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Time-Varying Spectrum of the Random String[1]

L. Galleani[a] and L. Cohen[b,2]

[a]Department of Electronics and Telecommunications, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy
[b]Department of Physics, Hunter College of the City University of New York, 695 Park Ave., New York, NY 10065, USA

Abstract: We consider the response of a finite string to white noise and obtain the exact time-dependent spectrum. The complete exact solution is obtained, that is, both the transient and steady-state solution. To define the time-varying spectrum we ensemble average the Wigner distribution. We obtain the exact solution by transforming the differential equation for the string into the phase space differential equation of time and frequency and solve it directly. We also obtain the exact solution by an impulse response method which gives a different form of the solution. Also, we obtain the time-dependent variance of the process at each position. Limiting cases for small and large times are obtained. As a special case we obtain the results of van Lear Jr. and Uhlenbeck and Lyon. A numerical example is given and the results plotted.

Keywords: random string; Wigner distribution; Brownian motion; time-varying spectrum

1 Introduction

Some time after the classical theory of Brownian motion of particles was developed [3], Ornstein [13] and van Lear Jr. and Uhlenbeck [9] considered the application of Brownian motion theory [16, 17] to extended bodies and in particular to the finite string. Since that time the random finite string has become a model problem for studying extended bodies acted on by random forces. Previous work involved the steady-state spectrum solution and that was generally achieved by starting the system at minus infinity wherein, therefore, an infinite time has gone by. In this paper we consider the random string problem and start at a finite time [9, 11]. We obtain the complete solution to the time-varying spectrum. To define the time-dependent spectrum we use the Wigner spectrum and obtain the exact solution.

This paper is organized as follows. In the next section we review the string equation with external forces. Then, we discuss how to handle nonstationary noise by way of the Wigner spectrum. Subsequently, we obtain the exact solution for the Wigner spectrum by two different methods and discuss how it approaches steady state.

2 String equation with external force

In preparation for solving the random case, in this section we review the deterministic string equation and its solution. We use the notation of Van Lear Jr. and Uhlenbeck and follow their approach. The only notational difference is that we use 2μ for their β.

The string equation with damping for unit string density is [9]

$$\frac{\partial^2 s}{\partial t^2} + 2\mu\frac{\partial s}{\partial t} = p'\frac{\partial s}{\partial x} + p\frac{\partial^2 s}{\partial x^2} + A(x,t) \tag{1}$$

[1]It is a pleasure to dedicate this paper to Prof. Igor Jex, who has made fundamental contributions to quantum dynamics and the interface with classical mechanics. In that spirit, this paper extends the results of van Lear Jr. and Uhlenbeck on the random string to the nonstationary case using the Wigner distribution.

[2]Corresponding author. Email: leon.cohen@hunter.cuny.edu

where $A(x,t)$ is the external applied force which subsequently we take to be the random force. The approach [9] to solve Eq. (1) is to first solve the homogeneous equation

$$p\frac{\partial^2 s_H}{\partial x^2} - \frac{\partial^2 s_H}{\partial t^2} - 2\mu\frac{\partial s_H}{\partial t} + p'\frac{\partial s_H}{\partial x} = 0 \tag{2}$$

by separation of variables. This results in a complete set of eigenfunctions $T_n(t)$ and $X_n(x)$ which are solutions of

$$pX_n''(x) + p'X_n'(x) = -\lambda_n X_n(x) \tag{3}$$

$$T_n''(t) + 2\mu T_n'(t) = -\lambda_n T_n(t) \tag{4}$$

and where $-\lambda_n$ are the eigenvalues. The eigenfunctions, $X_n(x)$, are orthogonal and complete and are assumed to be normalized to one[3]

$$\int X_n^*(x)X_m(x)dx = \delta_{nm} \tag{5}$$

where δ_{nm} is the Kronecker delta function. The general solution to the homogeneous equation is then

$$s_H(x,t) = \sum a_n T_n(t)X_n(x) \tag{6}$$

where

$$a_n = \int s_H(x,0)X_n^*(x)dx \tag{7}$$

2.1 Inhomogeneous equation

To solve the inhomogeneous case, Eq. (1), one expands the unknown solution, $s(x,t)$, and the known source term, $A(x,t)$, as

$$s(x,t) = \sum S_n(t)X_n(x) \tag{8}$$

$$A(x,t) = \sum A_n(t)X_n(x) \tag{9}$$

where $A_n(t)$ are known and are calculated by way of

$$A_n(t) = \int A(x,t)X_n^*(x)dx \tag{10}$$

The equations for the unknowns, $S_n(t)$, are obtained by substituting Eq. (8) and Eq. (9) into Eq. (1) to obtain an ordinary differential equation for each $S_n(t)$

$$S_n''(t) + 2\mu S_n'(t) + \lambda_n S_n(t) = A_n(t) \tag{11}$$

and which has to be solved for each $S_n(t)$. The solution is given by [9]

$$S_n(t) = \left[\frac{\mu S_n(0) + S_n'(0)}{\omega_n}\sin\omega_n t + S_n(0)\cos\omega_n t\right]e^{-\mu t} + \frac{1}{\omega_n}\int_0^t A_n(t')e^{-\mu(t-t')}\sin\omega_n(t-t')dt' \tag{12}$$

where

$$\omega_n = \sqrt{\lambda_n - \mu^2} \tag{13}$$

where one must take

$$\mu^2 < \lambda_n \tag{14}$$

Once one obtains the $S_n(t)$'s the general solution, $s(x,t)$, is given by Eq. (8). Following Van Lear and Uhlenbeck we consider the case of an initially flat string where $S_n(0) = S_n'(0) = 0$. Hence we take

$$S_n(t) = \frac{1}{\omega_n}\int_0^t A_n(t')e^{-\mu(t-t')}\sin\omega_n(t-t')dt' \tag{15}$$

We note that our method allows also to specify non-zero initial conditions of the string by setting proper values for $S_n(0)$ and $S_n'(0)$ in Eq. (12). Moreover, in Eq. (12), we take $A_n(t) = 0$ for $t < 0$, and, consequently, the initial conditions $S_n(0)$ and $S_n'(0)$ hold for $t \geq 0$ only.

