Addendum to “A formulation of volumetric growth as a mechanical problem subjected to non-holonomic and rheonomic constraint”

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Abstract

In this work, we critically review some aspects of our own article *“A formulation of volumetric growth as a mechanical problem subjected to non-holonomic and rheonomic constraint”* [Mathematics and Mechanics of Solids, DOI: 10.1177/10812865231152228], which has been recently published in this Journal. The reason for undertaking this critique is that, after exploring some fundamental literature, which was not included in our original paper, we have noticed that, if the “canonical doctrine” on non-holonomic and rheonomic constraints is followed, some of our conclusions should be partially rephrased, and the procedure adopted to obtain them can be shortened. In fact, some of the main results of our article, although remaining unaltered, can be retrieved in a more straightforward way, while some other results should be reconsidered, and some statements should be corrected. On the basis of these considerations, the scope of this work is to present the necessary amendments to our previous paper, and to recast the core messages of our article, which remain valid, in an alternative form that is more concise and consistent with the standard theory of non-holonomic constraints.

Keywords
Growth mechanics; Bilby-Kröner-Lee multiplicative decomposition; Non-holonomic constraints; Virtual displacements; Lagrange multipliers; Principle of Virtual Work; Dissipation; Cahn-Hilliard model.

1 Introduction

After a further inspection into the fundamental literature on Analytical Mechanics (see e.g. [9, 4]), we have reached the conclusion that some of the results presented in our recent article *“A formulation of volumetric growth as a mechanical problem subjected to non-holonomic and rheonomic constraint”* [5], although remaining valid, can/should be re-obtained by rephrasing them consistently with the classical approach to non-holonomic and rheonomic constraints, and with the classical definition of virtual displacements, contextualized to systems subjected to such constraints [9, 4, 1, 3, 2].

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1.1 Backstory

As for the original article [5], the point of departure of this work is the fact that, in several biomechanical problems dealing with the mechanics of volumetric growth, the mass balance law of a growing body can be put in the form of a *non-holonomic and rheonomic constraint*[1] on the so-called *growth tensor* $\boldsymbol{K}$, i.e.,

$$\check{\mathcal{C}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, \dot{\boldsymbol{K}}, \mathcal{X}, \mathcal{T}) := \boldsymbol{K}^{-\mathrm{T}} : \dot{\boldsymbol{K}} - \check{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \mathcal{X}, \mathcal{T}) = 0, \qquad \text{in } \mathscr{B} \times \mathscr{I} \tag{1}$$

(see Equation (8) of [5]). Here and in the sequel, the notation is the same as in [5]. However, we recall that: $\mathscr{B}$ is the reference placement of the body under consideration; $\mathscr{I}$ is the time line; $\boldsymbol{F}$ is the deformation gradient tensor; the auxiliary maps $\mathcal{X} : \mathscr{B} \times \mathscr{I} \to \mathscr{B}$ and $\mathcal{T} : \mathscr{B} \times \mathscr{I} \to \mathscr{I}$ are defined by $\mathcal{X}(X, t) = X$ and $\mathcal{T}(X, t) = t$, for each pair $(X, t) \in \mathscr{B} \times \mathscr{I}$; $R_{\gamma(\mathrm{ph})} \equiv \check{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \mathcal{X}, \mathcal{T})$ is referred to as *"growth law"* [5] and is assumed to be given from the outset, e.g. phenomenologically, in such a way that it can be expressed as a function of $\boldsymbol{F}$ and $\boldsymbol{K}$, as well as of $\mathcal{X}$ and $\mathcal{T}$, in order to account for its explicit dependence on the points of $\mathscr{B}$ and time. The growth tensor is introduced through the Bilby-Kröner-Lee (BKL) decomposition of $\boldsymbol{F}$, i.e., $\boldsymbol{F} = \boldsymbol{F}_{\mathrm{e}}\boldsymbol{K}$, into an elastic and a growth part, represented by the tensors $\boldsymbol{F}_{\mathrm{e}}$ and $\boldsymbol{K}$, respectively.

