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Cycles in Impulsive Goodwin's Oscillators of arbitrary order[☆]

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ABSTRACT

Existence of periodical solutions, i.e. cycles, in the Impulsive Goodwin's Oscillator (IGO) with the continuous part of an arbitrary order m is considered. The original IGO with a third-order continuous part is a hybrid model that portrays a chemical or biochemical system composed of three substances represented by their concentrations and arranged in a cascade. The first substance in the chain is introduced via an impulsive feedback where both the impulse frequency and weights are modulated by the measured output of the continuous part. It is shown that, under the standard assumptions on the IGO, a positive periodic solution with one firing of the pulse-modulated feedback in the least period also exists in models with any $m \geq 1$. Furthermore, the uniqueness of this 1-cycle is proved for the IGO with $m \leq 10$ whereas, for $m > 10$, the uniqueness can still be guaranteed under mild assumptions on the frequency modulation function.

1. Introduction

Analyzing the dynamics of systems that simultaneously operate in fast and slow time scale (slow–fast systems) is a classical problem leading to the theory of singularly-perturbed dynamical systems (Smith, 1985). Fast dynamics, i.e. rapid evolution occurring over shorter times, can be approximated by the impact of finite or infinite impulse sequences resulting in (state vector) jumps. The impulsive action is then modeled either as a feedback or an independent discrete process, e.g., a realization of a Markov chain. In the former case, one deals with pulse-modulated feedback (Gelig & Churilov, 1998) or impulsive event-triggered control (Heemels et al., 2012), whereas the latter leads to hybrid control with Markovian switching.

Theory of impulsive differential equations, pioneered by early works on stability of oscillatory solutions with impulses (Aymerich, 1955; Krasovskii & Lidskii, 1961; Milman & Myshkis, 1960), constitutes the mathematical ground for analysis and design of impulsive control systems (Lakshmikantham et al., 1989; Samoilenko & Perestyuk, 1995). Impulsive models organically arise in biomedical, mechanical, ecological, environmental applications and are present virtually in all fields of science where mathematical modeling is utilized. Predator–pray models with application to, e.g. pest control, make use of impulsive signals to represent human action (Zhang & Chen, 2005). Impulses (impacts) appear in non-smooth mechanics due to hard constraints on state variables and control signals. Numerous examples of practically important mechanical systems with impacts, including gear boxes, railway bogie, vibration table, are provided in Popp (2000). Periodical medical pharmacological treatments are another significant application area of impulsive dynamical systems, where modeling is typically aimed at optimizing the treatment protocol (Cacace et al., 2020). Impulses reflect the way drugs are administered, namely through injections or orally in tablet formulation. Pulsatile mode of drug administration also arises when a physiological behavior is mimicked by a treatment. A profound example of this concept is the pulsatile artificial pancreas. The physiological regulation exercised via the pancreas during a meal results in a series of insulin pulses whose frequency and amplitude are modulated by the blood glucose level (Bally et al., 2017). Therefore, there is increasing interest in impulsive control of the artificial pancreas (Huang et al., 2012).

Impulsive systems possess non-smooth dynamics and, thus, can exhibit complex nonlinear behaviors. Solutions converging to an equilibrium or an oscillative attractor are observed in linear time-invariant (LTI) systems under pulse-modulated feedback. The latter can correspond to sustained periodic or non-periodic (chaotic, quasiperiodic) solutions. The impulsive Goodwin's oscillator (IGO) (Churilov et al., 2009; Medvedev et al., 2006) is a hybrid system that generalizes the classical continuous Goodwin's oscillator (Goodwin, 1965) by substituting the original continuous static nonlinear feedback with a pulse-modulated one. The IGO

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lacks equilibria and admits solutions of high periodicity as well as chaotic and quasiperiodic ones (Zhusubaliyev et al., 2012a).

The rationale behind the IGO was originally to incorporate the experimentally observed principle of pulsatile endocrine regulation (Walker et al., 2010) into a widely used mathematical model of biochemical oscillation. However, the IGO can be seen as a general construct that illustrates how sustained oscillation can be obtained in a positive (continuous) LTI system by means of positive-valued feedback, no matter what the nature of the system is. From that perspective, the dynamics of the continuous part have to be as nonrestrictive as possible. Yet, in previous work on the IGO, only first-order (Zhusubaliyev et al., 2012c) and third-order continuous LTI dynamics have been addressed. In the latter case, the continuous dynamics augmented with point-wise (Churilov et al., 2014) or distributed delay (Churilov & Medvedev, 2016) were also considered.

The present paper generalizes the IGO structure to continuous LTI blocks of higher order than three. From an application point view, the order of an LTI model is a degree of freedom exploited by the designer to obtain a parsimonious description of essential model properties. Then setting the model order to a fixed constant is impractical. Further, when the model variables correspond to physical or chemical properties, the model order is defined by the number of variables whose time evolution has to be captured. Naturally, the number of dynamically interacting quantities in a concrete system can be arbitrarily large.

Sustained rhythmical behaviors are ubiquitous in nature (Glass & Mackey, 1988). It is debatable whether such a behavior is suitably modeled as a perturbed periodic solution of a dynamical system or a chaotic such. In the IGO, the main bifurcation mechanism leading till chaos is frequency doubling (Zhusubaliyev et al., 2012a). Therefore, the existence of a periodic solution is a central question in the IGO as it defines its very function. The focus here is, consequently, on the simplest kind of periodic solution (1-cycle) characterized by just one impulse in the pulse-modulation feedback in the least period.

In this paper, a generalization of the IGO to models with arbitrary continuous part order m, henceforth termed as IGO(m), is proposed. The existence and uniqueness of a 1-cycle for $m = 3$ were established by Churilov et al. (2009) and later extended to $m = 1$ (Zhusubaliyev et al., 2012c). We examine the general IGO(m) model for cycle existence and uniqueness, which constitutes the main contribution of this work.

First, we show that IGO(m) possesses at least one 1-cycle (Theorem 1). Furthermore, this property applies to a broad class of impulsive systems with Hurwitz stable and positive continuous-time part (Remark 8).

Second, we prove that the 1-cycle is unique for dimensions $m \leq 10$ (Theorem 2), thus generalizing Theorem 1 in Churilov et al. (2009). As discussed in Section 4, this development is far from being straightforward. It relies on the theory of divided differences and the Opitz formula allowing to compute an analytic function of a matrix with two-diagonal structure.

Third, we examine the problem of 1-cycle uniqueness in IGO(m) with $m \geq 11$. Surprisingly, in this situation, the uniqueness may fail to hold and an example of such a case is given in Section 6. The uniqueness is, however, ensured if the derivative of the frequency modulator function does not attain anomalously large values (Theorem 3).

The rest of the paper is organized as follows. After summarizing the notation, the IGO(m) model is introduced in Section 2. Section 3 formulates the problem at hand, namely the existence and the uniqueness of 1-cycles in IGO(m). Solutions to these problems are presented in Section 4, with the proofs following separately in Section 5. An example of IGO(11) with three distinct 1-cycles is given in Section 6. Appendices contain necessary information about divided differences and the Opitz formula (Appendix A) and a proof of one technical lemma on special functions (Appendix B).

Notation

The symbol $\triangleq$ henceforth means "defined as".

As usual, $\mathbb{R}$ and $\mathbb{R}_+$ stand, respectively, for the sets of all and nonnegative real numbers. The real vector space of dimension m is then $\mathbb{R}^m$. We use $\mathbb{N}_0$ to denote the set of nonnegative integers $\{0, 1, \ldots\}$.

As usual, $\dot{z}(t)$ denotes the derivative of the variable z at time $t \geq 0$. For a function f whose argument has meaning other than time, we denote the derivative as f'. The same symbol is used to denote the Jacobian matrices: For a mapping $Q : \mathbb{R}^m \to \mathbb{R}^m$, the symbol $Q'(x)$ denotes the Jacobian matrix evaluated at $x \in \mathbb{R}^m$.

2. The impulsive Goodwin's oscillator

Consider a continuous-time autonomous system

$$\dot{x}(t) = Ax(t), \quad y(t) = Cx(t) \tag{1}$$

with the state $x(t) \in \mathbb{R}^m$, the output $y(t) \in \mathbb{R}$, and the state–space matrices structured as

$$A = \begin{bmatrix} -a_1 & 0 & \ldots & & 0 \\ g_1 & -a_2 & 0 & & \vdots \\ 0 & g_2 & -a_3 & & \\ \vdots & & \ddots & \ddots & \\ 0 & \ldots & & g_{m-1} & -a_m \end{bmatrix}, C = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}^\top . \tag{2}$$

Assuming positive $a_i, i = 1, \ldots, m$ and $g_i, i = 1, \ldots, m-1$, the matrix A is both Hurwitz and Metzler.

Introduce an infinite sequence of time instants $t_n > 0, n \in \mathbb{N}_0$ generated by the recursion

$$t_{n+1} = t_n + T_n, \quad T_n = \Phi(y(t_n)). \tag{3}$$

The state vector of system (1) undergoes jumps at the times t_n governed by

$$\begin{aligned} x(t_n^+) &= x(t_n^-) + \lambda_n B,\ \lambda_n = F(y(t_n)), \\ B^\top &= \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}. \end{aligned} \tag{4}$$

Here $\Phi : \mathbb{R} \to \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ are known functions. In impulsive control systems (Gelig & Churilov, 1998), they are usually referred to as the frequency and amplitude modulation function, respectively. Interpreting the jumps as events, impulsive feedback (3),(4) can be seen as a *self-triggered* (Heemels et al., 2012) controller, because the output of the system at time t_n uniquely determines the subsequent jump instant t_{n+1}.