[3]Unless specified, all integrals are from $-\infty$ to ∞ and all summations are from 1 to ∞.

2.2 Statistics of $A_n(t)$

In the next section we will be finding the time-frequency spectrum for both the transient and steady state. We shall need the statistics of $A_n(t)$ as given by Eq. (10). In Appendix A we show that if we take the driving force, $A(x,t)$, to be

$$A(x,t) = u(t)\eta(x,t) \tag{16}$$

where $u(t)$ is the step function and $\eta(x,t)$ is white Gaussian noise in space and time, then $A_n(t)$ is white Gaussian noise starting at $t=0$, given by

$$A_n(t) = u(t)\xi_n(t) \tag{17}$$

where $\xi_n(t)$ is white Gaussian noise for each n, with zero mean and autocorrelation function

$$E\left[\xi_n(t')\xi_n(t'')\right] = \delta(t'-t'') \tag{18}$$

where E is the ensemble averaging operator. Note that, since we are dealing with continuous time, white Gaussian noise has infinite variance, as it can be seen by setting $t'=t''$ in Eq. (18).

3 Time-varying spectrum

Our aim is to calculate the time-varying spectrum for $s(x,t)$ as given by Eq. (8) and Eq. (11). The Wigner distribution W_z for a deterministic time-dependent function $z(t)$ is [18]

$$W_z(t,\omega) = \frac{1}{2\pi}\int z^*(t-\tfrac{1}{2}\tau)\,z(t+\tfrac{1}{2}\tau)\,e^{-i\tau\omega}\,d\tau \tag{19}$$

The Wigner distribution has found many applications in both time-frequency and position-momentum space [1, 2, 10, 14, 15]. For the time-varying spectrum we use the Wigner *spectrum,* which, for a stochastic process $Z(t)$ is defined by [12, 6, 2],

$$\overline{W}_Z(t,\omega) = \frac{1}{2\pi}\int E\left[Z^*(t-\tfrac{1}{2}\tau)\,Z(t+\tfrac{1}{2}\tau)\right]\,e^{-i\tau\omega}\,d\tau \tag{20}$$

where $E\left[Z^*(t_1)Z(t_2)\right]$ is the two-time autocorrelation function. Taking

$$Z = s(x,t) = \sum S_n(t)X_n(x) \tag{21}$$

then substituting Eq. (21) into Eq. (20), taking the ensemble average and also breaking up the terms into self terms and cross terms we obtain

$$\overline{W}(t,\omega;x) = \sum_{n=1}^{\infty}\left|X_n(x)\right|^2\overline{W}_{nn}(t,\omega) + \sum_{n\neq k=1}^{\infty} X_n^*(x)X_k(x)\overline{W}_{nk}(t,\omega) \tag{22}$$

where

$$\overline{W}_{nk}(t,\omega) = \int E\left[S_n^*(t-\tfrac{1}{2}\tau)S_k(t+\tfrac{1}{2}\tau)\right]e^{-i\tau\omega}d\tau \tag{23}$$

In Eq. (22) the prime signifies summation over all n,k, except for $n=k$.

Our aim is to obtain the time-frequency spectrum at position x on the string, that is $\overline{W}(t,\omega;x)$. We obtain both the transient and steady-state solution. In Appendix B we show that for white noise the cross Wigner spectrum is zero

$$\overline{W}_{nk}(t,\omega) = 0 \qquad \text{for } n\neq k \tag{24}$$

Therefore we only have to concern ourselves with the self terms

$$\overline{W}_{nn}(t,\omega) = \int E\left[S_n^*(t-\tfrac{1}{2}\tau)S_n(t+\tfrac{1}{2}\tau)\right]e^{-i\tau\omega}d\tau \tag{25}$$

and the position-dependent Wigner spectrum is then

$$\overline{W}(t,\omega;x) = \sum_{n=1}^{\infty} \overline{W}_{nn}(t,\omega)\, |X_n(x)|^2 \tag{26}$$

We note that if we integrate out frequency from $\overline{W}_{nn}(t,\omega)$ we obtain

$$\int \overline{W}_{nn}(t,\omega) d\omega = 2\pi E\left[|S_n(t)|^2\right] \tag{27}$$

and the time-dependent position spectrum becomes

$$\overline{W}(t;x) = 2\pi \int \sum_{n=1}^{\infty} E\left[|S_n(t)|^2\right] |X_n(x)|^2\, d\omega \tag{28}$$

which is the case considered by van Lear Jr. and Uhlenbeck [9].

3.1 Solution

There are three approaches we have developed to calculate $\overline{W}_{nn}(t,\omega)$. The first is to obtain the differential equation that $\overline{W}_{nn}(t,\omega)$ satisfies and solve it. The second is to use an impulse response method. The third is using the solution for S_k as given by Eq. (12), calculate $E\left[S_n^*(t-\frac{1}{2}\tau)S_n(t+\frac{1}{2}\tau)\right]$ and then obtain $\overline{W}_{nn}(t,\omega)$ by way of Eq. (12). We have found that the first two approaches are the most revealing and interesting.

