POLITECNICO DI TORINO
Repository ISTITUZIONALE

Local and non-local Poincaré inequalities on Lie groups

Original
Local and non-local Poincaré inequalities on Lie groups / Bruno, T., Peloso, M.M., Vallarino, M.. - In: BULLETIN OF THE LONDON MATHEMATICAL SOCIETY. - ISSN 0024-6093. - 54:6(2022), pp. 2162-2173. [10.1112/blms.12684]

Availability:
This version is available at: 11583/2982607 since: 2023-09-29T13:16:38Z

Publisher:
John Wiley and Sons

Published
DOI:10.1112/blms.12684

Terms of use:

This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

Publisher copyright

(Article begins on next page)

12 November 2026

Check for updates

Received: 19 July 2021 | Revised: 26 January 2022 | Accepted: 8 April 2022

DOI: 10.1112/blms.12684

RESEARCH ARTICLE

Bulletin of the London Mathematical Society

Local and non-local Poincaré inequalities on Lie groups

Tommaso Bruno[1] | **Marco M. Peloso**[2] | **Maria Vallarino**[3]

[1]Department of Mathematics: Analysis, Logic and Discrete Mathematics, Ghent University, Ghent, Belgium

[2]Dipartimento di Matematica, Università degli Studi di Milano, Milano, Italy

[3]Dipartimento di Scienze Matematiche "Giuseppe Luigi Lagrange", Politecnico di Torino, Torino, Italy

Correspondence
Tommaso Bruno, Dipartimento di Matematica, Università degli Studi di Genova, Via Dodecaneso 35, 16146 Genova, Italy.
Email: brunot@dima.unige.it

Funding information
Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni; FWO, Grant/Award Number: 12ZW120N

Abstract

We prove a local L^p-Poincaré inequality, $1 \leqslant p < \infty$, on non-compact Lie groups endowed with a sub-Riemannian structure. We show that the constant involved grows at most exponentially with respect to the radius of the ball, and that if the group is non-doubling, then its growth is indeed, in general, exponential. We also prove a non-local L^2-Poincaré inequality with respect to suitable finite measures on the group.

MSC 2020
26D10, 43A80 (primary)

1 | INTRODUCTION

The aim of this paper is to establish two forms of Poincaré inequality on non-compact connected Lie groups. On the one hand, we shall obtain the Lie group analogue of the classical inequality on $\mathbb{R}^d$

$$\|f - f_B\|_{L^p(B)} \leqslant Cr\|\nabla f\|_{L^p(B)}, \tag{1.1}$$

where $p \in [1, \infty)$, $f \in C^\infty(\mathbb{R}^d)$, B is a ball of radius r and f_B is the average of f on B. On the other hand, we shall consider a non-local L^2-version of such an inequality, which takes the form

$$\|f\|_{L^2(\mathbb{R}^d,\mu)} \leqslant C\|\nabla f\|_{L^2(\mathbb{R}^d,\mu)} \tag{1.2}$$

© 2022 The Authors. *Bulletin of the London Mathematical Society* is copyright © London Mathematical Society. This is an open access article under the terms of the Creative Commons Attribution License, which permits use, distribution and reproduction in any medium, provided the original work is properly cited.

for certain finite measures μ which are absolutely continuous with respect to the Lebesgue measure and whose densities satisfy a certain decay condition at infinity. One should think, for example, to the case when μ is a Gaussian measure.

Extensions of the classical Poincaré inequality (1.1) to non-Euclidean settings have widely been studied in the last decades. A thorough overview of the literature would go out of the scope of the present paper, so we refer the reader to the milestone [11] and the references therein. For what concerns Lie groups, a Poincaré inequality on unimodular groups can be obtained by combining [16, §8.3] and [11, Theorem 9.7]. In this paper we prove that a Poincaré inequality also holds on non-unimodular Lie groups endowed with a relatively invariant measure, and we also describe the behaviour of the Poincaré constant in a quantitative way. We show that this grows at most exponentially with respect to the radius of the ball, and that if the group is non-doubling, then such growth is, in general, exponential. More precisely, in a class of Lie groups including the real hyperbolic spaces as a subclass, we estimate from below the constant involved in the Poincaré inequality by a quantity which grows exponentially with respect to the radius of the ball.

Non-local inequalities such as (1.2) have been introduced more recently [2] on $\mathbb{R}^d$, for densities satisfying suitable differential inequalities expressed in terms of the Laplacian Δ (cf. [2, Corollary 1.6]) whose prototype is a Gaussian function. After its establishment on $\mathbb{R}^d$, such non-local inequalities were extended to unimodular Lie groups of polynomial growth in [15], where a sum-of-squares subelliptic sub-Laplacian plays the role of Δ. In this paper, we extend their method to the non-doubling regime where the sub-Laplacian has an additional drift term, in a setting where we previously studied various function spaces [4–6] and the Sobolev and Moser–Trudinger inequalities [7].

As a classical application of the local Poincaré inequality, we show the so-called local parabolic Harnack principle for the sub-Laplacian with drift. Another application of our inequality is given in [3] to the study of spectral properties of Schrödinger operators on Lie groups.

