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ON CANONICAL RADIAL KÄHLER METRICS

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Abstract

We prove that a radial Kähler metric g is Kähler-Einstein if and only if one of the following conditions is satisfied: 1. g is extremal and it is associated to a Kähler-Ricci soliton; 2. two different generalized scalar curvatures of g are constant; 3. g is extremal (not cscK) and one of its generalized scalar curvature is constant.

Contents

1. Introduction

Given a complex manifold M (compact or not) it is an interesting and well-studied problem to see when M can be endowed with some canonical metric. Undoubtedly the most studied and important are the Kähler-Einstein (KE) metrics.

Other prominent examples that generalize KE metrics and have attracted the attention of many mathematicians are the following three types of Kähler metrics.

1. *Extremal metrics*. Introduced by Calabi [1], are those metrics such that the (1,0)-part of the Hamiltonian vector field associated to the scalar curvature is holomorphic. The reader is referred to [10] and references therein for more details. We denote by $\mathcal{E}xt(M)$ the set of extremal metrics on M.

2. *The metrics associated to a Kähler-Ricci soliton (KRS)*. A KRS on a complex manifold M is a pair (g, X) consisting of a Kähler metric g and a holomorphic vector field X, called the *solitonic vector field*, such that

$$\rho = \lambda\omega + L_X\omega \tag{1}$$

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for some $\lambda \in \mathbb{R}$, called the *solitonic constant*. Here ω and ρ are respectively the Kähler form and the Ricci form of the metric g and $L_X\omega$ denotes the Lie derivative of ω with respect to X. KRS are special solutions of the Kähler-Ricci flow and they generalize Kähler–Einstein (KE) metrics[1] . Indeed any KE metric g on a complex manifold M gives rise to a trivial KRS by choosing $X = 0$ or X Killing with respect to g. Obviously if the automorphism group of M is discrete then a Kähler–Ricci soliton (g, X) is nothing but a KE metric g. We denote by $\mathcal{KRS}(M)$ the set of Kähler metrics g on M such that (g, X) is a KRS, for some solitonic vector field X.

3. *The k-generalized constant scalar curvature metrics,* $1 \leq k \leq n$ *(where n is the complex dimension of n).* Let g be a Kähler metric. By definition, the *k-generalized scalar curvature*, $1 \leq k \leq n$, $\rho_k(g)$ of g are defined as (see [13]):

$$\frac{\det\left(g_{i\bar{j}} + s\,\mathrm{Ric}_{i\bar{j}}\right)}{\det(g_{i\bar{j}})} = 1 + \sum_{k=1}^{n} \rho_k(g) s^k. \tag{2}$$

Notice that $\rho_1(g) = \mathrm{scal}_g$, where scal_g is the scalar curvature of the metric g. Denote by $C_k(M)$ the set of Kähler metrics g on M such that $\rho_k(g)$ is a constant.

For any complex manifold M one clearly has the following inclusions:

$$\mathcal{KRS}(M) \supseteq \mathcal{KE}(M) \subseteq C_k(M),\ \ \mathcal{KE}(M) \subseteq C_1(M) \subseteq \mathcal{E}xt(M) \tag{3}$$

where $\mathcal{KE}(M)$ is the set of KE metrics on M.

It is then interesting to study the following:

Problem. *Find conditions which ensure that a canonical Kähler metric of the types above is KE.*

In this regard we recall some results when M is *compact*, summarized in the following theorem.

Theorem A. *Let M be a compact complex manifold M. Then the following facts hold true.*

(a) $C_k(M) \cap \mathcal{KRS}(M) \subseteq \mathcal{KE}(M)$, *for all* $k \geq 1$.

(b) *if* $g \in \mathcal{E}xt(M) \cap \mathcal{KRS}(M)$ *and assume that one of the two following conditions holds true:*

 (b1) (M, g) *is toric;*

 (b2) *the holomorphic sectional curvature of g does not change sign.*

 Then g is KE.