For the first method, consider the Wigner distribution corresponding to S_n as given by Eq. (11). We have to transform the differential equation for S_n into a differential equation for the Wigner distribution. A general method to obtain the Wigner distribution that corresponds to the solution of an ordinary equation has been developed [4, 5]. When this method is applied to Eq. (11) one obtains that the differential equation for $\overline{W}_{nn}(t,\omega)$ is

$$\left(\frac{\partial}{\partial t} - p_1\right)\left(\frac{\partial}{\partial t} - p_1^*\right)\left(\frac{\partial}{\partial t} - p_2\right)\left(\frac{\partial}{\partial t} - p_2^*\right)\overline{W}_{nn}(t,\omega) = 16 W_{A_n}(t,\omega) \tag{29}$$

where $W_{A_n}(t,\omega)$ is the Wigner distribution of A_n

$$W_{A_n}(t,\omega) = \int E\left[A_n^*(t-\tfrac{1}{2}\tau)A_n(t+\tfrac{1}{2}\tau)\right] e^{-i\tau\omega} d\tau \tag{30}$$

and where p_1, p_2 are defined by the equation

$$p_k = 2\,\mathrm{Re}\,\gamma_k + 2i\,(\mathrm{Im}\,\gamma_k - \omega) \tag{31}$$

and

$$\gamma_{1,2} = -\mu \pm i\omega_n \tag{32}$$

The exact solution to Eq. (29) is

$$\begin{aligned}\overline{W}_{nn}(t,\omega) = {} & \frac{1}{2\pi}\frac{u(t)}{(\omega^2-\lambda_n)^2 + 4\mu^2\omega^2} \\ & \times \left[1 - e^{-2\mu t}\left[\frac{\mu^2 + (\omega+\omega_n)^2}{4\omega\omega_n}\left(\cos 2(\omega_n-\omega)t + \mu\frac{\sin 2(\omega_n-\omega)t}{(\omega_n-\omega)}\right)\right.\right. \\ & \left.\left. - \frac{\mu^2+(\omega-\omega_n)^2}{4\omega\omega_n}\left(\cos 2(\omega_n+\omega)t + \mu\frac{\sin 2(\omega_n+\omega)t}{(\omega_n+\omega)}\right)\right]\right]\end{aligned} \tag{33}$$

For the second method we consider Eq. (11) with $A_n(t)$ given by Eq. (17)

$$S_n''(t) + 2\mu S_n'(t) + \lambda_n S_n(t) = A_n(t) = u(t)\xi_n(t) \tag{34}$$

One can show that [7, 8]

$$\overline{W}_{nn}(t,\omega) = \int_0^t W_h(t-t',\omega)dt' \tag{35}$$

where

$$W_h(t,\omega) = \frac{1}{4\omega_n^2}\left[W_L(t,\omega-\omega_n) + W_L(t,\omega+\omega_n) - 2W_L(t,\omega)\cos 2\omega_n t\right] \tag{36}$$

and where $W_L(t,\omega)$ is given by

$$W_L(t,\omega) = u(t)e^{-2\mu t}\frac{\sin 2\omega t}{\pi\omega} \tag{37}$$

Evaluating Eq. (35) results in

$$\begin{aligned}\overline{W}_{nn}(t,\omega) = {} & \frac{1}{8\pi}\frac{u(t)}{\mu^2+(\omega-\omega_n)^2}\frac{1}{\omega\omega_n}\left[1-e^{-2\mu t}\left(\mu\frac{\sin 2(\omega-\omega_n)t}{\omega-\omega_n}+\cos 2(\omega-\omega_n)t\right)\right] \\ & -\frac{1}{8\pi}\frac{u(t)}{\mu^2+(\omega+\omega_n)^2}\frac{1}{\omega\omega_n}\left[1-e^{-2\mu t}\left(\mu\frac{\sin 2(\omega+\omega_n)t}{\omega+\omega_n}+\cos 2(\omega+\omega_n)t\right)\right]\end{aligned} \tag{38}$$

While it is not obvious that Eq. (33) is the same as Eq. (38) they are in fact equal. Both forms are useful depending on the issues being addressed.

3.2 Steady-state solution and variance

Taking the limit $t \to \infty$ in Eq. (33) we obtain the steady-state solution for $\overline{W}_{nn}(t,\omega)$,

$$\lim_{t\to\infty}\overline{W}_{nn}(t,\omega) = \frac{1}{2\pi}\frac{1}{(\omega^2-\lambda_n)^2+4\mu^2\omega^2} \tag{39}$$

Therefore the steady-state solution for the position-dependent Wigner spectrum is

$$\overline{W}(\omega;x) = \lim_{t\to\infty}\overline{W}(t,\omega;x) = \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{(\omega^2-\lambda_n)^2+4\mu^2\omega^2}\left|X_n(x)\right|^2 \tag{40}$$

The variance, $\sigma^2(x)$, at position x for the steady-state solution is given by

$$\sigma^2(x) = \int \lim_{t\to\infty}\overline{W}(t,\omega;x)d\omega = \frac{1}{2\pi}\sum_{n=1}^{\infty}\int\frac{1}{(\omega^2-\lambda_n)^2+4\mu^2\omega^2}d\omega\left|X_n(x)\right|^2 \tag{41}$$

The integral evaluates to

$$\int\frac{1}{(\omega^2-\lambda_n)^2+4\mu^2\omega^2} = \frac{\pi}{2\mu\lambda_n} \tag{42}$$

and therefore

$$\sigma^2(x) = \int \lim_{t\to\infty}\overline{W}(t,\omega;x)d\omega = \frac{1}{4\mu}\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left|X_n(x)\right|^2 \tag{43}$$

This is the result obtained by van Lear Jr. and Uhlenbeck [9]. However, in the next section we obtain the variance as a function of time.

4 The time-dependent variance

We now obtain the variance at each position as a function of time. For a general zero-mean process, $E[Z(t)] = 0$, the variance is given by

$$\sigma_Z^2(t) = E[Z(t)^2] = \int\overline{W}_Z(t,\omega)d\omega \tag{44}$$

For our case we take $Z(t) = S_n(t)$. Consider first

$$\sigma^2_{S_n}(t) = \int \overline{W}_{nn}(t,\omega)d\omega \tag{45}$$

Substituting Eq. (38) into Eq. (45) one has

$$\sigma^2_{S_n}(t) = \frac{1}{4\omega_n^2}\int_0^t\int_{-\infty}^{+\infty}\left[W_L(t-t',\omega-\omega_n) + W_L(t-t',\omega+\omega_n) - 2W_L(t-t',\omega)\cos 2\omega_n t'\right]dt'd\omega \tag{46}$$

