15, Proposition 49] for separable Frobenius functors and of [4, Proposition 2.7] for naturally full Frobenius functors.

Proposition 2.21. *Let $F : \mathcal{C} \to \mathcal{D}$ be a Frobenius functor, with left and right adjoint $G : \mathcal{D} \to \mathcal{C}$. Denote by η^l, ϵ^l and by η^r, ϵ^r the unit and the counit of the adjunctions (F, G) and (G, F), respectively. Then, the following assertions are equivalent:*

(i) *F is semiseparable.*

(ii) *There exists a natural transformation $\alpha : G \to G$ such that one of the following equivalent conditions holds:*

$$\eta^l \circ \epsilon^r \circ \alpha F \circ \eta^l = \eta^l; \qquad F\epsilon^r \circ F\alpha F \circ F\eta^l = \mathrm{Id}_F; \qquad \epsilon^r G \circ \alpha FG \circ \eta^l G = \mathrm{Id}_G.$$

(iii) *There exists a natural transformation $\beta : F \to F$ such that one of the following equivalent conditions holds:*

$$\eta^l \circ \epsilon^r \circ G\beta \circ \eta^l = \eta^l; \qquad F\epsilon^r \circ FG\beta \circ F\eta^l = \mathrm{Id}_F; \qquad \epsilon^r G \circ G\beta G \circ \eta^l G = \mathrm{Id}_G.$$

(iv) *There exists a natural transformation $\alpha' : G \to G$ such that one of the following equivalent conditions holds:*

$$\epsilon^r \circ \alpha' F \circ \eta^l \circ \epsilon^r = \epsilon^r; \qquad F\epsilon^r \circ F\alpha' F \circ F\eta^l = \mathrm{Id}_F; \qquad \epsilon^r G \circ \alpha' FG \circ \eta^l G = \mathrm{Id}_G.$$

(v) *There exists a natural transformation $\beta' : F \to F$ such that one of the following equivalent conditions holds:*

$$\epsilon^r \circ G\beta' \circ \eta^l \circ \epsilon^r = \epsilon^r; \qquad F\epsilon^r \circ FG\beta' \circ F\eta^l = \mathrm{Id}_F; \qquad \epsilon^r G \circ G\beta' G \circ \eta^l G = \mathrm{Id}_G.$$

Proof. $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ By [15, Proposition 10] applied to the adjunction (G, F) we have the following bijective correspondences:

$$\mathrm{Nat}(GF, \mathrm{Id}_{\mathcal{C}}) \cong \mathrm{Nat}(G, G) \cong \mathrm{Nat}(F, F) \cong \mathrm{Nat}(\mathrm{Id}_{\mathcal{D}}, FG).$$

Explicitly, for any natural transformation $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ there are unique natural transformations $\alpha : G \to G$, $\beta : F \to F$ such that

$$\epsilon^r \circ \alpha F = \nu = \epsilon^r \circ G\beta. \tag{3}$$

Apply Theorem 2.1 and Lemma 2.2 to the adjunction (F, G) and then (3) to the induced natural transformation $\nu^l : GF \to \mathrm{Id}_\mathcal{C}$ such that $\eta^l \circ \nu^l \circ \eta^l = \eta^l$.
$(i) \Leftrightarrow (iv) \Leftrightarrow (v)$ By [15, Proposition 10] applied to the adjunction (F, G), for any natural transformation $\gamma : \mathrm{Id}_\mathcal{C} \to GF$ there are unique natural transformations $\alpha' : G \to G$, $\beta' : F \to F$ such that

$$\alpha' F \circ \eta^l = \gamma = G\beta' \circ \eta^l. \tag{4}$$

Consider the adjunction (G, F) and apply Theorem 2.1 and Lemma 2.2. Then, apply (4) to the induced natural transformation $\gamma^r : \mathrm{Id}_\mathcal{C} \to GF$ such that $\epsilon^r \circ \gamma^r \circ \epsilon^r = \epsilon^r$. □

Remark 2.22. A semiseparable functor is not necessarily Frobenius. Indeed, from [15, Example 18, item 6, page 323] let G be a finite group and consider the group algebra $A = \Bbbk G$ over a field $\Bbbk$. Then A is a Frobenius $\Bbbk$-algebra and hence the restriction of scalars functor $\varphi_* : A\text{-Mod} \to \Bbbk\text{-Mod}$ is Frobenius, cf. [15, Theorem 28, item 3]. However if $\mathrm{char}(\Bbbk)$ divides $|G|$, the extension $A/\Bbbk$ is not separable so that φ_* is not separable and therefore not even semiseparable as it is faithful. See Subsection 3.1 for further results on φ_*.

Next aim is to study semiseparable (co)reflections. Recall that

- a functor admitting a fully faithful left adjoint is called a *coreflection*, see [6];
- a functor with a fully faithful right adjoint is called a *reflection*;
- a functor $G : \mathcal{D} \to \mathcal{C}$ is called a *bireflection* if it has a left and right adjoint equal, say $F : \mathcal{C} \to \mathcal{D}$, which is fully faithful and satisfies the coherent condition $\gamma \circ \epsilon = \mathrm{Id}$ where $\epsilon : FG \to \mathrm{Id}$ is the counit of $F \dashv G$ while $\gamma : \mathrm{Id} \to FG$ is the unit of $G \dashv F$, cf. [23, Definition 8].

Being a coreflection (respectively a reflection) is equivalent to the fact that the unit (respectively counit) of the corresponding adjunction is an isomorphism, see [9, Proposition 3.4.1]. The adjoint of the inclusion of a (co)reflective subcategory is a typical example of (co)reflection. In Theorem 2.24 we will see how semiseparable (co)reflections $G : \mathcal{D} \to \mathcal{C}$ naturally give rise to fully faithful adjoint triples. Bireflective subcategories of a given category $\mathcal{C}$ provide examples of bireflections. Recall that an idempotent $f : X \to X$ in a category $\mathcal{C}$ is *split* if there exist $g : X \to Y$ and $h : Y \to X$ such that $h \circ g = f$ and $g \circ h = \mathrm{Id}_Y$. The splitting is unique up to isomorphism. In [23] an endo-natural transformation whose components are all split idempotents is called a *split-idempotent natural transformation*. It is known that bireflective subcategories correspond bijectively to split-idempotent natural transformations $e : \mathrm{Id}_\mathcal{C} \to \mathrm{Id}_\mathcal{C}$ with specified splitting, [23, Theorem 13]. This fact is connected to the functor $H : \mathcal{C} \to \mathcal{C}_e$ of Theorem 1.15 which comes out to be a bireflection in case e splits naturally, as we will see in Proposition 2.27.

Let us see how the notions of (co)reflection and bireflection behave in connection to semiseparability. In particular, in the following proposition we observe their behaviour with respect to a semiseparable composition of functors, cf. [4, Proposition 2.4] for the naturally full case.

Proposition 2.23. *Let $G : \mathcal{D} \to \mathcal{C}$, $H : \mathcal{C} \to \mathcal{E}$ be functors, and assume that G is a (co)reflection. If $H \circ G : \mathcal{D} \to \mathcal{E}$ is semiseparable, then H is semiseparable.*

Proof. Assume that G is a coreflection with a fully faithful left adjoint F. If HG is semiseparable, since F is fully faithful (whence naturally full), then HGF is semiseparable by Lemma 1.12. The unit $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ of the adjunction (F, G) is an isomorphism, so that $H\eta : H \to HGF$ is an isomorphism. By Corollary 1.11, H is semiseparable. If G is a reflection, the proof is similar. □

The next result provides a characterization of semiseparable (co)reflections. Surprisingly it involves the notions of naturally full and Frobenius functor as well as the one of bireflection.

Theorem 2.24. *The following are equivalent for a functor $G : \mathcal{D} \to \mathcal{C}$.*

(1) *G is a naturally full coreflection.*
(2) *G is a semiseparable coreflection.*
(3) *G is a bireflection.*
(4) *G is a Frobenius coreflection.*
(5) *G is a naturally full reflection.*
(6) *G is a semiseparable reflection.*
(7) *G is a Frobenius reflection.*

Proof. We prove the equivalence between $(1), (2), (3)$ and (4). Assume that G is a coreflection. Denote by F the left adjoint of G, by $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ the unit and by $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$ the counit of the adjunction (F, G). Since F is fully faithful, we get that η is invertible. Therefore, from $\epsilon F \circ F\eta = \mathrm{Id}_F$ and $G\epsilon \circ \eta G = \mathrm{Id}_G$, we get $(F\eta)^{-1} = \epsilon F$ and $(\eta G)^{-1} = G\epsilon$.

$(1) \Leftrightarrow (2)$. Since η is invertible, the adjunction (F, G) is idempotent and Corollary 2.18 applies.

$(2) \Rightarrow (3)$. If G is semiseparable, by Theorem 2.1 (ii) there is a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $\epsilon \circ \gamma \circ \epsilon = \epsilon$. By Lemma 2.2, we have $\epsilon F \circ \gamma F = \mathrm{Id}_F$ and $G\epsilon \circ G\gamma = \mathrm{Id}_G$ so that,

$$F\left(\eta^{-1}\right) \circ \gamma F = (F\eta)^{-1} \circ \gamma F = \epsilon F \circ \gamma F = \mathrm{Id}_F,$$
$$\eta^{-1} G \circ G\gamma = (\eta G)^{-1} \circ G\gamma = G\epsilon \circ G\gamma = \mathrm{Id}_G.$$

This means that (G, F) is an adjunction with unit $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ and counit $\eta^{-1} : GF \to \mathrm{Id}_{\mathcal{C}}$. The equality $\eta^{-1}G = G\epsilon$ implies the coherent condition $\gamma \circ \epsilon = \mathrm{Id}$ by [23, Proposition 10].

(3) ⇒ (4). It is trivial.

(4) ⇒ (1). If G is Frobenius, then there are a unit $\eta' : \mathrm{Id}_{\mathcal{D}} \to FG$ and a counit $\epsilon' : GF \to \mathrm{Id}_{\mathcal{C}}$ of the adjunction (G, F). Set $\sigma := \epsilon' G \circ \eta G : G \to G$ and note that $\sigma \circ G\epsilon = \epsilon' G \circ \eta G \circ G\epsilon = \epsilon' G \circ \eta G \circ (\eta G)^{-1} = \epsilon' G$. If we set $\gamma := F\sigma \circ \eta'$, we obtain

$$\gamma \circ \epsilon = F\sigma \circ \eta' \circ \epsilon \overset{\text{nat.}\eta'}{=} F\sigma \circ FG\epsilon \circ \eta' FG = F(\sigma \circ G\epsilon) \circ \eta' FG = F\epsilon' G \circ \eta' FG = \mathrm{Id}_{FG}.$$

By [4, Theorem 2.6], we conclude that G is naturally full.

The implications (5) ⇔ (6) ⇒ (3) ⇒ (7) ⇒ (5) follow dually. □

Remark 2.25. It is known that a conservative (co)reflection is always an equivalence (see e.g. [6, Remark 1.4]). Since separable functors are conservative (cf. Remark 1.8), one recovers the fact that a separable (co)reflection is, actually, an equivalence, see e.g. in [35, Proposition 2.4]. Thus Theorem 2.24 can be seen as a semi-analogue of this result.

The following result will be useful in Subsection 3.5.

Proposition 2.26. *Let $F \dashv G \dashv H : \mathcal{C} \to \mathcal{D}$ be an adjoint triple with G fully faithful. Denote by η^l, ϵ^l and η^r, ϵ^r the unit and the counit of the adjunction $F \dashv G$ and of the adjunction $G \dashv H$, respectively. Consider the natural transformation $\sigma : H \to F$ defined by $\sigma := F\epsilon^r \circ (\epsilon^l H)^{-1} : H \to F$. Then, H is semiseparable if and only if σ is split-mono if and only if σ is invertible.*

Proof. Since G is fully faithful, then H is a coreflection so that, by Theorem 2.24, it is semiseparable if and only if it is naturally full if and only if it is Frobenius, i.e. $F \cong H$. By [35, Proposition 2.2], the condition $F \cong H$ is equivalent to the invertibility of σ. We now prove that G is naturally full if and only if σ is split-mono. We have a bijective correspondence $\mathrm{Nat}(F, H) \cong \mathrm{Nat}(\mathrm{Id}_{\mathcal{C}}, GH)$. Explicitly, for any natural transformation $\tau : F \to H$ there is a unique natural transformation $\gamma : \mathrm{Id}_{\mathcal{C}} \to GH$ given by $\gamma := G\tau \circ \eta^l$. Then $\gamma \circ \epsilon^r = G\tau \circ \eta^l \circ \epsilon^r = G\tau \circ GF\epsilon^r \circ \eta^l GH = G\tau \circ G(\sigma \circ \epsilon^l H) \circ \eta^l GH = G(\tau \circ \sigma) \circ G\epsilon^l H \circ \eta^l GH = G(\tau \circ \sigma)$ so that $\gamma \circ \epsilon^r = G(\tau \circ \sigma)$. Thus, $\gamma \circ \epsilon^r = \mathrm{Id}_{GH}$ if and only if $G(\tau \circ \sigma) = \mathrm{Id}_{GH}$ if and only if $\tau \circ \sigma = \mathrm{Id}_H$, as G is faithful. By Rafael-type Theorem for naturally full functors, the condition $\gamma \circ \epsilon^r = \mathrm{Id}_{GH}$ means that H is naturally full. □

Let us see how the quotient functor $H : \mathcal{C} \to \mathcal{C}_e$ results to be a bireflection in meaningful cases.

Proposition 2.27. *Let $\mathcal{C}$ be a category and let $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ be an idempotent natural transformation. Then, the quotient functor $H : \mathcal{C} \to \mathcal{C}_e$ is a bireflection if and only if e splits (e.g. when $\mathcal{C}$ is idempotent complete).*

Proof. Assume that $H : \mathcal{C} \to \mathcal{C}_e$ is a bireflection. Then, H has left and right adjoint functors equal, say $L : \mathcal{C}_e \to \mathcal{C}$, which is fully faithful, and such that the coherence condition $\eta \circ \epsilon' = \mathrm{Id}_{HL}$ is satisfied, where $\eta : \mathrm{Id} \to HL$ is the unit of adjunction $L \dashv H$, and $\epsilon' : HL \to \mathrm{Id}$ is the counit of $H \dashv L$. Denote by $\eta' : \mathrm{Id} \to LH$ and $\epsilon : LH \to \mathrm{Id}$ the unit and counit of the adjunctions $H \dashv L$ and $L \dashv H$, respectively. Since L is fully faithful, η is an isomorphism and hence, from the coherence condition and the triangular identity $L\epsilon' \circ \eta' L = \mathrm{Id}_L$, we get that $L\eta = (L\epsilon')^{-1} = \eta' L$. Therefore, by naturality of η', we have $\eta' \circ \epsilon = LH\epsilon \circ \eta' LH = LH\epsilon \circ L\eta H = L\mathrm{Id}_H = \mathrm{Id}_{LH}$. Similarly, from the latter condition and the triangular identity $H\epsilon \circ \eta H = \mathrm{Id}_H$, it follows that $H\eta' = (H\epsilon)^{-1} = \eta H$. Then, $H(\epsilon \circ \eta') = H\epsilon \circ H\eta' = \mathrm{Id}_H = H\mathrm{Id}$. Thus, for all $X \in \mathcal{C}$, we have $e_X = e_X \circ \epsilon_X \circ \eta'_X$. Now, recall $He = \mathrm{Id}_H$ as e is the idempotent natural transformation associated to H. Then, $e_X \circ \epsilon_X = \epsilon_X \circ LHe_X = \epsilon_X \circ LH\mathrm{Id}_X = \epsilon_X$ so that the equality $e_X = e_X \circ \epsilon_X \circ \eta'_X$ simplifies as $e_X = \epsilon_X \circ \eta'_X$ and hence e splits.

