POLITECNICO DI TORINO
Repository ISTITUZIONALE

Clearing payments in dynamic financial networks

(Article begins on next page)

28 April 2024

Clearing payments in dynamic financial networks☆

Giuseppe C. Calafiore [a,b], Giulia Fracastoro [a], Anton V. Proskurnikov [a,*]

[a] *Department of Electronics and Telecommunications, Polytechnic of Turin, Turin, Italy*
[b] *College of Engineering and Computer Science, VinUniversity, Hanoi, Viet Nam*

ARTICLE INFO

ABSTRACT

This paper proposes a novel dynamical model for determining clearing payments in financial networks. We extend the classical Eisenberg–Noe model of financial contagion to multiple time periods, allowing financial operations to continue after initial *pseudo defaults*, thus permitting nodes to possibly recover and eventually fulfill their liabilities. Dynamic optimal clearing payments in our model are computed by solving a suitable linear program, both in the full matrix payments case and in the pro-rata constrained case. We prove that the proposed model obeys the *priority of debt claims* requirement, that is, each node at every step either pays its liabilities in full, or it pays out all its balance. In the pro-rata case, the optimal dynamic clearing payments are unique, and can be determined via a time-decoupled sequential optimization approach.

1. Introduction

The global financial system forms a highly interconnected network of institutions linked together via a structure of mutual liabilities and common exposures to external risky investments. Such interconnected structure makes the system potentially prone to "cascading defaults", whereby a shock at a node (e.g., an expected incoming payment that gets cancelled or delayed for some reason) may provoke a default at that node, who hence cannot pay its liabilities to neighboring nodes, which may in turn default due to the missed payment, and so on in an avalanche fashion. The global financial crisis of 2008 is an example of such behavior, where the bankruptcy of Lehman-Brothers is identified as the watershed event that started the crisis. Since the consequences of these cascading events can be catastrophic, modeling and analyzing such behavior is of crucial importance. The seminal work (Eisenberg & Noe, 2001) introduced a simple model for studying financial contagion. In particular, it focused on defining a clearing procedure between financial entities. Clearing consists in an agreement for settling claims in the case of defaults, on the basis of a set of rules and prevailing regulations. In Eisenberg and Noe (2001), the authors showed that there exist a clearing vector which defines the mutual interbank payments, under certain assumptions. Among such assumptions, an important one is that the debts of all nodes of the system are paid simultaneously.

The basic model presented in Eisenberg and Noe (2001) has become a cornerstone in the analysis of financial contagion, and it has since been extended in various directions, also with the addition of non-trivial features in order to make the model more realistic. The models presented in Cifuentes, Ferrucci, and Shin (2005), Shin (2008), for instance, consider also the liquidity risk. Instead, in Elsinger et al. (2009), Suzuki (2002) cross-holdings and seniority of liabilities are introduced. Other works take into account costs of default (Rogers & Veraart, 2013), illiquid assets (Amini, Filipović, & Minca, 2016), mandatory disclosures (Alvarez & Barlevy, 2015), cross-ownership of equities and liabilities (Fischer, 2014), and decentralized clearing processes (Csóka & Jean-Jacques Herings, 2018).

The vast majority of the works based on the Eisenberg–Noe model, however, considers the problem only in a static, or single-period, setting. This assumption is quite unrealistic, since it supposes that all liabilities are claimed and due at the same time. In addition, static models are only able to capture the immediate consequences of a financial shock. For these reasons, several works recently proposed time-dynamic extensions of the Eisenberg–Noe model. In Sonin and Sonin (2020) a continuous-time model of clearing in financial networks is presented. This work has later been extended by considering liquid assets (Chen, Wang, & Yao, 2021), heterogeneous network structures over time and early defaults (Banerjee, Bernstein, & Feinstein, 2018). Other works, such as Feinstein and Søjmark (2021), propose to combine the interbank Eisenberg–Noe model and the dynamic mean field

---

approach. Instead, Feinstein (2020) uses a continuous-time model for price-mediated contagion.

A different line of research extended the Eisenberg–Noe model considering a discrete-time setting. In Capponi and Chen (2015), Ferrara, Langfield, Liu, and Ota (2019) a multi-period clearing framework is introduced. Using a similar approach, Kusnetsov, Veraart, and Luitgard (2019) consider the case where interbank liabilities can have multiple maturities, considering both long-term and short-term liabilities.

In the present work, we focus on a discrete-time setting and introduce a multi-period model whereby financial operations are allowed for a given number of time periods after the initial theoretical default (named here *pseudo default*). Contrary to the mainstream default models, which assume that regular financial transactions somehow *freeze* as soon as default is detected, in our proposed model we assume that when some nodes default, a *grace period* of T time intervals (e.g., days, or weeks) is granted to them for continuing regular operations, while new interbank claims are halted, with the objective of possibly repaying their outstanding debts and recovering from the pseudo default.[1] In practice, this model requires a prior agreement among the financial players to automatically re-negotiate in case of default the terms of the liability claims by extending them to T periods forward, were T is the agreed-upon grace period extension. Since some nodes may possibly recover with time, and eventually fulfill their debts, the proposed model allows for a potential significant reduction of the systemic effects of financial shocks and containment of default contagion, as demonstrated in the numerical experiments in Section 5.2.

We first consider the general case where payment matrices are unconstrained. This scenario has been introduced in the static case in Calafiore, Fracastoro, and Proskurnikov (2021a, 2021b), where its advantages over the proportional rule in terms of the overall system loss have been highlighted. We give a full characterization of clearing payments in a dynamic setting, in terms of a suitable dynamic optimization problem, and we prove that the optimal sequence of payment matrices satisfies the absolute priority of debt claims rule, hence the proposed method produces proper clearing matrices at each stage of the decision horizon.

We then consider the situation in which a proportionality rule is enforced, whereby nodes must pay the claimant institutions proportionally to their nominal claims (pro-rata rule). We prove that under the pro-rata rule the optimal dynamic payments are again proper clearing payments, they are unique and, further, the multi-stage optimization problem can be decoupled in time into an equivalent series of LP problems.

The present work focuses on the network of interbank liabilities and, as common to the mainstream approaches in this context (Acemoglu, Ozdaglar, & Tahbaz-Salehi, 2015; Eisenberg & Noe, 2001; Glasserman & Young, 2016), it assumes that the interbank liabilities are known. This is certainly a strong and possibly unrealistic assumption, since most interbank obligations are private and non disclosed to the public or even to the regulating authorities. However, on the one hand even if the precise liability network is unknown many studies suggest that there are typical network classes, such as core–periphery and scale-free networks (Craig & von Peter, 2014; Soramäki & Cook, 2022), which approximately match the structure of real interbank networks, and which can then be used as simulation proxies for studying the stylized phenomena we are interested in. On the other hand, approaches also exist in which the interbank liability matrix is *estimated* from the available partial information

via techniques of network reconstruction, see, e.g., Anand, Craig, and von Peter (2015), Billio, Getmansky, Lo, and Pelizzon (2012), Demirer, Diebold, Liu, and Yilmaz (2018), Diebold and Yilmaz (2014) and Petropoulos, Siakoulis, Lazaris, and Chatzis (2021). Thirdly, the emergence of literature highlighting the advantages for the society at large of making the interbank liability structure transparent to the regulatory authority may promote a transition towards a more safe and reliable banking system.

Finally, we observe that banks are connected not only via their mutual liabilities, but also via common exposures to external risky assets, see, e.g., Allen, Babus, and Carletti (2012). Banks on one side and assets on the other side form together a bipartite network of investments whereby each bank is linked to the investment assets it owns, and the projection of this network onto the bank nodes creates an interbank network of mutual exposures. Such bank network is clearly different from the interbank liability network we treat in this paper. However, we discuss in Remark 2 how systemic risk effects due to common exposures can be included in our model.

The remainder of the paper is organized as follows. Section 2 introduces some preliminary notions and the notation that will be used in the next sections. In Section 3 we introduce the Eisenberg–Noe financial network model. Then, in Section 4 we illustrate the proposed dynamic model, considering both the unrestricted case and the case with the pro-rata rule imposed. A schematic numerical example is proposed in Section 5.1 in order to illustrate the proposed model, while a randomized campaign of more realistic numerical simulations is presented in Section 5.2. Conclusions are drawn in Section 6. For ease of reading, we collected the proofs of all technical results in the appendix.

2. Preliminaries and notation

Given a finite set $\mathcal{V}$, the symbol $|\mathcal{V}|$ stands for its cardinality. The set of families $(a_\xi)_{\xi \in \Xi}$, $a_\xi \in \mathbb{R}$, is denoted by $\mathbb{R}^\Xi$. For two such families $(a_\xi), (b_\xi)$, we write $a \leq b$ (respectively, $a < b$) if $a_\xi \leq b_\xi$ (respectively, $a_\xi < b_\xi$) for all $\xi \in \Xi$. The symbols $\geq$ and $>$ are defined similarly. We write $a \lneq b$ if $a \leq b$ and $a \neq b$. If $a \leq b$, then $[a, b]$ stands for the set of such $c = (c_\xi)_{\xi \in \Xi}$ that $a \leq c \leq b$. The operations min, max are defined elementwise, e.g., $\min(a, b) \doteq (\min(a_\xi, b_\xi))_{\xi \in \Xi}$. These notation symbols apply to both vectors (usually, $\Xi = \{1, \ldots, n\}$) and matrices (usually, $\Xi = \{1, \ldots, n\} \times \{1, \ldots, n\}$).

Every nonnegative square matrix $A = (a_{ij})_{i,j \in \mathcal{V}}$ corresponds to a weighted digraph $\mathcal{G}[A] = (\mathcal{V}, \mathcal{E}[A], A)$ whose nodes are indexed by $\mathcal{V}$ and whose set of arcs is defined as $\mathcal{E}[A] = \{(i, j) \in \mathcal{V} \times \mathcal{V} : a_{ij} > 0\}$. The value a_{ij} can be interpreted as the weight of arc $i \to j$. A sequence of arcs $i_0 \to i_1 \to \ldots \to i_{s-1} \to i_s$ constitute a *walk* between nodes i_0 and i_s in graph $\mathcal{G}[A]$. The set of nodes $J \subseteq \mathcal{V}$ is *reachable* from node i if $i \in J$ or a walk from i to some element $j \in J$ exists; J is called *globally reachable* in the graph if it is reachable from every node $i \notin J$.

A graph is strongly connected (strong) if every two nodes i, j are mutually reachable. A graph that is not strong has several strongly connected (or simply strong) *components* (Fig. 1). A strong component is said to be non-trivial if it contains more than one node. A component is said to be a *sink* component if no arc leaves it and a *source* component if no arc enters it. A strong component can be *isolated* (see Fig. 1b), when it has neither incoming nor outcoming arcs, and thus it is both a source and a sink. Strong components of undirected graphs are always isolated.

3. The Eisenberg–Noe financial network model

We start by considering the "static" case introduced in the seminal work of Eisenberg and Noe (2001). In this setting, n

---

[1] As discussed in Banerjee et al. (2018), typical financial contracts distinguish defaults and pseudo-defaults, or *delinquencies*, of a firm, caused by temporal impossibility to satisfy its obligations due to insufficient liquidity. A delinquent firm can recover within some grace period.
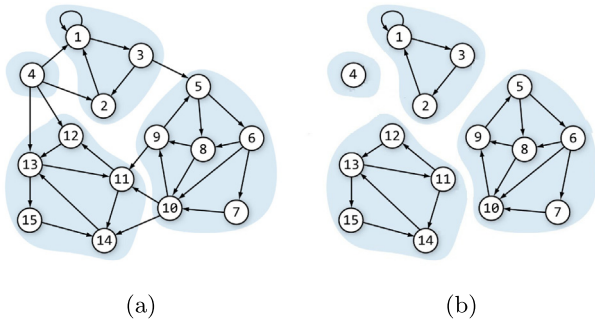
Fig. 1. Strong components of a directed graph: (a) non-isolated; (b) isolated. In (a), {4} is a (trivial) single source component, {11, ..., 15} is a single sink component.

nodes, representing financial entities (banks), are connected via a complex structure of mutual liabilities. The payment due from node i to node j is denoted by $\bar{p}_{ij} \geq 0$, and such liabilities are supposed to be due at the end of a fixed time period. These interbank liabilities form the *liability matrix* $\bar{P} \in \mathbb{R}^{n \times n}$, such that $[\bar{P}]_{ij} = \bar{p}_{ij}$ for $i \neq j = 1, \ldots, n$, and $[\bar{P}]_{ii} = 0$ for $i = 1, \ldots, n$.

