Stabilized Single Current Inverse Source Formulations Based on Steklov-Poincaré Mappings

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***Abstract*—The inverse source problem in electromagnetics has proved quite relevant for a large class of applications. When it is coupled with the equivalence theorem, the sources are often evaluated as electric and/or magnetic current distributions on an appropriately chosen equivalent surface. In this context, in antenna diagnostics in particular, Love solutions, i.e., solutions which radiate zero-fields inside the equivalent surface, are often sought at the cost of an increase of the dimension of the linear system to be solved. In this work, instead, we present a reduced-in-size single current formulation of the inverse source problem that obtains one of the Love currents via a stable discretization of the Steklov-Poincaré boundary operator leveraging dual functions. The new approach is enriched by theoretical treatments and by a further low-frequency stabilization of the Steklov-Poincaré operator based on the quasi-Helmholtz projectors that is the first of its kind in this field. The effectiveness and practical relevance of the new schemes are demonstrated via both theoretical and numerical results.**

***Index Terms*—Boundary-element method, inverse source problem, Love currents, low-frequency breakdown, Steklov-Poincaré operator.**

I. Introduction

The inverse source problem in electromagnetics, i.e., the recovery of a configuration of sources radiating a given field, has been adopted in a variety of applications ranging from antenna diagnostics to near-to-far-field reconstructions [1]–[3]. These sources are often electric and/or magnetic current distributions residing on a conveniently placed equivalent surface that can be tailored to scatter the target field by virtue of the equivalence theorem. These currents have traditionally been found within a boundary element framework on apertures or on arbitrary equivalent surfaces (see for example [4], [5]). Among inverse source strategies, single current solutions, that reconstruct only one family among electric or magnetic currents, are appealing because of the reduced dimensions of the linear systems to be solved and because of their reduced (numerical) nullspace that is limited to the intrinsic ill-posedness of the problem associated to the non-radiating modes. These strategies, however, have been reported to require more care in the solution process if further physical constraints are not used to ensure a simple relationship between equivalent currents and fields [6], [7]. On the other hand, the double current formulations have non-unique solutions due to the presence of non-radiating currents. Whereas the non-uniqueness can be addressed by selecting a particular solution [8]–[10], the numerical ill-conditioning of the matrix, inherited by the ill-posed nature of the inverse problem, remains to be addressed. To this end, truncated singular value decompositions (TSVD) or Tikhonov regularizations have been used to further regularize the problem [2], [11], [12].

Another feature of interest among inverse source schemes is their capacity to find equivalent Love currents—that are directly related to the tangential fields—which is considered in the literature particularly useful for antenna diagnostics [6], [12]. The Love currents can be obtained by adding further constraints to double current formulations [6], [13], [14] or by filtering any of the solution via Calderón projection [15]. Another interesting approach, leveraging Huygens radiators and valid for plane waves, has been proposed in [16] to reduce the size of the Love-constrained problem to that of a single current formulation, at the price of an approximation.

In this work we follow a different approach. While still targeting a single current formulation, we leveraged dual discretizations to avoid approximating the relationships linking electric and magnetic currents. The contribution of this paper is then twofold: we present a new single current formulation capable of obtaining Love currents by leveraging a stable discretization of the Steklov-Poincaré operator [17] without resorting to any approximations of the electromagnetic relations. This results in a single current formulation that delivers one of the Love currents. A similar equation has been used in a different context in [18] and [19]. Differently from what has been presented in those contributions, here we propose a discretization scheme based on dual elements which achieves an optimal conditioning despite a higher cost to generate the matrix entries. Moreover we present the first frequency stabilization of Steklov-Poincaré operators via quasi-Helmholtz projectors and we leverage on this new result to stabilize in frequency the new formulations. What we propose is then, to the best of our knowledge, the first low-frequency regularization of a full-wave inverse source scheme showing high level of accuracy and numerical stability till arbitrarily low-frequencies.

The paper is organized as follows: the main electromagnetic operators are introduced in Section II, the new formulations are presented in Section III, whereas Section IV presents the frequency stabilization of the Steklov-Poincaré operator and its application to the the new equations. Finally Section V illustrates the accuracy and stability of the new formulation through numerical test cases. Section VI concludes the latter. Very preliminary results from this work were presented in the

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conference contribution [20].

II. Background and Notation

Let Γ be a two-dimensional smooth manifold in $\mathbb{R}^3$ delimiting the internal and external domains Ω^- and Ω^+. Consider a time-harmonic source in Ω^- generating Maxwellian fields in $\Omega^- \cup \overline{\Omega^+} = \mathbb{R}^3$. In light of the equivalence theorem [21], there exist equivalent current densities $\boldsymbol{M}$ and $\boldsymbol{J}$ on Γ which radiate in Ω^+ the same fields as the original source and radiate in Ω^- possibly different electric and magnetic fields; these currents satisfy

$$\boldsymbol{M} = \left(\boldsymbol{E}^+ - \boldsymbol{E}'^-\right) \times \hat{\boldsymbol{n}}_{\boldsymbol{r}} \,, \tag{1}$$

$$\boldsymbol{J} = \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \left(\boldsymbol{H}^+ - \boldsymbol{H}'^-\right) \,, \tag{2}$$