2 | SETTING AND PRELIMINARIES

Let G be a non-compact connected Lie group with identity e. We denote by ρ a right Haar measure, by χ a continuous positive character of G and by μ_χ the measure with density χ with respect to ρ. As the modular function on G, which we denote by δ, is such a character, μ_δ is a left Haar measure on G. We denote it by λ. Observe also that $\mu_1 = \rho$.

Let $\mathbf{X} = \{X_1, \dots, X_\ell\}$ be a family of left-invariant linearly independent vector fields which satisfy Hörmander's condition. Let $d_C(\cdot, \cdot)$ be its associated left-invariant Carnot–Carathéodory distance. We let $|x| = d_C(x, e)$, and denote by B_r the ball centred at e of radius r. The measure of B_r with respect to ρ will be denoted by $V(r) = \rho(B_r)$; we recall that $V(r) = \lambda(B_r)$. It is well known, cf. [10, 19], that there exist $d \in \mathbb{N}^*$ depending on G and $\mathbf{X}$, and $C > 0$, such that

$$C^{-1} r^d \leqslant V(r) \leqslant C r^d \qquad \forall r \in (0, 1], \tag{2.1}$$

and $D_0, D > 0$ depending only on G, such that, either $C^{-1} r^D \leqslant V(r) \leqslant C r^D$ for all $r \geqslant 1$, or

$$C^{-1} \mathrm{e}^{D_0 r} \leqslant V(r) \leqslant C \mathrm{e}^{D r} \tag{2.2}$$

for all $r \geqslant 1$. In the former case, the group G is said to be of polynomial growth, while in the latter case of exponential growth.

14692120, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/blms.12684 by Politecnico Di Torino Sist. Bibl Del Polit Di Torino, Wiley Online Library on [29/09/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

For any character χ and $r > 0$, one has (see [12, Proposition 5.7])

$$\sup_{B_r} \chi = \mathrm{e}^{c(\chi)r}, \qquad \text{where} \quad c(\chi) = \left(\sum_{j=1}^{\ell} |X_j \chi(e)|^2 \right)^{1/2}. \tag{2.3}$$

Since χ is a character, by (2.3) one also has

$$\inf_{B_r} \chi = \mathrm{e}^{-c(\chi)r}. \tag{2.4}$$

Given a ball B with respect to d_C, we denote by c_B its center and by r_B its radius, and we write $B = B(c_B, r_B)$; we also set $2B = B(c_B, 2r_B)$. Moreover, for $R > 0$ let $\mathcal{B}_R$ be the family of all balls of radius $\leqslant R$ and

$$D(R, \chi) = \sup_{B \in \mathcal{B}_R} \frac{\mu_\chi(2B)}{\mu_\chi(B)} = \sup_{0<r\leqslant R} \frac{\mu_\chi(B_{2r})}{\mu_\chi(B_r)}, \tag{2.5}$$

where the latter equality holds since $\mu_\chi(B(c_B, r)) = (\chi\delta^{-1})(c_B)\mu_\chi(B_r)$ for all $r > 0$ and $c_B \in G$.

In the following lemma we estimate the local doubling constant $D(R, \chi)$.

Lemma 2.1. *The metric measure space (G, d_C, μ_χ) is doubling if and only if $\chi = 1$ and (G, d_C, ρ) is doubling, in which case there exists $C > 0$ such that $D(R, \chi) \leqslant C$. If $\chi \neq 1$ and (G, d_C, ρ) is doubling, then there exists $C > 0$ such that*

$$D(R, \chi) \leqslant C\mathrm{e}^{3c(\chi)R} \qquad \forall R > 0,$$

while if $\chi \neq 1$ and (G, d_C, ρ) is non-doubling, then there exists $C > 0$ such that

$$D(R, \chi) \leqslant C\mathrm{e}^{(2D-D_0+3c(\chi))R} \qquad \forall R > 0.$$

Proof. If $\chi = 1$, the first statement is obvious since $\mu_\chi = \rho$.

Assume then that $\chi \neq 1$, so that there is $x \in G$ with $\chi(x) > 1$. If N denotes the lowest integer such that $N \geqslant |x|$, then $B_N x^n \subseteq B_{(n+1)N}$. If $r > N$ and n is the largest integer such that $(n + 1)N \leqslant [r]$, then

$$\mu_\chi(B_r) \geqslant \mu_\chi(B_N x^n) = \chi(x)^n \mu_\chi(B_N) \geqslant \chi(x)^{[r]/N-2} \mu_\chi(B_N),$$

whence $\mu_\chi(B_r)$ grows exponentially with r and the space (G, d_C, μ_χ) is non-doubling.

