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THE YAKUBOVICH S-LEMMA REVISITED: STABILITY AND CONTRACTIVITY IN NON-EUCLIDEAN NORMS*

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Abstract. The celebrated S-Lemma was originally proposed to ensure the existence of a quadratic Lyapunov function in the Lur’e problem of absolute stability. A quadratic Lyapunov function is, however, nothing else than a squared Euclidean norm on the state space (that is, a norm induced by an inner product). A natural question arises as to whether squared *non-Euclidean* norms $V(x) = \|x\|^2$ may serve as Lyapunov functions in stability problems. This paper presents a novel non-polynomial S-Lemma that leads to constructive criteria for the existence of such functions defined by weighted ℓ_p norms. Our generalized S-Lemma leads to new absolute stability and absolute contractivity criteria for Lur’e-type systems, including, for example, a new simple proof of the Aizerman and Kalman conjectures for positive Lur’e systems.

Key words. S-Lemma, Contraction, Absolute stability, Positive Systems.

AMS subject classifications. 34H05, 93C15

1. Introduction. The history of the S-Lemma dates back to early works on stability of nonlinear control systems with partially uncertain dynamics [2, 42]. In many situations, such a system may be represented in the Lur’e form, that is, as a *feedback superposition* of two blocks as shown in Fig. 1. One block is a known linear time-invariant system, whereas the other block may be nonlinear (and is traditionally referred to as the “nonlinearity”) and uncertain or have no simple analytic representation as exemplified by neural network architectures [28] and lookup-table functions. A prototypical assumption on the uncertain block is that its input/output behavior satisfies some rough estimates. In the case of a static nonlinearity, such an estimate often takes the form of the *sector condition*

$$(1.1) \quad \alpha_1 \leq \frac{w(t)}{y(t)} \leq \alpha_2 \iff (y(t) - \alpha_1^{-1}w(t))(y(t) - \alpha_2^{-1}w(t)) \leq 0,$$

where $-\infty \leq \alpha_1 < \alpha_2 \leq +\infty$. The classical Lur’e problem [39, 42] was to find conditions on the coefficients of the known LTI block and the sector slopes $\{\alpha_1, \alpha_2\}$ that ensure global asymptotic stability of the closed-loop system for *all* nonlinearities in the sector. Later the term *absolute stability* has been coined for such problems; the term “absolute” emphasizes the applicability of the stability criteria to all unknown systems whose “nonlinear” parts belong to a certain class.

Historically, the first approach to absolute stability theory [7, 42] was based on quadratic Lyapunov functions and their extensions (e.g., a quadratic form plus a definite integral of the nonlinearity). The validation of the Lyapunov property (the Lyapunov function’s derivative along each trajectory is non-positive) leads to the following problem: *When is one quadratic inequality (the Lyapunov condition) implied by another quadratic inequality (the sector condition)?* More generally, when is a quadratic inequality implied by a system of quadratic inequalities?

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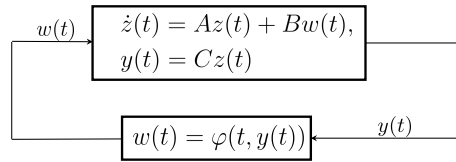


FIG. 1. A Lur'e system is the feedback superposition of an LTI system and a nonlinearity

A sufficient condition ensuring such an implication can be obtained by a simple trick, which was inspired by the idea of Lagrange multipliers and termed the “S-method” [2] or the S-procedure [29, 31]. The S-procedure reduces the problem of a quadratic Lyapunov function existence to an linear matrix inequality (LMI)-based condition, which can also be transcribed into a frequency-domain stability criteria by using the Kalman-Yakubovich-Popov (KYP) lemma. In [58], V. A. Yakubovich established the seminal *S-Lemma* showing that, under some conditions, the S-procedure does not introduce conservatism (i.e., the procedure is “lossless”). This result can also be interpreted as a strong duality theorem in a class of non-convex optimization problems [59, 60]. The S-procedure and S-Lemma are nowadays recognized among the most important tools of modern nonlinear control theory; their extensions and applications can be found in the excellent surveys [31, 47].

When the sector inequality (1.1) is replaced by the slope inequality

$$(1.2) \quad \alpha_1 \leq \frac{w_1(t) - w_2(t)}{y_1(t) - y_2(t)} \leq \alpha_2,$$

which is meant to hold for each two input-output pairs (w_1, y_1) and (w_2, y_2) and all t , the S-Lemma often allows us to derive not only the global asymptotic stability of the equilibrium, but in fact the stronger property of exponential, or strong *contractivity*. The strong contractivity property of a dynamical system implies highly ordered transient and asymptotic behavior, including (1) existence and global exponential stability of an equilibrium for time-invariant vector fields, (2) existence and global exponential stability of a limit cycle for time-varying periodic vector fields, (3) input-to-state stability and finite system gain for systems subject to state-independent disturbances as well as robustness with respect to unmodeled dynamics, (4) modularity and interconnection properties, and more. In other words, establishing the contractivity property is a worthy goal, independent from stability analysis alone. Remarkably, one of the first contractivity criterion based on the S-procedure was derived by V. A. Yakubovich [57] who formulated it, however, as a criterion for entrainment (or, using the terminology adopted in [57], for the existence and stability of forced periodic solutions). For comprehensive results on contractivity and incremental stability we refer to [12, 23, 40, 46, 55] and references therein.

Problem description and motivation. The classical S-procedure deals with quadratic inequalities, because its primary goal was to ensure the existence of a *quadratic* Lyapunov function. A quadratic Lyapunov function $V(z) = z^\top Pz$, where $P = P^\top$ is a constant positive definite matrix, is nothing else than the squared *Euclidean norm* on \mathbb{R}^n (e.g., the case where $P = I_n$ corresponds to $V(z) = \|z\|_2^2$), and all Euclidean norms (norms induced by inner products) can be represented in such a form. Stability and contraction analysis is usually performed by means of quadratic Lyapunov functions whose existence boils down to feasibility of LMIs [11].

Quadratic forms, however do not exhaust the list of possible Lyapunov functions. The interest for non-Euclidean norms (e.g., ℓ_1 , ℓ_∞ and polyhedral norms) is more recent and motivated by classes of network systems [12], such as biological transcriptional systems [50], Hopfield neural networks [20, 27, 48], chemical reaction networks [3], traffic networks [15–17], vehicle platoons [44], and coupled oscillators [5, 51].

A natural question thus arises as to whether a counterpart of the S-Lemma exists and allows us to transcribe sector-type and other standard constraints on the nonlinear blocks into *non-quadratic* Lyapunov inequalities arising when the Lyapunov function is chosen as $V(z) = \|z\|^2$, where the norm is non-Euclidean. In this paper, we give an affirmative answer and establish a *non-polynomial* counterpart of the S-Lemma, which allows us to obtain sufficient (and, in some situations, necessary) conditions ensuring the existence of the *non-quadratic* Lyapunov function $V(z) = \|Rz\|_p^2$, where R is an invertible matrix and $p \in [1, \infty]$. The theory developed in this paper is based on the techniques of logarithmic (log) norms and the *weak pairings* [19] associated to them. The theory of matrix logarithmic norms [53], or “matrix measures,” that has been extensively used in analysis of nonlinear circuits and systems since the 1970s [21, 22].

It should be noted that, in the context of non-Euclidean norms, the existence of a Lyapunov function $V(z) = \|Rz\|_1^2$ or $V(z) = \|Rz\|_\infty^2$, where R is a diagonal matrix, reduces to checking the Hurwitz stability of some Metzler matrix [20] (whose Perron-Frobenius eigenvector determines the weight R). As essentially argued for example by [49], from a computational viewpoint, checking the Hurwitzness of a Metzler matrix is much simpler problem (e.g., viable also for large scale problems) than solving LMIs. Regarding algorithmic and computational aspects, [13] and [56] analyze and compare efficient numerical algorithms to compute the Perron eigenvalue and eigenvector of a nonnegative irreducible matrix. Beside these computational simplifications, there are additional practical advantages of non-Euclidean ℓ_1/ℓ_∞ norms: (1) the ℓ_1 norm (respectively, the ℓ_∞ norm) is well suited for systems with conserved quantities (respectively, systems with translation invariance), e.g., see the theory of weakly contracting and monotone systems in [12, Chapter 4]; (2) contractivity with respect to non-Euclidean norms ensures robustness with respect to edge removals and structural perturbations, e.g., see the notion of connective stability in [52]; (3) ℓ_∞ contraction mappings are known to converge under the fully asynchronous distributed execution [8]; (4) in machine learning, analysis of the adversarial robustness of a neural network (NN) often needs to be performed in a non-Euclidean norm [34, 38, 61], because NNs are known to be vulnerable to small (in ℓ_∞ sense) disturbances [30].

Contributions. The contributions of this work are as follows:

- We extend the S-Lemma to a special class of non-polynomial functions that were introduced by Lumer [41] and later used in contraction analysis [19].
- Using the generalized S-Lemma, we derive novel criteria for absolute stability and contractivity of Lur’e systems. In other words, we provide a unifying framework for absolute stability and contractivity analysis of dynamical systems over normed vector spaces;
- We demonstrate that our criteria generalize some results available in the literature, for instance, stability and contractivity criteria exploring symmetrization [25] and the Aizerman conjecture for positive systems [14].

Structure of the paper. Section 2 introduces some mathematical concepts to be used in the subsequent sections, in particular, log norms and weak pairings associated to a norm. Section 3 presents our first main result (Theorem 3.1), which we call the non-polynomial S-Lemma. In Section 4, this result is applied to analysis

of Lur'e-type systems and establish new criteria of absolute stability and absolute contractivity in non-Euclidean norms; new proofs of the Aizerman and Kalman conjectures for positive Lur'e systems are given. Section 5 concludes the paper.

2. Technical preliminaries. We start with introducing notation. For $a \in \mathbb{R}$, let $a^+ = \max(a, 0)$. Unless otherwise stated, vectors from \mathbb{R}^n are considered as columns. For two vectors $x, y \in \mathbb{R}^n$, the relations $\leq, \geq, <, >$ are interpreted elementwise. The same rule applies to (equally dimensioned) matrices: $A \leq B$ if and only if $a_{ij} \leq b_{ij} \forall i, j$. The symbols \prec, \preceq apply to *symmetric* matrices: we write $A \prec B$ (respectively, $A \preceq B$) if $B - A$ is positive definite (respectively, semidefinite).

Given a matrix $A = (a_{ij})$, denote $|A| = (|a_{ij}|)$. For two identically sized matrices $A = (a_{ij}), B = (b_{ij})$, $A \circ B = (a_{ij}b_{ij})$ denotes their Hadamard (entry-wise) product.

