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Rank two bundles on $\mathbf{P}^n$ with isolated cohomology

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Abstract

The purpose of this paper is to study minimal monads associated to a rank two vector bundle $\mathcal{E}$ on $\mathbf{P}^n$. In particular, we study situations where $\mathcal{E}$ has $H^i_*(\mathcal{E}) = 0$ for $1 < i < n-1$, except for one pair of values $(k, n-k)$. We show that on $\mathbf{P}^8$, if $H^3_*(\mathcal{E}) = H^4_*(\mathcal{E}) = 0$, then $\mathcal{E}$ must be decomposable. More generally, we show that for $n \geqslant 4k$, there is no indecomposable bundle $\mathcal{E}$ for which all intermediate cohomology modules except for $H^1_*, H^k_*, H^{n-k}_*, H^{n-1}_*$ are zero.

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1 | INTRODUCTION

It has been difficult to disprove the existence of an indecomposable rank two bundle $\mathcal{E}$ on $\mathbf{P}^n$ for large n. Most known results have been obtained by imposing other conditions on $\mathcal{E}$ to show that $\mathcal{E}$ cannot exist or must be split. For example, the so-called Babylonian condition which requires $\mathcal{E}$ to be extendable to $\mathbf{P}^{n+m}$ for every m has been studied by a number of people including Barth and van de Ven [2] and Coanda and Trautmann [3]. Numerical criteria that force splitting are found again in Barth and van de Ven, where for a normalized rank two bundle with second Chern class a and with splitting type $\mathcal{O}_l(-b) \oplus \mathcal{O}_l(b)$ on the general line l, a function $f(a, b)$ is found such that if $n > f(a, b)$, then a bundle on $\mathbf{P}^n$ with these invariants must be split.

Cohomological criteria for forcing the splitting of $\mathcal{E}$ start with Horrocks [7]. If S is the polynomial ring corresponding to $\mathbf{P}^n$, then $H^i_*(\mathcal{E})$ (defined as $\oplus_\nu H^i(\mathbf{P}^n, \mathcal{E}(\nu))$) is an S-module. The intermediate cohomology modules $H^i_* sE), 1 \leqslant i \leqslant n-1$ are all graded modules of finite length and there is a strong relationship between $\mathcal{E}$ and its intermediate cohomology modules. He shows that if $H^i_*(\mathcal{E}) = 0$ for all i with $i \leqslant i \leqslant n-1$, then $\mathcal{E}$ is split. Moreover, Horrocks in [7] established that a vector bundle on $\mathbf{P}^n$ is determined up to isomorphism and up to a sum of line bundles (i.e., up to stable equivalence) by its collection of intermediate cohomology modules and also a certain collection of extension classes involving these modules. This correspondence has been generalized to any Arithmetically Cohen-Macaulay (ACM) varieties in [12]. The Syzygy Theorem ([5, 6]) shows that for a rank two bundle $\mathcal{E}$, it is enough to know that $H^1_*(\mathcal{E}) = 0$ to force splitting. In [14], it is shown that for a indecomposable rank two bundle on $\mathbf{P}^n$, in addition to $H^1_*(\mathcal{E})$ and $H^{n-1}_*(\mathcal{E})$ being nonzero, some intermediate cohomology module $H^k_*(\mathcal{E})$ $(1 < k < n-1)$ (and hence also $H^{n-k}_*(\mathcal{E})$) must be nonzero. Various calculations in [13] and [14] show that there are limitations on the module structure of $H^1_*(\mathcal{E})$ and $H^2_*(\mathcal{E})$ for some values of n.

In this paper, we study situations where a rank two bundle $\mathcal{E}$ on $\mathbf{P}^n$ has $H^i_*(\mathcal{E}) = 0$ for $1 < i < n-1$, except for one pair of values $(k, n-k)$. We describe the minimal monads associated to $\mathcal{E}$. We show that on $\mathbf{P}^8$, if $H^3_*(\mathcal{E}) = H^4_*(\mathcal{E}) = 0$, then $\mathcal{E}$ must be decomposable. More generally, we show that for $n \geqslant 4k$, there is no indecomposable bundle $\mathcal{E}$ for which all intermediate cohomology modules except for $H^1_*, H^k_*, H^{n-k}_*, H^{n-1}_*$ are zero. The proof utilizes the space between k and $n-k$ when $n \geqslant 4k$ for making cohomological computations.

2 | MONADS FOR RANK TWO VECTOR BUNDLES ON $\mathbf{P}^n$

Let $\mathcal{E}$ be an indecomposable rank two vector bundle on $\mathbf{P}^n$ over an algebraically closed field of characteristic different from two. If S is the polynomial ring on $n+1$ variables, let $N_i = H^i_*(\mathcal{E}) = \oplus_\nu H^i(\mathcal{E}(\nu))$ be the finite length graded S-module over S, for $1 \leqslant i \leqslant n-1$. By the Syzygy Theorem, both N_1 and N_{n-1} are nonzero modules. Horrocks ([8]) gives a brief description of the construction of a minimal monad for a bundle $\mathcal{E}$ of any rank on $\mathbf{P}^n$ by "killing both H^1_* and H^{n-1}_*." Barth and Hulek ([1]) use this idea to construct (with more detail) l-m minimal monads for a bundle $\mathcal{E}$ on $\mathbf{P}^n$, where only $H^1_{\geqslant l}(\mathcal{E})$ and $H^{n-1}_{<-m-n}(\mathcal{E})$ are killed. Horrocks' construction, which we use below, is obtained when both l and m are very negative.

The monad is a complex

$$0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{P} \xrightarrow{\beta} \mathcal{B} \to 0,$$

where $\mathcal{P}$ is a bundle with $H^i_*(\mathcal{P}) = 0$ for $i = 1$ and $i = n-1$, and where $\mathcal{A}, \mathcal{B}$ are free bundles. Let $\mathcal{G}$ be kernel β. We have two sequences

$$\begin{gathered} 0 \to \mathcal{G} \to \mathcal{P} \to \mathcal{B} \to 0, \\ 0 \to \mathcal{A} \to \mathcal{G} \to \mathcal{E} \to 0, \end{gathered} \tag{1}$$

from which we see that $H^i_*(\mathcal{E}) = H^i_*(\mathcal{G})$ for $1 \leqslant i \leqslant n-2$, while $H^{n-1}_*(\mathcal{G}) = 0$, and $H^i_*(\mathcal{E}) = H^i_*(\mathcal{P})$ for $2 \leqslant i \leqslant n-2$, while $H^1_*(\mathcal{P}) = H^{n-1}_*(\mathcal{P}) = 0$.

