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RESEARCH ARTICLE

Rank two bundles on \mathbf{P}^n with isolated cohomology

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Abstract

The purpose of this paper is to study minimal monads associated to a rank two vector bundle \mathcal{E} on \mathbf{P}^n . In particular, we study situations where \mathcal{E} has $H_*^i(\mathcal{E}) = 0$ for $1 < i < n - 1$, except for one pair of values $(k, n - k)$. We show that on \mathbf{P}^8 , if $H_*^3(\mathcal{E}) = H_*^4(\mathcal{E}) = 0$, then \mathcal{E} must be decomposable. More generally, we show that for $n \geq 4k$, there is no indecomposable bundle \mathcal{E} for which all intermediate cohomology modules except for $H_*^1, H_*^k, H_*^{n-k}, H_*^{n-1}$ are zero.

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1 | INTRODUCTION

It has been difficult to disprove the existence of an indecomposable rank two bundle \mathcal{E} on \mathbf{P}^n for large n . Most known results have been obtained by imposing other conditions on \mathcal{E} to show that \mathcal{E} cannot exist or must be split. For example, the so-called Babylonian condition which requires \mathcal{E} to be extendable to \mathbf{P}^{n+m} for every m has been studied by a number of people including Barth and van de Ven [2] and Coanda and Trautmann [3]. Numerical criteria that force splitting are found again in Barth and van de Ven, where for a normalized rank two bundle with second Chern class a and with splitting type $\mathcal{O}_l(-b) \oplus \mathcal{O}_l(b)$ on the general line l , a function $f(a, b)$ is found such that if $n > f(a, b)$, then a bundle on \mathbf{P}^n with these invariants must be split.

Cohomological criteria for forcing the splitting of \mathcal{E} start with Horrocks [7]. If S is the polynomial ring corresponding to \mathbf{P}^n , then $H_*^i(\mathcal{E})$ (defined as $\bigoplus_{\nu} H^i(\mathbf{P}^n, \mathcal{E}(\nu))$) is an S -module. The intermediate cohomology modules $H_*^i(\mathcal{E})$, $1 \leq i \leq n - 1$ are all graded modules of finite length and there is a strong relationship between \mathcal{E} and its intermediate cohomology modules. He shows that if $H_*^i(\mathcal{E}) = 0$ for all i with $1 \leq i \leq n - 1$, then \mathcal{E} is split. Moreover, Horrocks in [7] established

that a vector bundle on \mathbf{P}^n is determined up to isomorphism and up to a sum of line bundles (i.e., up to stable equivalence) by its collection of intermediate cohomology modules and also a certain collection of extension classes involving these modules. This correspondence has been generalized to any Arithmetically Cohen-Macaulay (ACM) varieties in [12]. The Syzygy Theorem ([5, 6]) shows that for a rank two bundle \mathcal{E} , it is enough to know that $H_*^1(\mathcal{E}) = 0$ to force splitting. In [14], it is shown that for an indecomposable rank two bundle on \mathbf{P}^n , in addition to $H_*^1(\mathcal{E})$ and $H_*^{n-1}(\mathcal{E})$ being nonzero, some intermediate cohomology module $H_*^k(\mathcal{E})$ ($1 < k < n - 1$) (and hence also $H_*^{n-k}(\mathcal{E})$) must be nonzero. Various calculations in [13] and [14] show that there are limitations on the module structure of $H_*^1(\mathcal{E})$ and $H_*^2(\mathcal{E})$ for some values of n .

In this paper, we study situations where a rank two bundle \mathcal{E} on \mathbf{P}^n has $H_*^i(\mathcal{E}) = 0$ for $1 < i < n - 1$, except for one pair of values $(k, n - k)$. We describe the minimal monads associated to \mathcal{E} . We show that on \mathbf{P}^8 , if $H_*^3(\mathcal{E}) = H_*^4(\mathcal{E}) = 0$, then \mathcal{E} must be decomposable. More generally, we show that for $n \geq 4k$, there is no indecomposable bundle \mathcal{E} for which all intermediate cohomology modules except for $H_*^1, H_*^k, H_*^{n-k}, H_*^{n-1}$ are zero. The proof utilizes the space between k and $n - k$ when $n \geq 4k$ for making cohomological computations.

2 | MONADS FOR RANK TWO VECTOR BUNDLES ON \mathbf{P}^n

Let \mathcal{E} be an indecomposable rank two vector bundle on \mathbf{P}^n over an algebraically closed field of characteristic different from two. If S is the polynomial ring on $n + 1$ variables, let $N_i = H_*^i(\mathcal{E}) = \bigoplus_{\nu} H^i(\mathcal{E}(\nu))$ be the finite length graded S -module over S , for $1 \leq i \leq n - 1$. By the Syzygy Theorem, both N_1 and N_{n-1} are nonzero modules. Horrocks ([8]) gives a brief description of the construction of a minimal monad for a bundle \mathcal{E} of any rank on \mathbf{P}^n by “killing both H_*^1 and H_*^{n-1} .” Barth and Hulek ([1]) use this idea to construct (with more detail) l - m minimal monads for a bundle \mathcal{E} on \mathbf{P}^n , where only $H_{\geq l}^1(\mathcal{E})$ and $H_{< -m-n}^{n-1}(\mathcal{E})$ are killed. Horrocks’ construction, which we use below, is obtained when both l and m are very negative.

The monad is a complex

$$0 \rightarrow \mathcal{A} \xrightarrow{\alpha} \mathcal{P} \xrightarrow{\beta} \mathcal{B} \rightarrow 0,$$

where \mathcal{P} is a bundle with $H_*^i(\mathcal{P}) = 0$ for $i = 1$ and $i = n - 1$, and where \mathcal{A}, \mathcal{B} are free bundles. Let \mathcal{G} be kernel β . We have two sequences

$$\begin{aligned} 0 \rightarrow \mathcal{G} \rightarrow \mathcal{P} \rightarrow \mathcal{B} \rightarrow 0, \\ 0 \rightarrow \mathcal{A} \rightarrow \mathcal{G} \rightarrow \mathcal{E} \rightarrow 0, \end{aligned} \tag{1}$$

from which we see that $H_*^i(\mathcal{E}) = H_*^i(\mathcal{G})$ for $1 \leq i \leq n - 2$, while $H_*^{n-1}(\mathcal{G}) = 0$, and $H_*^i(\mathcal{E}) = H_*^i(\mathcal{P})$ for $2 \leq i \leq n - 2$, while $H_*^1(\mathcal{P}) = H_*^{n-1}(\mathcal{P}) = 0$.