where $\hat{\boldsymbol{n}}_{\boldsymbol{r}}$ is the unit normal vector to Γ in $\boldsymbol{r}$ pointing towards Ω^+, $\boldsymbol{E}^+$, $\boldsymbol{H}^+$ are the original electric and magnetic field in Ω^+ and $\boldsymbol{E}'^-$, $\boldsymbol{H}'^-$ are the new fields in Ω^-. The $e^{-\mathrm{i}\omega t}$ time-harmonic dependence is assumed and suppressed throughout the paper. Solving the inverse source problem consists in finding a set of equivalent currents $\boldsymbol{M}$, $\boldsymbol{J}$ given the electric and/or magnetic fields' observations on a two-dimensional smooth and simply connected manifold $\Gamma_m \subset \Omega^+$. These observations are the output of the actual fields' measurement which includes possible probe compensation. We assume a sampling able to capture the degrees of freedom (defined as in [22]) and thus satisfy the equivalence theorem. The problem can be solved naturally by the boundary element method. In this framework, define the electric field integral operator (EFIO) on Γ

$$\mathcal{T}_{\boldsymbol{r}} \boldsymbol{f} = \mathrm{i}k\, \mathcal{T}_{s,\boldsymbol{r}} \boldsymbol{f} + \mathrm{i}k^{-1}\, \mathcal{T}_{h,\boldsymbol{r}} \boldsymbol{f} \tag{3}$$

with

$$\mathcal{T}_{s,\boldsymbol{r}} \boldsymbol{f} = \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \int_\Gamma \frac{e^{\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi\,|\boldsymbol{r}-\boldsymbol{r}'|} \boldsymbol{f}(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}' \,, \tag{4}$$

$$\mathcal{T}_{h,\boldsymbol{r}} \boldsymbol{f} = \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \nabla \int_\Gamma \frac{e^{\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi\,|\boldsymbol{r}-\boldsymbol{r}'|} \nabla_s \cdot \boldsymbol{f}(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}' \,, \tag{5}$$

and the magnetic field integral operator (MFIO) [23]

$$\mathcal{K}_{\boldsymbol{r}} \boldsymbol{f} = -\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times p.v. \int_\Gamma \nabla \times \frac{e^{\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi\,|\boldsymbol{r}-\boldsymbol{r}'|} \boldsymbol{f}(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}' \,, \tag{6}$$

where k is the wavenumber and $\boldsymbol{r}$ lies on any two-dimensional manifold in $\overline{\Omega^+}$ (possibly Γ or Γ_m), to which the definition of $\hat{\boldsymbol{n}}_{\boldsymbol{r}}$ is extended. In the case $\boldsymbol{r} \in \Gamma$ $\mathcal{T}_{\boldsymbol{r}}$, $\mathcal{K}_{\boldsymbol{r}}$ are denoted by $\mathcal{T}$, $\mathcal{K}$ respectively. When $\boldsymbol{r} \in \Gamma_m$, the radiation operator

$$\mathcal{R} = \begin{bmatrix} -\mathcal{K}_{\boldsymbol{r}} & \mathcal{T}_{\boldsymbol{r}} \\ -\mathcal{T}_{\boldsymbol{r}} & -\mathcal{K}_{\boldsymbol{r}} \end{bmatrix} \tag{7}$$

is a linear map between equivalent sources on Γ and observed tangential fields on Γ_m, meaning that

$$\mathcal{R} \begin{bmatrix} -\boldsymbol{M} \\ \eta \boldsymbol{J} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{E}^+ \\ \eta \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{H}^+ \end{bmatrix} , \tag{8}$$

with $\eta = \sqrt{\mu/\epsilon}$ and ϵ, μ being the permittivity and the permeability of the medium respectively. The inverse problem aims at finding unknown current distributions that satisfy (8), or part of it. Indeed, by selecting a single block of $\mathcal{R}$—either $\mathcal{K}_{\boldsymbol{r}}$ or $\mathcal{T}_{\boldsymbol{r}}$—and solving for the corresponding reduced right hand side—$\boldsymbol{E}^+$ or $\boldsymbol{H}^+$—four different single current formulations can be obtained. Alternatively, three double current formulations can be derived by considering the full radiator or one of its rows only. The latter systems of continuous equations admit several solutions because multiple equivalent currents can radiate the same external field in Ω^+ and the physical meaning of the solution depends on the type of implicit or explicit additional constraints used to select a particular solution. The Love currents $\boldsymbol{M}_L$, $\boldsymbol{J}_L$ are one of these particular solutions that are obtained by imposing the fields radiated in Ω^- to be identically $\mathbf{0}$ [6]. One way of enforcing this condition is to leverage the well-known Calderón projector [24]

$$\mathcal{P}^- = \begin{bmatrix} \frac{\mathcal{I}}{2} + \mathcal{K} & -\mathcal{T} \\ \mathcal{T} & \frac{\mathcal{I}}{2} + \mathcal{K} \end{bmatrix} , \tag{9}$$

where $\mathcal{I}$ is the identity operator, that can be added to the system of equations (8) [13] as

$$\begin{bmatrix} \mathcal{R} \\ \mathcal{P}^- \end{bmatrix} \cdot \begin{bmatrix} -\boldsymbol{M}_L \\ \eta \boldsymbol{J}_L \end{bmatrix} = \left[\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{E}^+,\, \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \eta \boldsymbol{H}^+,\, \mathbf{0},\, \mathbf{0}\right]^T \,. \tag{10}$$

III. Conforming Discretization of a Steklov-Poincaré-based Equation

In this section we introduce a single source method which enforces the Love condition without increasing the matrix system size with regards to standard single source formulations. Starting from the formulation in (10), consider the Love condition expressed with the inner Calderón projector

$$\mathcal{P}^- \begin{bmatrix} -\boldsymbol{M}_L \\ \eta \boldsymbol{J}_L \end{bmatrix} = \mathbf{0} \,. \tag{11}$$