Proof. Let g be the Kähler metric associated to a KRS and ω its Kähler form. Notice that the solitonic vector field of a KRS on a compact complex manifold is gradient and hence ω is cohomologically Einstein. Hence (a) follows by the first Corollary in [5] when $\rho_1(g)$ is constant and when $\rho_k(g)$ is constant and different from zero, for $k \geq 1$. If $\rho_k(g) = 0$ for $k \geq 1$ then [13, Theorem 1] yields that $c_1(M) = 0$, i.e. the KRS is steady and hence g is forced to be KE by [2].

The proofs of (b1) and (b2) can be found in [4] and [3] respectively. □

[1]For more information on KRS see references in [7].

Remark 1. We do not know if the assumptions (b1) and (b2) can be dropped. Notice that for the proof of (b2) one needs to use only that the KRS is gradient (always true in the compact case) and the holomorphic sectional curvature does not change sign.

Remark 2. Notice that the inclusion $\mathcal{KE}(M) \subseteq C_k(M) \cap C_1(M)$ (and hence the inclusion $\mathcal{KE}(M) \subseteq C_k(M) \cap \mathcal{E}xt(M)$) for $k \geq 1$ is strict for a compact complex manifold M even if one assumes (b2) in Theorem A. Indeed the metric g given by the product of the flat metric and the Fubini-Study metric on $T^{n-k+1} \times \mathbb{C}P^{k-1}$ (where T^{n-k+1} is the complex torus and $\mathbb{C}P^{k-1}$ the complex projective space) has constant scalar curvature, $\rho_k(g) = 0$, it is not KE and its holomorphic sectional curvature is non-negative (cfr. the final Remark in [5]). In light of (b1) in Theorem A it could be interesting to see if the equality $C_k(M) \cap \mathcal{E}xt(M) = \mathcal{KE}(M)$ holds true in the compact toric case.

When the manifold involved is noncompact the previous problem has been studied by the first and third author of the present paper for Hartogs domains. More precisely in [12] it is shown that if the Kähler metric g naturally associated to an Hartogs domain $D \subset \mathbb{C}^n$ belongs to one of the three types described above then g is forced to be KE (and hence (D, g) is holomorphically isometric to an open subset of the complex hyperbolic n-space).

In this paper we restrict to radial metrics, namely those Kähler metrics g on (noncompact) complex manifolds which admit a global Kähler potential which depends only on the sum $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$ of the local coordinates' moduli.

If M is a complex manifold we denote by

$$\mathcal{R}ad(M) = \{\text{radial Kähler metrics on } M\}$$

The main result of the paper is the following theorem which shows in particular that in the noncompact radial case the same conclusion of Theorem A can be achieved without any assumption on the curvature of the metric.

Theorem 1.1. *Let M be a complex manifold. Then the following facts hold true.*

(i) $\mathcal{E}xt(M) \cap \mathcal{KRS}(M) \cap \mathcal{R}ad(M) = \mathcal{KE}(M) \cap \mathcal{R}ad(M)$;
(ii) $C_k(M) \cap C_h(M) \cap \mathcal{R}ad(M) = \mathcal{KE}(M) \cap \mathcal{R}ad(M), \forall h, k \geq 1, h \neq k$;
(iii) $C_k(M) \cap \mathcal{E}xt(M) \cap \mathcal{R}ad(M) = \mathcal{KE}(M) \cap \mathcal{R}ad(M), \forall k > 1$;
(iv) $C_k(M) \cap \mathcal{KRS}(M) \cap \mathcal{R}ad(M) = \mathcal{KE}(M) \cap \mathcal{R}ad(M), \forall k \geq 1$.

In the next section we collect some results on radial metrics and we prove Theorem 1.1. In the final section we provide some explicit examples and compare Theorem 1.1 with Theorem A.