The minimality of the complex means that the rank of \mathcal{B} equals the number of generators of $H_*^1(\mathcal{G}) = N_1$ and the rank of \mathcal{A}^\vee equals the number of generators of $H_*^1(\mathcal{E}^\vee)$. When the rank of the bundle \mathcal{E} equals 2, we find that \mathcal{A} and \mathcal{B} have the same rank.

The two sequences give rise to

$$\begin{aligned} 0 \rightarrow \wedge^2 \mathcal{G} \rightarrow \wedge^2 \mathcal{P} \rightarrow \mathcal{B} \otimes \mathcal{P} \rightarrow S^2 \mathcal{B} \rightarrow 0, \\ 0 \rightarrow S^2 \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{G} \rightarrow \wedge^2 \mathcal{G} \rightarrow \wedge^2 \mathcal{E} \rightarrow 0. \end{aligned} \tag{2}$$

(Since the characteristic of the field is different from 2, we can assume that S^2 commutes with duals.)

Lemma 2.1. *If $H_*^2(\mathcal{E}) = 0$, then $H_*^1(\wedge^2 \mathcal{P})$ and $H_*^{n-1}(\wedge^2 \mathcal{P})$ are nonzero. If $H_*^l(\mathcal{E}) = 0$ for some l , with $2 \leq l \leq n-2$, then $H_*^l(\wedge^2 \mathcal{P}) = 0$.*

Proof. See [11, Theorem 2.2] for the first part. Next, suppose $H_*^l(\mathcal{E}) = 0$ for some l , with $2 \leq l \leq n-2$. So, $N_l = N_{n-l} = 0$ by Serre duality. In particular, \mathcal{G} and \mathcal{P} have $H_*^l = 0$ as well. It follows from Equation (2), that $H_*^l(\wedge^2 \mathcal{G}) = 0$ and hence $H_*^l(\wedge^2 \mathcal{P}) = 0$. \square

Lemma 2.2. *Let $2 \leq t \leq n-2$. Let $A = H_*^0(\mathcal{A})$, $B = H_*^0(\mathcal{B})$. There is an exact sequence*

$$A \otimes N_t \rightarrow H_*^t(\wedge^2 \mathcal{P}) \rightarrow B \otimes N_t,$$

which is injective on the left if $t \geq 3$ and $N_{t-1} = 0$, and is surjective on the right if $t \leq n-3$ and $N_{t+1} = 0$.

Proof. Break up the first sequence in 2 as $0 \rightarrow \wedge^2 \mathcal{G} \rightarrow \wedge^2 \mathcal{P} \rightarrow \mathcal{D} \rightarrow 0$, $0 \rightarrow \mathcal{D} \rightarrow B \otimes \mathcal{P} \rightarrow S^2 \mathcal{B} \rightarrow 0$. We get long exact sequences

$$H_*^{t-1}(\mathcal{D}) \rightarrow H_*^t(\wedge^2 \mathcal{G}) \rightarrow H_*^t(\wedge^2 \mathcal{P}) \rightarrow H_*^t(\mathcal{D}) \rightarrow H_*^{t+1}(\wedge^2 \mathcal{G}),$$

where $H_*^t(\mathcal{D}) \cong B \otimes N_t$ (always) and $H_*^{t-1}(\mathcal{D}) \cong B \otimes N_{t-1}$ provided $t \geq 3$. Likewise break up the second sequence as $0 \rightarrow S^2 \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{G} \rightarrow \mathcal{C} \rightarrow 0$, $0 \rightarrow \mathcal{C} \rightarrow \wedge^2 \mathcal{G} \rightarrow \wedge^2 \mathcal{E} \rightarrow 0$. We see that $H_*^i(\wedge^2 \mathcal{G}) \cong H_*^i(\mathcal{C})$ for $i = t, t+1$, $H_*^t(\mathcal{C}) \cong A \otimes N_t$ and when $t \leq n-3$, $H_*^{t+1}(\mathcal{C}) \cong A \otimes N_{t+1}$. \square

Moreover, when $H_*^l(\mathcal{E}) = 0$ for some l , with $2 \leq l \leq n-3$, from Equation (2) since $H_*^{l+1}(S^2 \mathcal{A}) = H_*^{l+2}(\mathcal{A} \otimes \mathcal{G}) = H_*^l(\wedge^2 \mathcal{E}) = H_*^{l+1}(\wedge^2 \mathcal{E}) = 0$, we get $H_*^{l+1}(\wedge^2 \mathcal{G}) \cong H_*^{l+1}(\mathcal{A} \otimes \mathcal{G})$. Since $H_*^{l-1}(S^2 \mathcal{B}) = H_*^l(B \otimes \mathcal{P}) = 0$, $H_*^{l+1}(\wedge^2 \mathcal{G}) \hookrightarrow H_*^{l+1}(\wedge^2 \mathcal{P})$. From

$$0 \rightarrow \mathcal{D} \rightarrow B \otimes \mathcal{P} \rightarrow S^2 \mathcal{B} \rightarrow 0,$$

we obtain $H_*^{l+1}(B \otimes \mathcal{P}) \cong H_*^{l+1}(\mathcal{D})$ giving an exact sequence

$$0 \rightarrow A \otimes N_{l+1} \rightarrow H_*^{l+1}(\wedge^2 \mathcal{P}) \rightarrow B \otimes N_{l+1}, \quad (3)$$

where $A = H_*^0(\mathcal{A})$, $B = H_*^0(\mathcal{B})$. Notice that the sequence is exact on the right if $N_{l+2} = 0$.

Likewise, if $H_*^l(\mathcal{E}) = 0$ for some l , with $3 \leq l \leq n-2$, we get the exact sequence

$$A \otimes N_{l-1} \rightarrow H_*^{l-1}(\wedge^2 \mathcal{P}) \rightarrow B \otimes N_{l-1} \rightarrow 0. \quad (4)$$

Notice that the sequence is exact on the right if $N_{l-2} = 0$.