Recall that a matrix is *Metzler* if all its off-diagonal terms are non-negative. Given a matrix $A \in \mathbb{R}^{n \times n}$, its *Metzler majorant* $[A]_{\text{Mzr}} \in \mathbb{R}^{n \times n}$ is defined by

$$([A]_{\text{Mzr}})_{ij} := \begin{cases} a_{ii}, & \text{if } i = j \\ |a_{ij}|, & \text{if } i \neq j. \end{cases}$$

Obviously, a matrix is Metzler if and only if it coincides with its Metzler majorant.

2.1. Log norms and weak pairings in normed spaces. Let $\|\cdot\|$ be a norm on \mathbb{R}^n ; the same symbol will be used to denote the induced operator norm on $\mathbb{R}^{n \times n}$. The *log norm* of $A \in \mathbb{R}^{n \times n}$ with respect to $\|\cdot\|$ is

$$(2.1) \quad \mu(A) := \lim_{h \rightarrow 0^+} \frac{\|I_n + hA\| - 1}{h}.$$

The *conic log norm* [33] of $A \in \mathbb{R}^{n \times n}$ is

$$\mu^+(A) := \lim_{h \rightarrow 0^+} \sup_{\substack{x \geq 0_n \\ x \neq 0_n}} \frac{\|(I_n + hA)x\| / \|x\| - 1}{h}.$$

We refer to [19, 21] and [33] for, the theory of log norms and conic log norms.

From [19], a *weak pairing* on \mathbb{R}^n is a map $[[\cdot, \cdot]] : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying

- (i) (Subadditivity and continuity of first argument) $[[x_1 + x_2, y]] \leq [[x_1, y]] + [[x_2, y]]$, for all $x_1, x_2, y \in \mathbb{R}^n$ and $[[\cdot, \cdot]]$ is continuous in its first argument,
- (ii) (Weak homogeneity) $[[\alpha x, y]] = [[x, \alpha y]] = \alpha [[x, y]]$ and $[[-x, -y]] = [[x, y]]$, for all $x, y \in \mathbb{R}^n, \alpha \geq 0$,
- (iii) (Positive definiteness) $[[x, x]] > 0$, for all $x \neq 0_n$,
- (iv) (Cauchy-Schwarz inequality) $|[[x, y]]| \leq [[x, x]]^{1/2} [[y, y]]^{1/2}$, for all $x, y \in \mathbb{R}^n$.

A weak pairing is *compatible* with a norm $\|\cdot\|$ if $[[x, x]] = \|x\|^2$ for all x . Table 1 contains weak pairings compatible with every ℓ_p norm, $p \in [1, \infty]$ [19, 33]. This list includes the *sign pairing* for ℓ_1 norm and the *max pairing* for the ℓ_∞ norm. Only unweighted ℓ_p norms are included since $\mu_{p,R}(A) = \mu_p(RAR^{-1})$ for any $p \in [1, \infty]$.

The pairings in Table 1 additionally satisfy Lumer's equalities and the curve norm derivative formulas [19, 33]. For all $A \in \mathbb{R}^{n \times n}$, *Lumer's equalities* state that

$$(2.2) \quad \mu(A) = \sup_{\|x\|=1} [[Ax, x]] = \sup_{x \neq 0_n} \frac{[[Ax, x]]}{\|x\|^2},$$

$$(2.3) \quad \mu^+(A) = \sup_{\|x\|=1, x \geq 0_n} [[Ax, x]] = \sup_{\substack{x \geq 0_n \\ x \neq 0_n}} \frac{[[Ax, x]]}{\|x\|^2}$$

Norm	Weak pairing	Log norms and Lumer's equality
$\ x\ _2 = \sqrt{x^\top x}$	$\llbracket x, y \rrbracket_2 = x^\top y$	$\mu_2(A) = \frac{1}{2} \lambda_{\max}(A + A^\top)$ $= \max_{\ x\ _2=1} x^\top Ax$
$\ x\ _p = \left(\sum_i x_i ^p \right)^{1/p}$, $1 < p < \infty$	$\llbracket x, y \rrbracket_p =$ $\ y\ _p^{2-p} (y \circ y ^{p-2})^\top x$	$\mu_p(A) = \max_{\ x\ _p=1} (x \circ x ^{p-2})^\top Ax$
$\ x\ _1 = \sum_i x_i $	$\llbracket x, y \rrbracket_1 = \ y\ _1 \text{sign}(y)^\top x$	$\mu_1(A) = \max_{j \in \{1, \dots, n\}} \left(a_{jj} + \sum_{i \neq j} a_{ij} \right)$ $= \sup_{\ x\ _1=1} \text{sign}(x)^\top Ax$ $\mu_1^+(A) = \max_{j \in \{1, \dots, n\}} \left(a_{jj} + \sum_{i \neq j} a_{ij}^+ \right)$ $= \sup_{\ x\ _1=1, x \geq 0_n} \text{sign}(x)^\top Ax$
$\ x\ _\infty = \max_i x_i $	$\llbracket x, y \rrbracket_\infty = \max_{i \in I_\infty(y)} x_i y_i$	$\mu_\infty(A) = \max_{i \in \{1, \dots, n\}} \left(a_{ii} + \sum_{j \neq i} a_{ij} \right)$ $= \sup_{\ x\ _\infty=1} \max_{i \in I_\infty(x)} (Ax)_i x_i$ $\mu_\infty^+(A) = \max_{i \in \{1, \dots, n\}} \left(a_{ii} + \sum_{j \neq i} a_{ij}^+ \right)$ $= \sup_{\ x\ _\infty=1, x \geq 0_n} \max_{i \in I_\infty(x)} (Ax)_i x_i$

TABLE 1

Table of norms, weak pairings, and log norms for ℓ_2 , ℓ_p for $p \in (1, \infty)$, ℓ_1 , and ℓ_∞ norms. We adopt the shorthand $I_\infty(x) = \{i \in \{1, \dots, n\} \mid |x_i| = \|x\|_\infty\}$.

For Metzler matrices and ℓ_1 norm, the log-norm and the conic-log norm are, obviously, coincident, furthermore, the closed simplex $\{x : \|x\| = 1, x \geq 0_n\}$ in (2.3) can be replaced by its (relative) interior as shown by the following proposition.

PROPOSITION 2.1. *For every $n \times n$ Metzler matrix M , one has*

$$(2.4) \quad \mu_1(M) = \mu_1^+(M) = \sup_{\substack{\|x\|_1=1 \\ x > 0}} \llbracket Mx, x \rrbracket_1 = \sup_{x > 0_n} \frac{\llbracket Mx, x \rrbracket}{\|x\|^2}.$$

The curve norm derivative formula states that for every differentiable $x : (a, b) \rightarrow \mathbb{R}^n$ and for almost every $t \in (a, b)$, that right upper Dini derivative of $\|x(t)\|^2$ at t is

$$(2.5) \quad D^+ \|x(t)\|^2 := \limsup_{h \rightarrow 0^+} \frac{\|x(t+h)\|^2 - \|x(t)\|^2}{h} = 2 \llbracket \dot{x}(t), x(t) \rrbracket.$$

The curve norm formula is typically used to differentiate a Lyapunov function $V(x) = \|x\|^2$ along the system's trajectories, proving stability or contraction [19]. Unless otherwise stated, the curve norm derivative formula is always supposed to hold. As shown in [19], all WPs from Table 1 enjoy this useful property. For these WPs, a useful equality holds [19, Appendix C]: for all $x, y \in \mathbb{R}^n$, and $c \in \mathbb{R}$

$$(2.6) \quad \llbracket x + cy, y \rrbracket = \llbracket x, y \rrbracket + c \|y\|^2.$$

Remark 2.2. Notice that (2.6) does not guarantee that the WP is linear in its first argument, that is,

$$(2.7) \quad \llbracket ax_1 + bx_2, y \rrbracket = a \llbracket x_1, y \rrbracket + b \llbracket x_2, y \rrbracket$$

for all vectors x_1, x_2, y and scalars $a, b \in \mathbb{R}$. For instance, $\llbracket \cdot, \cdot \rrbracket_\infty$, as can be seen from Table I, satisfies (2.6) yet fails to be linear in its first argument.

2.2. Non-polynomial 2-forms associated to a weak pairing. A standard quadratic form on \mathbb{R}^n admits two equivalent representations. On one hand, it can be considered as a homogeneous polynomial $q(x_1, \dots, x_n) = \sum_{i,j} q_{ij} x_i x_j$ of degree 2 also termed as a *polynomial 2-form*. The term *homogeneous* means that all (non-zero) terms have same degree d (in our situation $d = 2$), or, equivalently, for each scalar $\lambda \in \mathbb{R}$ one has $q(\lambda x_1, \dots, \lambda x_n) = \lambda^d q(x_1, \dots, x_n)$. On the other hand, the quadratic form can be considered as a function $q(x) = x^\top Q x = \llbracket Qx, x \rrbracket_2$, where Q is a matrix.

Given a weak pairing $\llbracket \cdot, \cdot \rrbracket$ and a matrix P , define the *non-polynomial 2-form*¹

$$p(x) = \llbracket Px, x \rrbracket, \quad x \in \mathbb{R}^n.$$

For the standard ℓ_1 , ℓ_2 , and ℓ_∞ norms, we have:

$$p_1(x) = \llbracket Px, x \rrbracket_1 = \|x\|_1 \operatorname{sign}(x)^\top P x,$$

$$p_2(x) = \llbracket Px, x \rrbracket_2 = x^\top P x,$$

$$p_\infty(x) = \llbracket Px, x \rrbracket_\infty = \max_{i \in I_\infty(x)} x_i (P x)_i, \quad I_\infty(x) = \{i \in \{1, \dots, n\} \mid |x_i| = \|x\|_\infty\}.$$

For brevity, we omit the term “non-polynomial” and call $p(x)$ simply 2-form.

3. Non-polynomial S-Lemma for general normed spaces. Consider a WP $\llbracket \cdot, \cdot \rrbracket$ on \mathbb{R}^n compatible with a norm $\|\cdot\|$ on vectors and log norm $\mu(\cdot)$ on matrices. We assume the weak pairing satisfies Lumer’s equality and the curve norm derivative formula. Consider a family of $s + 1$ matrices $P_0, \dots, P_s \in \mathbb{R}^{n \times n}$ and functions

$$(3.1) \quad p_i(x) = \llbracket P_i x, x \rrbracket, \quad i \in \{0, \dots, s\}.$$

Given a constant $\rho \in \mathbb{R}^s$, we define the primal optimization problem

$$(3.2) \quad \begin{aligned} & \sup_{x \in \mathbb{R}^n} p_0(x) \\ & \text{subject to } \|x\| = 1, \quad p_1(x) \leq \rho_1, \quad \dots, \quad p_s(x) \leq \rho_s, \end{aligned}$$

and the dual optimization problem:

$$(3.3) \quad \begin{aligned} & \inf_{\tau \in \mathbb{R}^s} \mu\left(P_0 - \sum_{j=1}^s \tau_j P_j\right) + \tau^\top \rho \\ & \text{subject to } \tau \geq \mathbb{0}_s. \end{aligned}$$

We note that the primal problem (3.2) is non-convex and the constraints, in general, are feasible only for sufficiently large ρ_i . By definition, let $-\infty$ be the value of the optimization problem (3.2) if the constraints are infeasible. On the other hand, (3.3) is a convex program with feasible constraints no matter how the norm is chosen. In the case of ℓ_2 norm (3.3) is a standard semidefinite program, whereas in ℓ_1/ℓ_∞ it turns out to be a linear program, which allows us to solve it efficiently.

