Application of Stochastic Averaging to Vibrating Electro-mechanical Systems for Piezoelectric Energy Harvesting

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***Abstract*—Electro-mechanical systems are ubiquitous in engineering. For example, they are used in energy harvesting systems to convert random mechanical excitations into usable electrical power. The analysis and design of these electro-mechanical systems is particularly challenging, because concepts and methods of stochastic analysis and nonlinear dynamics are required. In this work we present a methodology for the analysis of nonlinear electro-mechanical systems with small internal friction, and subject to random external mechanical excitations. The method is based on the combined application of a model order reduction technique, to reduce the number of dynamical variables, thus reducing the complexity, and of stochastic averaging, to calculate statistical relevant quantities, and in particular expectations of the output electrical variables. As an example, we apply the proposed technique to the analysis of a piezoelectric energy harvesting system.**

I. Introduction

Electro-mechanical systems can be found in several engineering applications, among which harvesting systems for powering Internet of Things systems [1] represent a fertile research area.

When embedded into a complex environment, electro-mechanical systems are subject to the action of environmental stimuli, representing the superposition of many different phenomena. As such, these external actions appear random in nature, and can be modeled as a white, Gaussian distributed noise source [2]–[6].

When the random nature of environmental disturbances is taken into account, together with the intrinsically nonlinear nature of electro-mechanical systems, the problem of analysis and design of such structures becomes particularly challenging, because methods of stochastic analysis and nonlinear dynamics must be applied [7]–[12].

This research has been partially conducted within the Italian inter university PhD program in Sustainable Development and Climate Change.

In this contribution, we present a novel technique for the analysis of nonlinear electro-mechanical systems with small internal friction, and subject to random external stimuli, modeled as white Gaussian noise. We start from a rather general model of an electro-mechanical system, where the mechanical part is characterized by a nonlinear elastic potential, and the electrical part, formed by interconnected linear two terminal elements, is described by a linear system of state equations. The resulting model is made of nonlinear stochastic differential equations (SDEs).

Generalizing the results in [7], we use a nonlinear reduction technique, that amounts to rewrite electrical variables as functions of the mechanical quantities, here approximated by a single harmonic representation, thus reducing the number of variables and the complexity of the problem. The reduced system can be transformed into energy-angle equation system, where the energy is a slow variable, while the angle is a fast one. Application of stochastic averaging permits to eliminate the fast variable, thus reducing further the system of SDEs to a single SDE for the energy. The stationary probability density function for the energy, which can be used to calculate expectations of any relevant quantity, is then calculated analytically solving the associated Fokker-Planck equation.

As an example of application of the method, we study the model of a nonlinear piezoelectric energy harvesting for ambient mechanical vibrations, and we calculate the expectation of the output voltage. The accuracy of the method is verified comparing theoretical predictions with the results of extensive numerical simulations.

II. Electro-mechanical system modeling

The electro-mechanical harvester can be represented as a a two-port network, excited by mechanical quantities, like forces and velocities, at the input port, and yielding electrical

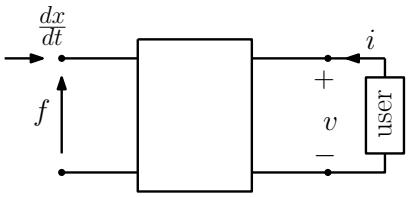

Fig. 1. Schematic representation of cascade connected two-port networks closed on a generic user.

quantities, like currents and voltages, at the output port, as schematized in figure 1. The internal structure of the electro-mechanical system can be quite complicated, being composed of different mechanical and electrical subsystems, as well as a transducer stage, that require additional variables, either mechanical, electrical or both, for a detailed description.