In [5], we have studied the constraint (1) by following an approach that we have developed by taking inspiration from a paper by Nadile [8], and from a consideration on non-holonomic and rheonomic constraints given in [6]. In particular, Lanczos [6] writes:

> *"Non-holonomic auxiliary conditions which are rheonomic [...] require particular care. Here it is necessary to know what conditions exist between the δq_k if the variation is not performed instantaneously but during the infinitesimal time δt. The auxiliary conditions now take the form $A_{i1}\delta q_1 + \ldots + A_{in}\delta q_n + B_i \delta t = 0$ [...]"* (see page 66 of [6])

although, few lines afterwards the text quoted above, he adds that the virtual displacements are taken "without *varying the time"* [6]. Accordingly, δt should be set equal to zero, and only the sum $\sum_{k=1}^{n} A_{ik}\delta q_k$ contributes to the equations of motion of the considered system. On the other hand, Nadile's study [8] treats time as a fictitious, additional Lagrangian parameter of his theory, and determines an Euler-Lagrange equation associated with it that is similar to Equation (29e) or (69b) of [5].

In the sequel, however, we show how the crucial point of our previous work [5] can be obtained also without considering time as a fictitious Lagrangian parameter and, for this purpose, we adhere to the classical formulation of non-holonomic and rheonomic constraints [9, 4, 1, 3, 2]. Therefore, the term $B_i\delta t$ reported above, which reads $[\check{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \mathcal{X}, \mathcal{T})]\delta\mathcal{T}$ in our framework [5], disappears from the formulation of the Principle of Virtual Work (PVW). Indeed, consistently with the standard definition of virtual displacements [9], given also for the constraints of the type shown in Equation (1), the virtual displacement $\delta\boldsymbol{K}$ is accompanied by the condition $\delta\mathcal{T} = 0$.

1.2 Main changes with respect to the original article [5]

In the remainder of this work, we review the most important results of [5], and we reformulate them in an alternative and more straightforward manner in light of the "canonical doctrine" on non-holonomic and rheonomic constraints [9, 4, 1, 2, 3], which *does not* require viewing time as

[1] Here, we are adopting the terminology of [6], in the same manner as we did in our work [5].

a *"fictitious Lagrangian parameter"* [5]. Moreover, to facilitate the comparison with the original article [5], we highlight the sentence(s) and/or mathematical expression(s) that must/can be rephrased, and we specify the section(s) of [5] in which they are to be found. Note that, in the blocks of reformulated text reported below, the references that feature in quotation marks refer to [5].

2 Review of Abstract and Sections 1, 2, 3, and 4 of [5]

If time is not viewed as a fictitious, additional Lagrangian parameter, then the following remarks apply:

1. At the fifth line of the abstract of [5], the sentence

   > *"For our purposes, [...] unitary."*

   should be rephrased as

   > "For our purposes, we put the constraint in Pfaffian form."

   Moreover, at the ninth line of the Abstract, the wording *"Lagrange multipliers"* should be replaced with "Lagrange multiplier".

2. The core messages reported in Section 1 (Introduction) of [5] remain identical to the ones announced in the original article.

3. If the standard approach to the study of non-holonomic and rheonomic constraints is followed (see e.g. [9, 4]), Sections 2 and 3 of [5] remain unaltered, while Section 4 of [5] is no longer necessary to obtain the boundary value (sub-)problem expressed by Equations (29a)–(29d) and (29g) of [5], which is indeed one of the crucial points of our work. Accordingly, in the sequel we introduce neither the fictitious Lagrangian parameter $\mathfrak{T}$, nor the virtual variations $\delta\mathfrak{T}$ and $\delta\mathcal{T}$, and we study the constraint on $\boldsymbol{K}$ either as assigned in Equation (1) or in the form

$$\hat{\mathcal{C}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, \dot{\boldsymbol{K}}, \omega) := \boldsymbol{K}^{-\mathrm{T}} : \dot{\boldsymbol{K}} - \hat{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega) = 0, \qquad \text{in } \mathscr{B} \times \mathscr{I}. \tag{2}$$

   Equation (2) is privileged if the dependence of the phenomenological growth law $\hat{R}_{\gamma(\mathrm{ph})}$ on the mass fraction of the nutrients is highlighted (see the discussion on this topic reported in [5]).