The minimality of the complex means that the rank of $\mathcal{B}$ equals the number of generators of $H^1_*(\mathcal{G}) = N_1$ and the rank of $\mathcal{A}^\vee$ equals the number of generators of $H^1_*(\mathcal{E}^\vee)$. When the rank of the bundle $\mathcal{E}$ equals 2, we find that $\mathcal{A}$ and $\mathcal{B}$ have the same rank.

The two sequences give rise to

$$\begin{gathered} 0 \to \wedge^2\mathcal{G} \to \wedge^2\mathcal{P} \to \mathcal{B} \otimes \mathcal{P} \to S^2\mathcal{B} \to 0, \\ 0 \to S^2\mathcal{A} \to \mathcal{A} \otimes \mathcal{G} \to \wedge^2\mathcal{G} \to \wedge^2\mathcal{E} \to 0. \end{gathered} \tag{2}$$

(Since the characteristic of the field is different from 2, we can assume that S^2 commutes with duals.)

Lemma 2.1. *If $H^2_*(\mathcal{E}) = 0$, then $H^1_*(\wedge^2\mathcal{P})$ and $H^{n-1}_*(\wedge^2\mathcal{P})$ are nonzero. If $H^l_*(\mathcal{E}) = 0$ for some l, with $2 \leqslant l \leqslant n-2$, then $H^l_*(\wedge^2\mathcal{P}) = 0$.*

Proof. See [11, Theorem 2.2] for the first part. Next, suppose $H^l_*(\mathcal{E}) = 0$ for some l, with $2 \leqslant l \leqslant n-2$. So, $N_l = N_{n-l} = 0$ by Serre duality. In particular, $\mathcal{G}$ and $\mathcal{P}$ have $H^l_* = 0$ as well. It follows from Equation (2), that $H^l_*(\wedge^2\mathcal{G}) = 0$ and hence $H^l_*(\wedge^2\mathcal{P}) = 0$. □

Lemma 2.2. *Let $2 \leqslant t \leqslant n-2$. Let $A = H^0_*(\mathcal{A}), B = H^0_*(\mathcal{B})$. There is an exact sequence*

$$A \otimes N_t \to H^t_*(\wedge^2\mathcal{P}) \to B \otimes N_t,$$

which is injective on the left if $t \geqslant 3$ and $N_{t-1} = 0$, and is surjective on the right if $t \leqslant n-3$ and $N_{t+1} = 0$.

Proof. Break up the first sequence in 2 as $0 \to \wedge^2\mathcal{G} \to \wedge^2\mathcal{P} \to \mathcal{D} \to 0$, $0 \to \mathcal{D} \to \mathcal{B} \otimes \mathcal{P} \to S^2\mathcal{B} \to 0$. We get long exact sequences

$$H^{t-1}_*(\mathcal{D}) \to H^t_*(\wedge^2\mathcal{G}) \to H^t_*(\wedge^2\mathcal{P}) \to H^t_*(\mathcal{D}) \to H^{t+1}_*(\wedge^2\mathcal{G}),$$

where $H^t_*(\mathcal{D}) \cong B \otimes N_t$ (always) and $H^{t-1}_*(\mathcal{D}) \cong B \otimes N_{t-1}$ provided $t \geqslant 3$. Likewise break up the second sequence as $0 \to S^2\mathcal{A} \to \mathcal{A} \otimes \mathcal{G} \to \mathcal{C} \to 0$, $0 \to \mathcal{C} \to \wedge^2\mathcal{G} \to \wedge^2\mathcal{E} \to 0$. We see that $H^i_*(\wedge^2\mathcal{G}) \cong H^i_*(\mathcal{C})$ for $i = t, t+1$, $H^t_*(\mathcal{C}) \cong A \otimes N_t$ and when $t \leqslant n-3$, $H^{t+1}_*(\mathcal{C}) \cong A \otimes N_{t+1}$. □

Moreover, when $H^l_*(\mathcal{E}) = 0$ for some l, with $2 \leqslant l \leqslant n-3$, from Equation (2) since $H^{l+1}_*(S^2\mathcal{A}) = H^{l+2}_*(\mathcal{A} \otimes \mathcal{G}) = H^l_*(\wedge^2\mathcal{E}) = H^{l+1}_*(\wedge^2\mathcal{E}) = 0$, we get $H^{l+1}_*(\wedge^2\mathcal{G}) \cong H^{l+1}_*(\mathcal{A} \otimes \mathcal{G})$. Since $H^{l-1}_*(S^2\mathcal{B}) = H^l_*(\mathcal{B} \otimes \mathcal{P}) = 0 H^{l+1}_*(\wedge^2\mathcal{G}) \hookrightarrow H^{l+1}_*(\wedge^2\mathcal{P})$. From

$$0 \to \mathcal{D} \to \mathcal{B} \otimes \mathcal{P} \to S^2\mathcal{B} \to 0,$$

we obtain $H^{l+1}_*(\mathcal{B} \otimes \mathcal{P}) \cong H^{l+1}_*(\mathcal{D})$ giving an exact sequence

$$0 \to A \otimes N_{l+1} \to H^{l+1}_*(\wedge^2\mathcal{P}) \to B \otimes N_{l+1}, \tag{3}$$

where $A = H^0_*(\mathcal{A}), B = H^0_*(\mathcal{B})$. Notice that the sequence is exact on the right if $N_{l+2} = 0$.

Likewise, if $H^l_*(\mathcal{E}) = 0$ for some l, with $3 \leqslant l \leqslant n-2$, we get the exact sequence

$$A \otimes N_{l-1} \to H^{l-1}_*(\wedge^2\mathcal{P}) \to B \otimes N_{l-1} \to 0. \tag{4}$$

Notice that the sequence is exact on the right if $N_{l-2} = 0$.

The following proposition is a typical one that shows that a minimal monad for a rank two bundle is built very minimally out of the cohomological data for $\mathcal{E}$. Other examples of such a result can be found in [15] and [13]. Decker ([4]) has conjectured such a minimality for rank two bundles on $\mathbf{P}^4$.

Proposition 2.3. *Suppose that $\mathcal{E}$ is a nonsplit rank two bundle on $\mathbf{P}^n$ ($n \geqslant 6$), with $H^l_*(\mathcal{E}) = 0$ for some l with $2 \leqslant l \leqslant n-2$. Then in the minimal monad for $\mathcal{E}$, the bundle $\mathcal{P}$ has no line bundle summands.*

Proof. Note that the statement is vacuous for $n = 4, 5$, since $\mathcal{E}$ will be split by [14]. So, assume that $n \geqslant 6$ and that $\mathcal{E}$ satisfies $H^l_*(\mathcal{E}) = 0$ for some $2 \leqslant l \leqslant n-2$. By [14], there must also be a j such that $H^j_*(\mathcal{E}) \neq 0$ for some $2 \leqslant j \leqslant n-2$.