As an example, we consider an electro-mechanical system whose mechanical domain is described by the Lagrangian function $\mathcal{L}(z,\dot{z}) = T(\dot{z}) - \hat{U}(z) = (1/2)m\dot{z}^2 - \hat{U}(z)$, where m is an inertial mass, and z, $\dot{z}$ are the generalized coordinate and velocity, respectively, corresponding to a one dimensional mechanical oscillator. Consequently, $T(\dot{z})$, $\hat{U}(z)$ are the kinetic and the potential energy functions. Introducing also the dissipation potential $\mathcal{D}(\dot{z}) = (1/2)\hat{\varepsilon}\dot{z}^2$, where $\hat{\varepsilon}$ is the friction coefficient, the Lagrange equation of motion for the mechanical system takes the form

$$m\ddot{z} + \hat{\varepsilon}\dot{z} + \hat{U}'(z) = F(t) \tag{1}$$

where $F(t)$ represents the resultant of the forces applied to the mechanical domain. In the following, we shall assume that F is the sum of two contributions: an external random force due to the action of the environment, modelled as a white Gaussian term, and a linear force due to the action of the electrical variables. For the sake of simplicity, we shall assume that the resultant force has the same order of magnitude of the internal friction.

The electrical domain can be described as a linear circuit, composed by the interconnection of linear two-terminal (or even multi-terminal) elements. The state equations are a linear system of ordinary differential equations (ODEs), which can be derived combining Kirchhoff voltage and current laws together with the characteristic relationships of the electrical elements [13], thereby obtaining the ODE system:

$$\dot{\mathbf{z}}_e = \hat{\mathbf{A}}_e\mathbf{z}_e + \hat{\mathbf{B}}_e[z,\dot{z}]^T \tag{2}$$

where $\mathbf{z}_e : \mathbb{R} \mapsto \mathbb{R}^n$ is the vector of the electrical variables, $\hat{\mathbf{A}}_e \in \mathbb{R}^{n,n}$, $\hat{\mathbf{B}}_e \in \mathbb{R}^{n,2}$ are real valued matrices, and ^{T} denotes the transpose.

Combining (1) and (2), and rewriting as a system of first order SDEs yields

$$\dot{z}_1 = z_2 \tag{3a}$$

$$\dot{z}_2 = \left(-\frac{1}{m}\hat{U}'(z_1) - \frac{\hat{\varepsilon}}{m}z_2 + \frac{\hat{\varepsilon}}{m}\hat{\mathbf{b}}_m^T\mathbf{z}_e\right) + \frac{\sqrt{\hat{\varepsilon}}}{m}\eta(t) \tag{3b}$$

$$\dot{\mathbf{z}}_e = \hat{\mathbf{A}}_e\mathbf{z}_e + \hat{\mathbf{B}}_e\mathbf{z}_m \tag{3c}$$

where $\hat{\varepsilon}\,\hat{\mathbf{b}}_m^T\mathbf{z}_e$ is the mechanical force due to the action of the electrical domain, $\hat{\mathbf{b}}_m = [b_{m,1},\ldots,b_{m,n}]^T$ is the vector of the elctro-mechanical coupling constants, $\eta(t)$ is a scalar white Gaussian noise process, and $\mathbf{z}_m = [z_1, z_2]^T = [z,\dot{z}]^T$ is the vector obtained collecting the mechanical variables. With a limited loss of generality, we have assumed that internal friction and electro-mechanical coupling are a factor $\sqrt{\hat{\varepsilon}}$ stronger than the external random force.

III. SDE DESCRIPTION AND MODEL ORDER REDUCTION

Let $(\Omega,\mathcal{F},P)$ be a probability space, where Ω is the sample space, $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ is a filtration, i.e. the σ-algebra of all the events, and P a probability measure. A vector valued stochastic process $\mathbf{X}_t$ is a function $\mathbf{X}_t : \Omega \times T \mapsto \mathbb{R}^d$, i.e. a vector of random variables taking values in Ω, and parameterized by $t \in T$. The parameter space T is usually the half-line $[0,+\infty[$.

In the following we adopt the standard notation used in probability: A capital letters denotes a random variable, while a lower case letter denotes its possible value. A scalar Wiener process $W_t = W(t)$, is characterized by the expectation value $\mathrm{E}[W_t] = 0$ (symbol $\mathrm{E}[X_t]$ denotes expectation of the stochastic process X_t), the covariance $\mathrm{cov}(W_t,W_s) = \mathrm{E}[W_t\,W_s] = \min(t,s)$ and $W_t \sim \mathcal{N}(0,t)$, where symbol $\sim$ means "distributed as", and $\mathcal{N}(0,t)$ denotes centered normal distribution.