   Note also that Appendix A2 of [5] fits in the context developed therein, but it is not necessary in the present framework.

4. Since the constraints (1) and (2) are affine in the generalized velocity $\dot{\boldsymbol{K}}$, they comply with Chetaev's conditions (see [4, 3, 2, 7]), which, in our case, read

$$\left[\frac{\partial \check{\mathcal{C}}_{\boldsymbol{K}}}{\partial \dot{\boldsymbol{K}}} \circ (\boldsymbol{F}, \boldsymbol{K}, \dot{\boldsymbol{K}}, \mathcal{X}, \mathcal{T})\right] : \delta\boldsymbol{K} = 0 \qquad \Rightarrow \qquad \boldsymbol{K}^{-\mathrm{T}} : \delta\boldsymbol{K} = 0, \tag{3a}$$

$$\left[\frac{\partial \hat{\mathcal{C}}_{\boldsymbol{K}}}{\partial \dot{\boldsymbol{K}}} \circ (\boldsymbol{F}, \boldsymbol{K}, \dot{\boldsymbol{K}}, \omega)\right] : \delta\boldsymbol{K} = 0 \qquad \Rightarrow \qquad \boldsymbol{K}^{-\mathrm{T}} : \delta\boldsymbol{K} = 0, \tag{3b}$$

where the tensor $\delta\boldsymbol{K}$ is the generalized virtual displacement associated with $\boldsymbol{K}$. In fact, with respect to the framework presented in [5], Equations (3a) and (3b) replace Equations (19) and (20b) of [5], and constitute the forms of the restriction that the components of $\delta\boldsymbol{K}$ must fulfill in order for $\delta\boldsymbol{K}$ to be compatible with the associated constraint consistently with the "canonical doctrine" on non-holonomic and rheonomic constraints [9, 4]. Here, thus, in accordance with Pars [9] and Gantmacher [4], $\delta\boldsymbol{K}$ is taken at *fixed time*, i.e., with $\delta\mathcal{T} = 0$ (see, in this respect, also the way in which Gantmacher [4], at pages 13 and 14 of his book, introduces the virtual displacements, and the motivation he gives for regarding them as the *"displacements in the case of 'frozen' constraints"* [4]).

In particular, by employing Equation (3a) or (3b) in the constrained version of the PVW, the *"technical difficulties"* mentioned at the beginning of Section 4 of [5] disappear. We also remark that the form of Chetaev's condition given in Equation (3a), or (3b), substitutes, in the present framework, the one supplied in the footnote 5 of [5].

In addition, it is worth to clarify the following points pertaining to the Introduction of [5]:

- In the Introduction (Section 1) of [5], the sentence six lines after the beginning of the section

  > *"However, [...] Kozlov [12–15]."*

  is incomplete, since some fundamental literature on these constraints was not cited. Therefore, by including some references on the topic, the sentence quoted above should read

  > "However, [...], the formulation of the PVW becomes less obvious when the considered constraints are non-holonomic *and* rheonomic, although [...] due to Kozlov [12–15]", and although there does exist classical literature on the topic (see e.g. [9, 4, 1, 3, 2]).

- Although in our opinion the conceptual novelty of [5] is preserved, the sentence in the second paragraph, third page of Section 1, i.e.,

  > *"Compared [...] Lagrange multiplier technique."*

  could be made clearer by reformulating it as follows:

  > "Compared with the formulation summarized above, [...] the approach that we are proposing is novel because it [...] provides a *constrained version* of the PVW, relying on the Lagrange multiplier technique", which may lead to deeper insights on the mechanics of inelastic processes, such as growth, remodeling, and aging. The principal advantage of our point of view is that it grants the ability to take into account a priori both experimentally observable growth laws and growth-conjugated generalized forces that could resolve other possible biological features specifically associated with growth itself.