We now show the two bounds on $D(R, \chi)$. First, observe that by (2.1), (2.3) and (2.4)

$$\frac{\mu_\chi(B_{2r})}{\mu_\chi(B_r)} \leqslant C \qquad \forall r \in (0, 1],$$

for some $C > 0$. Moreover, if (G, d_C, ρ) is doubling, then by (2.3) and (2.4)

$$\frac{\mu_\chi(B_{2r})}{\mu_\chi(B_r)} \leqslant \mathrm{e}^{3c(\chi)r} \frac{V(2r)}{V(r)} \leqslant C\mathrm{e}^{3c(\chi)r} \qquad \forall r \geqslant 1,$$

14692120, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/blms.12684 by Politecnico Di Torino Sist. Bibl Del Polit Di Torino, Wiley Online Library on [29/09/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

while if (G, d_C, ρ) is non-doubling, then the stated estimate follows similarly by (2.2), (2.3) and (2.4). □

3 | THE LOCAL POINCARÉ INEQUALITY ON LIE GROUPS

In this section we prove the L^p-Poincaré inequality for smooth functions on (G, d_C, μ_χ). Given a ball B and $f \in C^\infty(G)$, we denote by f_B^χ its average over B with respect to μ_χ,

$$f_B^\chi = \frac{1}{\mu_\chi(B)} \int_B f \, d\mu_\chi,$$

and we let $|\nabla f|^2 = \sum_{j=1}^{\ell} |X_j f|^2$. If S is a set of variables, we denote by $C(S)$ a constant depending only on the elements of S.

Theorem 3.1. *There exist a constant $C = C(G, \mathbf{X}) > 0$ and a universal constant $\alpha > 0$ such that, for all $p \in [1, \infty)$, $R > 0$, all balls B of radius $r \in (0, R]$ and $f \in C^\infty(G)$,*

$$\|f - f_B^\chi\|_{L^p(B,\mu_\chi)} \leqslant C \, e^{\frac{1}{p}[2c(\chi)+c(\chi\delta^{-1})]R} \, D(R, \chi)^\alpha \, r \, \||\nabla f|\|_{L^p(B,\mu_\chi)}. \tag{3.1}$$

Notice that the Poincaré constant grows at most exponentially with respect to the radius of the ball. The exponential term cannot, in general, be removed. After establishing the theorem, indeed, we show that when G is the so-called "$ax + b$" group and $\mu_\chi = \lambda$ is a left Haar measure, the growth of the constant is indeed exponential.

Proof. Let $p \in [1, \infty)$ be given. We shall prove that for every ball B of radius $r > 0$ and $f \in C^\infty(G)$

$$\int_B |f - f_B^\chi|^p \, d\mu_\chi \leqslant 2^p \, e^{c(\chi\delta^{-1})r} e^{2c(\chi)r} \, \frac{\mu_\chi(B_{2r})}{\mu_\chi(B_r)} \, r^p \int_{2B} |\nabla f|^p \, d\mu_\chi. \tag{3.2}$$

Once (3.2) is at disposal, the Poincaré inequality can be obtained by classical arguments, see, for example, [11, Theorem 9.7]. A careful inspection of [13, Section 5], in particular, shows how a Whitney decomposition of B brings to the constant given in the statement. We omit the details, which would be tedious and an almost verbatim repetition of the arguments that the reader can find in [13].

We then show (3.2). For $z \in G$, let $\gamma_z : [0, |z|] \to G$ be a C^1-geodesic such that $\gamma_z(0) = e$, $\gamma_z(|z|) = z$, $\gamma_z(s) \in B_{|z|}$ and $|\gamma_z'(s)| \leqslant 1$ for every $s \in [0, |z|]$.

Let B be a ball of radius $r > 0$. Observe that if $x, y \in B$, and $z = x^{-1}y$, then $|z| < 2r$. For every $x, z \in G$, by Hölder's inequality

$$|f(x) - f(xz)|^p \leqslant \left(\int_0^{|z|} |\nabla f(x\gamma_z(s))| \, ds \right)^p \leqslant |z|^{p-1} \int_0^{|z|} |\nabla f(x\gamma_z(s))|^p \, ds. \tag{3.3}$$

14692120, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/blms.12684 by Politecnico Di Torino Sist. Bibl Del Polit Di Torino, Wiley Online Library on [29/09/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

We then have

$$\int_B |f - f_B^\chi|^p \, d\mu_\chi = \int_B \left| \frac{1}{\mu_\chi(B)} \int_B (f(x) - f(y)) \, d\mu_\chi(y) \right|^p d\mu_\chi(x)$$
$$\leqslant \frac{1}{\mu_\chi(B)} \int_B \int_B |f(x) - f(y)|^p \, d\mu_\chi(y) \, d\mu_\chi(x),$$

and after the change of variables $y = xz$, we get

$$\int_B |f - f_B^\chi|^p \, d\mu_\chi \leqslant \frac{1}{\mu_\chi(B)} \int_G \int_G \mathbf{1}_B(x) \mathbf{1}_B(xz) \big| f(x) - f(xz) \big|^p (\chi\delta^{-1})(x) \, d\mu_\chi(x) \, d\mu_\chi(z).$$

Observe now that by (3.3) and Fubini's theorem, we get

$$\int_G \mathbf{1}_B(x) \mathbf{1}_B(xz) \big| f(x) - f(xz) \big|^p (\chi\delta^{-1})(x) \, d\mu_\chi(x)$$
$$\leqslant \frac{1}{\mu_\chi(B)} |z|^{p-1} \int_0^{|z|} \int_G \mathbf{1}_B(x) \mathbf{1}_B(xz) |\nabla f(x\gamma_z(s))|^p \, (\chi\delta^{-1})(x) \, d\mu_\chi(x) \, ds.$$