Clearly, for k different from resonant wavenumbers of Γ [25], (11) defines a relation between the two Love currents

$$\eta \boldsymbol{J}_{\boldsymbol{L}} = -\left(\frac{\mathcal{I}}{2} + \mathcal{K}\right)^{-1} \mathcal{T}\left(-\boldsymbol{M}_L\right) \tag{12}$$

where $\left(\frac{\mathcal{I}}{2} + \mathcal{K}\right)^{-1} \mathcal{T}$ is the Steklov-Poincaré operator [17]. By replacing (12) in the first row equation of (8), we obtain the equation

$$\left(-\mathcal{K}_{\boldsymbol{r}} - \mathcal{T}_{\boldsymbol{r}} \left(\frac{\mathcal{I}}{2} + \mathcal{K}\right)^{-1} \mathcal{T}\right)\left(-\boldsymbol{M}_L\right) = \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{E}^+ \,, \tag{13}$$

which is a single source equation that naturally yields the magnetic Love current $\boldsymbol{M}_L$. If instead of this current, the electric Love current $\boldsymbol{J}_L$ is desired as the first outcome of the procedure, a similar strategy can be applied obtaining

$$\left(\mathcal{T}_{\boldsymbol{r}} + \mathcal{K}_{\boldsymbol{r}} \mathcal{T}^{-1} \left(\frac{\mathcal{I}}{2} + \mathcal{K}\right)\right)\left(\eta \boldsymbol{J}_L\right) = \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{E}^+ . \tag{14}$$

An alternative approach to study (13) and (14) leverages the equivalence theorem, following a similar procedure to the one presented in chapter 3 of [26]. In this context, (13) and (14) can be the interpreted as the equations obtained after accordingly changing the material of the internal domain while imposing the Love condition as described in [27].

To numerically solve (13) and (14), the discretization scheme will require particular attention. Starting with (13), the

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magnetic current is expanded as $\boldsymbol{M}_L(\boldsymbol{r}) \approx \sum_{i=1}^{N_e} m_i \boldsymbol{f}_i(\boldsymbol{r})$ where $\{\boldsymbol{f}_i\}$ are Rao-Wilton-Glisson (RWG) basis functions (here used without edge normalization) and N_e is the number of mesh edges. The electric operator $\mathcal{T}$ is then tested with rotated RWG functions [28] which yields the matrix $\mathbf{T} = \mathrm{i}k\mathbf{T}_s + \mathrm{i}k^{-1}\mathbf{T}_h$, where $[\mathbf{T}_s]_{ij} = \langle \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{f}_i, \mathcal{T}_s \boldsymbol{f}_j\rangle_\Gamma$, $[\mathbf{T}_h]_{ij} = \langle \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{f}_i, \mathcal{T}_h \boldsymbol{f}_j\rangle_\Gamma$, and $\langle \boldsymbol{a}, \boldsymbol{b}\rangle_\Gamma = \int_\Gamma \boldsymbol{a}(\boldsymbol{r}) \cdot \boldsymbol{b}(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}$. As a consequence, the $\left(\frac{\mathcal{I}}{2} + \mathcal{K}\right)^{-1}$ term must be tested with rotated-RWGs, and to allow for a non-singular discretization of the identity, the source functions used for its discretization must be dual elements [29]—we will use in the following the Buffa-Christiansen (BC) basis functions, a definition of which can be found in [29], [30]. We define the Gram matrix $[\mathbb{G}]_{ij} = \langle \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{f}_i, \boldsymbol{g}_j\rangle_\Gamma$, where $\{\boldsymbol{g}_j(\boldsymbol{r})\}$ denote the BC functions and propose as matrix discretization for the $\mathcal{K}$ operator $[\mathbb{K}]_{ij} = \langle \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{f}_i, \mathcal{K}\boldsymbol{g}_j\rangle_\Gamma$. Finally, as a consequence of this choice, the source functions of $\mathcal{T}_{\boldsymbol{r}}$ must be BC functions and a possible choice for the testing functions are rotated-BC basis functions living on Γ_m. Thus, we define $\mathbb{T}_m = \mathrm{i}k\mathbb{T}_{s,m} + \mathrm{i}k^{-1}\mathbb{T}_{h,m}$ where $[\mathbb{T}_{s,m}]_{ij} = \langle \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{g}_i, \mathcal{T}_{s,\boldsymbol{r}}\boldsymbol{g}_j\rangle_{\Gamma_m}$ and $[\mathbb{T}_{h,m}]_{ij} = \langle \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{g}_i, \mathcal{T}_{h,\boldsymbol{r}}\boldsymbol{g}_j\rangle_{\Gamma_m}$. From the above choices the discretization of the leftmost $\mathcal{K}_{\boldsymbol{r}}$ is entirely determined as $[\mathbf{K}_m]_{ij} = \langle \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{g}_i, \mathcal{K}_{\boldsymbol{r}}\boldsymbol{f}_j\rangle_{\Gamma_m}$. By combining the previous discretization schemes we obtain the discretized equation

$$\left(-\mathbf{K}_m - \mathbb{T}_m \left(\mathbb{G}/2 + \mathbb{K}\right)^{-1} \mathbf{T}\right)(-\mathbf{m}) = \mathbf{e}_m \tag{15}$$

where $[\mathbf{e}_m]_i = \langle \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{g}_i, \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{E}^+\rangle_{\Gamma_m}$ is the discretization of the observed electric field and $\mathbf{m}$ is the vector of solution coefficients m_i. For (14), a similar reasoning leads to

$$\left(\mathbf{T}_m + \mathbb{K}_m \mathbb{T}^{-1}\left(-\mathbb{G}^{\mathrm{T}}/2 + \mathbf{K}\right)\right)(\eta \mathbf{j}) = \varepsilon_m\,, \tag{16}$$