¹Such functions were first introduced by Lumer [41] in the special case where $\llbracket \cdot, \cdot \rrbracket$ is a semi-inner product on a normed space (possibly, infinite-dimensional).

3.1. Non-polynomial S-Lemma: a weak duality result. The standard relation of *weak duality* (duality without zero-gap guarantee) [11] entails that the infimum in (3.2) is not less than the supremum in (3.3), provided that the WP is linear in the first argument in the sense of (2.7). In reality, the linearity requirement can be discarded as shown by the following lemma, which is a non-polynomial counterpart of the Yakubovich S-Lemma.

THEOREM 3.1 (Non-polynomial S-Lemma: Weak duality for Non-Euclidean norms).

Let $\|\cdot\|$ be a norm on \mathbb{R}^n with log norm $\mu(\cdot)$ and compatible weak pairing $\llbracket \cdot, \cdot \rrbracket$ satisfying Lumer's equality. Given $P_0, \dots, P_s \in \mathbb{R}^{n \times n}$ and $\rho \in \mathbb{R}^s$

- (i) if $\llbracket \cdot, \cdot \rrbracket$ is linear (2.7) in its first argument, then the optimization problem (3.3) is the Lagrangian dual problem to the optimization problem (3.2);
- (ii) for an arbitrary $\llbracket \cdot, \cdot \rrbracket$, let α and β denote, respectively, the supremum in (3.2) and the infimum in (3.3). Then $\alpha \leq \beta$.
- (iii) In particular, the following statement is valid:

$$(3.4) \quad p_0(x) \leq \beta \|x\|^2 \quad \forall x \in \mathbb{R}^n : p_1(x) \leq \rho_1 \|x\|^2, \dots, p_s(x) \leq \rho_s \|x\|^2.$$

Proof. Adopt the short-hand $P(\tau) = P_0 - \sum_{i=1}^s \tau_i P_i$. Following the notation from [60], define the Lagrangian function $L : \mathbb{R}^n \times \mathbb{R}^s \rightarrow \mathbb{R}$ by

$$L(x, \tau) = -p_0(x) + \sum_{j=1}^s \tau_j p_j(x) - \sum_{j=1}^s \tau_j \rho_j = -\tau^\top \rho - \llbracket P(\tau)x, x \rrbracket.$$

Applying Lumer's equality (2.2), the Lagrange dual function [11] becomes

$$g(\tau) = \inf_{x \in \mathbb{R}^n, \|x\|=1} L(x, \tau) = -\tau^\top \rho - \sup_{x \in \mathbb{R}^n, \|x\|=1} \llbracket P(\tau)x, x \rrbracket = -\tau^\top \rho - \mu(P(\tau)).$$

The Lagrange dual problem to (3.2) is written as

$$(3.5) \quad \sup_{\tau \in \mathbb{R}^s} g(\tau) \quad \text{subject to } \tau_1, \dots, \tau_s \geq 0,$$

or, equivalently, as problem (3.3). This concludes the proof of statement (i).

For each vector $x \in \mathbb{R}^n$ satisfying constraints (3.2) and every $\tau \geq 0_s$ one has

$$\begin{aligned} \llbracket P_0 x, x \rrbracket &= \left\llbracket P_0 x - \sum_{i=1}^s \tau_i P_i x + \sum_{i=1}^s \tau_i P_i x, x \right\rrbracket \\ &\leq \left\llbracket P_0 x - \sum_{i=1}^s \tau_i P_i x, x \right\rrbracket + \sum_{i=1}^s \tau_i \llbracket P_i x, x \rrbracket \quad (\text{by subadditivity of } \llbracket \cdot, \cdot \rrbracket) \\ &\leq \left\llbracket \left(P_0 - \sum_{i=1}^s \tau_i P_i \right) x, x \right\rrbracket + \sum_{i=1}^s \tau_i \llbracket P_i x, x \rrbracket \quad (\text{by weak homogeneity of } \llbracket \cdot, \cdot \rrbracket) \\ (3.6) \quad &\leq \mu(P(\tau)) + \sum_{i=1}^s \tau_i \rho_i. \quad (\text{by Lumer's equality for } \llbracket \cdot, \cdot \rrbracket) \end{aligned}$$

Taking the supremum of $\llbracket P_0 x, x \rrbracket$ over all feasible x and the infimum of the right-hand side over all $\tau \geq 0_s$, one proves that $\alpha \leq \beta$. This concludes the proof of statement (ii).

Statement (iii) is straightforward from (ii) by noticing that $p_0(x) \leq \alpha \|x\|^2$ for all $x \in \mathbb{R}^n$ satisfying the inequalities $p_i(x) \leq \rho_i \|x\|^2$: this statement is obvious for $x = 0$, otherwise, the normalized vector $\tilde{x} = x/\|x\|$ obeys the constraints in (3.2) and thus $p_0(x) = \|x\|^2 p_0(\tilde{x}) \leq \alpha \|x\|^2$ due to the definition of α . \square

Remark 3.2 (Conic constraints and more). Theorem 3.1 can be extended with a trivial modification to the optimization problems with an additional conic constraint $x \geq \mathbb{0}_s$. In this case, the primal problem is

$$(3.7) \quad \begin{aligned} & \sup_{x \in \mathbb{R}^n} p_0(x) \\ & \text{subject to } \|x\| = 1, x \geq \mathbb{0}_n, p_1(x) \leq \rho_1, \dots, p_s(x) \leq \rho_s, \end{aligned}$$

and the usual log norm in the dual problem (3.3) is replaced by the conic log norm:

$$(3.8) \quad \begin{aligned} & \inf_{\tau \in \mathbb{R}^s} \mu^+ \left(P_0 - \sum_{j=1}^s \tau_j P_j \right) + \tau^\top \rho \\ & \text{subject to } \tau \geq \mathbb{0}_s. \end{aligned}$$

Denoting the latter infimum by β^+ , the inequality (3.4) should be rewritten as

$$(3.9) \quad p_0(x) \leq \beta^+ \|x\|^2 \quad \forall x \in \mathbb{R}_{\geq 0}^n : p_1(x) \leq \rho_1 \|x\|^2, \dots, p_s(x) \leq \rho_s \|x\|^2.$$

Remark 3.3 (Equivalent primal constraints). From (2.6), we know $\llbracket y - \rho x, x \rrbracket = \llbracket y, x \rrbracket - \rho \|x\|^2$ for each $\rho \in \mathbb{R}$. This equality allows us to simplify the constraints in (3.2) by replacing $P_j \mapsto P_j - \rho_j I_j$ and $\rho_j \mapsto 0$. However, sometimes (see below the case of positive systems) it is more convenient to consider the general situation.

Remark 3.4 (No constraints implies no gap). If the primal problem is unconstrained, that is, $s = 0$, then $\alpha = \beta = \mu(P_0)$ because of Lumer's equality (2.2). We study two other special cases with zero-gap in Subsections 3.2 and 3.2. In general, the *duality gap* the optimal values exists: $\alpha < \beta$ (see the examples below).

3.2. The classic S-Lemma: the case of Euclidean norm ℓ_2 . The first case where there is no gap is where the primal problem has only one constraint ($s = 1$) and the norm is Euclidean (i.e., induced by the inner product). The following result is equivalent to [47, Theorem 5.17] and to the Yakubovich S-Lemma [59]. Without loss of generality (Remark 3.3), let $\rho_1 = 0$.

LEMMA 3.5 (The Yakubovich S-Lemma). *Suppose that $\llbracket \cdot, \cdot \rrbracket$ is an inner product, $s = 1$, and the primal constraint reads $\llbracket P_1 x, x \rrbracket \leq 0$. Suppose also that for some x the latter inequality is strict: $\llbracket P_1 x, x \rrbracket < 0$. Then*

(i) *the maximum and minimum points of primal and dual problems exist, and there is no duality gap: $\alpha = \beta$ or, equivalently*

$$(3.10) \quad \max\{\llbracket P_0 x, x \rrbracket \mid \|x\| = 1, \llbracket P_1 x, x \rrbracket \leq 0\} = \min\{\mu(P_0 - \tau P_1) \mid \tau \geq 0\},$$

(ii) *if x_* is a maximizer (generally, non-unique) in (3.2) and τ_* is a minimizer in (3.3), then the complementarity condition holds*

$$\tau_* \llbracket P_1 x_*, x_* \rrbracket = 0.$$

Proof. The proof retraces the proofs of Theorem 2 in [59], see also [60, Appendix A], and is based on the S-Lemma for two quadratic forms. Since the inequality $p_1(x) < 0$ has at least one solution, it also holds for some x from the sphere $\|x\|_2 \leq 1$. Hence, problem (3.2) (with $s = 1$) is feasible and $\alpha > -\infty$. The set of admissible vectors x in (3.2) is compact, and hence the supremum can be replaced by the maximum.² Since the functions p_0, p_1 are homogeneous,

$$p_1(x) \leq 0 \quad \implies \quad p_0(x) - \alpha \|x\|_2^2 < 0 \quad \forall x \in \mathbb{R}^n.$$

²Notice that the scalar product, unlike general WPs, is continuous in both arguments.

Recalling that the set $\{x \mid p_1(x) < 0\}$ is non-empty, the S-Lemma [59, Theorem 1] and [47, Theorem 2.2] ensures that a number $\tau_* \geq 0$ exists such that

$$p_0(x) - \alpha \|x\|_2^2 - \tau_* p_1(x) \leq 0,$$

whence we obtain that

$$\mu(P(\tau_*)) = \sup_{\|x\|_2=1} (p_0(x) - \tau_* p_1(x)) \leq \alpha.$$

On the other hand, $\mu(P(\tau)) \geq \alpha$ for any $\tau \geq 0$ due to the non-polynomial S-Lemma (Theorem 3.1). Hence, τ_* is a minimizer in (3.3) (where $\rho_1 = 0$). The last statement can be easily derived from (3.6), by noticing that $p_0(x_*) = \mu(P(\tau_*)) = \alpha$. \square

Remark 3.6 (Inhomogeneous quadratic functions). Lemma 3.5 remains valid [47] for more general inhomogeneous quadratic functions $\llbracket P_i x + q_i, x \rrbracket + r_i$, where P_i are constant matrices, q_i are vectors, and $r_i \in \mathbb{R}$, for $i \in \{0, 1\}$. Such quadratic constraints arise in many robust control and optimal control problems [6]. The generalization of Theorem 3.1 to such functions when $\llbracket \cdot, \cdot \rrbracket$ is a general weak pairing remains an open problem.