Evaluation leads to

$$\sigma^2_{S_n}(t) = \frac{1}{4\lambda_n\mu} - \frac{1}{\omega_n^2}e^{-2\mu t}\left[\frac{1}{4\mu} + \frac{1}{4\lambda_n}\left(\omega_n \sin 2\omega_n t - \mu\cos 2\omega_n t\right)\right] \tag{47}$$

The steady-state solution is given by

$$\sigma^2_{S_n}(\infty) = \frac{1}{4\lambda_n\mu} \tag{48}$$

For small times we have

$$\sigma^2_{S_n}(t) \sim \frac{1}{4\lambda_n\mu} - \frac{1}{\omega_n^2}(1-2\mu t)\left[\frac{1}{4\mu} + \frac{2\omega_n^2 t}{4\lambda_n}\right]$$

which simplifies to

$$\sigma^2_{S_n} \sim \frac{\mu^2}{2\omega_n^2\lambda_n}t \tag{49}$$

Therefore the variance as a function of position and time is

$$\sigma^2(t,x) = \sum_{n=1}^{\infty}\left\{\frac{1}{4\lambda_n\mu} - \frac{1}{\omega_n^2}e^{-2\mu t}\left[\frac{1}{4\mu} + \frac{1}{4\lambda_n}\left(\omega_n\sin 2\omega_n t - \mu\cos 2\omega_n t\right)\right]\right\}|X_n(x)|^2 \tag{50}$$

For the steady-state we have

$$\lim_{t\to\infty}\sigma^2(t,x) = \frac{1}{4\mu}\sum_{n=1}^{\infty}\frac{1}{\lambda_n}|X_n(x)|^2 \tag{51}$$

which is Eq. (43) and was obtained by van Lear Jr. and Uhlenbeck [9]. For small times

$$\lim_{t\to 0}\sigma^2(t,x) \to t\frac{\mu^2}{2}\sum_{n=1}^{\infty}\frac{1}{\omega_n^2\lambda_n}X_n(x)^2 \tag{52}$$

5 Undamped case

The undamped case is obtained by taking

$$\mu = 0 \tag{53}$$

and Eq. (1) becomes

$$\frac{\partial^2 s}{\partial t^2} = p'\frac{\partial s}{\partial x} + p\frac{\partial^2 s}{\partial x^2} + A(x,t) \tag{54}$$

with

$$\lambda_n = \omega_n^2 \tag{55}$$

The Wigner distribution then becomes

$$\overline{W}_{nn}(t,\omega) = \frac{1}{2\pi}\frac{u(t)}{(\omega^2-\omega_n^2)^2}\left(1 - \frac{(\omega+\omega_n)^2}{4\omega\omega_n}\left(\cos 2(\omega_n-\omega)t\right) - \frac{(\omega-\omega_n)^2}{4\omega\omega_n}\left(\cos 2(\omega_n+\omega)t\right)\right) \tag{56}$$

or

$$\overline{W}_{nn}(t,\omega) = \frac{1}{8\pi}\frac{u(t)}{\omega\omega_n}\left(\frac{1-\cos 2(\omega-\omega_n)t}{(\omega-\omega_n)^2} - \frac{1-\cos 2(\omega+\omega_n)t}{(\omega+\omega_n)^2}\right) \tag{57}$$

5.1 Steady-state solution for the undamped case

Taking the limit $t \to \infty$ in Eq. (57) we obtain the steady-state solution for $\overline{W}_{nn}(t,\omega)$,

$$\lim_{t\to\infty} \overline{W}_{nn}(t,\omega) = \frac{1}{2\pi}\frac{1}{(\omega^2-\omega_n^2)^2} \tag{58}$$

Therefore the position-dependent steady-state solution for the Wigner spectrum is

$$\lim_{t\to\infty} \overline{W}(t,\omega;x) = \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{(\omega^2-\omega_n^2)^2}\left|X_n(x)\right|^2 \tag{59}$$

5.2 Variance

For the damped case in Eq. (51) the time variance is

$$\sigma^2 = \int \lim_{t\to\infty} \overline{W}(t,\omega;x)d\omega = \frac{1}{4\mu}\sum_{n=1}^{\infty}\frac{1}{\omega_n^2}\left|X_n(x)\right|^2 \tag{60}$$

which goes to infinity for $\mu \to 0$

6 Example

Of particular interest is the case of a string with $p' = 0$ which was considered by van Lear Jr. and Uhlenbeck [9]. Without loss of generality we take $p = 1$ in which case we have for the string equation, Eq. (1),

$$\frac{\partial^2 s}{\partial t^2} + 2\mu\frac{\partial s}{\partial t} = \frac{\partial^2 s}{\partial x^2} + A(x,t) \tag{61}$$

and for Eq. (3) we have

$$X_n''(x) = -\lambda_n X_n(x) \tag{62}$$

The normalized solutions are given by

$$X_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi}{L}x \tag{63}$$

with

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \tag{64}$$

According to Eq. (14) we must choose μ so that

$$\mu^2 < \lambda_n = \left(\frac{n\pi}{L}\right)^2 \tag{65}$$

This must be the case for every n. Since the smallest λ_n is π/L we must take

$$\mu < \frac{\pi}{L} \tag{66}$$

In that case we have that

$$\omega_n = \sqrt{\left(\frac{n\pi}{L}\right)^2 - \mu^2} \tag{67}$$

The Wigner spectrum is then given by

$$\overline{W}(t,\omega;x) = \frac{2}{L}\sum_{n=1}^{\infty}\overline{W}_{nn}(t,\omega)\sin^2\frac{n\pi}{L}x \tag{68}$$

with

$$\begin{aligned}\overline{W}_{nn}(t,\omega) = {} & \frac{1}{8\pi}\frac{u(t)}{\mu^2+(\omega-\omega_n)^2}\frac{1}{\omega\omega_n}\left[1-e^{-2\mu t}\left(\mu\frac{\sin 2(\omega-\omega_n)t}{\omega-\omega_n}+\cos 2(\omega-\omega_n)t\right)\right] \\ & - \frac{1}{8\pi}\frac{u(t)}{\mu^2+(\omega+\omega_n)^2}\frac{1}{\omega\omega_n}\left[1-e^{-2\mu t}\left(\mu\frac{\sin 2(\omega+\omega_n)t}{\omega+\omega_n}+\cos 2(\omega+\omega_n)t\right)\right]\end{aligned} \tag{69}$$

