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RESEARCH ARTICLE

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Local and non-local Poincaré inequalities on Lie groups

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Abstract

We prove a local L^p-Poincaré inequality, $1 \leqslant p < \infty$, on non-compact Lie groups endowed with a sub-Riemannian structure. We show that the constant involved grows at most exponentially with respect to the radius of the ball, and that if the group is non-doubling, then its growth is indeed, in general, exponential. We also prove a non-local L^2-Poincaré inequality with respect to suitable finite measures on the group.

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1 | INTRODUCTION

The aim of this paper is to establish two forms of Poincaré inequality on non-compact connected Lie groups. On the one hand, we shall obtain the Lie group analogue of the classical inequality on $\mathbb{R}^d$

$$\|f - f_B\|_{L^p(B)} \leqslant Cr\|\nabla f\|_{L^p(B)}, \tag{1.1}$$

where $p \in [1, \infty)$, $f \in C^\infty(\mathbb{R}^d)$, B is a ball of radius r and f_B is the average of f on B. On the other hand, we shall consider a non-local L^2-version of such an inequality, which takes the form

$$\|f\|_{L^2(\mathbb{R}^d,\mu)} \leqslant C\|\nabla f\|_{L^2(\mathbb{R}^d,\mu)} \tag{1.2}$$

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for certain finite measures μ which are absolutely continuous with respect to the Lebesgue measure and whose densities satisfy a certain decay condition at infinity. One should think, for example, to the case when μ is a Gaussian measure.

Extensions of the classical Poincaré inequality (1.1) to non-Euclidean settings have widely been studied in the last decades. A thorough overview of the literature would go out of the scope of the present paper, so we refer the reader to the milestone [11] and the references therein. For what concerns Lie groups, a Poincaré inequality on unimodular groups can be obtained by combining [16, §8.3] and [11, Theorem 9.7]. In this paper we prove that a Poincaré inequality also holds on non-unimodular Lie groups endowed with a relatively invariant measure, and we also describe the behaviour of the Poincaré constant in a quantitative way. We show that this grows at most exponentially with respect to the radius of the ball, and that if the group is non-doubling, then such growth is, in general, exponential. More precisely, in a class of Lie groups including the real hyperbolic spaces as a subclass, we estimate from below the constant involved in the Poincaré inequality by a quantity which grows exponentially with respect to the radius of the ball.

Non-local inequalities such as (1.2) have been introduced more recently [2] on $\mathbb{R}^d$, for densities satisfying suitable differential inequalities expressed in terms of the Laplacian Δ (cf. [2, Corollary 1.6]) whose prototype is a Gaussian function. After its establishment on $\mathbb{R}^d$, such non-local inequalities were extended to unimodular Lie groups of polynomial growth in [15], where a sum-of-squares subelliptic sub-Laplacian plays the role of Δ. In this paper, we extend their method to the non-doubling regime where the sub-Laplacian has an additional drift term, in a setting where we previously studied various function spaces [4–6] and the Sobolev and Moser–Trudinger inequalities [7].

As a classical application of the local Poincaré inequality, we show the so-called local parabolic Harnack principle for the sub-Laplacian with drift. Another application of our inequality is given in [3] to the study of spectral properties of Schrödinger operators on Lie groups.

2 | SETTING AND PRELIMINARIES

Let G be a non-compact connected Lie group with identity e. We denote by ρ a right Haar measure, by χ a continuous positive character of G and by μ_χ the measure with density χ with respect to ρ. As the modular function on G, which we denote by δ, is such a character, μ_δ is a left Haar measure on G. We denote it by λ. Observe also that $\mu_1 = \rho$.

Let $\mathbf{X} = \{X_1, \dots, X_\ell\}$ be a family of left-invariant linearly independent vector fields which satisfy Hörmander's condition. Let $d_C(\cdot, \cdot)$ be its associated left-invariant Carnot–Carathéodory distance. We let $|x| = d_C(x, e)$, and denote by B_r the ball centred at e of radius r. The measure of B_r with respect to ρ will be denoted by $V(r) = \rho(B_r)$; we recall that $V(r) = \lambda(B_r)$. It is well known, cf. [10, 19], that there exist $d \in \mathbb{N}^*$ depending on G and $\mathbf{X}$, and $C > 0$, such that

$$C^{-1} r^d \leqslant V(r) \leqslant C r^d \qquad \forall r \in (0, 1], \tag{2.1}$$

and $D_0, D > 0$ depending only on G, such that, either $C^{-1} r^D \leqslant V(r) \leqslant C r^D$ for all $r \geqslant 1$, or

$$C^{-1} \mathrm{e}^{D_0 r} \leqslant V(r) \leqslant C \mathrm{e}^{D r} \tag{2.2}$$

for all $r \geqslant 1$. In the former case, the group G is said to be of polynomial growth, while in the latter case of exponential growth.