with $\mathbf{T}_m = \mathrm{i}k\mathbf{T}_{s,m} + \mathrm{i}k^{-1}\mathbf{T}_{h,m}$, $[\mathbf{T}_{s,m}]_{ij} = \langle \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{f}_i, \mathcal{T}_{s,\boldsymbol{r}}\boldsymbol{f}_j\rangle_{\Gamma_m}$, $[\mathbf{T}_{h,m}]_{ij} = \langle \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{f}_i, \mathcal{T}_{h,\boldsymbol{r}}\boldsymbol{f}_j\rangle_{\Gamma_m}$, $[\mathbb{K}_m]_{ij} = \langle \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{f}_i, \mathcal{K}_{\boldsymbol{r}}\boldsymbol{g}_j\rangle_{\Gamma_m}$, $\mathbb{T} = \mathrm{i}k\mathbb{T}_s + \mathrm{i}k^{-1}\mathbb{T}_h$, $[\mathbb{T}_s]_{ij} = \langle \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{g}_i, \mathcal{T}_s\boldsymbol{g}_j\rangle_\Gamma$, $[\mathbb{T}_h]_{ij} = \langle \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{g}_i, \mathcal{T}_h\boldsymbol{g}_j\rangle_\Gamma$, $[\mathbf{K}]_{ij} = \langle \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{g}_i, \mathcal{K}_{\boldsymbol{r}}\boldsymbol{f}_j\rangle_\Gamma$, $[\varepsilon_{mi}] = \langle \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{f}_i, \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{E}^+\rangle_{\Gamma_m}$ and $\mathbf{j}$ is the vector of coefficients j_i of the electric current expansion $\boldsymbol{J}_L(\boldsymbol{r}) \approx \sum_{i=1}^{N_e} j_i \boldsymbol{f}_i(\boldsymbol{r})$.

The reader should note that using RWG or BC as testing functions has an important theoretical value and it follows a consolidated practice in the literature (see [7], [13]). At the same time, however, it does not lead to formulations that can be applied directly to a realistic measurement setting to which, however, it can be extended. In fact, we observe that both the RWG and the BC testing functions can be used to interpolate a point-matching scenario. The point-matching strategy can then be handled similarly to previous works in the literature (as in [2], [6], [7] and references therein). Moreover, it should also be noted that it is not necessary to solve both (15) and (16) to obtain both currents: once one of the two currents has been computed (discretized with RWGs), the discretization of the other as a linear combination of BCs can be obtained after back substitution in (12). In addition, only one current is required to compute the probed field in the outside region by using (13) or (14), respectively, following the discretization strategies delineated above with the sole difference that the leftmost operators must be evaluated in the point of interest, and not tested with primal or dual functions. Finally, it is noted that the introduced single-source formulations need the additional inversion of first and second kind operators in (16) and (15) respectively. Thus the use of (15) should be preferred as the computational overhead of this formulation would be the inversion of $\mathbb{G}/2 + \mathbb{K}$, which is usually well-conditioned and therefore leads to better performances when using the standard iterative solvers present in the literature.

IV. Quasi-Helmholtz Stabilization

The linear system in (16) inherits the well-known low-frequency breakdown of the EFIO, that causes, among other things, the conditioning of the system to grow unbounded as the frequency decreases [31], [32]; at the same time the linear system in (15) will behave, frequency-wise, like an MFIO requiring low-frequency stabilization to avoid numerical cancellations due to a different behavior over frequency of the solenoidal and non-solenoidal components of fields and solutions [33]. Note that some of the standard inverse source formulations in the literature may also suffer from similar low-frequency problems and may benefit from a stabilization scheme similar to the one proposed below. In this contribution however, for the sake of brevity, we will limit the analysis to the low-frequency stabilization of our new formulations only. Define $\mathbf{P}_k = \mathbf{P}^{\Lambda H}k^{-1/2} + \mathrm{i}\mathbf{P}^{\Sigma}k^{1/2}$, $\mathbb{P}_k = \mathbb{P}^{\Sigma H}k^{-1/2} + \mathrm{i}\mathbb{P}^{\Lambda}k^{1/2}$, where $\mathbf{P}^{\Sigma} = \mathbf{\Sigma}(\mathbf{\Sigma}^T\mathbf{\Sigma})^{+}\mathbf{\Sigma}^T$, $\mathbf{P}^{\Lambda H} = \mathbf{I} - \mathbf{P}^{\Sigma}$, $\mathbb{P}^{\Lambda} = \mathbf{\Lambda}(\mathbf{\Lambda}^T\mathbf{\Lambda})^{+}\mathbf{\Lambda}^T$, $\mathbb{P}^{\Sigma H} = \mathbf{I} - \mathbb{P}^{\Lambda}$ are the quasi-Helmholtz projectors defined respectively in the RWG space and in the dual BC space, $\mathbf{I}$ is the identity matrix, and where $\mathbf{\Sigma}$, $\mathbf{\Lambda}$, are the star-to-RWG and loop-to-RWG transformation matrices, the definitions of which can be found in [32]. We indicate with $(\cdot)^+$ the Moore-Penrose (MP) pseudoinverse operator. These projectors allow us to separate the components of the solutions with a different behavior over frequency and to rescale them to avoid numerical cancellations. We propose the following regularization schemes for (15) and (16), respectively

$$\mathbb{P}_k\left(-\mathbf{K}_m - \mathbb{T}_m\left(\mathbb{G}/2 + \mathbb{K}\right)^{-1}\mathbf{T}\right)\mathbf{P}_k\mathbf{x} = \mathbb{P}_k\mathbf{e}_m, \tag{17}$$