The following proposition is a typical one that shows that a minimal monad for a rank two bundle is built very minimally out of the cohomological data for \mathcal{E} . Other examples of such a result can be found in [15] and [13]. Decker ([4]) has conjectured such a minimality for rank two bundles on \mathbf{P}^4 .

Proposition 2.3. *Suppose that \mathcal{E} is a nonsplit rank two bundle on \mathbf{P}^n ($n \geq 6$), with $H_*^l(\mathcal{E}) = 0$ for some l with $2 \leq l \leq n - 2$. Then in the minimal monad for \mathcal{E} , the bundle \mathcal{P} has no line bundle summands.*

Proof. Note that the statement is vacuous for $n = 4, 5$, since \mathcal{E} will be split by [14]. So, assume that $n \geq 6$ and that \mathcal{E} satisfies $H_*^l(\mathcal{E}) = 0$ for some $2 \leq l \leq n - 2$. By [14], there must also be a j such that $H_*^j(\mathcal{E}) \neq 0$ for some $2 \leq j \leq n - 2$.

We may choose l to be the lowest value with $H_*^l(\mathcal{E}) = 0$ and let us suppose that $l \geq 3$. Then $H_*^{l-1}(\mathcal{E}) = N_{l-1} \neq 0$. Consider the exact sequence using Lemma 2.2 (with $t = l - 1$)

$$A \otimes N_{l-1} \rightarrow H_*^{l-1}(\wedge^2 \mathcal{P}) \rightarrow B \otimes N_{l-1} \rightarrow 0.$$

Now if $\mathcal{P} \cong \mathcal{Q} \oplus \mathcal{O}_{\mathbf{P}}(a)$, then $H_*^{l-1}(\wedge^2 \mathcal{P}) \cong H_*^{l-1}(\wedge^2 \mathcal{Q}) \oplus [S(a) \otimes N_{l-1}]$, where $N_{l-1} \neq 0$. The map $S(a) \otimes N_{l-1} \rightarrow B \otimes N_{l-1}$ in the sequence is induced by the map $\mathcal{O}_{\mathbf{P}}(a) \otimes \mathcal{P}_k \xrightarrow{\beta_2 \otimes I} B \otimes \mathcal{P}_k$, where $\beta = [\beta_1, \beta_2]$ in the monad for \mathcal{E} .

The map $A \otimes N_{l-1} \rightarrow S(a) \otimes N_{l-1}$ is induced by the map $\mathcal{A} \otimes \mathcal{G} \rightarrow \wedge^2 \mathcal{G} \hookrightarrow \wedge^2 \mathcal{P} \rightarrow \mathcal{O}_{\mathbf{P}}(a) \otimes \mathcal{P}$, hence by $\mathcal{A} \otimes \mathcal{P} \xrightarrow{\alpha_2 \otimes I} \mathcal{L} \otimes \mathcal{P}$ if $\alpha = [\alpha_1, \alpha_2]^T$ in the monad.

The sequence above now reads

$$A \otimes N_{l-1} \xrightarrow{\begin{bmatrix} * \\ \alpha_2 \otimes I \end{bmatrix}} H_*^{l-1}(\wedge^2 \mathcal{Q}) \oplus [S(a) \otimes N_{l-1}] \xrightarrow{[* , \beta_2 \otimes I]} B \otimes N_{l-1} \rightarrow 0.$$

If we tensor the sequence by the quotient $k = S/(X_0, \dots, X_{n+1})$, since the matrix β_2 is a minimal matrix, $(\beta_2 \otimes I) \otimes k = 0$, hence $[S(a) \otimes N_{l-1} \otimes k]$ is inside the kernel of $[* , \beta_2 \otimes I] \otimes k$. By exactness, $S(a) \otimes N_{l-1} \otimes k$ is inside the image of $(\alpha_2 \otimes I) \otimes k$, which is not possible since α_2 is also a minimal matrix.

It remains to study the case where $l = 2$. There is a value l' between 3 and $n - 3$ for which $H_*^{l'}(\mathcal{E}) = N_{l'} \neq 0$ and $H_*^{l'+1}(\mathcal{E}) = 0$. We now have an exact sequence of nonzero S -modules

$$A \otimes N_{l'} \rightarrow H_*^{l'}(\wedge^2 \mathcal{P}) \rightarrow B \otimes N_{l'} \rightarrow 0,$$

and we repeat the earlier argument to get a contradiction. □

Definition 2.4. A rank two bundle \mathcal{E} on \mathbf{P}^n , $n \geq 6$, will be said to have isolated cohomology of type (n, k) if there exists an integer k , $1 < k \leq \frac{n}{2}$, with $H_*^k(\mathcal{E})$ and $H_*^{n-k}(\mathcal{E})$ nonzero modules, and $H_*^i(\mathcal{E}) = 0$ for $i \neq 1, k, n - k, n - 1$.

Remark 2.5. By Lemma 2.1, we get that if \mathcal{E} has isolated cohomology of type (n, k) , then $H_*^i(\wedge^2 \mathcal{P}) = 0$ for $i \neq 1, k, n - k, n - 1$.

A special case in the definition is when the middle cohomology is not zero, that is, of type (n, k) , where n is even, equal to $2k$, and the only nonzero cohomology modules are $H_*^1(\mathcal{E}), H_*^k(\mathcal{E}), H_*^{n-1}(\mathcal{E})$.

Note that the conditions that $H_*^1(\mathcal{E}), H_*^{n-1}(\mathcal{E})$ are both nonzero for an indecomposable rank two bundle follow from the Syzygy Theorem. In [14], it is proved that for an indecomposable rank two bundle on $\mathbf{P}^n, n \geq 4$, at least one cohomology module $H_*^l(\mathcal{E})$ must be nonzero with $1 < l < n - 1$. The reason n is chosen to be ≥ 6 in the definition is that first, the definition is vacuous for $n = 2, 3$ and second, for $n = 4, 5, k$ must be 2, and the definition made is always satisfied by any possible indecomposable rank two bundle on \mathbf{P}^4 or \mathbf{P}^5 , and hence imposes no restrictions.