We may choose l to be the lowest value with $H^l_*(\mathcal{E}) = 0$ and let us suppose that $l \geqslant 3$. Then $H^{l-1}_*(\mathcal{E}) = N_{l-1} \neq 0$. Consider the exact sequence using Lemma 2.2 (with $t = l-1$)

$$A \otimes N_{l-1} \to H^{l-1}_*(\wedge^2\mathcal{P}) \to B \otimes N_{l-1} \to 0.$$

Now if $\mathcal{P} \cong \mathcal{Q} \oplus \mathcal{O}_{\mathbf{P}}(a)$, then $H^{l-1}_*(\wedge^2\mathcal{P}) \cong H^{l-1}_*(\wedge^2\mathcal{Q}) \oplus [S(a) \otimes N_{l-1}]$, where $N_{l-1} \neq 0$. The map $S(a) \otimes N_{l-1} \to B \otimes N_{l-1}$ in the sequence is induced by the map $\mathcal{O}_{\mathbf{P}}(a) \otimes \mathcal{P}_k \xrightarrow{\beta_2 \otimes I} \mathcal{B} \otimes \mathcal{P}_k$, where $\beta = [\beta_1, \beta_2]$ in the monad for $\mathcal{E}$.

The map $A \otimes N_{l-1} \to S(a) \otimes N_{l-1}$ is induced by the map $\mathcal{A} \otimes \mathcal{G} \to \wedge^2\mathcal{G} \hookrightarrow \wedge^2\mathcal{P} \twoheadrightarrow \mathcal{O}_{\mathbf{P}}(a) \otimes \mathcal{P}$, hence by $\mathcal{A} \otimes \mathcal{P} \xrightarrow{\alpha_2 \otimes I} \mathcal{L} \otimes \mathcal{P}$ if $\alpha = [\alpha_1, \alpha_2]^T$ in the monad.

The sequence above now reads

$$A \otimes N_{l-1} \xrightarrow{\begin{bmatrix} * \\ \alpha_2 \otimes I \end{bmatrix}} H^{l-1}_*(\wedge^2\mathcal{Q}) \oplus [S(a) \otimes N_{l-1}] \xrightarrow{\left[*, \beta_2 \otimes I\right]} B \otimes N_{l-1} \to 0.$$

If we tensor the sequence by the quotient $k = S/(X_0, \dots, X_{n+1})$, since the matrix β_2 is a minimal matrix, $(\beta_2 \otimes I) \otimes k = 0$, hence $[S(a) \otimes N_{l-1} \otimes k]$ is inside the kernel of $\left[*, \beta_2 \otimes I\right] \otimes k$. By exactness, $S(a) \otimes N_{l-1} \otimes k$ is inside the image of $(\alpha_2 \otimes I) \otimes k$, which is not possible since α_2 is also a minimal matrix.

It remains to study the case where $l = 2$. There is a value l' between 3 and $n-3$ for which $H^{l'}_*(\mathcal{E}) = N_{l'} \neq 0$ and $H^{l'+1}_*(\mathcal{E}) = 0$. We now have an exact sequence of nonzero S-modules

$$A \otimes N_{l'} \to H^{l'}_*(\wedge^2\mathcal{P}) \to B \otimes N_{l'} \to 0,$$

and we repeat the earlier argument to get a contradiction. $\square$

Definition 2.4. A rank two bundle $\mathcal{E}$ on $\mathbf{P}^n$, $n \geqslant 6$, will be said to have isolated cohomology of type (n,k) if there exists an integer k, $1 < k \leqslant \frac{n}{2}$, with $H^k_*(\mathcal{E})$ and $H^{n-k}_*(\mathcal{E})$ nonzero modules, and $H^i_*(\mathcal{E}) = 0$ for $i \neq 1, k, n-k, n-1$.

Remark 2.5. By Lemma 2.1, we get that if $\mathcal{E}$ has isolated cohomology of type (n,k), then $H^i_*(\wedge^2\mathcal{P}) = 0$ for $i \neq 1, k, n-k, n-1$.

A special case in the definition is when the middle cohomology is not zero, that is, of type (n,k), where n is even, equal to $2k$, and the only nonzero cohomology modules are $H^1_*(\mathcal{E}), H^k_*(\mathcal{E}), H^{n-1}_*(\mathcal{E})$.

Note that the conditions that $H^1_*(\mathcal{E}), H^{n-1}_*(\mathcal{E})$ are both nonzero for an indecomposable rank two bundle follow from the Syzygy Theorem. In [14], it is proved that for an indecomposable rank two bundle on $\mathbf{P}^n$, $n \geqslant 4$, at least one cohomology module $H^l_*(\mathcal{E})$ must be nonzero with $1 < l < n-1$. The reason n is chosen to be $\geqslant 6$ in the definition is that first, the definition is vacuous for $n = 2, 3$ and second, for $n = 4, 5$, k must be 2, and the definition made is always satisfied by any possible indecomposable rank two bundle on $\mathbf{P}^4$ or $\mathbf{P}^5$, and hence imposes no restrictions.

Let $\mathcal{P}_k(N)$ be the kth syzygy bundle of the finite length module N. By this, we mean that in a minimal free resolution for N over the polynomial ring S:

$$0 \to L_{n+1} \xrightarrow{f_{n+1}} L_n \to \cdots \to L_{k+1} \xrightarrow{f_{k+1}} L_k \to \cdots \to L_1 \xrightarrow{f_1} L_0 \to N \to 0.$$

$P_k(N)$ will denote the image of f_{k+1} and $\mathcal{P}_k(N)$ will denote the sheafification of $P_k(N)$. Hence, $H^k_*(\mathcal{P}_k(N)) = N$, with $H^i_*(\mathcal{P}_k(N)) = 0$ when $i \neq 0, k, n$. According to [7], if $\mathcal{P}$ is any bundle on $\mathbf{P}^n$ with the property that $H^k_*(\mathcal{P}) = N$ and $H^i_*(\mathcal{P}) = 0$ when $i \neq 0, k, n$, then $\mathcal{P} \cong \mathcal{P}_k(N) \oplus \mathcal{F}$ where $\mathcal{F}$ is a direct sum of line bundles.

Lemma 2.6. *Let $\mathcal{P}$ be a vector bundle on $\mathbf{P}^n$ with nonzero cohomology modules $H^k_*(\mathcal{P}) = N$, $H^l_*(\mathcal{P}) = M$ for $1 \leqslant k < l \leqslant n-1$, and with $H^i_*(\mathcal{P}) = 0$ when $i \neq 0, k, l, n$. Then there is an exact sequence*

$$0 \to \mathcal{P}_k(N) \to \mathcal{P} \oplus \mathcal{F} \to \mathcal{P}_l(M) \to 0,$$

where $\mathcal{F}$ is some free bundle.