Remark 3.7 (Other zero-gap cases). In the Euclidean norm case some other situations (with $s \geq 2$) are known where the duality gap vanishes: this holds (under minor assumptions) when the image of the quadratic mapping $(p_0(x), \dots, p_s(x))$ is convex. This counter-intuitive convexity always takes place when $s = 1$ (two quadratic forms) [24], being also a feature of some specially structured quadratic functions when $s > 1$. For a detailed survey of recent achievements in the area, we refer the reader to [31, 47]. As noted in [58, 60], Lemma 3.5 can be extended to every such situation.

3.3. The S-Lemma for Metzler matrices in ℓ_1 norm. In this section we consider another situation where the duality gap vanishes. Consider the situation of ℓ_1 -norm with the sign weak pairing $\llbracket x, y \rrbracket_1 = \|y\|_1 \operatorname{sign}(y)^\top x$. Notice that this function is discontinuous in y . Along with the “non-negative” optimization problem (3.7) consider the problem with stricter constraints

$$(3.11) \quad \begin{aligned} & \sup_{x \in \mathbb{R}^n} \llbracket P_0 x, x \rrbracket_1 \\ & \text{subject to } \|x\|_1 = 1, x > 0_n, \llbracket P_1 x, x \rrbracket_1 < \rho_1, \dots, \llbracket P_s x, x \rrbracket_1 < \rho_s. \end{aligned}$$

The optimization problems (3.2) and (3.7) are non-convex, whereas the objective function and the constraints in (3.11) are linear, because $\llbracket P_i x, x \rrbracket_1 = \mathbb{1}^\top P_i x$ for all $x \in \mathbb{R}_{>0}^n$ with $\|x\|_1 = 1$. At the same time, (3.11) is not a standard LP problem due to the presence of strict inequalities.

LEMMA 3.8 (S-Lemma for Metzler matrices in ℓ_1 norm). *Assume that matrices P_0 and $(-P_1), (-P_2), \dots, (-P_s)$ are Metzler and that the constraints in (3.11) are feasible. Then the following values coincide:*

- (i) $\alpha := \text{supremum in (3.2) (where } \|\cdot\| = \|\cdot\|_1\text{)}$;
- (ii) $\alpha^+ := \text{supremum in (3.7) (where } \|\cdot\| = \|\cdot\|_1\text{)}$;
- (iii) $\alpha^{++} := \text{supremum in (3.11) (where } \|\cdot\| = \|\cdot\|_1\text{)}$;
- (iv) $\beta^+ := \text{infimum in (3.8) (where } \mu^+ = \mu_1^+\text{)}$;
- (v) $\beta := \text{infimum in (3.3) (where } \mu = \mu_1\text{)}$.

Furthermore, the minimum value in problem (3.3) exists.

Proof. Since $P(\tau)$ is Metzler for each $\tau \geq 0$, we have $\mu_1^+(P(\tau)) = \mu_1(P(\tau))$, and hence $\beta = \beta^+$. In view of evident inequalities $\alpha \geq \alpha^+ \geq \alpha^{++}$ and Theorem 3.1, ensuring that $\alpha \leq \beta$, it suffices to show that $\alpha^{++} \geq \beta^+$.

Introducing the simplex $\Delta = \{x > 0 \mid \|x\|_1 = 1\}$ and its closure $\bar{\Delta} = \{x \geq 0 \mid \|x\|_1 = 1\}$, one may easily notice that all functions $p_i(x) = \llbracket P_i x, x \rrbracket_1$ are linear on Δ . In particular, the set $\mathcal{P} = \{(p_0(x), \dots, p_s(x)) \mid x \in \Delta\}$ is convex in \mathbb{R}^{s+1} . By definition of α^{++} , the set $\mathcal{Y} = \{y = (y_0, \dots, y_s) \in \mathbb{R}^{s+1} \mid y_0 > \alpha^{++}, y_1 < \rho_1, \dots, y_s < \rho_s\}$ is disjoint with \mathcal{P} . Hence, a (non-strictly) separating hyperplane exists, i.e., there exists $\lambda \in \mathbb{R}^{s+1} \setminus \{0\}$ such that

$$\sum_{i=0}^s \lambda_i p_i(x) \leq \lambda^\top y \quad \text{for all } x \in \Delta, y \in \mathcal{Y}.$$

Obviously, the latter inequality may hold only when $\lambda_0 \geq 0$ and $\lambda_i \leq 0$ for $i \in \{1, \dots, s\}$. Furthermore, passing to the limit as $\mathcal{Y} \ni y \rightarrow (\beta^+, \rho_1, \dots, \rho^s)$, one has

$$\lambda_0(p_0(x) - \alpha^{++}) + \sum_{i=1}^s \lambda_i(p_i(x) - \rho_i) \leq 0.$$

Since the constraints in (3.11) are feasible, we know $\lambda_i(p_i(x) - \rho_i) \geq 0$ for some $x \in \Delta$ and, furthermore, the inequalities are strict unless $\lambda_i = 0$. Therefore, $\lambda_0 > 0$. Introducing now $\bar{\tau}_i = -\lambda_i/\lambda_0 \leq 0$, for all $i \in \{1, \dots, s\}$, one has

$$\alpha^{++} \geq p_0(x) - \sum_{i=1}^s \bar{\tau}_i p_i(x) + \tau^\top \rho = \llbracket P(\bar{\tau})x, x \rrbracket + \bar{\tau}^\top \rho, \quad \text{for all } x \in \Delta.$$

Taking the supremum over all $x \in \Delta$ and applying Proposition 2.1 to the Metzler matrix $P(\bar{\tau})$, one has $\alpha^{++} \geq \mu_1^+(P(\bar{\tau})) + \bar{\tau}^\top \rho \geq \beta^+$. This finishes the proof, showing also that $\bar{\tau}$ is a minimizer in (3.8) and (3.3). \square

3.4. Some counterexamples. Below we demonstrate that the conditions of Lemma 3.8, in fact, cannot be discarded. In the examples below, $x = (x_1, x_2)^\top \in \mathbb{R}^2$ and $\|x\| = \|x\|_1 = |x_1| + |x_2|$, we consider only one constraint $s = 1$.

Example 1. Our first example demonstrates that the feasibility of strict inequalities (3.11) is essential. Consider the parameters

$$P_0 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad \rho_1 = -1.$$

In view of Table 1, $p_0(x) = (x_1 + x_2, 0) \text{ sign } x = (x_1 + x_2) \text{ sign } x_1$ and $p_1(x) = (0, -x_2) \text{ sign } x = -|x_2|$. Hence the constraints in (3.7) are feasible and satisfied by the unique vector $x_1 = 0, x_2 = 1$, and the supremum in (3.7) is $\alpha^+ = 0$. At the same time, the constraints in (3.11) are infeasible. Matrices P_0 and $(-P_1)$ are, obviously, Metzler. Notice now that the cost function in (3.8) is constant, since

$$\mu_1^+(P_0 - \tau P_1) + \tau \rho_1 = \mu_1^+ \begin{pmatrix} 1 & 1 \\ 0 & \tau \end{pmatrix} - \tau = (1 + \tau) - \tau = 1 \quad \forall \tau \geq 0,$$

and hence $\beta^+ = 1 > 0 = \alpha^+$.

Example 2. Our next example shows that the requirement of Metzler matrices in Lemma 3.8 is essential.

A) Consider the parameters

$$P_0 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad \rho_1 \in (-1, 0).$$

Here, matrix $(-P_1)$ is Metzler, whereas P_0 is not. Then, $p_1(x) = -x_1 \operatorname{sign} x_2$ and the constraints in (3.11) are feasible and satisfied, e.g., by $x_1 = 1 - \varepsilon$, $x_2 = \varepsilon$ for $\varepsilon > 0$ being small. Notice, now that if $p_1(x) \leq \rho_1 < 0$, then

$$p_0(x) = x_1 \operatorname{sign} x_1 - x_1 \operatorname{sign} x_2 \stackrel{x_1 \operatorname{sign} x_2 > 0}{=} |x_1| - |x_1| = 0,$$

and hence $\alpha^+ = 0$. At the same time, we have

$$\mu_1^+(P_0 - \tau P_1) + \tau \rho_1 = \mu_1^+ \begin{pmatrix} 1 & 0 \\ \tau - 1 & 0 \end{pmatrix} + \tau \rho_1 = \begin{cases} 1 + \tau \rho_1, & \tau \leq 1, \\ \tau(1 + \rho_1), & \tau > 1. \end{cases}$$

so the infimum in (3.8) is $\beta^+ = 1 + \rho_1 > 0 = \alpha^+$.

B) Consider now the parameters

$$P_0 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \rho_1 \in (0, 1/2).$$

Now P_0 is Metzler, where $(-P_1)$ is not. Obviously, $p_1(x) = x_1 \operatorname{sign} x_2$ and the constraints in (3.11) are feasible (being satisfied, e.g., by $x_1 = \rho_1/2$, $x_2 = 1 - x_1$). Also, $p_0(x) = |x_1| + x_1 \operatorname{sign} x_2$ achieves its maximum at $x_1 = 1, x_2 = 0$, so the supremum in (3.7) is $\alpha^+ = 1$ (indeed, if $x_2 > 0$, then $p_0(x) = 2x_1 \leq 2\rho_1 < 1$). However,

$$\mu_1^+(P_0 - \tau P_1) + \tau \rho_1 = \mu_1^+ \begin{pmatrix} 1 & 0 \\ 1 - \tau & 0 \end{pmatrix} + \tau \rho_1 = \begin{cases} 2 - \tau(1 - \rho_1), & \tau \leq 1, \\ \tau(1 + \rho_1), & \tau > 1 \end{cases},$$

that is, $\beta^+ = 1 + \rho_1 > \alpha^+$.

Remark 3.9. Example 2 also demonstrates the duality gap in the pair of dual problems (3.2), (3.3): in both situations **A)** and **B)**, the suprema in (3.2) and (3.7) coincide $\alpha = \alpha^+$, and hence $\beta \geq \beta^+ > \alpha$, because $\mu_1(A) \geq \mu_1^+(A)$ for each matrix A .