$$\mathbf{P}_k\left(\mathbf{T}_m + \mathbb{K}_m\mathbb{T}^{-1}\left(-\mathbb{G}^T/2 + \mathbf{K}\right)\right)\mathbf{P}_k\mathbf{y} = \mathbf{P}_k\varepsilon_m \tag{18}$$

where $-\mathbf{m} = \mathbf{P}_k\mathbf{x}$ and $\eta\mathbf{j} = \mathbf{P}_k\mathbf{y}$. The frequency stability of the above equations will now be demonstrated in two steps: the stabilization of the Steklov-Poincaré operators used in (15), (16) and the one of equations (15), (16) themselves. First we will show that quasi-Helmholtz projectors can successfully regularize the Steklov-Poincaré operators in both discretizations presented here. This is proven in (19) and (20) where we exploited standard cancellation properties of projectors on solenoidal spaces [23] (i.e., $\mathbf{P}^{\Lambda H}\mathbf{T}_h = \mathbf{T}_h\mathbf{P}^{\Lambda H} = \mathbb{P}^{\Sigma H}\mathbb{T}_h = \mathbb{T}_h\mathbb{P}^{\Sigma H} = 0$) from which $\mathbf{T}_h = \mathbf{P}^{\Sigma}\mathbf{T}_h\mathbf{P}^{\Sigma}$ and $\mathbb{T}_h = \mathbb{P}^{\Lambda}\mathbb{T}_h\mathbb{P}^{\Lambda}$. In addition in (20) we used the result $\|\mathbf{P}^{\Sigma}\left(-\mathbb{G}^T/2 + \mathbf{K}\right)^{-1}\mathbb{P}^{\Lambda}\| = \mathcal{O}\left(k^2\right)$ which follows from $\|\mathbf{P}^{\Sigma}\left(-\mathbb{G}^T/2 + \mathbf{K}\right)\mathbb{P}^{\Lambda}\| = \mathcal{O}\left(k^2\right)$ (proven in Section IV.B.1 of [23]) after following a similar procedure as the one in Appendix B of [23]; in (19) the result $\|\mathbb{P}^{\Lambda}\left(\mathbb{G}/2 + \mathbb{K}\right)^{-1}\mathbf{P}^{\Sigma}\| = \mathcal{O}\left(k^2\right)$ which can be proven in a similar and dual way. This ends the proof of the stabilization

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$$\begin{aligned}\mathbb{P}_k^{-1}\left(\left(\mathbb{G}/2+\mathbb{K}\right)^{-1}\mathbf{T}\right)\mathbf{P}_k &= \left(\sqrt{k}\mathbb{P}^{\Sigma H}+\frac{1}{\mathrm{i}\sqrt{k}}\mathbb{P}^{\Lambda}\right)\left(\left(\mathbb{G}/2+\mathbb{K}\right)^{-1}\left(\mathrm{i}k\mathbf{T}_s+\frac{\mathrm{i}}{k}\mathbf{T}_h\right)\right)\left(\frac{1}{\sqrt{k}}\mathbf{P}^{\Lambda H}+\mathrm{i}\sqrt{k}\mathbf{P}^{\Sigma}\right)\\
&= \mathbb{P}^{\Sigma H}\left(\mathbb{G}/2+\mathbb{K}\right)^{-1}\left(\mathrm{i}k\mathbf{T}_s\right)\mathbf{P}^{\Lambda H}+ik\mathbb{P}^{\Sigma H}\left(\mathbb{G}/2+\mathbb{K}\right)^{-1}\left(\mathrm{i}k\mathbf{T}_s+\mathrm{i}k^{-1}\mathbf{T}_h\right)\mathbf{P}^{\Sigma}\\
&+(ik)^{-1}\mathbb{P}^{\Lambda}\left(\mathbb{G}/2+\mathbb{K}\right)^{-1}\left(\mathrm{i}k\mathbf{T}_s\right)\mathbf{P}^{\Lambda H}+\mathbb{P}^{\Lambda}\left(\mathbb{G}/2+\mathbb{K}\right)^{-1}\left(\mathrm{i}k\mathbf{T}_s+\mathrm{i}k^{-1}\mathbf{T}_h\right)\mathbf{P}^{\Sigma}\\
&= -\mathbb{P}^{\Sigma H}\left(\mathbb{G}/2+\mathbb{K}\right)^{-1}\mathbf{T}_h\mathbf{P}^{\Sigma}+\mathbb{P}^{\Lambda}\left(\mathbb{G}/2+\mathbb{K}\right)^{-1}\mathbf{T}_s\mathbf{P}^{\Lambda H}\\
&+\mathrm{i}k^{-1}\mathbb{P}^{\Lambda}\left(\mathbb{G}/2+\mathbb{K}\right)^{-1}\mathbf{P}^{\Sigma}\mathbf{T}_h\mathbf{P}^{\Sigma}+\mathcal{O}\left(k\right)\\
&= -\mathbb{P}^{\Sigma H}\left(\mathbb{G}/2+\mathbb{K}\right)^{-1}\mathbf{T}_h\mathbf{P}^{\Sigma}+\mathbb{P}^{\Lambda}\left(\mathbb{G}/2+\mathbb{K}\right)^{-1}\mathbf{T}_s\mathbf{P}^{\Lambda H}+\mathcal{O}\left(k\right)\end{aligned}\tag{19}$$