6.1 Steady-state solution

The steady-state solution is when $t \to \infty$ in which case

$$\overline{W}_{nn}(t=\infty,\omega) = \frac{1}{2\pi}\frac{1}{(\omega^2-\lambda_n)^2+4\mu^2\omega^2} \tag{70}$$

and hence the Wigner spectrum is

$$\overline{W}(t=\infty,\omega;x) = \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{(\omega^2-\lambda_n)^2+4\mu^2\omega^2}\left|X_n(x)\right|^2 \tag{71}$$

6.2 Undamped case

For the undamped case where we take $\mu = 0$

$$\omega_n = \sqrt{\lambda_n} = \frac{n\pi}{L} \tag{72}$$

and the Wigner spectrum is given by

$$\overline{W}(t,\omega;x) = \frac{2}{L}\sum_{n=1}^{\infty}\overline{W}_{nn}(t,\omega)\sin^2\frac{n\pi}{L}x \tag{73}$$

where now

$$\overline{W}_{nn}(t,\omega) = \frac{1}{8\pi}\frac{u(t)}{\omega\omega_n}\left(\frac{1-\cos 2(\omega-\omega_n)t}{(\omega-\omega_n)^2} - \frac{1-\cos 2(\omega+\omega_n)t}{(\omega+\omega_n)^2}\right) \tag{74}$$

Using Eq. (71) the steady-state solution is

$$\overline{W}(\infty,\omega;x) = \frac{1}{\pi L}\sum_{n=1}^{\infty}\frac{1}{(\omega^2-\left(\frac{n\pi}{L}\right)^2)^2}\sin^2\frac{n\pi}{L}x \tag{75}$$

6.3 Numerical example

Following the example of van Lear and Uhlenbeck [9] we take $L=1$ and $\mu = \frac{\pi}{8}$ in which case

$$X_n(x) = \sqrt{2}\sin n\pi x \tag{76}$$

with

$$\lambda_n = \pi^2 n^2 \tag{77}$$

and

$$\omega_n = \frac{\pi}{8}\sqrt{64n^2-1} \tag{78}$$

The Wigner spectrum is then

$$\overline{W}(t,\omega;x) = 2\sum_{n=1}^{\infty}\overline{W}_{nn}(t,\omega)\sin^2 n\pi x \tag{79}$$

with

$$\begin{aligned}\overline{W}_{nn}(t,\omega) = {} & \frac{8}{\pi}\frac{u(t)}{\pi^2+64(\omega-\omega_n)^2}\frac{1}{\omega\omega_n}\left[1-e^{-\pi t/4}\left(\frac{\pi}{8}\frac{\sin 2(\omega-\omega_n)t}{\omega-\omega_n}+\cos 2(\omega-\omega_n)t\right)\right] \\ & -\frac{8}{\pi}\frac{u(t)}{\pi^2+64(\omega+\omega_n)^2}\frac{1}{\omega\omega_n}\left[1-e^{-\pi t/4}\left(\frac{\pi}{8}\frac{\sin 2(\omega+\omega_n)t}{\omega+\omega_n}+\cos 2(\omega+\omega_n)t\right)\right]\end{aligned} \tag{80}$$

The steady-state solution is when $t \to \infty$

$$\overline{W}_{nn}(t=\infty,\omega) = \frac{1}{2\pi}\frac{1}{(\omega^2-\pi^2n^2)^2+\pi^2\omega^2/16} \tag{81}$$

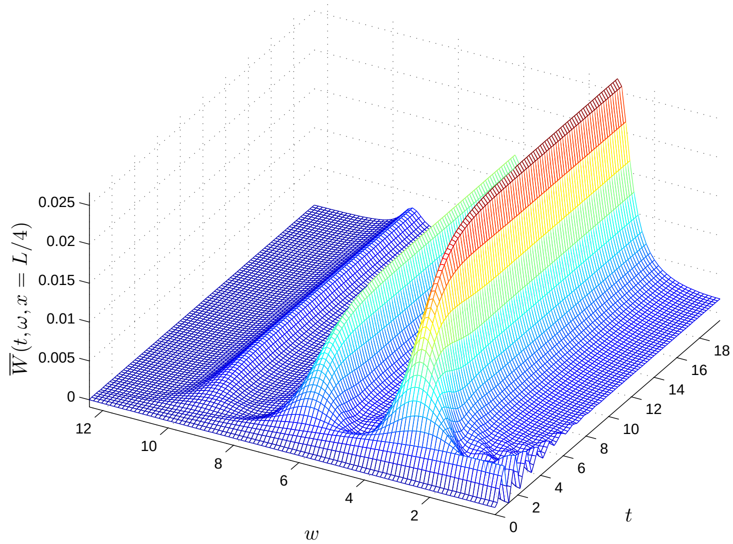

Figure 1: The first 3 modes of the Wigner spectrum $\overline{W}(t,\omega,x)$, as given by Eq. (79) computed at $x = L/4$.

and the Wigner spectrum at infinity is given by

$$\overline{W}(t=\infty,\omega;x) = \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{(\omega^2-\pi^2n^2)^2+\pi^2\omega^2/16}\,|X_n(x)|^2 \tag{82}$$

In Fig. 1 we show the Wigner spectrum at position $x = L/4$ as given by Eq. (79). At $t = 0$ the Wigner spectrum is zero, then the energy increases for each mode and concentrates on the modal frequencies ω_n, for $n = 1, 2, 3$. After the initial transient, the Wigner spectrum reaches a steady state, as given by Eq. (82).