With $m = 3$, model (1)–(4) is known as the impulsive Goodwin's oscillator (IGO) (Churilov et al., 2016). Below, a generalization of the IGO to an arbitrary order m of the continuous part (1), i.e. IGO(m), is treated.

Notice that Φ, F are not generally required to be continuous to guarantee a unique solution to hybrid system (1)–(4). Nevertheless, their continuity will be assumed to prove the existence of periodic solutions. Following Churilov et al. (2009), we also assume that

$$\forall y \geq 0 \quad \Phi_1 \leq \Phi(y) \leq \Phi_2,\ F_1 \leq F(y) \leq F_2, \tag{5}$$

where Φ_1, Φ_2, F_1, F_2 are *positive* constant numbers. This entails a number of important properties of the IGO that are proved similarly to the case of $m = 3$ (Churilov et al., 2009; Zhusubaliyev et al., 2012a). Namely, IGO(m) is a *positive* system also for any order m, i.e., for positive initial conditions $\forall i : x_i(0) > 0$, the solution remains positive $\forall i : x_i(t) > 0$. Furthermore, a solution $x(t), t \in [0, \infty)$ admits the following ultimate bounds

$$V_i \leq \liminf_{t\to\infty} x_i(t) \leq \limsup_{t\to\infty} x_i(t) \leq H_i, \tag{6}$$

$$V_1 = \frac{F_1}{e^{a_1\Phi_2}-1}, \quad H_1 = \frac{F_2}{1-e^{-a_1\Phi_1}},$$
$$V_i = \frac{g_{i-1}}{a_i}V_{i-1}, \quad H_i = \frac{g_{i-1}}{a_i}H_{i-1}, \quad \forall i = 2,\dots,m. \tag{7}$$

In this paper, we focus on *periodic* solutions such that $x(t+T) = x(t)$, for some $T > 0$. For such a solution, lim inf and lim sup in (6) can be omitted.

Motivated by application to feedback endocrine regulation, additional monotonicity restrictions were imposed on the frequency and amplitude modulation functions of IGO in Churilov et al. (2009). It was in particular assumed that Φ is *non-decreasing* and F is *non-increasing*. These assumptions are consistent with the experimentally observed behavior of the pulse-modulated feedback loop in testosterone (Te) regulation (Keenan & Veldhuis, 1998; Wu et al., 1987). A decrease in the concentration of Te increases both the frequency and amplitude of the gonadotropin-releasing hormone pulses, which in turn stimulate the Te production. In fact, as will be shown in this paper, the existence of periodic solutions does not require the monotonicity assumption. Moreover, we will prove that a certain periodic solution termed as 1-cycle always exists. At the same time, the monotonicity of nonlinear characteristics allows to prove, under certain conditions, the uniqueness of 1-cycle.

3. Problem formulation: 1-cycle

A fundamental property of IGO(3) (Churilov et al., 2009) is that it always possesses a unique periodic solution featuring only one jump over the (minimal) period $T > 0$, i.e. a *1-cycle*. Then, (3) becomes

$$t_{n+1} = t_n + T, \quad \Phi(y(t_n)) = T, \quad \forall n \in \mathbb{N}_0.$$

With the notation $X_n = x(t_n^-)$, the return map $X_{n+1} = Q(X_n)$, for $n = 0, 1, \dots$, of IGO(m) is given (Churilov, 2020; Churilov et al., 2009) by

$$Q(x) \triangleq e^{\Phi(Cx)A}(x + F(Cx)B), \quad x \in \mathbb{R}^m_+. \tag{8}$$

As shown in Churilov (2020), Churilov et al. (2009), a 1-cycle corresponds to a *fixed point* of the map Q. For such a point $x_* = Q(x_*)$, the corresponding 1-cycle is found as

$$\begin{aligned} x(t) &= e^{(t-t_n)A}(x_* + F(Cx_*)B), \quad t \in (t_n, t_{n+1}),\\ X_n &= x_*, \ x(t_n^+) = x_* + F(Cx_*)B, \\ t_{n+1} &= nT, \quad T = \Phi(Cx_*), \quad n \in \mathbb{N}_0. \end{aligned} \tag{9}$$

In view of the positivity of the IGO(m), admissible 1-cycles correspond to fixed points $x_* \in \mathbb{R}^m_+$. For such a solution, periodic solution (9) will stay in $\mathbb{R}^m_+$ for all $t \geq 0$.

In this paper, we address the problems of existence and uniqueness of feasible (positive) fixed points:

Problem A. Does IGO(m) always have a feasible 1-cycle? Equivalently, does the corresponding mapping Q have a fixed point $x_* = Q(x_*) \in \mathbb{R}^m_+$?

Below, in Theorem 1, we give an affirmative answer to Problem A for an *arbitrary* m. This existence property is actually valid for a much more general class of impulsive systems (Remark 8).

A natural question of how many distinct 1-cycles an IGO(m) might have then arises:

Problem B. Is the feasible 1-cycle of IGO(m) (equivalently, the fixed point $x_* \in \mathbb{R}^m_+$ of Q) unique?

The uniqueness of 1-cycle for IGO(3) established in Churilov et al. (2009, Theorem 1) is generalized to $m \leq 10$ in Theorem 2 of the present paper. For $m = 11$, however, it is possible to find parameter values $a_i, g_i > 0$ and functions F, Φ such that the corresponding IGO has three distinct 1-cycles (see counterexample in Section 6). Multiple 1-cycles are although highly uncommon. As Theorems 2 and 3 show, the uniqueness can always be secured by limiting Φ' or by letting the impulses to be sufficiently sparse, i.e. bounding Φ_1 from below.

4. Main results

In this section, we state the main result of the paper providing complete solutions to Problems A and B and formulated in Theorems 1–3. Their proofs are summarized separately in Section 5.

Problem A: Existence of 1-cycles in IGO(m)

The so-called "equation of periods" (Churilov et al., 2009) characterizes the feasible fixed points of Q introduced in (8)

$$y = R(y) \triangleq F(y)C(e^{-\Phi(y)A} - I)^{-1}B, \quad y \in \mathbb{R}_+. \tag{10}$$

Since $\Phi(y) > 0$ for $y \geq 0$ thanks to (5) and A is Hurwitz, the inverse matrix in (10) is well-defined.

Note that for $x \in \mathbb{R}^m_+$, the equation $Q(x) = x$ can be equivalently written as

$$x = F(Cx)(e^{-\Phi(Cx)A} - I)^{-1}B, \tag{11}$$

and, therefore, $y = Cx$ obeys (10). Conversely, if y is a root of Eq. (10), then $x = F(y)(e^{-\Phi(y)A} - I)^{-1}B$ obeys (11), entailing $Q(x) = x$. However, it is not obvious that such a vector x is positive and the latter fact is ensured by one of the statements in the theorem below.

Theorem 1. *For all values of the parameters $a_1, \dots, a_m > 0$, $g_1, \dots, g_{m-1} > 0$ and* continuous *functions Φ, F obeying (5), the following statements are valid:*

(a) The function $R(\cdot)$ defined in (10) is uniformly strictly positive and bounded on $[0, \infty)$;
(b) Eq. (10) has at least one solution; all its solutions are strictly positive.
(c) For every solution y of (10), the vector $x = F(y)(e^{-\Phi(y)A} - I)^{-1}B$ is a positive fixed point of return map (8);

Hence, IGO(m) always has at least one positive 1-cycle.

Noticeably, Theorem 1 does not impose any monotonicity restrictions on F and Φ. As will be shown (Remark 8), this theorem generalizes to a broad class of impulsive systems with positive and stable continuous-time part (1), whose matrices A, B, C may differ in structure from .

Problem B: Uniqueness of 1-cycles in IGO(m)

An elegant result established in Churilov et al. (2009, Theorem 1) states that, in the case $m = 3$, the solution to (10) is *unique*, because the function R is *non-increasing* on $[0, \infty)$. This monotonicity property, proved in Churilov et al. (2009) for $m = 3$ by evoking the Jensen inequality, remains valid for $1 \leq m \leq 10$, as shown below.

Theorem 2. *For all $1 \leq m \leq 10$, positive parameter values $a_i, g_i > 0$, and continuous non-increasing functions F and non-decreasing functions Φ satisfying (5), the function R defined in (10) is non-increasing on $[0, \infty)$. In particular, (10) has a unique positive solution, and the corresponding IGO(m) has a unique 1-cycle.*

These statements retain their validity if one replaces the condition $m \leq 10$ by the inequality

$$\frac{m-1}{a_{\min}} \leq \Phi_1, \quad a_{\min} \triangleq \min\{a_i : i = 1, \dots, m\} > 0. \tag{12}$$

In Section 5, we will show that the uniqueness of 1-cycle cannot be generally established for $m = 11$ and it is possible to find an IGO(11) with at least three different 1-cycles. The numerical example of this in Section 6 requires the function Φ to possess very large derivative at some points (violating also (12)). By forbidding excessive values of Φ', one can guarantee uniqueness for the order $m \geq 11$ as stated by the next theorem.