The converse essentially follows from the dual of [23, Proof of Theorem 13]. We give a slightly different proof here. Assume that e splits. Since we know that H is naturally full (see Subsection 1.5), it is in particular semiseparable. Thus, in order to conclude, by Theorem 2.24, it is enough to check that H is a coreflection. Choose a splitting $\mathrm{Id}_{\mathcal{C}} \overset{\pi}{\twoheadrightarrow} P \overset{\epsilon}{\hookrightarrow} \mathrm{Id}_{\mathcal{C}}$ of the idempotent e such that $\pi \circ \epsilon = \mathrm{Id}_P$. Note that $Pe = Pe \circ \pi \circ \epsilon \overset{\text{nat.}\pi}{=} \pi \circ e \circ \epsilon = \pi \circ \epsilon \circ \pi \circ \epsilon = \mathrm{Id}_P$ and hence $Pe = \mathrm{Id}_P$. Thus, by Lemma 1.14, there is a unique functor $P_e : \mathcal{C}_e \to \mathcal{C}$ such that $P_e \circ H = P$. It is now straightforward to check that $P_e \dashv H$ with counit ϵ and invertible unit $\eta : \mathrm{Id}_{\mathcal{C}_e} \to HP_e$ defined by the equality $\eta H = H\pi$ i.e. by setting $\eta_X := (\overline{\pi_X})_{X \in \mathcal{C}}$. Thus H is a coreflection. $\square$

As a consequence of Theorem 1.15 and Proposition 2.27, we have the following corollary.

Corollary 2.28. *A functor $F : \mathcal{C} \to \mathcal{D}$ factors as a bireflection followed by a separable functor if and only if it is semiseparable and the associated natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ splits. Moreover, any such a factorization is the same given by the coidentifier within Theorem 1.15, up to a category equivalence.*

Proof. Assume that $F = S \circ N$ where $N : \mathcal{C} \to \mathcal{E}$ is a bireflection and $S : \mathcal{E} \to \mathcal{D}$ is a separable functor. Since N is in particular naturally full, we get that $F = S \circ N$ is semiseparable and hence by Theorem 1.15, there is a unique functor $N_e : \mathcal{C}_e \to \mathcal{E}$ (necessarily fully faithful) such that $N_e \circ H = N$ and $S \circ N_e = F_e$. If we denote by $L : \mathcal{E} \to \mathcal{C}$ the left adjoint of N, then it is fully faithful and hence, since the unit $\eta : \mathrm{Id} \to NL$ is an isomorphism, we have that $\mathrm{Id} \cong N \circ L = N_e \circ H \circ L$. Therefore N_e is essentially surjective on objects. Since it is also fully faithful, we get N_e is an equivalence of categories and, from $\mathrm{Id} \cong N_e \circ H \circ L$, it has quasi-inverse $H \circ L$. Thus, $(H \circ L) \circ N_e \cong \mathrm{Id}$ and hence $(H \circ L) \circ N = (H \circ L \circ N_e) \circ H \cong \mathrm{Id} \circ H = H$, from which it follows that H is a bireflection as N is. Thus, by Proposition 2.27, the idempotent e splits. Moreover the factorization $F = S \circ N$, up to the category equivalence N_e, is the same given by the coidentifier within Theorem 1.15.

Conversely, if the natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ attached to the semiseparable functor F splits, then by Proposition 2.27 the quotient functor $H : \mathcal{C} \to \mathcal{C}_e$ results to be a bireflection. Thus, since by Theorem 1.15 the semiseparable functor F factors as $H : \mathcal{C} \to \mathcal{C}_e$ followed by a separable functor $F_e : \mathcal{C}_e \to \mathcal{D}$, we achieve the desired factorization of F into a bireflection followed by a separable functor. □

Remark 2.29. By Theorem 1.15, any semiseparable functor $F : \mathcal{C} \to \mathcal{D}$ factors as $H : \mathcal{C} \to \mathcal{C}_e$ followed by a separable functor $F_e : \mathcal{C}_e \to \mathcal{D}$, where e is the associated idempotent natural transformation. Assume that e splits. Then, by Proposition 2.27, $H : \mathcal{C} \to \mathcal{C}_e$ is a bireflection. In particular H is a coreflection and F_e is conservative. This is what is called an *image-factorization* of F in [6, Definition 1.1]. Image-factorizations are unique up to an equivalence of categories, see [6, Lemma 1.2]. As a consequence, if we can write $F = S \circ N$ where $S : \mathcal{E} \to \mathcal{D}$ is conservative (e.g. separable) and $N : \mathcal{C} \to \mathcal{E}$ is a coreflection (e.g. a bireflection) then there is an equivalence $\mathcal{E} \cong \mathcal{C}_e$.

3. Applications and examples

In this section we test the notion of semiseparability on relevant functors attached to ring and coalgebra morphisms, corings, bimodules and Hopf modules.

We start in Subsection 3.1 by considering the restriction of scalars functor $\varphi_* : S\text{-Mod} \to R\text{-Mod}$, the extension of scalars functor $\varphi^* = S \otimes_R (-) : R\text{-Mod} \to S\text{-Mod}$ and the coinduction functor $\varphi^! = {}_R\mathrm{Hom}(S, -) : R\text{-Mod} \to S\text{-Mod}$ associated to a ring morphism $\varphi : R \to S$. On the one hand, since φ_* is faithful, its semiseparability falls back to its separability. On the other hand, the functors above form an adjoint triple $\varphi^* \dashv \varphi_* \dashv \varphi^!$ so that the semiseparability of $\varphi^!$ is equivalent to the one of φ^*. The latter is characterized in Proposition 3.1 in terms of the regularity of φ as a morphism of R-bimodules and in Proposition 3.6 in terms of the existence of a suitable central idempotent element $z \in R$. In a similar fashion in Subsection 3.2 we investigate the semiseparability of two adjoint functors attached to a coalgebra map $\psi : C \to D$. The main result here is Proposition 3.8.

In Subsection 3.3, we turn our attention to the induction functor $(-) \otimes_R \mathcal{C} : \text{Mod-}R \to \mathcal{M}^{\mathcal{C}}$, attached to an R-coring $\mathcal{C}$. Here we highlight Theorem 3.10 where this functor is proved to be semiseparable if and only if the coring counit $\varepsilon_{\mathcal{C}} : \mathcal{C} \to R$ is a regular morphism of R-bimodules. In Subsection 3.4, given an (R, S)-bimodule M, we consider the coinduction functor $\sigma_* := \mathrm{Hom}_S(M, -) : \text{Mod-}S \to \text{Mod-}R$ together with its left adjoint $\sigma^* := (-) \otimes_R M : \text{Mod-}R \to \text{Mod-}S$. In Theorem 3.18 we show that the semiseparability of σ_* can be completely rewritten both in terms of the regularity of the evaluation map plus a mild condition that is redundant when M is projective as a right S-module, and in terms of a property of M that will lead us to introduce the new notion of M-semiseparability over R for the ring S, a right semiseparable version of the one given in [37]. In Corollary 3.20 we prove that S is M-separable over R if and only if S is M-semiseparable over R and M is a generator in Mod-S. A different characterization of M-semiseparability of S over R, that will allow us to exhibit an example where S is M-semiseparable but not M-separable over R (see Example 3.23), is obtained in Proposition 3.22. Then, if we add the assumption that M is finitely generated and projective as a right S-module, further characterizations of the semiseparability of σ_* and σ^* are provided in Proposition 3.26 and Proposition 3.27, respectively.

It is worth noticing that the above functors φ^*, $(-)\otimes_R \mathcal{C}$, and σ_* have sources which are idempotent complete categories so that, by Corollary 2.28, they always admit a factorization as a bireflection followed by a separable functor, when they are semiseparable. In Proposition 3.5, Corollary 3.12, and Proposition 3.24, we explicitly provide such factorizations.

Theorem 3.31 concerns the semiseparability of the coinvariant functor $(-)^{\mathrm{co}B} : \mathfrak{M}_B^B \to \mathfrak{M}$, from the category of right Hopf modules over a $\Bbbk$-bialgebra B to the category of $\Bbbk$-vector spaces over a field $\Bbbk$, proving that it is semiseparable if and only if B is a right Hopf algebra with anti-multiplicative and anti-comultiplicative right antipode.

Finally, Subsection 3.6 provides particular examples of (co)reflections that highlight the connections between the types of functors we have studied in this paper.

3.1. Extension and restriction of scalars

A morphism of rings $\varphi : R \to S$ induces

- the restriction of scalars functor $\varphi_* : S\text{-Mod} \to R\text{-Mod}$;
- the extension of scalars (or induction) functor $\varphi^* := S \otimes_R (-) : R\text{-Mod} \to S\text{-Mod}$;
- the coinduction functor $\varphi^! := {}_R\mathrm{Hom}(S, -) : R\text{-Mod} \to S\text{-Mod}$.

All together these functors form an adjoint triple $\varphi^* \dashv \varphi_* \dashv \varphi^!$.

The unit η and the counit ϵ of the adjunction (φ^*, φ_*), are respectively defined by

$$\eta_M = \varphi \otimes_R M : M \to S \otimes_R M,\ m \mapsto 1_S \otimes_R m, \qquad \text{and} \qquad \epsilon_N : S \otimes_R N \to N,\ s \otimes_R n \mapsto sn,$$

while the unit $\eta^!$ and the counit $\epsilon^!$ of the adjunction $(\varphi_*, \varphi^!)$, are defined by

$$\eta^!_N : N \to {}_R\mathrm{Hom}(S, N),\ n \mapsto [s \mapsto sn], \qquad \text{and} \qquad \epsilon^!_M : {}_R\mathrm{Hom}(S, M) \to M,\ f \mapsto f(1_S),$$

for every $M \in R\text{-Mod}$ and $N \in S\text{-Mod}$. In the literature we can find results either on the separability or on the natural fullness of these functors. For instance we know that

- φ_* is separable if and only if S/R is separable, see [33, Proposition 1.3];
- φ_* is naturally full if and only if it is full, see [4, Proposition 3.1 (1)], if and only if it is fully faithful (in fact it is always faithful being a forgetful functor) if and only if φ is an epimorphism in the category of rings, cf. [36, Proposition XI.1.2];
- φ^* is separable if and only if φ is split-mono as an R-bimodule map, i.e. if there is $E \in {}_R\mathrm{Hom}_R(S, R)$ such that $E \circ \varphi = \mathrm{Id}$, see [33, Proposition 1.3];

- φ^* is naturally full if and only if φ is split-epi as an R-bimodule map, i.e. if there is $E \in {}_R\mathrm{Hom}_R(S, R)$ such that $\varphi \circ E = \mathrm{Id}$, see [4, Proposition 3.1 (2)];
- $\varphi^!$ is separable if and only if so is φ^* [17, Corollary 3.10].

We now investigate the semiseparability of these three functors. Indeed, from the general characterization given in Proposition 2.19, we know that $\varphi^!$ is semiseparable (resp. separable, naturally full) if and only if so is φ^*. For this reason we are only dealing with the functors φ_* and φ^*.

Concerning φ_*, since it is always faithful, we have that φ_* is semiseparable if and only if φ_* is separable, that is, S/R is separable. Thus, although we are tempted to name S/R "semiseparable" whenever φ_* is semiseparable, by the foregoing, this would bring us back to S/R separable.

In the next results we investigate when the functor φ^* is semiseparable.

Proposition 3.1. *Let $\varphi : R \to S$ be a morphism of rings. Then, the extension of scalars functor $\varphi^* = S \otimes_R (-) : R\text{-Mod} \to S\text{-Mod}$ is semiseparable if and only if φ is a regular morphism of R-bimodules, i.e. there is $E \in {}_R\mathrm{Hom}_R(S, R)$ such that $\varphi \circ E \circ \varphi = \varphi$, i.e., such that $\varphi E(1_S) = 1_S$.*

Proof. It is known that there is a bijective correspondence $\mathrm{Nat}(\varphi_*\varphi^*, \mathrm{Id}_{R\text{-Mod}}) \cong {}_R\mathrm{Hom}_R(S, R)$, see [15, Theorem 27]. Now, by Theorem 2.1, φ^* is semiseparable if and only if there exists a natural transformation $\nu \in \mathrm{Nat}(\varphi_*\varphi^*, \mathrm{Id}_{R\text{-Mod}})$ such that $\eta \circ \nu \circ \eta = \eta$. So, given ν for φ^*, we consider the corresponding $E \in {}_R\mathrm{Hom}_R(S, R)$, $E(s) := \nu_R(s \otimes_R 1_R)$, for every $s \in S$. Then, for every $r \in R$, we get $(\varphi \circ E \circ \varphi)(r) = \varphi(E(\varphi(r))) = \varphi(\nu_R(\varphi(r) \otimes_R 1_R)) = \varphi(\nu_R(\eta_R(r))) = r_S\eta_R(\nu_R(\eta_R(r))) = r_S\eta_R(r) = \varphi(r)$ where $r_S : S \otimes_R R \to S, s \otimes_R r \mapsto s\varphi(r)$, is the canonical isomorphism. Conversely, given $E \in {}_R\mathrm{Hom}_R(S, R)$ such that $\varphi \circ E \circ \varphi = \varphi$, define $\nu_M : S \otimes_R M \to M$, $\nu_M(s \otimes_R m) = E(s)m$, for every $M \in R\text{-Mod}$, $m \in M$ and $s \in S$. Then,

$$\begin{aligned}(\eta_M \circ \nu_M \circ \eta_M)(m) &= \eta_M(\nu_M(1_S \otimes_R m)) = \eta_M(\nu_M(\varphi(1_R) \otimes_R m) = \eta_M(E(\varphi(1_R))m) \\ &= 1_S \otimes_R E(\varphi(1_R))m = 1_S E(\varphi(1_R)) \otimes_R m = \varphi(E(\varphi(1_R))) \otimes_R m \\ &\overset{\varphi E\varphi=\varphi}{=} \varphi(1_R) \otimes_R m = 1_S \otimes_R m = \eta_M(m).\end{aligned}$$

Now, note that, since E is a morphism of R-bimodules, we get $(\varphi \circ E \circ \varphi)(r) = \varphi(E(\varphi(r))) = \varphi(E(r1_S)) = \varphi(rE(1_S)) = \varphi(r)\varphi E(1_S)$. As a consequence, the condition $(\varphi \circ E \circ \varphi)(r) = \varphi(r)$ is equivalent to $\varphi E(1_S) = 1_S$. $\square$

We now give an example of a semiseparable functor which is neither separable nor naturally full.

Example 3.2. Let $\varphi : R \to S$ and $\psi : Q \to R$ be morphisms of rings whose induction functors φ^* and ψ^* are separable and naturally full respectively. This means there is

$E \in {}_R\mathrm{Hom}_R(S,R)$ such that $E \circ \varphi = \mathrm{Id}$ (in particular φ is injective) and there is $D \in {}_Q\mathrm{Hom}_Q(R,Q)$ such that $\psi \circ D = \mathrm{Id}$ (in particular ψ is surjective). By Corollary 1.16, the composition $\varphi^* \circ \psi^* \cong (\varphi \circ \psi)^*$ is semiseparable. The map corresponding to $\varphi \circ \psi$ via Proposition 3.1 is $D \circ E \in {}_Q\mathrm{Hom}_Q(S,Q)$. Note that, if $\varphi \circ \psi$ is neither injective nor surjective, we can conclude that $(\varphi \circ \psi)^*$ is neither separable nor naturally full. For instance, let $\varphi : \mathbb{Q} \to \mathbb{Q}[X]$ be the canonical injection of the field of rational numbers into the polynomial ring over it and let $\psi : \mathbb{Q} \times \mathbb{Z} \to \mathbb{Q}$ be given by $\psi((q,z)) = q$. Then we can define D by setting $D(q) = (q,0)$ and E to be the evaluation at 0 of the given polynomial. Then $(\varphi \circ \psi)^*$ is semiseparable but it is neither separable nor naturally full.

In a similar way as in Example 3.2, the following example shows that semiseparable functors are not closed under composition.