We make the change of variables $\zeta = x\gamma_z(s)$ and observe that by (2.3), if $x \in B$, then

$$(\chi\delta^{-1})(x) \leqslant (\chi\delta^{-1})(c_B) \sup_{B_r} (\chi\delta^{-1}) \leqslant (\chi\delta^{-1})(c_B) \, \mathrm{e}^{c(\chi\delta^{-1})r},$$

and $\chi(\gamma_z(s)) \leqslant \mathrm{e}^{2c(\chi)r}$. We obtain

$$\int_G \mathbf{1}_B(x) \mathbf{1}_B(xz) (\chi\delta^{-1})(x) |\nabla f(x\gamma_z(s))|^p \, d\mu_\chi(x)$$
$$\leqslant (\chi\delta^{-1})(c_B) \, \mathrm{e}^{c(\chi\delta^{-1})r} \, \mathrm{e}^{2c(\chi)r} \int_G \mathbf{1}_{B\gamma_z(s)}(\zeta) \mathbf{1}_{Bz^{-1}\gamma_z(s)}(\zeta) |\nabla f(\zeta)|^p \, d\mu_\chi(\zeta).$$

Notice that $B\gamma_z(s) \cap Bz^{-1}\gamma_z(s) \subseteq 2B$ for all $s \in [0, |z|]$. This is straightforward by the triangle inequality when $|z| < r$. Otherwise, let $s_0 \in [0, |z|]$ be such that $|\gamma_z(s_0)| = r$. Then $B\gamma_z(s) \subseteq 2B$ for $s \in [0, s_0]$ and $Bz^{-1}\gamma_z(s) \subseteq 2B$ for $s \in (s_0, |z|]$. Therefore

$$\int_G \mathbf{1}_B(x) \mathbf{1}_B(xz) (\chi\delta^{-1})(x) |\nabla f(x\gamma_z(s))|^p \, d\mu_\chi(x)$$
$$\leqslant (\chi\delta^{-1})(c_B) \, \mathrm{e}^{c(\chi\delta^{-1})r} \, \mathrm{e}^{2c(\chi)r} \mathbf{1}_{B_{2r}}(z) \int_{2B} |\nabla f(\zeta)|^p \, d\mu_\chi(\zeta).$$

Since $(\chi\delta^{-1})(c_B)\mu_\chi(B_{2r}) = \mu_\chi(2B)$, by integrating with respect to z we get

$$\int_B |f - f_B^\chi|^p \, d\mu_\chi \leqslant \mathrm{e}^{2c(\chi)r} \mathrm{e}^{c(\chi\delta^{-1})r} (2r)^p \, \frac{\mu_\chi(2B)}{\mu_\chi(B)} \int_{2B} |\nabla f(\zeta)|^p \, d\mu_\chi(\zeta),$$

which concludes the proof. □

14692120, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/blms.12684 by Politecnico Di Torino Sist. Bibl Del Polit Di Torino, Wiley Online Library on [29/09/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

As a corollary, we obtain the so-called local parabolic Harnack principle. We introduce the operator

$$\Delta_\chi = -\sum_{j=1}^{\ell}(X_j^2 + (X_j\chi)(e)X_j), \tag{3.4}$$

which is essentially self-adjoint on $L^2(\mu_\chi)$ and non-negative; see, for example, [4, 12]. We say that Δ_χ satisfies the local parabolic Harnack principle up to distance $R > 0$ if there is $C(R) > 0$ such that, for all $x \in G$, $r \in (0, R]$, $s \in \mathbb{R}$, and any positive solutions u of $(\partial_t + \Delta_\chi)u = 0$ on $(s, s + r^2) \times B(x, r)$, we have that

$$\sup_{Q_-} u \leqslant C(R) \inf_{Q_+} u,$$

where

$$Q_- = \left(s + r^2/6, s + r^2/3\right) \times B(x, r/2), \qquad Q_+ = \left(s + 2r^2/3, s + r^2\right) \times B(x, r/2).$$

The following result follows at once from Theorem 3.1 and [16, Theorem 2.1].

Corollary 3.2. *For every $R > 0$, Δ_χ satisfies the local parabolic Harnack principle up to distance R. In particular, the positive Δ_χ-harmonic functions satisfy the local elliptic Harnack inequality.*

3.1 | Exponential growth of the constant

For $r > 0$ and $p \in [1, \infty)$, define

$$C(r, p) = \inf \frac{\int_{B_r} |f - f_B^\chi|^p \, d\mu_\chi}{\int_{B_r} |\nabla f|^p \, d\mu_\chi}, \tag{3.5}$$

where the infimum runs over all functions $f \in C^\infty(G)$. In this section we show that the exponential bound of $C(r, p)$ appearing in inequality (3.1) is in general optimal, in the sense that such constant cannot grow less than exponentially with respect to r. Indeed, in the particular case of $ax + b$ groups of arbitrary dimension, we provide a lower bound of exponential type for $C(r, p)$. For notational convenience, we shall write $A \lesssim B$ to indicate that there is a constant C such that $A \leqslant CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$.