2. Radial canonical Kähler metrics

Let g be a radial Kähler metric on a connected complex manifold M, equipped with complex coordinates $z_1, \ldots, z_n$ and let ω and ρ be respectively the Kähler form and the Ricci form associated to g. Then there exists a smooth function

$$f : (r_{\inf}, r_{\sup}) \to \mathbb{R},\ 0 \leq r_{\inf} < r_{\sup} \leq \infty,$$

where $(r_{\inf}, r_{\sup})$ is the maximal domain where $f(r)$ is defined such that

$$\omega = \frac{i}{2}\partial\bar{\partial}f(r),\ r = |z|^2 = |z_1|^2 + \cdots + |z_n|^2, \tag{4}$$

i.e. $f(r)$ is a radial potential for the metric g.

One can easily see that the matrix of the metric g and of the Ricci form ρ read as

$$\omega_{i\bar{j}} = f'(r)\delta_{ij} + f''(r)\bar{z}_i z_j, \tag{5}$$

$$\rho_{i\bar{j}} = L'(r)\delta_{ij} + L''(r)\bar{z}_i z_j, \tag{6}$$

where $L(r) = -\log(\det g)(r)$.

Set

$$y(r) := rf'(r) \tag{7}$$

and

$$\psi(r) := ry'(r). \tag{8}$$

Then

$$\psi(r) = \frac{dy}{dt},\ r = e^t. \tag{9}$$

The fact that g is a metric is equivalent to $y(r) > 0$ and $\psi(r) > 0$, $\forall r \in (r_{\inf}, r_{\sup})$. Then

$$\lim_{r\to r_{\inf}^+} y(r) = y_{\inf} \tag{10}$$

is a non negative real number. Similarly set

$$\lim_{r\to r_{\sup}^-} y(r) = y_{\sup} \in (0, +\infty]. \tag{11}$$

Therefore we can invert the map

$$(r_{\inf}, r_{\sup}) \to (y_{\inf}, y_{\sup}),\ r \mapsto y(r) = rf'(r)$$

on $(r_{\inf}, r_{\sup})$ and think r as a function of y, i.e. $r = r(y)$.

Hence we can set

$$\psi(y) := \psi(r(y)). \tag{12}$$

Finally, from (5), we easily get

$$(\det g_{i\bar{j}})(r) = \frac{(y(r))^{n-1}\psi(y(r))}{r^n}. \tag{13}$$

The following three propositions (Proposition 2.1, Proposition 2.2 and Proposition 2.3) are the key tools for the proof of Theorem 1.1 and provide us with the explicit expressions of radial extremal metrics, radial KRS and radial generalized cscK metrics respectively, in terms of the functions y and $\psi(y)$ defined by (7) and (8).

Proposition 2.1. *A radial Kähler metric g is extremal if and only if*

$$\psi(y) = y - \frac{A}{y^{n-1}} - \frac{B}{y^{n-2}} - Cy^2 - Dy^3. \tag{14}$$

for some $A, B, C, D \in \mathbb{R}$. Moreover,

(a) *if $n = 1$, g is KE (i.e. a complex space form) iff $D = 0$. Moreover, its (constant) scalar curvature is given by $2C$;*

(b) *if $n \geq 2$, g is KE iff $B = D = 0$ with Einstein constant $2C(n + 1)$. Moreover, the metric is flat iff $A = B = C = D = 0$.*

Proof. See [10, Lemma 2.1] for a proof. □

Remark 3. From (14) we easily deduce that if a Kähler-Einstein metric is defined at the origin $r = 0$ then it is a complex space form. Indeed, the metric is Einstein if and only if

$$\psi(y) = y - \frac{A}{y^{n-1}} - Cy^2,$$

which immediately implies that $A = 0$ if the metric is defined at the origin since in that case $y(r) = rf'(r) = 0$ and $\psi(r) = r(rf'(r))' = 0$ for $r = 0$.