Following the notation introduced in Glasserman and Young (2016, Section 5), we let $c \in \mathbb{R}^n_+$ be the vector whose ith component $c_i \geq 0$ represents the total payments due to node i from non-financial entities (i.e., from any other entity, different from the n banks), plus the liquidation value of any assets or reserves in the belonging of bank i. Typically, c_i consists of cash, securities, mortgages, and other assets outside the network. Payments from banks to the external sector are instead modeled by introducing a fictitious node that represents the external sector and owes no liability to the other nodes (the corresponding row of $\bar{P}$ is zero).

The book value of the *asset* side of node i's balance sheet is given by

$$\bar{\phi}_i \doteq c_i + \sum_{k \neq i} \bar{p}_{ki}, \tag{1}$$

while the *liability* side is given by

$$\bar{p}_i \doteq \sum_{k \neq i} \bar{p}_{ik}. \tag{2}$$

The nominal net worth of node i is given by $\bar{w}_i = c_i + \sum_{k \neq i} \bar{p}_{ki} - \bar{p}_i$. In regular operations, the asset side of each bank is no smaller than its liability side (i.e., $\bar{\phi}_i \geq \bar{p}_i$), which means that each bank remains solvable and is able to pay its liabilities in full; in such case $\bar{w}_i \geq 0$ and it represents the book value of bank i's *equity*.

A critical situation occurs instead when (due to, e.g., a drop in the external assets c_i) some bank i has not enough incoming liquidity to fully pay its liabilities. In this situation, the actual payments to other banks have to be remodulated to lesser values than their nominal values $\bar{p}_{ij}$. The *clearing payments* are a set of mutual payments which settle the mutual claims in case of defaults, by enforcing a set of rules (Csóka & Jean-Jacques Herings, 2018; Eisenberg & Noe, 2001), which are: *(i)* payments cannot exceed the corresponding liabilities, *(ii)* limited liability, i.e., the balance at each node cannot be negative, *(iii)* absolute priority (i.e., each node either pays its liabilities in full, or it pays out all its balance).

We let $p_{ij} \in [0, \bar{p}_{ij}]$, $i \neq j = 1, \ldots, n$, denote the actual inter-bank payments executed at the end of the period, which we shall collect in matrix $P \in \mathbb{R}^{n,n}$. At each node i we write a balance equation, involving the actual in-flow and out-flow, defined respectively as

$$\phi_i \doteq c_i + \sum_{k \neq i} p_{ki}, \tag{3}$$

$$p_i \doteq \sum_{k \neq i} p_{ik}. \tag{4}$$

The balance represents the net worth w_i of the ith bank, which is defined as

$$w_i \doteq \phi_i - p_i = c_i + \sum_{k \neq i} p_{ki} - \sum_{k \neq i} p_{ik}. \tag{5}$$

The limited liability rule *(ii)* requires that $w_i \geq 0$, $\forall i$.

In vector notation, the vectors of actual and nominal in/outflows and the vector of net worths are

$$\phi = c + P^\top \mathbf{1}, \quad \bar{\phi} = c + \bar{P}^\top \mathbf{1} \tag{6}$$

$$p = P\mathbf{1}, \quad \bar{p} = \bar{P}\mathbf{1} \tag{7}$$

$$w = \phi - p = (c + P^\top \mathbf{1}) - P\mathbf{1}, \tag{8}$$

where $\mathbf{1}$ denotes a vector of ones of suitable dimension.

The above mentioned conditions *(i), (ii)* on the payments are written in compact vector form as $0 \leq P \leq \bar{P}$ and $P\mathbf{1} \leq c + P^\top \mathbf{1}$, that is the payment matrix P is restricted to belong to the following convex polytope

$$\mathcal{P}(c, \bar{P}) \doteq \big\{ P \in \mathbb{R}^{n \times n} : 0 \leq P \leq \bar{P}, \\ P\mathbf{1} \leq c + P^\top \mathbf{1}, \; P_{ii} = 0, \; i = 1, \ldots, n \big\}. \tag{9}$$

A payment matrix $P \in \mathcal{P}(c, \bar{P})$ is a *clearing matrix*, or matrix of clearing payments, if it complies with the *absolute priority* of debt claims rule *(iii)*, that is,

$$P\mathbf{1} = \min(\bar{P}\mathbf{1}, c + P^\top \mathbf{1}). \tag{10}$$

It can be shown (Calafiore et al., 2021b; Csóka & Jean-Jacques Herings, 2018) that a clearing matrix can be found by solving an optimization problem of the form

$$\min_P \quad f(P) \tag{11}$$

$$\text{subject to:} \quad P \in \mathcal{P}(c, \bar{p})$$

where f is a decreasing function of the matrix argument P on $[0, \bar{P}]$, i.e., a function such that $\bar{P} \geq P^{(2)} > P^{(1)} \geq 0$, $P^{(2)} \neq P^{(1)}$, implies $f(P^{(2)}) < f(P^{(1)})$. It can also be shown that for any choice of f the solution to (11) is automatically a clearing matrix, that is, (10) holds. Possible choices for f in (11) are for instance $f(P) = \|\bar{\phi} - \phi\|_1 = \mathbf{1}^\top(\bar{\phi} - \phi)$ and $f(P) = \|\bar{\phi} - \phi\|_2^2$, where $\phi = c + P^\top \mathbf{1}$. The optimal solution of (11), however, is generally non-unique (Calafiore et al., 2021b).

Remark 1. In a regular situation the external asset vector $c = c_{nom}$ is such that all nodes meet their liabilities, hence all payments p_{ij} are equal to their book values $\bar{p}_{ij}$, and no default occurs. If an external shock hits the system and makes the external assets drop to lower values $c = c_{shock} \leq c_{nom}$, then the application of the clearing mechanism defines clearing payments p_{ij} such that $p_{ij} \leq \bar{p}_{ij}$. The defaulted nodes are those nodes i for which $p_i < \bar{p}_i$. The number of defaulted nodes, relative to the total number n of nodes, gives a measure of the spread of the default contagion. The corresponding total loss of asset values over all entities of the network $\sum_{ij}(\bar{p}_{ij} - p_{ij}) = \mathbf{1}^\top(\bar{P} - P)\mathbf{1} = \mathbf{1}^\top(\bar{\phi} - \phi)$ quantifies the currency value of the global losses due to defaults; this loss measure is also called *"the systemic loss in value"* (Glasserman & Young, 2016). ◊

Remark 2. When performing stress-test simulations, various shock scenarios c_{shock} are generated and the corresponding default outcomes are recorded. Often, the shock is assumed to affect one single node at a time. However, since the banks are typically exposed to common external risky assets, it is usual that when one of these assets plummets then the c_i terms of all nodes

that are invested in that asset drop *simultaneously*, in proportion to the amount of that asset held by each node; it is said in this case that the financial system is experiencing a *systematic shock* (Amini & Feinstein, 2023). The possibility of such shocks is also in harmony with the game-theoretic analysis by Banerjee and Feinstein (2022), proving that, under the Eisenberg–Noe model setting, all banks will invest with comonotonic payoffs. Knowledge of the banks' common investment network may thus help to devise meaningful shock scenarios (i.e., c vectors) to be used in stress-test simulations. ◇

3.1. The pro-rata rule

In practice, payments under default are subject to additional prevailing regulations. A common one is the so called *proportionality* (or, pro-rata) rule, according to which payments are made in proportion to the original outstanding claims. Denoting by

$$a_{ij} \doteq \begin{cases} \dfrac{\bar{p}_{ij}}{\bar{p}_i} & \text{if } \bar{p}_i > 0 \\ 1 & \text{if } \bar{p}_i = 0 \text{ and } i = j \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

the relative proportion of payment due nominally by node i to node j, we form the *relative liability* matrix $A = [a_{ij}]$. By definition, A is row-stochastic, that is $A\mathbf{1} = \mathbf{1}$. The pro-rata rule imposes the relations

$$p_{ij} = a_{ij} p_i, \quad \forall i, j, \tag{13}$$

where p_i is the out-flow defined in (4). In matrix notation, the pro-rata rule corresponds to a linear equality constraint on the entries of P, that is $P = \text{diag}(P\mathbf{1})A$. Under pro-rata rule, the problem of clearing payments can be rewritten in terms of the total out-payments vector $p = P\mathbf{1}$, which is said to be feasible if it belongs to

$$\mathcal{P}_{\text{pr}}(c, \bar{p}) \doteq \{p \in \mathbb{R}^n : 0 \le p \le \bar{p}, \ p \le c + A^\top p\}, \tag{14}$$

where $\bar{p} \doteq \bar{P}\mathbf{1}$. Among the feasible payment vectors $p \in \mathcal{P}_{\text{pr}}(c, \bar{p})$, a vector of clearing payments, or simply *clearing vector* is a vector $p \in \mathcal{P}_{\text{pr}}(c, \bar{p})$ such that

$$p = \min(\bar{p}, c + A^\top p). \tag{10a}$$

A clearing vector p^* can be found (Glasserman & Young, 2016) by solving an optimization problem of the form

$$\min_p f(p) \tag{15}$$
$$\text{subject to: } p \in \mathcal{P}_{\text{pr}}(c, \bar{p})$$

where $f : [0, \bar{p}] \to \mathbb{R}$ is any decreasing function, that is, a function such that $p^{(1)}, p^{(2)} \in [0, \bar{p}]$ and $p^{(1)} \le p^{(2)}$ imply $f(p^{(1)}) \ge f(p^{(2)})$, and the latter inequality is strict unless $p^{(1)} = p^{(2)}$. Possible choices for f are for instance $f(p) = \|\bar{\phi} - \phi(p)\|_2^2$, and $f(p) = \sum_{i=1}^n (\bar{\phi}_i - \phi_i(p))$, where $\phi(p) = c + A^\top p$. The following proposition holds.

Lemma 1 (*Calafiore et al., 2021b, Lemma 1*). *The solution $p^* = p^*[A, c, \bar{p}]$ to (15) exists, is unique and does not depend on the choice of f, provided that f is decreasing. Additionally,*

(a) *p^* satisfies (10a) (being thus a clearing vector);*
(b) *p^* dominates any other admissible payment vector*

$$p^*[A, c, \bar{p}] \ge p \quad \forall p \in \mathcal{P}_{\text{pr}}(c, \bar{p});$$

(c) *each strongly connected component being a sink (without outcoming arcs) of graph $\mathcal{G}[A]$ contains at least one node i such that $p_i^* = \bar{p}_i$;*

(d) *p^* is the unique solution of (10a) enjoying the property from statement (c);*

Lemma 1, in fact, is valid for *every* stochastic matrix $A \in \mathbb{R}^{\mathcal{V} \times \mathcal{V}}$, because its proof (available in Calafiore et al. (2021b)) does not rely on (12).

4. Dynamic financial networks

A key observation is that the default and clearing model discussed in the previous section, which coincides with the mainstream one studied in the literature (Glasserman & Young, 2016), is an *instantaneous* one. By instantaneous we mean that the described process assumes that at one point in time (say, at the end of a day), all liabilities are claimed and due simultaneously, and that the entire network of banks becomes aware of the claims and possible defaults and instantaneously agrees on the clearing payments. On the one hand such an instantaneous model may be quite unrealistic, and on the other hand the implied default mechanism is such that all financial operations of defaulted nodes are instantaneously frozen, which possibly induces propagation of the default to other neighboring nodes, in an avalanche fashion, see, e.g., Massai, Como, and Fagnani (2022).

One motivation for the dynamic model we propose in this paper is that one may expect that if financial operations are allowed for a given number of time periods after the initial theoretical defaults (grace period), then some nodes may actually *recover* and eventually manage to fulfill their obligations. The overall system-level advantage of such strategy is that the catastrophic effects of avalanche defaults are possibly mitigated, as shown by numerical examples in Section 5.