Let $G = \mathbb{R}^{n-1} \rtimes \mathbb{R}^+$ and let (x, a) be its generic element. Recall that

$$d\lambda(x, a) = \frac{dx\, da}{a^n} \qquad \text{and} \qquad d\rho(x, a) = \frac{dx\, da}{a},$$

since $\delta(x, a) = a^{-n+1}$; all positive characters of G are of the form $\chi_\gamma(x, a) = a^\gamma$ for some $\gamma \in \mathbb{R}$. We shall write μ_γ for the measure μ_{χ_γ}. In particular, $\lambda = \mu_{1-n}$ is the hyperbolic measure. We consider the left-invariant vector fields $X_i = a\partial_i$, $i = 1, \dots, n-1$, and $X_0 = a\partial_a$ which form a basis of the Lie algebra of G. The distance induced by such vector fields is the hyperbolic metric which is given by

$$\cosh |(x, a)| = \tfrac{1}{2}(a + a^{-1} + a^{-1}|x|^2),$$

14692120, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/blms.12684 by Politecnico Di Torino Sist. Bibl Del Polit Di Torino, Wiley Online Library on [29/09/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

where $|x|$ is the Euclidean norm of $x \in \mathbb{R}^{n-1}$ (see [1, (2.18)], [17, (1.1)]). Then

$$B_r = \left\{(x,a) : \ \mathrm{e}^{-r} < a < \mathrm{e}^{r}, \ |x|^2 < 2a(\cosh r - \cosh \log a)\right\}.$$

In the case of the real hyperbolic space, that is, the $ax+b$ group endowed with the measure λ and the metric defined above, the constant $C(r,p)$ in (3.5) was estimated from above in [11, Section 10.1]. We now estimate such constant from below.

Consider the function $\phi : G \to \mathbb{R}$ defined by

$$\phi(x,a) = x_1, \quad (x,a) \in G.$$

Observe that $\int_{B_r} \phi \, d\mu_\gamma = 0$ for all $\gamma \in \mathbb{R}$ and $|\nabla \phi(x,a)|^p = a^p$. Moreover,

$$\int_{B_r} |\phi|^p \, d\mu_\gamma \approx \int_{\mathrm{e}^{-r}}^{\mathrm{e}^{r}} a^{\gamma - 1 + \frac{p+n-1}{2}} (\cosh r - \cosh \log a)^{\frac{p+n-1}{2}} \, da,$$

while

$$\int_{B_r} |\nabla \phi|^p \, d\mu_\gamma \approx \int_{\mathrm{e}^{-r}}^{\mathrm{e}^{r}} a^{\gamma - 1 + p + \frac{n-1}{2}} (\cosh r - \cosh \log a)^{\frac{n-1}{2}} \, da.$$

Lemma 3.3. *Let $\delta \in \mathbb{R}$ and $\epsilon > 0$. Then*

$$\int_{\mathrm{e}^{-r}}^{\mathrm{e}^{r}} a^{\delta} (\cosh r - \cosh \log a)^{\epsilon} \, da \approx \mathrm{e}^{r(|\delta+1|+\epsilon)}.$$

Proof. We first make a change of variables

$$\int_{\mathrm{e}^{-r}}^{\mathrm{e}^{r}} a^{\delta} (\cosh r - \cosh \log a)^{\epsilon} \, da = \int_{-r}^{r} \mathrm{e}^{t(\delta+1)} (\cosh r - \cosh t)^{\epsilon} \, dt.$$

Since $\cosh r - \cosh t \approx \mathrm{e}^{r}$ if $|t| < r-1$, while $\cosh r - \cosh t \approx (r - |t|)\mathrm{e}^{r}$ if $r-1 < |t| < r$, we get

$$\int_{-r+1}^{r-1} \mathrm{e}^{t(\delta+1)} (\cosh r - \cosh t)^{\epsilon} \, dt \approx \mathrm{e}^{\epsilon r} \int_{-r+1}^{r-1} \mathrm{e}^{t(\delta+1)} \, dt \approx \mathrm{e}^{(\epsilon + |\delta+1|)r}$$

while

$$\int_{r-1<|t|<r} \mathrm{e}^{t(\delta+1)} (\cosh r - \cosh t)^{\epsilon} \, dt \approx \mathrm{e}^{(\epsilon+|\delta+1|)r} \int_{r-1<|t|<r} (r - |t|)^{\epsilon} \, dt \approx \mathrm{e}^{(\epsilon+|\delta+1|)r},$$

as required. □

From the lemma above, we get that

$$\int_{B_r} |\phi|^p \, d\mu_\gamma \approx \mathrm{e}^{(|\gamma + \frac{p+n-1}{2}| + \frac{p+n-1}{2})r},$$

14692120, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/blms.12684 by Politecnico Di Torino Sist. Bibl Del Polit Di Torino, Wiley Online Library on [29/09/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

while

$$\int_{B_r} |\nabla\phi|^p \, d\mu_\gamma \approx \mathrm{e}^{(|\gamma+p+\frac{n-1}{2}|+\frac{n-1}{2})r}.$$

We observe that, if $\gamma < -\frac{p+n-1}{2}$, then

$$\left|\gamma + \frac{p+n-1}{2}\right| + \frac{p+n-1}{2} > \left|\gamma + p + \frac{n-1}{2}\right| + \frac{n-1}{2}.$$

Thus for such γ

$$C(r,p) \geqslant C\mathrm{e}^{r\left(\left|\gamma+\frac{p+n-1}{2}\right|+\frac{p}{2}-\left|\gamma+p+\frac{n-1}{2}\right|\right)}.$$