Example 3.3. Let $\varphi : R \to S$ and $\psi : S \to Q$ be morphisms of rings whose induction functors φ^* and ψ^* are separable and naturally full respectively (in particular both semiseparable by Proposition 1.3). This means there is $E \in {}_R\mathrm{Hom}_R(S,R)$ such that $E \circ \varphi = \mathrm{Id}$ and there is $D \in {}_S\mathrm{Hom}_S(Q,S)$ such that $\psi \circ D = \mathrm{Id}$. The results we have proved so far do not allow us to conclude that the composition $\psi^* \circ \varphi^* \cong (\psi \circ \varphi)^*$ is semiseparable. Indeed we can provide a specific example where this is not true. Let $\varphi : \mathbb{Z} \to \mathbb{Q} \times \mathbb{Z}, z \mapsto (z,z)$, and let $\psi : \mathbb{Q} \times \mathbb{Z} \to \mathbb{Q}$ be given by $\psi((q,z)) = q$. Then we can define D by setting $D(q) = (q,0)$ and E by setting $E((q,z)) = z$. In this way we get that φ^* and ψ^* are separable and naturally full respectively. Let us show that $(\psi \circ \varphi)^*$ is not semiseparable. Otherwise, by Proposition 3.1 there exists $E' \in {}_\mathbb{Z}\mathrm{Hom}_\mathbb{Z}(\mathbb{Q},\mathbb{Z})$ such that $\psi\varphi(E'(1_\mathbb{Q})) = 1_\mathbb{Q}$. Since ${}_\mathbb{Z}\mathrm{Hom}_\mathbb{Z}(\mathbb{Q},\mathbb{Z}) = \{0\}$, this means $0_\mathbb{Z} = 1_\mathbb{Z}$, a contradiction.

Let us see that all morphisms of rings $\varphi : R \to S$ whose induction functor $\varphi^* := S \otimes_R (-) : R\text{-Mod} \to S\text{-Mod}$ is semiseparable are of the form given in Example 3.2. More precisely, we will get that φ^* factors as a bireflection followed by a separable functor. First we need the next remark.

Remark 3.4. Let $\varphi : R \to S$ be an epimorphism in the category of rings. By [36, Proposition 1.2] the faithful functor φ_* is also full, and hence its left adjoint $\varphi^* = S \otimes_R (-)$ is a reflection, whereas its right adjoint $\varphi^! = {}_R\mathrm{Hom}(S,-)$ is a coreflection. Thus, Theorem 2.24 applies in this case to get that φ^* is naturally full if and only if it is semiseparable if and only if it is Frobenius, that is, in the same way $\varphi^!$ is naturally full if and only if it is semiseparable if and only if it is Frobenius. In particular, in this case φ^* and $\varphi^!$ are isomorphic bireflections.

Proposition 3.5. *Let $\varphi : R \to S$ be a morphism of rings. Write $\varphi = \iota \circ \overline{\varphi}$ where $\iota : \varphi(R) \to S$ is the canonical inclusion and $\overline{\varphi} : R \to \varphi(R)$ is the corestriction of φ to its image $\varphi(R)$.*

Then, the induction functor $\varphi^* := S \otimes_R (-) : R\text{-Mod} \to S\text{-Mod}$ is semiseparable if and only if ι^* is separable and $\overline{\varphi}^*$ is a bireflection.

Proof. If φ^* is semiseparable, by Proposition 3.1, there is $E \in {}_R\mathrm{Hom}_R(S, R)$ such that $\varphi \circ E \circ \varphi = \varphi$, i.e. $\iota \circ \overline{\varphi} \circ E \circ \iota \circ \overline{\varphi} = \iota \circ \overline{\varphi}$. Since ι is injective and $\overline{\varphi}$ is surjective, we get $\overline{\varphi} \circ E \circ \iota = \mathrm{Id}_{\varphi(R)}$ which implies that ι^* is separable. On the other hand, $\overline{\varphi}^*$ is a bireflection in view of Remark 3.4 and surjectivity of $\overline{\varphi}$. Conversely, if ι^* is separable and $\overline{\varphi}^*$ is a bireflection, whence naturally full, then the composition $\iota^* \circ \overline{\varphi}^* \cong (\iota \circ \overline{\varphi})^* = \varphi^*$ is semiseparable by Corollary 1.16. □

In the proof of Proposition 3.5 we obtained a factorization $\iota^* \circ \overline{\varphi}^* \cong \varphi^*$ in case φ^* is semiseparable. In view of Corollary 2.28, this factorization is the same given by the coidentifier within Theorem 1.15, up to a category equivalence.

Next proposition provides a further characterization of the semiseparability of φ^*. We point out that its current proof, more direct than the original one, was suggested by P. Saracco.

First recall that, given a central idempotent element z in a ring R, then $zRz = Rz$ is a ring with addition and multiplication those of R restricted to zR and with identities $0_{Rz} = 0_R z = 0_R$ and $1_{Rz} = 1_R z = z$, and there is a surjective ring homomorphism $R \to Rz, r \mapsto rz$, see [1, 1.16].

Proposition 3.6. *Let $\varphi : R \to S$ be a ring homomorphism. Then, the induction functor $\varphi^* = S \otimes_R (-) : R\text{-Mod} \to S\text{-Mod}$ is semiseparable if and only if there is a central idempotent $z \in R$ (necessarily unique) such that $\varphi(z) = 1_S$ and the ring map $\tau := \varphi_{|Rz} : Rz \to S$ is split-mono as an Rz-bimodule map.*

Proof. By Proposition 3.1, the functor φ^* is semiseparable if and only if there exists $E \in {}_R\mathrm{Hom}_R(S, R)$ such that $\varphi E(1_S) = 1_S$. Assume that there is E as above and set $z := E(1_S) \in R$. Clearly $\varphi(z) = \varphi E(1_S) = 1_S$, i.e. $\varphi(z) = 1_S$. For any $r \in R$ we have $rz = rE(1_S) = E(\varphi(r)1_S) = E(\varphi(r))$ and similarly $zr = E(\varphi(r))$ so that $rz = zr$, i.e. z is central. Taking $z = r$ in the computation above, we get $zz = E(\varphi(z)) = E(1_S) = z$, thus z is an idempotent. Concerning the ring map $\tau := \varphi_{|Rz} : Rz \to S$, consider the canonical projection $\psi : R \to Rz, r \mapsto rz$. Since z is central, we get that ψ is R-bilinear so that the map $\pi := \psi \circ E : S \to Rz, s \mapsto E(s)z$ is R-bilinear as a composition of bilinear maps. In particular π is Rz-bilinear. Then, for any $r \in R$ we have $\pi\tau(rz) = \pi\varphi(rz) = \pi(\varphi(r)\varphi(z)) = \pi(r1_S) = r\pi(1_S) = rE(1_S)z = rzz = rz$, hence $\pi \circ \tau = \mathrm{Id}_{Rz}$ i.e. τ is a split-monomorphism of Rz-bimodules.

Conversely, assume there is a central idempotent $z \in S$ such that $\varphi(z) = 1_S$ and the ring map $\tau := \varphi_{|Rz} : Rz \to S$ is split-mono through a Rz-bimodule map $\pi : S \to Rz$. Set $E : S \to R, s \mapsto \pi(s)$. Then $rs = \varphi(r)s = \varphi(r)\varphi(z)s = \varphi(rz)s = (rz)s$ so that $E(rs) = E((rz)s) = \pi((rz)s) = rz\pi(s) = r\pi(s) = rE(s)$ where the second-last equality follows from the fact that $\pi(s) \in Rz$ and z is a central idempotent. Similarly one gets $E(sr) = E(s)r$ so that E is R-bilinear. Finally, we have $E(1_S) = \pi(1_S) = \pi(\varphi(z)) = \pi(\tau(z)) = z$ and hence $\varphi E(1_S) = \varphi(z) = 1_S$. Assume there is another idempotent $z' \in R$

as in the statement. Then $zz' = E(1_S)z' = E(1_S z') = E(\varphi(z')) = E(1_S) = z$. By exchanging the roles of z and z' we get $z'z = z'$ and hence $z = z'$. $\square$

Remark 3.7. In the proof of Proposition 3.6 we considered the maps $\tau := \varphi_{|Rz} : Rz \to S$ and $\psi : R \to Rz, r \mapsto rz$. Since $\varphi(z) = 1_S$, we get the equality $\varphi = \tau \circ \psi$ which provides the factorization $\varphi^* \cong \tau^* \circ \psi^*$. On the other hand, in Proposition 3.5 we obtained the equality $\varphi = \iota \circ \overline{\varphi}$, where $\iota : \varphi(R) \to S, s \mapsto s$, and $\overline{\varphi} : R \to \varphi(R), r \mapsto \varphi(r)$, which yields the factorization $\varphi^* \cong \iota^* \circ \overline{\varphi}^*$. Define the morphism $\lambda : Rz \to \varphi(R), rz \mapsto \varphi(r)$. Clearly the following diagrams commute.

$$\begin{tikzcd}
R \arrow[d, "\varphi"'] \arrow[dr, two heads, "\psi"] \arrow[drr, two heads, bend left, "\overline{\varphi}"] & & \\
& Rz \arrow[r, dotted, two heads, "\lambda"] \arrow[dl, hook, "\tau"'] & \varphi(R) \arrow[dll, hook, bend left, "\iota"] \\
S & &
\end{tikzcd}
\qquad
\begin{tikzcd}
R\text{-Mod} \arrow[dd, "\varphi^*"'] \arrow[dr, "\psi^*"] \arrow[drr, bend left, "\overline{\varphi}^*"] & & \\
& Rz\text{-Mod} \arrow[r, "\lambda^*"] \arrow[dl, "\tau^*"'] & \varphi(R)\text{-Mod} \arrow[dll, bend left, "\iota^*"] \\
S\text{-Mod} & &
\end{tikzcd}$$

The first diagram entails that λ is both injective and surjective whence bijective. As a consequence the given factorizations are the same up to the equivalence λ^*.

3.2. Coinduction and corestriction of coscalars

Let $\Bbbk$ be a field. We simply denote the tensor product over $\Bbbk$ by the unadorned $\otimes$. A $\Bbbk$-*coalgebra* C is a vector space C over $\Bbbk$ equipped with two $\Bbbk$-linear maps $\Delta_C : C \to C \otimes C$ and $\varepsilon_C : C \to \Bbbk$ such that Δ_C is coassociative and counital, i.e. the equalities

$$(\Delta_C \otimes C) \circ \Delta_C = (C \otimes \Delta_C) \circ \Delta_C \quad \text{and} \quad (\varepsilon_C \otimes C) \circ \Delta_C = (C \otimes_{\Bbbk} \varepsilon_C) \circ \Delta_C = C$$

hold true. A right C-comodule M is a $\Bbbk$-vector space together with a $\Bbbk$-linear map $\rho_M : M \to M \otimes C$, called the *coaction*, that is coassociative and right counital i.e.

$$(\rho_M \otimes C) \circ \rho_M = (M \otimes \Delta_C) \circ \rho_M \quad \text{and} \quad (M \otimes \varepsilon_C) \circ \rho_M = M.$$

A coalgebra C can be seen as a right C-comodule with $\rho_C = \Delta_C$. Both for Δ and ρ_M we adopt the usual Sweedler notations $\Delta(c) = \sum c_1 \otimes c_2$ and $\rho_M(m) = \sum m_0 \otimes m_1$ for every $c \in C, m \in M$. A *morphism of right C-comodules* (or a C-*colinear* morphism) is a $\Bbbk$-linear map $f : M \to N$ between right C-comodules such that $\rho_N \circ f = (f \otimes C) \circ \rho_M$. The category of right C-comodules and their morphisms is denoted by $\mathcal{M}^C$. Analogously, the category of left C-comodules and their morphisms is denoted by ${}^C\mathcal{M}$. Recall from [39] that, given a right C-comodule M and a left C-comodule N, the *cotensor product* $M \square_C N$ is the kernel of the $\Bbbk$-linear map

$$\rho_M \otimes N - M \otimes \rho_N : M \otimes N \to M \otimes C \otimes N,$$

where ρ_M and ρ_N are the right and the left C-comodule structures of M and N, respectively.

Now, let $\psi : C \to D$ be a *morphism of coalgebras*, i.e. a $\Bbbk$-linear map $\psi : C \to D$ such that $\Delta_D \circ \psi = (\psi \otimes \psi) \circ \Delta_C$ and $\varepsilon_D \circ \psi = \varepsilon_C$. Since any right C-comodule M with coaction $\rho_M : M \to M \otimes C$ can be viewed as a right D-comodule with coaction $(M \otimes \psi) \circ \rho_M : M \to M \otimes D$ and C can be considered as a (D, C)-bicomodule, ψ induces

- the corestriction of coscalars functor $\psi_* : \mathcal{M}^C \to \mathcal{M}^D$,
- the coinduction functor $\psi^* := (-)\Box_D C : \mathcal{M}^D \to \mathcal{M}^C$,

which form an adjunction $\psi_* \dashv \psi^* : \mathcal{M}^D \to \mathcal{M}^C$, with unit $\eta : \mathrm{Id}_{\mathcal{M}^C} \to \psi^*\psi_*$ and counit $\epsilon : \psi_*\psi^* \to \mathrm{Id}_{\mathcal{M}^D}$, given by

$$\eta_M : M \to M\Box_D C,\ m \mapsto \sum m_0 \Box_D m_1, \quad \text{and} \quad \epsilon_N : N\Box_D C \to N,\ n\Box_D c \mapsto n\varepsilon_C(c),$$

for any $M \in \mathcal{M}^C$ and $N \in \mathcal{M}^D$, see [13, 11.10]. Note that, in the definition of ψ^*, for any right D-comodule N, the cotensor product $N\Box_D C$ is regarded as a right C-comodule via $\rho_{N\Box_D C} : N\Box_D C \to (N\Box_D C) \otimes C, n\Box_D c \mapsto \sum (n\Box_D c_1) \otimes c_2$. Furthermore, the coaction ρ_M of M as a right C-comodule induces a morphism of right C-comodules $\bar{\rho}_M = \eta_M : M \to M\Box_D C$ such that $\rho_M = i \circ \bar{\rho}_M$, where $i : M\Box_D C \to M \otimes C$ is the canonical inclusion. In particular, if $M = C$ then $\bar{\rho}_C = \bar{\Delta}_C = \eta_C : C \to C\Box_D C$. It is known that

- ψ_* is separable if and only if the canonical morphism $\bar{\Delta}_C : C \to C\Box_D C$ is split-mono as a C-bicomodule map, see [16, Theorem 2.4];
- ψ_* is naturally full if and only if $\bar{\Delta}_C$ is split-epi as a C-bicomodule map, see [4, Examples 3.23 (1)];
- ψ^* is separable if and only if ψ is split-epi as a D-bicomodule map, see [16, Theorem 2.7];
- ψ^* is naturally full if and only if ψ is split-mono as a D-bicomodule map, see [4, Examples 3.23 (1)].

Since ψ_* is faithful, we have that ψ_* is semiseparable if and only if it is separable. The semiseparability of ψ^* is investigated in the following result.

Proposition 3.8. *Let $\psi : C \to D$ be a morphism of coalgebras. Then, the coinduction functor $\psi^* = (-)\Box_D C : \mathcal{M}^D \to \mathcal{M}^C$ is semiseparable if and only if ψ is a regular morphism of D-bicomodules if and only if there is a D-bicomodule morphism $\chi : D \to C$ such that $\varepsilon_C \circ \chi \circ \psi = \varepsilon_C$.*

Proof. Assume that ψ^* is semiseparable. By Theorem 2.1, there exists a natural transformation $\gamma : \mathrm{Id}_{\mathcal{M}^D} \to \psi_*\psi^*$ such that $\epsilon_N \circ \gamma_N \circ \epsilon_N = \epsilon_N$, for any $N \in \mathcal{M}^D$. Since D is a right D-comodule, consider the right D-comodule map $\gamma_D : D \to D\Box_D C$ and define the map $\chi : D \to C$ as $\chi := l_C \circ \gamma_D$, where $l_C : D\Box_D C \to C, \sum_i d_i \otimes c_i \mapsto \sum_i \varepsilon_D(d_i)c_i$, is the canonical isomorphism. Note that ψ is a morphism of D-bicomodules and $\psi = \epsilon_D \circ l_C^{-1}$. We show that χ is a morphism of D-bicomodules. For any $f \in D^* = \mathrm{Hom}_{\Bbbk}(D, \Bbbk)$, consider the morphism of right D-comodules $\hat{f} : N \to D$, $n \mapsto \sum f(n_0)n_1$. For any $n \in N$, denote $\gamma(n)$ by $\sum_i n_i \otimes c_i$. Then, by naturality of γ, we have that $l_C\gamma_D\hat{f}(n) = \chi\hat{f}(n) = \sum f(n_0)\chi(n_1)$ is equal to $l_C(\hat{f}\Box_D C)\gamma_N(n) = l_C(\hat{f}\Box_D C)(\sum_i n_i \otimes c_i) = l_C(\sum_i f(n_{i_0})n_{i_1} \otimes c_i) = \sum_i f(n_i)c_i$. Since f is arbitrary, it follows that for any $N \in \mathcal{M}^D$ and for all $n \in N$, $\gamma_N(n) = \sum_i n_i \otimes c_i = \sum n_0 \otimes \chi(n_1)$. In particular, consider $\gamma_D : D \to D\Box_D C$. Since $\sum d_1 \otimes \chi(d_2) = \gamma_D(d) \in D\Box_D C$, we have $\sum d_1 \otimes d_2 \otimes \chi(d_3) = \sum d_1 \otimes \psi(\chi(d_2)_1) \otimes \chi(d_2)_2$. If we apply on both sides $\varepsilon_D \otimes \mathrm{Id}$, we get $\sum d_1 \otimes \chi(d_2) = \sum \psi(\chi(d)_1) \otimes \chi(d)_2$ which means that χ is a morphism of left D-comodules whence of D-bicomodules. Moreover, we have $\psi \circ \chi \circ \psi = (\epsilon_D \circ l_C^{-1}) \circ (l_C \circ \gamma_D) \circ (\epsilon_D \circ l_C^{-1}) = \epsilon_D \circ \gamma_D \circ \epsilon_D \circ l_C^{-1} = \epsilon_D \circ l_C^{-1} = \psi$, hence χ is a regular morphism of D-bicomodules.