In our dynamic multi-period model described below, if a theoretical default condition (we shall call this a *pseudo-default*) happens at some time $t < T$, where T is the final time, we do not freeze operations. Instead, we allow for a grace period and carry over the residual liabilities for the next time slot and let the nodes continue their mutual payments operations, and so on until the final time T. The key elements of this model are the following:

- $t = 0, 1, \ldots, T$, denote discrete time instants delimiting periods of fixed length (e.g., one day, one month, etc.);
- $T \ge 0$ denotes the final horizon and length of the grace period;
- $c(t) \in \mathbb{R}^n \ge 0$ represents the external assets at the nodes at the beginning of period t;
- matrix $\bar{P}(t) = (\bar{p}_{ij}(t)) \in \mathbb{R}^{n,n}$ describes the liabilities (i.e., the mutual payment obligations) among the nodes at period t, i.e., $\bar{p}_{ij}(t)$ is the nominal amount due from i to j at the end of period t. $\bar{P} \doteq \bar{P}(0)$ denotes the initial liabilities at $t = 0$;
- matrix $P(t) = (p_{ij}(t)) \in \mathbb{R}^{n,n}$ contains the actual payments from i to j performed at the end of period t;
- the vectors of actual and nominal in-flows and out-flows $\phi(t), p(t), \bar{\phi}(t), \bar{p}(t)$ at period $t = 0, \ldots, T-1$, are defined similarly to (6) and (7);
- the net worth $w_i(t)$ of node i at the beginning of period t evolves in accordance with

$$w_i(t+1) = w_i(t) + \phi_i(t) - p_i(t) \tag{16}$$

or, in the equivalent vector form

$$w(t+1) = w(t) + c(t) + P(t)^\top \mathbf{1} - P(t)\mathbf{1}. \tag{17}$$

Similar to the single-period case discussed in Section 3, the limited liability condition requires that $w(t) \ge 0$ at all t. It may therefore happen that a payment $p_{ij}(t)$ has to be lower than the corresponding liability $\bar{p}_{ij}(t)$ in order to guarantee $w_i(t) \ge 0$. When this happens at some $t < T$, instead of declaring default

and freezing the financial system, we allow operations to continue up to the final time T, updating the due payments according to the equation

$$\bar{p}_{ij}(t+1) = \alpha \left(\bar{p}_{ij}(t) - p_{ij}(t)\right), \tag{18}$$

where $\alpha \geq 1$ is the interest rate applied on past due payments. The previous relation can be written as

$$\bar{P}(t+1) = \alpha \left(\bar{P}(t) - P(t)\right), \quad t \in \mathcal{T}, \tag{19}$$

where $\mathcal{T} \doteq \{0, \ldots, T-1\}$. The meaning of Eq. (19) is that if a due payment at t is not paid in full, then the residual debt is transferred to the next period, increased by an interest factor $\alpha \geq 1$. Note that the banks cannot acquire new financial obligations (other than the accrued interest rate), in particular, if $\bar{p}_{ij}(t) = 0$ (the debt of bank i to bank j has been fully paid at time t, then $\bar{p}_{ij}(t+1) = \cdots = \bar{p}_{ij}(T-1) = 0$).

This mechanism allows for a node which is technically in default at a time t to continue operations and (possibly) repay its dues in subsequent periods. Notice that time-varying $\bar{P}(t)$ depends on the *actual* payment matrices $P(0), \ldots, P(t-1)$. The final nominal matrix $\bar{P}(T)$ contains the residual debts at the end of the final period. The recursions (17) and (19) are initialized with

$$w(0) = 0, \quad \bar{P}(0) = \bar{P}, \tag{20}$$

where $\bar{P}$ is the initial liability matrix.

Vectors of external assets $c(t)$ are considered as given inputs, while actual payments matrices $P(t)$ are to be determined, being subject to the constraints

$$P(t) \geq 0, \quad P(t) \leq \bar{P}(t), \quad t \in \mathcal{T} \tag{21}$$

$$P(t)\mathbf{1} \leq w(t) + c(t) + P(t)^\top \mathbf{1}, \quad t \in \mathcal{T}, \tag{22}$$

where (21) represents the requirement that actual payments never exceed the nominal liabilities, and (22) represents the requirement that $w(t+1)$, as given in (17), remains nonnegative at all t. Conditions (21), (22) can be made explicit by eliminating the variables $w(t)$ and $\bar{P}(t)$, which by using (17)–(20) can be expressed as

$$\bar{P}(t) = \alpha^t \bar{P}(0) - \sum_{k=0}^{t-1} \alpha^{t-k} P(k), \tag{23}$$

$$w(t) = C(t-1) + \sum_{k=0}^{t-1} \left(P^\top(k) - P(k)\right)\mathbf{1}, \tag{24}$$

$$C(t) \doteq \sum_{k=0}^{t} c(k), \quad t = 0, \ldots, T. \tag{25}$$

Conditions (21), (22) can thus be rewritten as

$$P(t) \geq 0, \tag{26}$$

$$\sum_{k=0}^{t} \alpha^{t-k} P(k) \leq \alpha^t \bar{P} \tag{27}$$

$$C(t) + \sum_{k=0}^{t} \left(P(k)^\top - P(k)\right)\mathbf{1} \geq 0 \tag{28}$$

$\forall t \in \mathcal{T}$.

For brevity, we denote

$$[P] \doteq (P(0), \ldots, P(T-1)), \quad [c] \doteq (c(0), \ldots, c(T-1)).$$

Definition 1. We call a sequence of payment matrices $[P]$ *admissible* if conditions (26)–(28) hold. Let

$$\mathcal{P}([c], \bar{P}) \doteq \{[P] : (26)-(28) \text{ hold}\}$$

stand for the polyhedral set of all admissible matrix sequences $[P]$ that correspond to the given sequence of vectors $[c]$ and initial liability matrix $\bar{P}$.

The system-level cost that we consider is the cumulative sum of deviations of the actual in-flows at nodes from the nominal ones, that is

$$L_T([P]) \doteq \sum_{t=0}^{T-1} \sum_{i=1}^{n} (\bar{\phi}_i(t) - \phi_i(t)). \tag{29}$$

From the definition (6) of in-flow vectors and from (23) we obtain that

$$\begin{aligned} L_T([P]) &= \sum_{t=0}^{T-1} \mathbf{1}^\top (\bar{\phi}(t) - \phi(t)) = \sum_{t=0}^{T-1} \mathbf{1}^\top (\bar{P}(t) - P(t))\mathbf{1} \\ &= \sum_{t=0}^{T-1} \mathbf{1}^\top \left(\alpha^t \bar{P} - \sum_{k=0}^{t} \alpha^{t-k} P(k)\right)\mathbf{1} \\ &= a_0 \mathbf{1}^\top \bar{P}\mathbf{1} - \sum_{t=0}^{T-1} a_t \mathbf{1}^\top P(t)\mathbf{1}, \end{aligned}$$

where the constants $a_0 > a_1 > \cdots > a_{T-1}$ are defined as

$$a_t \doteq \sum_{j=0}^{T-t-1} \alpha^j = \begin{cases} \frac{\alpha^{T-t}-1}{\alpha-1}, & \text{if } \alpha > 1 \\ T-t, & \text{if } \alpha = 1. \end{cases} \tag{30}$$

Remark 3. In view of Remark 1, the quantity $L^{(t)} \doteq \mathbf{1}^\top (\bar{\phi}(t) - \phi(t))$ characterizes the total loss after step t, and $L_T([P])$ is the total loss accumulated over T periods. Notice that without interest rate ($\alpha = 1$), the instant losses can only decrease but not increase with each new iteration of clearing process:

$$\begin{aligned} L_T^{(t+1)} &= \mathbf{1}^\top (\bar{P}(t+1) - P(t+1))\mathbf{1} = \\ &\quad \mathbf{1}^\top (\bar{P}(t) - P(t) - P(t+1))\mathbf{1} = \\ &= L_T^{(t)} - \mathbf{1}^\top P(t+1)\mathbf{1} \leq L_T^{(t)}, \end{aligned}$$

where the last inequality holds since $P(t+1) \geq 0$. $\diamond$

The optimal payment matrices are thus obtained as a solution to the following maximization problem

$$\max_{[P]} \sum_{t=0}^{T-1} a_t \mathbf{1}^\top P(t)\mathbf{1} \quad \text{s.t.:} \quad [P] \in \mathcal{P}([c], \bar{P}), \tag{31}$$

which is equivalent to minimization of the overall system loss $L_T([P])$ over the set of admissible payment matrices.

Observe that, from a numerical point of view, finding an optimal sequence of payment matrices amounts to solving the linear programming (LP) problem (31). In the case $T = 1$ the set $\mathcal{P}([c])$ reduces to the polytope of matrices (9), and the optimization problem (31) is a special case of (11), where $f(P) = -\mathbf{1}^\top P(0)\mathbf{1}$.

We next establish a fundamental property of the payment matrices resulting from (31).

4.1. The absolute priority rule

Recall that in the static (single period) case the optimal payment matrix automatically satisfies the absolute priority rule (10). A natural question arises whether a counterpart of this rule can be proved for the dynamical model in question: is it true that a bank failing to meet the nominal obligation has to nevertheless pay the maximal possible amount? Mathematically, this means that for all $t = 0, \ldots, T-1$ the following implication holds:

$$p_i(t) < \bar{p}_i(t) \Longrightarrow p_i(t) = \phi_i(t) + w(t). \tag{32}$$

The affirmative answer is given by the following theorem.

Theorem 1. *Suppose that $[P] = (P(t))_{t=0}^{T-1}$ is an optimal solution of* (31)*, and let* $(\bar{P}(t))_{t=0}^{T}$ *be the corresponding sequence of nominal liability matrices, defined in accordance to* (19)*. For a given bank i, let $t_* = t_*(i)$ be the first instant when i pays its debt to the other banks*

$$p_{ij}(t_*) = \bar{p}_{ij}(t_*) \quad \forall j \neq i$$

(if such an instant fails to exist, we formally define $t_* = T$). Then, either $t_* = 0$ (the debt is paid immediately) or

$$p_i(t) = \phi_i(t) + w_i(t) \ \forall t = 0, \ldots, (t_* - 1). \tag{33}$$

In particular, the implication (32) *holds for any optimal sequence of payments matrices $[P]$. Furthermore, for each $t \geq 1$ the graph $\mathcal{G}[P(t)]$ contains no directed cycles.*

A proof of Theorem 1 is provided in Appendix A.2.

Remark 4. Implication (32) implies that each bank pays its nominal liability at the earliest instant t when such a payment is possible: $w_i(t) + \phi_i(t) \geq \bar{p}_i(t)$. The requirement of minimal system loss thus prevents unnecessary deferral of payments and pushes the banks towards paying the claims as early as possible. Since the payment matrices resulting from the solution of (31) satisfy rules *(i), (ii), (iii)* from Section 3, they are guaranteed to be proper clearing matrices at each stage. ◊

Theorem 1 entails the following important corollary, proved in Appendix A.5.

Corollary 1. *If $c(t_0) = 0$ at some step $t_0 \geq 1$, then $P(t_0) = 0$ for every optimal solution of* (31)*.*

Corollary 1 implies, in particular, that if all external assets become available at $t = 0$ (so that $c(1) = \cdots = c(T-1) = 0$), then the clearing process terminates at $t = 0$ and $P(1) = \cdots = P(T-1) = 0$. Hence, the objective (31) equals $\mathbf{1}^\top P(0)\mathbf{1}$ and attains its maximal value if and only if $P(0)$ is a minimizer in (11) with $f(P) = -\mathbf{1}^\top P \mathbf{1}$ and $c = c(0)$. Problem (31) thus reduces to the static clearing problem with $T = 1$.

4.2. A sub-optimal sequential approach

Looking at the objective function in problem (31), we observe that this function is linear and separable in the $P(t)$ variables, $t = 0, \ldots, T-1$. Also, looking at the constraints of (31), given by (26)–(28), we see that at each $t = 0, \ldots, T-1$ the variable $P(t)$ is constrained as

$$0 \leq P(t) \leq \bar{P}(t),$$
$$w(t) + c(t) + \left(P^\top(t) - P(t)\right)\mathbf{1} \geq 0,$$

where

$$\bar{P}(t) \doteq \alpha^t \bar{P} - \sum_{k=0}^{t-1} \alpha^{t-k} P(k) \tag{34}$$

$$w(t) \doteq C(t-1) + \sum_{k=0}^{t-1} \left(P^\top(k) - P(k)\right)\mathbf{1}, \tag{35}$$

and $\bar{P}(t)$, $w(t)$ depend only on the variables $P(0), \ldots, P(t-1)$ and external payments $c(0), \ldots, c(t-1)$ at periods preceding t. This suggests the following recursive relaxation of problem (31) where, at each $t = 0, \ldots, T-1$ we solve a problem in the $P(t)$ variable only

$$\tilde{P}^*(t) = \arg\max_{P(t)} \mathbf{1}^\top P(t)\mathbf{1} \tag{36}$$
$$\text{s.t.: } w^*(t) + c(t) + \left(P^\top(t) - P(t)\right)\mathbf{1} \geq 0,$$

$$\bar{P} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
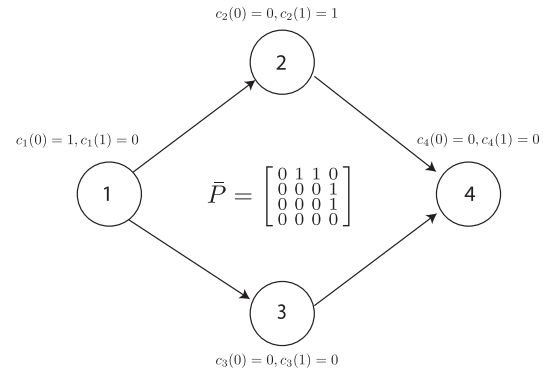
Fig. 2. A four-node liability network.

$$0 \leq P(t) \leq \bar{P}^*(t-1),$$

where $\bar{P}^*(t)$, $w^*(t)$ are given by (34), (35) evaluated at the previous optimal values $\tilde{P}^*(0), \ldots, \tilde{P}^*(t-1)$, and initialized so that $w^*(0) \doteq 0$, $\bar{P}^*(0) \doteq \bar{P}$.