$$\begin{aligned}\left(\mathbb{P}_k^{-1}\left(\mathbb{T}^{-1}\left(-\mathbb{G}^T/2+\mathbf{K}\right)\right)\mathbf{P}_k\right)^{-1} &= \left(\sqrt{k}\mathbf{P}^{\Lambda H}+\frac{1}{\mathrm{i}\sqrt{k}}\mathbf{P}^{\Sigma}\right)\left(\left(-\mathbb{G}^T/2+\mathbf{K}\right)^{-1}\left(\mathrm{i}k\mathbb{T}_s+\frac{\mathrm{i}}{k}\mathbb{T}_h\right)\right)\left(\frac{1}{\sqrt{k}}\mathbb{P}^{\Sigma H}+\mathrm{i}\sqrt{k}\mathbb{P}^{\Lambda}\right)\\
&= \mathbf{P}^{\Lambda H}\left(-\mathbb{G}^T/2+\mathbf{K}\right)^{-1}\left(\mathrm{i}k\mathbb{T}_s\right)\mathbb{P}^{\Sigma H}+ik\mathbf{P}^{\Lambda H}\left(-\mathbb{G}^T/2+\mathbf{K}\right)^{-1}\left(\mathrm{i}k\mathbb{T}_s+\mathrm{i}k^{-1}\mathbb{T}_h\right)\mathbb{P}^{\Lambda}\\
&+(ik)^{-1}\mathbf{P}^{\Sigma}\left(-\mathbb{G}^T/2+\mathbf{K}\right)^{-1}\left(\mathrm{i}k\mathbb{T}_s\right)\mathbb{P}^{\Sigma H}+\mathbf{P}^{\Sigma}\left(-\mathbb{G}^T/2+\mathbf{K}\right)^{-1}\left(\mathrm{i}k\mathbb{T}_s+\mathrm{i}k^{-1}\mathbb{T}_h\right)\mathbb{P}^{\Lambda}\\
&= -\mathbf{P}^{\Lambda H}\left(-\mathbb{G}^T/2+\mathbf{K}\right)^{-1}\mathbb{T}_h\mathbb{P}^{\Lambda}+\mathbf{P}^{\Sigma}\left(-\mathbb{G}^T/2+\mathbf{K}\right)^{-1}\mathbb{T}_s\mathbb{P}^{\Sigma H}\\
&+\mathrm{i}k^{-1}\mathbf{P}^{\Sigma}\left(-\mathbb{G}^T/2+\mathbf{K}\right)^{-1}\mathbb{P}^{\Lambda}\mathbb{T}_h\mathbb{P}^{\Lambda}+\mathcal{O}\left(k\right)\\
&= -\mathbf{P}^{\Lambda H}\left(-\mathbb{G}^T/2+\mathbf{K}\right)^{-1}\mathbb{T}_h\mathbb{P}^{\Lambda}+\mathbf{P}^{\Sigma}\left(-\mathbb{G}^T/2+\mathbf{K}\right)^{-1}\mathbb{T}_s\mathbb{P}^{\Sigma H}+\mathcal{O}\left(k\right)\end{aligned}\tag{20}$$

of the Steklov-Poincaré operator. As a second step, we demonstrate the frequency regularity of (17) noticing that $\mathbb{P}_k\mathbf{K}_m\mathbf{P}_k$ is frequency stable [33] and that $\mathbb{P}_k\mathbb{T}_m\left(\mathbb{G}/2+\mathbb{K}\right)^{-1}\mathbf{T}\mathbf{P}_k = \left(\mathbb{P}_k\mathbb{T}_m\mathbb{P}_k\right)\left(\mathbb{P}_k^{-1}\left(\mathbb{G}/2+\mathbb{K}\right)^{-1}\mathbf{T}\mathbf{P}_k\right)$ which, following the above developments and the regularity of $\mathbb{P}_k\mathbb{T}_m\mathbb{P}_k$, is the product of two frequency regular operators and thus is frequency regular. Dually the stability and well-conditioning of (18) is proved with $\mathbf{P}_k\mathbb{K}_m\mathbb{T}^{-1}\left(-\mathbb{G}^T/2+\mathbf{K}\right)\mathbf{P}_k = \left(\mathbf{P}_k\mathbb{K}_m\mathbb{P}_k\right)\left(\mathbb{P}_k^{-1}\mathbb{T}^{-1}\left(-\mathbb{G}^T/2+\mathbf{K}\right)\mathbf{P}_k\right)$ and the frequency regularity of $\mathbf{P}_k\mathbb{K}_m\mathbb{P}_k$ (on simply-connected geometries), $\mathbb{P}_k^{-1}\mathbb{T}^{-1}\left(-\mathbb{G}^T/2+\mathbf{K}\right)\mathbf{P}_k$, and $\mathbf{P}_k\mathbf{T}_m\mathbf{P}_k$. We conclude this section by noticing that the proposed strategies hold for plane wave sources, but they can be adapted for different excitations by modifying the coefficients of $\mathbf{P}_k$ and $\mathbb{P}_k$ in an analogous way to what would be needed for the EFIO and the MFIO [34]. The extension to different excitation has been omitted from this paper for the sake of clarity and brevity. Finally, we highlight that in the implementation of (17) and (18) we explicitly set to 0 the static component of the terms $\mathbf{P}^{\Sigma}\mathbf{K}_m\mathbb{P}^{\Lambda}$, $\mathbb{P}^{\Lambda}\mathbb{K}_m\mathbf{P}^{\Sigma}$, $\mathbf{P}^{\Sigma}\left(-\mathbb{G}^T/2+\mathbf{K}\right)\mathbb{P}^{\Lambda}$, and $\mathbb{P}^{\Lambda}\left(\mathbb{G}/2+\mathbb{K}\right)\mathbf{P}^{\Sigma}$.