Proof. This too follows from [7]. Letting P denote $H^0_*(\mathcal{P})$, form an exact sequence (by partially resolving $P^\vee$)

$$0 \to P \to L_k \to L_{k-1} \to \dots L_1 \to A \to N \to 0,$$

where A is not a free module. Compare this with a truncated minimal free resolution of N:

$$0 \to P_k(N) \to L'_k \to L'_{k-1} \to \dots L'_1 \to L'_0 \to N \to 0.$$

The induced map $P_k(N) \to P$ gives a map $\mathcal{P}_k(N) \to \mathcal{P}$ that is an isomorphism at the cohomology level H^k_*. Minimally add a free module F to P to force a surjection $P^\vee \oplus F^\vee \to P_k(N)^\vee$. This gives an inclusion of bundles $\mathcal{P}_k(N) \to \mathcal{P} \oplus \mathcal{F}$ whose cokernel is $\mathcal{P}_l(M) \oplus \mathcal{F}'$ where $\mathcal{F}'$ is a free bundle (since it has only H^l_* intermediate cohomology). We notice that both for $k = 1$ and for $k > 1$, the map $H^1_*(\mathcal{P}_k(N)) \to H^1_*(\mathcal{P} \oplus \mathcal{F})$ is an isomorphism, so we get a surjection from $H^0_*(\mathcal{P} \oplus \mathcal{F})$ to $H^0_*(\mathcal{P}_l(M) \oplus \mathcal{F}')$. By the minimality of F, we may conclude that $\mathcal{F}' = 0$ □

Summarizing this below, we get the following.

Proposition 2.7. *Let $\mathcal{E}$ be a rank two bundle on $\mathbf{P}^n$, $n \geqslant 6$ with isolated cohomology of type (n, k) with $H^k_*(\mathcal{E}) = N$, for some k strictly between 1 and $\frac{n}{2}$. Then $\mathcal{E}$ has the monad*

$$0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{P} \xrightarrow{\beta} \mathcal{B} \to 0,$$

where

- $\mathcal{P}$ *satisfies an exact sequence* $0 \to \mathcal{P}_k(N) \to \mathcal{P} \oplus \mathcal{F} \to \mathcal{P}_{n-k}(M) \to 0$, *where* $\mathcal{F}$ *is some free bundle,* $M = H^{n-k}_*(\mathcal{E})$ *(which can be identified with* $N^\vee$ *up to twist).*
- $H^i_*(\wedge^2\mathcal{P}) = 0$ *for* $i \neq 1, k, n-k, n-1$.
- $H^1_*(\wedge^2\mathcal{P})$ *and* $H^{n-1}_*(\wedge^2\mathcal{P})$ *are nonzero if* $k \neq 2$.

In the case left out in the above proposition, where $\mathcal{E}$ has isolated middle cohomology with $n = 2k$ and with $H^k_*(\mathcal{E}) = N \neq 0$ equal to the only nonzero cohomology module in the range $1 < i < n-1$, the monad for $\mathcal{E}$ has the form

$$0 \to \mathcal{A} \to \mathcal{P}_k(N) \to \mathcal{B} \to 0.$$

Also, there is a short exact sequence

$$0 \to A \otimes N \to H^k_*(\wedge^2\mathcal{P}_k(N)) \to B \otimes N \to 0.$$

Thus,

Proposition 2.8. *Let* $\mathcal{E}$ *be a rank two bundle on* $\mathbf{P}^n$, $n = 2k, n \geqslant 6$, *with* $H^k_*(\mathcal{E}) = N$, $H^i_*(\mathcal{E}) = 0, i \neq 1, k, n$. *Let* $\mathcal{P}_k$ *be the kth syzygy bundle of N where* $\mathcal{P}_k$ *is the sheafification of* P_k *with* $P_k =$ *Image of* $(f_{k+1} : L_{k+1} \to L_k)$ *in a minimal free resolution of N. Then* $\mathcal{E}$ *has the monad*

$$0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{P}_k \xrightarrow{\beta} \mathcal{B} \to 0,$$

where $\mathcal{A}, \mathcal{B}$ *are sheafifications of free summands* A, B *of* L_{k+1} *and* L_k*, respectively, and where* α, β *are induced by* f_{k+1}*. Furthermore,*

- $H^i_*(\wedge^2\mathcal{P}_k) = 0$ *for* $i \neq 1, k, n-1$,
- *the induced sequence* $0 \to A \otimes N \to H^k_*(\wedge^2\mathcal{P}_k) \to B \otimes N \to 0$ *is exact,*
- $H^1_*(\wedge^2\mathcal{P}_k)$ *and* $H^{n-1}_*(\wedge^2\mathcal{P}_k)$ *are nonzero.*

Proof. The only item to verify is that $\mathcal{A}, \mathcal{B}$ are sheafifications of free summands A, B of L_{k+1} and L_k, respectively, and that α, β are induced by f_{k+1}. Since $L_{k+1} \to P_k$ is surjective, $\alpha : A \to P_k$ factors through $\tilde{\alpha} : A \to L_{k+1}$. Likewise, since $L_k^\vee \to P_k^\vee$ is surjective, $\beta^\vee : B^\vee \to P_k^\vee$ factors through $\tilde{\beta}^\vee : B^\vee \to L_k^\vee$. It remains to show that the matrices $\tilde{\alpha}, \tilde{\beta}$ have full rank when tensored by k.

The map $H^k_*(\wedge^2\mathcal{P}_k) \to B \otimes N \to 0$ in the short sequence above is obtained from $\wedge^2\mathcal{P}_k \to \mathcal{B} \otimes \mathcal{P}_k$ where $p \wedge q$ maps to $\beta(p) \otimes q - \beta(q) \otimes p$. This factors through $\mathcal{L}_k \otimes \mathcal{P}_k$ via the lift $\tilde{\beta}$. In particular, the map $L_k \otimes N \to B \otimes N$, given by $\tilde{\beta} \otimes I$, is onto. Hence so is $(\tilde{\beta} \otimes k) \otimes I$, a map of vector spaces. Hence, the matrix $\tilde{\beta} \otimes k$ has rank equal to the rank of B. So, B is a direct summand of L_k.

The map $0 \to A \otimes N \to H^k_*(\wedge^2\mathcal{P}_k)$ is obtained from $H^k_*(\mathcal{A} \otimes \mathcal{G}) \cong H^k_*(\wedge^2\mathcal{G}) \hookrightarrow H^k_*(\wedge^2\mathcal{P}_k)$, which, in turn, is obtained from $\mathcal{A} \otimes \mathcal{G} \to \wedge^2\mathcal{G} \hookrightarrow \wedge^2\mathcal{P}_k$, where $a \otimes g$ maps to $\alpha(a) \wedge g$ in $\wedge^2\mathcal{P}_k$. This map $\mathcal{A} \otimes \mathcal{G} \to \wedge^2\mathcal{P}_k$ factors through $\mathcal{L}_{k+1} \otimes \mathcal{G}$, vial the lift $\tilde{\alpha}$.