Proposition 2.2. *Let g be a radial KRS with solitonic constant λ. Then the following facts hold true.*

If $n = 1$ then there exist $\mu, k \in \mathbb{R}$ such that

$$\dot{\psi}(y) = \mu\psi(y) + k + 1 - \lambda y \tag{15}$$

and if $\mu = 0$ then the soliton is trivial (i.e. a complex space form). If $\mu \neq 0$ then

$$\psi(y) = \nu e^{\mu y} + \frac{\lambda}{\mu} y + \left(\frac{\lambda}{\mu^2} - \frac{k+1}{\mu}\right) \tag{16}$$

and the soliton is trivial iff it is flat iff $\nu = 0$.

If $n \geq 2$ then there exists $\mu \in \mathbb{R}$ such that

$$\dot{\psi}(y) = \left(\mu - \frac{n-1}{y}\right)\psi(y) + n - \lambda y \tag{17}$$

and if $\mu = 0$ the soliton is trivial (i.e. KE). If $\mu \neq 0$ then

$$\psi(y) = \frac{\nu e^{\mu y}}{y^{n-1}} + \frac{\lambda}{\mu} y + \frac{\lambda - \mu}{\mu^{1+n}} \sum_{j=0}^{n-1} \frac{n!}{j!} \mu^j y^{j+1-n} \tag{18}$$

and the soliton is trivial iff it is flat iff $\nu = 0$ and $\mu = \lambda$.

Proof. See either [11, Proposition 2.2] or [6] for a proof. □

Proposition 2.3. *Let g be a radial Kähler metric and set*

$$\sigma(y) := \frac{1}{y^{n-1}} \frac{d}{dy}\left[y^{n-1}\psi(y)\right] = \dot{\psi}(y) + \frac{(n-1)\psi(y)}{y}. \tag{19}$$

Then its k-th generalized scalar curvature $\rho_k(g)$, $1 \leq k \leq n$, is constant, i.e. $\rho_k(g) = \rho_k$, if and only if

$$\sigma(y) = n - y\left(A_k + \frac{B_k}{y^n}\right)^{1/k}, \tag{20}$$

where $A_k = \rho_k \frac{k!(n-k)!}{n!}$ and B_k is constant (depending on k).

Moreover, g is KE with Einstein constant λ if and only if $\sigma(y) = n - \frac{\lambda}{2}y$.

Proof. Let g be radial with Kähler potential $f(r)$, where $r = |z_1|^2 + \ldots + |z_n|^2$. By (7) and (8) we immediately get $f'(r) = \frac{y(r)}{r}$ and $f''(r) = \frac{\psi(y(r))-y(r)}{r^2}$, which combined with (5) yields

$$g_{i\bar{j}}(r) = \frac{\psi(y(r)) - y(r)}{r^2}\bar{z}_i z_j + \frac{y(r)}{r}\delta_{ij}.$$

Also by (13) we have

$$L(r) = -\log(\det g)(r) = -(n-1)\log y(r) - \log\psi(y(r)) + n\log r$$

and then, by using $y'(r) = \frac{\psi(y(r))}{r}$ and (19),

$$L'(r) = -\frac{n-1}{y}\frac{\psi}{r} - \frac{\dot{\psi}(y)}{r} + \frac{n}{r} = \frac{n - \sigma(y(r))}{r}.$$

By (6), then finally one gets (cfr. also [8])

$$\mathrm{Ric}_{i\bar{j}}(r) = \frac{-\dot{\sigma}(y(r))\psi(y(r)) + \sigma(y(r)) - n}{r^2}\bar{z}_i z_j + \frac{n - \sigma(y(r))}{r}\delta_{ij}.$$