If in particular $\gamma = -n+1$, hence μ_γ is the left measure, and $n > p+1$, then

$$C(r,p) \geqslant \mathrm{e}^{\left(\left|\frac{-n+p+1}{2}\right|-\left|\frac{-n+1}{2}+p\right|+\frac{p}{2}\right)r} = \begin{cases} \mathrm{e}^{pr} & n \geqslant 2p+1 \\ \mathrm{e}^{(n-p-1)r} & p+1 < n \leqslant 2p+1. \end{cases}$$

4 | NON-LOCAL POINCARÉ INEQUALITY

In this second part of the paper we prove a non-local L^2-Poincaré inequality for suitable finite measures on G in the spirit of [14, 15]. More precisely, let M be a positive C^2-function in $L^1(\mu_\chi)$ and $\mu_{\chi,M}$ be the finite measure whose density is M with respect to μ_χ. We shall prove L^2-global Poincaré inequalities for the measure $\mu_{\chi,M}$ for a large family of functions M. In order to do this, we let

$$L^2_1(\mu_{\chi,M}) = \{f \in L^2(\mu_{\chi,M}) : \ |\nabla f| \in L^2(\mu_{\chi,M})\}$$

and introduce the operator

$$\Delta_{\chi,M} = \Delta_\chi - \nabla(\log M)\cdot\nabla, \tag{4.1}$$

where Δ_χ is that of (3.4), $\mathrm{Dom}(\Delta_{\chi,M}) = \{f \in L^2_1(\mu_{\chi,M}) : \ \Delta_{\chi,M} f \in L^2(\mu_{\chi,M})\}$ and the derivatives are meant in the distributional sense. Observe that $\Delta_{\chi,M}$ is symmetric on $L^2(\mu_{\chi,M})$; in particular, for all $f \in \mathrm{Dom}(\Delta_{\chi,M})$ and $g \in L^2_1(\mu_{\chi,M})$,

$$\int_G \nabla f\cdot\nabla g \, d\mu_{\chi,M} = \int_G \Delta_{\chi,M} f \cdot g \, d\mu_{\chi,M},$$

where $\nabla f\cdot\nabla g = \sum_{j=1}^{\ell}(X_j f)(X_j g)$.

We say that the couple (Δ_χ, M) admits a Lyapunov function if there exist a C^2 function $W : G \to [1,\infty)$ and constants $\theta > 0$, $b \geqslant 0$, $R > 0$ such that

$$-\Delta_{\chi,M} W(x) \leqslant -\theta W(x) + b\mathbf{1}_{B_R}(x) \qquad \forall x \in G. \tag{4.2}$$

14692120, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/blms.12684 by Politecnico Di Torino Sist. Bibl Del Polit Di Torino, Wiley Online Library on [29/09/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Observe that the existence of a Lyapunov function depends on G, $\mathbf{X}$, χ and M. For $f \in L^1(\mu_{\chi,M})$ we let

$$f_{\chi,M} = \frac{1}{\mu_{\chi,M}(G)} \int_G f \, d\mu_{\chi,M}.$$

Our second main result is the following global L^2-Poincaré inequality for $\mu_{\chi,M}$.

Theorem 4.1. *If (Δ_χ, M) admits a Lyapunov function, then there exists a constant $C = C(G, \mathbf{X}, \chi, M)$ such that for all $f \in L^2_1(\mu_{\chi,M})$*

$$\|f - f_{\chi,M}\|_{L^2(\mu_{\chi,M})} \leqslant C \| |\nabla f| \|_{L^2(\mu_{\chi,M})}. \tag{4.3}$$

Theorem 4.1 is a generalization to any connected non-compact possibly non-unimodular Lie group of the non-local L^2-Poincaré inequalities proved in [2, Theorem 1.4] in the Euclidean setting and in [15, Theorem 1.1] in unimodular Lie groups of polynomial growth, by which our proof is inspired. The validity of other versions of non-local Poincaré inequalities in the current setting, such as those in [15, Theorem 1.4], is still an open problem. More general versions of non-local Poincaré inequalities of this kind were proved in [9] in the setting of a topological measure space endowed with a family of sets which play the role of unit balls and satisfy suitable assumptions. Recently in [8] non-local L^p-Poincaré inequalities were obtained on Carnot groups of Engel type in the case when the density of the measure depends on a homogeneous norm of the group; we note, however, that the case $p = 2$ is always excluded.

Proof. Let $f \in L^2_1(\mu_{\chi,M})$, and observe first that

$$\int_G \left| f - f_{\chi,M} \right|^2 d\mu_{\chi,M} = \min_{c \in \mathbb{R}} \int_G |f - c|^2 d\mu_{\chi,M}. \tag{4.4}$$

Let now $g = f - c$ for a positive c to be determined, and W be a Lyapunov function for (Δ_χ, M). By (4.2)

$$\int_G |g|^2 \, d\mu_{\chi,M} \leqslant \int_G |g|^2 \frac{\Delta_{\chi,M} W}{\theta W} \, d\mu_{\chi,M} + \int_{B_R} |g|^2 \frac{b}{\theta W} \, d\mu_{\chi,M}. \tag{4.5}$$

We treat the two terms separately.