4. Absolute stability and absolute contractivity of Lur'e systems. Consider first the *Lur'e system* shown in Fig. 1

$$(4.1) \quad \dot{z}(t) = Az(t) + Bw(t) \in \mathbb{R}^d, \quad y(t) = Cz(t) \in \mathbb{R}$$

$$(4.2) \quad w(t) = \varphi(t, y(t)) \in \mathbb{R}.$$

The function φ is traditionally called “nonlinearity” (although one can also consider linear feedback as well). The nonlinear block need not be static, moreover, it can have a more general form than (4.2) and be a nonlinear operator on the whole trajectory $y(\cdot)|_0^t$. In absolute stability theory, φ is usually uncertain and belongs to the class of functions that obey certain *quadratic constraints* [29, 31, 43], the simplest of which are the *sector condition*³

$$(4.3) \quad \zeta \leq \frac{\varphi(t, y)}{y} \leq \varkappa \quad \forall y \neq 0$$

³In the case of static continuous nonlinearity $w(t) = \phi(y(t))$ and $-\infty < \zeta \varkappa < \infty$, the sector condition implies that $\phi(0) = 0$ and $\{(y, w) | w = \phi(y)\}$ of the nonlinearity lies in the sector bounded by two lines $w = \varkappa y$ and $w = \zeta y$.

and the more sophisticated *slope* condition⁴

$$(4.4) \quad \zeta \leq \frac{\varphi(t, y_1) - \varphi(t, y_2)}{y_1 - y_2} \leq \varkappa \quad \forall y_1 \in \mathbb{R} \forall y_2 \neq y_1.$$

Here $\zeta \geq -\infty$ and $\varkappa \leq \infty$, where at least one of the latter inequalities is strict. Note that the slope condition, generally does not entail the sector condition as exemplified, e.g., by a non-zero constant function; this implication is valid only when $\varphi(t, 0) = 0$.

4.1. Sector and slope constraints. In fact, the results on stability and contractivity of Lur'e system do not use the explicit form of the signal $w(t)$; the only information used to derive stability/contractivity criteria are the sector and slope constraints imposed on the pair of signals $w(t), y(t)$. We give a formal definition.

DEFINITION 4.1. *A pair of signals $(w(t), y(t))$ of the linear system (4.1) obeys the sector constraint with sector slopes ζ, \varkappa if*

$$(4.5) \quad \zeta y(t)^2 \leq w(t)y(t) \leq \varkappa y(t)^2 \quad \forall t.$$

Two input/output pairs $(w_1(t), y_1(t))$ and $(w_2(t), y_2(t))$ of (4.1) obeys the slope constraint with slopes ζ, \varkappa if

$$(4.6) \quad \zeta \Delta y(t)^2 \leq \Delta w(t) \Delta y(t) \leq \varkappa \Delta y(t)^2,$$

(where, for brevity $\Delta w = w_1 - w_2$ and $\Delta y = y_1 - y_2$).

Obviously, if the function φ obeys the sector condition (4.3) (respectively, the slope condition (4.4)), then every solution (respectively, pair of solutions) of the Lur'e system (4.1), (4.2) obeys the sector constraint (4.5) (respectively, the slope constraint (4.6)).

Without loss of generality, we may confine ourselves to sector and slope constraints with $\zeta = 0$, referring the upper slope $\varkappa \in (0, \infty]$ to as the *rate* of the constraint:

$$(4.7) \quad 0 \leq wy \leq \varkappa y^2 \iff w(\varkappa^{-1}w - y) \leq 0,$$

$$(4.8) \quad 0 \leq \Delta w \Delta y \leq \varkappa \Delta y^2 \iff \Delta w(\varkappa^{-1} \Delta w - \Delta y) \leq 0.$$

Otherwise, system (4.1) and ζ, \varkappa can be replaced by the system

$$\dot{z}(t) = A'z(t) + B'v(t), \quad y(t) = Cz(t),$$

and the new slopes $\zeta' = 0, \varkappa'$, which are defined as follows

- (i) in the case where $\zeta > -\infty$, we denote $v(t) = w(t) - \zeta y(t)$ and $A' = A + \zeta BC$, $B' = B$, $\varkappa' = \varkappa - \zeta$;
- (ii) in the case where $\zeta = -\infty, \varkappa < \infty$, we denote $v(t) = \varkappa y(t) - w(t)$ and $A' = A - \varkappa BC$, $B' = B$, $\varkappa' = \infty$.

4.2. Problem setup. Inspired by the curve norm derivative formula (2.5), we consider the following stability analysis problem.

Problem 4.2 (Global convergence of a Lur'e system with scalar nonlinearity). Given a norm $\|\cdot\|$ with compatible weak pairing $\llbracket \cdot, \cdot \rrbracket$, find conditions on (A, B, C, \varkappa) ensuring the existence of $c > 0$ satisfying the following *Lyapunov inequality*:

$$(4.9) \quad \llbracket Az + Bw, z \rrbracket \leq -c\|z\|^2 \quad \text{for all } z \in \mathbb{R}^d, w \in \mathbb{R} \text{ such that } w(\varkappa^{-1}w - Cz) \leq 0.$$

⁴In the case of static differentiable nonlinearity $w(t) = \phi(y(t))$, this condition means that the minimal and maximal slope of the curve $w = \phi(y)$ are bounded, respectively, by ζ and \varkappa .

The inequality (4.9) allows to use the function $z \mapsto \|z\|^2$ as a global Lyapunov function in order to prove the exponential convergence of each solution to 0 or the exponential contractivity (both with rate c) as shown by the following simple lemma.

LEMMA 4.3. *Suppose that the WP $[[\cdot, \cdot]]$ satisfies the curve norm derivative formula (2.5) and the Lyapunov inequality (4.9). Then, each solution (z, w, y) of (4.1) that obeys the sector constraint (4.7) admits the following estimate*

$$(4.10) \quad \|z(t)\| \leq \|z(0)\| e^{-ct} \quad \forall t \geq 0.$$

Similarly, the deviation between two solutions (z_i, w_i, y_i) (where $i = 1, 2$) that obey the slope constraint (4.8) can be estimated as follows

$$(4.11) \quad \|\Delta z(t)\| \leq e^{-ct} \|\Delta z(0)\| \quad \forall t \geq 0.$$

In both situations, t varies on the maximal interval where the solution(s) exist(s).

Proof. The proof is straightforward by noticing that (4.9) entails that $D^+V(t) \leq -2cV(t)$, where $V(t) = \|z(t)\|^2$ in the case of sector constraint (4.7) and $V(t) = \|\Delta z(t)\|^2$ in the case of slope constraint (4.8). \square

Remark 4.4 (Stability and contractivity). In the case of Lur'e system (4.1),(4.2), the point $z = \mathbb{0}_d$ is usually supposed to be an equilibrium (this is automatically implied by (4.7) provided that $\varkappa < \infty$). In this situation, (4.10) is the stronger form of global exponential stability of this equilibrium, which is called *absolute stability*. The term *absolute* emphasizes that the exponential stability (with a known rate) is ensured for *every* nonlinear feedback φ that obeys the sector condition and, more generally, for every input $w(t)$ that is restricted at any time by sector constraint (4.7). The inequality (4.11) is guaranteed for each nonlinear feedback φ that obeys the slope condition; in this sense it may be called the *absolute contractivity*.

Remark 4.5 (Necessary condition for Problem 4.2). Since $w = 0$ obviously obeys the sector constraint (4.7), the Lyapunov inequality (4.9) can hold only if $[[Az, z]] \leq -c\|z\|^2$, for all $z \in \mathbb{R}^d$. By Lumer's equality (2.2), the latter inequality is equivalent to $\mu(A) \leq -c < 0$. In sum, a necessary condition for Problem 4.2 is $\mu(A) \leq -c$. Note, additionally, that this condition implies A is Hurwitz, since the log norm is an upper bound on the spectral abscissa.

Remark 4.6. In fact, (4.9) can be of interest even for $c \leq 0$; in this case, we cannot guarantee convergence/contractivity of solutions, however, we still can get an estimate of the solution (respectively, the deviation of two solutions). The key result provided below (Theorem 4.10) retains its validity for an arbitrary choice of $c \in \mathbb{R}$.

4.3. Transcription via weak pairings. In this section we aim to transcribe equation (4.9) in Problem 4.2. To this end, we introduce two matrices

$$(4.12) \quad P_0 = \begin{bmatrix} A + cI_d & B \\ \mathbb{0}_{1 \times d} & 0 \end{bmatrix}, \quad P_1 = \begin{bmatrix} \mathbb{0}_{d \times d} & \mathbb{0}_{d \times 1} \\ -C & \varkappa^{-1} \end{bmatrix}$$

and the column vector $x = \begin{bmatrix} z \\ w \end{bmatrix} \in \mathbb{R}^{d+1}$. Obviously,

$$(4.13) \quad P_0 x = \begin{bmatrix} (A + cI_d)z + Bw \\ 0 \end{bmatrix} \quad \text{and} \quad P_1 x = \begin{bmatrix} \mathbb{0}_d \\ \varkappa^{-1}w - Cz \end{bmatrix}.$$

We will show that, in the case of ℓ_p norms, (4.9) boils down to the implication

$$\llbracket P_1 x, x \rrbracket_p \leq 0 \implies \llbracket P_0 x, x \rrbracket_p \leq 0$$

which can be tested via the non-polynomial S-Lemma (Theorem 3.1, statement (iii)). To prove this, we need two technical lemmas.

LEMMA 4.7 (Sector-constraint transcription, for ℓ_p). *The following conditions are equivalent for each $p \in [1, \infty)$ and $x = \begin{bmatrix} z \\ w \end{bmatrix} \in \mathbb{R}^{d+1}$:*

- (i) scalars w and $y = Cz$ obey the inequality (4.7);
- (ii) $\llbracket P_1 x, x \rrbracket_p \leq 0$.

In the case $p = \infty$, implication (i) \implies (ii) holds. The inverse implication (ii) \implies (i) is valid when $|w| > \|z\|_\infty$.

Proof. Consider first the case $p = 1$. Noting $P_1 x = \begin{bmatrix} \mathbb{0}_d \\ -Cz + \varkappa^{-1}w \end{bmatrix}$, we compute

$$\begin{aligned} \llbracket P_1 x, x \rrbracket_1 &= (\|z\|_1 + |w|) \text{sign}(x)^\top P_1 x \\ &= (\|z\|_1 + |w|) \begin{bmatrix} \text{sign}(z) \\ \text{sign}(w) \end{bmatrix}^\top \begin{bmatrix} \mathbb{0}_d \\ \varkappa^{-1}w - Cz \end{bmatrix} = (\|z\|_1 + |w|) \text{sign}(w)(\varkappa^{-1}w - Cz). \end{aligned}$$

Clearly, $\text{sign}(w)(-Cz + \varkappa^{-1}w) \leq 0$ if and only if $w(\varkappa^{-1}w - Cz) \leq 0$.