Let $\mathcal{P}_k(N)$ be the k th syzygy bundle of the finite length module N . By this, we mean that in a minimal free resolution for N over the polynomial ring S :

$$0 \rightarrow L_{n+1} \xrightarrow{f_{n+1}} L_n \rightarrow \dots \rightarrow L_{k+1} \xrightarrow{f_{k+1}} L_k \rightarrow \dots \rightarrow L_1 \xrightarrow{f_1} L_0 \rightarrow N \rightarrow 0.$$

$P_k(N)$ will denote the image of f_{k+1} and $\mathcal{P}_k(N)$ will denote the sheafification of $P_k(N)$. Hence, $H_*^k(\mathcal{P}_k(N)) = N$, with $H_*^i(\mathcal{P}_k(N)) = 0$ when $i \neq 0, k, n$. According to [7], if \mathcal{P} is any bundle on \mathbf{P}^n with the property that $H_*^k(\mathcal{P}) = N$ and $H_*^i(\mathcal{P}) = 0$ when $i \neq 0, k, n$, then $\mathcal{P} \cong \mathcal{P}_k(N) \oplus \mathcal{F}$ where \mathcal{F} is a direct sum of line bundles.

Lemma 2.6. *Let \mathcal{P} be a vector bundle on \mathbf{P}^n with nonzero cohomology modules $H_*^k(\mathcal{P}) = N, H_*^l(\mathcal{P}) = M$ for $1 \leq k < l \leq n - 1$, and with $H_*^i(\mathcal{P}) = 0$ when $i \neq 0, k, l, n$. Then there is an exact sequence*

$$0 \rightarrow \mathcal{P}_k(N) \rightarrow \mathcal{P} \oplus \mathcal{F} \rightarrow \mathcal{P}_l(M) \rightarrow 0,$$

where \mathcal{F} is some free bundle.

Proof. This too follows from [7]. Letting P denote $H_*^0(\mathcal{P})$, form an exact sequence (by partially resolving P^\vee)

$$0 \rightarrow P \rightarrow L_k \rightarrow L_{k-1} \rightarrow \dots \rightarrow L_1 \rightarrow A \rightarrow N \rightarrow 0,$$

where A is not a free module. Compare this with a truncated minimal free resolution of N :

$$0 \rightarrow P_k(N) \rightarrow L'_k \rightarrow L'_{k-1} \rightarrow \dots \rightarrow L'_1 \rightarrow L'_0 \rightarrow N \rightarrow 0.$$

The induced map $P_k(N) \rightarrow P$ gives a map $\mathcal{P}_k(N) \rightarrow \mathcal{P}$ that is an isomorphism at the cohomology level H_*^k . Minimally add a free module F to P to force a surjection $P^\vee \oplus F^\vee \rightarrow P_k(N)^\vee$. This gives an inclusion of bundles $\mathcal{P}_k(N) \rightarrow \mathcal{P} \oplus F$ whose cokernel is $\mathcal{P}_l(M) \oplus \mathcal{F}'$ where \mathcal{F}' is a free bundle (since it has only H_*^l intermediate cohomology). We notice that both for $k = 1$ and for $k > 1$, the map $H_*^1(\mathcal{P}_k(N)) \rightarrow H_*^1(\mathcal{P} \oplus F)$ is an isomorphism, so we get a surjection from $H_*^0(\mathcal{P} \oplus F)$ to $H_*^0(\mathcal{P}_l(M) \oplus \mathcal{F}')$. By the minimality of F , we may conclude that $\mathcal{F}' = 0$ □

Summarizing this below, we get the following.

Proposition 2.7. *Let \mathcal{E} be a rank two bundle on $\mathbf{P}^n, n \geq 6$ with isolated cohomology of type (n, k) with $H_*^k(\mathcal{E}) = N$, for some k strictly between 1 and $\frac{n}{2}$. Then \mathcal{E} has the monad*

$$0 \rightarrow \mathcal{A} \xrightarrow{\alpha} \mathcal{P} \xrightarrow{\beta} \mathcal{B} \rightarrow 0,$$

where

- \mathcal{P} satisfies an exact sequence $0 \rightarrow \mathcal{P}_k(N) \rightarrow \mathcal{P} \oplus \mathcal{F} \rightarrow \mathcal{P}_{n-k}(M) \rightarrow 0$, where \mathcal{F} is some free bundle, $M = H_*^{n-k}(\mathcal{E})$ (which can be identified with N^\vee up to twist).
- $H_*^i(\wedge^2 \mathcal{P}) = 0$ for $i \neq 1, k, n - k, n - 1$.
- $H_*^1(\wedge^2 \mathcal{P})$ and $H_*^{n-1}(\wedge^2 \mathcal{P})$ are nonzero if $k \neq 2$.

In the case left out in the above proposition, where \mathcal{E} has isolated middle cohomology with $n = 2k$ and with $H_*^k(\mathcal{E}) = N \neq 0$ equal to the only nonzero cohomology module in the range $1 < i < n - 1$, the monad for \mathcal{E} has the form

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{P}_k(N) \rightarrow \mathcal{B} \rightarrow 0.$$

Also, there is a short exact sequence

$$0 \rightarrow A \otimes N \rightarrow H_*^k(\wedge^2 \mathcal{P}_k(N)) \rightarrow B \otimes N \rightarrow 0.$$

Thus,

Proposition 2.8. *Let \mathcal{E} be a rank two bundle on \mathbf{P}^n , $n = 2k, n \geq 6$, with $H_*^k(\mathcal{E}) = N, H_*^i(\mathcal{E}) = 0, i \neq 1, k, n$. Let \mathcal{P}_k be the k th syzygy bundle of N where \mathcal{P}_k is the sheafification of P_k with $P_k = \text{Image of } (f_{k+1} : L_{k+1} \rightarrow L_k)$ in a minimal free resolution of N . Then \mathcal{E} has the monad*

$$0 \rightarrow \mathcal{A} \xrightarrow{\alpha} \mathcal{P}_k \xrightarrow{\beta} \mathcal{B} \rightarrow 0,$$

where \mathcal{A}, \mathcal{B} are sheafifications of free summands A, B of L_{k+1} and L_k , respectively, and where α, β are induced by f_{k+1} . Furthermore,

- $H_*^i(\wedge^2 \mathcal{P}_k) = 0$ for $i \neq 1, k, n - 1$,
- the induced sequence $0 \rightarrow A \otimes N \rightarrow H_*^k(\wedge^2 \mathcal{P}_k) \rightarrow B \otimes N \rightarrow 0$ is exact,
- $H_*^1(\wedge^2 \mathcal{P}_k)$ and $H_*^{n-1}(\wedge^2 \mathcal{P}_k)$ are nonzero.