V. Numerical Results and Discussion

A series of tests is now presented to demonstrate reconstruction, enforcement of the Love condition, and frequency behavior of the formulation. First the reconstruction capability of the Steklov-Poincaré approach (15) is tested: it maps magnetic currents to electric fields, a most relevant setting for real case scenarios. The electric field of a combination of Hertzian dipoles at frequency $f = 5\,\mathrm{GHz}$ is sampled with ideal probes on a spherical surface Γ_m at $1\lambda = 2\pi/k$ distance from a spherical equivalent surface Γ of radius $a = 6\,\mathrm{cm}$. The surfaces Γ and Γ_m are discretized with an average mesh edge length of $\lambda/10$ as common in the literature. Similarly to what is done in [12], noise is added to the sampled fields sampled to obtain a signal to noise ratio $SNR = 60\,\mathrm{dB}$. Our work is

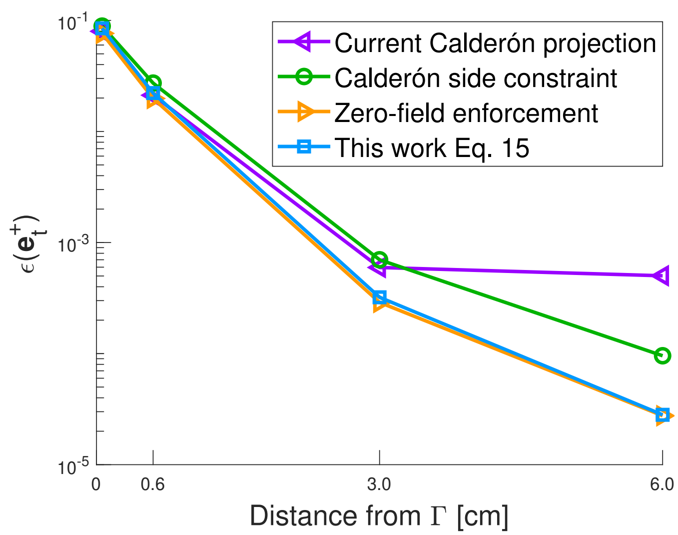

Fig. 1. Field reconstruction error ϵ for the different Love formulations. The fields are obtained from a combination of Hertzian dipoles oscillating at $f = 5\,\mathrm{GHz}$ and noise has been applied to obtain a $SNR = 60\,\mathrm{dB}$. The field observations are performed on a spherical surface of same center as Γ and situated 1λ away from Γ. The evaluation of ϵ is then performed on spherical surfaces concentric to Γ with different radii.

then compared to other Love formulations analyzed in [6], [7], [13] which are three of the several possible approaches that can be found in the literature. The reconstruction capabilities of the formulations are evaluated on several spherical surfaces concentric to Γ, which we define according to the difference between their radius and the one of Γ. On these surfaces, we compute the fields $\mathbf{e}_t$ reconstructed by the different formulations and their error $\epsilon\left(\mathbf{e}_t\right)$ with respect to the original noiseless field $\mathbf{e}_{ref}$ radiated by the source. The error is defined as

$$\epsilon\left(\mathbf{e}_t\right) := \sqrt{\frac{\sum_{n=1}^{N}\left|[\mathbf{e}_t]_n-[\mathbf{e}_{ref}]_n\right|^2}{\sum_{n=1}^{N}\left|[\mathbf{e}_{ref}]_n\right|^2}}\tag{21}$$

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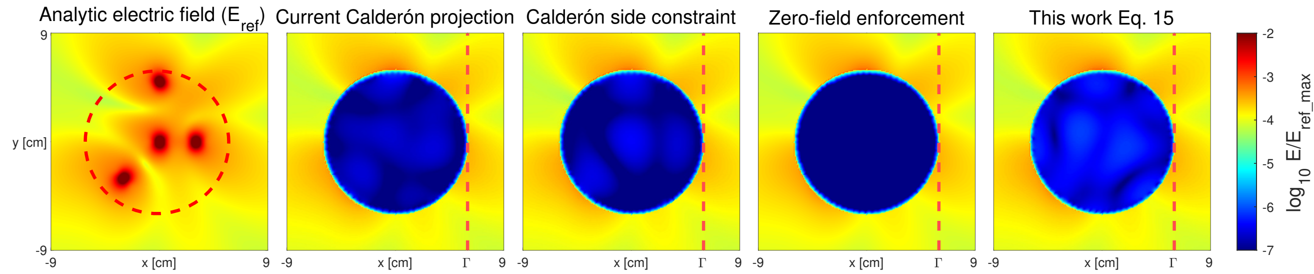

Fig. 2. Modulus of the electric field in a xy planar section of $\mathbb{R}^3$: fields are normalized on the maximum value of the reference field and are obtained from a combination of Hertzian dipoles oscillating at $f = 5\,\mathrm{GHz}$.

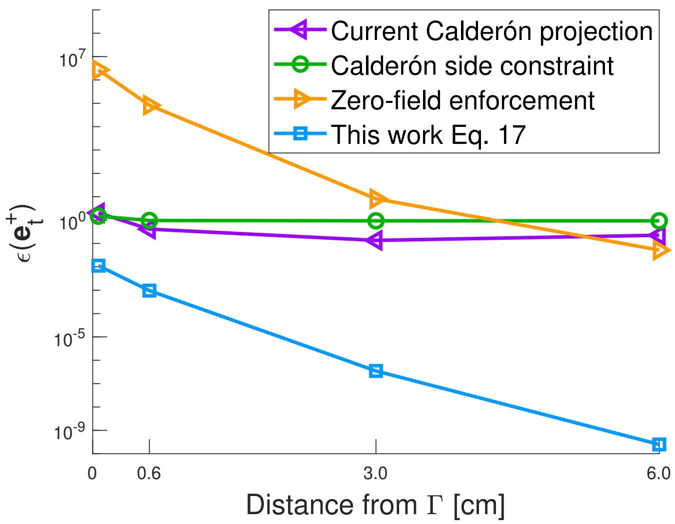

Fig. 3. Field reconstruction error ϵ for the different Love formulations. The equivalent currents and the field samplings are defined on the same meshes used in Fig. 1. The fields are scattered from a $1\,\mathrm{cm}$-radius PEC sphere concentric to Γ illuminated by a plane wave oscillating at $f = 5 \cdot 10^{-20}\mathrm{Hz}$.

where N is here used to represent the number of edges of the meshes on which the field is tested. The reconstruction errors obtained in this way in Ω^+ are reported in Fig. 1. We can observe that in this setting all the the considered formulations manage to reconstruct the field up to noise level.