The case $p \in (1, \infty)$ is considered in a similar way:

$$\begin{aligned} \llbracket P_1 x, x \rrbracket_p &= \|x\|_p^{2-p} (x \circ |x|^{p-2})^\top P_1 x \\ &= \|x\|_p^{2-p} \begin{bmatrix} \text{sign}(z) \circ |z|^{2-p} \\ \text{sign}(w) |w|^{p-1} \end{bmatrix}^\top \begin{bmatrix} \mathbb{0}_d \\ -Cz + \varkappa^{-1}w \end{bmatrix} \\ &= \|x\|_p^{2-p} |w|^{p-1} \text{sign}(w)(-Cz + \varkappa^{-1}w), \end{aligned}$$

which expression is ≤ 0 if and only if $w(\varkappa^{-1}w - Cz) \leq 0$.

In the case $p = \infty$, one has $\llbracket P_1 x, x \rrbracket_p = \max_{i \in I_\infty(x)} (P_1 x)_i x_i$. Obviously,

$$(P_1 x)_i x_i = \begin{cases} 0, & i = 1, \dots, n; \\ w(\varkappa^{-1}w - Cz), & i = n + 1. \end{cases}$$

Therefore, (4.7) entails inequality $\llbracket P_1 x, x \rrbracket_\infty \leq 0$, whereas the inverse statement is, generally, incorrect except for the case where $I_\infty(x) = \{n + 1\}$, that is, $|w| > \|z\|_\infty$. \square

To show that the first (Lyapunov) inequality in (4.9) shapes into $\llbracket P_0 x, x \rrbracket \leq 0$, an additional technical lemma is required. Notice that for two vectors in \mathbb{R}^{d+1}

$$(4.14) \quad x^1 = \begin{bmatrix} z^1 \\ 0 \end{bmatrix}, \quad x^2 = \begin{bmatrix} z^2 \\ w^2 \end{bmatrix} \in \mathbb{R}^{d+1},$$

and $p \neq 2$, generally, $\llbracket x^1, x^2 \rrbracket_p \neq \llbracket z^1, z^2 \rrbracket_p$. However, the following result holds.

LEMMA 4.8 (Useful property for weak pairing).

- (i) For the vectors (4.14) (where $z^i \in \mathbb{R}^d$ and $w^2 \in \mathbb{R}$) and $1 \leq p < \infty$, we have

$$(4.15) \quad \llbracket x^1, x^2 \rrbracket_p \leq 0 \iff \llbracket z^1, z^2 \rrbracket_p \leq 0.$$

(ii) If, additionally, $|w^2| < \|z^2\|_\infty$ or $w^2 = \|z^2\|_\infty = 0$, then $\llbracket z^1, z^2 \rrbracket_\infty = \llbracket x^1, x^2 \rrbracket_\infty$.

Proof. The proof is straightforward from Table 1. In the case where $x^2 = 0$ the statement is trivial, so we always suppose that $\|x^2\| \neq 0$.

If $p < \infty$, then $\llbracket z^1, z^2 \rrbracket_p = k \llbracket x^1, x^2 \rrbracket_p$, where $k > 0$ is some positive number. In the case $p = \infty$ (and thus $|w^2| < \|z^2\|_\infty$), one may notice that $\llbracket x^1, x^2 \rrbracket_\infty = \max_{i \in I_\infty(x^2)} x_i^1 x_i^2$, where $I_\infty(x^2) = \{j : |x_j^2| = \|x^2\|_\infty\} = I_\infty(z^2) \not\equiv d+1$. Therefore, $\llbracket x^1, x^2 \rrbracket_\infty = \max_{i \in I_\infty(z^2)} z_i^1 z_i^2 = \llbracket z^1, z^2 \rrbracket_\infty$. \square

COROLLARY 4.9 (Lyapunov-inequality transcription). *For all $p \in [1, \infty)$, the inequalities $\llbracket Az + Bw, z \rrbracket_p \leq -c\|z\|_p^2$ and $\llbracket P_0 x, x \rrbracket_p \leq 0$ are equivalent. If $p = \infty$, these inequalities are also equivalent for such vectors x that $x = 0$ or $|w| < \|z\|_\infty$.*

Proof. In view of (2.6), the inequality $\llbracket Az + Bw, z \rrbracket_p \leq -c\|z\|_p^2$ can be written as $\llbracket (A + cI)z + Bw, z \rrbracket_p \leq 0$. The statement is now obvious from Lemma 4.8, applied to $z^1 = (A + cI)z + Bw$, $z^2 = z$ and $w^2 = w$. \square

4.4. Sufficient conditions for stability. We are now ready to give a sufficient condition for Problem 4.2 in the special situation where $\llbracket \cdot, \cdot \rrbracket = \llbracket \cdot, \cdot \rrbracket_p$.

THEOREM 4.10. *[Sufficient conditions for Problem 4.2 and special cases] Consider a Lur'e system with sector constraint defined by (A, B, C, \varkappa) . Fix a norm ℓ_p , $p \in [1, \infty]$ with log norm μ_p and weak pairing $\llbracket \cdot, \cdot \rrbracket_p$. If $p = \infty$, assume also that $\varkappa\|C\|_\infty < 1$. Then the following statements hold:*

(i) *the Lyapunov inequality (4.9) holds if*

$$(4.16) \quad \llbracket P_0 x, x \rrbracket_p \leq 0 \quad \text{for all } x \in \mathbb{R}^{d+1} \text{ such that } \llbracket P_1 x, x \rrbracket_p \leq 0;$$

furthermore, (4.9) and (4.16) are equivalent when $p < \infty$;

(ii) *the Lyapunov inequality (4.9) holds if*

$$(4.17) \quad \exists \tau \geq 0 \text{ such that } \mu_p(P(\tau)) \leq 0 \text{ with } P(\tau) = \begin{pmatrix} A + cI_d & B \\ \tau C & -\tau \varkappa^{-1} \end{pmatrix}$$

(iii) *if $p = 2$, then condition (4.17) is not only sufficient but also necessary for the Lyapunov inequality (4.9);*

(iv) *if $p = 1$, A is Metzler, and B, C are non-negative, then condition (4.17) is not only sufficient but also necessary for the Lyapunov inequality (4.9).*

Notice that (4.17) with $p = \infty$ in principle can hold only when $\varkappa\|C\|_\infty \leq 1$. In view of this, the assumption $\varkappa\|C\|_\infty < 1$ is not very restrictive.

Proof. Statement (i) in the case where $p \in [1, \infty)$ is straightforward from Lemma 4.7 and Corollary 4.9. To prove it for $p = \infty$, one has to use the assumption $\varkappa\|C\|_\infty < 1$ entailing that $|w| \leq \varkappa\|Cz\| \leq \varkappa\|C\|_\infty\|z\|_\infty$, that is, $|w| < \|z\|_\infty$ unless $w = \|z\|_\infty = 0$. Corollary 4.9 entails that the first inequality in (4.9) is also equivalent to $\llbracket P_0 x, x \rrbracket_\infty \leq 0$. The second inequality in (4.9) also implies the relation $\llbracket P_1 x, x \rrbracket_\infty \leq 0$ (Lemma 4.7). Hence, (4.16) is sufficient (yet not necessary) for (4.9) also for $p = \infty$.

Statement (ii) is now straightforward from Theorem 3.1.

Statements (iii) and (iv) follow, respectively, from Lemmas 3.5 and 3.8 (one can easily notice that P_0 and $(-P_1)$ are Metzler matrices). \square

Theorem 4.10 can be generalized to the weighted norm case. Introducing a new variable $\tilde{z} = Rz$, one easily shows that (4.9) holds for $\llbracket \cdot, \cdot \rrbracket = \llbracket \cdot, \cdot \rrbracket_{p,R}$, where $R \in \mathbb{R}^{d \times d}$

is an invertible matrix, if and only if it holds for $[[\cdot, \cdot]] = [[\cdot, \cdot]]_p$ and matrices $\tilde{A} = RAR^{-1}$, $\tilde{B} = RB$, $\tilde{C} = CR^{-1}$. Introducing the block-diagonal matrix $\tilde{R} = \text{diag}(R, 1)$, one obtains the following corollary.

COROLLARY 4.11. *Suppose that $[[\cdot, \cdot]] = [[\cdot, \cdot]]_{p,R}$, where $1 \leq p \leq \infty$ and, if $p = \infty$ one has $\varkappa \|CR^{-1}\|_\infty < 1$. Then (4.9) holds if $\tau \geq 0$ exists such that*

$$(4.18) \quad \mu_{p,\tilde{R}}(P(\tau)) = \mu_p \begin{pmatrix} RAR^{-1} + cI_d & RB \\ \tau CR^{-1} & -\tau \varkappa^{-1} \end{pmatrix} \leq 0.$$

If $p = 2$, (4.18) is also necessary for (4.9); the same holds when $p = 1$, RAR^{-1} is a Metzler matrix and matrices RB, CR^{-1} are nonnegative.

Remark 4.12 (Convergence and contractivity in a single criterion). In view of Lemma 4.3, Theorem 4.10 and Corollary 4.11 give simultaneously a criterion of global exponential convergence for solutions that obey the sector constraint (4.7) and a criterion of exponential contraction for solutions obeying the slope constraint (4.8).

We would like to notice that, for fixed $p \in [1, \infty]$ and fixed invertible matrix R , the feasibility of (4.18) reduces to a *convex program*

$$(4.19) \quad \inf_{c,\tau} \mu_{p,R} \begin{pmatrix} A + cI_d & B \\ \tau C & -\tau \varkappa^{-1} \end{pmatrix},$$

subject to: $c > 0, \tau \geq 0$.

The condition (4.18) holds if and only if the answer is non-positive. Unfortunately, the logarithmic norm can be computed efficiently only for $p = 1, 2, \infty$, where its computation in the case of general p is a difficult operation.⁵ Regarding R as a parameter, the problem becomes in general *non-convex* as the function $\mu_{p,R}$ is non-convex in R . However, when $p = 2$ or when $p \in \{1, \infty\}$ and R is diagonal, one can show that the problem is quasiconvex in R (after a transcription), at fixed c and τ . Therefore, the combined problem of computing R and searching for the contraction factor c can be solved via a bisection algorithm. A discussion of the $p = 2$ case is given in the classic text on LMIs in control [10, Section 5.1] and a discussion of the $p \in \{1, \infty\}$ for the case of diagonal weights R is given in [12, Section 2.7]. We discuss both cases in some detail below.

4.5. The ℓ_2 norm case: comparison with known results. In the case $p = 2$, Theorem 4.10 and Corollary 4.11 lead to several known results.

4.5.1. Analytic form of the condition (4.17). Recall that for an arbitrary square matrix M one has $\mu_2(M) = \lambda_{\max}(M^s)$, where $M^s = (M + M^\top)/2$ is the symmetric part of M . In other words, $\mu_2(M) \leq 0$ (respectively, < 0) if and only if $M^s \preceq 0$ (respectively, $M^s \prec 0$). This allows us to simplify the condition (4.17). It can be easily shown that (4.17) (with $p = 2$) cannot hold with $\tau = 0$, except for the trivial situation where $B = 0$. For this reason, we are interested only in $\tau > 0$.