Proof. The only item to verify is that \mathcal{A}, \mathcal{B} are sheafifications of free summands A, B of L_{k+1} and L_k , respectively, and that α, β are induced by f_{k+1} . Since $L_{k+1} \rightarrow P_k$ is surjective, $\alpha : A \rightarrow P_k$ factors through $\tilde{\alpha} : A \rightarrow L_{k+1}$. Likewise, since $L_k^\vee \rightarrow P_k^\vee$ is surjective, $\beta^\vee : B^\vee \rightarrow P_k^\vee$ factors through $\tilde{\beta}^\vee : B^\vee \rightarrow L_k^\vee$. It remains to show that the matrices $\tilde{\alpha}, \tilde{\beta}$ have full rank when tensored by k .

The map $H_*^k(\wedge^2 \mathcal{P}_k) \rightarrow B \otimes N \rightarrow 0$ in the short sequence above is obtained from $\wedge^2 \mathcal{P}_k \rightarrow \mathcal{B} \otimes \mathcal{P}_k$ where $p \wedge q$ maps to $\beta(p) \otimes q - \beta(q) \otimes p$. This factors through $\mathcal{L}_k \otimes \mathcal{P}_k$ via the lift $\tilde{\beta}$. In particular, the map $L_k \otimes N \rightarrow B \otimes N$, given by $\tilde{\beta} \otimes I$, is onto. Hence so is $(\tilde{\beta} \otimes k) \otimes I$, a map of vector spaces. Hence, the matrix $\tilde{\beta} \otimes k$ has rank equal to the rank of B . So, B is a direct summand of L_k .

The map $0 \rightarrow A \otimes N \rightarrow H_*^k(\wedge^2 \mathcal{P}_k)$ is obtained from $H_*^k(\mathcal{A} \otimes \mathcal{G}) \cong H_*^k(\wedge^2 \mathcal{G}) \hookrightarrow H_*^k(\wedge^2 \mathcal{P}_k)$, which, in turn, is obtained from $\mathcal{A} \otimes \mathcal{G} \rightarrow \wedge^2 \mathcal{G} \hookrightarrow \wedge^2 \mathcal{P}_k$, where $a \otimes g$ maps to $\alpha(a) \wedge g$ in $\wedge^2 \mathcal{P}_k$. This map $\mathcal{A} \otimes \mathcal{G} \rightarrow \wedge^2 \mathcal{P}_k$ factors through $\mathcal{L}_{k+1} \otimes \mathcal{G}$, via the lift $\tilde{\alpha}$.

It follows that the injection $A \otimes N \rightarrow H_*^k(\wedge^2 \mathcal{P}_k)$ factors through $A \otimes N \rightarrow L_{k+1} \otimes N$, by the map $\tilde{\alpha} \otimes I$. This must also be injective. Choose a socle element n in N (an element that is annihilated by all linear forms in S). The submodule generated by $n, \langle n \rangle$, is a one-dimensional vector space and $A \otimes \langle n \rangle$ is mapped injectively by $\tilde{\alpha} \otimes I$ to $L_{k+1} \otimes N$. Since the image of $\tilde{\alpha} \otimes I$ on $A \otimes \langle n \rangle$ is the same as the image of $(\tilde{\alpha} \otimes k) \otimes I$ on $(A \otimes k) \otimes \langle n \rangle$, it follows that the rank of the matrix $\tilde{\alpha} \otimes k$ has rank equal to the rank of A . Thus, A is a direct summand of L_{k+1} . \square

We now review a result of Jyotilingam [9] about cohomology modules of tensor products, applying it to the special case of syzygy bundles for our purposes. In the theorem below, N and M will be graded finite length S -modules where $S = k[X_0, X_1, \dots, X_n]$ corresponding to \mathbf{P}^n . $\mathcal{P}_k(N)$ and $\mathcal{Q}_i(M)$ will indicate syzygy bundles obtained from minimal free resolutions of N and M . Note that in the minimal free resolution,

$$0 \rightarrow L_{n+1} \rightarrow L_n \rightarrow \dots \rightarrow L_1 \rightarrow L_0 \rightarrow N \rightarrow 0,$$

when we tensor by M , the map $L_{n+1} \otimes M \rightarrow L_n \otimes M$, cannot be injective since M has finite length, hence $\text{Tor}_{n+1}^S(N, M) \neq 0$, and by Lichtenbaum’s theorem [10] $\text{Tor}_i^S(N, M) \neq 0$ for all $i \leq n + 1$.

Theorem 2.9. *Let N be a finite S -module and let \mathcal{P}_k be its k th syzygy bundle on \mathbf{P}^n , with $k \geq 1$. Let \mathcal{Q} be a bundle on \mathbf{P}^n with $H_*^l(\mathcal{Q}) = M \neq 0$, with $k \leq l \leq n - 2$, and with $H_*^i(\mathcal{Q}) = 0$ for $i = l - 1, l - 2, \dots, l - k + 2$. Then $H_*^{l+1}(\mathcal{P}_k \otimes \mathcal{Q}) \neq 0$.*