It follows that the injection $A \otimes N \to H^k_*(\wedge^2\mathcal{P}_k)$ factors through $A \otimes N \to L_{k+1} \otimes N$, by the map $\tilde{\alpha} \otimes I$. This must also be injective. Choose a socle element n in N (an element that is annihilated by all linear forms in S). The submodule generated by n, $\langle n \rangle$, is a one-dimensional vector space and $A \otimes \langle n \rangle$ is mapped injectively by $\tilde{\alpha} \otimes I$ to $L_{k+1} \otimes N$. Since the image of $\tilde{\alpha} \otimes I$ on $A \otimes \langle n \rangle$ is the same as the image of $(\tilde{\alpha} \otimes k) \otimes I$ on $(A \otimes k) \otimes \langle n \rangle$, it follows that the rank of the matrix $\tilde{\alpha} \otimes k$ has rank equal to the rank of A. Thus, A is a direct summand of L_{k+1}. ☐

We now review a result of Jyotilingam [9] about cohomology modules of tensor products, applying it to the special case of syzygy bundles for our purposes. In the theorem below, N and M will be graded finite length S-modules where $S = k[X_0, X_1, \dots, X_n]$ corresponding to $\mathbf{P}^n$. $\mathcal{P}_k(N)$ and $\mathcal{Q}_l(M)$ will indicate syzygy bundles obtained from minimal free resolutions of N and M. Note that in the minimal free resolution,

$$0 \to L_{n+1} \to L_n \to \cdots \to L_1 \to L_0 \to N \to 0,$$

when we tensor by M, the map $L_{n+1} \otimes M \to L_n \otimes M$, cannot be injective since M has finite length, hence $\mathrm{Tor}^S_{n+1}(N, M) \neq 0$, and by Lichtenbaum's theorem [10] $\mathrm{Tor}^S_i(N, M) \neq 0$ for all $i \leqslant n+1$.

Theorem 2.9. *Let N be a finite S-module and let $\mathcal{P}_k$ be its kth syzygy bundle on $\mathbf{P}^n$, with $k \geqslant 1$. Let $\mathcal{Q}$ be a bundle on $\mathbf{P}^n$ with $H^l_*(\mathcal{Q}) = M \neq 0$, with $k \leqslant l \leqslant n-2$, and with $H^i_*(\mathcal{Q}) = 0$ for $i = l-1, l-2, \dots, l-k+2$. Then $H^{l+1}_*(\mathcal{P}_k \otimes \mathcal{Q}) \neq 0$.*

Proof. The cases $k = 1$ and $k = 2$ require no conditions on $H^{l-1}_*(\mathcal{Q})$. When $k = 1$, we get the sequence $H^l_*(\mathcal{L}_1 \otimes \mathcal{Q}) \to H^l_*(\mathcal{L}_0 \otimes \mathcal{Q}) \to H^{l+1}_*(\mathcal{P}_1 \otimes \mathcal{Q}) \to 0$ and the map $L_1 \otimes M \to L_0 \otimes M$ can never be surjective. When $k > 1$, consider the diagram obtained from the sequences $0 \to \mathcal{P}_i \otimes \mathcal{Q} \to \mathcal{L}_i \otimes \mathcal{Q} \to \mathcal{P}_{i-1} \otimes \mathcal{Q} \to 0$, $i = k, k-1, k-2$ (with $\mathcal{P}_j = 0$ if $j < 0$ and $\mathcal{P}_0 = \mathcal{L}_0$):

$$\begin{array}{ccccc}
L_k \otimes M & = & L_k \otimes M & & H^{l-1}_*(\mathcal{P}_{k-3} \otimes \mathcal{Q}) \\
\downarrow & & \downarrow \gamma & & \downarrow \\
H^l_*(\mathcal{P}_{k-1} \otimes \mathcal{Q}) & \xrightarrow{\alpha} & L_{k-1} \otimes M & \xrightarrow{\beta} & H^l_*(\mathcal{P}_{k-2} \otimes \mathcal{Q}) \\
\downarrow \mu & & \downarrow \delta & & \downarrow \\
H^{l+1}_*(\mathcal{P}_k \otimes \mathcal{Q}) & & L_{k-2} \otimes M & = & L_{k-2} \otimes M
\end{array}$$

The vanishing conditions on $H^i_*(\mathcal{Q})$ show that $H^{l-1}_*(\mathcal{P}_{k-3} \otimes \mathcal{Q}) = H^{l-2}_*(\mathcal{P}_{k-4} \otimes \mathcal{Q}) = \cdots = H^{l-k+2}_*(\mathcal{L}_0 \otimes \mathcal{Q}) = 0$. So, $\ker \delta = \operatorname{im} \alpha$ and the diagram induces a surjection $\operatorname{im} \mu \to \mathrm{Tor}_{k-1}(N, M)$. By Lichtenbaum's theorem, $H^{l+1}_*(\mathcal{P}_k \otimes \mathcal{Q}) \neq 0$. □

3 | ISOLATED COHOMOLOGY OF TYPE (n, k), WITH $n \geqslant 4k$

In this section, we will prove that there are no indecomposable rank two bundles on $\mathbf{P}^n$ with isolated cohomology of type (n, k), where $n \geqslant 4k$. We study the sequence $0 \to \mathcal{P}_k(N) \to \mathcal{P} \oplus \mathcal{F} \to \mathcal{P}_{n-k}(M) \to 0$ of Proposition 2.7. We will need to pay special attention to the case where N is a cyclic module. Hence the following lemma.