Then, to verify the Love condition, we check whether the internal fields radiated by equivalent currents obtained are zero (within the discretization error) inside the equivalent surface. Results are shown in Fig. 2 where the magnitude of the radiated electric field is displayed on the plane $z = 0$ for the different formulations and qualitatively confirm that all Love formulations find $\sum_{i=1}^{N_e} m_i \boldsymbol{f}_i \approx \boldsymbol{M}_L = -\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{E}^+$ on Γ. The better Love condition achieved by the zero-field enforcement method can be attributed to the stronger constraining of the system. Still, a better Love constraining does not imply a better reconstruction of the external fields, as the internal and the external problems are decoupled.

To evaluate the low-frequency behavior of (17), we fix the geometries Γ and Γ_m and we decrease the frequency to $f = 5 \cdot 10^{-20}\mathrm{Hz}$. The reader should note that, differently from the previous one, the importance of this test is a purely theoretical one. By stably reconstructing a quasi static setting, in fact, we show that the impact of our new technology encompasses low-frequency scenarios, that however will require specific measurement settings [35]. The application of this scheme to these scenarios, however, will be the topic of specific future investigations.

Moreover, as a right-hand side we use the fields scattered by perfect electric conductor (PEC) illuminated by a plane wave. The EFIO is used to evaluate the electric currents on a spherical surface Γ_s, concentric to Γ and with a radius of $1\,\mathrm{cm}$, discretized with a triangle mesh composed of 120 edges. The magnetic currents are here not considered as Γ_s is assumed to be a PEC object. Also in this case quasi-Helmoltz projectors are exploited to cure the low-frequency breakdown, resulting in

$$\mathbf{P}_k \mathbf{T} \mathbf{P}_k \mathbf{y} = \mathbf{P}_k \mathbf{e}^i \tag{22}$$

where $\mathbf{e}^i$ is the incident field obtained from a plane wave and tested on Γ_s using RWG basis functions. This equation can be solved by means of standard techniques [23], which include the extraction of the static contribution of the plane wave and the cancellation of $\mathbf{T}_h$ in solenoidal spaces. The solution of (22) can then be used to scatter the fields on Γ_m and on the previously test spheres, whose distance from Γ has not been changed with respect to the previous test. Finally, we employ these fields in an analogous way to what we did in Fig. 1 to study the reconstruction capabilities of the considered formulations also in this setting. As expected, the results in Fig. 3 show that our formulation is the only one still able to correctly reconstruct the field, as the low-frequency breakdown and the numerical cancellations are successfully handled. Similarly to the previous test, by studying the magnitude of the electric field on the plane $z = 0$ we can observe in Fig. 4 that our formulation is still able to enforce the Love condition.

VI. Conclusion

We have presented a new single current approach that naturally yields Love solutions of the inverse source problem and we have shown that the Love condition is satisfied. Although the presented strategy is currently considered for non-resonant settings, the extension to the resonant setting is possible and will be the focus of further investigations. The technique is enriched by the first frequency stabilization of the Steklov-Poincaré operator via quasi-Helmholtz projectors then used to stabilize the new formulation till arbitrary low frequency. This was then confirmed both by theoretical treatments and by numerical results.

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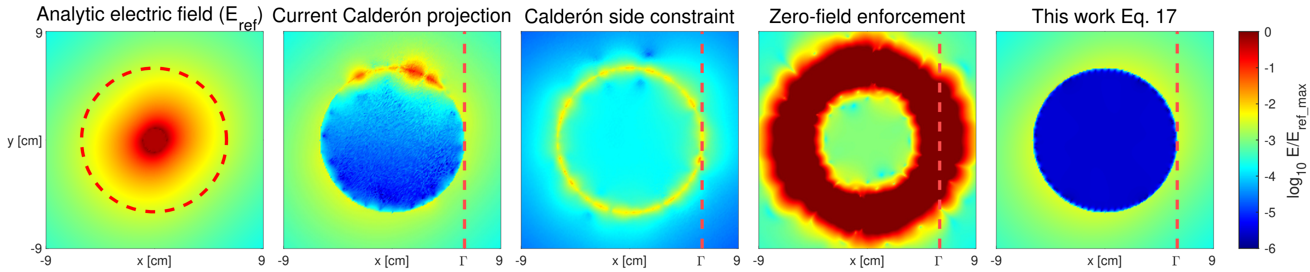

Fig. 4. Modulus of the electric field in a xy planar section of $\mathbb{R}^3$: fields are normalized on the maximum value of the reference field and are obtained from a $1\,\mathrm{cm}$-radius PEC sphere concentric to Γ, excited by a plane wave oscillating at $f = 5 \cdot 10^{-20}\mathrm{Hz}$.

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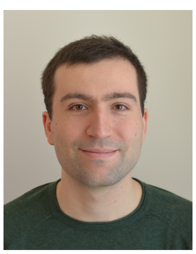

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