Proof. The cases $k = 1$ and $k = 2$ require no conditions on $H_*^{l-1}(\mathcal{Q})$. When $k = 1$, we get the sequence $H_*^l(\mathcal{L}_1 \otimes \mathcal{Q}) \rightarrow H_*^l(\mathcal{L}_0 \otimes \mathcal{Q}) \rightarrow H_*^{l+1}(\mathcal{P}_1 \otimes \mathcal{Q}) \rightarrow 0$ and the map $L_1 \otimes M \rightarrow L_0 \otimes M$ can never be surjective. When $k > 1$, consider the diagram obtained from the sequences $0 \rightarrow \mathcal{P}_i \otimes \mathcal{Q} \rightarrow \mathcal{L}_i \otimes \mathcal{Q} \rightarrow \mathcal{P}_{i-1} \otimes \mathcal{Q} \rightarrow 0$, $i = k, k - 1, k - 2$ (with $\mathcal{P}_j = 0$ if $j < 0$ and $\mathcal{P}_0 = \mathcal{L}_0$):

$$\begin{array}{ccccc} L_k \otimes M & = & L_k \otimes M & H_*^{l-1}(\mathcal{P}_{k-3} \otimes \mathcal{Q}) & \\ \downarrow & & \downarrow \gamma & \downarrow & \\ H_*^l(\mathcal{P}_{k-1} \otimes \mathcal{Q}) & \xrightarrow{\alpha} & L_{k-1} \otimes M & \xrightarrow{\beta} & H_*^l(\mathcal{P}_{k-2} \otimes \mathcal{Q}) \\ \downarrow \mu & & \downarrow \delta & \downarrow & \\ H_*^{l+1}(\mathcal{P}_k \otimes \mathcal{Q}) & L_{k-2} \otimes M = & L_{k-2} \otimes M & & \end{array}$$

The vanishing conditions on $H_*^i(\mathcal{Q})$ show that $H_*^{l-1}(\mathcal{P}_{k-3} \otimes \mathcal{Q}) = H_*^{l-2}(\mathcal{P}_{k-4} \otimes \mathcal{Q}) = \dots = H_*^{l-k+2}(\mathcal{L}_0 \otimes \mathcal{Q}) = 0$. So, $\ker \delta = \text{im } \alpha$ and the diagram induces a surjection $\text{im } \mu \rightarrow \text{Tor}_{k-1}^S(N, M)$. By Lichtenbaum’s theorem, $H_*^{l+1}(\mathcal{P}_k \otimes \mathcal{Q}) \neq 0$. \square

3 | ISOLATED COHOMOLOGY OF TYPE (n, k) , WITH $n \geq 4k$

In this section, we will prove that there are no indecomposable rank two bundles on \mathbf{P}^n with isolated cohomology of type (n, k) , where $n \geq 4k$. We study the sequence $0 \rightarrow \mathcal{P}_k(N) \rightarrow \mathcal{P} \oplus \mathcal{F} \rightarrow \mathcal{P}_{n-k}(M) \rightarrow 0$ of Proposition 2.7. We will need to pay special attention to the case where N is a cyclic module. Hence the following lemma.

Lemma 3.1. *Let N be a graded cyclic S -module. For the corresponding syzygy bundle $\mathcal{P}_2(N)$ on \mathbf{P}^n , $H_*^3(S^2\mathcal{P}_2(N)) = 0$ and $H_*^3(\wedge^2\mathcal{P}_2(N)) \neq 0$.*

Proof. From the sequence $0 \rightarrow \mathcal{P}_2 \rightarrow \mathcal{L}_2 \rightarrow \mathcal{P}_1 \rightarrow 0$ obtained from a minimal resolution of N , it suffices to show that the map $H_*^1(\mathcal{L}_2 \otimes \mathcal{P}_1) \rightarrow H_*^1(\wedge^2\mathcal{P}_1)$ is surjective to prove that

$H_*^3(S^2\mathcal{P}_2(N)) = 0$. This map can be studied using the natural commuting diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{L}_2 \otimes \mathcal{P}_1 & \rightarrow & \mathcal{L}_2 \otimes \mathcal{L}_1 & \rightarrow & \mathcal{L}_2 \otimes \mathcal{L}_0 & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \wedge^2 \mathcal{P}_1 & \rightarrow & \wedge^2 \mathcal{L}_1 & \rightarrow & \mathcal{L}_1 \otimes \mathcal{L}_0 & \rightarrow & S^2 \mathcal{L}_0 \end{array}$$

It simplifies when \mathcal{L}_0 has rank one, where without loss of generality, we can take \mathcal{L}_0 to be $\mathcal{O}_{\mathbb{P}^n}$, yielding

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{L}_2 \otimes \mathcal{P}_1 & \rightarrow & \mathcal{L}_2 \otimes \mathcal{L}_1 & \rightarrow & \mathcal{L}_2 & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \wedge^2 \mathcal{P}_1 & \rightarrow & \wedge^2 \mathcal{L}_1 & \rightarrow & \mathcal{P}_1 & \rightarrow & 0 \end{array}$$

Since \mathcal{L}_2 surjects onto the global sections of \mathcal{P}_1 , it follows from the diagram of long exact sequences of cohomology modules that $H_*^1(\mathcal{L}_2 \otimes \mathcal{P}_1) \rightarrow H_*^1(\wedge^2 \mathcal{P}_1)$ is onto.

For the second part, we will show that $H_*^3(\mathcal{P}_2 \otimes \mathcal{P}_2) \neq 0$. (This argument will be repeated later in a slightly different setting.) With $H_*^3(S^2\mathcal{P}_2) = 0$, since $H_*^3(\mathcal{P}_2 \otimes \mathcal{P}_2) = H_*^3(S^2\mathcal{P}_2) \oplus H_*^3(\wedge^2 \mathcal{P}_2)$, the conclusion of the lemma follows.