Lemma 3.1. *Let N be a graded cyclic S-module. For the corresponding syzygy bundle $\mathcal{P}_2(N)$ on $\mathbf{P}^n$, $H^3_*(S^2\mathcal{P}_2(N)) = 0$ and $H^3_*(\wedge^2\mathcal{P}_2(N)) \neq 0$.*

Proof. From the sequence $0 \to \mathcal{P}_2 \to \mathcal{L}_2 \to \mathcal{P}_1 \to 0$ obtained from a minimal resolution of N, it suffices to show that the map $H^1_*(\mathcal{L}_2 \otimes \mathcal{P}_1) \to H^1_*(\wedge^2\mathcal{P}_1)$ is surjective to prove that

$H^3_*(S^2\mathcal{P}_2(N)) = 0$. This map can be studied using the natural commuting diagram

$$\begin{array}{ccccccccc} 0 \to & \mathcal{L}_2 \otimes \mathcal{P}_1 & \to & \mathcal{L}_2 \otimes \mathcal{L}_1 & \to & \mathcal{L}_2 \otimes \mathcal{L}_0 & \to 0 & & \\ & \downarrow & & \downarrow & & \downarrow & & & \\ 0 \to & \wedge^2\mathcal{P}_1 & \to & \wedge^2\mathcal{L}_1 & \to & \mathcal{L}_1 \otimes \mathcal{L}_0 & \to & S^2\mathcal{L}_0 & \end{array}$$

It simplifies when $\mathcal{L}_0$ has rank one, where without loss of generality, we can take $\mathcal{L}_0$ to be $\mathcal{O}_{\mathbf{P}^n}$, yielding

$$\begin{array}{ccccccc} 0 \to & \mathcal{L}_2 \otimes \mathcal{P}_1 & \to & \mathcal{L}_2 \otimes \mathcal{L}_1 & \to & \mathcal{L}_2 & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & . \\ 0 \to & \wedge^2\mathcal{P}_1 & \to & \wedge^2\mathcal{L}_1 & \to & \mathcal{P}_1 & \to 0 \end{array}$$

Since $\mathcal{L}_2$ surjects onto the global sections of $\mathcal{P}_1$, it follows from the diagram of long exact sequences of cohomology modules that $H^1_*(\mathcal{L}_2 \otimes \mathcal{P}_1) \to H^1_*(\wedge^2\mathcal{P}_1)$ is onto.

For the second part, we will show that $H^3_*(\mathcal{P}_2 \otimes \mathcal{P}_2) \neq 0$. (This argument will be repeated later in a slightly different setting.) With $H^3_*(S^2\mathcal{P}_2) = 0$, since $H^3_*(\mathcal{P}_2 \otimes \mathcal{P}_2) = H^3_*(S^2\mathcal{P}_2) \oplus H^3_*(\wedge^2\mathcal{P}_2)$, the conclusion of the lemma follows.

Consider $0 \to \mathcal{P}_2 \otimes \mathcal{P}_2 \to \mathcal{L}_2 \otimes \mathcal{P}_2 \to \mathcal{L}_1 \otimes \mathcal{P}_2 \to \mathcal{L}_0 \otimes \mathcal{P}_2 \to 0$. From $0 \to \mathcal{P}_1 \otimes \mathcal{P}_2 \to \mathcal{L}_1 \otimes \mathcal{P}_2 \to \mathcal{L}_0 \otimes \mathcal{P}_2 \to 0$, we get

$$H^2_*(\mathcal{P}_1 \otimes \mathcal{P}_2) = \ker(L_1 \otimes N \to L_0 \otimes N) = L_1 \otimes N$$

since N is cyclic. Hence, we get

$$H^3_*(\mathcal{P}_2 \otimes \mathcal{P}_2) = \text{ coker } (L_2 \otimes N \to L_1 \otimes N),$$

which is clearly nonzero. □

Proposition 3.2. *Suppose that $\mathcal{E}$ on $\mathbf{P}^n$ is a rank two bundle of type (n, k) with $n \geqslant 7$, k strictly less than $\frac{n}{2}$. Then the sequence $0 \to \mathcal{P}_k(N) \to \mathcal{P} \oplus \mathcal{F} \to \mathcal{P}_l(M) \to 0$, in Proposition 2.7, is not-split.*

Proof. Suppose $\mathcal{P} \oplus \mathcal{F} = \mathcal{P}_k(N) \oplus \mathcal{P}_{n-k}(M)$. Neither $\mathcal{P}_k(N)$ nor $\mathcal{P}_{n-k}(M)$ has any line bundle summands, hence $\mathcal{P} = \mathcal{P}_k(N) \oplus \mathcal{P}_{n-k}(M)$. So, $\wedge^2\mathcal{P}$ has summands $\mathcal{P}_k(N) \otimes \mathcal{P}_{n-k}(M)$ and $\wedge^2\mathcal{P}_k(N)$. If $k > 2$, then using Proposition 2.9, $H^{n-k+1}_*(\mathcal{P}_k(N) \otimes \mathcal{P}_{n-k}(M))$ is nonzero which contradicts the requirement in Proposition 2.7 that $H^{n-k+1}_*(\wedge^2\mathcal{P}) = 0$.

If $k = 2$, there are two cases: if N is cyclic, then $H^3_*(\wedge^2\mathcal{P}_2(N)) \neq 0$ by Lemma 3.1, which contradicts Proposition 2.7 since $n - k > 3$ when $n \geqslant 6$.

If N is noncyclic, then from the sequences $0 \to \mathcal{P}_2(N) \to \mathcal{L}_2 \to \mathcal{P}_1(N) \to 0$ and $0 \to \mathcal{P}_1 \to \mathcal{L}_1 \to \mathcal{L}_0 \to 0$, we get $H^4_*(\wedge^2\mathcal{P}_2(N)) \neq 0$. This a contradiction to Proposition 2.7 when $n \geqslant 7$. □

Remark 3.3. The case $n = 6, k = 2$ is not answered above. A weaker argument can be made here that even though $\mathcal{P} = \mathcal{P}_k(N) \oplus \mathcal{P}_{n-k}(M)$, N itself is neither cyclic nor a direct sum of submodules $N_1 \oplus N_2$.

Theorem 3.4. *Let $\mathcal{E}$ be a rank two vector bundle on $\mathbf{P}^8$ with $H^3_*(\mathcal{E}) = H^4_*(\mathcal{E}) = 0$, then $\mathcal{E}$ splits.*

Proof. Let $N = H^2_*(\mathcal{E})$ and $M = H^6_*(\mathcal{E})$. Both are nonzero unless $\mathcal{E}$ splits. By Proposition 3.2 (with $k = 2$), we know that the sequence below is nonsplit.

$$0 \to \mathcal{P}_2(N) \to \mathcal{P} \oplus \mathcal{F} \to \mathcal{P}_6(M) \to 0. \tag{5}$$

The proof will analyze the consequences of the two sequences below obtained from sequence.

$$0 \to S^2\mathcal{P}_2(N) \to \mathcal{P}_2(N) \otimes [\mathcal{P} \oplus \mathcal{F}] \to \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \to \wedge^2\mathcal{P}_6(M) \to 0, \tag{6}$$

$$0 \to \wedge^2\mathcal{P}_2(N) \to \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \to \mathcal{P}_6(M) \otimes [\mathcal{P} \oplus \mathcal{F}] \to S^2\mathcal{P}_6(M) \to 0. \tag{7}$$

Case 1 If N is cyclic, we look at the sequence (6).
It breaks into

$$\begin{aligned} &0 \to S^2\mathcal{P}_2(N) \to \mathcal{P}_2(N) \otimes [\mathcal{P} \oplus \mathcal{F}] \to \mathcal{D} \to 0, \\ &0 \to \mathcal{D} \to \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \to \wedge^2\mathcal{P}_6(M) \to 0, \end{aligned} \tag{8}$$

$H^3_*(\mathcal{P}_2(N) \otimes [\mathcal{P} \oplus \mathcal{F}]) \neq 0$ by the same argument in the second part of the proof of Lemma 3.1, and by the same lemma, $H^3_*(S^2\mathcal{P}_2(N)) = 0$. Hence, $H^3_*(\mathcal{D}) \neq 0$ from the first sequence in (8).