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For any character χ and $r > 0$, one has (see [12, Proposition 5.7])

$$\sup_{B_r} \chi = \mathrm{e}^{c(\chi)r}, \qquad \text{where} \quad c(\chi) = \left(\sum_{j=1}^{\ell} |X_j \chi(e)|^2 \right)^{1/2}. \tag{2.3}$$

Since χ is a character, by (2.3) one also has

$$\inf_{B_r} \chi = \mathrm{e}^{-c(\chi)r}. \tag{2.4}$$

Given a ball B with respect to d_C, we denote by c_B its center and by r_B its radius, and we write $B = B(c_B, r_B)$; we also set $2B = B(c_B, 2r_B)$. Moreover, for $R > 0$ let $\mathcal{B}_R$ be the family of all balls of radius $\leqslant R$ and

$$D(R, \chi) = \sup_{B \in \mathcal{B}_R} \frac{\mu_\chi(2B)}{\mu_\chi(B)} = \sup_{0<r\leqslant R} \frac{\mu_\chi(B_{2r})}{\mu_\chi(B_r)}, \tag{2.5}$$

where the latter equality holds since $\mu_\chi(B(c_B, r)) = (\chi\delta^{-1})(c_B)\mu_\chi(B_r)$ for all $r > 0$ and $c_B \in G$.

In the following lemma we estimate the local doubling constant $D(R, \chi)$.

Lemma 2.1. *The metric measure space (G, d_C, μ_χ) is doubling if and only if $\chi = 1$ and (G, d_C, ρ) is doubling, in which case there exists $C > 0$ such that $D(R, \chi) \leqslant C$. If $\chi \neq 1$ and (G, d_C, ρ) is doubling, then there exists $C > 0$ such that*

$$D(R, \chi) \leqslant C\mathrm{e}^{3c(\chi)R} \qquad \forall R > 0,$$

while if $\chi \neq 1$ and (G, d_C, ρ) is non-doubling, then there exists $C > 0$ such that

$$D(R, \chi) \leqslant C\mathrm{e}^{(2D-D_0+3c(\chi))R} \qquad \forall R > 0.$$

Proof. If $\chi = 1$, the first statement is obvious since $\mu_\chi = \rho$.

Assume then that $\chi \neq 1$, so that there is $x \in G$ with $\chi(x) > 1$. If N denotes the lowest integer such that $N \geqslant |x|$, then $B_N x^n \subseteq B_{(n+1)N}$. If $r > N$ and n is the largest integer such that $(n + 1)N \leqslant [r]$, then

$$\mu_\chi(B_r) \geqslant \mu_\chi(B_N x^n) = \chi(x)^n \mu_\chi(B_N) \geqslant \chi(x)^{[r]/N-2} \mu_\chi(B_N),$$

whence $\mu_\chi(B_r)$ grows exponentially with r and the space (G, d_C, μ_χ) is non-doubling.

We now show the two bounds on $D(R, \chi)$. First, observe that by (2.1), (2.3) and (2.4)

$$\frac{\mu_\chi(B_{2r})}{\mu_\chi(B_r)} \leqslant C \qquad \forall r \in (0, 1],$$

for some $C > 0$. Moreover, if (G, d_C, ρ) is doubling, then by (2.3) and (2.4)

$$\frac{\mu_\chi(B_{2r})}{\mu_\chi(B_r)} \leqslant \mathrm{e}^{3c(\chi)r} \frac{V(2r)}{V(r)} \leqslant C\mathrm{e}^{3c(\chi)r} \qquad \forall r \geqslant 1,$$

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while if (G, d_C, ρ) is non-doubling, then the stated estimate follows similarly by (2.2), (2.3) and (2.4). □

3 | THE LOCAL POINCARÉ INEQUALITY ON LIE GROUPS

In this section we prove the L^p-Poincaré inequality for smooth functions on (G, d_C, μ_χ). Given a ball B and $f \in C^\infty(G)$, we denote by f_B^χ its average over B with respect to μ_χ,

$$f_B^\chi = \frac{1}{\mu_\chi(B)} \int_B f \, d\mu_\chi,$$

and we let $|\nabla f|^2 = \sum_{j=1}^{\ell} |X_j f|^2$. If S is a set of variables, we denote by $C(S)$ a constant depending only on the elements of S.