Recall the Riemann ζ-function

$$\zeta(s) \triangleq \sum_{k=1}^{\infty} k^{-s}, \quad s > 1. \tag{13}$$

Theorem 3. *Consider an IGO(m), $m \geq 11$, whose modulation functions Φ, F are, respectively, non-increasing and non-decreasing. Assume also that Φ, F are absolutely continuous and, furthermore, for each $y > 0$, one has*

$$\Phi'(y) \leq \frac{C_m}{g_1 \dots g_{m-1} F(0)}, \; C_m \triangleq \frac{(2\pi)^m}{2(m-1)\zeta(m)}. \tag{14}$$

Then, the function $y - R(y)$ is strictly increasing on $\mathbb{R}_+$, Eq. (10) has only one solution, and thus the IGO(m) has a unique 1-cycle.

The sequence $\zeta(m)$, $m = 1, 2, \dots$ is decreasing and $\zeta(m) \geq 1$. Hence, for all $m \geq 11$, $\zeta(m) \leq \zeta(11) \approx 1.005$ and, consequently, C_m grows exponentially as $m \to \infty$. Numerical evaluation yields $C_{11} \approx 3.01 \cdot 10^7$, $C_{12} \approx 1.72 \cdot 10^8$, $C_{13} \approx 9.91 \cdot 10^8$. Condition (14) is thus not very restrictive for large m but, yet, cannot be fully abolished (see the example in Section 6).

Discussion

Remarkably, none of Theorems 1–3 requires the nonlinearities F, Φ to be differentiable everywhere. In the case of Theorem 3, we only need absolute continuity, which ensures existence of $\Phi'(y), F'(y)$ at *almost every* point $y \in \mathbb{R}_+$. If F and Φ are *continuously differentiable* in a vicinity of the fixed point $y_* = R(y_*)$ in (10), then the (local exponential) *orbital stability* of the corresponding 1-cycle can be examined (see Churilov (2020), where the underlying stability definitions can be found). Namely, the 1-cycle defined by a fixed point of the map Q (as stated in Theorem 1) is orbitally stable if and only if the Jacobian matrix $Q'(x_*)$ is Schur stable (Churilov, 2020, Theorem 3). Obviously, $Q'(x_*)$ is fully determined by the parameters of continuous part and the values $F(y_*)$, $\Phi(y_*)$, $F'(y_*)$, $\Phi'(y_*)$.

It should be noticed that the method of proving Theorem 1 in Churilov et al. (2009), although it yields the results of Theorems 1 and 2 for $m = 3$, is not applicable to a general IGO(m) for several reasons. First, both existence and uniqueness are derived in Churilov et al. (2009) from the monotonicity of the function R, which, as proved above, does not hold for $m > 10$ without additional assumptions, whereas Theorem 1 (existence of 1-cycle) retains its validity. Second, the method of proving this monotonicity property is based on an analytic representation of $R(y)$ and its derivative $R'(y)$ in $m = 3$ case, which was obtained in the proof of Churilov et al. (2009, Theorem 1) by a straightforward computation.[1] In the general case considered in the present paper, these two functions are computed by using the Opitz formula and the method of divided differences (Appendix A). These tools have not been exploited in Churilov et al. (2009). Third, the closed-form representation of the derivative $R'(y)$ allows to derive its positivity from the Jensen inequality, which trick, to the best of our knowledge, cannot be applied for $m > 3$. Hence, while following the same line of reasoning as Churilov et al. (2009), this paper substantially generalizes the results of the latter by applying a different set of mathematical tools.

5. Proofs and auxiliary results

This section summarizes the proofs of the Theorems formulated in Section 4 and also establishes necessary auxiliary technical statements that might be of use elsewhere.

5.1. Lemmas and proof of Theorem 1

The key step in proving Theorem 1 is to show that the column vector $(e^{-\xi A} - I)^{-1}B$ is strictly positive for all $\xi > 0$ (see Corollary 6). One way of proving this is to use the Opitz formula, as will be discussed in Remark 12. However, we give a simpler direct proof, which remains valid for a broader class of Metzler matrices and is based on the definition of the matrix exponential, on one hand, and the duality between Metzler matrices and weighted directed graphs, on the other hand. Combining these, it will be shown (Lemma 4) that the matrix $(e^{-\xi A} - I)^{-1}$, for each $\xi > 0$, encodes, in some sense, all possible walks in the graph of matrix A. Namely, the off-diagonal (i, j)-entry is positive if and only if a walk from i to j exists in the matrix's graph.

Recall that the matrix is called *nonnegative* (respectively, Metzler) if all its entries (respectively, all its off-diagonal entries) are nonnegative. Hence, if A is a Metzler matrix, then $A + nI$ is nonnegative for some $n \in \mathbb{R}$ being large enough.

Following Horn and Johnson (2012), we introduce the *graph* $\Gamma(P)$ of a nonnegative square $n \times n$ matrix $P = (p_{ij})$. In this graph, the nodes are indexed 1 through n, and a directed arc $i \to j$ is present if and only if $p_{ij} > 0$. Positive diagonal entries stand for self-arcs. For each $k = 1, 2, \dots$, the matrix B^k has a positive entry $(P^k)_{ij} > 0$ if and only if $\Gamma(P)$ contains a directed walk of length k connecting i to j (this walk may contain self-loops and visit some vertices multiple times). We may formally generalize the definition of the graph to Metzler square matrices: given such a matrix $A = (a_{ij})_{i,j \in I}$, we connect two nodes i, j if and only if $a_{ij} > 0$.

The proof of Theorem 1 is based on the following positivity result.

Lemma 4. *For every Metzler and Hurwitz matrix $A \in \mathbb{R}^{n \times n}$, the matrix-valued function $\Theta(\xi) = (e^{-\xi A} - I)^{-1}$ exists and is nonnegative for all $\xi > 0$ with strictly positive diagonal entries $\Theta_{ii}(\xi) > 0$ $i = 1, \dots, n$. Each off-diagonal entry $\Theta_{ij}(\xi)$, $i \neq j$ is positive $\forall \xi > 0$ if and only if the graph $\Gamma[A]$ contains a path from i to j; otherwise, $\Theta_{ij}(\xi) \equiv 0$.*

Proof. Notice first that the matrix exponential

$$e^P = \sum_{k=0}^{\infty} \frac{1}{k!} P^k$$

of a *nonnegative* matrix P is also a nonnegative matrix. Furthermore, $(e^P)_{ij} > 0$ if and only if either $i = j$ or a directed walk from i to j exists in $\Gamma(B)$.

For an arbitrary $\xi \in \mathbb{R}$, the graphs of A and $A + \xi I$ may differ only by the presence of self-arcs, which do not influence connectivity. Hence, two nodes i and $j \neq i$ are connected by a directed walk in $\Gamma(A)$ if and only if they are connected by such in $\Gamma(A + \xi I)$. Choosing $\xi > 0$ large enough, the matrix $A + \xi I$ is nonnegative. Hence, $(e^A)_{ij} = e^{-\xi}(e^{A+\xi I})_{ij}$ is nonnegative for all i, j, being positive if and only if $i = j$ or a walk leads from i to j in $\Gamma(A)$. The same statements hold true for $e^{\xi A}$ if $\xi > 0$, because matrix ξA is Hurwitz and Metzler for any $\xi > 0$, having same graph as A.

By noticing that $e^{\xi A}$ has the eigenvalues $e^{\xi \lambda_j(A)}$, where $\lambda_j(A)$ are the eigenvalues of A, one concludes that the exponential $e^{\xi A}$ has

[1] Churilov et al. (2009) adopt a modeling assumption that a_1, a_2, a_3 are pairwise distinct, which is abolished here.

the spectral radius $\rho(e^{\xi A}) = \max_j |e^{\lambda_j(A)}| = e^{\max_j \operatorname{Re} \lambda_j(A)} < 1$. Hence

$$\Theta(\xi) = e^{\xi A}(I - e^{\xi A})^{-1} = e^{\xi A}\sum_{k=0}^{\infty}(e^{\xi A})^k = \sum_{k=1}^{\infty}(e^{\xi A})^k$$

is well-defined and nonnegative for all $\xi > 0$. Furthermore, $\Theta_{ij}(\xi) > 0$ if and only if $i = j$ or i is connected to j by a walk in $\Gamma(A)$ (otherwise, the (i, j)-entry of all summands vanishes), which completes the proof □

Corollary 5. *Let $A = (a_{ij})_{i,j\in I}$ be a Hurwitz and Metzler matrix and b, c be two nonnegative column vectors of same dimension as A. Then $c^\top\Theta(\xi)b > 0$ for all $\xi > 0$ if and only if there exist indices $i, j \in I$ such that the elements $c_i > 0$, $b_j > 0$ and either $i = j$ or $\Gamma(A)$ contains a directed walk from i to j. If the latter condition is violated, then $c^\top\Theta(\xi)b \equiv 0$.*

Proof. Notice that $c^\top\Theta(\xi)b = \sum_{i,j} c_i\Theta_{ij}(\xi)b_j$. For $\xi > 0$, all summands in the latter sum are nonnegative, and thus $c^\top\Theta(\xi)b \geq 0$. The latter inequality is strict if and only if at least one summand is positive $c_i\Theta_{ij}(\xi)b_j > 0$, which is possible if and only if $c_i, b_j > 0$ and $\Theta_{ij}(\xi) > 0$. The statement now follows from Lemma 4 □

Corollary 6. *For the matrix A in and the column B in (4), the column $(e^{-\xi A} - I)^{-1}B$ is positive for $\xi > 0$.*

Proof. The graph $\Gamma(A)$ contains a unidirectional chain $n \to (n-1) \to \dots \to 1$ thanks to inequalities $g_i > 0\,\forall i$. Hence, each node $i = 2, \dots, n$ is connected to 1 by a directed walk. Applying Corollary 5 to A, $b = B$ and the coordinate vectors $c = e_1, e_2, \dots, e_n$, one concludes that all elements of $(e^{-\xi A}-I)^{-1}B$ are positive for $\xi > 0$ □

Proof of Theorem 1

Now all the auxiliary results are in place to prove the claim of Theorem 1.