- Moreover, the sentence four lines after the one quoted previously

  > *"To the best of our knowledge, [...] in completely different frameworks."*

  should be clarified as follows

> "To the best of our knowledge, this procedure is not standard for the case of non-holonomic and rheonomic constraints", although it applies also to systems subjected to such constraints with some clarifications about the way in which virtual displacements comply with the given constraints [3, 2]. In our work, we propose an extension of the standard procedure, thereby generalizing some results put forward by Nadile [8] in a completely different framework (in fact, for discrete systems) to the context of Continuum Mechanics.

3 Review of Section 5 of [5]

With respect to Section 5 of [5], we discuss the following modifications, which apply if time is not viewed as a fictitious, additional Lagrangian parameter.

1. Consistently with the present framework, the sentence starting three lines after the beginning of Section 5:

   > *"First, [...] as follows:"*

   and ending with Equation (23) of [5] should be reformulated as (from here on, for the sake of readability, in the sentences taken from [5] that feature in the blocks of reformulated text, we do not report the mathematical symbols and words that are related to viewing time as a fictitious Lagrangian parameter):

   > "First, we recall that the kinematic descriptors of the present theory, which is of grade one in χ, and of grade zero in $\boldsymbol{K}$ [54], are given as follows:
   >
   > $$(\chi, \boldsymbol{F}, \boldsymbol{K}, \delta\chi, \mathrm{Grad}\delta\chi, \delta\boldsymbol{K})." \tag{4}$$

   This amounts to avoiding the introduction of $\delta\mathfrak{T}$ and $\delta\mathcal{T}$, or, equivalently, to setting $\delta\mathfrak{T} = \delta\mathcal{T} = 0$, which means that the virtual displacements $\delta\chi$ and $\delta\boldsymbol{K}$ are taken here at fixed time.

2. The duality pairings in Equations (24a) and (24b) should be re-considered in light of the fact that the Lagrange multiplier $\mu_{\mathfrak{T}}$ and its virtual variation $\delta\mu_{\mathfrak{T}}$ are not present in the "classical doctrine". Accordingly, the sentence immediately after Equation (23) of [5]

   > *"Then, [...] duality:"*

   should be reformulated as (the emphasized text highlights the modifications of the original text)

   > "Then, since we are going to append the *constraint*, both in the rescaled *form*" $t_{\rm c}[\hat{\mathcal{C}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, \dot{\boldsymbol{K}}, \omega)]$ and in the Chetaev form $\boldsymbol{K}^{-\mathrm{T}} : \delta\boldsymbol{K} = 0$ [7], "to the expression of the PVW that one would have in the absence of constraints, we introduce the Lagrange *multiplier* $\mu_{\boldsymbol{K}}$, along with *its* virtual *variation* $\delta\mu_{\boldsymbol{K}}$, so that the following duality pairings apply:"
   >
   > $$\mu_{\boldsymbol{K}} \div \boldsymbol{K}^{-\mathrm{T}} : \delta\boldsymbol{K}, \tag{5a}$$
   >
   > $$\delta\mu_{\boldsymbol{K}} \div t_{\rm c}[\hat{\mathcal{C}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, \dot{\boldsymbol{K}}, \omega)], \tag{5b}$$
   >
   > "where the symbol "$\div$" indicates the conjugation induced by duality."

3. Within the "classical doctrine", the generalized forces $\mathcal{Y}_\mathrm{u}$ and $\mathcal{Z}$ are not introduced, so that Equation (25) of [5] becomes

$$\boldsymbol{P} \div \mathrm{Grad}\delta\chi, \quad \boldsymbol{Y}_\mathrm{u} \div \boldsymbol{K}^{-1}\delta\boldsymbol{K}, \tag{6}$$

and the text two lines after Equation (25)

> *"; and $\mathcal{Y}_\mathrm{u}$ [...] $\delta\mathfrak{T}$."*

is no longer necessary. Moreover, the sentence three lines after Equation (25)

> *"The subscript [...], respectively."*

should be reformulated as (the emphasized text highlights the modifications of the original text)

> "The subscript "u" in $\boldsymbol{Y}_\mathrm{u}$ indicates that *this force is* "unconstrained", in the sense that, because of the presence of the Lagrange *multiplier* $\mu_{\boldsymbol{K}}$, *it is* associated with arbitrary (and, thus, "unconstrained") variations $\delta\boldsymbol{K}$."