Theorem 3.1. *There exist a constant $C = C(G, \mathbf{X}) > 0$ and a universal constant $\alpha > 0$ such that, for all $p \in [1, \infty)$, $R > 0$, all balls B of radius $r \in (0, R]$ and $f \in C^\infty(G)$,*

$$\|f - f_B^\chi\|_{L^p(B,\mu_\chi)} \leqslant C \, e^{\frac{1}{p}[2c(\chi)+c(\chi\delta^{-1})]R} \, D(R, \chi)^\alpha \, r \, \||\nabla f|\|_{L^p(B,\mu_\chi)}. \tag{3.1}$$

Notice that the Poincaré constant grows at most exponentially with respect to the radius of the ball. The exponential term cannot, in general, be removed. After establishing the theorem, indeed, we show that when G is the so-called "$ax + b$" group and $\mu_\chi = \lambda$ is a left Haar measure, the growth of the constant is indeed exponential.

Proof. Let $p \in [1, \infty)$ be given. We shall prove that for every ball B of radius $r > 0$ and $f \in C^\infty(G)$

$$\int_B |f - f_B^\chi|^p \, d\mu_\chi \leqslant 2^p \, e^{c(\chi\delta^{-1})r} e^{2c(\chi)r} \, \frac{\mu_\chi(B_{2r})}{\mu_\chi(B_r)} \, r^p \int_{2B} |\nabla f|^p \, d\mu_\chi. \tag{3.2}$$

Once (3.2) is at disposal, the Poincaré inequality can be obtained by classical arguments, see, for example, [11, Theorem 9.7]. A careful inspection of [13, Section 5], in particular, shows how a Whitney decomposition of B brings to the constant given in the statement. We omit the details, which would be tedious and an almost verbatim repetition of the arguments that the reader can find in [13].

We then show (3.2). For $z \in G$, let $\gamma_z : [0, |z|] \to G$ be a C^1-geodesic such that $\gamma_z(0) = e$, $\gamma_z(|z|) = z$, $\gamma_z(s) \in B_{|z|}$ and $|\gamma_z'(s)| \leqslant 1$ for every $s \in [0, |z|]$.

Let B be a ball of radius $r > 0$. Observe that if $x, y \in B$, and $z = x^{-1}y$, then $|z| < 2r$. For every $x, z \in G$, by Hölder's inequality

$$|f(x) - f(xz)|^p \leqslant \left(\int_0^{|z|} |\nabla f(x\gamma_z(s))| \, ds \right)^p \leqslant |z|^{p-1} \int_0^{|z|} |\nabla f(x\gamma_z(s))|^p \, ds. \tag{3.3}$$

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We then have

$$\int_B |f - f_B^\chi|^p \, d\mu_\chi = \int_B \left| \frac{1}{\mu_\chi(B)} \int_B (f(x) - f(y)) \, d\mu_\chi(y) \right|^p d\mu_\chi(x)$$
$$\leqslant \frac{1}{\mu_\chi(B)} \int_B \int_B |f(x) - f(y)|^p \, d\mu_\chi(y) \, d\mu_\chi(x),$$

and after the change of variables $y = xz$, we get

$$\int_B |f - f_B^\chi|^p \, d\mu_\chi \leqslant \frac{1}{\mu_\chi(B)} \int_G \int_G \mathbf{1}_B(x) \mathbf{1}_B(xz) \big| f(x) - f(xz) \big|^p (\chi\delta^{-1})(x) \, d\mu_\chi(x) \, d\mu_\chi(z).$$