Proof of statement (a). Corollary 6 ensures that function $r(\xi) = C(e^{-\xi A} - I)^{-1}B$ is positive for $\xi > 0$. Also, $r(\xi)$ is continuous at every point $\xi \in (0, \infty)$. Notice that $R(y) = F(y)r(\Phi(y))$ due to (10). In view of (5), for every $y \geq 0$, one has

$$0 < F_1 \min_{\xi\in[\Phi_1,\Phi_2]} r(\xi) \leq R(y) \leq F_2 \max_{\xi\in[\Phi_1,\Phi_2]} r(\xi) < \infty,$$

where the minimum and the maximum exist due to the Weierstrass extreme value theorem. This proves statement (a) of the Theorem: function r is uniformly positive and bounded.

Proof of statement (b). Recall that F, Φ are assumed to be continuous, and hence $y - R(y)$ is also continuous function on $\mathbb{R}_+$ attaining a negative value at $y = 0$ and positive values where y is large enough. The existence of a solution in (10) is now straightforward from the intermediate value theorem.

Proof of statement (c). Recall that the fixed-point equation can be re-written as (11) or, equivalently, $x = F(y)(e^{-\Phi(y)A} - I)^{-1}B$ where y is a root of scalar equation (10). Since $F(y) \geq F_1 > 0$, Corollary 6 entails that this fixed point has positive coordinates in the state space of (1). □

Remark 7. The proof of Theorem 1 implies that all the roots of (10) belong, in fact, to the closed interval

$$F_1 \min_{\xi\in[\Phi_1,\Phi_2]} r(\xi) \leq y = R(y) \leq F_2 \max_{\xi\in[\Phi_1,\Phi_2]} r(\xi).$$

The minimum and maximum can, in turn, be estimated by using the explicit representation of r provided by (18) and Lemma 10 in Section 5.2. This facilitates the numerical solution of (10) by e.g. the bisection method. We omit the technical details here for brevity.

Remark 8. Theorem 1 can be generalized to guarantee the existence of a 1-cycle in a broader class of impulsive systems (1), (3), (4) than those with the matrix structures specified in . Corollary 6 remains valid for any Hurwitz and Metzler matrix A and a column B such that each node i of the graph $\Gamma(A)$ either corresponds to $B_i > 0$ or is connected by a path to some node j such that $B_j > 0$. In such a case, statements (1)-(3) remain valid.

Remark 9. Notice also that, assuming that F, Φ are continuous and (5) holds, the map Q admits a nonnegative fixed point (whose components, however, may be zero) whenever A is Hurwitz and Metzler and B, C are nonnegative. Indeed, Corollary 5 states that either $r(\xi) > 0\,\forall\xi > 0$ or $r(\xi) \equiv 0$ (in this degenerate case, the output $y(t)$ is decoupled from the input $u(t)$). In the former case, Statement (1) and Statement (2) of Theorem 1 are valid; in the latter case, the equation of periods $y = R(y)$ has the unique solution $y = 0$. In both cases, vector $x = F(y)(e^{-\Phi(y)A} - I)^{-1}B$ is a nonnegative (Lemma 4) fixed point of the map Q.

5.2. Lemmas and proofs of Theorems 2 and 3

To prove Theorems 2–3, we will need a more thorough analysis of the "equation of periods" in (10). In fact, we will prove that the assumptions of each theorem imply that the function $y - R(y)$ is increasing on $\mathbb{R}_+$. Thanks to Theorem 1, this function is negative as $y \to 0+$ and positive as $y \to +\infty$, which implies the existence of a unique solution to (10). In order to examine the equation of periods, it is convenient to write its right-hand side $R(y)$ as $R(y) = r(\Phi(y))F(y)$, where

$$r(\xi) \triangleq C(e^{-\xi A} - I)^{-1}B. \tag{15}$$

It will be shown that under the assumptions of Theorem 2, the function $r(\Phi(y))$ is decreasing, and hence $R(y)$ is also decreasing as a product of two decreasing positive functions. The situation of Theorem 3 is more delicate; in this case R may be non-monotone, however, we prove that $R'(y) < 1$, and hence $y - R(y)$ is nevertheless increasing. Both results rely on Lemma 11, which provides representations of $r(\xi)$ and its derivative as divided differences of some known functions, and the subsequent technical Lemma 14, providing upper estimates for $r'(\xi)$.

Divided differences and opitz formula

In this subsection, we intensively use divided differences (DD) and the Opitz formula (see Appendix A where the necessary background is summarized).

Given a function $f : I \to \mathbb{R}$ on the interval $I \subseteq \mathbb{R}$ and $k + 1$ points $x_0, \dots, x_k$, $k \in \mathbb{N}_0$, $f[x_0, \dots, x_k]$ stands for the kth order DD (briefly, k-DD) evaluated at $x_0, \dots, x_k$.

A useful property of k-DD in the present context is the following extension of the mean-value theorem.

Lemma 10 (de Boor (2005), Section 8)**.** *Suppose that $f : I \to \mathbb{R}$ is k times differentiable on I and let $x_0, \dots, x_k \in I$. Then a point $\bar{x} \in [\min_i x_i, \max_i x_i]$ exists such that*

$$f[x_0, \dots, x_k] = \frac{1}{k!}f^{(k)}(\bar{x}). \tag{16}$$

By substituting $x_0 = \dots = x_k = \xi$, one thus has

$$f[\underbrace{\xi, \dots, \xi}_{k+1}] = \frac{1}{k!}f^{(k)}(\xi)\ \forall\xi \in I. \tag{17}$$

Representation of $r(\xi)$ as a DD

Introduce two auxiliary functions

$$\varphi(x) = 1/(e^x - 1), \quad \psi(x) = -x\varphi'(x) = xe^x/(e^x - 1)^2.$$

The Opitz formula (Appendix A, Eq. (A.6)) leads to the following result.

Lemma 11. *Consider state–space matrices , and let $\bar{g} = g_1 \dots g_{m-1} > 0$. Then, for all $\xi > 0$, the function $r(\xi)$ and its derivative are found as*

$$r(\xi) = (-\xi)^{m-1}\bar{g}\varphi[\xi a_1, \dots, \xi a_m], \tag{18}$$

$$r'(\xi) = (-\xi)^{m-2}\bar{g}\psi[\xi a_1, \dots, \xi a_m]. \tag{19}$$

Proof.

Step 1: Let $\bar{g}_i \triangleq g_1 \dots g_{i-1}$ for $i = 2, \dots, m$, (hence, $\bar{g}_m = \bar{g}$), and $\bar{g}_1 \triangleq 1$. Notice first that

$$A = S\Lambda S^{-1}, \quad S = \mathrm{diag}(\bar{g}_1, \bar{g}_2, \bar{g}_3, \dots, \bar{g}_m).$$

The matrix Λ is two-diagonal with the eigenvalues $-a_i$, $i = 1, \dots, m$ on the main diagonal and ones on the diagonal below. One can check that

$$CS = \bar{g}_m C = \bar{g}C, \quad S^{-1}B = \bar{g}_1 B = B.$$

Step 2: For a function f analytic in a vicinity of the eigenvalues $-a_1, \dots, -a_m$, one thus has $f(A) = Sf(\Lambda)S^{-1}$, furthermore, $Cf(A)B = \bar{g}f(\Lambda)_{m,1}$ (the subscript denotes the $(m, 1)$ entry of the matrix $f(\Lambda)$). In virtue of (A.6), it follows

$$Cf(A)B = \bar{g}f[-a_1, \dots, -a_m].$$

Then Lemma 18 implies

$$Cf(-\xi A)B = Cf_{-\xi}(A)B = (-\xi)^{m-1}\bar{g}f[\xi a_1, \dots, \xi a_m],$$

(in accordance with Lemma 18, $f_{-\xi}(x) = f(-\xi x)$).

Step 3: Equality (18) is now straightforward by noticing that $r(\xi) = C\varphi(-\xi A)B$. To prove (19), recall that, for any differentiable invertible matrix function X, one has $(X(\xi)^{-1})' = -X(\xi)^{-1}X'(\xi)X(\xi)^{-1}$. Therefore,

$$\frac{d}{d\xi}(e^{-\xi A} - I)^{-1} = (e^{-\xi A} - I)^{-1}Ae^{-\xi A}(e^{-\xi A} - I)^{-1} = \\ = Ae^{-\xi A}(e^{-\xi A} - I)^{-2} = (-\xi)^{-1}\psi(-\xi A).$$

Hence, $r'(\xi) = (-\xi)^{-1}C\psi(-\xi A)B$, entailing (19) $\square$

Introducing the *polylogarithm* (Wei & Guo, 2014; Wood, 1992) of order $s \in \mathbb{R}$

$$\mathrm{Li}_s(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^s}, \quad z \in \mathbb{C}, |z| < 1, \tag{20}$$

it can be checked that $\varphi(y) = -1 + 1/(1 - e^{-y}) = \mathrm{Li}_0(e^{-y})$ and, by using induction over k,

$$\varphi^{(k)}(y) = (-1)^k \mathrm{Li}_{-k}(e^{-y}). \tag{21}$$

Remark 12. Notice that $\mathrm{Li}_s(z) > 0$ for z being a real number from $(0, 1)$. Equality (21) thus shows that φ is *completely monotonic* (Miller & Samko, 2001): $(-1)^k\varphi^{(k)}(y) > 0$ for all $y > 0$. In agreement with Corollary 6, $r(\xi) > 0$ for all $\xi > 0$, thanks to (16) and (18), for any order m and every choice of parameters $a_i, g_i > 0$.