In Fig. 2 we take the position at $x = L/2$. For the range of ω plotted one would expect the first four modes as given by Eq. (79). However, the second and fourth mode are zero because $\sin^2 n\pi/2$ is zero when $n = 2$ or 4. In fact for that value of $x = L/2$ all the even modes are zero.

In Fig. 3 we take the position at $x = L/\sqrt{5}$ and now we see all the modes for the range of frequencies plotted. Not all of the modes are visible, as some are very small.

In Fig. 4 we take $x = L/4$ and show the first 12 modes of the Wigner spectrum $\overline{W}(t,\omega,x)$. The fourth, eighth, and twelfth mode are zero because $\sin^2 n\pi/4$ is zero for all of the modes whose n is a multiple of 4. In Fig. 5 we show the Wigner spectrum at position $x = L/10$. For a given time, the energy of the higher modes decreases slower than for the cases shown in Figs. 1-3. Note that the tenth mode is zero because $\sin^2 n\pi/10$ is zero. Finally, in Fig. 6 the left plot is the Wigner spectrum at $x = L/4$, and the right plot is for $x = 3L/4$. As expected, the two Wigner spectra are identical because of the symmetry with respect to $x = L/2$ of the $\sin^2$ term in Eq. (79). This shows the consistency of the calculation.

7 Summary of results

For the string equation

$$\frac{\partial^2 s}{\partial t^2} + 2\mu\frac{\partial s}{\partial t} = p'\frac{\partial s}{\partial x} + p\frac{\partial^2 s}{\partial x^2} + A(x,t) \tag{83}$$

where $A(x,t)$ is the external applied random force the solution is

$$s(x,t) = \sum S_n(t)X_n(x) \tag{84}$$

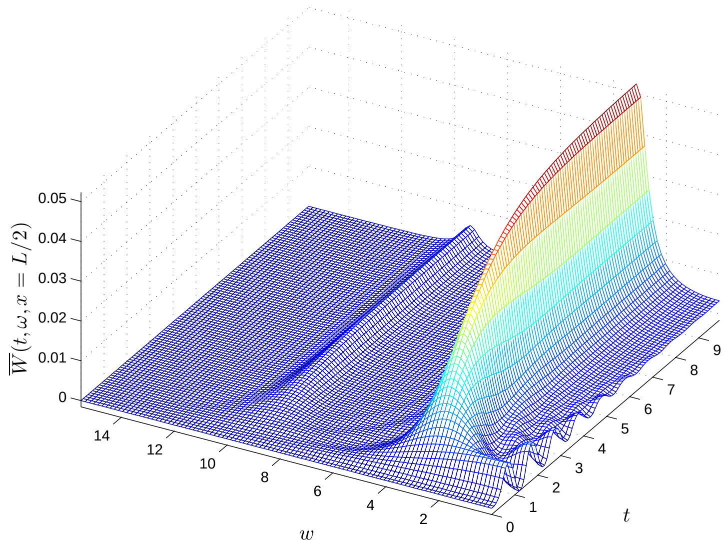

Figure 2: The first 4 modes of the Wigner spectrum computed at $x = L/2$. The second and fourth mode are zero because of the $\sin^2$ term in Eq. (79), which zeros all of the even modes.

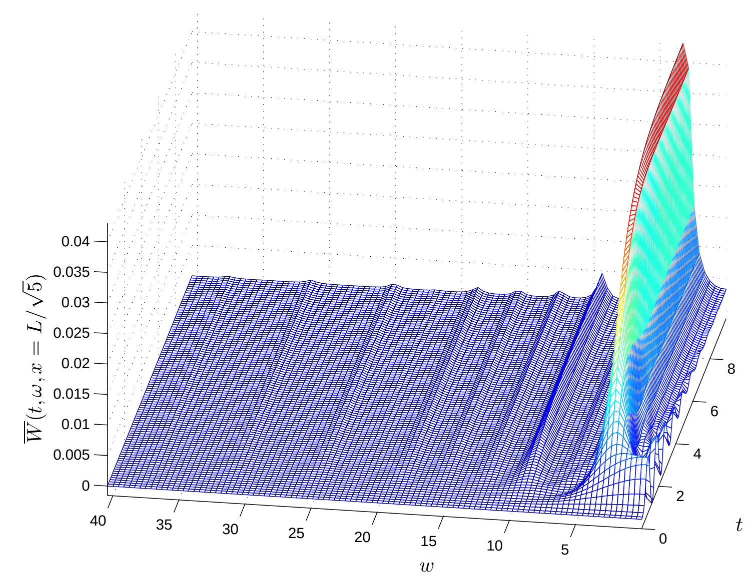

Figure 3: The Wigner spectrum at $x = L/\sqrt{5}$.

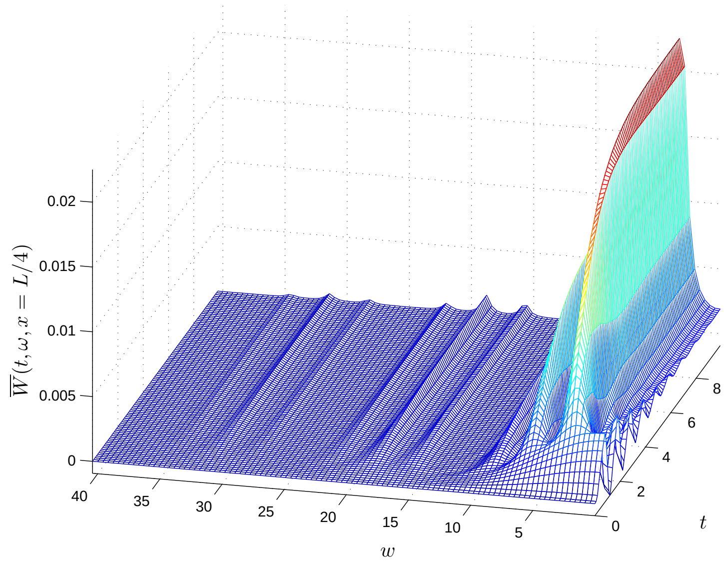

Figure 4: The first 12 modes of the Wigner spectrum $\overline{W}(t,\omega,x)$, computed for $x = L/4$. The fourth, eighth, and twelfth mode are zero because of the $\sin^2$ term in Eq. (79).