In the second sequence in (8), $H^3_*(\mathcal{P}) = 0$. Hence so is $H^3_*(\wedge^2\mathcal{P})$. Finally, $\mathcal{P}_6(M)$ fits into a sequence with free bundles

$$0 \to \mathcal{L}'_9 \to \mathcal{L}'_8 \to \mathcal{L}'_7 \to \mathcal{P}_6 \to 0.$$

This yields two exact sequences

$$\begin{aligned} &0 \to S^2\mathcal{P}_7 \to S^2\mathcal{L}'_7 \to \mathcal{L}'_7 \otimes \mathcal{P}_6 \to \wedge^2\mathcal{P}_6 \to 0, \\ &0 \to \wedge^2\mathcal{L}'_9 \to \wedge^2\mathcal{L}'_8 \to \mathcal{L}'_8 \otimes \mathcal{P}_7 \to S^2\mathcal{P}_7 \to 0. \end{aligned} \tag{9}$$

From these, we can chase down $H^2_*(\wedge^2\mathcal{P}_6)$ to be equal to zero since $H^2_*(\mathcal{P}_6) = 0, H^4_*(\mathcal{P}_7) = 0, H^6_*(\wedge^2\mathcal{L}'_9) = 0$. Hence, $H^3_*(\mathcal{D})$ is both zero and nonzero, a contradiction.

Case 2 If N is noncyclic, we look at the sequence (7)

$$0 \to \wedge^2\mathcal{P}_2(N) \to \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \to \mathcal{P}_6(M) \otimes [\mathcal{P} \oplus \mathcal{F}] \to S^2\mathcal{P}_6(M) \to 0.$$

It breaks into

$$\begin{aligned} &0 \to \wedge^2\mathcal{P}_2(N) \to \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \to \mathcal{D} \to 0, \\ &0 \to \mathcal{D} \to \mathcal{P}_6(M) \otimes [\mathcal{P} \oplus \mathcal{F}] \to S^2\mathcal{P}_6(M) \to 0. \end{aligned} \tag{10}$$

From

$$0 \to S^2\mathcal{P}_1(N) \to S^2\mathcal{L}_1 \to \mathcal{L}_1 \otimes \mathcal{L}_0 \to \wedge^2\mathcal{L}_0 \to 0,$$

$$0 \to \wedge^2\mathcal{P}_2(N) \to \wedge^2\mathcal{L}_2 \to \mathcal{L}_2 \otimes \mathcal{L}_1 \to S^2\mathcal{P}_1(N) \to 0,$$

we get $H^2_*(S^2\mathcal{P}_1(N)) \neq 0$ and $H^4_*(\wedge^2\mathcal{P}_2(N)) \neq 0$. Since $H^4_*(\mathcal{P})$ and $H^4_*(\wedge^2\mathcal{P})$ are zero, we obtain $H^3_*(\mathcal{D}) \neq 0$.

Again, in the second sequence in (10), $H^3_*(\mathcal{P}_6(M) \otimes \mathcal{F}) = 0$ and $H^3_*(\mathcal{P}_6(M) \otimes \mathcal{P})$ can be studied using a resolution for $\mathcal{P}_6(M)$ and tensoring with $\mathcal{P}$.

$$0 \to \mathcal{L}'_9 \otimes \mathcal{P} \to \mathcal{L}'_8 \otimes \mathcal{P} \to \mathcal{L}'_7 \otimes \mathcal{P} \to \mathcal{P}_6(M) \otimes \mathcal{P} \to 0.$$

Then $H^3_*(\mathcal{P}_6(M) \otimes \mathcal{P}) = 0$ since $H^3_*(\mathcal{P}), H^4_*(\mathcal{P}), H^5_*(\mathcal{P})$ are all zero.

We compute $H^2_*(S^2\mathcal{P}_6(M))$, breaking up the resolution of $\mathcal{P}_6$ (suppressing the letter M) into short exact sequences:

$$\begin{aligned} &0 \to \wedge^2\mathcal{P}_7 \to \wedge^2\mathcal{L}'_7 \to \mathcal{L}'_7 \otimes \mathcal{P}_6 \to S^2\mathcal{P}_6 \to 0, \\ &0 \to S^2\mathcal{L}'_9 \to S^2\mathcal{L}'_8 \to \mathcal{L}'_8 \otimes \mathcal{P}_7 \to \wedge^2\mathcal{P}_7 \to 0. \end{aligned} \tag{11}$$

$H^2_*(S^2\mathcal{P}_6(M))$ will vanish since $H^2_*(\mathcal{P}_6)$, $H^4_*(\mathcal{P}_7)$ and $H^6_*(S^2\mathcal{L}'_9)$ are all zero. □

Corollary 3.5. *Let $n \geqslant 8$. Let $\mathcal{E}$ be a rank two vector bundle on $\mathbf{P}^n$ with $H^i_*(\mathcal{E}) = 0$ for $i = 3, \dots n-3$. Then $\mathcal{E}$ splits.*

Proof. Use induction on n. The case $n = 8$ is proved in the above theorem. Assume the result for $n-1$. Let $\mathcal{E}$ be a rank two vector bundle on $\mathbf{P}^n$ with $H^i_*(\mathcal{E}) = 0$ for $i = 3, \dots n-3$. For a hyperplane H, by the restriction sequence in cohomology,

$$H^i_*(\mathcal{E}) \to H^i_*(\mathcal{E}_H) \to H^{i+1}_*(\mathcal{E}(-1)),$$

we get that $H^i_*(\mathcal{E}_H) = 0$ for $i = 3, \dots n-4$ on $\mathbf{P}^{n-1}$. So, $\mathcal{E}_H$ splits and hence also $\mathcal{E}$. □

The theorem above can be generalized to arbitrary k using the similar calculations.