As follows from Lemma 14, the function ψ, is *not* completely monotonic, and hence (19) does not allow to establish that $r'(\xi) < 0$ for all $\xi > 0$. Nevertheless, for a low order m, the derivative $r'(\xi)$ is indeed sign-preserving, which allows to prove Theorem 2. For an exact formulation, we state a corollary.

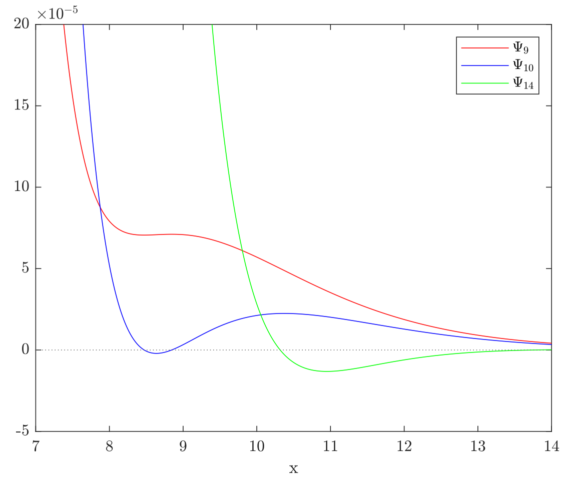


Fig. 1. Graphs of $\Psi_k(x)$ for $k = 9, 10, 14$.

Corollary 13. *If $(-1)^{m-1}\psi^{(m-1)}(\zeta) > 0$ at all $\zeta > 0$, then $r' < 0$ (i.e., r is decreasing) on $(0, \infty)$. More generally, r is decreasing on any interval (ξ_0, ξ_1) provided that $(-1)^{m-1}\psi^{(m-1)}(\zeta) > 0$ for $\zeta \in (\xi_0 \min_i a_i, \xi_1 \max_i a_i)$.*

Proof. The proof is immediate from (16) and (19) (applied to $k = m - 1$) $\square$

Corollary 13 implies, e.g., that, for $m = 3$, the function r is decreasing, because ψ is convex (Churilov et al., 2009).

The derivatives $\psi^{(k)}$ and their estimates

The derivatives of the function ψ, in fact, are also closely related to polylogarithm (20) as summarized in the following lemma.

Lemma 14. *For each $k = 1, 2, \dots$, one has*

$$\Psi_k(x) \triangleq (-1)^k\psi^{(k)}(x) = \\ = x\mathrm{Li}_{-k-1}(e^{-x}) - k\mathrm{Li}_{-k}(e^{-x}), \tag{22}$$

where Ψ_k possesses the following properties:

(i) *$\Psi_k(x) > 0$ for $0 \le x < \bar{x}(k) \triangleq 2\pi / \sqrt[k+1]{2k\zeta(k+1)}$;*
(ii) *$\Psi_k(x) > 0$ for $x \ge k$;*
(iii) *$\Psi_k(x) > 0$ for all $x > 0$ if $k \le 9$;*
(iv) *in general, the following inequality holds*

$$\Psi_k(x) \ge -2k\frac{k!}{(2\pi)^{(k+1)}}\zeta(k+1) \quad \forall x \ge 0, \tag{23}$$

where ζ is the Riemann ζ-function (13);

(v) *however, Ψ_{10} attains negative values at some points $x > 0$.*

The proof of Lemma 14 is quite technical and given in Appendix B. Numerical simulation shows, in fact, that Ψ_k is negative for all $k > 9$. Fig. 1 illustrates the behavior of Ψ_9 (positive) and functions Ψ_{10}, Ψ_{14} that can attain negative values.

Combining the mean-value formula (Lemma 10) with (19) and (23), the following corollary is immediate.

Corollary 15. *For each $\xi > 0$, the derivative $r'(\xi)$ admits the following upper bound*

$$r'(\xi) < \frac{\bar{g}}{C_m}, \tag{24}$$

where C_m is defined in (14).

As discussed in Section 4, C_m^{-1} decays exponentially as $m \to \infty$. Nevertheless, the right-hand side of (24) is positive, and for some parameters of IGO(m), $m \geq 11$, it is possible that r' attains positive values. In such a situation, the IGO may possess multiple 1-cycles (Section 6).

Proof of Theorem 2

If $m \leq 10$, Lemma 14 implies that $(-1)^{m-1}\psi^{(m-1)}(\xi) = \Psi_{m-1}(\xi) > 0$ for all $\xi > 0$; in general, $\Psi_{m-1}(\xi) > 0$ for $\xi \geq m-1$. Applying Corollary 13, one proves that $r(\cdot)$ decreases on the interval (ρ, ∞), where

$$\rho \triangleq \begin{cases} 0, & 1 \leq m \leq 10, \\ \frac{m-1}{\min_i a_i}, & \text{otherwise.} \end{cases}$$

When either $1 \leq m \leq 10$ or inequality (12) holds, then, obviously, $\Phi(y) > \rho$ for all $y \in \mathbb{R}_+$. Recalling that Φ, F are, respectively, non-decreasing and non-increasing, $R(y) = r(\Phi(y))F(y)$ is thus a non-increasing function, which means that Eq. (10) has only one solution on $\mathbb{R}_+$, and thus the IGO(m) has a unique 1-cycle □

Proof of Theorem 3

Theorem 3 is straightforward from Corollary 15. Indeed, the composition $r(\Phi(y))$ of a continuously differentiable (thus, locally Lipschitz) function and an absolutely continuous function is absolutely continuous, and one has

$$\frac{d}{dy} r(\Phi(y)) = r'(\Phi(y))\Phi'(y) < \frac{1}{F(0)},$$

for almost all $y > 0$ in view of (14) and (24). The function $R(y) = r(\Phi(y))F(y)$ is now also absolutely continuous as a product of two absolutely continuous functions. Recalling that $F'(y) \leq 0$ at almost all $y > 0$, one has $(y - R(y))' = 1 - r'(\Phi(y))\Phi'(y)F(y) - r(\Phi(y))F'(y) > 0$, hence, $y - R(y)$ is increasing on $(0, \infty)$. Here, we used the fact that $0 \leq F(y) \leq F(0)$ for all $y > 0$ □

6. An example of the IGO with multiple 1-cycles

In this subsection, we construct IGO(m) with at least three distinct 1-cycles for every m such that $\Psi_{m-1}(v_0) < 0$ at some point v_0. This holds, e.g., for $m = 11$ (Lemma 14). Let Φ_{σ,y_*} be the Gaussian density distribution function with variance σ^2 and expectation y_*, that is,

$$\Phi_{\sigma,y_*}(y) \triangleq \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{(s-y_*)^2}{2\sigma^2}} ds.$$

For each $\sigma > 0$, one has $0 < \Phi_{\sigma,y_*}(y_*) = 1/2 < \Phi_{\sigma,y_*}(\infty) = 1$; also, Φ_{σ,y_*} is strictly increasing. By construction, the derivative

$$\Phi'_{\sigma,y_*}(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-y_*)^2}{2\sigma^2}}$$

attains its maximum $1/(\sigma\sqrt{2\pi})$ at $y = y_*$.

The existence of multiple 1-cycles is established by the following lemma.

Lemma 16. *Choose numbers $y_* > 0$, $\sigma > 0$, and let $v_0 > 0$ be a point where $\Psi_{m-1}(v_0) < 0$. Define IGO(m) with the following parameters:*

- *a non-increasing differentiable function F obeying (5);*
- *$\Phi = \Phi_{\sigma,y_*}$ (this function is strictly increasing on $\mathbb{R}$);*
- *$a_1 = \cdots = a_m = a \triangleq v_0/\Phi_{\sigma,y_*}(y_*) = 2v_0$;*
- *finally, $g_1, \ldots, g_{m-1} > 0$ are such that,*[2]

$$\bar{g} = (-1)^{(m-1)} \frac{2^{m-1}(m-1)!y_*}{\varphi^{(m-1)}(v_0)F(y_*)}. \tag{25}$$

Then, for a small enough $\sigma > 0$, this IGO possesses at least three distinct positive 1-cycles.

Proof. Combining (17) and (18), one has

$$r(\Phi(y_*)) = \frac{(-1)^{(m-1)}\Phi(y_*)^{m-1}\bar{g}}{(m-1)!}\varphi^{(m-1)}(a(\sigma)\Phi(y_*)) = \\ = \frac{(-1)^{(m-1)}\bar{g}}{2^{m-1}(m-1)!}\varphi^{(m-1)}(v_0) = \frac{y_*}{F(y_*)}.$$

Recalling that $R(y) = r(\Phi(y))F(y)$, one shows that $R(y_*) = y_*$.