COROLLARY 4.13. *Inequality (4.9) with $p = 2$, $\varkappa < \infty$, $\tau > 0$ holds if and only if*

$$(4.20) \quad \exists \tau > 0 : A^s + cI + \frac{\varkappa}{4\tau} (B + \tau C^\top)(B^\top + \tau C) \preceq 0.$$

⁵Note that computation of the matrix operator norm $\|\cdot\|_p$ with $p \in (1, \infty) \setminus \{2\}$ (even up to a constant factor) proves to be an NP-hard problem [9].

Proof. The left-hand side of (4.20) is nothing else than the Schur complement of the bottom-right diagonal entry in the matrix

$$P(\tau)^s = \begin{bmatrix} A^s + cI & \frac{1}{2}(B + \tau C^\top) \\ * & -\tau \varkappa^{-1} \end{bmatrix}.$$

Since $\tau \varkappa^{-1} < 0$, the Haynsworth [32] theorem implies that $P(\tau)^s \preceq 0$ (i.e., (4.17) holds with $p = 2$) if and only if this Schur complement is negative semidefinite. \square

In stability and contraction problems, one is primarily interested in the *existence* of such a convergence/contractivity rate $c > 0$ that (4.20) holds, that is, in the validity of the strict inequality as follows

$$(4.21) \quad \exists \tau > 0 : A^s + \frac{\varkappa}{4\tau}(B + \tau C^\top)(B^\top + \tau C) \prec 0.$$

Notice that (4.20) can be rewritten as

$$(4.22) \quad \begin{aligned} \exists \tau > 0 : \quad \mathcal{A}(\varkappa) + \frac{\varkappa}{4\tau}(B - \tau C^\top)(B^\top - \tau C) \prec 0, \\ \mathcal{A}(\varkappa) := A^s + \varkappa(BC)^s. \end{aligned}$$

In particular, (4.20) entails that $\mathcal{A}(\varkappa) \prec 0$. This condition, however, is only necessary yet not sufficient for (4.13). A necessary and sufficient condition is provided by the following lemma, relating the result of our Theorem 4.10 and [25, Theorems 1 and 3].

LEMMA 4.14. *The following statements are equivalent:*

- (i) condition (4.21) (equivalently, (4.22)) holds;
- (ii) the family of inequalities (4.23) is valid

$$(4.23) \quad \mathcal{A}(\kappa) \prec 0 \quad \forall \kappa \in [0, \varkappa].$$

- (iii) $\mathcal{A}(\varkappa) \prec 0$ and the matrix $S = (-\mathcal{A}(\varkappa))^{-1/2} \succ 0$ satisfies the condition

$$(4.24) \quad \|\tilde{B}\|\|\tilde{C}\| - \tilde{C}\tilde{B} \leq 2\varkappa^{-1}, \quad \tilde{B} := SB, \tilde{C} := CS.$$

Proof. To show that (i) implies (ii), notice first that the validity of (4.20) with rate \varkappa entails its validity with every smaller rate $\kappa \in [0, \varkappa]$. Using the decomposition (4.22) (with κ instead of \varkappa), one shows that $\mathcal{A}(\kappa) \prec 0$.

To prove the implication (ii) \implies (iii), notice that inequalities (4.23) can be written as follows: $S^{-2} + (\varkappa - \kappa)(BC)^s \succ 0 \forall \kappa \in [0, \varkappa]$ or, equivalently, $I + k(\tilde{B}\tilde{C})^s \succ 0 \forall k \in [0, \varkappa]$. The latter condition can be formulated as follows: the minimal eigenvalue $\lambda_{\min}((\tilde{B}\tilde{C})^s) \geq -\varkappa^{-1}$. The eigenvalues of $(\tilde{B}\tilde{C})^s$ are easy to compute: two of them equal $(\tilde{C}\tilde{B} \pm \|\tilde{B}\|\|\tilde{C}\|)/2$ and the $(n - 2)$ are zero (if B and C are parallel, then matrix has only one non-zero eigenvalue). Hence, (4.23) can be written as follows: $\tilde{C}\tilde{B} - \|\tilde{B}\|\|\tilde{C}\| \geq -2\varkappa^{-1}$, which is equivalent to (4.24).

To prove the final implication (iii) \implies (i), recall that $\mathcal{A}(\varkappa) = -S^2$. Hence (4.22) can be equivalently written as follows: $\tau > 0$ exists such that

$$-I + \frac{\varkappa}{4\tau}(\tilde{B} - \tau\tilde{C}^\top)(\tilde{B}^\top - \tau\tilde{C}) \prec 0,$$

which holds if and only if $\|\tilde{B} - \tau\tilde{C}^\top\|^2 < 4\tau\varkappa^{-1}$. It can be easily checked that (4.24) entails the latter inequality with $\tau = \|\tilde{B}\|/\|\tilde{C}\|$. This finishes the proof. \square

The inequalities (4.23) were introduced in [25] as a condition for the absolute stability (in the case of nonlinearities with sector condition) and the absolute contractivity (in the case of slope condition). Lemma 4.14 shows that this condition is a special case of the standard S-Lemma for the ℓ_2 -norm. Statement (iii) gives an efficient way to test the condition (4.23). Furthermore, the classical Yakubovich S-Lemma (Lemma 3.5) shows that, in fact, (4.17) is *necessary* for the Lyapunov property (4.9), and hence (4.23) are in fact minimal conditions under which absolute stability (4.10) (with sector constraint) and absolute contractivity (4.11) (with slope constraint) in ℓ_2 norm can be established.

4.5.2. The LMI stability criterion and KYP lemma. In this subsection, we consider the standard reformulation of Corollary 4.11 in the case $p = 2$. Without loss of generality we may assume that R in Corollary 4.11 is symmetric positive definite, and thus it can be written as $R = H^{1/2}$, where $H = H^\top \succ 0$. Indeed, $\llbracket x, y \rrbracket_{2,R} = \llbracket R^\top R x, y \rrbracket_2 = \llbracket x, y \rrbracket_{2,H^{1/2}}$, where $H = R^\top R \succ 0$ and where R is invertible). The inequality (4.18) with $p = 2$ is therefore equivalent to

$$\begin{bmatrix} RAR^{-1} + cI_d & RB \\ \tau CR^{-1} & -\tau \varkappa^{-1} \end{bmatrix} + \begin{bmatrix} RAR^{-1} + cI_d & RB \\ \tau CR^{-1} & -\tau \varkappa^{-1} \end{bmatrix}^\top \leq 0,$$

which, recalling that $R = H^{1/2} > 0$, further simplifies to the condition [18]

$$(4.25) \quad \begin{bmatrix} HA + A^\top H + 2cH & HB + \tau C \\ * & -\tau \varkappa^{-1} \end{bmatrix} \leq 0.$$

This condition, obviously, cannot hold for $\tau = 0$ (except for the degenerate case where $B = 0$). If we are interested in the existence of some $H \succ 0, c > 0$, then we may assume, without loss of generality that $\tau = 1$. If the triple (A, B, C) is controllable and observable, the Kalman-Yakubovich-Popov lemma [29] implies that the existence of $H = H^\top \succ 0, c > 0$ obeying (4.25) is equivalent⁶ to the frequency-domain condition

$$(4.26) \quad \max_{\omega \in \mathbb{R}} \operatorname{Re} (C(j\omega I - A)^{-1} B) < \varkappa^{-1}.$$

This is a special case of the so-called ‘‘circle criterion’’ for absolute stability [29], in which the circle degenerates to a half-plane on \mathbb{C} .

4.6. A sufficient condition for the diagonally weighted ℓ_p norm. This subsection offers a simple condition ensuring (4.18) for an arbitrary $p \in [1, \infty]$ and some *diagonal* weight matrix R . This condition is inspired by properties of Metzler matrices and their logarithmic norms. While this Metzler-based approach is potentially conservative for general triples of matrices (A, B, C) , we refer the reader to the discussion in the introduction about the advantages of non-Euclidean norms.

Recall that $[\cdot]_{\text{Mzr}}$ stands for the Metzler majorant of a matrix and $|\cdot|$ stands for the entry-wise absolute value; we also use $\alpha(A) = \max \operatorname{Re} \lambda_j(A)$ to denote the spectral abscissa of matrix A . The following lemma is not widely known in the control literature, but has been established in a sequence of paper on linear algebra, see [54, Theorem 2], [4], and [45, Lemma 3]. We provide its proof for readers’ convenience.

⁶The KYP lemma (relying only on the controllability and observability) guarantees, in fact, that the strict inequality in (4.25) holds for appropriate $c > 0, H = H^\top$; formally, H is not guaranteed to be positive definite. However, the strict inequality in (4.25) entails that $H \succ 0$ when A is Hurwitz.

LEMMA 4.15. For each $\varkappa < \infty$, consider the Metzler matrix $\mathfrak{A}(\varkappa) = [A]_{\text{Mzr}} + \varkappa|B||C|$. The following two statements hold:

- (i) If $\alpha(\mathfrak{A}(\varkappa)) < -c$ (that is, $[A]_{\text{Mzr}} + \varkappa|B||C| + cI$ is a Hurwitz matrix), then for every $p \in [1, \infty]$, $\tau > 0$ there exists a diagonal matrix $R = R(p, \tau) \succ 0$ such that (4.18) holds and, in the case of $p = \infty$, $\varkappa\|CR^{-1}\|_\infty < 1$.
- (ii) On the other hand, if (4.18) holds with some diagonal $R \succ 0$ and $p \in \{1, \infty\}$, then $\alpha(\mathfrak{A}(\varkappa)) \leq -c'$, i.e., the condition from (i) holds for each $c' \in (0, c)$.

Proof of Lemma 4.15. To prove (i), we first apply [26, Lemma 2] (Schur complements for Metzler Hurwitz matrices) to the Metzler majorant of the matrix $P(\tau)$

$$[P(\tau)]_{\text{Mzr}} = \begin{bmatrix} [A]_{\text{Mzr}} + cI & |B| \\ \tau|C| & -\tau\varkappa^{-1} \end{bmatrix}, \quad \tau > 0$$

showing that $[P(\tau)]_{\text{Mzr}}$ is Hurwitz if (and in fact, only if) the Metzler matrix $\mathfrak{A}(\varkappa) + cI = [A]_{\text{Mzr}} + \varkappa|B|(\tau\varkappa^{-1})^{-1}\tau|C| + cI$ is Hurwitz, that is, $\alpha(\mathfrak{A}(\varkappa)) < -c$.