Let us consider the first term and prove that

$$\int_G \frac{\Delta_{\chi,M} W}{W} g^2 d\mu_{\chi,M} \leqslant \int_G |\nabla g|^2 d\mu_{\chi,M}. \tag{4.6}$$

We prove it by density, and firstly assume that g is compactly supported. By definition of $\Delta_{\chi,M}$,

$$\begin{aligned} \int_G \frac{\Delta_{\chi,M} W}{W} g^2 \, d\mu_{\chi,M} &= \int_G \nabla\left(\frac{g^2}{W}\right) \cdot \nabla W \, d\mu_{\chi,M} \\ &= 2 \int_G \frac{g}{W} \nabla g \cdot \nabla W \, d\mu_{\chi,M} - \int_G \frac{g^2}{W^2} |\nabla W|^2 \, d\mu_{\chi,M} \end{aligned}$$

14692120, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/blms.12684 by Politecnico Di Torino Sist. Bibl Del Polit Di Torino, Wiley Online Library on [29/09/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$$= \int_G |\nabla g|^2 \, d\mu_{\chi,M} - \int_G |\nabla g - \frac{g}{W} \nabla W|^2 \, d\mu_{\chi,M}$$
$$\leqslant \int_G |\nabla g|^2 d\mu_{\chi,M}.$$

Let now $g \in L^2_1(\mu_{\chi,M})$, and consider a non-decreasing sequence of functions $\psi_n \in C^\infty_c(G)$ such that

$$\mathbf{1}_{B_{nR}} \leqslant \psi_n \leqslant 1, \quad |\nabla \psi_n| \leqslant 1.$$

By applying (4.6) to $g\psi_n$, the monotone convergence theorem in the left-hand side and the dominated convergence theorem in the right-hand side, one gets (4.6).

To deal with the second term, we choose c such that $\int_{B_R} g \, d\mu_\chi = 0$. By (3.2) applied to g on B_R, and the fact that M is bounded from above and below on B_{2R}, one has

$$\int_{B_R} |g|^2 \, d\mu_{\chi,M} \leqslant C \int_{B_R} |g|^2 \, d\mu_\chi \leqslant C \int_{B_{2R}} |\nabla g|^2 \, d\mu_\chi \leqslant C \int_{B_{2R}} |\nabla g|^2 \, d\mu_{\chi,M},$$

where the constant C depends on R and M. Therefore, since $W \geqslant 1$,

$$\int_{B_R} |g|^2 \frac{b}{\theta W} \, d\mu_{\chi,M} \leqslant C \int_{B_{2R}} |\nabla g|^2 \, d\mu_{\chi,M} \leqslant C \int_G |\nabla g|^2 \, d\mu_{\chi,M},$$

which completes the proof. □

Corollary 4.2. *Let $v = -\log M$. If there exist $a \in (0,1)$, $c > 0$ and $R > 0$ such that*

$$a|\nabla v|^2(x) + \Delta_\chi v(x) \geqslant c \qquad \forall\, x \in B_R^c, \tag{4.7}$$

then (Δ_χ, M) admits a Lyapunov function, and (4.3) holds.

Proof. Let $W(x) = \mathrm{e}^{(1-a)(v(x)-\inf_G v)}$, so that

$$-\Delta_{\chi,M} W = (1-a)W\big(-\Delta_\chi v - a|\nabla v|^2\big).$$

Then W is a Lyapunov function with $\theta = c(1-a)$ and $b = \max_{B_R}(-\Delta_{\chi,M} W + \theta W)$. □

One can actually show that if (4.7) holds with $a < 1/2$, then (4.3) self-improves as follows.

Proposition 4.3. *Let $v = -\log M$. If there exist $c > 0$, $R > 0$ and $\epsilon \in (0,1)$ such that*

$$\frac{1-\epsilon}{2}|\nabla v|^2(x) + \Delta_\chi v(x) \geqslant c \qquad \forall\, x \in B_R^c, \tag{4.8}$$

then there exists $C > 0$ such that for all $f \in L^2_1(\mu_{\chi,M})$

$$\| |f - f_{\chi,M}|(1 + |\nabla v|)\|_{L^2(\mu_{\chi,M})} \leqslant C \| |\nabla f| \|_{L^2(\mu_{\chi,M})}. \tag{4.9}$$

14692120, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/blms.12684 by Politecnico Di Torino Sist. Bibl Del Polit Di Torino, Wiley Online Library on [29/09/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Proof. Observe firstly that, since v is C^2 and (4.8) holds,

$$\frac{1-\epsilon}{2}|\nabla v|^2 - \Delta_\chi v \geqslant \alpha \tag{4.10}$$

for some $\alpha \in \mathbb{R}$. Let $f \in L^2_1(\mu_{\chi,M})$ and let $g = f\sqrt{M}$. Since

$$\nabla f = \frac{1}{\sqrt{M}}\nabla g + \frac{1}{2} g \frac{1}{\sqrt{M}} \nabla v,$$

by (4.10)

$$\begin{aligned}
\int_G |\nabla f|^2 \, d\mu_{\chi,M} &= \int_G \left(|\nabla g|^2 + \frac{1}{4}|g|^2|\nabla v|^2 + g\nabla g \cdot \nabla v \right) d\mu_\chi \\
&= \int_G \left(|\nabla g|^2 + \frac{1}{4}|g|^2|\nabla v|^2 + \frac{1}{2}\nabla(|g|^2) \cdot \nabla v \right) d\mu_\chi \\
&\geqslant \int_G |g|^2 \left(\frac{1}{4}|\nabla v|^2 + \frac{1}{2}\Delta_\chi v \right) d\mu_\chi \\
&\geqslant \frac{1}{2}\int_G |f|^2 \left(\frac{\epsilon}{2}|\nabla v|^2 + \alpha \right) d\mu_{\chi,M}.
\end{aligned}$$