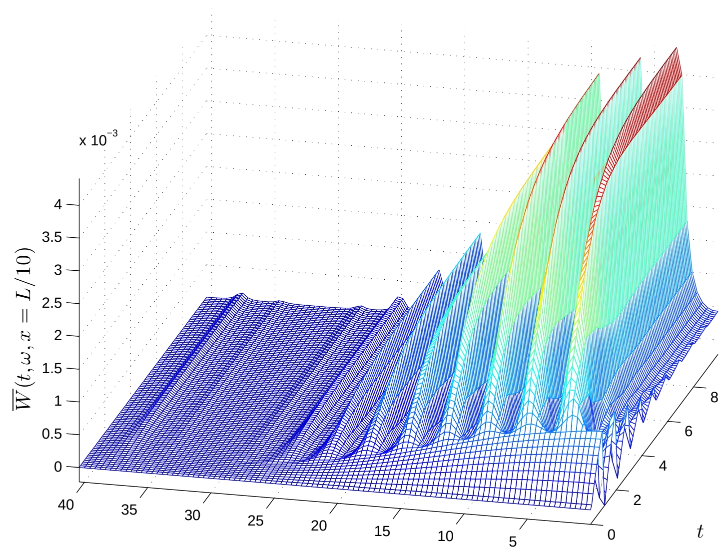

Figure 5: The plot shows the first 12 modes of the Wigner spectrum $\overline{W}(t,\omega,x)$, computed for $x = L/10$.

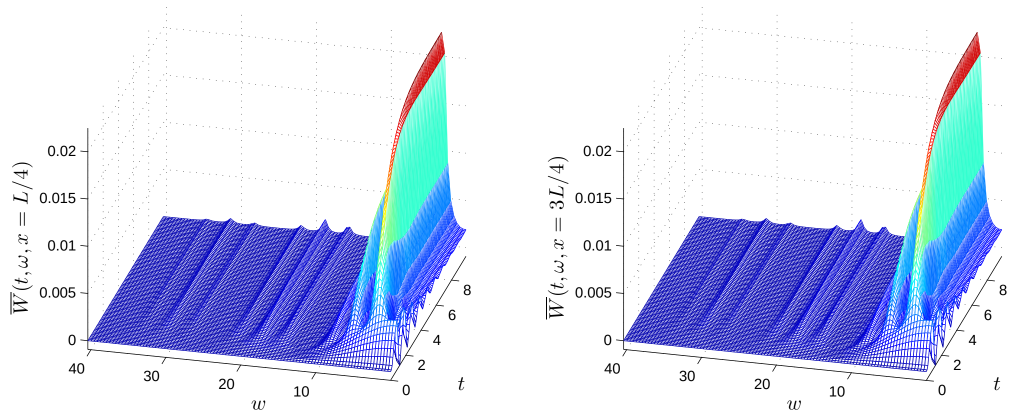

Figure 6: The left plot is the Wigner spectrum at $x = L/4$, whereas the right plot is for $x = 3L/4$. The two spectra are identical because of the symmetry with respect to $x = L/2$ of the $\sin^2$ in Eq. (70).

where X_n are the eigenfunctions of the eigenvalue equation

$$pX_n''(x) + p'X_n'(x) = -\lambda_n X_n(x) \tag{85}$$

$$T_n''(t) + 2\mu T_n'(t) = -\lambda_n T_n(t) \tag{86}$$

The equation that $S_n(t)$ satisfies is

$$S_n''(t) + 2\mu S_n'(t) + \lambda_n S_n(t) = A_n(t) \tag{87}$$

where

$$A_n(t) = \int A(x,t) X_n^*(x) dx \tag{88}$$

The exact Wigner spectrum is

$$\overline{W}(t,\omega;x) = \sum_{n=1}^{\infty} |X_n(x)|^2 \, \overline{W}_{nn}(t,\omega) \tag{89}$$

where

$$\begin{aligned}\overline{W}_{nn}(t,\omega) = {} & \frac{1}{2\pi}\frac{u(t)}{(\omega^2-\lambda_n)^2+4\mu^2\omega^2} \\ & \times \left[1 - e^{-2\mu t}\left[\frac{\mu^2+(\omega+\omega_n)^2}{4\omega\omega_n}\left(\cos 2(\omega_n-\omega)t + \mu\frac{\sin 2(\omega_n-\omega)t}{(\omega_n-\omega)}\right)\right.\right. \\ & \left.\left. - \frac{\mu^2+(\omega-\omega_n)^2}{4\omega\omega_n}\left(\cos 2(\omega_n+\omega)t + \mu\frac{\sin 2(\omega_n+\omega)t}{(\omega_n+\omega)}\right)\right]\right]\end{aligned} \tag{90}$$

or

$$\begin{aligned}\overline{W}_{nn}(t,\omega) = {} & \frac{1}{8\pi}\frac{u(t)}{\mu^2+(\omega-\omega_n)^2}\frac{1}{\omega\omega_n}\left[1 - e^{-2\mu t}\left(\mu\frac{\sin 2(\omega-\omega_n)t}{\omega-\omega_n} + \cos 2(\omega-\omega_n)t\right)\right] \\ & - \frac{1}{8\pi}\frac{u(t)}{\mu^2+(\omega+\omega_n)^2}\frac{1}{\omega\omega_n}\left[1 - e^{-2\mu t}\left(\mu\frac{\sin 2(\omega+\omega_n)t}{\omega+\omega_n} + \cos 2(\omega+\omega_n)t\right)\right]\end{aligned} \tag{91}$$

and where

$$\omega_n = \sqrt{\lambda_n - \mu^2} \tag{92}$$

The steady-state solution is given by

$$\overline{W}(\omega;x) = \lim_{t\to\infty} \overline{W}(t,\omega;x) = \frac{1}{2\pi}\sum_{n=1}^{\infty} \frac{1}{(\omega^2-\lambda_n)^2+4\mu^2\omega^2}\left|X_n(x)\right|^2 \tag{93}$$

The variance as a function of position and time is

$$\sigma^2(t,x) = \sum_{n=1}^{\infty}\left\{\frac{1}{4\lambda_n\mu} - \frac{1}{\omega_n^2}e^{-2\mu t}\left[\frac{1}{4\mu} + \frac{1}{4\lambda_n}\left(\omega_n \sin 2\omega_n t - \mu\cos 2\omega_n t\right)\right]\right\}\left|X_n(x)\right|^2 \tag{94}$$

The undamped case is obtained by letting $\mu \to 0$ and the explicit expressions are given in Sect. 5.