Hence, [45, Lemma 3] (the lemma on optimally diagonally weighted log norms for Metzler matrices) ensures the existence of a diagonal matrix $\tilde{R} \succ 0$ such that $\mu_{p, \tilde{R}}([P(\tau)]_{\text{Mzr}}) < 0$. Rescaling \tilde{R} , one may assume without loss of generality that $\tilde{R} = \text{diag}(R, 1)$, where $R \succ 0$. Noticing that the norm $\|\cdot\|_{p, \tilde{R}}$ is *monotone*, that is, for each two vectors $x, y \in \mathbb{R}^{d+1}$ such that $|x_i| \leq |y_i| \forall i$, one has $\|x\|_{p, \tilde{R}} \leq \|y\|_{p, \tilde{R}}$ and applying [12, Theorem 2.23], one proves that (4.18):

$$\mu_p \begin{pmatrix} RAR^{-1} + cI_d & RB \\ \tau CR^{-1} & -\tau\varkappa^{-1} \end{pmatrix} = \mu_{p, \tilde{R}}(P(\tau)) \stackrel{\text{monotonicity}}{\leq} \mu_{p, \tilde{R}}([P(\tau)]_{\text{Mzr}}) < 0.$$

Finally, if $p = \infty$, one can also notice that

$$-\tau\varkappa^{-1} + \tau\|CR^{-1}\|_\infty \leq -\tau\varkappa^{-1} + \tau\|CR^{-1}\|_1 \stackrel{(*)}{\leq} \mu_\infty \begin{pmatrix} RAR^{-1} + cI_d & RB \\ \tau CR^{-1} & -\tau\varkappa^{-1} \end{pmatrix} < 0.$$

Here the inequality $(*)$ holds due to the representation of μ_∞ (Table 1).

To prove (ii), notice that $\alpha([P(\tau)]_{\text{Mzr}}) \leq \mu_{p, \tilde{R}}([P(\tau)]_{\text{Mzr}}) = \mu_{p, \tilde{R}}(P(\tau)) \leq 0$ when $p \in \{1, \infty\}$ and $\tilde{R} \succ 0$ is a diagonal matrix. Hence, for each $\varepsilon > 0$ the matrix

$$[P(\tau)]_{\text{Mzr}} - (\tau\varkappa^{-1}\varepsilon)I_{d+1} = \begin{bmatrix} [A]_{\text{Mzr}} + (c - \tau\varkappa^{-1}\varepsilon)I_d & |B| \\ \tau|C| & -\tau\varkappa^{-1}(1 + \varepsilon) \end{bmatrix}$$

is Hurwitz. Invoking [26, Lemma 2], one proves that $A + (c - \tau\varkappa^{-1}\varepsilon)I_d + \varkappa(1 + \varepsilon)^{-1}|B||C| = \mathfrak{A}(\varkappa) + cI_d + O(\varepsilon)$ is a Hurwitz matrix for an arbitrary small $\varepsilon > 0$. Statement (ii) is now proved by taking the limit as $\varepsilon \rightarrow 0$. \square

Remark 4.16. Recall that, in view of Corollary 4.11, for the case where A is Metzler and B, C are nonnegative, the Lyapunov condition (4.9) holds for the norm $\|\cdot\|_{1, R}$, where $R \succ 0$ is diagonal matrix, *if and only if* (4.18) is valid. The condition $\alpha(\mathfrak{A}(\varkappa)) < 0$ is hence the *necessary and sufficient* condition for the absolute stability (4.10) (with sector constraint) and absolute contractivity (4.11) (with slope constraint) with respect to diagonally weighted ℓ_1 -norms.

4.7. A discussion on Aizerman and Kalman conjectures. It is interesting to compare the results of Lemma 4.14 and Lemma 4.15 with the two famous conjectures that were formulated at the dawn of nonlinear control theory. Aizerman [1]

conjectured that for the global stability of the equilibrium $x = 0$ in the Lur'e system (4.1),(4.2) with an arbitrary continuous nonlinearity in sector (4.5) it suffices to prove stability with all linear feedback functions $\varphi(y) = ky$, where $k \in [\zeta, \varkappa]$, that is,

$$(4.27) \quad \alpha(A + kBC) < 0 \quad \forall k \in [\zeta, \varkappa].$$

(all matrices $A + kBC$ are Hurwitz). Notice that (4.27) can be efficiently tested, e.g., via the Nyquist criterion. Later, Kalman [35] conjectured that (4.27) guarantees global stability of (4.1),(4.2) with every differentiable nonlinearity such that $\zeta \leq \varphi'(y) \leq \varkappa$.

In general, neither Aizerman's nor even Kalman's conjecture proves to be valid when the dimension of the state vector is $d \geq 3$; the reader is referred to [25, 37] for the survey of main historical milestones and recent achievements in this area. However, these conjectures may be valid for special triples of matrices (A, B, C) .

The criterion from Lemma 4.14 (partly available in [25]) shows that the Aizerman conjecture is valid in the situation where A is symmetric and B, C^\top are parallel (in this situation, BC is also a symmetric matrix). Note that formally we have considered only the sector with $\zeta = 0$, however, the transformation introduced in Subsection 4.1 allows to discard this assumption. Lemma 4.15 shows that the Aizerman conjecture is valid for the case where A is Metzler and B, C are nonnegative and the sector's lower slope is $\zeta = 0$. Notice that this fact is typically proved by using the diagonally weighted ℓ_2 norm as a Lyapunov function [14]; Lemma 4.15 shows that, in fact, one can use diagonally weighted ℓ_p norm with an arbitrary choice of $p \in [1, \infty]$. In both situation, the stronger version of Kalman's conjecture proves to be valid: if the nonlinearity is slope-restricted in the sense that $0 \leq \varphi'(y) \leq \varkappa$, then the Lur'e system (4.1),(4.2) enjoys the exponential global contractivity property.

4.8. Generalization to MIMO nonlinear blocks. To keep the presentation in this paper simple, we have confined ourselves to the classical Lur'e system with a scalar nonlinear block. However, the general form of Theorem 3.1 with $s > 1$ constraints allows us to extend the results of these paper to many kinds of multidimensional nonlinearities, e.g., *diagonal* nonlinearities $w(t) = \Phi(y(t))$ where $\dim y = \dim w = p$ and $\Phi(y)$ is a diagonal matrix whose i th diagonal entry $\Phi(y)_{ii} = \varphi_i(y_i)$ depends only on y_i . Assuming that all scalar functions $\varphi_i(y_i)$ obey the sector or slope constraint (4.7) or (4.8) (where w, y have to be replaced by w_i, y_i), Theorem 4.10 retains its validity with the only difference that in (4.17) τ is not a scalar but a diagonal matrix $\tau = \text{diag}(\tau_1, \dots, \tau_p)$; the corollaries of Theorem 4.10 thus also can be generalized to the case of Lur'e-type systems with MIMO nonlinearities. Such systems arise in a broad range of applications, e.g., in dynamical models of neural circuits [20, 36].

5. Conclusions and future works. One of keystones of modern nonlinear control theory, the S-Lemma serves a convenient tool for Lyapunov stability and contractivity analysis of nonlinear systems. This lemma enables one to transform quadratic constraints on nonlinearities (e.g., sector or slope constraints) into the Lyapunov condition $\dot{V} \leq -cV$, where V is a positive definite quadratic form of the state vector (or, in contraction analysis, state increment) and $c \geq 0$ is the convergence or contraction rate. Quadratic Lyapunov functions allow us to estimate the convergence and contraction rate in some *Euclidean* norm (that is, a norm induced by an inner product). To obtain such estimates in *non-Euclidean* norms, e.g., the ℓ_p norms, we generalize the classical S-Lemma to non-quadratic functions that are induced by the *weak pairing* associated with the norm. Using this generalized S-Lemma, we derive novel criteria for absolute stability and contractivity that are based on ℓ_p norms (possibly, weighted)

and give alternative proofs of some recent results, in particular, symmetrization-based stability criteria from [25] and the Aizerman conjecture for positive Lur'e systems [14].

A topic of ongoing research is to tighten the criteria of absolute stability and contractivity by accounting additional properties of nonlinearities, for instance, their boundedness as in [28]; we believe that this extension will be helpful in ℓ_1/ℓ_∞ robustness, reachability and safety analysis of neural networks. Another direction of ongoing research is to obtain efficient numerical algorithms for validation of the conditions (4.17) for $p \neq 1, 2, \infty$. Although the exact computation of the log norm is troublesome, some upper estimates can potentially be employed.

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Appendix A. Proof of Proposition 2.1. Without loss of generality, we may assume that M is non-negative (otherwise, replace M by the matrix $M + \alpha I$ with α chosen large enough; obviously, this operation increases all three parts of (2.4) by α).

The first equality in (2.4) is straightforward from Table 1: $|M_{ij}| = M_{ij}^+ \geq 0 \forall i \neq j$.

Considering the open simplex $\Delta = \{x > 0 \mid \|x\|_1 = 1\}$ and its closure $\bar{\Delta} = \{x \geq 0 \mid \|x\|_1 = 1\}$, we know (Table 1) that $\mu_1^+(M) = \sup_{x \in \bar{\Delta}} \llbracket Mx, x \rrbracket \geq \sup_{x \in \Delta} \llbracket Mx, x \rrbracket$. To prove the inverse inequality, notice that for M being non-negative, the function $\llbracket Mx, x \rrbracket$ is *concave* on $\bar{\Delta}$. Indeed, denoting $I(x) = \{i : x_i > 0\}$, one has

$$\llbracket Mx, x \rrbracket_1 = \sum_{i \in I(x)} (Mx)_i.$$

Given two vectors $x^0, x^1 \in \Delta$, $\theta \in (0, 1)$ and denoting $x^\theta = \theta x^1 + (1 - \theta)x^0$, one has $I(x^\theta) = I(x^0) \cup I(x^1)$, $Mx^\theta \geq \theta(Mx^1) \geq 0$ and $Mx^\theta \geq (1 - \theta)Mx^0 \geq 0$. Thus

$$f(\theta) := \sum_{i=1}^n (Mx^\theta)_i \geq (1 - \theta) \sum_{i \in I(x^0)} (Mx^0)_i + \theta \sum_{i \in I(x^1)} (Mx^1)_i = (1 - \theta)f(0) + \theta f(1).$$

In particular, $\liminf_{\theta \rightarrow 0^+} f(\theta) \geq f(0)$. Choosing x^0 and x^1 in such a way that $x^\theta > 0$ for each $\theta \in (0, 1)$ (that is, $I(x^0) \cup I(x^1) = \{1, \dots, n\}$), one proves that

$$\llbracket Mx^0, x^0 \rrbracket_1 \leq \liminf_{\theta \rightarrow 0^+} \llbracket Mx^\theta, x^\theta \rrbracket \leq \sup_{x \in \Delta} \llbracket Mx, x \rrbracket_1.$$

Since, $x^0 \in \bar{\Delta}$ can be arbitrary, we have $\mu_1^+(M) = \sup_{x \in \bar{\Delta}} \llbracket Mx, x \rrbracket \leq \sup_{x \in \Delta} \llbracket Mx, x \rrbracket$, which proves the second equality in (2.4).