Observe now that by (3.3) and Fubini's theorem, we get

$$\int_G \mathbf{1}_B(x) \mathbf{1}_B(xz) \big| f(x) - f(xz) \big|^p (\chi\delta^{-1})(x) \, d\mu_\chi(x)$$
$$\leqslant \frac{1}{\mu_\chi(B)} |z|^{p-1} \int_0^{|z|} \int_G \mathbf{1}_B(x) \mathbf{1}_B(xz) |\nabla f(x\gamma_z(s))|^p \, (\chi\delta^{-1})(x) \, d\mu_\chi(x) \, ds.$$

We make the change of variables $\zeta = x\gamma_z(s)$ and observe that by (2.3), if $x \in B$, then

$$(\chi\delta^{-1})(x) \leqslant (\chi\delta^{-1})(c_B) \sup_{B_r} (\chi\delta^{-1}) \leqslant (\chi\delta^{-1})(c_B) \, \mathrm{e}^{c(\chi\delta^{-1})r},$$

and $\chi(\gamma_z(s)) \leqslant \mathrm{e}^{2c(\chi)r}$. We obtain

$$\int_G \mathbf{1}_B(x) \mathbf{1}_B(xz) (\chi\delta^{-1})(x) |\nabla f(x\gamma_z(s))|^p \, d\mu_\chi(x)$$
$$\leqslant (\chi\delta^{-1})(c_B) \, \mathrm{e}^{c(\chi\delta^{-1})r} \, \mathrm{e}^{2c(\chi)r} \int_G \mathbf{1}_{B\gamma_z(s)}(\zeta) \mathbf{1}_{Bz^{-1}\gamma_z(s)}(\zeta) |\nabla f(\zeta)|^p \, d\mu_\chi(\zeta).$$

Notice that $B\gamma_z(s) \cap Bz^{-1}\gamma_z(s) \subseteq 2B$ for all $s \in [0, |z|]$. This is straightforward by the triangle inequality when $|z| < r$. Otherwise, let $s_0 \in [0, |z|]$ be such that $|\gamma_z(s_0)| = r$. Then $B\gamma_z(s) \subseteq 2B$ for $s \in [0, s_0]$ and $Bz^{-1}\gamma_z(s) \subseteq 2B$ for $s \in (s_0, |z|]$. Therefore

$$\int_G \mathbf{1}_B(x) \mathbf{1}_B(xz) (\chi\delta^{-1})(x) |\nabla f(x\gamma_z(s))|^p \, d\mu_\chi(x)$$
$$\leqslant (\chi\delta^{-1})(c_B) \, \mathrm{e}^{c(\chi\delta^{-1})r} \, \mathrm{e}^{2c(\chi)r} \mathbf{1}_{B_{2r}}(z) \int_{2B} |\nabla f(\zeta)|^p \, d\mu_\chi(\zeta).$$

Since $(\chi\delta^{-1})(c_B)\mu_\chi(B_{2r}) = \mu_\chi(2B)$, by integrating with respect to z we get

$$\int_B |f - f_B^\chi|^p \, d\mu_\chi \leqslant \mathrm{e}^{2c(\chi)r} \mathrm{e}^{c(\chi\delta^{-1})r} (2r)^p \, \frac{\mu_\chi(2B)}{\mu_\chi(B)} \int_{2B} |\nabla f(\zeta)|^p \, d\mu_\chi(\zeta),$$

which concludes the proof. □

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As a corollary, we obtain the so-called local parabolic Harnack principle. We introduce the operator

$$\Delta_\chi = -\sum_{j=1}^{\ell}(X_j^2 + (X_j\chi)(e)X_j), \tag{3.4}$$

which is essentially self-adjoint on $L^2(\mu_\chi)$ and non-negative; see, for example, [4, 12]. We say that Δ_χ satisfies the local parabolic Harnack principle up to distance $R > 0$ if there is $C(R) > 0$ such that, for all $x \in G$, $r \in (0, R]$, $s \in \mathbb{R}$, and any positive solutions u of $(\partial_t + \Delta_\chi)u = 0$ on $(s, s + r^2) \times B(x, r)$, we have that

$$\sup_{Q_-} u \leqslant C(R) \inf_{Q_+} u,$$

where

$$Q_- = \left(s + r^2/6, s + r^2/3\right) \times B(x, r/2), \qquad Q_+ = \left(s + 2r^2/3, s + r^2\right) \times B(x, r/2).$$

The following result follows at once from Theorem 3.1 and [16, Theorem 2.1].