It is clear by construction that any optimal sequence of solutions $\tilde{P}^*(0), \ldots, \tilde{P}^*(T-1)$ of (36) is feasible for problem (31). However, this "greedy" sequential solution is in general not optimal for problem (31), as highlighted by the following example.

Example 1. Consider a group of four banks with initial liability matrix $\bar{P}$ and liability graph shown in Fig. 2.

We assume that $\alpha = 1$ and consider a time horizon $T = 2$, with external payments $c(0) = (1, 0, 0, 0)^\top$, $c(1) = (0, 1, 0, 0)^\top$. The unique optimal strategy in (31) can be easily found: at stage 0, node 1 pays its maximum possible to node 3, i.e., $p_{13}(0) = \bar{p}_{13} = 1$, and node 3 transfers it to node 4: $p_{34}(0) = \bar{p}_{34} = 1$. Node 2 receives and pays nothing at period $t = 0$, while at $t = 1$ node 2 receives an external payment and hence pays its liability to node 4: $p_{24}(1) = \bar{p}_{24} = 1$. This optimal strategy leads to the optimal loss $L = 3$, and at the end of the time horizon only node 1 is in default (owing 1 to node 2).

If we consider the sequential approach instead, we see that the objective function (36) at $t = 0$ is $(p_{12}(0) + p_{13}(0)) + p_{24}(0) + p_{34}(0)$, hence it is insensitive to how node 1 divides its asset $c_1(0) = 1$ between nodes 2 and 3. An optimal solution to (36) at $t = 0$ is for instance $\tilde{p}_{12}^*(0) = \bar{p}_{12} = 1$, $\tilde{p}_{24}^*(0) = \bar{p}_{24} = 1$. With this solution in place, problem (36) at $t = 1$ leads to a network in which no further payments can be made (i.e., $\tilde{p}_{ij}^*(1) = 0$ for all i, j), and the loss function under this sub-optimal solution is $L = 4$, with two defaulted nodes at the end of the horizon: node 1, which still owes 1 to node 3, and node 3, which still owes 1 to node 4.

The point here is that the correct choice at $t = 0$ cannot be made in general unless one knows the *future* external payments at all nodes and at all $t > 0$. The sequential solution hence remains sub-optimal, since it does not exploit this information (it only uses, at each t, the observed external payments $c(k)$, $k = 0, \ldots, t$, up to that t). On the one hand, this fact highlights that the solution to the "full" problem (31) is in general superior in terms of optimal loss to the solution of the sequential problem. On the other hand, however, it also underlines that the whole stream of future external payments must be known at $t = 0$ in order to being able to solve (31). If, at each t, one has total uncertainty about the future payments $c(\tau)$, $\tau > t$, then the full approach is not viable while the sequential approach still is. ★

4.3. Dynamic networks with pro-rated payments

The pro-rata rule discussed in Section 3.1 can be introduced also in the dynamic network setting. Here, we let the pro-rata

matrix be fixed according to the *initial* liabilities, that is the A matrix is given by (12) with $\bar{P} = \bar{P}(0)$. Then, the pro-rata rule is nothing but a linear equality constraint on the payment matrices, that is

$$P(t) = \text{diag}(P(t)\mathbf{1})A, \quad t = 0, \ldots, T-1. \tag{37}$$

In view of the definition of A, one has $\bar{P}(0) = \text{diag}(\bar{P}(0)\mathbf{1})A$. Using induction on t and Eq. (23), it can be easily shown that (37) entails the equations

$$\bar{P}(t) = \text{diag}(\bar{P}(t)\mathbf{1})A, \quad t = 0, \ldots, T.$$

Hence, payment matrices $P(t)$ and $\bar{P}(t)$ are uniquely determined by the actual and nominal payment vectors

$$p(t) \doteq P(t)\mathbf{1}, \quad \bar{p}(t) \doteq \bar{P}(t)\mathbf{1}. \tag{38}$$

Also, it holds that $\phi = P^\top(t)\mathbf{1} = A^\top p(t)$. Conditions (21), (22) can be now rewritten as

$$p(t) \geq 0, \tag{39}$$

$$\sum_{k=0}^{t} \alpha^{t-k} p(k) \leq \alpha^t \bar{p} \tag{40}$$

$$C(t) + \sum_{k=0}^{t} \left(A^\top p(k) - p(k)\right) \geq 0 \tag{41}$$

$\forall t \in \mathcal{T}$.

Definition 2. We call a sequence of payment vectors $[p] \doteq (p(0), \ldots, p(T-1))$ *admissible* (under the pro-rata requirement) if conditions (39)–(41). Let

$$\mathcal{P}_{\text{pr}}([c], \bar{p}) \doteq \{[p] = (p(0), \ldots, p(T-1)) : (39)-(41) \text{ hold}\}$$

stand for the convex polytope of all admissible sequences.

The optimization problem (31) can be now rewritten as

$$\max_{[p]} \sum_{k=0}^{T-1} a_k \mathbf{1}^\top p(k) \quad \text{s.t.:} \quad [p] \in \mathcal{P}_{\text{pr}}([c], \bar{p}). \tag{42}$$

This is again an LP problem, which may be solved numerically with great efficiency. The pro-rata rule drastically reduces the number of unknown variables (each zero-diagonal payment $n \times n$ matrix reduces to n-dimensional vector). Furthermore, unlike the original problem (31), the optimization problem (42) admits a *unique* maximizer $[p^*]$. Also, the solution abides by the absolute priority rule (32). These properties are summarized in the following theorem.

Theorem 2. *For each given sequence $[c]$, the optimization problem (42) has a unique solution $[p^*]$. Furthermore, at each period $t = 0, \ldots, T-1$, the optimal vector $p^*(t)$ is the unique solution of the LP:*

$$p^*(t) = \arg\max_p \ \mathbf{1}^\top p \tag{43}$$

$$\text{s.t.: } 0 \leq p \leq \bar{p}^*(t), \ p \leq c(t) + w^*(t) + A^\top p, \tag{44}$$

where $w^*(0) \doteq 0$, $\bar{p}^*(0) \doteq \bar{p}$, and, for $t = 1, \ldots, T-1$,

$$\bar{p}^*(t) \doteq \alpha^t \bar{p} - \sum_{k=0}^{t-1} \alpha^{t-k} p^*(k) \tag{45}$$

$$w^*(t) \doteq C(t-1) + \sum_{k=0}^{t-1} \left(A^\top p^*(k) - p^*(k)\right). \tag{46}$$

In particular, $p^*(t) \geq 0$ obeys the absolute priority rule

$$p^*(t) = \min(\bar{p}^*(t), c(t) + w^*(t) + A^\top p^*(t)). \tag{47}$$

The proof of Theorem 2 is based on Lemma 1, and it is detailed in Appendix A.3.

Remark 5. A few observations are in order regarding Theorem 2. First, we observe that the "full" multi-period problem (42) is equivalent to the sequence of problems (43). Therefore, in the pro-rata case the sequential approach is *optimal*, and not only sub-optimal, as it instead happened in the case with unrestricted payment matrices discussed in Section 4.2. Thus, the system-level objective in the full optimization problem (42) is minimized by finding regular clearing payments at each step t, whereby the liabilities among nodes are updated at each step by considering the residual payments due to pseudo-defaults at the previous step.

Further, we observe that, for each t, problem (43)–(44) has the same structure as problem (15), with $c = c(t) + w^*(t)$. Hence, in view of the maximality of the vector $p^*(t)$, we have that the objective (43) can be replaced by any other *increasing* function of p. In view of Lemma 1, the relations (43), (44) can be rewritten as follows

$$p^*(t) = p^*[A, c(t) + w^*(t), \bar{p}^*(t)], \tag{48}$$

which also entails (47) due to Lemma 1, statement (a). ◊

Eq. (48) allows us to prove a counterpart of Corollary 1 for problem (42), showing that, in the case where all external assets are available at $t = 0$, the dynamic model reduces to the Eisenberg–Noe model.

Corollary 2. *If $c(t_0) = 0$, then $p^*(t_0) = 0$. In particular, if $c(1) = \cdots = c(T-1) = 0$, then $p^*(0) = p^*[A, c(0), \bar{p}]$ is the maximal clearing vector in the Eisenberg–Noe model with $c = c(0)$ and $p^*(1) = \cdots = p^*(T-1) = 0$.*

5. Numerical examples and simulations

5.1. A simple network

We first consider a variation on the simplified network given in Glasserman and Young (2016). This network, displayed in Fig. 3, contains $n = 5$ nodes, including the fictitious sink node representing the external sector, with initial liability matrix

$$\bar{P} = \begin{bmatrix} 0 & 180 & 0 & 0 & 180 \\ 0 & 0 & 100 & 0 & 100 \\ 90 & 0 & 0 & 100 & 50 \\ 150 & 0 & 0 & 0 & 150 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where the last row refers to the sink node.

In both the pro-rata case and the full matrix case, the idea implied by the single-period (static) approach is that in case of default the financial operations of a node are frozen, that is, defaulted nodes cannot operate even if there are cash in-flows that are foreseen in the immediate future. A classical situation arises when there is a liquidity crisis, i.e., due payments from the external sector are late and their lateness provokes defaults at some nodes, which freeze and may propagate further defaults over the network. Suppose that the in-flows in the nominal vector $c \doteq [121, 21, 130, 204, 0]^\top$ do not arrive simultaneously at time $t = 0$, due to delays. Instead, external assets are available progressively as

$$c(0) = [60, 10, 120, 0, 0]^\top$$

$$c(1) = [60, 8, 0, 200, 0]^\top,$$

$$c(2) = [1, 3, 10, 4, 0]^\top.$$
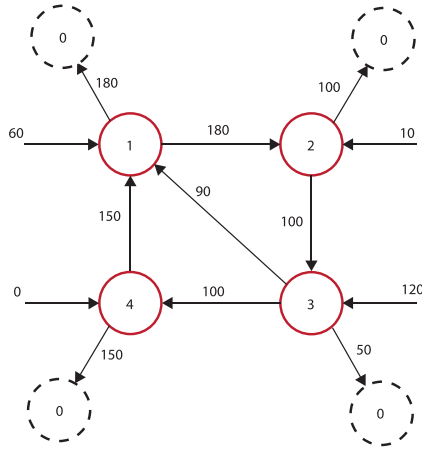
Fig. 3. A simple network with four nodes, plus external fictitious node (node 0). Numbers on the graph edges denote the initial liabilities $\bar{P}(0) = \bar{P}$, the incoming arrows are labeled with the values $c(0)$ of the external assets at time $t = 0$.

A static approach at time $t = 0$, with unrestricted payment matrix, would result in

$$P = \begin{bmatrix} 0 & 180 & 0 & 0 & 70 \\ 0 & 0 & 100 & 0 & 90 \\ 90 & 0 & 0 & 100 & 30 \\ 100 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

with all nodes in default and a total default loss of 343.40. If we allow operations to continue over an horizon $T = 3$, according to the model described in Section 4, assuming an interest rate $\alpha = 1.01$ (i.e., 1% interest per period), and solving the multi-period problem (31) with full payment matrices, we obtain $P(0) = P$, and

$$P(1) = \begin{bmatrix} 0 & 0 & 0 & 0 & 110.5 \\ 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 \\ 50.5 & 0 & 0 & 0 & 149.5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$P(2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.61 \\ 0 & 0 & 0 & 0 & 2.12 \\ 0 & 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 2.02 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

After these three rounds of clearing payments, only node 3 is in default, owing a residual 10.51 to the external sector. Notice that, if we used the pro-rata rule, thus solving the multi-step problem (42), we would obtain a different set of (pro-rata) clearing payments, leading to a final situation of default at *all* nodes, with a total defaulted value of 21.07.