A d-dimensional system of SDEs driven by a scalar Wiener process takes the form

$$\mathrm{d}\mathbf{Z}_t = \mathbf{a}(\mathbf{Z}_t)\mathrm{d}t + \mathbf{B}(\mathbf{Z}_t)\mathrm{d}W_t \tag{4}$$

where $\mathbf{Z}_t : \Omega \times [0,T] \mapsto \mathbb{R}^d$ is a vector valued stochastic process. The d-dimensional vector $\mathbf{a} : \mathbb{R}^d \mapsto \mathbb{R}^d$ is called the drift function, while the vector valued function $\mathbf{B} : \mathbb{R}^d \mapsto \mathbb{R}^d$ is the diffusion. For a constant diffusion noise is additive (or unmodulated), while the general case $\mathbf{B}(\mathbf{Z}_t)$ corresponds to a modulated, or multiplicative, noise. The SDE system (3) can be rewritten in the form

$$\mathrm{d}\mathbf{Z}_t = \left(\hat{\mathbf{A}}\,\mathbf{Z}_t + \hat{\mathbf{n}}(\mathbf{Z}_t)\right)\mathrm{d}t + \hat{\mathbf{B}}\;\mathrm{d}W_t \tag{5}$$

where the matrix $\hat{\mathbf{A}} \in \mathbb{R}^{d,d}$, and $\hat{\mathbf{n}} : \mathbb{R}^d \mapsto \mathbb{R}^d$, with $d = n+2$, collect the linear and the nonlinear terms of the drift, respectively, and $\hat{\mathbf{B}} \in \mathbb{R}^d$ is a constant diffusion vector.

After application of the linear transformation described in [14] to normalize the SDE system (3), we find

$$\mathrm{d}X_1 = X_2\;\mathrm{d}t \tag{6a}$$

$$\mathrm{d}X_2 = \left(-U'(X_1) - \varepsilon X_2 + \varepsilon\,\mathbf{b}_m^T\mathbf{X}_e\right)\mathrm{d}t + \sqrt{\varepsilon}\;\mathrm{d}W_t \tag{6b}$$

$$\mathrm{d}\mathbf{X}_e = \left(\mathbf{A}_e\mathbf{X}_e + \mathbf{b}_1X_1 + \mathbf{b}_2X_2\right)\mathrm{d}t \tag{6c}$$

where X_1, X_2 and $\mathbf{X}_e$ are dimensionless variables corresponding to z_1, z_2 and $\mathbf{z}_e$, respectively, $\mathbf{b}_1 \in \mathbb{R}^n$ and $\mathbf{b}_2 \in \mathbb{R}^n$ are the columns of the scaled matrix $\mathbf{B}_e \in \mathbb{R}^{n,n}$, and $\varepsilon = \hat{\varepsilon}/m$.

Our goal is to derive a reduced order model, "eliminating" the electrical variables. To this end, we apply a method similar to that used in [7]. Let's introduce the normalized mechanical

energy $E = (1/2)X_2^2 + U(X_1)$. Application of Itô formula yields

$$\mathrm{d}E = \varepsilon \left(\frac{1}{2} - X_2^2 + \mathbf{b}_e^T \mathbf{X}_e \right) \mathrm{d}t + \sqrt{\varepsilon} X_2 \, \mathrm{d}W_t$$

Then, for $\varepsilon \ll 1$, the mecha nical energy is a slow, or nearly constant, variable.

According to [7] we assume $X_1(t) = A(E)\sin(\Omega(E)t)$, where $A(E)$ and $\Omega(E)$ are an unknown amplitude and angular frequency, respectively. Because the energy is nearly constant, (6a) implies $X_2(t) \simeq \Omega(E)A(E)\cos(\Omega(E)t)$, and

$$\frac{\mathrm{d}X_2}{\mathrm{d}t} = -\Omega^2(E)A(E)\sin(\Omega(E)t) = -\Omega^2(E)X_1$$