Assume that ψ is a regular morphism of D-bicomodules, i.e. there is a D-bicomodule morphism $\chi : D \to C$ such that $\psi\circ\chi\circ\psi = \psi$. Then $\varepsilon_C\circ\chi\circ\psi = \varepsilon_D\circ\psi\circ\chi\circ\psi = \varepsilon_D\circ\psi = \varepsilon_C$.

Assume now there is a D-bicomodule morphism $\chi : D \to C$ such that $\varepsilon_C \circ \chi \circ \psi = \varepsilon_C$ and let us prove that ψ^* is semiseparable. For any $N \in \mathcal{M}^D$ define $\gamma_N : N \to N\Box_D C$ as $\gamma_N(n) = \sum n_0 \otimes \chi(n_1)$, for every $n \in N$. Using that χ is a left D-comodule morphism, one easily checks that the image of γ_N is really contained in $N\Box_D C$. Moreover γ_N comes out to be a right D-comodule morphism, since χ is a morphism of right D-comodules, and natural in N. For every $n \in N$, $c \in C$, we have $\gamma_N\epsilon_N(n\Box_D c) = \gamma_N(n\varepsilon_C(c)) = \gamma_N(n)\varepsilon_C(c) = \sum n_0 \otimes \chi(n_1)\varepsilon_C(c) = \sum n \otimes \chi\psi(c_1)\varepsilon_C(c_2) = \sum n \otimes \chi\psi(c)$, where in the second-last equality we used that $n\Box_D c$ belongs to $N\Box_D C$. Thus $\epsilon_N\gamma_N\epsilon_N(n\Box_D c) = \epsilon_N(\sum n \otimes \chi\psi(c)) = \sum n\varepsilon_C\chi\psi(c) = n\varepsilon_C(c) = \epsilon_N(n\Box_D c)$. Therefore, by Theorem 2.1, ψ^* is semiseparable. $\square$

Example 3.9. It is known that the Axiom of Choice is equivalent to require that, for any function $f : A \to B$, there is a function $g : B \to A$ such that $f \circ g \circ f = f$. Consider the group-like coalgebras $\Bbbk A$ and $\Bbbk B$ and the coalgebra map $\psi := \Bbbk f : \Bbbk A \to \Bbbk B$ defined by setting $\psi(a) = f(a)$, for every $a \in A$. Define also the linear map $\chi : \Bbbk B \to \Bbbk A$ by setting $\chi(b) = g(b)$ if $b \in \mathrm{Im}(f)$ and $\chi(b) = 0$ otherwise, for all $b \in B$. It is easy to check that χ is a $\Bbbk B$-bicomodule morphism such that $\varepsilon_{\Bbbk A} \circ \chi \circ \psi = \varepsilon_{\Bbbk A}$. Thus, by Proposition 3.8, the functor $\psi^* = (-)\Box_{\Bbbk B}\Bbbk A : \mathcal{M}^{\Bbbk B} \to \mathcal{M}^{\Bbbk A}$ is semiseparable. However it is neither separable nor naturally full in general. Indeed, if ψ^* is separable, then ψ is split-epi whence surjective. In this case f must be surjective too. Similarly, if ψ^* is naturally full, then ψ is split-mono whence injective. In this case f must be injective too.

3.3. Corings

Let R be a ring. Recall that an *R-coring* [38] is an R-bimodule $\mathcal{C}$ together with R-bimodule maps $\Delta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_R \mathcal{C}$ and $\varepsilon_{\mathcal{C}} : \mathcal{C} \to R$, called the comultiplication and the counit, respectively, such that $\Delta_{\mathcal{C}}$ is coassociative and counital similarly to the case of coalgebras. Given an R-coring $\mathcal{C}$, a *right $\mathcal{C}$-comodule* is a right R-module M together with a right R-linear map $\rho_M : M \to M \otimes_R \mathcal{C}$, called the coaction, that is coassociative and right counital. A map between right $\mathcal{C}$-comodules is defined in the expected way. Let $\mathcal{M}^{\mathcal{C}}$ denote the category of right $\mathcal{C}$-comodules and consider the induction functor

$$G := (-) \otimes_R \mathcal{C} : \text{Mod-}R \to \mathcal{M}^{\mathcal{C}}, \quad M \mapsto M \otimes_R \mathcal{C}, \quad f \mapsto f \otimes_R \mathcal{C},$$

which is the right adjoint of the forgetful functor $F : \mathcal{M}^{\mathcal{C}} \to \text{Mod-}R$, see e.g. [12, Lemma 3.1]. The right $\mathcal{C}$-comodule structure of $M \otimes_R \mathcal{C}$ is given by $M \otimes_R \Delta_{\mathcal{C}}$. The unit and counit of the adjunction are given by $\eta_M = \rho_M : M \to M \otimes_R \mathcal{C}$, for every $M \in \mathcal{M}^{\mathcal{C}}$, and $\epsilon_N = N \otimes_R \varepsilon_{\mathcal{C}} : N \otimes_R \mathcal{C} \to N$, $\epsilon_N(n \otimes_R c) = n\varepsilon_{\mathcal{C}}(c)$, for every $N \in \text{Mod-}R$, $n \in N$, $c \in \mathcal{C}$, respectively.

Denote by $\mathcal{C}^R = \{c \in \mathcal{C} \mid rc = cr,\ \forall r \in R\}$ the set of invariant elements in $\mathcal{C}$.

Concerning the separability and natural fullness of F and G we know that

- F is separable if and only if the coring $\mathcal{C}$ is coseparable, see [12, Corollary 3.6];
- F is naturally full if and only if $\Delta_{\mathcal{C}}$ is surjective, see [4, Proposition 3.13];
- G is separable if and only if there exists an invariant element $z \in \mathcal{C}^R$ such that $\varepsilon_{\mathcal{C}}(z) = 1_R$, see [12, Theorem 3.3]; if G is separable, the coring $\mathcal{C}$ is said to be *cosplit* [13, 26.12];
- G is naturally full if and only if there exists an invariant element $z \in \mathcal{C}^R$ such that $c = \varepsilon_{\mathcal{C}}(c)z$, for every $c \in \mathcal{C}$, see [4, Proposition 3.13].

Since $F : \mathcal{M}^{\mathcal{C}} \to \text{Mod-}R$ is faithful, by Proposition 1.3 (i), it is semiseparable if and only if it is separable so, although we are led to name a coring "semicoseparable" whenever F is semiseparable, this would bring us back to the notion of coseparable coring. Let us study when the induction functor $G = (-) \otimes_R \mathcal{C} : \text{Mod-}R \to \mathcal{M}^{\mathcal{C}}$ is semiseparable. In this case, we say that the R-coring $\mathcal{C}$ is **semicosplit**. Note that, since separable functors are in particular semiseparable, it is obvious that cosplit corings are in particular semicosplit.

Theorem 3.10. *Let $\mathcal{C}$ be an R-coring. Then, the following are equivalent.*

(1) $\mathcal{C}$ is semicosplit.
(2) The coring counit $\varepsilon_{\mathcal{C}} : \mathcal{C} \to R$ is regular as a morphism of R-bimodules.
(3) There exists an invariant element $z \in \mathcal{C}^R$ such that $\varepsilon_{\mathcal{C}}(z)\varepsilon_{\mathcal{C}}(c) = \varepsilon_{\mathcal{C}}(c)$ (equivalently such that $\varepsilon_{\mathcal{C}}(z)c = c$), for every $c \in \mathcal{C}$.

Proof. (1) ⇒ (2). Assume that $\mathcal{C}$ is semicosplit, i.e. the induction functor $G = (-) \otimes_R \mathcal{C}$ is semiseparable. Then, by Theorem 2.1, there exists a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $\epsilon \circ \gamma \circ \epsilon = \epsilon$. Consider the canonical isomorphism $l_{\mathcal{C}} : R \otimes_R \mathcal{C} \to \mathcal{C}$. Since

R is a right R-module, consider the right R-linear map $\gamma_R : R \to R \otimes_R \mathcal{C}$. Let us check it is also left R-linear. For any $r \in R$ define the morphism $f_r : R \to R$ by $f_r(r') = rr'$. Since γ_R is natural, we have

$$\gamma_R(rr') = (\gamma_R \circ f_r)(r') = ((f_r \otimes_R \mathcal{C}) \circ \gamma_R)(r') = r\gamma_R(r').$$

Thus γ_R is a morphism of R-bimodules. Define the R-bimodule map $\alpha := l_{\mathcal{C}} \circ \gamma_R : R \to \mathcal{C}$. By noting that $\varepsilon_{\mathcal{C}} = \epsilon_R \circ l_{\mathcal{C}}^{-1}$, we get

$$\varepsilon_{\mathcal{C}} \circ \alpha \circ \varepsilon_{\mathcal{C}} = (\epsilon_R \circ l_{\mathcal{C}}^{-1}) \circ (l_{\mathcal{C}} \circ \gamma_R) \circ (\epsilon_R \circ l_{\mathcal{C}}^{-1}) = \epsilon_R \circ \gamma_R \circ \epsilon_R \circ l_{\mathcal{C}}^{-1} = \epsilon_R \circ l_{\mathcal{C}}^{-1} = \varepsilon_{\mathcal{C}}$$

so that $\varepsilon_{\mathcal{C}}$ is a regular morphism of R-bimodules.

(2) $\Rightarrow$ (3). Assuming the regularity of $\varepsilon_{\mathcal{C}}$, i.e. the existence of an R-bimodule map α such that $\varepsilon_{\mathcal{C}} \circ \alpha \circ \varepsilon_{\mathcal{C}} = \varepsilon_{\mathcal{C}}$, we can set $z = \alpha(1_R) \in \mathcal{C}$. For $r \in R$, we have $rz = r\alpha(1_R) = \alpha(r) = \alpha(1_R)r = zr$ so that z is in $\mathcal{C}^R$. Moreover, from $\varepsilon_{\mathcal{C}}(c) = \varepsilon_{\mathcal{C}}\alpha\varepsilon_{\mathcal{C}}(c) = \varepsilon_{\mathcal{C}}\alpha(1_R\varepsilon_{\mathcal{C}}(c)) = \varepsilon_{\mathcal{C}}\alpha(1_R)\varepsilon_{\mathcal{C}}(c) = \varepsilon_{\mathcal{C}}(z)\varepsilon_{\mathcal{C}}(c)$ it follows that $\varepsilon_{\mathcal{C}}(c) = \varepsilon_{\mathcal{C}}(z)\varepsilon_{\mathcal{C}}(c)$, for every $c \in \mathcal{C}$.

(3) $\Rightarrow$ (1). Suppose there exists $z \in \mathcal{C}^R$ such that $\varepsilon_{\mathcal{C}}(c) = \varepsilon_{\mathcal{C}}(z)\varepsilon_{\mathcal{C}}(c)$, for every $c \in \mathcal{C}$. For any $N \in$ Mod-R define $\gamma_N : N \to N \otimes_R \mathcal{C}$, $\gamma_N(n) = n \otimes_R z$, for every $n \in N$. Since $z \in \mathcal{C}^R$, for every $n \in N$, $r \in R$, we have $\gamma_N(nr) = nr \otimes_R z = n \otimes_R rz = n \otimes_R zr = \gamma_N(n)r$, so γ_N is a right R-module morphism, and it is also natural in N: indeed, for any morphism $f : N \to M$ in Mod-R, $(\gamma_M \circ f)(n) = f(n) \otimes_R z = ((f \otimes_R \mathcal{C}) \circ \gamma_N)(n)$. Moreover, for every $n \in N$, $c \in \mathcal{C}$, we have

$$\begin{aligned}(\epsilon_N \circ \gamma_N \circ \epsilon_N)(n \otimes_R c) &= \epsilon_N\gamma_N(n\varepsilon_{\mathcal{C}}(c)) = \epsilon_N(n \otimes_R \varepsilon_{\mathcal{C}}(c)z) = n\varepsilon_{\mathcal{C}}(z)\varepsilon_{\mathcal{C}}(c) \overset{(*)}{=} n\varepsilon_{\mathcal{C}}(c) \\ &= \epsilon_N(n \otimes_R c),\end{aligned}$$

where $(*)$ follows from the assumption $\varepsilon_{\mathcal{C}}(c) = \varepsilon_{\mathcal{C}}(z)\varepsilon_{\mathcal{C}}(c)$. Therefore, by Theorem 2.1 G is semiseparable and $\mathcal{C}$ is semicosplit.

Finally, assume that $\varepsilon_{\mathcal{C}}(z)\varepsilon_{\mathcal{C}}(c) = \varepsilon_{\mathcal{C}}(c)$, for every $c \in \mathcal{C}$. Then $\varepsilon_{\mathcal{C}}(z)c = \varepsilon_{\mathcal{C}}(z)\varepsilon_{\mathcal{C}}(c_{(1)})c_{(2)} = \varepsilon_{\mathcal{C}}(c_{(1)})c_{(2)} = c$ and hence $\varepsilon_{\mathcal{C}}(z)c = c$. Conversely, if $\varepsilon_{\mathcal{C}}(z)c = c$, for every $c \in \mathcal{C}$, then $\varepsilon_{\mathcal{C}}(z)\varepsilon_{\mathcal{C}}(c) = \varepsilon_{\mathcal{C}}(\varepsilon_{\mathcal{C}}(z)c) = \varepsilon_{\mathcal{C}}(c)$. □

Remark 3.11. At the beginning of Subsection 3.3 we mentioned that the functor $G := (-) \otimes_R \mathcal{C} : \text{Mod-}R \to \mathcal{M}^{\mathcal{C}}$ is naturally full if and only if there exists $z \in \mathcal{C}^R$ such that $c = \varepsilon_{\mathcal{C}}(c)z$, for every $c \in \mathcal{C}$. We expect this characterization to be a particular case of Theorem 3.10, as a naturally full functor is semiseparable by Proposition 1.3 (ii). Indeed, if there exists $z \in \mathcal{C}^R$ such that $c = z\varepsilon_{\mathcal{C}}(c)$ for every $c \in \mathcal{C}$, then $\varepsilon_{\mathcal{C}}(c) = \varepsilon_{\mathcal{C}}(z\varepsilon_{\mathcal{C}}(c)) = \varepsilon_{\mathcal{C}}(z)\varepsilon_{\mathcal{C}}(c)$, for every $c \in \mathcal{C}$, and equivalently, $\varepsilon_{\mathcal{C}}(z)c = \varepsilon_{\mathcal{C}}(z)z\varepsilon_{\mathcal{C}}(c) = \varepsilon_{\mathcal{C}}(z\varepsilon_{\mathcal{C}}(c))z = \varepsilon_{\mathcal{C}}(c)z = c$. Analogously, we recalled that G is separable if and only if there exists an invariant element $z \in \mathcal{C}^R$ such that $\varepsilon_{\mathcal{C}}(z) = 1_R$ and hence the equality $\varepsilon_{\mathcal{C}}(c) = \varepsilon_{\mathcal{C}}(z)\varepsilon_{\mathcal{C}}(c)$ trivially holds true in this case.