5.2. Randomized simulations

In this section, we show the advantages of the proposed dynamic setting with respect to a static approach.

In the experiments, we considered an horizon of $T = 2$ time periods and unconstrained payment matrices. The random graphs used for simulations are constructed using a technique inspired by Nier, Yang, Yorulmazer, and Alentorn (2007) with minor modifications proposed in Calafiore et al. (2021b). We generate scale-free random graphs with $n = 50$ nodes using the Barabasi–Albert model. The interbank liabilities $\bar{P}_{ij}$ for every edge (i, j) of the random graph are then found by sampling from a

uniform distribution $\bar{P}_{ij} \sim \mathcal{U}(0, P_{max})$, where P_{max} is the maximum possible value of a single interbank payment. In the experiments we set $P_{max} = 200$. Unlike (Nier et al., 2007), the values $\bar{P}_{ij}$ can thus be heterogeneous.

Following (Nier et al., 2007), we choose the total amount of the external assets at time $t = 0$ as $E = \frac{\beta}{1-\beta} I$, where $I = \sum_{i,j=1}^{n} \bar{P}_{ij}$ is the total amount of the interbank liabilities and $\beta = E/(E + I)$ is a parameter representing the percentage of external assets in total assets at the system level; in our experiments $\beta = 0.05$. The nominal asset vector $\bar{c}(0)$ is then computed in two steps. First, each bank is given the minimal value of external assets under which its balance sheet is equal to zero. At the second step, the remainder of the aggregated external assets is evenly distributed among all banks.

The financial shock is modeled by randomly choosing a subset of $n_s = 15$ banks of the system and nullifying their external financial assets at time $t = 0$. We then suppose that at time $t = 1$ new external assets $c(1)$ are available and the total amount of such external assets is equal to 30% of the system loss at time $t = 0$, that is,

$$\sum_{i=1}^{n} c_i(1) = 0.3 \sum_{i=1}^{n} (\bar{\phi}_i(0) - \phi_i(0)).$$

We also assume that there is no interest rate: $\alpha = 1$. We next compare the two-stage dynamic setting with a static clearing mechanisms that considers only the events at time $t = 0$ and thus finds the matrix of payments by solving (11) with $c = c(0)$ and $f(P) = \mathbf{1}^\top (\bar{\phi} - \phi)$.

In order to quantify the advantage of the proposed dynamic clearing mechanisms, we consider the decrease R in the system's loss *relative* to the newly injected external resources in the grace period. Formally, in the considered two-period setup, R is computed as

$$R \doteq \frac{L_1^{(0)} - L_2^{(1)}}{\sum_{i=1}^{n} c_i(1)},$$

where $L_1^{(0)} \doteq \sum_{i=1}^{n} (\bar{\phi}_i(0) - \phi_i(0))$ is the loss in the static model with $T = 1$, whereas $L_2^{(1)} \doteq \sum_{i=1}^{n} (\bar{\phi}_i(1) - \phi_i(1))$ is the final loss in the dynamic model with $T = 2$ steps computed at time $t = 1$. When $L_2^{(1)} \leq L_1^{(0)}$, the R ratio thus measures the *efficiency* of the loss reduction (loss reduction per unit injected cash).

Table 1 summarizes the results of the simulations considering graph topologies with different values of average node degree d. The results are averaged over 250 runs. We observe that the ratio R is larger than one for all the connectivity levels considered in the experiments. This means that the proposed dynamic clearing mechanism is always advantageous over a static approach, since the reduction in the system loss is always larger than the total amount of additional cash injected at time $t = 1$, with an efficiency ranging from 11% ($R = 1.1$) for networks with higher average relative degree, up to 70% ($R = 1.70$) for networks with lower average relative degree. It is also relevant to observe that the average number nd(0) of defaults in the static case is from 47% to 65% higher than the average number of defaulted nodes at the end of the grace period, thus showing that a dynamic approach can effectively reduce the default contagion.

6. Conclusions

In this paper we explored dynamic clearing mechanisms in financial networks, under both pro-rata payment rules and unrestricted matrix payments. In both cases, we showed that proper clearing payments can be obtained as optimal solutions to suitable multi-stage linear optimization problems, namely problem

Table 1
System losses and defaults with and without the grace period. $nd(t)$ denotes the average number of defaulted nodes at time t in the static problem ($t = 0$) and in the dynamic problem (at period $t = 1$).

$\frac{d}{n}$	R	$L_1^{(0)}$	$L_2^{(1)}$	$\sum_i c_i(1)$	$nd(0)$	$nd(1)$
0.039	1.70	714	350	214	10.64	7.19
0.075	1.45	921	519	276	11.81	8.08
0.111	1.31	1146	694	344	11.66	7.31
0.147	1.19	1196	767	359	12.30	7.61
0.183	1.11	1348	901	405	12.10	7.34

(31) for the unrestricted case, and problem (42) for the pro-rata case. Theorem 1 establishes some fundamental properties of the solution in the unrestricted case stating, in particular, that payments are not unnecessarily delayed when they are feasible (absolute priority of debt claims), so that the solutions are indeed clearing matrices at each stage. Unrestricted optimal payments, however, are possibly non-unique and need be computed in a centralized way, since knowledge of the whole network structure is necessary. Theorem 2 establishes instead key properties of the optimal pro-rated payments. The solution is in this case unique and, moreover, it can be computed by solving sequentially a series of LP problems (43)–(44). In turn, under mild hypotheses (for instance, when graph $\mathcal{G}[A]$ has a unique and globally reachable sink node), the LP solutions coincide with the solution of a series of fixed-point equations of the form (47) whose solution is unique (Calafiore et al., 2021b) and can be found by means of a decentralized algorithm known as the *fictitious default algorithm* of Eisenberg and Noe (2001). Hence, the optimal multi-stage payments in the pro-rata case can be obtained by decentralized iterations among neighboring nodes. Numerical investigations, see, e.g., Calafiore et al. (2021b), however, suggested that proportional payments may lead to severely suboptimal clearings, and may be a concurring cause of cascaded defaults: removing the pro-rata rule, both in the static and in the dynamic case, generally improves the high-level objective of reducing the systemic effects of defaults.

Appendix. Proofs

A.1. Technical preliminaries

We start with some auxiliary material, which will be used in the proofs.

The following proposition follows, e.g., from Harary, Norman, and Cartwright (1965, Corollaries 4.3 and 4.3a').

Proposition 1. *Among strongly connected components of a directed graph, at least one component is a sink and at least one component is a source. If a strongly connected component is not a sink (respectively, not a source), then some directed path connects it to one of sink components (respectively, connects one of source components to it).*

We will also employ several technical propositions, dealing with substochastic[2] matrices.

Proposition 2. *Let $A = (a_{ij})_{i,j \in \mathcal{V}}$ be a stochastic matrix and $\mathcal{V}' \subsetneq \mathcal{V}$. Then, submatrix $A' = (a_{ij})_{i,j \in \mathcal{V}'}$ is **not** Schur stable[3] if and only if $\mathcal{V}'$ contains all nodes of some strongly connected sink component of $\mathcal{G}[A]$.*

---

[2] A *nonnegative* square matrix $A \in \mathbb{R}^{\mathcal{V} \times \mathcal{V}}$ is *substochastic* if $\sum_{j \in \mathcal{V}} a_{ij} \leq 1 \, \forall i \in \mathcal{V}$.

[3] A square matrix is *Schur stable* (or simply *Schur*) if all its eigenvalues are less than 1 in absolute value; A is Schur stable if and only if $A^k \to 0$ as $k \to \infty$.

Proof. The "if" part is obvious. If $\mathcal{V}^0 \subseteq \mathcal{V}'$ is the set of nodes of some sink component, then $A^0 = (a_{i,j})_{i,j \in \mathcal{V}^0}$ is a stochastic matrix and A is decomposed as

$$A = \begin{pmatrix} A^0 & \mathbf{O} \\ * & * \end{pmatrix}, \tag{A.1}$$

where $\mathbf{O}$ is the block of zeros and symbols $*$ denote some sub-matrices. Therefore, A has eigenvalue 1 and is not Schur stable. The "only if" part is implied by Calafiore et al. (2021b, Lemma 6). Thanks to this lemma, A' is not Schur stable if and only if $\mathcal{V}^0 \subseteq \mathcal{V}'$ exists such that $A^0 = (a_{i,j})_{i,j \in \mathcal{V}^0}$ is a stochastic matrix, which also implies that A is decomposed as in (A.1). In other words, the set of nodes $\mathcal{V}^0 \subseteq \mathcal{V}'$ is "closed": each arc of $\mathcal{G}[A]$ starting in $\mathcal{V}^0$ ends also in $\mathcal{V}^0$. Hence, strong components of graph $\mathcal{G}[A^0]$ are also strong components of $\mathcal{G}[A]$, and (due to Proposition 1) at least one of them is a sink. $\square$

Proposition 3. *Suppose that a substochastic matrix $A = (a_{ij})_{i,j \in \mathcal{V}' \cup \mathcal{V}''}$ is Schur stable, where $\mathcal{V}' \cap \mathcal{V}'' = \emptyset$. Then a vector $\xi \geq 0$ exists such that*

$$\xi_i - (A^\top \xi)_i \begin{cases} > 0, \; i \in \mathcal{V}', \\ = 0, \; i \in \mathcal{V}''. \end{cases}$$

Proof. If A is Schur stable, then $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k \geq 0$ exists. Choosing an arbitrary vector e such that $e_i > 0 \, \forall i \in \mathcal{V}'$ and $e_i = 0 \, \forall i \in \mathcal{V}''$, the vector $\xi = (I - A^\top)^{-1} e$ thus is also nonnegative. By construction, $(\xi - A^\top \xi)_i = e_i > 0$ if and only if $i \in \mathcal{V}'$. $\square$

We also need a special form of the Perron–Frobenius theorem.

Lemma 2. *Let $A \in \mathbb{R}^{\mathcal{V} \times \mathcal{V}}$ be a stochastic matrix and $\mathcal{V}^0$ be the set of nodes of some sink component in $\mathcal{G}[A]$. Then, vector $\pi \in \mathbb{R}^{\mathcal{V}}$ exists such that*

$$A^\top \pi = \pi, \; \pi^\top \mathbf{1} = 1, \; \pi_i \begin{cases} > 0, \; \forall i \in \mathcal{V}^0, \\ = 0, \; \forall i \notin \mathcal{V}^0. \end{cases} \tag{A.2}$$

Proof. The special case where $\mathcal{G}[A]$ is a strongly connected graph (A is irreducible) and $\mathcal{V}^0 = \mathcal{V}$ is immediate from Perron–Frobenius theorem for irreducible matrices (Berman & Plemmons, 1994, Ch.2, Th. 1.3). Otherwise, matrix A has structure (A.1), where $A^0 = (a_{ij})_{i,j \in \mathcal{V}^0}$ is irreducible (graph $\mathcal{G}[A^0]$ is strongly connected by the definition of a strongly connected component). Introducing the Perron–Frobenius eigenvector $\pi^0 > 0$ of matrix A^0, vector π can be defined as follows:

$$\pi_i = \pi_i^0 \, \forall i \in \mathcal{V}^0, \quad \pi_i = 0 \, \forall i \notin \mathcal{V}^0.$$

Lemmas 1 and 2 have a simple corollary, which will be used in the proof of Theorem 2.

Corollary 3. *Given a stochastic matrix A and nonnegative vectors $c, \bar{p} \geq 0$, consider the maximal payment vector $p^* = p^*[A, c, \bar{p}]$ from Lemma 1. Suppose that graph $\mathcal{G}[A]$ has a strongly connected component with set of nodes $\mathcal{V}^0 \subset \mathcal{V}$, which is a sink (no arc leaves it) and is such that $\bar{p}_i > 0 \, i \in \mathcal{V}^0$. Then, $p_i^* > 0 \, \forall i \in \mathcal{V}^0$.*

Proof. Choosing π as in (A.2), one has $\varepsilon \pi \in \mathcal{P}_{pr}(c, \bar{p})$ for $\varepsilon > 0$ small enough (so small that $\varepsilon \pi_i < \bar{p}_i \, \forall i \in \mathcal{V}^0$). Since p^* is the maximal element of $\mathcal{P}_{pr}(c, \bar{p})$, we have $p_i^* \geq \varepsilon \pi_i > 0 \, \forall i \in \mathcal{V}^0$. $\square$

A.2. Proof of Theorem 1

We introduce the following notation: for a pair of banks $i, j \neq i$ let

$$\delta_{ij}(t) = \bar{p}_{ij}(t) - p_{ij}(t) \geq 0, \; t \in \mathcal{T}$$

be the amount bank i owes to bank j before period $t + 1$. In view of (18), $\bar{p}_{ij}(t + 1) > 0$ if and only if $\delta_{ij}(t) > 0$.