Corollary 3.2. *For every $R > 0$, Δ_χ satisfies the local parabolic Harnack principle up to distance R. In particular, the positive Δ_χ-harmonic functions satisfy the local elliptic Harnack inequality.*

3.1 | Exponential growth of the constant

For $r > 0$ and $p \in [1, \infty)$, define

$$C(r, p) = \inf \frac{\int_{B_r} |f - f_B^\chi|^p \, d\mu_\chi}{\int_{B_r} |\nabla f|^p \, d\mu_\chi}, \tag{3.5}$$

where the infimum runs over all functions $f \in C^\infty(G)$. In this section we show that the exponential bound of $C(r, p)$ appearing in inequality (3.1) is in general optimal, in the sense that such constant cannot grow less than exponentially with respect to r. Indeed, in the particular case of $ax + b$ groups of arbitrary dimension, we provide a lower bound of exponential type for $C(r, p)$. For notational convenience, we shall write $A \lesssim B$ to indicate that there is a constant C such that $A \leqslant CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$.

Let $G = \mathbb{R}^{n-1} \rtimes \mathbb{R}^+$ and let (x, a) be its generic element. Recall that

$$d\lambda(x, a) = \frac{dx\, da}{a^n} \qquad \text{and} \qquad d\rho(x, a) = \frac{dx\, da}{a},$$

since $\delta(x, a) = a^{-n+1}$; all positive characters of G are of the form $\chi_\gamma(x, a) = a^\gamma$ for some $\gamma \in \mathbb{R}$. We shall write μ_γ for the measure μ_{χ_γ}. In particular, $\lambda = \mu_{1-n}$ is the hyperbolic measure. We consider the left-invariant vector fields $X_i = a\partial_i$, $i = 1, \dots, n-1$, and $X_0 = a\partial_a$ which form a basis of the Lie algebra of G. The distance induced by such vector fields is the hyperbolic metric which is given by

$$\cosh |(x, a)| = \tfrac{1}{2}(a + a^{-1} + a^{-1}|x|^2),$$

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where $|x|$ is the Euclidean norm of $x \in \mathbb{R}^{n-1}$ (see [1, (2.18)], [17, (1.1)]). Then

$$B_r = \left\{(x,a) : \ \mathrm{e}^{-r} < a < \mathrm{e}^{r}, \ |x|^2 < 2a(\cosh r - \cosh \log a)\right\}.$$

In the case of the real hyperbolic space, that is, the $ax+b$ group endowed with the measure λ and the metric defined above, the constant $C(r,p)$ in (3.5) was estimated from above in [11, Section 10.1]. We now estimate such constant from below.

Consider the function $\phi : G \to \mathbb{R}$ defined by

$$\phi(x,a) = x_1, \quad (x,a) \in G.$$

Observe that $\int_{B_r} \phi \, d\mu_\gamma = 0$ for all $\gamma \in \mathbb{R}$ and $|\nabla \phi(x,a)|^p = a^p$. Moreover,

$$\int_{B_r} |\phi|^p \, d\mu_\gamma \approx \int_{\mathrm{e}^{-r}}^{\mathrm{e}^{r}} a^{\gamma - 1 + \frac{p+n-1}{2}} (\cosh r - \cosh \log a)^{\frac{p+n-1}{2}} \, da,$$

while

$$\int_{B_r} |\nabla \phi|^p \, d\mu_\gamma \approx \int_{\mathrm{e}^{-r}}^{\mathrm{e}^{r}} a^{\gamma - 1 + p + \frac{n-1}{2}} (\cosh r - \cosh \log a)^{\frac{n-1}{2}} \, da.$$

Lemma 3.3. *Let $\delta \in \mathbb{R}$ and $\epsilon > 0$. Then*

$$\int_{\mathrm{e}^{-r}}^{\mathrm{e}^{r}} a^{\delta} (\cosh r - \cosh \log a)^{\epsilon} \, da \approx \mathrm{e}^{r(|\delta+1|+\epsilon)}.$$

Proof. We first make a change of variables

$$\int_{\mathrm{e}^{-r}}^{\mathrm{e}^{r}} a^{\delta} (\cosh r - \cosh \log a)^{\epsilon} \, da = \int_{-r}^{r} \mathrm{e}^{t(\delta+1)} (\cosh r - \cosh t)^{\epsilon} \, dt.$$