We look for a solution to (6c) in the form

$$\mathbf{X}_e = \mathbf{m}_1(E)X_1 + \mathbf{m}_2(E)X_2 \tag{7}$$

where $\mathbf{m}_1 \in \mathbb{R}^n$ and $\mathbf{m}_2 \in \mathbb{R}^n$ are unknown vectors to be determined. Taking the derivative and equating with (6c) gives

$$\begin{aligned} \left(\mathbf{A}_e\mathbf{m}_1 + \Omega^2\mathbf{I}_{n,n}\,\mathbf{m}_2 + \mathbf{b}_1\right) X_1 \\ + \left(-\mathbf{m}_1 + \mathbf{A}_e\mathbf{m}_2 + \mathbf{b}_2\right) X_2 = 0 \end{aligned}$$

where $\mathbf{I}_{n,n}$ is the identity matrix of size n. Since the equality holds for any t, we find

$$\begin{aligned} \mathbf{A}_e\mathbf{m}_1 + \Omega^2\mathbf{I}_{n,n}\,\mathbf{m}_2 + \mathbf{b}_1 &= 0 \\ -\mathbf{m}_1 + \mathbf{A}_e\mathbf{m}_2 + \mathbf{b}_2 &= 0 \end{aligned}$$

Solving the system yields

$$\mathbf{m}_1(E) = -\,\mathbf{A}_e \left(\mathbf{A}_e^2 + \Omega^2\mathbf{I}_{n,n}\right)^{-1} (\mathbf{b}_1 + \mathbf{A}_e\mathbf{b}_2) + \mathbf{b}_2 \tag{8a}$$

$$\mathbf{m}_2(E) = -\left(\mathbf{A}_e^2 + \Omega^2\mathbf{I}_{n,n}\right)^{-1} (\mathbf{b}_1 + \mathbf{A}_e\mathbf{b}_2) \tag{8b}$$

We would like to stress that because the energy is a function of X_1 and X_2, the vector valued function $\mathbf{X}_e$ is a nonlinear function of X_1 and X_2. Substituting (7) into (6b) we find the reduced SDE system

$$\mathrm{d}X_1 = X_2 \, \mathrm{d}t \tag{9a}$$

$$\mathrm{d}X_2 = \left(-U'(X_1) + \varepsilon f(X_1, X_2)\right) \mathrm{d}t + \sqrt{\varepsilon} \, \mathrm{d}W_t \tag{9b}$$

with

$$\begin{aligned} f(X_1, X_2) = &- X_2 + \mathbf{b}_m^T \big(\mathbf{m}_1(X_1, X_2)X_1 \\ &+ \mathbf{m}_2(X_1, X_2)X_2\big) \end{aligned} \tag{10}$$

IV. Stochastic averaging

For $\varepsilon = 0$ system (9) is *Hamiltonian*, i.e. it admits a first integral that corresponds to the total (normalized) energy $E = (1/2)X_2^2 + U(X_1)$. Hamiltonian systems admit a representation in terms of energy and angle coordinates, such that the state equations take the form [15]

$$\dot{E} = 0 \tag{11a}$$

$$\dot{\theta} = \Omega(E) \tag{11b}$$

where θ is the angle function and $\Omega(E)$ the angular frequency.

The energy-angle representation can be extended to the perturbed system (9). In fact, since the Jacobian of the coordinate transformation $(x, y) \to (E, \theta)$ is regular, by the implicit function theorem the coordinate transformation is invertible for small values of ε, at least locally.

Theorem 1 (Energy-angle SDEs given $E = E(X_1, X_2)$ and $\theta = \theta(X_1, X_2)$.):

Consider the nonlinear oscillator described by the SDE system (9). The energy and the angle are Itô processes. If an explicit expression for the angle variable as a function of the old coordinate is available in the form $\theta = \theta(X_1, X_2)$, the the energy and the angle are solutions of the SDE system

$$\mathrm{d}E = \varepsilon a_E(E, \theta)\mathrm{d}t + \sqrt{\varepsilon} B_E(E, \theta)\mathrm{d}W_t \tag{12a}$$

$$\mathrm{d}\theta = \big(\Omega(E) + \varepsilon a_\theta(E, \theta)\big)\mathrm{d}t + \sqrt{\varepsilon} B_\theta(E, \theta)\mathrm{d}W_t \tag{12b}$$

where

$$a_E(E, \theta) = \frac{1}{2} + X_2 \, f(X_1, X_2) \tag{13a}$$

$$B_E(E, \theta) = X_2 \tag{13b}$$

$$\Omega(E) = \frac{\partial \theta}{\partial X_1} X_2 - \frac{\partial \theta}{\partial X_2} U'(X_1) \tag{13c}$$