Consider $0 \rightarrow \mathcal{P}_2 \otimes \mathcal{P}_2 \rightarrow \mathcal{L}_2 \otimes \mathcal{P}_2 \rightarrow \mathcal{L}_1 \otimes \mathcal{P}_2 \rightarrow \mathcal{L}_0 \otimes \mathcal{P}_2 \rightarrow 0$. From $0 \rightarrow \mathcal{P}_1 \otimes \mathcal{P}_2 \rightarrow \mathcal{L}_1 \otimes \mathcal{P}_2 \rightarrow \mathcal{L}_0 \otimes \mathcal{P}_2 \rightarrow 0$, we get

$$H_*^2(\mathcal{P}_1 \otimes \mathcal{P}_2) = \ker(L_1 \otimes N \rightarrow L_0 \otimes N) = L_1 \otimes N$$

since N is cyclic. Hence, we get

$$H_*^3(\mathcal{P}_2 \otimes \mathcal{P}_2) = \text{coker}(L_2 \otimes N \rightarrow L_1 \otimes N),$$

which is clearly nonzero. □

Proposition 3.2. *Suppose that \mathcal{E} on \mathbb{P}^n is a rank two bundle of type (n, k) with $n \geq 7$, k strictly less than $\frac{n}{2}$. Then the sequence $0 \rightarrow \mathcal{P}_k(N) \rightarrow \mathcal{P} \oplus \mathcal{F} \rightarrow \mathcal{P}_1(M) \rightarrow 0$, in Proposition 2.7, is not-split.*

Proof. Suppose $\mathcal{P} \oplus \mathcal{F} = \mathcal{P}_k(N) \oplus \mathcal{P}_{n-k}(M)$. Neither $\mathcal{P}_k(N)$ nor $\mathcal{P}_{n-k}(M)$ has any line bundle summands, hence $\mathcal{P} = \mathcal{P}_k(N) \oplus \mathcal{P}_{n-k}(M)$. So, $\wedge^2 \mathcal{P}$ has summands $\mathcal{P}_k(N) \otimes \mathcal{P}_{n-k}(M)$ and $\wedge^2 \mathcal{P}_k(N)$. If $k > 2$, then using Proposition 2.9, $H_*^{n-k+1}(\mathcal{P}_k(N) \otimes \mathcal{P}_{n-k}(M))$ is nonzero which contradicts the requirement in Proposition 2.7 that $H_*^{n-k+1}(\wedge^2 \mathcal{P}) = 0$.

If $k = 2$, there are two cases: if N is cyclic, then $H_*^3(\wedge^2 \mathcal{P}_2(N)) \neq 0$ by Lemma 3.1, which contradicts Proposition 2.7 since $n - k > 3$ when $n \geq 6$.

If N is noncyclic, then from the sequences $0 \rightarrow \mathcal{P}_2(N) \rightarrow \mathcal{L}_2 \rightarrow \mathcal{P}_1(N) \rightarrow 0$ and $0 \rightarrow \mathcal{P}_1 \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_0 \rightarrow 0$, we get $H_*^4(\wedge^2 \mathcal{P}_2(N)) \neq 0$. This a contradiction to Proposition 2.7 when $n \geq 7$. □

Remark 3.3. The case $n = 6, k = 2$ is not answered above. A weaker argument can be made here that even though $\mathcal{P} = \mathcal{P}_k(N) \oplus \mathcal{P}_{n-k}(M)$, N itself is neither cyclic nor a direct sum of submodules $N_1 \oplus N_2$.

Theorem 3.4. *Let \mathcal{E} be a rank two vector bundle on \mathbf{P}^8 with $H_*^3(\mathcal{E}) = H_*^4(\mathcal{E}) = 0$, then \mathcal{E} splits.*

Proof. Let $N = H_*^2(\mathcal{E})$ and $M = H_*^6(\mathcal{E})$. Both are nonzero unless \mathcal{E} splits. By Proposition 3.2 (with $k = 2$), we know that the sequence below is nonsplit.

$$0 \rightarrow \mathcal{P}_2(N) \rightarrow \mathcal{P} \oplus \mathcal{F} \rightarrow \mathcal{P}_6(M) \rightarrow 0. \quad (5)$$

The proof will analyze the consequences of the two sequences below obtained from sequence.

$$0 \rightarrow S^2\mathcal{P}_2(N) \rightarrow \mathcal{P}_2(N) \otimes [\mathcal{P} \oplus \mathcal{F}] \rightarrow \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \rightarrow \wedge^2\mathcal{P}_6(M) \rightarrow 0, \quad (6)$$

$$0 \rightarrow \wedge^2\mathcal{P}_2(N) \rightarrow \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \rightarrow \mathcal{P}_6(M) \otimes [\mathcal{P} \oplus \mathcal{F}] \rightarrow S^2\mathcal{P}_6(M) \rightarrow 0. \quad (7)$$

Case 1 If N is cyclic, we look at the sequence (6).

It breaks into

$$\begin{aligned} 0 \rightarrow S^2\mathcal{P}_2(N) \rightarrow \mathcal{P}_2(N) \otimes [\mathcal{P} \oplus \mathcal{F}] \rightarrow \mathcal{D} \rightarrow 0, \\ 0 \rightarrow \mathcal{D} \rightarrow \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \rightarrow \wedge^2\mathcal{P}_6(M) \rightarrow 0, \end{aligned} \quad (8)$$

$H_*^3(\mathcal{P}_2(N) \otimes [\mathcal{P} \oplus \mathcal{F}]) \neq 0$ by the same argument in the second part of the proof of Lemma 3.1, and by the same lemma, $H_*^3(S^2\mathcal{P}_2(N)) = 0$. Hence, $H_*^3(\mathcal{D}) \neq 0$ from the first sequence in (8).

In the second sequence in (8), $H_*^3(\mathcal{P}) = 0$. Hence so is $H_*^3(\wedge^2\mathcal{P})$. Finally, $\mathcal{P}_6(M)$ fits into a sequence with free bundles

$$0 \rightarrow \mathcal{L}'_9 \rightarrow \mathcal{L}'_8 \rightarrow \mathcal{L}'_7 \rightarrow \mathcal{P}_6 \rightarrow 0.$$

This yields two exact sequences

$$\begin{aligned} 0 \rightarrow S^2\mathcal{P}_7 \rightarrow S^2\mathcal{L}'_7 \rightarrow \mathcal{L}'_7 \otimes \mathcal{P}_6 \rightarrow \wedge^2\mathcal{P}_6 \rightarrow 0, \\ 0 \rightarrow \wedge^2\mathcal{L}'_9 \rightarrow \wedge^2\mathcal{L}'_8 \rightarrow \mathcal{L}'_8 \otimes \mathcal{P}_7 \rightarrow S^2\mathcal{P}_7 \rightarrow 0. \end{aligned} \quad (9)$$

From these, we can chase down $H_*^2(\wedge^2\mathcal{P}_6)$ to be equal to zero since $H_*^2(\mathcal{P}_6) = 0$, $H_*^4(\mathcal{P}_7) = 0$, $H_*^6(\wedge^2\mathcal{L}'_9) = 0$. Hence, $H_*^3(\mathcal{D})$ is both zero and nonzero, a contradiction.