Retracing the arguments from the proof of Theorem 3, one has

$$(y - R(y))'|_{y=y_*} = 1 - \underbrace{r'(\Phi(y_*))\Phi'(y_*)F(y_*)}_{P_1} - \\ - \underbrace{r(\Phi(y_*))F'(y_*)}_{P_2}.$$

In view of (18) and (25), P_2 does not depend on σ, being determined by y_* and F only:

$$P_2 = \frac{y_* F'(y_*)}{F(y_*)}.$$

Recalling that $\Phi'(y_*) = 1/(\sigma\sqrt{2\pi})$ and applying (19),

$$P_1 = r'(\Phi(y_*))\Phi'(y_*)F(y_*) = \\ = \frac{(-1)^{m-2}\bar{g}\Phi(y_*)^{m-2}\psi^{(m-1)}(a(\sigma)\Phi(y_*))\Phi'(y_*)F(y_*)}{(m-1)!} = \\ = -\frac{\bar{g}F(y_*)\Psi_{m-1}(v_0)}{2^{m-2}(m-1)!\sigma\sqrt{2\pi}} = \\ = \frac{2(-1)^{m-1}y_*}{\varphi^{(m-1)}(v_0)} \frac{(-\Psi_{m-1}(v_0))}{\sigma\sqrt{2\pi}} > 0.$$

One notices that P_1 can be arbitrarily large for small $\sigma > 0$; In particular, it is possible to choose $\sigma > 0$ in such a way that $1 - R'(y_*) < 0$. Since $y_* - R(y_*) = 0$, in there exists $\varepsilon \in (0, y_*)$ such that

$$y - R(y) > 0, \quad y \in (y_* - \varepsilon, y_*),$$
$$y - R(y) < 0, \quad y \in (y_*, y_* + \varepsilon).$$

On the other hand, $y - R(y) < 0$ as $y \to 0+$ and $y - R(y) \to +\infty$ as $y \to \infty$ (see the proof of Theorem 1). Hence, (10) has at least two additional solutions $y_1 \in (0, y_*)$ and $y_2 \in (y_*, \infty)$. In view of Theorem 1, y_1, y_*, y_2 correspond to three distinct 1-cycles of the IGO □

Remark 17. One may suspect that the existence of multiple 1-cycles is caused by the multiplicity of the eigenvalues $a_i = a = 2v_0$, however, this is not the case. The construction in Lemma 16 can be generalized to the case where a_i are close enough to $2v_0$ yet pairwise distinct. We omit this for brevity.

6.1. Numerical example

The existence of multiple 1-cycles for the IGO of order $m = 11$ is demonstrated now numerically by computations in Matlab, following the IGO construction method in Lemma 16. Set $y_* = 2$, $\sigma = 2 \cdot 10^{-4}$, $F(y) = 1$ (constant) and $v_0 = 8.64$, which corresponds to $\Psi_{m-1}(v_0) = \Psi_{10}(v_0) < 0$. We consider matrices , where $a_1 = \cdots = a_{11} = a = 17.28$ and $g_1 = \cdots = g_{10} = 22.6486$, which correspond, in view of (21), to the values

$$\bar{g} = 3.5515 \cdot 10^{13}, \quad P_1 = 1.1257, \quad P_2 = 0.$$

In particular

$$(y - R(y))'|_{y=y_*} = -0.1257,$$

[2] Due to Remark 12 the right-hand side of (25) is positive.

which indicates the existence of three solutions y_1, y_*, y_2 to the equation $y - R(y) = 0$ and three corresponding 1-cycles. y_1, y_2 are found numerically to have the values

$$y_1 = 1.9998234, \quad y_2 = 2.0002739.$$

The fixed points of $Q(x)$ corresponding to y_1, y_*, y_2 are calculated according to Theorem 1 to

$$x_1 = \begin{bmatrix} 0.00019 \\ 0.00216 \\ 0.01213 \\ 0.04535 \\ 0.12731 \\ 0.28635 \\ 0.53835 \\ 0.87280 \\ 1.25282 \\ 1.63477 \\ 1.99982 \end{bmatrix}, \quad x_* = \begin{bmatrix} 0.00018 \\ 0.00200 \\ 0.01135 \\ 0.04287 \\ 0.12155 \\ 0.27608 \\ 0.52400 \\ 0.85724 \\ 1.24048 \\ 1.62906 \\ 2 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0.00015 \\ 0.00178 \\ 0.01024 \\ 0.03927 \\ 0.11305 \\ 0.26064 \\ 0.50199 \\ 0.83277 \\ 1.22041 \\ 1.61921 \\ 2.00027 \end{bmatrix}.$$

Stability of the corresponding 1-cycles is determined by the Schur stability of the Jacobian matrix

$$Q'(x) = e^{A\Phi(Cx)}(I + F'(Cx)BC) + \Phi'(Cx)AQ(x)C,$$

evaluated at the fixed points. The numerical calculation shows that all three 1-cycles are unstable, and the spectral radii of the corresponding Jacobian matrices are:

$$\rho(Q'(x_1)) = 68.64, \rho(Q'(x_*)) = 64.91, \rho(Q'(x_2)) = 58.47.$$

7. Conclusions and future work

A special case of periodic solutions in the impulsive Goodwin's oscillator (IGO) characterized by one impulse generated by the pulse-modulated feedback in the least period, i.e. a 1-cycle, is considered. The continuous part of the IGO is allowed to be of arbitrary order, in contrast with the established in the literature case of third-order dynamics. The structure of the continuous part is still assumed to be a chain of first-order blocks. It is proved that a 1-cycle always exists in the IGO, regardless of the continuous part order. Further, when the continuous part order is at most ten, the 1-cycle is unique. It is demonstrated, by a constricting an example, that uniqueness does not generally apply to higher orders of the continuous part, e.g. for order eleven. Uniqueness of 1-cycle can however be recovered by restricting the slopes of the modulation functions of the IGO or even by restricting the feedback impulses to be sufficiently sparse.

In conclusion, we mention two possible directions for future research.

Whereas this work is limited to analysis of 1-cycles, periodic solutions with multiple impulses in the least period may exist in the IGO as demonstrated by bifurcation analysis (Zhusubaliyev et al., 2012b). It can be proved (Churilov, 2020, Theorem 1) that every such solution corresponds to a Nth-order fixed point of mapping Q for some $N \geq 1$, that is, a point x_* such that

$$Q^N(x_*) \triangleq \underbrace{Q \circ Q \circ \ldots \circ Q}_{m \text{ times}}(x_*) = x_*,$$
$$Q^k(x_*) \neq x_* \quad \forall k = 1, \ldots, N-1.$$

Such a solution is called N-cycle (Churilov, 2020; Churilov et al., 2009; Zhusubaliyev et al., 2012b). For a given N-cycle, the local stability can be studied by checking the Schur stability of the Jacobian matrix

$$(Q^N)'(x_*) = Q'(Q^{N-1}(x_*))Q'(Q^{N-2}(x_*)) \ldots Q'(x_*).$$

The equation $Q^N(x) = x$ can be reduced to a cyclic system of N transcendental equations

$$x_1 = Q(x_N), x_2 = Q(x_1), \ldots, x_N = Q(x_{N-1}),$$

which can be simplified by means of the Opitz formula; the simplified form of this system for $N = 2$ and IGO(3) is presented in Theorem 3 from Churilov et al. (2009). Even in this special case, however, the unique solvability of these equations seems to be a challenging problem. Bifurcation analysis in Zhusubaliyev et al. (2012b) suggests that, in general, an IGO may have several N-cycles for $N > 1$, however, most of these cycles are unstable.

Unstable cycles are primarily of theoretical interest, and all current applications of the IGO are concerned with only stable solutions. This leads to another important problem of *monostability*, that is, existence of at most one stable solution for each combination of the parameters. Whereas bifurcation analysis of IGO(3) suggests that the model is monostable, the mathematical proof of this is currently lacked even in the scalar IGO(1) case (Zhusubaliyev et al., 2012c). Furthermore, the monostability apparently does not hold for more complex dynamics of the continuous part, as demonstrated by the IGO with time delay (Churilov et al., 2016; Zhusubaliyev et al., 2015).

Appendix A. Divided differences and opitz formula

Divided differences (DD) are widely used in numerical analysis and employed in this work to compute matrix functions. Here we review some basic properties of the DDs, referring the reader to Berezin and Zhidkov (1965), de Boor (2005), Horn and Johnson (1991) for further details.

Definitions of DD

Throughout this section, we deal with functions $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is some interval (possibly, open). The standard definition of the kth order DD (briefly, k-DD) for such a function at a sequence of pairwise distinct points $x_0, \ldots, x_k \in I$ is as follows. We formally define the 0-DD as $f[x_0] \triangleq f(x_0)$ and, subsequently, the 1-DD as

$$f[x_0, x_1] \triangleq \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

For $k \geq 2$, the k-DD is constructed inductively as

$$f[x_0, \ldots, x_k] = \frac{f[x_1, \ldots, x_k] - f[x_0, \ldots, x_{k-1}]}{x_k - x_0}. \tag{A.1}$$

An equivalent and more compact definition of the k-DD is based on the concept of *interpolation polynomial*, which can be written in the Lagrange or Newton form. By definition, the interpolation polynomial of f at the points $x_0, \ldots, x_k$ (where $x_i \neq x_j \, \forall i \neq j$) is the (unique) polynomial $L = L_{f,x_0,\ldots,x_k}$ of degree $\leq k$ such that all x_i are roots of the equation

$$L(x) = f(x). \tag{A.2}$$

It can be proven (Berezin & Zhidkov, 1965) that L admits the form

$$\begin{aligned} L_{f,x_0,\ldots,x_k}(x) = f[x_0] + f[x_0, x_1](x - x_0) + \cdots \\ + f[x_0, x_1, \ldots, x_k] \prod_{j=0}^{k-1}(x - x_j), \end{aligned} \tag{A.3}$$

known as Newton's form of the interpolation polynomial. This leads to an alternative definition of the k-DD $f[x_0, x_1, \ldots, x_k]$, which is the *lead* (degree k) coefficient of the interpolation polynomial.