Then (2) reads as

(21)

$$1 + \sum_{k=1}^{n}\rho_k s^k = \frac{(\psi(y) - s\dot{\sigma}(y)\psi(y))(y + sn - s\sigma(y))^{n-1}}{\psi(y)y^{n-1}} = (1 - s\dot{\sigma}(y))\left(1 + s\frac{n-\sigma(y)}{y}\right)^{n-1}$$
$$= 1 + \sum_{k=1}^{n-1}\left(\frac{n-\sigma(y)}{y}\right)^{k-1}\left[\binom{n-1}{k}\frac{n-\sigma(y)}{y} - \binom{n-1}{k-1}\dot{\sigma}(y)\right]s^k - \dot{\sigma}(y)\left(\frac{n-\sigma(y)}{y}\right)^{n-1}s^n.$$

Then n-th generalized scalar curvature ρ_n is constant if and only if

$$-\dot{\sigma}(y)\,(n - \sigma(y))^{n-1} = \rho_n\, y^{n-1}$$

which integrates to

$$\sigma(y) = n - y\left(\rho_n + \frac{B_n}{y^n}\right)^{1/n},$$

i.e. (20) for $k = n$.

On the other hand the k-th generalized scalar curvature ρ_k, $1 \le k \le n-1$, is constant if and only if

(22) $$R_k y^k - (n-k)(n-\sigma(y))^k + ky\dot{\sigma}(y)(n-\sigma(y))^{k-1} = 0,$$

where $R_k = \rho_k\frac{k!(n-k)!}{(n-1)!}$.

If $R_k y^k - n(n-\sigma(y))^k = 0$ then

$$\sigma(y) = n - \left(\frac{R_k}{n}\right)^{1/k} y,$$

i.e. (20) with $B_k = 0$.

If $R_k y^k - n(n-\sigma(y))^k \neq 0$ then (22) gives

$$\frac{n-k}{y} + \frac{R_k k y^{k-1} + kn(n-\sigma(y))^{k-1}\dot{\sigma}(y)}{R_k y^k - n(n-\sigma(y))^k} = 0$$

which integrates to

$$\sigma(y) = n - \left(\frac{R_k}{n} y^k + B_k y^{k-n}\right)^{1/k}.$$

For the last assertion of the proposition we assume $n \geq 2$ (the case $n = 1$ is obtained similarly). We know by Proposition 2.1 that the metric is KE with Einstein constant λ if and only if $\psi(y) = y - \frac{\lambda}{2(n+1)} y^2 - \frac{A}{y^{n-1}}$. It is immediate to see that this is equivalent to $\frac{1}{y^{n-1}} \frac{d}{dy}\left[y^{n-1}\psi(y)\right] = n - \frac{\lambda}{2} y$, which by (19) proves the assertion. □

Remark 4. Equation (20) combined with (19), together with a choice of initial values $y_0 > 0$ and $\psi(y_0) > 0$, yield a Cauchy problem for $\psi(y)$ whose solution is a k-generalized cscK which is not cscK. For an explicit example, take $k = n$ and $A_n = 0$ in (20): then $\sigma(y) = c := n - (B_n)^{1/n}$ which by (19) yields $\psi(y) = \frac{c}{n} y + \frac{d}{y^{n-1}}$. Notice that if either $n \neq 1$ or $c \neq n$, i.e. $B_n \neq 0$, this is not an extremal metric. In particular, for $d = 0$, by $\psi(y) = \frac{dy}{dt}$ and by recalling that $r = e^t$ and $y(r) = r f'(r)$, one gets the potential $f(r) = \beta r^{c/n}$, for some $\beta \in \mathbb{R}$.

We are now in the position to prove Theorem 1.1.

Proof of Theorem 1.1. To show (i) let us assume that a radial metric is both extremal and KRS. Let us distinguish the cases $n = 1$ and $n \geq 2$.