Next aim is to show that, when G is semiseparable, then we can provide an explicit factorization of it as a bireflection followed by a separable functor. By Corollary 2.28, this factorization amounts to the one given by the coidentifier, up to a category equivalence.

Corollary 3.12. *Let $\mathcal{C}$ be an R-coring. Then, $\mathcal{C}$ is semicosplit if and only if the induction functor $G = (-) \otimes_R \mathcal{C} : \text{Mod-}R \to \mathcal{M}^{\mathcal{C}}$ factors up to isomorphism as $\psi_* \circ G'$ where $\psi_* = (-)\square_I \mathcal{C} : \mathcal{M}^I \to \mathcal{M}^{\mathcal{C}}$ is separable and $G' = (-) \otimes_R I : \text{Mod-}R \to \mathcal{M}^I$ is a bireflection for some morphism of corings $\psi : \mathcal{C} \to I$.*

Proof. Assume $\mathcal{C}$ is semicosplit, i.e. $G = (-) \otimes_R \mathcal{C} : \text{Mod-}R \to \mathcal{M}^{\mathcal{C}}$ is semiseparable. Then, by Theorem 3.10, there exists an invariant element $z_{\mathcal{C}} \in \mathcal{C}^R$ such that $\varepsilon_{\mathcal{C}}(c) = \varepsilon_{\mathcal{C}}(z_{\mathcal{C}})\varepsilon_{\mathcal{C}}(c)$, for every $c \in \mathcal{C}$. We observe that, since $\varepsilon_{\mathcal{C}}$ is a morphism of bimodules, $I := \mathrm{Im}(\varepsilon_{\mathcal{C}})$ is an ideal of R with multiplicative identity $z := \varepsilon_{\mathcal{C}}(z_{\mathcal{C}})$. Indeed, for any $r \in I$ there is $c \in \mathcal{C}$ such that $r = \varepsilon_{\mathcal{C}}(c)$ and hence $rz = \varepsilon_{\mathcal{C}}(c)\varepsilon_{\mathcal{C}}(z_{\mathcal{C}}) = \varepsilon_{\mathcal{C}}(c) = r$. Therefore the morphism $\varphi : R \to I, r \mapsto rz$, is a ring epimorphism (in fact it is surjective) and hence the map $m_I : I \otimes_R I \to I$ is bijective, see [36, Proposition XI.1.2]. Thus we can consider $\Delta_I = m_I^{-1} : I \to I \otimes_R I$, $\Delta_I(i) = i \otimes_R z = z \otimes_R i$, so that $(I, \Delta_I, \varepsilon_I)$ is an R-coring, where the counit $\varepsilon_I : I \hookrightarrow R$ is the canonical inclusion. Note that $\psi : \mathcal{C} \to I, c \mapsto \varepsilon_{\mathcal{C}}(c)$, is a morphism of corings and consider the corresponding coinduction functor $\psi_* = (-)\square_I \mathcal{C} : \mathcal{M}^I \to \mathcal{M}^{\mathcal{C}}$. We recall from [25] that, given M a $(\mathcal{C}', \mathcal{C})$-bicomodule and N a $(\mathcal{C}, \mathcal{C}'')$-bicomodule, where $\mathcal{C}', \mathcal{C}, \mathcal{C}''$ are corings over the rings R', R, R'', respectively, then $M \square_{\mathcal{C}} N$ is the kernel of the $(\mathcal{C}', \mathcal{C}'')$-bicomodule map

$$M \otimes_R N \xrightarrow{\omega_{M,N} := \rho_M \otimes_R N - M \otimes_R \lambda_N} M \otimes_R \mathcal{C} \otimes_R N,$$

where ρ_M and λ_N are the right and the left $\mathcal{C}$-comodule structures of M and N, respectively.

Consider also the induction functor $G' := (-) \otimes_R I : \text{Mod-}R \to \mathcal{M}^I$. Our aim is to prove that G factors as $G \cong \psi_* \circ G'$, that ψ_* is separable and G' is a bireflection.

First let us check that $G \cong \psi_* \circ G'$. In fact, for every $T \in \text{Mod-}R$,

$$(\psi_* \circ G')(T) = \psi_*(T \otimes_R I) = (T \otimes_R I)\square_I \mathcal{C} \overset{(*)}{\cong} T \otimes_R (I \square_I \mathcal{C}) \cong T \otimes_R \mathcal{C} = G(T),$$

where we note that the above isomorphism $(*)$ follows e.g. from [13, 10.6, page 95] once observed that $\lambda_{\mathcal{C}}(c) = \psi(c_{(1)}) \otimes_R c_{(2)} = z \otimes_R \varepsilon_{\mathcal{C}}(c_{(1)})c_{(2)} = z \otimes_R c$ and hence $\omega_{I,\mathcal{C}}(i \otimes_R c) = \rho_I(i) \otimes_R c - i \otimes_R \lambda_{\mathcal{C}}(c) = i \otimes_R z \otimes_R c - i \otimes_R z \otimes_R c = 0$ so that $\omega_{I,\mathcal{C}}$ is the zero map whence trivially T-*pure* [13, 40.13]. Let us check that G' is a bireflection. To this aim, first note that, since $i = \varepsilon_I(i)z$, for every $i \in I$, then G' is naturally full by the characterization of natural fullness of the induction functors we recalled at the beginning of the present subsection.

The functor G' is right adjoint of the forgetful functor F' and the unit is $\eta'_M = \rho_M : M \to M \otimes_R I$ for every (M, ρ_M) in $\mathcal{M}^I$. Since $I = Rz$, for every $m \in M$ there is $m' \in M$

such that $\rho_M(m) = m' \otimes_R z$ and hence $m = \sum m_0 \varepsilon_I(m_1) = \sum m_0 m_1 = m'z$. As a consequence $\rho_M(m) = m' \otimes_R z = m' \otimes_R zz = m'z \otimes_R z = m \otimes_R z$ for every $m \in M$. Now, given $w \in M \otimes_R I$, there is $m \in M$ such that $w = m \otimes_R z = \rho_M(m)$ and hence ρ_M is surjective. Since it is also split-mono via $r_M \circ (M \otimes_R \varepsilon_I)$, where $r_M : M \otimes_R R \to M$ is the canonical isomorphism, we get that $\eta'_M = \rho_M$ is invertible and hence F' is fully faithful. Hence G' is a naturally full coreflection thus a bireflection by Theorem 2.24.

It remains to check that ψ_* is separable. If we see $\mathcal{C}$ as an I-bicomodule with left structure $\lambda_{\mathcal{C}} : \mathcal{C} \to I \otimes_R \mathcal{C}, c \mapsto z \otimes c$, and right structure $\rho_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_R I, c \mapsto c \otimes z$, the map $\nu : I \to \mathcal{C}, i \mapsto iz_{\mathcal{C}} = z_{\mathcal{C}} i$, is an I-bicomodule morphism which satisfies $\psi \circ \nu = \mathrm{Id}_I$. Indeed, $\psi(\nu(i)) = \psi(iz_{\mathcal{C}}) = \varepsilon_{\mathcal{C}}(iz_{\mathcal{C}}) = i\varepsilon_{\mathcal{C}}(z_{\mathcal{C}}) = iz = i$. The existence of ν implies that $\psi_* : (-)\square_I \mathcal{C} : \mathcal{M}^I \to \mathcal{M}^C$ is separable by [25, Theorem 5.8] in case $A = B = R$ and $\mathcal{D} = I$ once we have checked its hypothesis, namely that both ${}_R R$ and ${}_R \mathcal{C}$ preserve the equalizer of $(\rho_M \otimes_R \mathcal{C}, M \otimes_R \lambda_{\mathcal{C}})$ for every (M, ρ_M) in $\mathcal{M}^I$. By the foregoing, for such an (M, ρ_M), one has $\rho_M(m) = m \otimes z$ so that $\omega_{M,\mathcal{C}}(m \otimes_R c) = \rho_M(m) \otimes_R c - m \otimes_R \lambda_{\mathcal{C}}(c) = m \otimes_R z \otimes_R c - m \otimes_R z \otimes_R c = 0$ so that $\omega_{M,\mathcal{C}} = \rho_M \otimes_R \mathcal{C} - M \otimes_R \lambda_{\mathcal{C}}$ is the zero map. Thus both ${}_R R$ and ${}_R \mathcal{C}$ trivially preserve the equalizer of $(\rho_M \otimes_R \mathcal{C}, M \otimes_R \lambda_{\mathcal{C}})$ for every (M, ρ_M) in $\mathcal{M}^I$ as desired. $\square$

Remark 3.13. Consider the R-coring I of Corollary 3.12. By construction it is also a ring with unit z. Since the comultiplication Δ_I of I is invertible, then I is a coseparable R-coring. Thus, by [8, Proposition 2.17] there is a category isomorphism between the category $\mathcal{M}^I$ of right comodules over the coring I and the category Mod-I of right modules over the ring I.

We already mentioned that a cosplit coring is always semicosplit. We now give a concrete example of a semicosplit coring $\mathcal{C}$ which is not cosplit.

Example 3.14. 1) Let $\varphi : R \to S$ be a morphism of rings such that the induction functor $\varphi^* = S \otimes_R (-)$ is naturally full. As recalled in Subsection 3.1, there exists $\varepsilon \in {}_R\mathrm{Hom}_R(S, R)$ such that $\varphi \circ \varepsilon = \mathrm{Id}_S$. Since, in particular, $\varphi : R \to S$ is an epimorphism in the category of rings, by [36, Proposition XI.1.2], the multiplication $m : S \otimes_R S \to S$ is bijective and hence we can set $\Delta := m^{-1}$ so that $\Delta(s) = s \otimes_R 1_S = 1_S \otimes_R s$. We compute

$$(\varepsilon \otimes_R S)\Delta(s) = (\varepsilon \otimes_R S)(1_S \otimes_R s) = \varepsilon(1_S) \otimes_R s = 1_R \otimes_R \varepsilon(1_S)s = 1_R \otimes_R \varphi\varepsilon(1_S)s = 1_R \otimes_R s$$

and similarly $(S \otimes_R \varepsilon)\Delta(s) = (S \otimes_R \varepsilon)(s \otimes_R 1_S) = s \otimes_R 1_S$. As a consequence (S, Δ, ε) is an R-coring. Now $\varepsilon(1_S)s = \varphi\varepsilon(1_S)s = 1_S s = s$ so that $z := 1_S \in S^R$ fulfills the conditions of Theorem 3.10 guaranteeing that the functor $G := (-) \otimes_R S : \text{Mod-}R \to \mathcal{M}^S$ is semiseparable and hence S is a semicosplit R-coring. Nevertheless S is not cosplit in general. In fact if G is separable, as observed at the beginning of the present subsection, there exists $w \in S^R$ such that $1_R = \varepsilon(w)$ and hence, for every $r \in R$, we have $r = r1_R =$

$r\varepsilon(w) = \varepsilon(rw)$ so that ε is surjective which, together with the condition $\varphi \circ \varepsilon = \mathrm{Id}_S$, implies that φ and ε are mutual inverses.

2) To get an example of 1) with φ not invertible, consider S and T rings, set $R := S \times T$, take $\varphi : R \to S, (s,t) \mapsto s$ and $\varepsilon : S \to R, s \mapsto (s,0)$. Then S is a semicosplit but not cosplit R-coring.

Example 3.15. Let R be a commutative ring and consider an idempotent ideal I of R, assumed to be a pure right R-submodule. We recall that a submodule N of an R-module M is said to be *pure* [13, 40.13] if the inclusion $N \hookrightarrow M$ remains injective after tensoring by any right R-module. Since I is pure, we get that the multiplication $m_I : I \otimes_R I \to I$, $m_I(a \otimes_R a') = aa'$, is injective as it is obtained as the composition $I \otimes_R I \overset{I\otimes_R \varepsilon_I}{\to} I \otimes_R R \overset{\cong}{\to} I$, where $\varepsilon_I : I \to R$ is the canonical inclusion. Since I is idempotent, i.e. $I^2 = I$, we get that m_I is also surjective whence bijective. Thus we can consider $\Delta_I = m_I^{-1} : I \to I \otimes_R I$ and write $\Delta_I(a) = \sum a_1 \otimes_R a_2$ by means of Sweedler's notation. Then $\sum \varepsilon_I(a_1)a_2 = \sum a_1 a_2 = m_I(\sum a_1 \otimes_R a_2) = m_I(\Delta_I(a)) = a$ and similarly $\sum a_1 \varepsilon_I(a_2) = a$ so that $(I, \Delta_I, \varepsilon_I)$ is an R-coring. Now, the condition in Theorem 3.10 for this coring is the existence of an element $z \in I^R$ such that $c = \varepsilon_I(z)c$ i.e., by definition of ε_I, the existence of $z \in I$ such that $c = zc = cz$ for every $c \in I$. This means that z is the multiplicative identity in I. This goes back to a particular case of the ideal I constructed in Corollary 3.12 by taking $\mathcal{C} = I$ and noting that $\mathrm{Im}(\varepsilon_I) = I$. Moreover, in Example 3.14 2) we can identify S with the idempotent ideal $I = S \times \{0\}$ of the ring $R = S \times S$, through the isomorphism $S \overset{\cong}{\to} I : s \to (s,0)$. In this case, we can take $z = (1,0)$ (note that $z \neq (1,1) = 1_R$) and $\Delta_I(x) := x \otimes_R z = z \otimes_R x$.

3.4. Bimodules

Let R and S be rings, and let ${}_R\mathcal{M}_S$ denote the category of (R,S)-bimodules. For an (R,S)-bimodule M we often write ${}_RM$, M_S, ${}_RM_S$ to indicate the left R-module, the right S-module, the (R,S)-bimodule structure used, respectively, and morphisms in the corresponding categories are denoted by ${}_R\mathrm{Hom}(-,-)$, $\mathrm{Hom}_S(-,-)$ and ${}_R\mathrm{Hom}_S(-,-)$.

We recall from [3,4] that every $M \in {}_R\mathcal{M}_S$ defines an adjunction $\sigma^* \dashv \sigma_*$ formed by

- the induction functor $\sigma^* := (-) \otimes_R M : \text{Mod-}R \to \text{Mod-}S$,
- the coinduction functor $\sigma_* := \mathrm{Hom}_S(M, -) : \text{Mod-}S \to \text{Mod-}R$.

The unit η and the counit ϵ of this adjunction are given for all $X \in \text{Mod-}R$ and $Y \in \text{Mod-}S$ by

$$\eta_X : X \to \mathrm{Hom}_S(M, X \otimes_R M),\ x \mapsto [m \mapsto x \otimes_R m],$$

$$\epsilon_Y : \mathrm{Hom}_S(M, Y) \otimes_R M \to Y,\ f \otimes_R m \mapsto f(m).$$

Given bimodules ${}_RM_S$ and ${}_{R'}N_S$, where R' is a ring, the abelian group $\mathrm{Hom}_S(M,N)$ is an (R',R)-bimodule via the multiplication defined by

$$(r'fr)(m) := r'f(rm), \quad \text{for every } f \in \mathrm{Hom}_S(M,N), r \in R, m \in M, r' \in R'.$$

In particular, the endomorphism ring $\mathcal{E} := \mathrm{End}_S(M)$ belongs to ${}_R\mathcal{M}_R$. We denote by

$${}^*M = {}_R\mathrm{Hom}(M,R) \quad \text{and} \quad M^* = \mathrm{Hom}_S(M,S)$$

the left dual and the right dual of M, respectively, which both belong to ${}_S\mathcal{M}_R$.

Given an (R,S)-bimodule M, in [37] R is said to be *M-separable over S* if the evaluation map

$$\mathrm{ev}_M : M \otimes_S {}^*M \to R, \quad \mathrm{ev}_M(m \otimes_S f) = f(m), \tag{5}$$

is a split epimorphism of R-bimodules. By [15, Theorem 34], this is equivalent to say that the functor ${}_R\mathrm{Hom}(M,-) : R\text{-Mod} \to S\text{-Mod}$ is separable. Hereafter, we consider the right version of this definition. Explicitly, we say that S is *M-separable over R*, if the evaluation map

$$\mathrm{ev}_M : M^* \otimes_R M \to S, \quad \mathrm{ev}_M(f \otimes_R m) = f(m), \tag{6}$$

is a split epimorphism of S-bimodules. This means there is a central element $\sum_i f_i \otimes_R m_i \in (M^* \otimes_R M)^S$ such that $\sum_i f_i(m_i) = 1_S$.