$$a_\theta(E, \theta) = \frac{\partial \theta}{\partial X_2} f(X_1, X_2) + \frac{1}{2} \frac{\partial^2 \theta}{\partial X_2^2} \tag{13d}$$

$$B_\theta(E, \theta) = \frac{\partial \theta}{\partial X_2} \tag{13e}$$

Proof: That the energy and angle variables are Itô processes is a direct consequence of the fact that, for $\varepsilon = 0$, the coordinate transformation is invertible and of the implicit function theorem. The SDE system for the energy and angle are obtained by straightforward application of Itô formula, using the definition of the normalized energy. □

For $\varepsilon \ll 1$, equation (12) shows a time scale separation, with the energy being a slow (or nearly constant) variable with repect to the fast angle variable.

A classical theorem by Khasminskii [16] states that the slow varying process E converges weakly, e.g. in probability, to a one dimensional Markov process as $\varepsilon \to 0$, in a time interval $0 \le t \le T$ with $T = \mathcal{O}(1/\varepsilon)$. The Itô SDE system for the one dimensional Markov process is obtained by averaging the original SDEs (12a) with respect to the fast variable, while the slow one is kept constant [9], [16], thereby obtaining

$$\mathrm{d}E \simeq \varepsilon \overline{a}_E(E)\mathrm{d}t + \sqrt{\varepsilon}\, \overline{B}_E(E)\mathrm{d}W_t \tag{14}$$

where the averaged coefficients are

$$\overline{a}_E(E) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T a_E(E, \theta(t)) \, \mathrm{d}t \tag{15a}$$

$$\overline{B}_E(E) = \sqrt{\lim_{T \to +\infty} \frac{1}{T} \int_0^T B_E^2(E, \theta(t)) \, \mathrm{d}t} \tag{15b}$$

and $\theta(t)$ is the solution of the fast equation, calculated keeping the slow variable constant. In particular, for $\varepsilon \to 0$, the fast

equation admits the trivial solution $\theta = \Omega(E)t + \theta_0$, where θ_0 is the initial condition, and the averaged coefficients become

$$\overline{a}_E(E) = \frac{1}{2\pi}\int_0^{2\pi} a_E(E,\theta)\,\mathrm{d}\theta \tag{16a}$$

$$\overline{B}_E(E) = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} B_E^2(E,\theta)\,\mathrm{d}\theta} \tag{16b}$$

V. Probability density function and expected quantities

The energy-angle equations can be exploited to derive an accurate approximation for the probability density function, which can be used to find expectations.

The time scale separation between the slow energy and the fast angle variables, suggests that the stationary probability density function can be factorized in the form

$$p_{st}(E,\theta) = \hat{\rho}(E,\theta)\,\rho_{st}(E) \tag{17}$$

Now we take advantage of the fact that, for $\varepsilon \to 0$, the SDE for the angle becomes independent on the energy. The stationary solution to the Fokker-Planck equation (FPE) corresponding to (12b) is found solving

$$-\Omega(E)\frac{\partial \hat{\rho}(E,\theta)}{\partial \theta} = 0 \tag{18}$$

implying $\hat{\rho}(E,\theta) = h(E)$, where $h(E)$ is an arbitrary function of the energy. Imposing the normalizing condition

$$\int_0^{2\pi} \hat{\rho}(E,\theta)\mathrm{d}\theta = 1 \tag{19}$$

yields $\hat{\rho}(E,\theta\ = (2\pi)^{-1}$.

Similarly, the stationary FPE for the energy variable is

$$-\frac{\partial}{\partial E}\left(\overline{a}_E(E)\rho_{st}(E)\right) + \frac{1}{2}\frac{\partial^2}{\partial E^2}\left(\overline{B}_E^2(E)\rho_{st}(E)\right) = 0 \tag{20}$$

By separation of variables and imposing the null boundary conditions

$$\lim_{E\to+\infty}\rho_{st}(E) = \lim_{E\to+\infty}\frac{\partial \rho_{st}(E)}{\partial E} = 0 \tag{21}$$

we find the solution

$$\rho_{st}(E) = \frac{\mathcal{N}}{\overline{B}_E^2(E)}\exp\left(2\int\frac{\overline{a}_E(E)}{\overline{B}_E^2(E)}\mathrm{d}E\right) \tag{22}$$

where $\mathcal{N}$ is a normalizing constant whose value is determined imposing $\int_0^{+\infty}\rho_{st}(E)\mathrm{d}E = 1$.