If $n = 1$ the extremal condition (14) and its derivative read as

$$\psi(y) = (1 - B)y - A - Cy^2 - Dy^3, \tag{23}$$

$$\dot\psi(y) = (1 - B) - 2Cy - 3Dy^2. \tag{24}$$

By inserting (23) into the soliton equation (15) (for $n = 1$) we get

$$\dot\psi(y) = [\mu(1 - B) - \lambda]\, y - \mu C y^2 - \mu D y^3 - \mu A + k + 1,$$

which compared with (24) forces the coefficient of y^3 to vanish, i.e. $\mu D = 0$. If $D = 0$ or $\mu = 0$ the metric is KE respectively by Proposition 2.1 and Proposition 2.2. Let us now assume $n \geq 2$. By inserting equation for extremal metrics (14) into the soliton equation (17) we obtain

$$\dot\psi(y) = 1 + [C(n-1) + \mu - \lambda]\, y + [D(n-1) - C\mu]\, y^2 - D\mu y^3 + \frac{A(n-1)}{y^n} + \frac{B(n-1) - \mu A}{y^{n-1}} - \frac{B\mu}{y^{n-2}}.$$

On the other hand, derivating (14) we get

$$\dot\psi(y) = 1 - \frac{A(1-n)}{y^n} - \frac{B(2-n)}{y^{n-1}} - 2Cy - 3Dy^2.$$

Comparing these two last expressions and observing that in the first one there are the terms in y^3 and $\frac{1}{y^{n-2}}$ which are not in the second one, one finds either $\mu = 0$ and then the soliton is trivial by Proposition 2.2, or $B = D = 0$, which by Proposition 2.1, again implies that the metric is KE. Hence (i) is proved.

In order to prove (ii), assume that the generalized curvatures $\rho_k(g)$ and $\rho_h(g)$ are constant for some $h, k \geq 1$, $h \neq k$. By (20) in Proposition 2.3, we must have

$$\left(A_k + \frac{B_k}{y^n}\right)^{1/k} = \left(A_h + \frac{B_h}{y^n}\right)^{1/h}$$

which clearly implies that $B_k = B_h = 0$ and $(A_k)^{1/k} = (A_h)^{1/h} = A$.

Then, $\sigma(y) = n - Ay$ and the metric is KE by the last assertion of Proposition 2.3.

We now prove (iii). If a radial Kähler metric g is extremal then by combining (14) and (19) one gets:

$$\sigma(y) = n - \frac{B}{y^{n-1}} - C(n+1)y - D(n+2)y^2. \tag{25}$$

Assume that the k-th generalized scalar curvature $\rho_k(g)$ (with $k > 1$) is constant: by Proposition 2.3 and by comparing (20) with (25) we see that $\left(A_k + \frac{B_k}{y^n}\right)^{1/k}$ must be a rational function. This is possible only if either $B_k = 0$ (and hence the metric is KE by Proposition 2.3) or $A_k = 0$ and $\frac{n}{k} \in \mathbb{Z}$. In the latter $\sigma(y) = n - \frac{(B_k)^{1/k}}{y^{\frac{n}{k}-1}}$ which compared with (25) and recalling that $k > 1$ yields again $B_k = 0$.

Finally we prove (iv). By the equations (16) and (18) of a radial non trivial KRS one easily gets that (19) reads as

$$\sigma(y) = \frac{\mu\nu e^{\mu y}}{y^{n-1}} + n\frac{\lambda}{\mu} + \frac{\lambda - \mu}{\mu^{1+n}} \sum_{j=1}^{n-1} \frac{n!}{(j-1)!}\mu^j y^{n-j}. \tag{26}$$

By comparing the previous equation with (20), we easily get that if a radial non trivial KRS has constant k-th generalized scalar curvature (with $1 \leq k \leq n$), then $\nu = 0$ and $\lambda = \mu$, which by the last assertion of Proposition 2.2 means that g is KE (actually Ricci flat), yielding the desired contradiction and proving (iv). □