Remark 3.16. As we will see in Claim 3.25, when M_S is finitely generated and projective, the Eilenberg-Moore category $(\text{Mod-}S)^{\sigma^*\sigma_*}$ results to be equivalent to the category $\mathcal{M}^{\mathcal{C}}$ of right comodules over the comatrix S-coring $\mathcal{C}$ which was defined in [22] as $\mathcal{C} := M^* \otimes_R M$. This explains our choice to use the right version of (5), since otherwise we would have achieved the less usual coring $M \otimes_S {}^*M$. Moreover, our choice is also motivated by the fact that the map

$$\varphi : R \to \mathcal{E}, \quad r \mapsto [m \mapsto rm], \tag{7}$$

results to be a ring homomorphism. If we had taken (5), we would have been forced to choose $\mathcal{E} = {}_R\mathrm{End}(M) = {}_R\mathrm{Hom}(M,M) \in {}_S\mathcal{M}_S$ and to consider the ring homomorphism $\varphi : S \to \mathcal{E}^{\mathrm{op}}$, $s \mapsto [m \mapsto ms]$ into the opposite ring.

Given an (R,S)-bimodule M, concerning the separability and natural fullness of $\sigma_* = \mathrm{Hom}_S(M,-)$ and $\sigma^* = (-) \otimes_R M$, we know that

- σ_* is separable if and only if S is M-separable over R (right version of [15, Theorem 34]);

- σ_* is naturally full if and only if there is $\sum_i f_i \otimes_R m_i \in (M^* \otimes_R M)^S$ satisfying $\mathrm{Id}_M \otimes_R m = \sum_i m f_i(-) \otimes_R m_i$ for all $m \in M$ (right version of [4, Theorem 3.8 (1)]);
- if σ^* is separable, then (right version of [34, Proposition 2.5]) there is $E \in {}_R\mathrm{Hom}_R(\mathcal{E}, R)$ such that $E \circ \varphi = \mathrm{Id}_R$, i.e. φ^* is separable, where φ is the map in (7).
- Assume M is finitely generated and projective as a right S-module. Then, σ^* is naturally full if and only if there is $E \in {}_R\mathrm{Hom}_R(\mathcal{E}, R)$ such that $\varphi \circ E = \mathrm{Id}_{\mathcal{E}}$ (right version of [4, Theorem 3.8 (2)]). By what we recalled at the beginning of Subsection 3.1, this is equivalent to say that φ^* is naturally full.

Now, we investigate the semiseparability of σ_* and, in the finitely generated and projective case, the one of σ^*. To this aim we first introduce the following definition, which will be mainly used in its first part by the same reasons discussed in Remark 3.16.

Definition 3.17. Let R, S be rings and M an (R, S)-bimodule. We say that S is **M-semiseparable over** R if there exists an element $\sum_i f_i \otimes_R m_i \in (M^* \otimes_R M)^S$ such that $\sum_i m f_i(m_i) = m$ for every $m \in M$. In a similar way, it is possible to define R is **M-semiseparable over** S.

Given an (R, S)-bimodule M, the equivalence between (1) and (3) in the following result is the semiseparable counterpart of [15, Theorem 34].

Theorem 3.18. *Let R, S be rings and M an (R, S)-bimodule. Then, the following are equivalent.*

(1) The functor $\sigma_* = \mathrm{Hom}_S(M, -) : \mathrm{Mod}\text{-}S \to \mathrm{Mod}\text{-}R$ is semiseparable.
(2) $\mathrm{ev}_M : M^* \otimes_R M \to S$ is regular as a morphism of S-bimodules and $M \otimes_S \mathrm{ev}_M$ is surjective.
(3) S is M-semiseparable over R.

Proof. It is known (e.g. the right version of [15, Lemma 11]) that there is a bijective correspondence

$$\mathrm{Nat}(\mathrm{Id}_{\mathrm{Mod}\text{-}S}, \sigma^*\sigma_*) \cong (M^* \otimes_R M)^S. \tag{8}$$

Explicitly, a natural transformation $\gamma : \mathrm{Id}_{\mathrm{Mod}\text{-}S} \to \sigma^*\sigma_*$ is mapped to $\gamma_S(1) \in (M^* \otimes_R M)^S$ while an element $\sum_i f_i \otimes_R m_i \in (M^* \otimes_R M)^S$ is mapped, for every $Y \in \mathrm{Mod}\text{-}S$, to

$$\gamma_Y : Y \to \mathrm{Hom}_S(M, Y) \otimes_R M, \quad \gamma_Y(y) = \sum_i y f_i(-) \otimes_R m_i. \tag{9}$$

(1) $\Rightarrow$ (2). If the functor σ_* is semiseparable, then by Theorem 2.1 there exists a natural transformation $\gamma : \mathrm{Id}_{\mathrm{Mod}\text{-}S} \to \sigma^*\sigma_*$ such that $\epsilon \circ \gamma \circ \epsilon = \epsilon$. Consider the right S-module

map $\gamma_S : S \to M^* \otimes_R M$. For every $s \in S$ set $f_s : S \to S, s' \mapsto ss'$. Since γ_S is natural in S, we have $\gamma_S(ss') = (\gamma_S \circ f_s)(s') = ((\mathrm{Hom}_S(M, f_s) \otimes_R M) \circ \gamma_S)(s') = s\gamma_S(s')$ so that γ_S is S-bilinear. Since $\epsilon_S = \mathrm{ev}_M$, from $\epsilon \circ \gamma \circ \epsilon = \epsilon$ we get $\mathrm{ev}_M \circ \gamma_S \circ \mathrm{ev}_M = \mathrm{ev}_M$ and hence ev_M is regular as a morphism of S-bimodules. Note that any $m \in M$ is of the form $m = \mathrm{Id}(m) = \epsilon_M(\mathrm{Id} \otimes_R m)$ so that ϵ_M is surjective. Thus, from $\epsilon_M \circ \gamma_M \circ \epsilon_M = \epsilon_M$, we get $\epsilon_M \circ \gamma_M = \mathrm{Id}$. From (8) we have that γ_M is defined by (9) for $Y = M$, where $\sum_i f_i \otimes_R m_i = \gamma_S(1_S) \in (M^* \otimes_R M)^S$. Thus

$$m = \mathrm{Id}(m) = (\epsilon_M \circ \gamma_M)(m) = \sum_i m f_i(m_i) = r_M(M \otimes_S \mathrm{ev}_M)(\sum_i m \otimes_S f_i \otimes_R m_i)$$

where $r_M : M \otimes_S S \to M$ is the canonical isomorphism. Thus, $r_M \circ (M \otimes_S \mathrm{ev}_M)$ is surjective and hence also $M \otimes_S \mathrm{ev}_M$ is surjective.

(2) ⇒ (3). Assume that ev_M is regular as a morphism of S-bimodules, i.e. that there is an S-bimodule map $\gamma_S : S \to M^* \otimes_R M$ such that $\mathrm{ev}_M \circ \gamma_S \circ \mathrm{ev}_M = \mathrm{ev}_M$. Thus $(M \otimes_S \mathrm{ev}_M) \circ (M \otimes_S \gamma_S) \circ (M \otimes_S \mathrm{ev}_M) = (M \otimes_S \mathrm{ev}_M)$. If $M \otimes_S \mathrm{ev}_M$ is surjective, we get $(M \otimes_S \mathrm{ev}_M) \circ (M \otimes_S \gamma_S) = \mathrm{Id}_{M \otimes_S S}$. Now set $\sum_i f_i \otimes_R m_i = \gamma_S(1_S) \in (M^* \otimes_R M)^S$. Thus, S is M-semiseparable over R as

$$m = r_M \mathrm{Id}_{M \otimes_S S}(m \otimes_S 1) = r_M(M \otimes_S \mathrm{ev}_M)(M \otimes_S \gamma_S)(m \otimes_S 1) = \sum_i m f_i(m_i).$$

(3) ⇒ (1). Assume S is M-semiseparable over R. By definition, there exists an element $\sum_i f_i \otimes_R m_i \in (M^* \otimes_R M)^S$ such that $\sum_i f_i(m_i)m = m$ for every $m \in M$ and the corresponding natural transformation $\gamma : \mathrm{Id}_{\text{Mod-}S} \to \sigma^*\sigma_*$ from (8) is given, for every $Y \in \text{Mod-}S$, by (9). Moreover, for every $Y \in \text{Mod-}S$, $m \in M$, $f \in \mathrm{Hom}_S(M, Y)$, we have $\epsilon_Y \gamma_Y \epsilon_Y(f \otimes_R m) = \epsilon_Y \gamma_Y(f(m)) = \epsilon_Y(\sum_i f(m) f_i(-) \otimes_R m_i) = \sum_i f(m) f_i(m_i) = f(\sum_i m f_i(m_i)) = f(m) = \epsilon_Y(f \otimes_R m)$. Thus ϵ is regular and by Theorem 2.1 (ii) σ_* is semiseparable. □

Remark 3.19. In the setting of Theorem 3.18, assume further that M_S is projective. Then, the requirement that $M \otimes_S \mathrm{ev}_M$ is surjective is superfluous. Indeed, there is a dual basis formed by elements $m_i \in M, f_i \in M^*$, with $i \in I$, such that, for every $m \in M$, we have $m = \sum_{i \in I} m_i f_i(m)$. By definition, $f_i(m) = 0$ for almost all i. Thus there is a finite subset $I(m)$ of I such that $m = \sum_{i \in I(m)} m_i f_i(m) = r_M(M \otimes_S \mathrm{ev}_M)(\sum_{i \in I(m)} m_i \otimes_S f_i \otimes_R m)$, whence $M \otimes_S \mathrm{ev}_M$ is surjective.

As a consequence of Theorem 3.18, we have the following characterization of M-separability, for an (R, S)-bimodule M, which extends some known results, see e.g. [37, Theorem 1], [34, Corollary 2.4] and [3, Proposition 4.3].

Corollary 3.20. *Let R, S be rings and M an (R, S)-bimodule. Then, S is M-separable over R if and only if S is M-semiseparable over R and M_S is a generator.*

Proof. By what we recalled at the beginning of this subsection, S is M-separable over R if and only if $\sigma_* = \mathrm{Hom}_S(M,-) : \text{Mod-}S \to \text{Mod-}R$ is a separable functor. By Proposition 1.3 (i), this is equivalent to require that σ_* is semiseparable and faithful. The semiseparability of σ_* is equivalent to S being M-semiseparable over R, by Theorem 3.18. Since the forgetful functor $U : \text{Mod-}R \to \mathsf{Set}$ is faithful, the faithfulness of σ_* is equivalent to the faithfulness of the composition $U \circ \sigma_* = \mathrm{Hom}_S(M,-) : \text{Mod-}S \to \mathsf{Set}$, i.e. to M_S being a generator, see e.g. [36, Section 6]. $\square$

Remark 3.21. Let R, S be rings and M an (R,S)-bimodule. If M_S is a generator and $\varphi : R \to \mathcal{E} = \mathrm{End}_S(M)$ is a ring epimorphism, then by the right version of [4, Proposition 3.11], the functor σ_* is fully faithful, i.e. σ^* is a reflection. Thus, by Theorem 2.24, σ^* results to be semiseparable if and only if it is naturally full if and only if it is Frobenius in this case.

We now obtain a different characterization of M-semiseparability of S over R, for an (R,S)-bimodule M, that will allow us to exhibit an example where S is M-semiseparable but not M-separable over R, see Example 3.23.

Proposition 3.22. *Let R, S be rings and let M be an (R,S)-bimodule. Then S is M-semiseparable over R if and only if there is a central idempotent $z \in S$ (necessarily unique) such that M is obtained by restriction of scalars from an (R, Sz)-bimodule N and Sz is N-separable over R, via $\varphi : S \to Sz, s \mapsto sz$. Furthermore, S is M-separable over R if and only if $z = 1_S$.*

Proof. Assume that S is M-semiseparable over R, i.e. that there is a central element $\sum_i f_i \otimes_R m_i \in (M^* \otimes_R M)^S$ such that $\sum_i m f_i(m_i) = m$, for every $m \in M$. Set $z := \sum_i f_i(m_i) \in S$ so that $mz = m$, for every $m \in M$. Since $\mathrm{ev}_M : M^* \otimes_R M \to S$ is a morphism of S-bimodules, it induces a morphism $\mathrm{ev}_M^S : (M^* \otimes_R M)^S \to S^S$ so that $z = \mathrm{ev}_M(\sum_i f_i \otimes_R m_i) \in S^S$, i.e. z is central. Moreover $zz = \sum_i f_i(m_i)z = \sum_i f_i(m_i z) = \sum_i f_i(m_i) = z$, so that z is idempotent. Since for every $m \in M$ one has $mz = m$, then M becomes a right Sz-module, via $\mu_M : M \times Sz \to M, (m, sz) \mapsto ms$. Let us write N for M regarded as an (R, Sz)-bimodule so that $M = \varphi_* N$ where $\varphi : S \to Sz, s \mapsto sz$. Set $N^* := \mathrm{Hom}_{Sz}(N, Sz)$. Then $\sum_i \varphi f_i \otimes_R m_i \in (N^* \otimes_R N)^{Sz}$ and $\sum_i \varphi f_i(m_i) = \varphi(z) = zz = z = 1_{Sz}$ so that Sz is N-separable over R. Conversely, assume there is a central idempotent $z \in S$ such that M is obtained by restriction of scalars from an (R, Sz)-bimodule N and Sz is N-separable over R, via $\varphi : S \to Sz, s \mapsto sz$. This implies $mz = m$ for every $m \in M$. Since Sz is N-separable over R, there is $\sum_i g_i \otimes_R m_i \in (N^* \otimes_R N)^S$ such that $\sum_i g_i(m_i) = 1_{Sz} = z$. Let $j : Sz \to S$ be the canonical injection. Then $f_i := j \circ g_i \in M^*$ and $\sum_i f_i \otimes_R m_i \in (M^* \otimes_R M)^S$. Moreover $\sum_i m f_i(m_i) = \sum_i m j g_i(m_i) = mj(z) = mz = m$ so that S is M-semiseparable over R. Assume there is another central idempotent $z' \in S$ such that $M = \varphi_* N'$ for some (R, Sz')-bimodule N' and Sz is N'-separable over R via the ring homomorphism

$\varphi' : S \to Sz', s \mapsto sz'$. Then $zz' = \sum_i f_i(m_i)z' = \sum_i f_i(m_i z') = \sum_i f_i(m_i) = z$. Exchanging the roles of z and z', we also get $z'z = z'$ so that $z = z'$. Let $z \in S$ be a central idempotent such that $M = \varphi_* N$ where Sz is N-separable over R via $\varphi : S \to Sz, s \mapsto sz$. If $z = 1_S$, then S is N-separable over R and $\varphi = \mathrm{Id}_S$ so that S is $M = \varphi_* N$-separable over R as well. Conversely, if S is M-separable over R, then $z = 1_S$ is an idempotent as in the statement, whence the unique one. □

The following is an instance of an (R, S)-bimodule M such that S is M-semiseparable but not M-separable over R.