From (7) we have

$$\mathbf{X}_e(E,\theta) = \mathbf{m}_1(E)X_1(E,\theta) + \mathbf{m}_2(E)X_2(E,\theta)$$

Therefore, expectation for any arbitrary function $F(\mathbf{X}_e(E,\theta))$ can be calculated as

$$E[F(\mathbf{X}_e(E,\theta))] = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{+\infty} F(\mathbf{X}_e(E,\theta))\rho_{st}(E)\,\mathrm{d}E\mathrm{d}\theta$$

VI. Application to energy harvesting

As an example of application, we consider the model of a cantilever beam piezoelectric energy harvester with Duffing type nonlinearity, such as the one discussed in [10], [12], [17], connected to a resistive electrical load in the absence of any intermediate stage that might enhance the energy transfer to the load itself. The normalized equations take the form

$$\mathrm{d}X_1 = X_2\,\mathrm{d}t \tag{23a}$$

$$\mathrm{d}X_2 = (-U'(X_1) - \varepsilon X_2 - \varepsilon X_3)\,\mathrm{d}t + \sqrt{\varepsilon}\mathrm{d}W_t \tag{23b}$$

$$\mathrm{d}X_3 = (\alpha X_2 - \gamma X_3)\,\mathrm{d}t \tag{23c}$$

where X_1 is proportional to the cantilever position, X_2 to the cantilever velocity, X_3 represents the normalized output voltage, and α, γ are real positive parameters linked to the electro-mechanical coupling and to the load, respectively. Nonlinearity is introduced by means of a normalized elastic potential of the form $U(X_1) = X_1^2/2 + X_1^4/4$. For $\varepsilon = 0$ the solution of the Hamiltonian system takes the form [10]

$$X_1(t) = \left(\frac{4E^2}{1+4E}\right)^{1/4}\mathrm{sd}(\theta,k) \tag{24a}$$

$$X_2(t) = \sqrt{2E}\,\mathrm{cd}(\theta,k)\,\mathrm{nd}(\theta,k) \tag{24b}$$

where $E = X_2^2(0)/2 + U(X_1(0))$ is the initial value of the energy, $\mathrm{sd}(\theta,k)$, $\mathrm{cd}(\theta,k)$ and $\mathrm{nd}(\theta,k)$ are the Jacobi elliptic functions and

$$k^2 = \frac{1}{2}\left(1 - \frac{1}{\sqrt{1+4E}}\right) \tag{25}$$

is the elliptic modulus [18]. The angle is $\theta(t) = \Omega(E)t$ with angular frequency

$$\Omega(E) = (1+4E)^{1/4} \tag{26}$$

Applying the method of Section III yields

$$m_1(E) = \frac{\alpha\Omega^2(E)}{\gamma^2+\Omega^2(E)}, \qquad m_2(E) = \frac{\alpha\gamma}{\gamma^2+\Omega^2(E)} \tag{27}$$

To calculate the coefficients of the averaged equation it is necessary to evaluate

$$\frac{1}{2\pi}\int_0^{2\pi} X_2^2(E,\theta)\,\mathrm{d}\theta = \frac{E}{2\mathcal{K}(E)}\int_0^{4\mathcal{K}(E)}\mathrm{cd}^2(u)\,\mathrm{nd}^2(u)\,\mathrm{d}u \tag{28}$$

where $\mathcal{K}(E)$ is the complete elliptic integral of the first kind [18]. The integral can be calculated using the relationships between the squares of the Jacobi elliptic functions:

$$k^2\mathrm{sd}^2u = \mathrm{nd}^2u - 1 \tag{29a}$$

$$k^2\mathrm{cd}^2(u) = 1 - k'^2\mathrm{nd}^2(u) \tag{29b}$$

where $k'^2 = 1 - k^2$ is the complementary modulus. Thus evaluating integral (34) is reduced to compute integrals of the type

$$I_n = \int \mathrm{nd}^n(u)\,\mathrm{d}u \tag{30}$$

which evaluation can be carried out using the Fourier series [19] and the recursive relationships between I_n, I_{n+2} and I_{n+4}