Theorem 3.6. *Let $n \geqslant 4k$, with $k > 1$. Then there cannot exist a rank two bundle $\mathcal{E}$ on $\mathbf{P}^n$, for which the only nonzero intermediate cohomology modules are $H^1_*(\mathcal{E})$, $H^k_*(\mathcal{E}) = N$, $H^{n-k}_*(\mathcal{E}) = M$, and $H^{n-1}_*(\mathcal{E})$.*

Proof. The case $k = 2$ was done in the corollary above. So, we assume that $k > 2$. The proof will analyze the consequences of the sequence

$$0 \to \mathcal{P}_k(N) \to \mathcal{P} \oplus \mathcal{F} \to \mathcal{P}_{n-k}(M) \to 0, \tag{12}$$

which is nonsplit by Proposition 3.2. We get the collateral sequence:

$$0 \to \wedge^2\mathcal{P}_k(N) \to \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \to \mathcal{P}_{n-k}(M) \otimes [\mathcal{P} \oplus \mathcal{F}] \to S^2\mathcal{P}_{n-k}(M) \to 0. \tag{13}$$

We will prove it using several cases.

Case 1 The case where N is cyclic, k is even and > 2.

We look at the sequence (13) which breaks into

$$\begin{aligned} &0 \to \wedge^2\mathcal{P}_k(N) \to \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \to \mathcal{D} \to 0, \\ &0 \to \mathcal{D} \to \mathcal{P}_{n-k}(M) \otimes [\mathcal{P} \oplus \mathcal{F}] \to S^2\mathcal{P}_{n-k}(M) \to 0, \end{aligned} \tag{14}$$

$H^3_*(\wedge^2\mathcal{P}_2(N)) \neq 0$. This yields $H^{2k-1}_*(\wedge^2\mathcal{P}_k(N)) \neq 0$, since $n > 2k-1$. On the other hand, $H^{2k-1}_*(\mathcal{P})$ and $H^{2k-1}_*(\wedge^2\mathcal{P})$ are zero, since $k < 2k-1 < n-k$ when $n \geqslant 4k$. Hence $H^{2k-2}_*(\mathcal{D}) \neq 0$ using the first short exact sequence in (14).

In the second sequence in (14), $H^{2k-2}_*(\mathcal{P}_{n-k}(M) \otimes \mathcal{F}) = 0$ since $2k-2 \neq n-k$. $H^{2k-2}_*(\mathcal{P}_{n-k}(M) \otimes \mathcal{P})$ can be studied using a resolution for $\mathcal{P}_{n-k}(M)$ and tensoring with $\mathcal{P}$.

$$0 \to \mathcal{L}'_{n+1} \otimes \mathcal{P} \to \mathcal{L}'_n \otimes \mathcal{P} \to \dots \mathcal{L}'_{n-k+2} \otimes \mathcal{P} \to \mathcal{L}'_{n-k+1} \otimes \mathcal{P} \to \mathcal{P}_{n-k}(M) \otimes \mathcal{P} \to 0.$$

Then $H^{2k-2}_*(\mathcal{P}_{n-k}(M) \otimes \mathcal{P}) = 0$ provided $H^{2k-2}_*(\mathcal{P}), H^{2k-1}_*(\mathcal{P}), \dots, H^{3k-2}_*(\mathcal{P})$ are all zero. Since $n \geqslant 4k$, $n-k > 3k-2$ and since $k > 2$, $k < 2k-2$. Hence, these vanishings hold.

We compute $H^{2k-3}_*(S^2\mathcal{P}_{n-k}(M))$, breaking up the resolution of $\mathcal{P}_{n-k}$ (suppressing the letter M) into short exact sequences:

$$\begin{aligned}
0 \to \wedge^2\mathcal{P}_{n-k+1} \to \wedge^2\mathcal{L}'_{n-k+1} \to \mathcal{L}'_{n-k+1} \otimes \mathcal{P}_{n-k} \to S^2\mathcal{P}_{n-k}\\
0 \to S^2\mathcal{P}_{n-k+2} \to \wedge^2\mathcal{L}'_{n-k+2} \to \mathcal{L}'_{n-k+2} \otimes \mathcal{P}_{n-k+1} \to \wedge^2\mathcal{P}_{n-k+1}\\
0 \to \wedge^2\mathcal{P}_{n-k+3} \to \wedge^2\mathcal{L}'_{n-k+3} \to \mathcal{L}'_{n-k+3} \otimes \mathcal{P}_{n-k+2} \to S^2\mathcal{P}_{n-k+2}\\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots\\
0 \to S^2\mathcal{L}'_{n+1} \to S^2\mathcal{L}'_n \to \mathcal{L}'_n \otimes \mathcal{P}_{n-1} \to \wedge^2\mathcal{P}_{n-1}.
\end{aligned} \tag{15}$$

$H^{2k-3}_*(S^2\mathcal{P}_{n-k}(M))$ will vanish provided $H^{2k-3}_*(\mathcal{P}_{n-k})$, $H^{2k-1}_*(\mathcal{P}_{n-k+1})$, …, $H^{4k-5}_*(\mathcal{P}_{n-1})$ and $H^{4k-3}_*(S^2\mathcal{L}'_{n+1})$ are all zero. $H^{4k-3}_*(S^2\mathcal{L}'_{n+1}) = 0$ since $n > 4k-3$. For the others, $H^{2k-3+2i}_*(\mathcal{P}_{n-k+i}) = 0$ since $n-k+i > 2k-3+2i$ when $0 \leqslant i \leqslant k-1$. We have concluded that $H^{2k-2}_*(\mathcal{D}) = 0$ from the second sequence, contradicting the earlier result of being nonzero.

Case 2 The case where N is noncyclic, $k > 2$ is even.

This is very similar to Case 1. We use the same sequence (13). Now $H^4_*(\wedge^2\mathcal{P}_2(N)) \neq 0$. Hence, $H^{2k}_*(\wedge^2\mathcal{P}_k(N)) \neq 0$, since $n > 2k$. $H^{2k}_*(\mathcal{P})$ and $H^{2k}_*(\wedge^2\mathcal{P})$ are zero, since $k < 2k < n-k$, hence $H^{2k-1}_*(\mathcal{D}) \neq 0$.

Again, $H^{2k-1}_*(\mathcal{P}_{n-k}(M) \otimes \mathcal{F}) = 0$ since $2k-1 \neq n-k$ and $H^{2k-1}_*(\mathcal{P}_{n-k}(M) \otimes \mathcal{P}) = 0$ since $n-k > 3k-1$ and $k < 2k-1$. Lastly, $H^{2k-2}_*(S^2\mathcal{P}_{n-k}(M)) = 0$ since $n > 4k-2$ and $n-k+i > 2k-2+2i$ when $0 \leqslant i \leqslant k-1$. Hence, $H^{2k-1}_*(\mathcal{D})$ is also equal to 0.

Case 3 The case where k is odd.

$H^{2k}_*(\wedge^2\mathcal{P}_k(N)) \neq 0$ as in Case 2. We use sequence (13) and copy the proof in Case 2. □

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