3. Some final remarks

The assertion (i) in Theorem 1.1 should be compared with (b2) of Theorem A in the introduction. Hence it is worth to exhibit radial extremal metrics and non trivial radial KRS with sign-changing holomorphic sectional curvature. This is done in the following two examples. We first recall that in [9] we have shown that, given a radial metric, in the point $p = (z_1, 0, \ldots, 0)$ the only non vanishing components of the Riemann tensor $R_{i\bar{j}k\bar{l}}$ are

$$R_{1\bar{1}1\bar{1}} = \frac{\ddot{\psi}(y)\psi^2(y)}{r^2},$$

$$R_{1\bar{1}i\bar{i}} = \frac{\dot{\psi}(y)y - \psi(y)}{yr^2},$$

$$R_{i\bar{i}i\bar{i}} = 2R_{i\bar{i}j\bar{j}} = 2\frac{\psi(y) - y}{r^2}.$$

Then, in p, the holomorphic sectional curvature along $Z = \sum_k \xi_k \frac{\partial}{\partial z_k}$ is

$$R(Z, \bar{Z}, Z, \bar{Z}) = \frac{\ddot{\psi}(y)\psi^2(y)}{r^2}|\xi_1|^4 + \frac{\dot{\psi}(y)y - \psi(y)}{yr^2}|\xi_1|^2 \sum |\xi_i|^2$$

$$+\frac{\psi(y)-y}{r^2}\sum|\xi_i|^2|\xi_j|^2+2\frac{\psi(y)-y}{r^2}\sum|\xi_i|^4.$$

If we assume that $\xi_2=\cdots=\xi_n=0$ this formula (always in p) reduces to

$$R(Z,\bar{Z},Z,\bar{Z})=\frac{\ddot{\psi}(y)\psi^2(y)}{r^2}|\xi_1|^4. \tag{27}$$

Thus to find radial extremal metrics or radial KRS with sign-changing holomorphic sectional curvature, it will be enough to find metrics for which $\ddot{\psi}$ changes sign in its domain of definition.

Example 1. Take the radial extremal metric in dimension $n\geq 2$ with $A=B=0$, $C=1$, $D=-1$ in (14), i.e.

$$\psi(y)=y-y^2+y^3.$$

Since $\ddot{\psi}(y)=-2+6y$ we have that $\ddot{\psi}(y)$ changes sign in a neighbourhood of $y=\frac{1}{3}$; moreover, being $\psi(\frac{1}{3})>0$ the local solution $y(t)$ of the Cauchy problem $\frac{dy}{dt}=\psi(y(t))$, $y(t_0)=\frac{1}{3}$, for any $t_0\in\mathbb{R}$, satisfies the conditions $y>0$ and $\psi(y)>0$ to represent a metric and then ψ defines an extremal metric which, by (27), has sign-changing holomorphic sectional curvature.

Example 2. In order to find a nontrivial radial KRS with sign-changing holomorphic sectional curvature, take for example

$$n=3,\ \ \nu=0,\ \ \mu<0,\ \ \lambda<-\frac{5}{4}\mu,\ \lambda\neq\mu$$

in (18), i.e.

$$\psi(y)=\frac{\lambda}{\mu}y+\frac{\lambda-\mu}{\mu^4}\left(\frac{6}{y^2}+\frac{6\mu}{y}+3\mu^2\right). \tag{28}$$

One gets

$$\ddot{\psi}(y)=\frac{12(\lambda-\mu)}{\mu^4y^4}(3+\mu y) \tag{29}$$

and then $\ddot{\psi}(y)$ changes sign in a neighbourhood of $y=-\frac{3}{\mu}$, which is positive by the assumptions. Moreover, one finds

$$\psi\left(-\frac{3}{\mu}\right)=\frac{-4\lambda-5\mu}{3\mu^2}.$$

which is positive by the assumptions. Then, we conclude as in the previous example that ψ yields a non-trivial Ricci soliton which, by (27), has sign-changing holomorphic sectional curvature (the non-triviality is guaranteed by $\lambda\neq\mu$).

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