If f is differentiable k times on I, then the latter approach allows to define the k-DD to an *arbitrary* sequence $x_0, \ldots, x_k$. If some number ξ occurs s times in this sequence ($1 \leq s \leq k$), then ξ is a root of (A.2) with multiplicity s: $L^{(p)}(\xi) = f^{(p)}(\xi) \quad p = 0, \ldots, s-1$. Adopting such a convention, the interpolation polynomial remains uniquely determined (Berezin & Zhidkov, 1965), and hence its lead coefficient $f[x_0, x_1, \ldots, x_k]$ is well defined.

Example. If $x_0 = \cdots = x_k = \xi$, then the interpolation polynomial is nothing else than the Taylor sum

$$L(x) = \sum_{j=0}^{k} \frac{f^{(j)}(\xi)}{j!}(x-\xi)^j, \tag{A.4}$$

whose lead coefficient is

$$f[\xi, \dots, \xi] = f^{(k)}(\xi)/k!.$$

Technical properties of DDs

In the next subsections, we will use the following simple property of the DD.

Lemma 18 (*Scaling*). *Given a function $f : (a, \infty) \to \mathbb{R}$ and a number $\xi \neq 0$, denote $f_\xi(x) \triangleq f(\xi x)$. Then*

$$f_\xi[x_0, \dots, x_k] = \xi^k f[\xi x_0, \dots, \xi x_k].$$

Proof. Notice that if $L(x) = L_{f,x_0,\dots,x_k}$ is the interpolation polynomial for f, then $L(\xi x)$ is the interpolation polynomial for f_ξ. Recalling that $f_\xi[x_0, \dots, x_k]$ and $f[x_0, \dots, x_k]$ are the lead coefficients of respectively $L(\xi x)$, $L(x)$, one obtains the desired relation □

Finally, we notice that the DDs linearly depend on f, that is, for two functions f_1, f_2 defined on (a, b) and two coefficients α_1, α_2, one has $(\alpha_1 f_1 + \alpha_2 f_2)[x_0, \dots, x_k] = \alpha_1 f_1[x_0, \dots, x_k] + \alpha_2 f_2[x_0, \dots, x_k]$.

Functions on matrices and the opitz formula

Let $\mathcal{D} \subseteq \mathbb{C}$ be an open simply connected set containing the eigenvalues λ_j of the matrix A and $f : \mathbb{C} \to \mathbb{C}$ be holomorphic on $\mathcal{D}$. Then, for any simple closed curve $\Gamma \subset \mathcal{D}$ that encircles all λ_j in the counter-clockwise direction (Horn & Johnson, 1991, Section 6.2),

$$f(A) \triangleq \frac{1}{2\pi\imath} \oint_\Gamma f(z)(zI - A)^{-1} dz. \tag{A.5}$$

In particular, if S is an invertible matrix, then $f(SAS^{-1}) = Sf(A)S^{-1}$. Also, for every two functions f, g, the matrices $f(A)$ and $g(A)$ commute.

Consider now the two-diagonal matrix below

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 1 & \lambda_2 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & 1 & \lambda_m \end{bmatrix}.$$

Assuming f complex analytic in vicinity of $\lambda_1, \dots, \lambda_n$, the matrix $f(\Lambda)$ admits an elegant representation, known as the *Opitz formula*[3] (Eller, 1987). Namely, $f(\Lambda)$ is the lower-triangular matrix whose entries are

$$(f(\Lambda))_{ij} = \begin{cases} f[\lambda_i, \dots, \lambda_j], & i \geq j, \\ 0, & i < j. \end{cases} \tag{A.6}$$

For instance, the left-bottom corner entry is the $(m-1)$-DD of function f, that is, $f(\Lambda)_{m1} = f[\lambda_1, \dots, \lambda_m]$.

[3] Usually, the Opitz formula is given for upper-triangular two-diagonal matrices, the case of lower triangular is straightforward by noticing that $f(\Lambda^\top) = f(\Lambda^\top)^\top$.

Appendix B. Proof of Lemma 14

To obtain the expression for Ψ_k, note that $\psi(x)$ can be expressed as a series:

$$\psi(x) = \frac{xe^x}{(e^x-1)^2} = -x\left(\frac{1}{1-e^{-x}}\right)' =$$
$$= -x\left(\sum_{j=0}^{\infty} e^{-jx}\right)' = \sum_{j=0}^{\infty} xje^{-jx} = \sum_{j=1}^{\infty} xje^{-jx},$$

which implies the expression for the kth derivative

$$\psi^{(k)}(x) = (-1)^k \sum_{j=1}^{\infty} j^k (xj - k)e^{-jx} =$$
$$= (-1)^k (x\mathrm{Li}_{-k-1}(e^{-x}) - k\mathrm{Li}_{-k}(e^{-x})),$$

resulting in (22).

Statement (ii) follows from Wei and Guo (2014, Theorem 8).

To prove statements (i) and (iv), we need a representation of the polylogarithm of order $(-k) < 0$ (Gradshteyn & Ryzhik, 2014, 9.553)

$$\mathrm{Li}_{-k}(e^{-x}) = k! \sum_{l=-\infty}^{\infty} (2\pi li + x)^{-k-1},$$

which leads to an alternative representation of Ψ_k:

$$\Psi_k(x) = x\mathrm{Li}_{-k-1}(e^{-x}) - k\mathrm{Li}_{-k}(e^{-x}) =$$
$$(k+1)! \sum_{l=-\infty}^{\infty} x(2\pi li + x)^{-k-2} - kk! \sum_{l=-\infty}^{\infty} (2\pi li + x)^{-k-1} =$$
$$= k! \sum_{l=-\infty}^{\infty} ((k+1)x(2\pi li + x)^{-k-2} - k(2\pi li + x)^{-k-1}) =$$
$$= k! \sum_{l=-\infty}^{\infty} (2\pi li + x)^{-k-2}((k+1)x - k(2\pi li + x)) =$$
$$= k!\Big(\frac{1}{x^{k+1}} + \underbrace{\sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \frac{x - 2\pi kli}{(x + 2\pi li)^{k+2}}}_{=h_k(x)}\Big).$$

Notice that for each $x > 0$, one has

$$\left|\frac{x - 2\pi kli}{(x + 2\pi li)^{k+2}}\right| \leq \frac{1}{|x + 2\pi li|^{k+1}} \left|\frac{x - 2\pi kli}{x + 2\pi li}\right|,$$

where the multipliers are, obviously, less, than $(2\pi|l|)^{-k-1}$ and k, respectively. Statement (iv) and (23) are now straightforward from the following estimate:

$$|h_k(x)| \leq 2\sum_{l=1}^{\infty} \left|\frac{x - 2\pi kli}{(x + 2\pi li)^{k+2}}\right| \leq \frac{2k}{(2\pi)^{k+1}} \sum_{l=1}^{\infty} \frac{1}{l^{k+1}}.$$

To prove statement (i), it suffices to notice that $x = \bar{x}(k)$ is the unique real positive solution to the equation

$$\frac{1}{x^{k+1}} - \frac{2k}{(2\pi)^{k+1}} \sum_{l=1}^{\infty} \frac{1}{l^{k+1}} = 0;$$

Obviously, $\Psi_k(x) > 0$ as $0 < x < \bar{x}(k)$.

Statement (iii) is proved similarly, refining the estimate for the term $h_k(x)$. Notice that for $k \leq 4$, statement (iii) follows from statements (i) and (ii), because $\bar{x}(k) < k$. For $k = 5, \dots, 8$, one can use a more precise estimate:

$$h_k(x) = \underbrace{\sum_{\substack{l=-\infty \\ l\neq 0,\pm 1}}^{\infty} \frac{x - 2\pi kli}{(x + 2\pi li)^{k+2}}}_{\tilde{h}_k(x)} + \\ + \frac{x - 2\pi ki}{(x + 2\pi i)^{k+2}} + \frac{x + 2\pi ki}{(x - 2\pi i)^{k+2}},$$

where $\tilde{h}_k(x)$ is estimated similarly to $h_k(x)$, that is,

$$|\tilde{h}_k(x)| \leq \frac{2k}{(2\pi)^{k+1}} \sum_{l=2}^{\infty} \frac{1}{l^{k+1}} = \frac{2k(\zeta(k+1) - 1)}{(2\pi)^{k+1}}.$$

Therefore, one obtains the following estimate for Ψ_k:

$$\frac{\Psi_k(x)}{k!} \geq \frac{1}{x^{k+1}} - \frac{2k}{(2\pi)^{k+1}}(\zeta(k+1) - 1) + \\ + \frac{x - 2\pi ki}{(x + 2\pi i)^{k+2}} + \frac{x + 2\pi ki}{(x - 2\pi i)^{k+2}} = \\ = \frac{p_k(x)}{x^{k+1}(x^2 + 4\pi^2)^{k+2}},$$

where $p_k(x)$ is a polynomial of degree $3k + 5$ such that $p_k(0) = 4\pi^2 > 0$. To prove that $\Psi_k(x) > 0$ for $x > 0$, in view of statement (ii), it suffices to check that p_k has no real roots on $[0, k]$. This is indeed the case for $k = 5, \ldots, 8$, as reported in Table B.1 (the roots were found numerically using Matlab), however, for $k = 9$ this condition is violated.