Finally, Equation (26) becomes

$$\boldsymbol{f}, \boldsymbol{\tau} \div \delta\chi, \quad \boldsymbol{Z} \div \boldsymbol{K}^{-1}\delta\boldsymbol{K}, \tag{7}$$

while the sentence two lines after Equation (26)

> *"[...] from here on, $\boldsymbol{Z}$ and $\mathcal{Z}$ [...] respectively."*

should be reformulated as

> "[...] from here on, $\boldsymbol{Z}$ *is* said to be *external growth-conjugated stress-like force.*"

4. Remark 4 is no longer necessary.

5. In light of the comments above, the constrained expressions of the Principle of Virtual Work (PVW) reported in Equations (27) and (28) of [5] become

$$\begin{aligned}
&\int_{\mathscr{B}} \boldsymbol{P} : \mathrm{Grad}\delta\chi \;+\; \int_{\mathscr{B}} \boldsymbol{Y}_\mathrm{u} : \boldsymbol{K}^{-1}\delta\boldsymbol{K} \;+\; \int_{\mathscr{B}} \mu_{\boldsymbol{K}}[\boldsymbol{K}^{-\mathrm{T}} : \delta\boldsymbol{K}] \;+\; \int_{\mathscr{B}} t_\mathrm{c}\delta\mu_{\boldsymbol{K}}[\hat{\mathcal{C}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, \dot{\boldsymbol{K}}, \omega)] \\
&= \int_{\mathscr{B}} \boldsymbol{f}\delta\chi \;+\; \int_{\partial^\chi_\mathrm{N}\mathscr{B}} \boldsymbol{\tau}\delta\chi \;+\; \int_{\mathscr{B}} \boldsymbol{Z} : \boldsymbol{K}^{-1}\delta\boldsymbol{K},
\end{aligned} \tag{8a}$$

$$\begin{aligned}
&\int_{\partial^\chi_\mathrm{N}\mathscr{B}} \{\boldsymbol{\tau} - \boldsymbol{P}\boldsymbol{N}\}\delta\chi \;+\; \int_{\mathscr{B}} \{\mathrm{Div}\boldsymbol{P} + \boldsymbol{f}\}\delta\chi + \int_{\mathscr{B}} \left\{\boldsymbol{Z} - \mu_{\boldsymbol{K}}\boldsymbol{I}^\mathrm{T} - \boldsymbol{Y}_\mathrm{u}\right\} : \boldsymbol{K}^{-1}\delta\boldsymbol{K} \\
&- \int_{\mathscr{B}} t_\mathrm{c}\delta\mu_{\boldsymbol{K}}[\hat{\mathcal{C}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, \dot{\boldsymbol{K}}, \omega)] \;= 0.
\end{aligned} \tag{8b}$$

Clearly, since neither the generalized forces dual to $\delta\mathfrak{T}$, i.e., $\mathcal{Y}_\mathrm{u}$ and $\mathcal{Z}$, nor the Lagrange multiplier $\mu_{\mathfrak{T}}$ are introduced in the present framework, the boundary value problem (BVP) in Equations (29a)–(29h) of [5] reduces to

$$
\begin{aligned}
&\mathrm{Div}\boldsymbol{P} + \boldsymbol{f} = \boldsymbol{0}, && \text{in } \mathscr{B}, && \text{(9a)}\\
&\chi = \chi_{\mathrm{b}}, && \text{on } \partial^{\chi}_{\mathrm{D}}\mathscr{B}, && \text{(9b)}\\
&\boldsymbol{P}\boldsymbol{N} = \boldsymbol{\tau}, && \text{on } \partial^{\chi}_{\mathrm{N}}\mathscr{B}, && \text{(9c)}\\
&\left(\boldsymbol{Y}_{\mathrm{u}} + \mu_{\boldsymbol{K}}\boldsymbol{I}^{\mathrm{T}}\right) - \boldsymbol{Z} = \boldsymbol{0}, && \text{in } \mathscr{B}, && \text{(9d)}\\
&\hat{\mathcal{C}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, \dot{\boldsymbol{K}}, \omega) = 0, && \text{in } \mathscr{B}. && \text{(9e)}
\end{aligned}
$$