Since (4.3) holds by (4.8) and Corollary 4.2, the conclusion follows. □

ACKNOWLEDGEMENTS

The authors wish to thank the anonymous referee for their careful reading of the paper and interesting comments. All authors are partially supported by the GNAMPA 2020 project "Fractional Laplacians and subLaplacians on Lie groups and trees" and are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). T. Bruno acknowledges support by the Research Foundation–Flanders (FWO) through the postdoctoral grant 12ZW120N.

Open Access Funding provided by Universita degli Studi di Genova within the CRUI-CARE Agreement.

JOURNAL INFORMATION

The *Bulletin of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

REFERENCES

1. J.-Ph. Anker, E. Damek, and C. Yacoub, *Spherical analysis on harmonic AN groups*, Ann. Sc. Norm. Super. Pisa **33** (1996), 643–679.
2. D. Bakry, F. Barthe, P. Cattiaux, and A. Guillin, *A simple proof of the Poincaré inequality for a large class of probability measures including the log-concave case*, Electron. Comm. Probab. **13** (2008), 60–66.
3. T. Bruno and M. Calzi, *Schrödinger operators on Lie groups with purely discrete spectrum*, Adv. Math. **404** (2022) n. 108444.

14692120, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/blms.12684 by Politecnico Di Torino Sist. Bibl Del Polit Di Torino, Wiley Online Library on [29/09/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

4. T. Bruno, M. M. Peloso, A. Tabacco, and M. Vallarino, *Sobolev spaces on Lie groups: embedding theorems and algebra properties*, J. Funct. Anal. **276** (2019), no. 10, 3014–3050.
5. T. Bruno, M. M. Peloso, and M. Vallarino, *Besov and Triebel–Lizorkin spaces on Lie groups*, Math. Ann. **377** (2020), no. 1–2, 335–377.
6. T. Bruno, M. M. Peloso, and M. Vallarino, *Potential spaces on Lie groups*. Geometric Aspects of Harmonic Analysis, Springer INdAM Series, vol. 45, 2021, pp. 149–192.
7. T. Bruno, M. M. Peloso, and M. Vallarino, *The Sobolev embedding constant on Lie groups*, Nonlinear Anal. **216** (2022), 112707.
8. M. Chatzakou, S. Federico, and B. Zegarlinski, *q-Poincarè inequalities on Carnot Groups with a filiform Lie algebra*, arXiv:2007.04689.
9. P. T. Gressman, *Fractional Poincaré and logarithmic Sobolev inequalities for measure spaces*, J. Funct. Anal. **265** (2013), no. 6, 867–889.
10. Y. Guivarc'h, *Croissance polynomiale et périodes des fonctions harmoniques*, Bull. Soc. Math. France **101** (1973), 333–379.
11. P. Hajlasz and P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. **145** (2000), no. 688, x+101.
12. W. Hebisch, G. Mauceri, and S. Meda, *Spectral multipliers for Sub-Laplacians with drift on Lie groups*, Math. Z. **251** (2005), no. 4, 899–927.
13. D. Jerison, *The Poincaré inequality for vector fields satisfying Hörmander's condition*, Duke Math. J. **53** (1986), no. 2, 503–523.
14. C. Mouhot, E. Russ, and Y. Sire, *Fractional Poincaré inequalities for general measures*, J. Math. Pure Appl. (9) **95** (2011), no. 1, 72–84.
15. E. Russ and Y. Sire, *Nonlocal Poincaré inequalities on Lie groups with polynomial volume growth and Riemannian manifolds*, Studia Math. **203** (2011), no. 2, 105–127.
16. L. Saloff-Coste, *Parabolic Harnack inequality for divergence-form second-order differential operators. Potential theory and degenerate partial differential operators*, Potential Anal. **4** (1995), no. 4, 429–467.
17. P. Sjögren and M. Vallarino, *Boundedness from H^1 to L^1 of Riesz transforms on a Lie group of exponential growth*, Ann. Inst. Fourier, Grenoble **58** (2008), no. 4, 1117–1151.
18. N. Th. Varopoulos, *Fonctions harmoniques sur les groupes de Lie*, C. R. Acad. Sci. Paris, Série I, Math. **309** (1987), 519–521.
19. N. Th. Varopoulos, *Analysis on Lie groups*, J. Funct. Anal. **76** (1988), no. 2, 346–410.

14692120, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/blms.12684 by Politecnico Di Torino Sist. Bibl Del Polit Di Torino, Wiley Online Library on [29/09/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License