To prove statement (iii) for $k = 9$, one needs an even more refined estimate of h_k as follows:

$$h_k(x) = \underbrace{\sum_{\substack{l=-\infty \\ l\neq 0,\pm 1,\pm 2}}^{\infty} \frac{x - 2\pi kli}{(x + 2\pi li)^{k+2}}}_{\bar{h}_k(x)} + \\ + \sum_{l=1}^{2} \frac{x - 2\pi kli}{(x + 2\pi li)^{k+2}} + \sum_{l=1}^{2} \frac{x + 2\pi kli}{(x - 2\pi li)^{k+2}},$$

where $\bar{h}_k(x)$ can be estimated similar to h_k, $\tilde{h}_k$:

$$|\bar{h}_k(x)| \leq \frac{2k}{(2\pi)^{k+1}} \sum_{l=3}^{\infty} \frac{1}{l^{k+1}} = \frac{2k(\zeta(k+1) - 1 - 2^{-k-1})}{(2\pi)^{k+1}}.$$

This entails a more refined estimate for Ψ_k:

$$\frac{\Psi_k(x)}{k!} \geq \frac{1}{x^{k+1}} - \frac{2k}{(2\pi)^{k+1}}(\zeta(k+1) - 1 - \frac{1}{2^{k+1}}) \\ + \frac{x - 2\pi ki}{(x + 2\pi i)^{k+2}} + \frac{x + 2\pi ki}{(x - 2\pi i)^{k+2}} \\ + \frac{x - 4\pi ki}{(x + 4\pi i)^{k+2}} + \frac{x + 4\pi ki}{(x - 4\pi i)^{k+2}} = \\ = \frac{q_k(x)}{x^{k+1}(4\pi^2 + x^2)^{k+2}(16\pi^2 + x^2)^{k+2}},$$

where $q_k(x)$ is a polynomial of order $5k + 9$ satisfying $q_k(0) = (64\pi^2)^{k+2} > 0$. As shown in Table B.1, the real roots of $q_9(x)$ are located outside the interval $(0, 9]$.

Finally, statement (v) can be validated by computing the polylogarithmic functions in Matlab:

$$\Psi_{10}(8.64) \approx -2.087496 \cdot 10^{-6} \quad \square$$

Table B.1
Real roots of polynomials p_k (for $k = 5, \ldots, 8$) and q_9.

k	Real roots
5	$\pm 9.563\ldots$
6	$10.115\ldots$
7	$\pm 10.369\ldots$
8	$10.291\ldots$
9	$\pm 15.456\ldots$

References

Aymerich, G. (1955). Cicli di prima e di seconda specie di un sistema meccanico autosostenuto impulsivamente. *Rendiconti del Seminario della Facolta di Scienze dell'Universita di Cagliari*, *25*, 26–36.

Bally, L., Thabit, H., & Hovorka, R. (2017). Closed-loop for type 1 diabetes - an introduction and appraisal for the generalist. *BMC Medicine*, *15*(1).

Berezin, I., & Zhidkov, N. (1965). *Computing methods, vol. 1*. Pergamon Press.

de Boor, C. (2005). Divided differences. *Surveys in Approximation Theory*, *1*, 46–69.

Cacace, F., Cusimano, V., & Palumbo, P. (2020). Optimal impulsive control with application to antiangiogenic tumor therapy. *IEEE Transactions on Control Systems Technology*, *28*(1), 106–117.

Churilov, A. N. (2020). Orbital stability of periodic solutions of an impulsive system with a linear continuous-time part. *AIMS Mathematics*, [ISSN: 2473-6988] *5*(1), 96–110.

Churilov, A., & Medvedev, A. (2016). Discrete-time map for an impulsive goodwin oscillator with a distributed delay. *Mathematics of Control, Signals, and Systems*, *28*(9).

Churilov, A., Medvedev, A., & Mattsson, P. (2014). Periodical solutions in a pulse-modulated model of endocrine regulation with time-delay. *IEEE Transactions on Automatic Control*, *59*(3), 728–733.

Churilov, A., Medvedev, A., & Shepeljavyi, A. (2009). Mathematical model of non-basal testosterone regulation in the male by pulse modulated feedback. *Automatica*, *45*(1), 78–85.

Churilov, A., Medvedev, A., & Zhusubaliyev, Z. T. (2016). Impulsive Goodwin oscillator with large delay: Periodic oscillations, bistability, and attractors. *Nonlinear Analysis. Hybrid Systems*, *21*, 171–183.

Eller, J. (1987). On functions of companion matrices. *Linear Algebra and its Applications*, *96*, 191–210.

Gelig, A. K., & Churilov, A. N. (1998). *Stability and oscillations of nonlinear pulse-modulated systems*. Boston: Birkhäuser.

Glass, L., & Mackey, M. C. (1988). *From clocks to chaos: The rhythms of life*. Princeton University Press.

Goodwin, B. C. (1965). Oscillatory behavior in enzymatic control processes. In G. Weber (Ed.), *Advances of enzime regulation, vol. 3* (pp. 425–438). Oxford: Pergamon.

Gradshteyn, I. S., & Ryzhik, I. M. (2014). *Table of integrals, series, and products*. Academic Press.

Heemels, W., Johansson, K., & Tabuada, P. (2012). An introduction to event-triggered and self-triggered control. In *Proc. of the 51st IEEE conference on decision and control* (pp. 3270–3285).

Horn, R. A., & Johnson, C. R. (1991). *Topics in matrix analysis*. Cambridge Univ. Press.

Horn, R. A., & Johnson, C. R. (2012). *Matrix analysis* (2nd ed.). Cambridge Univ. Press.

Huang, M., Li, J., Song, X., & Guo, H. (2012). Modeling impulsive injections of insulin: towards artificial pancreas. *SIAM Journal on Applied Mathematics*, *72*(5), 1524–1548.

Keenan, D. M., & Veldhuis, J. D. (1998). A biomathematical model of time-delayed feedback in the human male hypothalamic-pituitary-Leydig cell axis. *Americab Journal of Physiology-Endocrinology and Metabolism*, *275*(1), E157–E176.

Krasovskii, N. N., & Lidskii, E. A. (1961). Analytical design of controllers in systems with jump parameters. *Automation and Remote Control*, *22*, 1021–1025.

Lakshmikantham, V., Bainov, D. D., & Simeonov, P. S. (1989). *Theory of impulsive differential equations*. Singapore: World Scientific.

Medvedev, A., Churilov, A., & Shepeljavyi, A. (2006). Mathematical models of testosterone regulation. In *Stochastic optimization in informatics, no. 2* (pp. 147–158). Saint Petersburg State University (in Russian).

Miller, K. S., & Samko, S. G. (2001). Completely monotonic functions. *Integral Transforms and Special Functions*, *12*(4), 389–402.

Milman, V. D., & Myshkis, A. D. (1960). On the stability of motion in the presence of impulses. *Siberian Mathematical Journal*, *1*(2), 233–237 (in Russian).

Popp, K. (2000). Non-smooth mechanical systems. *Journal of Applied Mathematics and Mechanics*, [ISSN: 0021-8928] *64*(5), 765–772.

Samoilenko, A. M., & Perestyuk, N. A. (1995). *Impulsive differential equations*. World Scientific.

Smith, D. R. (1985). *Singular-perturbation theory an introduction with applications*. Cambridge: Cambridge University Press.

Walker, J., Terry, J. R., Tsaneva-Atanasova, K., Armstrong, S., McArdle, C., & Lightman, S. L. (2010). Encoding and decoding mechanisms of pulsatile hormone secretion. *Journal of Neuroendocrinology*, *22*(12), 1226–1238.

Wei, C.-F., & Guo, B.-N. (2014). Complete monotonicity of functions connected with the exponential function and derivatives. *Abstract and Applied Analysis*, Article 851213.

Wood, D. C. (1992). *The computation of polylogarithms*: *Technical report*, University of Kent, Canterbury, UK: UKC, University of Kent, Computing Laboratory.

Wu, F. C. W., Irby, D. C., Clarce, I. J., Cummins, J. T., & de Kretse, D. M. (1987). Effects of gonadotropin-releasing hormone pulse-frequency modulation on luteinizing hormone, foilicle-stimulating hormone and testosterone in hypothalamo/pituitary-disconnected rams. *Biology of Reproduction*, *37*(10), 501–505.

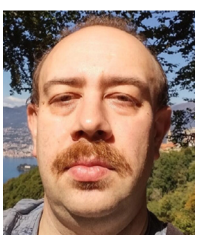
Zhang, S., & Chen, L. (2005). Chaos in three species food chain system with impulsive perturbations. *Chaos, Solitons & Fractals*, *24*(1), 73–83.

Zhusubaliyev, Z. T., Churilov, A., & Medvedev, A. (2012a). Bifurcation phenomena in an impulsive model of non-basal testosterone regulation. *Chaos*, *22*(1), 013121–1–013121–11.

Zhusubaliyev, Z., Churilov, A., & Medvedev, A. (2012b). Complex dynamics and chaos in a scalar linear continuous system with impulsive feedback. In *Proceedings of the American control conference*.

Zhusubaliyev, Z. T., Churilov, A., & Medvedev, A. (2012c). Complex dynamics and chaos in a scalar linear continuous system with impulsive feedback. In *Proceedings of the 2012 American control conference* (pp. 2419–2424).

Zhusubaliyev, Z., Mosekilde, E., Churilov, A., & A., M. (2015). Multistability and hidden attractors in an impulsive goodwin oscillator with time delay. *The European Physical Journal Special Topics*, *224*, 1519–1539.



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