Note that the comments on Equations (29a) and (29d) of [5], which are identical to Equations (9a) and (9d), apply to the latter equations. Moreover, Equation (9e) is consistent with Equation (29g), while Equations (29e), (29f), and (29h) of [5] disappear from the present framework. We also remark that, in the current context, Equations (9a), (9d), and (9e) constitute a set of 13 scalar equations in the 13 scalar unknowns identified with the components of χ and $\boldsymbol{K}$, and with $\mu_{\boldsymbol{K}}$, respectively.

We emphasize that Equations (9a)–(9e) are identical to those of the original article [5] (see Equations (29a)–(29d) and (29g)), and, thus, the core message contained in them remains unchanged.

Analogously to the original article [5], the BVP (9a)–(9e) admits the equivalent formulation

$$
\begin{aligned}
&\mathrm{Div}\boldsymbol{P} + \boldsymbol{f} = \boldsymbol{0}, && \text{in } \mathscr{B}, && \text{(10a)}\\
&\chi = \chi_{\mathrm{b}}, && \text{on } \partial^{\chi}_{\mathrm{D}}\mathscr{B}, && \text{(10b)}\\
&\boldsymbol{P}\boldsymbol{N} = \boldsymbol{\tau}, && \text{on } \partial^{\chi}_{\mathrm{N}}\mathscr{B}, && \text{(10c)}\\
&\mathrm{dev}\boldsymbol{Y}_{\mathrm{u}} = \mathrm{dev}\boldsymbol{Z}, && \text{in } \mathscr{B}, && \text{(10d)}\\
&\boldsymbol{K}^{-\mathrm{T}} : \dot{\boldsymbol{K}} = \hat{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega), && \text{in } \mathscr{B}, && \text{(10e)}
\end{aligned}
$$

in which Equation (9d) is replaced by its deviatoric part, and the constraint (9e) is written explicitly, while $\mu_{\boldsymbol{K}}$ is computed *a posteriori* as

$$
\mu_{\boldsymbol{K}} = \tfrac{1}{3}\mathrm{tr}\boldsymbol{Z} - \tfrac{1}{3}\mathrm{tr}\boldsymbol{Y}_{\mathrm{u}}, \qquad \text{in } \mathscr{B}. \qquad (11)
$$

Again, it is important to remark that Equations (10a)–(10e) and (11) remain unchanged with respect to [5], and, indeed, correspond to Equations (34a)–(34e) and (33c) of [5], respectively.

6. In the present setting, the comments in the last two paragraphs of Section 5.1, and Equations (30a), (30b), and (31) do not come into play.

4 Review of Section 6 of [5]

The study of the dissipation inequality and of the constitutive laws as well as the considerations on the *"final form of the IBVP"* (initial and boundary value problem) (42a)–(42k), reported in Section 6 of [5], are unaffected by the present reformulation, with the exception of the preliminary discussion on the forces $\mathcal{Y}_{\mathrm{u}}$ and $\mathcal{Z}$, and the results presented in Equations (44b) and (45), which are no longer necessary. Moreover, the content of Subsection 6.4 of [5] remains unchanged.

5 Review of Section 7 of [5]

The content and the core message of Section 7 of [5] remain essentially unchanged within the present framework. However, since it holds that $\delta\mathcal{T} = 0$, the extension of the PVW expressed in Equation (8a) to the context of a theory of grade one in $J_{\boldsymbol{K}} := \det \boldsymbol{K}$, as is the case for the Cahn-Hilliard model, reads