The proof is based on a simple transformation, which we call the transformation of advance payment (TAP). Let J be a subset of arcs in graph $\mathcal{G}[P(t_0)]$, where $1 \le t_0 \le T$, and $\varepsilon > 0$. For $(i, j) \in J$, one has $\bar{p}_{ij}(t_0) \ge p_{ij}(t_0) > 0$ and, thus $\delta_{ij}(t_0 - 1) > 0$. The TAP with parameters (t_0, ε, J) modifies matrices $P(t_0 - 1)$ and $P(t_0)$ as follows:

- at time $t_0 - 1$, payment on each arc from J is increased

$$p_{ij}(t_0 - 1) \mapsto p_{ij}(t_0 - 1) + \alpha^{-1}\varepsilon \quad \forall (i, j) \in J;$$

- at time t_0, payment on each arc from J is decreased

$$p_{ij}(t_0) \mapsto p_{ij}(t_0) - \varepsilon \quad \forall (i, j) \in J;$$

- all other entries of $P(t_0 - 1)$ and $P(t_0)$ and remaining matrices $P(t)$, $t \ne t_0 - 1, t_0$ remain unchanged.

Obviously, this transformation increases the objective function (31) by $\varepsilon |J|(\alpha^{-1}a_{t_0-1} - a_{t_0})$. For $\varepsilon > 0$ being sufficiently small, the TAP transformation preserves constraints (26): it suffices to choose $\varepsilon < \min\{p_{ij}(t_0) : (i, j) \in J\}$. Conditions (27) also retain their validity, provided that $\varepsilon < \alpha \min\{\delta_{ij}(t_0 - 1) : (i, j) \in J\}$. Notice that the nominal payment matrices $\bar{P}(0), \dots \bar{P}(t_0)$ remain unchanged, and hence the condition $P(t) \le \bar{P}(t)$ (equivalent to (27)) holds for all $t \le t_0$. For $t \ge t_0$, the sum in the left-hand side of (27) is invariant under the TAP transformation, so the constraint is also not violated. Finally, constraints (28) (equivalent to $w_i(t + 1) \ge 0$) also hold for all t except for, possibly, $t = t_0 - 1$ and $t = t_0$, because other matrices $P(t)$ remain unchanged.

In view of the optimality of sequence $[P]$, the TAP transformation with parameters (t_0, J, ε), where $\varepsilon > 0$ is sufficiently small, violate (28) at $t = t_0 - 1$ or at $t = t_0$.

Step 1. We first prove the last statement of Theorem 1. Assume that this statement is not valid and a cycle $i_1 \to \dots \to i_s \to i_1$ exists in $\mathcal{G}[P(t_0)]$, where $t_0 \ge 1$. Choosing the set of arcs $J = \{i_1 \to i_2, \dots, i_{s-1} \to i_s, i_s \to i_1\}$, the TAP transformation with parameters (t_0, J, ε) (with $\varepsilon > 0$ small enough), obviously, leaves the vectors $p(t) - \phi(t) = P(t)\mathbf{1} - P(t)^\top \mathbf{1}$ unchanged, and thus constraints (28) are not violated, which leads one to a contradiction with the optimality of $[P]$.

Step 2. Suppose now that $t_* = t_*(i) \le T$ is defined as described in Theorem 1 yet (33) fails to hold at some period $0 \le t < t_*$. Let $t_+ < t_*$ be the *last* period when (33) fails, that is, the maximum of $t < t_*$ such that $w_i(t + 1) = \phi_i(t) + w_i(t) - p_i(t) > 0$.

Notice first that $t_+ < T - 1$. Otherwise, one would have $w_i(T) > 0$ and $t_* = T$, in particular, $\delta_{ij}(T - 1) > 0$ for some $j \ne i$. Increasing $p_{ij}(T - 1)$ by a sufficiently small value $\varepsilon > 0$, one could obviously preserve all constraints and also increase the objective function.

Denoting for brevity $t_0 \doteq 1 + t_+ \le t_*$, one thus has $t_0 < T$. The definition of t_+ and t_0 implies that $p_i(t_0) > 0$. Indeed, if $t_0 = t_*$, then one has $p_i(t_0) = \bar{p}_i(t_*) > 0$ by definition of t_*. Otherwise, $p_i(t_0) = \phi_i(t_0) + w_i(t_0) \ge w_i(t_0) = w_i(1 + t_+) > 0$ due to the choice of t_0.

We know that graph $\mathcal{G}[P(t_0)]$ contains no cycles and, in particular, all its strongly connected components are trivial (single-node) graphs. Since $p_i(t_0) > 0$, node i is not a sink node. Proposition 1 ensures that i is connected to a sink node k by a path $i \to j_1 \to \dots \to j_s \to k$ (all nodes $i, j_1, \dots, j_s, k, s \ge 0$ are mutually different). Let J be the set of arcs in this path. The TAP with parameters (t_0, J, ε) with $\varepsilon > 0$ small enough, obviously, preserves (28) (equivalent to $w(t + 1) \ge 0$) at $t = t_0 - 1$ or at $t = t_0$. Indeed, the TAP leaves the components $w_j(t_0)$, $w_j(t_0+1)$ for each $j \ne i, k$ invariant. The component $w_k(t_0)$ increases (becoming

thus positive), and hence $w_k(t_0 + 1)$ is also positive (recall that $p_k(t_0) = 0$). The TAP transformation decreases $w_i(t_0)$ by $\alpha^{-1}\varepsilon$ (providing that $w_i(t_0) > 0$ for ε being small), however, $p_i(t_0)$ is decreased by ε, so that $w_i(t_0 + 1)$ is increased by $(1 - \alpha^{-1})\varepsilon \ge 0$, and inequality $w_i(t_0+1) \ge 0$ is preserved. Hence, constraints (28) are not violated, and we arrive at a contradiction with optimality of $[P]$.

Step 3. The proof of implication (32) is now straightforward. Suppose that $p_i(t) < \bar{p}_i(t)$ at some period $t \in \mathcal{T}$. Then, obviously, $t < t_*(i)$, and hence $\phi_i(t) + w_i(t) - p_i(t) = 0$ due to (33). $\square$

A.3. Proof of Theorem 2

In the proof, we will use a *transformation of advanced payment* (TAP), which is similar to the transformation used in the proof of Theorem 1. The TAP is determined by time instant t_*, scalar $\varepsilon > 0$ and non-negative vector $\zeta \ge 0$; it replaces sequence $[p]$ by the sequence $[\hat{p}]$, where

$$\hat{p}(t) = \begin{cases} p(t), & t \ne t_*, t_* + 1, \\ p(t_*) + \varepsilon\alpha^{-1}\zeta, & t = t_*, \\ p(t_+) - \varepsilon\zeta, & t = t_* + 1. \end{cases} \quad \text{(A.3)}$$

In other words, some payments are transferred (taking into account the interest rate $\alpha \ge 1$) from period $t_* + 1$ to the previous period t_*.

If $[p]$ satisfies constraints (39)–(41), then $[\hat{p}]$ also obeys all constraints, except for, possibly: 1) constraint (39) at $t_* + 1$ (at other periods, $\hat{p}(t) \ge p(t)$); 2) constraint (40) at $t = t_*$ (at other periods, the left-hand side of (40) remains invariant under the TAP); 3) constraints (41) at periods $t = t_*, \dots, T - 1$ (for $t < t_*$, the left-hand side of (41) remains invariant under the TAP). Also, for any $\zeta \ne 0$ and $\varepsilon > 0$ the TAP always increases the value of the objective function (42) by $\varepsilon(\alpha^{-1}a_{t_*} - a_{t_*+1})\mathbf{1}^\top \zeta$.

For the optimal sequence of payment vectors $[p^*]$, we are going to prove that $p^*(t)$ (at each t) is a maximizer at problem (43), (44), or, equivalently, (48) holds, via backward induction on $t = T - 1, T - 2, \dots, 0$. Here $w^*(t)$ is the net worth (24) corresponding to $p^*(t)$.

The induction base $t = T - 1$ is obvious, recalling that constraints (44) are equivalent to (40), (41). If $p^*(T - 1)$ were not a maximizer in (43), (44) with $T - 1$, the value of objective function in (42) could be increased.

The induction step. Suppose that our statement has been proved for $t = t_* + 1, \dots, T - 1$. In particular, at each $t > t_*$ vector $p^*(t)$ obeys Eq. (47). We are now going to prove that (48) holds at $t = t_*$. The proof is based on Lemma 1 and is performed in two steps.

Step 1. We first show that each strongly connected sink component of $\mathcal{G}[A]$ contains node i such that $p_i^*(t_*) = \bar{p}_i^*(t_*)$. Suppose that the statement is not correct and consider such a sink strong component of $\mathcal{G}[A]$ with the set of nodes $\mathcal{V}^0 \subseteq \mathcal{V}$ that $\bar{p}_i^*(t_*) > p_i^*(t_*) \forall i \in \mathcal{V}^0$, or, equivalently, $\bar{p}_i^*(t_* + 1) > 0 \forall i \in \mathcal{V}^0$. Applying Corollary 3 to $\bar{p} = \bar{p}(t_* + 1)$ and recalling that (48) holds at $t = t_* + 1$, one has $p_i(t + 1) > 0 \forall i \in \mathcal{V}^0$.

Introducing the eigenvector from Lemma 2, consider the TAP (A.3) with $p = p^*$, $\zeta = \pi$ and $\varepsilon > 0$ sufficiently small. Since $\zeta = A^\top \zeta$, the left-hand side of (41) remains invariant under the TAP, and hence $[\hat{p}]$ obeys constraints (41). Since $\zeta_i = \pi_i = 0$ for $i \notin \mathcal{V}^0$ and $p_i(t + 1) > 0 \forall i \in \mathcal{V}^0$, constraint (39) at $t = t_* + 1$ is also preserved by the TAP when $\varepsilon > 0$ is so small that $\hat{p}_i(t_*+1) = p_i^*(t_* + 1) - \varepsilon\zeta_i > 0 \forall i \in \mathcal{V}^0$. Finally, constraint (40) at $t = t_*$ can be rewritten as $p_i(t_*) \le \bar{p}_i(t_*)$. Recalling that $\zeta_i = \pi_i = 0$ for $i \notin \mathcal{V}^0$ and $p_i^*(t_*) < \bar{p}_i^*(t_*)$ for $i \in \mathcal{V}^0$, it is obvious that the TAP does not violate this constraint for $\varepsilon > 0$ sufficiently small. As has

been noticed, the remaining constraints are always preserved by the TAP. The new sequence of payment vectors $[\hat{p}]$ thus satisfies all the constraints (39)–(41) and corresponds to a *larger* value of the objective function, which leads to a contradiction with the optimality of $[p^*]$. The contradiction shows that inequality $p_i^*(t_*) < \bar{p}_i^*(t_*)$ is violated for at least one index $i \in \mathcal{V}^0$.

Step 2. In view of Lemma 1, statement (d) (applied for $\bar{p} = \bar{p}^*(t_*)$ and $c = c(t_*) + w(t_*)$), to prove (48) at $t = t_*$ it remains to prove (47) at $t = t_*$.

Assume that (47) fails to hold, that is, index $s \in \mathcal{V}$ exists such that $p_s^*(t_*) < \bar{p}_s^*(t_*)$ and $p_s^*(t_*) < c_s(t_*) + w_s^*(t_*) + (A^\top p^*(t_*))_s$. We are going to show that this leads to a contradiction with the assumption that sequence $[p^*]$ is optimal, using the TAP (A.3).