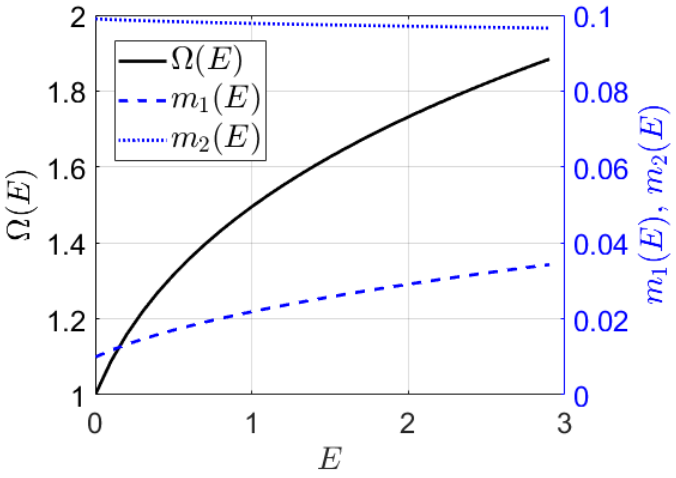

Fig. 2. Coefficients $\Omega(E)$ (solid black line, left scale), $m_1(E)$ and $m_2(E)$ (dashed and dotted blue lines, respectively, right scale) as a function of the energy. Parameters are $\alpha = 1$ and $\gamma = 10$.

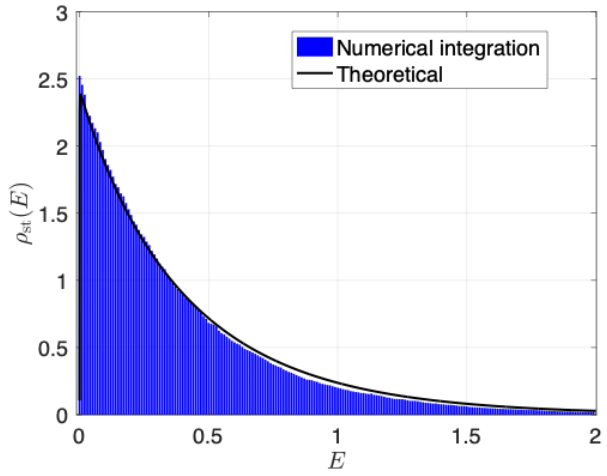

Fig. 3. Stationary probability density function for the SDEs system (23). Blue vertical bars are the result of numerical integration, black line is the theoretical prediction. Parameters values are $\alpha = 1$, $\gamma = 10$, and $\varepsilon = 0.25$.

[9]. The final result, after defining the complete elliptic integral of the second kind $\mathcal{E}(E)$ [18], reads

$$I_2 = \frac{4\mathcal{E}(E)}{k'^2} \tag{31a}$$

$$I_4 = \frac{8(2-k^2)\mathcal{E}(E) - 4(1-k^2)\mathcal{K}(E)}{3k'^4} \tag{31b}$$

Combining all these remarks together yields the following expressions for the averaged coefficients:

$$\overline{a}_E(E) = \frac{1}{2} - (1+m_2(E))\frac{E}{2\mathcal{K}(E)}\left(\frac{1}{k^2}I_2 - \frac{k'^2}{k^2}I_4\right) \tag{32a}$$

$$\overline{B}_E(E) = \left(\frac{E}{2\mathcal{K}(E)}\left(\frac{1}{k^2}I_2 - \frac{k'^2}{k^2}I_4\right)\right)^{1/2} \tag{32b}$$

These coefficients are finally used to calculate the stationary probability density function in (22).

The (normalized) energy dependence of the unperturbed angular frequency $\Omega(E)$ (right scale) and of the coefficients $m_1(E)$ and $m_2(E)$ (left scale) is shown in Figure 2 for $\alpha = 1$ and $\gamma = 10$.