Since $\cosh r - \cosh t \approx \mathrm{e}^{r}$ if $|t| < r-1$, while $\cosh r - \cosh t \approx (r - |t|)\mathrm{e}^{r}$ if $r-1 < |t| < r$, we get

$$\int_{-r+1}^{r-1} \mathrm{e}^{t(\delta+1)} (\cosh r - \cosh t)^{\epsilon} \, dt \approx \mathrm{e}^{\epsilon r} \int_{-r+1}^{r-1} \mathrm{e}^{t(\delta+1)} \, dt \approx \mathrm{e}^{(\epsilon + |\delta+1|)r}$$

while

$$\int_{r-1<|t|<r} \mathrm{e}^{t(\delta+1)} (\cosh r - \cosh t)^{\epsilon} \, dt \approx \mathrm{e}^{(\epsilon+|\delta+1|)r} \int_{r-1<|t|<r} (r - |t|)^{\epsilon} \, dt \approx \mathrm{e}^{(\epsilon+|\delta+1|)r},$$

as required. □

From the lemma above, we get that

$$\int_{B_r} |\phi|^p \, d\mu_\gamma \approx \mathrm{e}^{(|\gamma + \frac{p+n-1}{2}| + \frac{p+n-1}{2})r},$$

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while

$$\int_{B_r} |\nabla\phi|^p \, d\mu_\gamma \approx \mathrm{e}^{(|\gamma+p+\frac{n-1}{2}|+\frac{n-1}{2})r}.$$

We observe that, if $\gamma < -\frac{p+n-1}{2}$, then

$$\left|\gamma + \frac{p+n-1}{2}\right| + \frac{p+n-1}{2} > \left|\gamma + p + \frac{n-1}{2}\right| + \frac{n-1}{2}.$$

Thus for such γ

$$C(r,p) \geqslant C\mathrm{e}^{r\left(\left|\gamma+\frac{p+n-1}{2}\right|+\frac{p}{2}-\left|\gamma+p+\frac{n-1}{2}\right|\right)}.$$

If in particular $\gamma = -n+1$, hence μ_γ is the left measure, and $n > p+1$, then

$$C(r,p) \geqslant \mathrm{e}^{\left(\left|\frac{-n+p+1}{2}\right|-\left|\frac{-n+1}{2}+p\right|+\frac{p}{2}\right)r} = \begin{cases} \mathrm{e}^{pr} & n \geqslant 2p+1 \\ \mathrm{e}^{(n-p-1)r} & p+1 < n \leqslant 2p+1. \end{cases}$$

4 | NON-LOCAL POINCARÉ INEQUALITY

In this second part of the paper we prove a non-local L^2-Poincaré inequality for suitable finite measures on G in the spirit of [14, 15]. More precisely, let M be a positive C^2-function in $L^1(\mu_\chi)$ and $\mu_{\chi,M}$ be the finite measure whose density is M with respect to μ_χ. We shall prove L^2-global Poincaré inequalities for the measure $\mu_{\chi,M}$ for a large family of functions M. In order to do this, we let

$$L_1^2(\mu_{\chi,M}) = \{f \in L^2(\mu_{\chi,M}) : \, |\nabla f| \in L^2(\mu_{\chi,M})\}$$

and introduce the operator

$$\Delta_{\chi,M} = \Delta_\chi - \nabla(\log M)\cdot\nabla, \tag{4.1}$$

where Δ_χ is that of (3.4), $\mathrm{Dom}(\Delta_{\chi,M}) = \{f \in L_1^2(\mu_{\chi,M}) : \, \Delta_{\chi,M} f \in L^2(\mu_{\chi,M})\}$ and the derivatives are meant in the distributional sense. Observe that $\Delta_{\chi,M}$ is symmetric on $L^2(\mu_{\chi,M})$; in particular, for all $f \in \mathrm{Dom}(\Delta_{\chi,M})$ and $g \in L_1^2(\mu_{\chi,M})$,

$$\int_G \nabla f \cdot \nabla g \, d\mu_{\chi,M} = \int_G \Delta_{\chi,M} f \cdot g \, d\mu_{\chi,M},$$

where $\nabla f \cdot \nabla g = \sum_{j=1}^{\ell} (X_j f)(X_j g)$.