8 Conclusion

We have obtained the exact time-dependent spectrum for the random string where the forcing term is white noise. The time spectrum is defined by way of the ensemble average of the Wigner distribution. Explicit expressions have been derived which involve the transient and steady-state solution. The limiting value for time going to infinity gives the standard spectrum and agrees with the result of van Lear and Uhlenbeck [9]. The variance at each position was explicitly given.

9 Appendix A: The statistics of $A_n(t)$

Following van Lear and G. E. Uhlenbeck, we take the random force $A(x,t)$ to be white Gaussian noise in space and time starting at $t=0$,

$$A(x,t) = u(t)\eta(x,t) \tag{95}$$

where $u(t)$ is the step function and $\eta(x,t)$ is a white Gaussian noise in space and time with autocorrelation function

$$E\left[\eta(x',t')\eta(x'',t'')\right] = \delta(x'-x'')\delta(t'-t'') \tag{96}$$

Consequently, the autocorrelation function of the random force is given by

$$E\left[A(x',t')A(x'',t'')\right] = u(t')\delta(x'-x'')\delta(t'-t'') \tag{97}$$

where we have used the simplification $u(t')u(t'')\delta(t'-t'') = u(t')\delta(t'-t'')$.

We now obtain the statistics of $A_n(t)$ as given by Eq. (10). For convenience we take $X_n(x)$ to be real and write Eq. (10) as

$$A(x,t) = \sum A_n(t)X_n(x) \tag{98}$$

Now consider the autocorrelation function of the $A_n(t)$ $'s$

$$E\left[A_n(t')A_n(t'')\right] = E\left[\int A(x',t')X_n(x')dx' \int A(x'',t'')X_n(x'')dx''\right] \tag{99}$$

$$= \iint E\left[A(x',t')A(x'',t'')\right]X_n(x')X_n(x'')dx'dx'' \tag{100}$$

Using Eq. (95), and performing the delta function integrations and also using Eq. (5) we obtain that

$$E\left[A_n(t')A_n(t'')\right] = u(t')\delta(t'-t'') \tag{101}$$

Moreover, since $A(x,t)$ is Gaussian with mean zero, it also follows that $A_n(t)$ is Gaussian with mean zero

$$E\left[A_n(t)\right] = 0 \tag{102}$$

Consequently, $A_n(t)$ is a white Gaussian noise starting at $t=0$, and we can write it as

$$A_n(t) = u(t)\xi_n(t) \tag{103}$$

where $\xi_n(t)$ is a white Gaussian noise for each n, with zero mean and autocorrelation function

$$E\left[\xi_n(t')\xi_n(t'')\right] = \delta(t'-t'') \tag{104}$$

10 Appendix B: Proof of $\overline{W}_{nk}(t,\omega) = 0$

$\overline{W}_{nk}(t,\omega)$ is given by

$$\overline{W}_{nk}(t,\omega) = \frac{1}{2\pi}\int E\left[S_n\left(t-\tfrac{1}{2}\tau\right) S_k\left(t+\tfrac{1}{2}\tau\right)\right] e^{-i\tau\omega} d\tau \tag{105}$$

For zero initial conditions where

$$S_n(t) = \frac{1}{\omega_n}\int_0^t A_n(t')e^{-\mu(t-t')}\sin\omega_n(t-t')dt' \tag{106}$$

we have

$$\overline{W}_{nk}(t,\omega) = \frac{1}{2\pi}\frac{1}{\omega_n^2}\int E\left[\int_0^{t-\tau/2} A_n(t')e^{-\mu(t-\tau/2-t')}\sin\omega_n(t-\tau/2-t')dt'\right. \tag{107}$$

$$\left.\times\int_0^{t+\tau/2} A_k(t'')e^{-\mu(t+\tau/2-t'')}\sin\omega_n(t+\tau/2-t'')dt''\right] e^{-i\tau\omega}d\tau \tag{108}$$

$$= \frac{1}{2\pi}\frac{1}{\omega_n^2}\int\int_0^{t-\tau/2}\int_0^{t+\tau/2} E\left[A_n(t')A_k(t'')\right] e^{-\mu(t-\tau/2-t')}e^{-\mu(t+\tau/2-t'')} \tag{109}$$

$$\times\sin\omega_n(t-\tau/2-t')\sin\omega_n(t+\tau/2-t'')e^{-i\tau\omega}dt'dt''d\tau \tag{110}$$

But

$$E\left[A_n(t')A_k(t'')\right] = E\left[\int A(x',t')X_n(x')dx'\int A(x'',t'')X_k(x'')dx''\right] \tag{111}$$

$$= \int\int E\left[A(x',t')A(x'',t'')\right] X_n(x')X_k(x'')dx'dx'' \tag{112}$$

$$= \int\int u(t')\delta(x'-x'')\delta(t'-t'')X_n(x')X_k(x'')dx'dx'' \tag{113}$$

$$= u(t')\delta(t'-t'')\int X_n(x')X_k(x')dx' \tag{114}$$

$$= 0 \tag{115}$$

and hence

$$\overline{W}_{nk}(t,\omega) = 0 \tag{116}$$

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