Figure 3 shows the comparison between the stationary probability density function obtained using the approach described in sections III to V, and the result obtained through a numerical integration of the SDE system (23). The excellent matching between the two approaches clearly validates the adopted stochastic averaging procedure. For the numerical integration, we used a strong order one stochastic Runge-Kutta method. We performed relatively long simulations ($\Delta T = 10^4$), using a small (normalized) time integration step ($\delta t \simeq 37 \times 10^{-6}$). After integration, the initial transient interval was removed from the data, while we averaged over 10 simulations carried out for different realizations of the Wiener process. The probability to find the system in the energy range between E and $E + \mathrm{d}E$ was evaluated as the number of samples in that interval, normalized to the total number of samples.

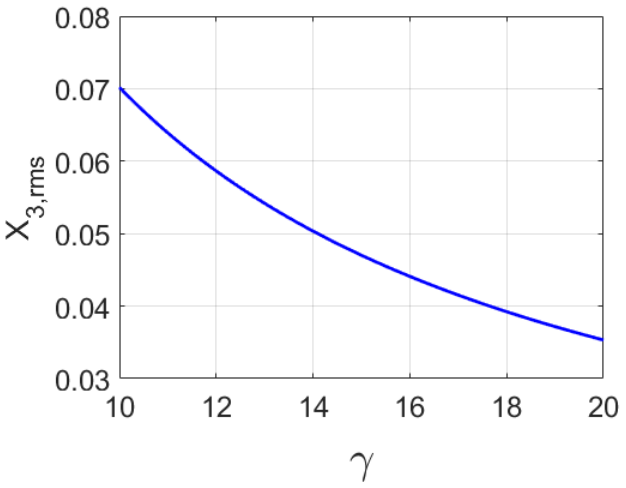

Fig. 4. Root mean square value of the normalized output voltage $X_{3,rms} = \sqrt{\mathrm{E}[X_3^2]}$ as a function of parameter γ. Other parameters are $\alpha = 1$, and $\varepsilon = 0.25$.

Finally, figure 4 shows the root mean square value of the normalized output voltage $X_{3,rms} = \sqrt{\mathrm{E}[X_3^2]}$ as a function of parameter γ. According to (7), the expectation is calculated as follows:

$$\mathrm{E}[X_3^2] = \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^{+\infty} \Big(m_1(E)X_1(E,\theta) + m_2 X_2(E,\theta)\Big)^2 \rho_{st}(E) \tag{33}$$

The integral of $\int_0^{2\pi} X_2^2(E,\theta)d\theta$ has already been discussed

(see (34)-(31)). For $\int_0^{2\pi} X_1^2(E,\theta)d\theta$, using (29) we have

$$\frac{1}{2\pi}\int_0^{2\pi} X_2^2(E,\theta)\,\mathrm{d}\theta = \frac{1}{4\mathcal{K}(E)}\left(\frac{4E^2}{1+4E}\right)^{1/2}\int_0^{4\mathcal{K}(E)} (\mathrm{nd}^2(u)-1)\,\mathrm{d}u \tag{34}$$

which can be solved using (30) and (31).

VII. Conclusions

In a complex environment, the external stimuli applied to an electro-mechanical systems can be represented as random excitations. If the system is also nonlinear, the analysis and design may become particularly challenging, because methods from stochastic analysis and nonlinear dynamics are necessary.

We have presented a methodology for the analysis of complex nonlinear electro-mechanical systems subject to random external forces, modeled as a white Gaussian noise. The method combines a nonlinear model order reduction technique, aimed at reducing the number of dynamic variables, and a stochastic averaging method, that allows a description of the reduced system in terms of a slow energy and a fast angle variables. The model order reduction technique is based on representing the electrical quantities as functions of mechanical variables, through a first harmonic approximation, thus reducing the number of variables and the complexity of the problem. The stochastic averaging method corresponds to averaging the stochastic dynamics with respect to the fast variable, under the assumption that the slow variable remains almost constant. The final result is a single SDE for the energy, whose probability density function can be calculated analytically integrating the corresponding Fokker-Planck equation. The knowledge of the probability density function allows to calculate all expected quantities analytically. The method is shown to provide accurate results for electro-mechanical systems with small internal friction.

As an example of application of the proposed technique, we have analysed the model of a cantilever beam piezoelectric energy harvesting system, that is frequently used in research on energy harvesting.

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