We say that the couple (Δ_χ, M) admits a Lyapunov function if there exist a C^2 function $W : G \to [1,\infty)$ and constants $\theta > 0$, $b \geqslant 0$, $R > 0$ such that

$$-\Delta_{\chi,M} W(x) \leqslant -\theta W(x) + b\mathbf{1}_{B_R}(x) \qquad \forall x \in G. \tag{4.2}$$

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Observe that the existence of a Lyapunov function depends on G, $\mathbf{X}$, χ and M. For $f \in L^1(\mu_{\chi,M})$ we let

$$f_{\chi,M} = \frac{1}{\mu_{\chi,M}(G)} \int_G f \, d\mu_{\chi,M}.$$

Our second main result is the following global L^2-Poincaré inequality for $\mu_{\chi,M}$.

Theorem 4.1. *If (Δ_χ, M) admits a Lyapunov function, then there exists a constant $C = C(G, \mathbf{X}, \chi, M)$ such that for all $f \in L^2_1(\mu_{\chi,M})$*

$$\|f - f_{\chi,M}\|_{L^2(\mu_{\chi,M})} \leqslant C \| |\nabla f| \|_{L^2(\mu_{\chi,M})}. \tag{4.3}$$

Theorem 4.1 is a generalization to any connected non-compact possibly non-unimodular Lie group of the non-local L^2-Poincaré inequalities proved in [2, Theorem 1.4] in the Euclidean setting and in [15, Theorem 1.1] in unimodular Lie groups of polynomial growth, by which our proof is inspired. The validity of other versions of non-local Poincaré inequalities in the current setting, such as those in [15, Theorem 1.4], is still an open problem. More general versions of non-local Poincaré inequalities of this kind were proved in [9] in the setting of a topological measure space endowed with a family of sets which play the role of unit balls and satisfy suitable assumptions. Recently in [8] non-local L^p-Poincaré inequalities were obtained on Carnot groups of Engel type in the case when the density of the measure depends on a homogeneous norm of the group; we note, however, that the case $p = 2$ is always excluded.

Proof. Let $f \in L^2_1(\mu_{\chi,M})$, and observe first that

$$\int_G \left| f - f_{\chi,M} \right|^2 d\mu_{\chi,M} = \min_{c \in \mathbb{R}} \int_G |f - c|^2 d\mu_{\chi,M}. \tag{4.4}$$

Let now $g = f - c$ for a positive c to be determined, and W be a Lyapunov function for (Δ_χ, M). By (4.2)

$$\int_G |g|^2 \, d\mu_{\chi,M} \leqslant \int_G |g|^2 \frac{\Delta_{\chi,M} W}{\theta W} \, d\mu_{\chi,M} + \int_{B_R} |g|^2 \frac{b}{\theta W} \, d\mu_{\chi,M}. \tag{4.5}$$

We treat the two terms separately.

Let us consider the first term and prove that

$$\int_G \frac{\Delta_{\chi,M} W}{W} g^2 d\mu_{\chi,M} \leqslant \int_G |\nabla g|^2 d\mu_{\chi,M}. \tag{4.6}$$

We prove it by density, and firstly assume that g is compactly supported. By definition of $\Delta_{\chi,M}$,

$$\begin{aligned} \int_G \frac{\Delta_{\chi,M} W}{W} g^2 \, d\mu_{\chi,M} &= \int_G \nabla\left(\frac{g^2}{W}\right) \cdot \nabla W \, d\mu_{\chi,M} \\ &= 2 \int_G \frac{g}{W} \nabla g \cdot \nabla W \, d\mu_{\chi,M} - \int_G \frac{g^2}{W^2} |\nabla W|^2 \, d\mu_{\chi,M} \end{aligned}$$

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$$= \int_G |\nabla g|^2 \, d\mu_{\chi,M} - \int_G |\nabla g - \frac{g}{W} \nabla W|^2 \, d\mu_{\chi,M}$$
$$\leqslant \int_G |\nabla g|^2 d\mu_{\chi,M}.$$

Let now $g \in L^2_1(\mu_{\chi,M})$, and consider a non-decreasing sequence of functions $\psi_n \in C^\infty_c(G)$ such that

$$\mathbf{1}_{B_{nR}} \leqslant \psi_n \leqslant 1, \quad |\nabla \psi_n| \leqslant 1.$$

By applying (4.6) to $g\psi_n$, the monotone convergence theorem in the left-hand side and the dominated convergence theorem in the right-hand side, one gets (4.6).