$$\begin{aligned}&\int_{\mathscr{B}} \boldsymbol{P} : \mathrm{Grad}\delta\chi \; + \; \int_{\mathscr{B}} q_{\mathrm{u}} \frac{\delta J_{\boldsymbol{K}}}{J_{\boldsymbol{K}}} \; + \; \int_{\mathscr{B}} \mu_{\boldsymbol{K}} \frac{\delta J_{\boldsymbol{K}}}{J_{\boldsymbol{K}}} \; + \; \int_{\mathscr{B}} t_{\mathrm{c}} \delta\mu_{\boldsymbol{K}} \left\{ \frac{\dot{J}_{\boldsymbol{K}}}{J_{\boldsymbol{K}}} + \left[\frac{1}{J_{\boldsymbol{K}}} \mathrm{Div} \mathfrak{v} - R_{\gamma(\mathrm{ph})} \right] \right\} \\ &+ \int_{\mathscr{B}} \mathfrak{f} \frac{\mathrm{Grad}\, \delta J_{\boldsymbol{K}}}{J_{\boldsymbol{K}}} = \int_{\mathscr{B}} \boldsymbol{f} \delta\chi \; + \; \int_{\partial^{\chi}_{\mathrm{N}}\mathscr{B}} \boldsymbol{\tau} \delta\chi \; + \; \int_{\mathscr{B}} z \frac{\delta J_{\boldsymbol{K}}}{J_{\boldsymbol{K}}},\end{aligned} \tag{12}$$

with $R_{\gamma(\mathrm{ph})} \equiv \hat{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega)$, $\boldsymbol{K} = J_{\boldsymbol{K}}^{1/3} \boldsymbol{I}$, $\boldsymbol{I}$ identity tensor, and $z := \frac{1}{3}\mathrm{tr}\boldsymbol{Z}$. Moreover, $\mu_{\mathfrak{T}}$ does not appear in the present formulation.

6 Review of Section 8 of [5]

The comments summarized in Section 8 of [5] remain unchanged with respect to the original article, even though, in the present setting, the *"'constrained version' of the PVW"* [5] must be understood as in Equation (8a) above, i.e., with the virtual displacements $\delta\boldsymbol{K}$ taken at fixed time, i.e., with $\delta\mathfrak{T} = \delta\mathcal{T} = 0$. Moreover, Subsection 8.1 only requires to set $\delta\mathfrak{T} = \delta\mathcal{T} = 0$, and to disregard $\mathfrak{T}$, $\mathcal{Y}_{\mathrm{u}}$ and $\mu_{\mathfrak{T}}$, while Subsections 8.2 and 8.3 need no changes.

7 Review of Appendix A1 of [5]

Appendix A1 of [5], as it stands, suggests that the formulation that we have proposed therein is necessary for contextualizing the study of the non-holonomic and rheonomic constraint considered in our work to the framework developed in it. However, since this *is not* the case, we review here Appendix A1 accordingly, and we highlight below the sentences and equations that should be amended.

For our purposes, let us consider a mechanical system subjected to $m \in \mathbb{N}$ non-holonomic and rheonomic constraints[2], given by (see Equation (65) of [5])

$$\check{\mathcal{C}}^i(q(t), \dot{q}(t), t) := \sum_{k=1}^{n} [a^i{}_k(q(t), t)] \dot{q}^k(t) + b^i(q(t), t) = 0, \quad i = 1, \ldots, m. \tag{13}$$

In the jargon of [9], a system of this type is said to be *"acatastatic"* because of the presence of the terms $b^i(q(t), t)$, with $i = 1, \ldots, m$. By adhering to the classical approach to the study of such systems (see e.g. [9, 4, 3, 2]), the virtual displacements $\delta q^1, \ldots, \delta q^n$ that are *admissible* for the given constraints are, by definition, those that comply with the Pfaffian form of Equations (13) in the following way

$$\sum_{k=1}^{n} [a^i{}_k(q(t), t)] \delta q^k(t) + b^i(q(t), t) \delta\mathcal{T}(t) = \sum_{k=1}^{n} [a^i{}_k(q(t), t)] \delta q^k(t) = 0, \quad i = 1, \ldots, m, \tag{14}$$

[2] See Lanczos [6].