Example 3.23. Let $\varphi : S \to T$ be a ring homomorphism and assume that there is $E \in {}_S\mathrm{Hom}_S(T, S)$ such that $\varphi \circ E = \mathrm{Id}_T$. If we set $z := E(1_T) \in S$, then z is a central idempotent in S, the map $\varphi_{|Sz} : Sz \to T$ is a ring isomorphism and $\varphi : S \to T \cong Sz$ is the projection $s \mapsto sz$, see [4, Proposition 3.1]. By Proposition 3.22, if N is a (R, T)-bimodule such that T is N-separable over R, then $M := \varphi_* N$ is an (R, S)-bimodule such that S is M-semiseparable over R. Moreover, if S is also M-separable over R, then $z = 1_S$, whence φ is bijective. As a consequence, S will not be M-separable over R unless $\varphi : S \to T$ is bijective. As an example, let $\psi : \mathbb{Q} \times \mathbb{Z} \to \mathbb{Q}, (q, z) \mapsto q$ and $D : \mathbb{Q} \to \mathbb{Q} \times \mathbb{Z}, q \mapsto (q, 0)$ be as in Example 3.2. Then, if N is a $(R, \mathbb{Q})$-bimodule such that $\mathbb{Q}$ is N-separable over R, then the $(R, \mathbb{Q} \times \mathbb{Z})$-bimodule $M := \psi_* N$ is such that $\mathbb{Q} \times \mathbb{Z}$ is M-semiseparable but not M-separable over R. For instance consider the $\mathbb{Q}$-vector space $N := \mathbb{Q}^n$, with $n > 1$, and take $R := \mathbb{Q}$. Let us check that N is a $(\mathbb{Q}, \mathbb{Q})$-bimodule such that $\mathbb{Q}$ is N-separable over $\mathbb{Q}$, with $nq = qn$ for all $n \in N, q \in \mathbb{Q}$. Since N is a free left $\mathbb{Q}$-module, then it is a generator. Moreover $\mathcal{E} = \mathrm{End}_{\mathbb{Q}}(N) = \mathrm{End}_{\mathbb{Q}}(\mathbb{Q}^n) \cong \mathrm{M}_n(\mathrm{End}_{\mathbb{Q}}(\mathbb{Q})) \cong \mathrm{M}_n(\mathbb{Q})$ is a separable $\mathbb{Q}$-algebra. Therefore by the right version of [37, Theorem 1(1)] (see also Proposition 3.26 below), $\mathbb{Q}$ is N-separable over $\mathbb{Q}$. Thus the $(\mathbb{Q}, \mathbb{Q} \times \mathbb{Z})$-bimodule $M := \psi_* N = \mathbb{Q}^n$ is such that $\mathbb{Q} \times \mathbb{Z}$ is M-semiseparable but not M-separable over $\mathbb{Q}$. For a direct computation by means of Definition 3.17, set $m := (1, 0, \ldots, 0)$ and define $f \in M^* = \mathrm{Hom}_{\mathbb{Q} \times \mathbb{Z}}(\mathbb{Q}^n, \mathbb{Q} \times \mathbb{Z})$ by $f(q_1, \ldots, q_n) := (q_1, 0)$. Then $f \otimes_{\mathbb{Q}} m \in (M^* \otimes_{\mathbb{Q}} M)^{\mathbb{Q} \times \mathbb{Z}}$ and for every $m' \in M$ one has $m' f(m) = m'(1, 0) = m' \psi(1, 0) = m'$.

Next result provides an explicit factorization as a bireflection followed by a separable functor for the coinduction functor σ_* attached to an (R, S)-bimodule M in case it is semiseparable. By Corollary 2.28, this factorization amounts to the one given by the coidentifier.

Proposition 3.24. *Let M be an (R, S)-bimodule. The coinduction functor $\sigma_* = \mathrm{Hom}_S(M, -) :$ $\text{Mod-}S \to \text{Mod-}R$ is semiseparable if and only if there is an S-coring I with a grouplike element $z \in I^S$ such that $\sigma_* \cong \tilde{\sigma}_* \circ G_I$ where $\tilde{\sigma}_* := \mathrm{Hom}^I(M, -) : \mathcal{M}^I \to \text{Mod-}R$ is separable and the induction functor $G_I := (-) \otimes_S I : \text{Mod-}S \to \mathcal{M}^I$ is a bireflection. Here M is in $\mathcal{M}^I$ via $\rho_M(m) = m \otimes_S z$.*

Proof. Assume that σ_* is semiseparable. Then, by Theorem 3.18 S is M-separable over R through some $c := \sum_i f_i \otimes_R m_i \in (M^* \otimes_R M)^S$. Since $\mathrm{ev}_M : M^* \otimes_R M \to S$ is a morphism of S-bimodules, then $I := \mathrm{Im}(\mathrm{ev}_M)$ is an ideal of S with multiplicative identity $z := \mathrm{ev}_M(c) = \sum_i f_i(m_i)$. Indeed, for all $s \in S$, $zs = \mathrm{ev}_M(c)s = \mathrm{ev}_M(cs) = \mathrm{ev}_M(sc) = s\mathrm{ev}_M(c) = sz$ and hence $z \in I^S$. For all $m \in M, f \in M^*$, we have $zf(m) = \sum_i f(m)f_i(m_i) = f(\sum_i mf_i(m_i)) = f(m)$ and hence $zi = i$ for every $i \in I$. Moreover, since the morphism $\varphi : S \to I, s \mapsto sz$, is a ring epimorphism, the map $m_I : I \otimes_S I \to I$ is bijective. Thus we can consider $\Delta_I = m_I^{-1} : I \to I \otimes_S I$, $\Delta_I(i) = i \otimes_S z = z \otimes_S i$, so that $(I, \Delta_I, \varepsilon_I)$ becomes an S-coring, where $\varepsilon_I : I \hookrightarrow S$ is the canonical inclusion. By the foregoing $z \in I^S$ and, for every $i \in I$, we have $\varepsilon_I(i)z = iz = i$. By what we recalled at the beginning of Subsection 3.3, the induction functor $G_I := (-) \otimes_S I : \text{Mod-}S \to \mathcal{M}^I$ is naturally full. Consider its left adjoint, the forgetful functor $F_I : \mathcal{M}^I \to \text{Mod-}S$, and the corresponding unit η defined on each N in $\mathcal{M}^I$ by setting $\eta_N := \rho_N : N \to N \otimes_S I$. Given $n \in N$ write $\rho_N(n) = \sum_t n_t \otimes_S i_t$. By applying $N \otimes_S \varepsilon_I$ we get $n = \sum_t n_t i_t$. Thus $\rho_N(n) = \sum_t n_t \otimes_S i_t = \sum_t n_t \otimes_S i_t z = \sum_t n_t i_t \otimes_S z = n \otimes_S z$. We have so proved that $\rho_N(n) = n \otimes_S z$, for every $n \in N$. By applying $N \otimes_S \varepsilon_I$ to this equality we get $n = nz$, for every $n \in N$. Therefore ρ_N is invertible with inverse given by $n \otimes_S i \mapsto ni$, and then the unit η of $F_I \dashv G_I$ is invertible, i.e. G_I is a coreflection. By Theorem 2.24, G_I is a bireflection. As in [25, Example 4.3], once noticed that $M \in {}^R\mathcal{M}^I$ (this just means that ρ_M is left R-linear), we can consider the functor $\tilde{\sigma}^* := (-)\otimes_R M : \text{Mod-}R = \mathcal{M}^R \to \mathcal{M}^I$. By [13, 18.10.2] we have that $\tilde{\sigma}^* \dashv \tilde{\sigma}_* = \mathrm{Hom}^I(M, -)$ with unit and counit given by

$$\tilde{\eta}_X : X \to \mathrm{Hom}^I(M, X \otimes_R M),\ x \mapsto [m \mapsto x \otimes_R m],$$
$$\tilde{\epsilon}_Y : \mathrm{Hom}^I(M, Y) \otimes_R M \to Y,\ f \otimes_R m \mapsto f(m).$$

Thus, by Rafael Theorem, $\tilde{\sigma}_*$ is separable if and only if there is a natural transformation $\tilde{\gamma} : \mathrm{Id} \to \tilde{\sigma}^*\tilde{\sigma}_*$ such that $\tilde{\epsilon}\circ\tilde{\gamma} = \mathrm{Id}$. For Y in $\mathcal{M}^I$, define $\tilde{\gamma}_Y : Y \to \mathrm{Hom}^I(M, Y)\otimes_R M$, $y \mapsto \sum_i yf_i(-) \otimes_R m_i$. It is easy to check it defines a natural transformation $\tilde{\gamma} : \mathrm{Id} \to \tilde{\sigma}^*\tilde{\sigma}_*$. Moreover $\tilde{\epsilon}_Y\tilde{\gamma}_Y(y) = \sum_i yf_i(m_i) = yz$ but we already proved that $yz = y$, hence $\tilde{\epsilon}\circ\tilde{\gamma} = \mathrm{Id}$ and $\tilde{\sigma}_*$ is separable. Let us check that $G \cong \tilde{\sigma}_* \circ G_I$. Note that $\varphi \circ \varepsilon = \mathrm{Id}_I$ and both φ and ε are both left S-linear. As a consequence I is projective, whence flat, as a left S-module. Thus, by [13, 22.12] applied in case $\mathcal{D}$ is the S-coring S, for every N in Mod-R we have a functorial isomorphism of abelian groups

$$\tilde{\sigma}_* G_I(N) = \mathrm{Hom}^I(M, N \otimes_S I) \to \mathrm{Hom}_S(M, N) = \sigma_*(N), \quad f \mapsto (N \otimes_S \varepsilon_I) \circ f.$$

This isomorphism is easily checked to be right R-linear. Thus it yields $\tilde{\sigma}_* \circ G_I \cong \sigma_*$ as desired. Conversely, if $\sigma_* \cong \tilde{\sigma}_* \circ G_I$, where G_I is a bireflection, whence naturally full by Theorem 2.24, and $\tilde{\sigma}_*$ is separable, then σ_* is semiseparable in view of Lemma 1.12(ii). $\square$

Claim 3.25. *As already mentioned, given an (R, S)-bimodule M, in order to characterize the semiseparability of the induction functor $\sigma^* = (-) \otimes_R M : \text{Mod-}R \to \text{Mod-}S$*

we need, as in the separable case, the further assumption that M_S is finitely generated and projective. It is well-known that this hypothesis implies that the map $N \otimes_S M^* \to \mathrm{Hom}_S(M, N), n \otimes f \mapsto [m \mapsto nf(m)]$, is an isomorphism natural in N, for every right S-module N, where $M^* = \mathrm{Hom}_S(M, S)$. As a consequence, the right adjoint of σ^* can be chosen to be $\sigma_* = (-) \otimes_S M^* : \mathrm{Mod}\text{-}S \to \mathrm{Mod}\text{-}R$. If we let $\{e_i^*, e_i\} \subseteq M^* \times M$ be a finite dual basis, the unit η and the counit ϵ of this adjunction are given for all $X \in \mathrm{Mod}\text{-}R$, $Y \in \mathrm{Mod}\text{-}S$ by

$$\eta_X : X \to X \otimes_R M \otimes_S M^*,\ x \mapsto \sum_i x \otimes_R e_i \otimes_S e_i^*,$$
$$\epsilon_Y : Y \otimes_S M^* \otimes_R M \to Y,\ y \otimes_R f \otimes_S m \mapsto yf(m).$$

It turns out that, see e.g. [30, page 30], the Eilenberg-Moore category $(\mathrm{Mod}\text{-}R)_{\sigma_*\sigma^*}$ is equivalent to the category $\mathrm{Mod}\text{-}\mathcal{E}$, where $\mathcal{E} := \mathrm{End}_S(M) \cong M \otimes_S M^*$ is the endomorphism ring with canonical morphism $\varphi : R \to \mathcal{E}$, $\varphi(r)(m) = rm$, for all $r \in R$ and $m \in M$. Dually the Eilenberg-Moore category $(\mathrm{Mod}\text{-}S)^{\sigma^*\sigma_*}$ is equivalent to the category $\mathcal{M}^{\mathcal{C}}$ of right comodules over the comatrix S-coring $\mathcal{C} := M^* \otimes_R M$, see e.g. [30, page 36]. Comatrix corings have been introduced in [22] and they generalize the Sweedler's canonical coring. See also [14] for further investigations. The diagram (1) becomes:

$$\begin{tikzcd}
(\mathrm{Mod}\text{-}S)^{\sigma^*\sigma_*} \cong \mathcal{M}^{\mathcal{C}} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathrm{Mod}\text{-}S \arrow[l, shift left, "G=(-)\otimes_R \mathcal{C}"] \arrow[d, shift left, "\sigma_*=(-)\otimes_S M^*"] \arrow[d, phantom, "\dashv"] \arrow[dr, bend left, "K_{\sigma_*\sigma^*}"] & \\
& \mathrm{Mod}\text{-}R \arrow[u, shift left, "\sigma^*=(-)\otimes_R M"] \arrow[r, shift left, "\varphi^*=(-)\otimes_R \mathcal{E}"] \arrow[r, phantom, "\perp"] \arrow[ul, bend left, "K^{\sigma^*\sigma_*}"] & \mathrm{Mod}\text{-}\mathcal{E} \cong (\mathrm{Mod}\text{-}R)_{\sigma_*\sigma^*} \arrow[l, shift left, "\varphi_*"]
\end{tikzcd}$$

where $F \dashv G$ is the adjunction as in Subsection 3.3, given by the forgetful functor $F : \mathcal{M}^{\mathcal{C}} \to \mathrm{Mod}\text{-}S$ and the induction functor $G := (-) \otimes_S \mathcal{C} : \mathrm{Mod}\text{-}S \to \mathcal{M}^{\mathcal{C}}$; the functors $K_{\sigma_*\sigma^*}$ and $K^{\sigma^*\sigma_*}$ are the comparison and the cocomparison functor, respectively. From the diagram, we have

$$\varphi_* \circ K_{\sigma_*\sigma^*} = \sigma_* \qquad F \circ K^{\sigma^*\sigma_*} = \sigma^* \qquad K_{\sigma_*\sigma^*} \circ \sigma^* = \varphi^* \qquad K^{\sigma^*\sigma_*} \circ \sigma_* = G.$$

The above refinement of the functors involved allows us to obtain a different characterization also for the semiseparability of σ_*

Proposition 3.26. *In the setting of Claim 3.25, the following assertions are equivalent.*

(i) *S is M-semiseparable over R;*
(ii) *$\sigma_* = (-) \otimes_S M^* : \mathrm{Mod}\text{-}S \to \mathrm{Mod}\text{-}R$ is semiseparable;*
(iii) *the comatrix S-coring $\mathcal{C}$ is semicosplit;*

(iv) *there exists an invariant element* $z \in \mathcal{C}^S$ *such that for every* $c \in \mathcal{C}$, $c = \varepsilon_{\mathcal{C}}(z)c$, *where* $\varepsilon_{\mathcal{C}}$ *is the counit of the comatrix* S*-coring* $\mathcal{C}$*;*

(v) $\varphi_* : \text{Mod-}\mathcal{E} \to \text{Mod-}R$ *is separable (that is,* $\mathcal{E}/R$ *is separable) and* $K_{\sigma_*\sigma^*}$ *is naturally full.*

Proof. (i)$\Leftrightarrow$ (ii). It is Theorem 3.18.
(ii)$\Leftrightarrow$ (iii). By Remark 2.5 3), σ_* is semiseparable if and only if so is $V^{\sigma^*\sigma_*} = G$.
(iii)$\Leftrightarrow$ (iv). It follows by Theorem 3.10.
(ii)$\Leftrightarrow$ (v). It follows by Theorem 2.9 applied to the adjunction (σ^*, σ_*). $\square$

We now obtain the announced characterization of the semiseparability of σ^*.

Proposition 3.27. *In the setting of Claim 3.25, the following assertions are equivalent.*

(i) $\sigma^* = (-) \otimes_R M : \text{Mod-}R \to \text{Mod-}S$ *is semiseparable;*
(ii) $\varphi^* = (-) \otimes_R \mathcal{E} : \text{Mod-}R \to \text{Mod-}\mathcal{E}$ *is semiseparable;*
(iii) *there exists an* $E \in {}_R\mathrm{Hom}_R(\mathcal{E}, R)$ *such that* $\varphi E(1_{\mathcal{E}}) = 1_{\mathcal{E}}$*;*
(iv) $F : \mathcal{M}^{\mathcal{C}} \to \text{Mod-}S$ *is separable (i.e.* $\mathcal{C}$ *is coseparable) and* $K^{\sigma^*\sigma_*}$ *is naturally full.*

Proof. (i)$\Leftrightarrow$ (ii). By Remark 2.5 4), σ^* is semiseparable if and only if so is $V_{\sigma_*\sigma^*} = \varphi^*$.
(ii)$\Leftrightarrow$ (iii). It follows by Proposition 3.1.
(i)$\Leftrightarrow$ (iv). It follows by Theorem 2.14 applied to the adjunction (σ^*, σ_*). $\square$

Remark 3.28. At the beginning of this subsection we recalled that the separability of σ^* implies the one of φ^*. The other implication is also true if M_S is finitely generated and projective. Indeed, from $K_{\sigma_*\sigma^*} \circ \sigma^* = \varphi^*$, by Remark 2.5 4), we get that σ^* is separable if and only if φ^* is separable.