We first define the following sets of indices. Let $\mathcal{V}^0 \neq \emptyset$ consist of such nodes i that $p_i^*(t_*) = \bar{p}_i^*(t_*)$ (at Step 1, we have shown every strongly connected sink component of $\mathcal{G}[A]$ contains an element from $\mathcal{V}^0$) and $\tilde{\mathcal{V}} \doteq \mathcal{V} \setminus \mathcal{V}^0$. Obviously, $s \in \tilde{\mathcal{V}}$. We introduce the submatrix $\tilde{A} = (a_{ij})_{i,j \in \tilde{\mathcal{V}}}$ and the corresponding graph $\tilde{\mathcal{G}} = \mathcal{G}[\tilde{A}]$. Let $\mathcal{V}^1$ stand for all nodes $i \in \tilde{\mathcal{V}}, i \neq k$ that i *are not* reachable from s in $\tilde{G}$, $\mathcal{V}^2$ stand for all nodes $i \in \tilde{\mathcal{V}}, i \neq s$ that i *are* reachable from s in $\tilde{G}$.

Step 2a. We first show that $p_i^*(t_* + 1) > 0$ for all $i \in \mathcal{V}^2$.

By construction, $\bar{p}_i^*(t_* + 1) = \alpha[\bar{p}_i^*(t_*) - p_i^*(t_*)] > 0\, \forall i \in \tilde{\mathcal{V}}$. The induction hypothesis entails now that $p_i^*(t_* + 1) > 0$ for $i \in \mathcal{V}^2 \cup \{s\}$. Indeed, by assumption $w_s^*(t_* + 1) = c_s(t_*) + w_s^*(t_*) + (A^\top p^*(t_*))_s - p_s^*(t_*) > 0$. Recalling that (47) holds at $t = t_* + 1$, one shows that $p_s^*(t_*+1) > 0$. If node $\ell \in \mathcal{V}^2$ is directly accessible from s (that is, $a_{s\ell} > 0$) in $\tilde{\mathcal{G}}$, then (47) at $t = t_* + 1$ implies that $p_\ell^*(t_* + 1) > 0$, because $(A^\top p^*)_\ell \geq a_{s\ell} p_s^*(t_* + 1) > 0$. Similarly, if a path $s \to \ell \to m$ exists in $\tilde{\mathcal{G}}$, then $p_m^*(t_* + 1) > 0$ due to (47), because $a_{\ell m} p_m^*(t_* + 1) > 0$, and so on: via induction of the length of the path connecting s to $i \in \mathcal{V}^2$, one shows that $p_i^*(t_* + 1) > 0$ for all $i \in \mathcal{V}^2$.

Step 2b. As has been shown at Step 1, set $\tilde{\mathcal{V}}$ does not contain any strongly connected sink component of $\mathcal{G}[A]$, and hence matrix $\tilde{A}$ and all its submatrices are Schur stable (Proposition 2). Applying Proposition 3 to $\mathcal{V}' = \{s\}$ and $\mathcal{V}'' = \mathcal{V}^2$, a vector $\xi \in \mathbb{R}^{\mathcal{V}^2 \cup \{s\}}$ exists such that

$$\xi \geq 0 \text{ and } \xi_i - \sum_{j \in \mathcal{V}^2 \cup \{s\}} a_{ji}\xi_j \begin{cases} > 0, & i = s; \\ = 0, & i \in \mathcal{V}^2. \end{cases} \tag{A.4}$$

Define the vector ζ as follows: $\zeta_i \doteq 0$ for $i \in \mathcal{V}^0 \cup \mathcal{V}^1$ and $\zeta_i \doteq \xi_i$ for $i \in \mathcal{V}^2 \cup \{s\}$. Then,

$$\zeta_s - (A^\top \zeta)_s > 0$$
$$\zeta_i - (A^\top \zeta)_i = 0 \quad \forall i \in \mathcal{V}^1 \cup \mathcal{V}^2. \tag{A.5}$$

Indeed, for $i \in \mathcal{V}^2 \cup \{s\}$ one has

$$(A^\top \zeta)_i = \sum_{j \in \mathcal{V}^2 \cup \{s\}} a_{ji}\zeta_j + \sum_{j \in \mathcal{V}^0 \cup \mathcal{V}^1} a_{ji} \underbrace{\zeta_j}_{=0} \leq \zeta_i$$

due to (A.4), which inequality can be strict only when $i = s$. Obviously, if $i \in \mathcal{V}^1$ and $j \in \mathcal{V}^2 \cup \{s\}$, then $a_{ji} = 0$ (otherwise, i would be reachable from s in graph $\tilde{G}$, contradiction to the definition of $\mathcal{V}^1$). Thus,

$$(A^\top \zeta)_i = \sum_{j \in \mathcal{V}^2 \cup \{s\}} \underbrace{a_{ji}}_{=0} \zeta_j + \sum_{j \in \mathcal{V}^0 \cup \mathcal{V}^1} a_{ji} \underbrace{\zeta_j}_{=0} = 0 = \zeta_i.$$

for all $i \in \mathcal{V}^1$.

Step 2c. We are now ready to show that the transformation (A.3) with the constructed vector ξ and $p = p^*$ does not violate constraints (39)–(41) if $\varepsilon > 0$ is small.

As we know, constraint (39) has to be checked only at $t = t_* + 1$. By construction, $\zeta_i = 0$ unless $i \in \mathcal{V}^2 \cup \{s\}$. As we have seen at Step 2a, for such indices one has $p_i^*(t_* + 1) > 0$. Hence, $\hat{p}_i(t_*+1) = p_i^*(t_*+1) - \varepsilon\zeta_i \geq 0\, \forall i \in \mathcal{V}$ for $\varepsilon > 0$ being sufficiently small.

Constraint (40) has to be tested only at $t = t_*$. For each $i \in \mathcal{V}^2 \cup \{s\}$ one has $0 < \bar{p}_i^*(t_*) - p_i^*(t_*) = \alpha^{t_*} \bar{p}_i - \sum_{k=0}^{t_*} \alpha^{t_*-k} p_i^*(k)$, which inequality, obviously, remains valid also when p^* is replaced by $[\hat{p}]$ (provided that $\varepsilon > 0$ is small enough). On the other hand, for $i \in \mathcal{V}^0 \cup \mathcal{V}$,

$$0 \leq \alpha^{t_*} \bar{p}_i - \sum_{k=0}^{t_*} \alpha^{t_*-k} p_i^*(k) = \alpha^{t_*} \bar{p}_i - \sum_{k=0}^{t_*} \alpha^{t_*-k} \hat{p}_i^*(k).$$

Hence, $[\hat{p}]$ satisfies constraints (40) if $\varepsilon > 0$ is small.

Finally, we have to check constraints (41) for all $t \geq t_*$. Recall that $w(t+1) = C(t) + \sum_{k=0}^{t} (A^\top p(k) - p(k))$ due to (24) and (37), and (41) is equivalent to the inequality $w(t + 1) \geq 0$. Denoting the net worth vectors corresponding to $[\hat{p}]$ by

$$\hat{w}(t + 1) \doteq C(t) + \sum_{k=0}^{t} (A^\top \hat{p}(k) - \hat{p}(k)) = \\ = \hat{w}(t) + c(t) + A^\top \hat{p}(t) - \hat{p}(t), \tag{A.6}$$

our goal is to show that $\hat{w}(t + 1) \geq 0$ for $t = t_*, \ldots, T - 1$ and $\varepsilon > 0$ being small.

By assumption, for each $i \in \mathcal{V}^0$ and each $t > t_*$ one has $\hat{p}_i(t) = p_i(t) = 0$ (node i pays full debt at period t_*). As has been already shown, we have $\hat{p}(t) \geq 0$ for all $t \geq 0$. In view of (A.6), for every such index the sequence $\hat{w}_i(t)$ is non-decreasing as $t = t_*, t_* + 1, \ldots, T - 1$:

$$\hat{w}_i(t + 1) = \hat{w}_i(t) + c_i(t) + (A^\top \hat{p}(t))_i - \hat{p}_i(t) \geq \hat{w}_i(t)$$
$$\forall t \geq t_* + 1, \, \forall i \in \mathcal{V}^0.$$

On the other hand, $\hat{p}(t_*) \geq p^*(t_*)$ and $\hat{p}_i(t_*) = p_i^*(t_*)\, \forall i \in \mathcal{V}^0$, whereas $p^*(t) = \hat{p}(t)$ for $t < t_*$. In view of this, $\hat{w}_i(t_* + 1) = w_i^*(t_* + 1) \geq 0\, \forall i \in \mathcal{V}^0$, which shows that $\hat{w}_i(t + 1) \geq 0$ for $t = t_*, \ldots, (T - 1)$ and $i \in \mathcal{V}^0$.

On the other hand, (A.3) entails that

$$\hat{w}(t + 1) - w^*(t + 1) = \begin{cases} 0, & t < t_*, \\ -\alpha^{-1}\varepsilon(\zeta - A\zeta), & t = t_* \\ (1 - \alpha^{-1})\varepsilon(\zeta - A\zeta), & t > t_*. \end{cases}$$

Since $1 - \alpha^{-1} \geq 0$, inequalities (A.5) entails that $\hat{w}_i(t + 1) \geq w_i^*(t + 1) \geq 0$ for $i \in \mathcal{V}^1 \cup \mathcal{V}^2 \cup \{k\}$ and $t > t_*$. Furthermore, $\hat{w}_i(t_*+1) = w_i^*(t_*+1) \geq 0$ for $i \in \mathcal{V}^1 \cup \mathcal{V}^2$. Finally, by assumption $w_s^*(t_* + 1) > 0$ entails that $\hat{w}_s(t_* + 1) > 0$ provided that $\varepsilon > 0$ is small enough. We have demonstrated that $\hat{w}(t + 1) \geq 0$ (equivalently, $[\hat{p}]$ satisfies (41)) for $t = t_*, \ldots, (T - 1)$, provided that $\varepsilon > 0$ is chosen sufficiently small.

The assumption about the existence of index s such that $p_s^*(t_*) < \bar{p}_s^*(t_*)$ and $p_s^*(t_*) < c_s(t_*) + w_s^*(t_*) + (A^\top p^*(t_*))_s$ has led us to the contradiction with optimality of sequence $[p]$, because the new sequence $[\hat{p}]$ satisfies all constraints and corresponds to a large value of the objective function. Therefore, (47) should hold at $t = t_*$, which, along with the statement proved at Step 1 and Lemma 1, ensures that (48) holds at $t = t_*$. This finishes the proof of induction step.

The uniqueness of the optimal solution is now trivial. Lemma 1 ensures the uniqueness of $p^*(0)$, which is the maximizer at (43), (44) with $t = 0$ (and depends only on $c(0)$). Similarly, $p^*(1)$ (depending on $c(1)$ and $p^*(0)$) is uniquely found as the maximizer at (43), (44) with $t = 1$, and so on; using induction on $t = 0, \ldots, T-1$, one shows that $p^*(t)$ is defined uniquely and depends on $c(t)$ and $p^*(0), \ldots, p^*(t - 1)$. $\square$

A.4. A remark on the structure of the cost function

Note that the proofs in the previous subsections do not use the representation of coefficients a_t in (31) and (42). The constants (30) can be replaced by any positive numbers $a_0, \ldots, a_{T-1}$ such that $a_{t-1} > \alpha a_t \; \forall t = 1, \ldots, T-1$.

In particular, instead of minimizing the loss function (29), one may minimize a more general function

$$J([P]) = (1-\eta)L_T([P]) + \eta \mathbf{1}^\top \bar{P}(T)\mathbf{1}, \tag{A.7}$$

$$\bar{P}(T) \overset{(23)}{=} \alpha^T \bar{P} - \sum_{k=0}^{T-1} \alpha^{T-k} P(k),$$

where $\eta \in [0,1)$. Such functions, e.g., have been considered in our recent work (Calafiore, Fracastoro, & Proskurnikov, 2022). The loss function (29) corresponds to $\eta = 0$; the weight $\eta > 0$ corresponds to the additional penalty on unpaid liabilities (recall that $\bar{P}(T) = 0$ if and only if there is no default at the terminal time). To minimize the cost function (A.7), one has to maximize the function (31) (or, in the case of pro-rata payments, (42)) with the weights

$$a_t = \eta \alpha^{T-t} + (1-\eta)\sum_{k=0}^{T-t-1} \alpha^k. \tag{A.8}$$

Theorems 1 and 2 retain their validity, replacing the coefficients (30) by (A.8).

A.5. Proof of Corollaries 1 and 2

Proof of Corollary 1 is based on Theorem 1 and Proposition 1. To prove the first statement of Corollary 1, suppose that $t_0 > 0$, $c(t_0) = 0$ yet some entry of $P(t_0)$ is non-zero, that is, at least one node i_0 in $\mathcal{G}[P(t_0)]$ is a not a source node. According to Theorem 1, graph $\mathcal{G}[P(t_0)]$ is acyclic, in particular, all its strongly connected components are trivial. Proposition 1 ensures that some *source* node j is not a sink, being connected to i_0. Being a source, node j does not receive any payment from other bank at time t_0; by assumption, it also has no outside asset: $c_j(t_0) = 0$. Since some arcs come out node j, one, on one hand, has $\bar{p}_j(t_0) > 0$ (node j has not paid its debt before step t_0) and, on the other hand, $w_j(t_0) > 0$ in view of the constraint (22). We arrive at a contradiction with Theorem 1 and the absolute priority rule (33): a bank must pay all its balance until its debt is cleaned.