i.e., with the variation $\delta\mathcal{T}(t) = 0$. In this sense, the virtual displacements are taken at fixed time, and Equation (14) replaces Equation (66) of [5] (see [4], page 14). Note that, since each $\check{\mathcal{C}}^i(q(t), \dot{q}(t), t)$ is affine in the *true* generalized velocities, Equation (14), in fact, coincides with the set of Chetaev's conditions [3, 2, 7]

$$\sum_{k=1}^{n} \left[\frac{\partial \check{\mathcal{C}}^i}{\partial \dot{q}^k}(q(t), \dot{q}(t), t) \right] \delta q^k(t) = \sum_{k=1}^{n} [a^i{}_k(q(t), t)] \delta q^k(t) = 0, \quad i = 1, \ldots, m. \tag{15}$$

Moreover, since the virtual velocities of the system under consideration must satisfy Equations (14), or (15), with $\delta q^k(t)$ replaced by the corresponding virtual velocity $\nu^k(t)$, for $k = 1, \ldots, n$, as pointed out in [9] (page 16), *the class of the true velocities does not coincide with the class of the virtual velocities.* Therefore, in this respect, the sentence of Appendix A1 of [5], eight lines after Equation (65), i.e.,

> *"The relations [...] constraints."*

is not consistent with the standard definition of virtual displacements or virtual velocities (see [9]), and it should be turned into

> "The relations obtained this way must be respected also by the virtual velocities of the considered mechanical system," but *as if* the terms $b^1(q(t), t), \ldots, b^m(q(t), t)$ were absent [4], "since, by definition, they [the virtual velocities] must be instantaneously in harmony with the imposed constraints."

Analogously, the text *"(be they virtual or real)"* [5] in the subsequent sentence should be turned into "(in fact, the real ones)". Furthermore, since the virtual displacements are taken at $\delta\mathcal{T} = 0$, Equation (66) of [5] reduces to Equation (15) above, while Equation (67) becomes

$$\sum_{k=1}^{n} \mathcal{Q}_k(t) \delta q^k(t) + \sum_{i=1}^{m} \mu_i(t) \left\{ \sum_{k=1}^{n} [a^i{}_k(q(t), t)] \delta q^k(t) \right\} = 0. \tag{16}$$

Therefore, the PVW "sees" the constraints (13) *as if* the terms $b^1(q(t), t), \ldots, b^m(q(t), t)$ were absent. Finally, Equations (68) and (69a)–(69c) of [5] become

$$\sum_{k=1}^{n} \left\{ \mathcal{Q}_k(t) + \sum_{i=1}^{m} \mu_i(t) [a^i{}_k(q(t), t)] \right\} \delta q^k(t) = 0, \tag{17a}$$

$$\mathcal{Q}_k(t) + \sum_{i=1}^{m} \mu_i(t) [a^i{}_k(q(t), t)] = 0, \qquad k = 1, \ldots, n. \tag{17b}$$

8 Conclusions

In this work, we have reviewed the main results of a previous article of ours [5] that were determined by regarding time as a fictitious Lagrangian parameter. In particular, we have reformulated some sentences, some equations, and some conclusions of [5] in light of an analysis of the constraint (2) that complies with the standard doctrine on non-holonomic and rheonomic constraints, and with the classical interpretation of the virtual displacements associated with this type of constraints [9].

In fact, we have shown that our main results are valid even though the virtual variations $\delta\mathfrak{T}$ and $\delta\mathcal{T}$ introduced in [5] are taken identically equal to zero, so that the constrained version of the PVW is written in the form expressed in Equations (8a), or (8b), and (12). Moreover, we have clarified some points of our previous work [5], and we have amended some statements of Appendix A1 of [5] as well as some aspects of its formulation.

In summary, we would like to remark that having regarded time as a fictitious Lagrangian parameter is not necessary for determining the crucial conclusions reported in [5], which we confirm here. However, the approach presented in [5] could be useful for further research on growth and on its connections with other biomechanical phenomena that are explicitly time dependent and typically studied within the framework of Continuum Mechanics, as is the case e.g. for aging. For such processes, indeed, a suitable adaptation of Nadile's procedure [5] could lead to an insightful interpretation of their physics.

Conflict of Interests

The Authors declare that they have no conflict of interests.

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