Now, as a particular case of Claim 3.25, given a morphism of rings $\varphi : R \to S$ consider the (R, S)-bimodule $M := {}_R S_S$, with left action induced by φ, which is trivially finitely generated and projective as a right S-module. In this case $\sigma^* = (-) \otimes_R S = \varphi^* :$ Mod-$S \to$ Mod-R is the induction functor of Section 3.1 As a consequence, the right adjoint σ_* of σ^* is isomorphic to the restriction of scalars functor $\varphi_* : \text{Mod-}S \to \text{Mod-}R$ and since it is faithful, it follows that S is S-semiseparable over R if and only if S is S-separable over R.

In this case, the comatrix S-coring $\mathcal{C}$ is the Sweedler coring $S \otimes_R S$, we have $\mathcal{E} = \mathrm{End}_S(M) \cong M \otimes_S M^* \cong S$, $K_{\varphi_*\varphi^*} = \mathrm{Id}_{\text{Mod-}S}$, i.e. φ_* is *strictly monadic*, and $K^{\varphi^*\varphi_*} = (-) \otimes_R S : \text{Mod-}R \to \text{Mod-}S$. Consider the induction functor $G = (-) \otimes_S \mathcal{C} : \text{Mod-}S \to \mathcal{M}^{\mathcal{C}}$ and the forgetful functor $F : \mathcal{M}^{\mathcal{C}} \to \text{Mod-}S$. In this setting, as a consequence of Proposition 3.26 and Proposition 3.27 we have the following corollaries, relating the functors φ_*, φ^*, F, G and the Sweedler coring $\mathcal{C}$. We just point out that, since the coring counit $\varepsilon_{\mathcal{C}}$ is the multiplication $S \otimes_R S \to S$ and we can choose $c = 1_S \otimes_R 1_S \in \mathcal{C}$, the existence of $z \in \mathcal{C}^S$ such that $c = \varepsilon_{\mathcal{C}}(z)c$, for every $c \in \mathcal{C}$, is equivalent to the existence of $z \in \mathcal{C}^S$ such that $1_S = \varepsilon_{\mathcal{C}}(z)$ i.e. of a separability idempotent of S/R.

Corollary 3.29. *In the above setting, the following assertions are equivalent.*

(i) *S is S-separable over R;*
(ii) *$\varphi_* : \mathrm{Mod}\text{-}S \to \mathrm{Mod}\text{-}R$ is separable, i.e. S/R is separable;*
(iii) *the Sweedler S-coring $S \otimes_R S$ is semicosplit;*
(iv) *S/R has a separability idempotent.*

Corollary 3.30. *In the above setting, the following assertions are equivalent.*

(i) *φ^* is semiseparable;*
(ii) *there exists an $E \in {}_R\mathrm{Hom}_R(S, R)$ such that $\varphi E(1_S) = 1_S$;*
(iii) *F is separable (i.e. the Sweedler S-coring $S \otimes_R S$ is coseparable) and $K^{\varphi^*\varphi_*}$ is a bireflection.*

Clearly the equivalence between (i), (ii) and (iv) of Corollary 3.29 above is well-known while the equivalence between (i) and (ii) of Corollary 3.30 is just Proposition 3.1.

3.5. Right Hopf algebras

Let B be a bialgebra over a field $\Bbbk$, let $\mathfrak{M}$ denote the category of vector spaces over $\Bbbk$ and let $\mathfrak{M}_B^B$ denote the category of right Hopf modules over B. Consider the coinvariant functor $(-)^{\mathrm{co}B} : \mathfrak{M}_B^B \to \mathfrak{M}$ which, for every object M in $\mathfrak{M}_B^B$, is defined by setting $M^{\mathrm{co}B} := \{m \in M \mid \rho_M(m) = m \otimes 1_B\}$. It is known that it fits into an adjoint triple $\overline{(-)}^B \dashv (-) \otimes B \dashv (-)^{\mathrm{co}B}$, see e.g. [35, Section 3], where $\overline{M}^B = \frac{M}{MB^+}$ and $B^+ = \ker(\varepsilon_B)$. The unit and counit are given by

$$\eta_M : M \to \overline{M}^B \otimes B,\ m \mapsto \sum \overline{m_0} \otimes m_1, \qquad \epsilon_V : \overline{(V \otimes B)}^B \overset{\cong}{\to} V,\ \overline{v \otimes b} \mapsto v\varepsilon_B(b)$$

$$\nu_V : V \overset{\cong}{\to} (V \otimes B)^{\mathrm{co}B},\ v \mapsto v \otimes 1_B, \qquad \theta_M : M^{\mathrm{co}B} \otimes B \to M,\ m \otimes b \mapsto mb.$$

By Proposition 2.19, the functor $(-)^{\mathrm{co}B}$ is semiseparable (resp. separable, naturally full) if and only if so is $\overline{(-)}^B$. Moreover, by [9, Proposition 3.4.1], the functor $(-) \otimes B$ is fully faithful so that $(-)^{\mathrm{co}B}$ is a coreflection. Thus, by Theorem 2.24 it follows that $(-)^{\mathrm{co}B}$ is semiseparable, if and only if it is naturally full, if and only if it is Frobenius. Our aim here is to characterize the semiseparability of $(-)^{\mathrm{co}B}$. Note that there is a natural transformation $\sigma : (-)^{\mathrm{co}B} \to \overline{(-)}^B$ defined on components by $\sigma_M : M^{\mathrm{co}B} \to \overline{M}^B, m \mapsto \overline{m} := m + MB^+$, see [35, Section 3].

(1) $\Leftrightarrow$ (2) in the following result is a semi-analogue of (1) $\Leftrightarrow$ (6) in [35, Theorem 3.13].

Theorem 3.31. *Let B be a bialgebra over a field $\Bbbk$ and consider the coinvariant functor $(-)^{\mathrm{co}B} : \mathfrak{M}_B^B \to \mathfrak{M}$. The following assertions are equivalent.*

(1) $(-)^{\mathrm{co}B}$ *is semiseparable.*
(2) B *is a right Hopf algebra with anti-multiplicative and anti-comultiplicative right antipode.*
(3) *The canonical natural transformation* $\sigma : (-)^{\mathrm{co}B} \to \overline{(-)}^B$ *is invertible.*
(4) *The canonical natural transformation* $\sigma : (-)^{\mathrm{co}B} \to \overline{(-)}^B$ *is split-mono.*

Proof. (1) $\Leftrightarrow$ (2). We already noticed that $(-)^{\mathrm{co}B}$ is semiseparable if and only if it is Frobenius. Moreover $(-)^{\mathrm{co}B}$ is Frobenius if and only if the natural transformation σ is invertible, cf. [35, Lemma 2.3] applied to the adjoint triple $\overline{(-)}^B \dashv (-) \otimes B \dashv (-)^{\mathrm{co}B}$.

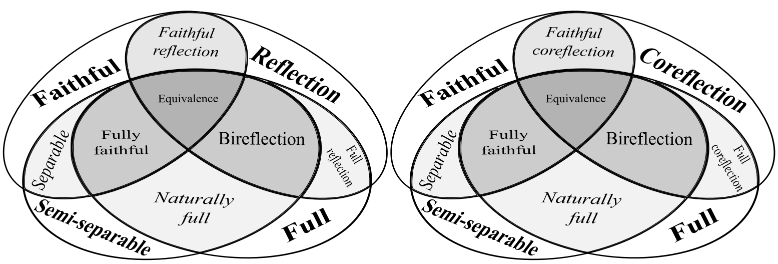
(2) $\Leftrightarrow$ (3). The equivalence follows by [35, Theorem 3.7].

(1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). It follows from Proposition 2.26. $\square$

Remark 3.32. As mentioned, the functor $(-)^{\mathrm{co}B} : \mathfrak{M}_B^B \to \mathfrak{M}$ fits into an adjoint triple $\overline{(-)}^B \dashv (-) \otimes B \dashv (-)^{\mathrm{co}B}$. Thus, $(-)^{\mathrm{co}B}$ is Frobenius if and only if $(-)^{\mathrm{co}B} \dashv (-) \otimes B$, if and only if $\overline{(-)}^B \cong (-)^{\mathrm{co}B}$. Note that there are bialgebras B which are not right Hopf algebras and hence $(-)^{\mathrm{co}B}$ needs not to be a Frobenius functor in general. For instance let G be a monoid and consider the monoid algebra $B = \Bbbk G$ over a field $\Bbbk$. If B is a right Hopf algebra, then it has a right antipode $S_B : B \to B$ and hence, for every $x \in G$, one has $xS_B(x) = \sum x_{(1)} S_B\left(x_{(2)}\right) = \varepsilon_B(x) 1_B = 1_G$. In particular each element in G is right invertible and hence G must be a group, which is not always the case. Moreover, see [35, Example 3.9], there are bialgebras B satisfying the equivalent conditions of Theorem 3.31 that are not Hopf algebras, i.e. such that the coreflection $(-)^{\mathrm{co}B}$ is semiseparable but not separable. Indeed, B is a Hopf algebra if and only if $(-)^{\mathrm{co}B}$ is an equivalence if and only if it is separable, cf. Remark 2.25.

3.6. Examples of (co)reflections

The connections between some type of functors we have considered in this paper are summarized in the following diagrams.



(10)

Indeed we know that separable functors are the "intersection" of semiseparable and faithful functors and that naturally full functors are the "intersection" of semiseparable and full functors, see Proposition 1.3. At the very beginning, we observed that a functor is fully faithful if and only if it is at the same time separable and naturally full. We know that a (co)reflection is semiseparable if and only if it is naturally full if and only if it is a bireflection, see Theorem 2.24. We also observed that a separable (co)reflection is necessarily an equivalence, see Remark 2.25.

In order to completely justify the coherence of diagrams (10) we need examples of a

- (co)reflection which is neither full nor faithful;
- faithful (co)reflection which is neither semiseparable nor full;
- full (co)reflection which is neither semiseparable nor faithful.

Now, we have observed in Remark 1.2, that a functor F is semiseparable (resp. separable, naturally full, full, faithful, fully faithful) if and only if so is F^{op}. On the other hand, since the opposite switches the functors of an adjunction, one has that F is a reflection (resp. coreflection) if and only if F^{op} is a coreflection (resp. reflection). As a consequence we can focus on coreflections as the corresponding examples for reflections can be obtained by duality.

Moreover, a fully faithful coreflection is an equivalence, whence in particular semiseparable. Thus we can omit the last option in the second and in the third item above. Moreover, by Theorem 2.24, we know that a coreflection is semiseparable if and only if it is naturally full if and only if it is a bireflection if and only if it is Frobenius. Thus a faithful coreflection, which is also semiseparable, must be full whence an equivalence. Summing up, our problem reduces in finding examples of a

- coreflection which is neither full nor faithful;
- faithful coreflection which is not an equivalence;
- full coreflection which is not a bireflection.

We start by including an example of coreflection which is neither full nor faithful.

Example 3.33. Let $\Bbbk$ be a field, let Coalg be the category of coalgebras over $\Bbbk$ and let Set be the category of sets. The functor $G : \mathsf{Coalg} \to \mathsf{Set}$ that associates to a coalgebra C the set $G(C)$ of grouplike elements in C, is a coreflection. In fact it has a fully faithful left adjoint F that takes a set S to the group-like coalgebra $\Bbbk S$. The unit and counit components are the canonical bijection $\eta_S : S \to GFS = G(\Bbbk S)$ and the canonical injection $\epsilon_C : FGC = \Bbbk G(C) \hookrightarrow C$, respectively. Let us check that G is not full. Let D be the matrix coalgebra $M_2^c(\Bbbk)$. Note that $GD = G(M_2^c(\Bbbk)) = \emptyset$ which is the initial object in Set.

Note also that, if we denote by 0 the zero coalgebra, then we also have $G0 = \emptyset$ so that $GD = G0$. If G is full then there is a coalgebra map $f : D \to 0$. In particular we have

$\varepsilon_D = \varepsilon_0 \circ f = 0$, a contradiction as ε_D is the map that assigns to a matrix its trace. Thus G is not full. Let us check it is not even faithful. Otherwise, ϵ_C would be an epimorphism in Coalg, for every coalgebra C, but, by [32, Theorem 3.1], an epimorphism in Coalg is necessarily surjective whence ϵ_C would be invertible and hence every coalgebra C would be isomorphic to $\Bbbk G(C)$, a contradiction.

Remark 3.34. We already observed that a conservative (co)reflection is always an equivalence. It is known (see e.g. [24, A1.2]) that a faithful functor from a balanced category (i.e. a category where every monomorphism which is an epimorphism is necessarily an isomorphism) is always conservative. As a consequence a faithful (co)reflection from a balanced category is always an equivalence. Since the category R-Mod of left modules over a ring R is abelian, it is in particular balanced and hence any faithful (co)reflection from R-Mod is always an equivalence.

An instance of a faithful coreflection which is not an equivalence is obtained by duality from the following example. Other examples arise as full epi-coreflective subcategories, see [26].

Example 3.35. Let Dom be the category of integral domains and injective ring homomorphisms. Let Field be the category of fields. The forgetful functor $G : \mathsf{Field} \to \mathsf{Dom}$ has a left adjoint F that takes every integral domain D to its quotient field $Q(D)$. Given an integral domain D, denote by $j_D : D \to Q(D), d \mapsto \frac{d}{1}$, the canonical injection. Then the unit on an integral domain D is the injection $\eta_D = j_D : D \to GFD = Q(D)$, while the counit is the isomorphism $\epsilon_F = j_K^{-1} : FGK = Q(K) \to K$. Note that, in general, η is just a monomorphism but not an isomorphism on components. Thus, F is a faithful reflection which is not an equivalence.

We finally provide an example of a full coreflection which is not a bireflection.

Example 3.36. Let $\Bbbk$ be an arbitrary field, let $\mathsf{Coalg}_\bullet$ be the full subcategory of Coalg whose objects are pointed coalgebras over $\Bbbk$ and let Set be the category of sets. The functor $G : \mathsf{Coalg}_\bullet \to \mathsf{Set}$, that associates to a coalgebra C the set $G(C)$ of grouplike elements in C, is a coreflection. In fact, it has a fully faithful left adjoint F that takes a set S to the group-like coalgebra $\Bbbk S$. The unit and counit components are the canonical bijection $\eta_S : S \to GFS = G(\Bbbk S)$ and the canonical injection $\epsilon_C : FGC = \Bbbk G(C) \hookrightarrow C$, respectively. By the dual Wedderburn-Malcev Theorem [31, Theorem 5.4.2], since C is pointed, there exists a coalgebra projection $\pi : C \to C_0 = \Bbbk G(C)$ such that $\pi \circ \epsilon_C = \mathrm{Id}$. Thus ϵ_C is a split monomorphism for each pointed coalgebra C and hence G is full. Therefore G is a full coreflection. Let us check it is not a bireflection in general. Otherwise G would be Frobenius and hence from $F \dashv G$ we should deduce $G \dashv F$. Consider the Sweedler's 4-dimensional Hopf algebra $H = \Bbbk\langle g, x \mid g^2 = 1, x^2 = 0, gx + xg = 0\rangle$ with coalgebra structure given by $\Delta(g) = g \otimes g$ and $\Delta(x) = x \otimes 1 + g \otimes x$ and set $S := G(H) = \{1, g\}$. We have

$$\mathrm{Hom}_{\mathsf{Set}}(S,S) = \mathrm{Hom}_{\mathsf{Set}}(G(H),S) \cong \mathrm{Hom}_{\mathsf{Coalg}_\bullet}(H,FS) = \mathrm{Hom}_{\mathsf{Coalg}}(H,\Bbbk S).$$

Since $\mathrm{Hom}_{\mathsf{Set}}(S,S)$ has cardinality 4, we get that $\mathrm{Hom}_{\mathsf{Coalg}}(H,\Bbbk S)$ must contain exactly 4 elements. For every $k \in \Bbbk$ define $f_k : H \to \Bbbk S$ by setting $f_k(1) = 1, f_k(g) = g, f_k(x) = k(1-g) = f_k(xg)$. Then f_k is a coalgebra map. By linear independence of grouplike elements, we have that $f_k \neq f_l$ for every $k,l \in \Bbbk$ such that $k \neq l$. Since $\mathrm{Hom}_{\mathsf{Coalg}}(H,\Bbbk S)$ contains 4 elements we deduce that the field $\Bbbk$ has at most 4 elements, a contradiction.

Data availability

No data was used for the research described in the article.

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