Proof of Corollary 2. Assume that the statement of Corollary 1 is not correct: $c(t_0) = 0$ for some $t_0 > 0$, however, the set $I = \{i : p_i^*(t_0) > 0\}$ is non-empty. Since $\bar{p}_i^*(t_0) = \alpha(\bar{p}_i(t_0-1) - p_i(t_0-1)) > 0$, the absolute priority rule (47) (applied at $t = t_0 - 1$) entails that banks $i \in I$ pay all their balance at $t = t_0 - 1$, so $w_i^*(t_0) = 0$. By assumption, also $c_i(t_0) = 0$. Substituting $t = t_0$ into (47), one arrives at the inequalities

$$0 < p_i^*(t_0) \le \sum_{j\in\mathcal{V}} a_{ji} p_j^*(t_0) = \sum_{j\in I} a_{ji} p_j^*(t_0) \quad \forall i \in I.$$

(recall that, by definition of set I, $p_j^*(t_0) = 0$ when $j \notin I$). Consider now the Perron–Frobenius left eigenvector $\tilde{\zeta}$ of submatrix $\tilde{A} = (a_{ij})_{i,j\in I}$, that is, $\tilde{\zeta}^\top \tilde{A} = \rho\tilde{\zeta}$, where $\rho \le 1$ is the spectral radius of $\tilde{A}$. Obviously,

$$0 < \sum_{i\in I}\zeta_i p_i^*(t_0) \le \sum_{j\in I}\left(\sum_{i\in I}\zeta_i a_{ji}\right)p_j^* = \rho\sum_{j\in I}\zeta_j p_j^*(t_0).$$

Therefore, the substochastic matrix $\tilde{A}$ is not Schur and has the spectral radius $\rho = 1$, which means that I contains at least one

strongly connected sink component of graph $\mathcal{G}[A]$ (Proposition 2). Substituting $t = t_0 - 1$ into (48) and using Lemma 1, one proves that for some $i \in I$ one has $p_i^*(t_0-1) = \bar{p}_i^*(t_0-1)$, which implies that bank i has paid at its debt at period $t_0 - 1$ and thus $p_i^*(t_0) = \bar{p}_i^*(t_0) = 0$. We arrive at a contradiction with the definition of set I.

References

Acemoglu, D., Ozdaglar, A., & Tahbaz-Salehi, A. (2015). Systemic risk and stability in financial networks. *American Economic Review, 105*(2), 564–608.

Allen, F., Babus, A., & Carletti, E. (2012). Asset commonality, debt maturity and systemic risk. *Journal of Financial Economics, 104*(3), 519–534.

Alvarez, F., & Barlevy, G. (2015). *Mandatory disclosure and financial contagion: Technical report*, National Bureau of Economic Research.

Amini, H., & Feinstein, Z. (2023). Optimal network compression. *European Journal of Operational Research*, [ISSN: 0377-2217] *306*(3), 1439–1455. http://dx.doi.org/10.1016/j.ejor.2022.07.026.

Amini, H., Filipović, D., & Minca, A. (2016). To fully net or not to net: Adverse effects of partial multilateral netting. *Operations Research, 64*(5), 1135–1142.

Anand, K., Craig, B., & von Peter, G. (2015). Filling in the blanks: Network structure and interbank contagion. *Quantitative Finance, 15*(4), 625–636.

Banerjee, T., Bernstein, A., & Feinstein, Z. (2018). Dynamic clearing and contagion in financial networks. arXiv preprint arXiv:1801.02091.

Banerjee, T., & Feinstein, Z. (2022). Pricing of debt and equity in a financial network with comonotonic endowments. *Operations Research, 70*(4), 2085–2100. http://dx.doi.org/10.1287/opre.2022.2275.

Berman, A., & Plemmons, R. J. (1994). *Nonnegative matrices in the mathematical sciences*. Philadelphia,PA: SIAM.

Billio, M., Getmansky, M., Lo, A. W., & Pelizzon, L. (2012). Econometric measures of connectedness and systemic risk in the finance and insurance sectors. *Journal of Financial Economics, 104*(3), 535–559.

Calafiore, G., Fracastoro, G., & Proskurnikov, A. (2021a). On optimal clearing payments in financial networks. In *IEEE conf. decision and control* (pp. 4804–4810).

Calafiore, G., Fracastoro, G., & Proskurnikov, A. (2021b). Optimal clearing payments in a financial contagion model. (submitted for publication). arXiv:2103.10872v2.

Calafiore, G., Fracastoro, G., & Proskurnikov, A. (2022). Control of dynamic financial networks. *IEEE Control Systems Letters, 6*, 3206–3211.

Capponi, A., & Chen, P.-C. (2015). Systemic risk mitigation in financial networks. *Journal of Economic Dynamics & Control, 58*, 152–166.

Chen, H., Wang, T., & Yao, D. D. (2021). Financial network and systemic risk - A dynamic model. *Production and Operations Management, 30*(8), 2441–2466.

Cifuentes, R., Ferrucci, G., & Shin, H. S. (2005). Liquidity risk and contagion. *Journal of the European Economic Association, 3*(2–3), 556–566.

Craig, B., & von Peter, G. (2014). Interbank tiering and money center banks. *Journal of Financial Intermediation, 23*(3), 322–347.

Csóka, P., & Jean-Jacques Herings, P. (2018). Decentralized clearing in financial networks. *Management Science, 64*(10), 4681–4699.

Demirer, M., Diebold, F. X., Liu, L., & Yilmaz, K. (2018). Estimating global bank network connectedness. *Journal of Applied Econometrics, 33*(1), 1–15.

Diebold, F. X., & Yilmaz, K. (2014). On the network topology of variance decompositions: Measuring the connectedness of financial firms. *Journal of Econometrics, 182*(1), 119–134.

Eisenberg, L., & Noe, T. H. (2001). Systemic risk in financial systems. *Management Science, 47*(2), 236–249.

Elsinger, H., et al. (2009). *Financial networks, cross holdings, and limited liability*. Oesterreichische Nationalbank Austria.

Feinstein, Z. (2020). Capital regulation under price impacts and dynamic financial contagion. *European Journal of Operational Research, 281*(2), 449–463.

Feinstein, Z., & Søjmark, A. (2021). Dynamic default contagion in heterogeneous interbank systems. *SIAM Journal on Financial Mathematics, 12*(4), SC83–SC97.

Ferrara, G., Langfield, S., Liu, Z., & Ota, T. (2019). Systemic illiquidity in the interbank network. *Quantitative Finance, 19*(11), 1779–1795.

Fischer, T. (2014). No-arbitrage pricing under systemic risk: Accounting for cross-ownership. *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics, 24*(1), 97–124.

Glasserman, P., & Young, H. P. (2016). Contagion in financial networks. *Journal of Economic Literature, 54*(3), 779–831.

Harary, F., Norman, R., & Cartwright, D. (1965). *Structural models. An introduction to the theory of directed graphs*. New York, London, Sydney: Wiley & Sons.

Kusnetsov, M., Veraart, M., & Luitgard, A. (2019). Interbank clearing in financial networks with multiple maturities. *SIAM Journal on Financial Mathematics, 10*(1), 37–67.

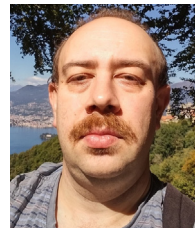Massai, L., Como, G., & Fagnani, F. (2022). Equilibria and systemic risk in saturated networks. *Mathematics of Operations Research, 47*(3), 1781–1801. http://dx.doi.org/10.1287/moor.2021.1188.

Nier, E., Yang, J., Yorulmazer, T., & Alentorn, A. (2007). Network models and financial stability. *Journal of Economic Dynamics & Control*, *31*(6), 2033–2060.

Petropoulos, A., Siakoulis, V., Lazaris, P., & Chatzis, S. (2021). Re-constructing the interbank links using machine learning techniques. an application to the greek interbank market. *Intelligent Systems with Applications, 12*, Article 200055.

Rogers, L. C., & Veraart, L. A. (2013). Failure and rescue in an interbank network. *Management Science*, *59*(4), 882–898.

Shin, H. S. (2008). Risk and liquidity in a system context. *Journal of Financial Intermediation*, *17*(3), 315–329.

Sonin, I. M., & Sonin, K. (2020). *A continuous-time model of financial clearing*: *Becker Friedman Institute for Economics Working Paper No. 2020-101*, online at SSRN, http://dx.doi.org/10.2139/ssrn.3664060.

Soramäki, K., & Cook, S. (2022). *Nonnegative matrices in the mathematical sciences* (second ed.). Risk Books.

Suzuki, T. (2002). Valuing corporate debt: The effect of cross-holdings of stock and debt. *Journal of the Operations Research Society of Japan*, *45*(2), 123–144.

Giulia Fracastoro received the B.Sc. and M.Sc. degrees in applied mathematics and the Ph.D. degree in electronics and telecommunications engineering from the Politecnico di Torino, Turin, Italy, in 2011, 2013, and 2017, respectively. In 2016, she was a Visiting Student with the Signal Processing Laboratory, École Polytechnique Fédérale de Lausanne (EPFL), Lausanne, Switzerland. She is currently an Assistant Professor with the Department of Electronics and Telecommunications (DET), Politecnico di Torino. She is a member of the ELLIS Society for the advancement of artificial intelligence in Europe and cofounder of the Torino ELLIS Research Unit. Her main research interests include machine learning, signal processing, and optimization, with a special focus on graphs and networks. She has been serving as an Associate Editor for the IEEE Transactions on Image Processing, since 2022, and the IEEE Transactions on Signal and Information Processing over Networks, since 2023.

Giuseppe C. Calafiore is a full professor at DET, Politecnico di Torino, where he coordinates the Control Systems and Data Science group. He is also Professor of Mechanical Engineering at CECS, VinUniversity, Vietnam, and an associate fellow of the IEIIT-CNR, Italy. Dr. Calafiore held visiting positions at the Information Systems Laboratory (ISL), Stanford University, California, in 1995; at the Ecole Nationale Supérieure de Techniques Avancées (ENSTA), Paris, in 1998; and at the University of California at Berkeley, in 1999, 2003, 2007, 2017-2019, and 2021 where he lately taught a Master course on Financial Data Science. He was a Senior Fellow at the Institute of Pure and Applied Mathematics (IPAM), University of California at Los Angeles, in 2010. Dr. Calafiore is the author of about 210 journal and conference proceedings papers, and of eight books. He is a Fellow of the IEEE since 2018. He received the IEEE Control System Society "George S. Axelby" Outstanding Paper Award in 2008. His research interests are in the fields of convex optimization, identification and control of uncertain systems, with applications ranging from finance and economic systems to robust control, machine learning, and data science. Dr. Calafiore has over twenty-five years of teaching experience in Master-level and Ph.D. courses in the areas of Systems and Control Theory, Convex Optimization and Machine Learning.

Anton V. Proskurnikov received his M.Sc. (Specialist) degree in 2003 and his Ph.D. (Candidate of Sciences) degree in 2005, both from St. Petersburg State University, Russia. Currently he is an Associate Professor with the Department of Electronics and Telecommunications (DET), Politecnico di Torino, Turin, Italy. Previously he used to be an assistant professor and senior researcher at St. Petersburg State University (2003–2010), Russian Academy of Sciences (2006–2021), postdoctoral researcher at the University of Groningen (2014–2016) and Delft University of Technology (2016–2018).

Anton Proskurnikov is an associate editor at IEEE Transactions on Automatic Control, IFAC World Congress 2023 and EUCA Conference Board. He was the Program Committee Co-Chair of 7th International Conference on Event-based Control, Communication, and Signal Processing (EBCCSP 2021) and invited semi-plenary speaker of IFAC World Congress 2017. His works have been awarded the Medal for Prominent Young Researchers of the Russian Academy of Sciences (2009), IFAC and Elsevier paper prize award (2020) and O. Hugo Schuck Best Paper Award of the American Automatic Control Council (2023). His research interests are focused on dynamical complex networks, multi-agent systems and agent-based models in social, economical and biological sciences.