To deal with the second term, we choose c such that $\int_{B_R} g \, d\mu_\chi = 0$. By (3.2) applied to g on B_R, and the fact that M is bounded from above and below on B_{2R}, one has

$$\int_{B_R} |g|^2 \, d\mu_{\chi,M} \leqslant C \int_{B_R} |g|^2 \, d\mu_\chi \leqslant C \int_{B_{2R}} |\nabla g|^2 \, d\mu_\chi \leqslant C \int_{B_{2R}} |\nabla g|^2 \, d\mu_{\chi,M},$$

where the constant C depends on R and M. Therefore, since $W \geqslant 1$,

$$\int_{B_R} |g|^2 \frac{b}{\theta W} \, d\mu_{\chi,M} \leqslant C \int_{B_{2R}} |\nabla g|^2 \, d\mu_{\chi,M} \leqslant C \int_G |\nabla g|^2 \, d\mu_{\chi,M},$$

which completes the proof. □

Corollary 4.2. *Let $v = -\log M$. If there exist $a \in (0,1)$, $c > 0$ and $R > 0$ such that*

$$a|\nabla v|^2(x) + \Delta_\chi v(x) \geqslant c \qquad \forall\, x \in B_R^c, \tag{4.7}$$

then (Δ_χ, M) admits a Lyapunov function, and (4.3) holds.

Proof. Let $W(x) = \mathrm{e}^{(1-a)(v(x)-\inf_G v)}$, so that

$$-\Delta_{\chi,M} W = (1-a)W\big(-\Delta_\chi v - a|\nabla v|^2\big).$$

Then W is a Lyapunov function with $\theta = c(1-a)$ and $b = \max_{B_R}(-\Delta_{\chi,M} W + \theta W)$. □

One can actually show that if (4.7) holds with $a < 1/2$, then (4.3) self-improves as follows.

Proposition 4.3. *Let $v = -\log M$. If there exist $c > 0$, $R > 0$ and $\epsilon \in (0,1)$ such that*

$$\frac{1-\epsilon}{2}|\nabla v|^2(x) + \Delta_\chi v(x) \geqslant c \qquad \forall\, x \in B_R^c, \tag{4.8}$$

then there exists $C > 0$ such that for all $f \in L^2_1(\mu_{\chi,M})$

$$\| |f - f_{\chi,M}|(1 + |\nabla v|)\|_{L^2(\mu_{\chi,M})} \leqslant C \| |\nabla f| \|_{L^2(\mu_{\chi,M})}. \tag{4.9}$$

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Proof. Observe firstly that, since v is C^2 and (4.8) holds,

$$\frac{1-\epsilon}{2}|\nabla v|^2 - \Delta_\chi v \geqslant \alpha \tag{4.10}$$

for some $\alpha \in \mathbb{R}$. Let $f \in L^2_1(\mu_{\chi,M})$ and let $g = f\sqrt{M}$. Since

$$\nabla f = \frac{1}{\sqrt{M}}\nabla g + \frac{1}{2} g \frac{1}{\sqrt{M}} \nabla v,$$

by (4.10)

$$\begin{aligned}
\int_G |\nabla f|^2 \, d\mu_{\chi,M} &= \int_G \left(|\nabla g|^2 + \frac{1}{4}|g|^2|\nabla v|^2 + g\nabla g \cdot \nabla v \right) d\mu_\chi \\
&= \int_G \left(|\nabla g|^2 + \frac{1}{4}|g|^2|\nabla v|^2 + \frac{1}{2}\nabla(|g|^2) \cdot \nabla v \right) d\mu_\chi \\
&\geqslant \int_G |g|^2 \left(\frac{1}{4}|\nabla v|^2 + \frac{1}{2}\Delta_\chi v \right) d\mu_\chi \\
&\geqslant \frac{1}{2}\int_G |f|^2 \left(\frac{\epsilon}{2}|\nabla v|^2 + \alpha \right) d\mu_{\chi,M}.
\end{aligned}$$

Since (4.3) holds by (4.8) and Corollary 4.2, the conclusion follows. □

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