Case 2 If N is noncyclic, we look at the sequence (7)

$$0 \rightarrow \wedge^2\mathcal{P}_2(N) \rightarrow \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \rightarrow \mathcal{P}_6(M) \otimes [\mathcal{P} \oplus \mathcal{F}] \rightarrow S^2\mathcal{P}_6(M) \rightarrow 0.$$

It breaks into

$$\begin{aligned} 0 \rightarrow \wedge^2\mathcal{P}_2(N) \rightarrow \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \rightarrow \mathcal{D} \rightarrow 0, \\ 0 \rightarrow \mathcal{D} \rightarrow \mathcal{P}_6(M) \otimes [\mathcal{P} \oplus \mathcal{F}] \rightarrow S^2\mathcal{P}_6(M) \rightarrow 0. \end{aligned} \quad (10)$$

From

$$\begin{aligned} 0 \rightarrow S^2\mathcal{P}_1(N) \rightarrow S^2\mathcal{L}_1 \rightarrow \mathcal{L}_1 \otimes \mathcal{L}_0 \rightarrow \wedge^2\mathcal{L}_0 \rightarrow 0, \\ 0 \rightarrow \wedge^2\mathcal{P}_2(N) \rightarrow \wedge^2\mathcal{L}_2 \rightarrow \mathcal{L}_2 \otimes \mathcal{L}_1 \rightarrow S^2\mathcal{P}_1(N) \rightarrow 0, \end{aligned}$$

we get $H_*^2(S^2\mathcal{P}_1(N)) \neq 0$ and $H_*^4(\wedge^2\mathcal{P}_2(N)) \neq 0$. Since $H_*^4(\mathcal{P})$ and $H_*^4(\wedge^2\mathcal{P})$ are zero, we obtain $H_*^3(\mathcal{D}) \neq 0$.

Again, in the second sequence in (10), $H_*^3(\mathcal{P}_6(M) \otimes \mathcal{F}) = 0$ and $H_*^3(\mathcal{P}_6(M) \otimes \mathcal{P})$ can be studied using a resolution for $\mathcal{P}_6(M)$ and tensoring with \mathcal{P} .

$$0 \rightarrow \mathcal{L}'_9 \otimes \mathcal{P} \rightarrow \mathcal{L}'_8 \otimes \mathcal{P} \rightarrow \mathcal{L}'_7 \otimes \mathcal{P} \rightarrow \mathcal{P}_6(M) \otimes \mathcal{P} \rightarrow 0.$$

Then $H_*^3(\mathcal{P}_6(M) \otimes \mathcal{P}) = 0$ since $H_*^3(\mathcal{P}), H_*^4(\mathcal{P}), H_*^5(\mathcal{P})$ are all zero.

We compute $H_*^2(S^2\mathcal{P}_6(M))$, breaking up the resolution of \mathcal{P}_6 (suppressing the letter M) into short exact sequences:

$$\begin{aligned} 0 \rightarrow \wedge^2\mathcal{P}_7 \rightarrow \wedge^2\mathcal{L}'_7 \rightarrow \mathcal{L}'_7 \otimes \mathcal{P}_6 \rightarrow S^2\mathcal{P}_6 \rightarrow 0, \\ 0 \rightarrow S^2\mathcal{L}'_9 \rightarrow S^2\mathcal{L}'_8 \rightarrow \mathcal{L}'_8 \otimes \mathcal{P}_7 \rightarrow \wedge^2\mathcal{P}_7 \rightarrow 0. \end{aligned} \tag{11}$$

$H_*^2(S^2\mathcal{P}_6(M))$ will vanish since $H_*^2(\mathcal{P}_6), H_*^4(\mathcal{P}_7)$ and $H_*^6(S^2\mathcal{L}'_9)$ are all zero. □

Corollary 3.5. *Let $n \geq 8$. Let \mathcal{E} be a rank two vector bundle on \mathbf{P}^n with $H_*^i(\mathcal{E}) = 0$ for $i = 3, \dots, n - 3$. Then \mathcal{E} splits.*

Proof. Use induction on n . The case $n = 8$ is proved in the above theorem. Assume the result for $n - 1$. Let \mathcal{E} be a rank two vector bundle on \mathbf{P}^n with $H_*^i(\mathcal{E}) = 0$ for $i = 3, \dots, n - 3$. For a hyperplane H , by the restriction sequence in cohomology,

$$H_*^i(\mathcal{E}) \rightarrow H_*^i(\mathcal{E}_H) \rightarrow H_*^{i+1}(\mathcal{E}(-1)),$$

we get that $H_*^i(\mathcal{E}_H) = 0$ for $i = 3, \dots, n - 4$ on \mathbf{P}^{n-1} . So, \mathcal{E}_H splits and hence also \mathcal{E} . □

The theorem above can be generalized to arbitrary k using the similar calculations.

Theorem 3.6. *Let $n \geq 4k$, with $k > 1$. Then there cannot exist a rank two bundle \mathcal{E} on \mathbf{P}^n , for which the only nonzero intermediate cohomology modules are $H_*^1(\mathcal{E}), H_*^k(\mathcal{E}) = N, H_*^{n-k}(\mathcal{E}) = M$, and $H_*^{n-1}(\mathcal{E})$.*

Proof. The case $k = 2$ was done in the corollary above. So, we assume that $k > 2$. The proof will analyze the consequences of the sequence

$$0 \rightarrow \mathcal{P}_k(N) \rightarrow \mathcal{P} \oplus \mathcal{F} \rightarrow \mathcal{P}_{n-k}(M) \rightarrow 0, \tag{12}$$

which is nonsplit by Proposition 3.2. We get the collateral sequence:

$$0 \rightarrow \wedge^2\mathcal{P}_k(N) \rightarrow \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \rightarrow \mathcal{P}_{n-k}(M) \otimes [\mathcal{P} \oplus \mathcal{F}] \rightarrow S^2\mathcal{P}_{n-k}(M) \rightarrow 0. \tag{13}$$

We will prove it using several cases.

Case 1 The case where N is cyclic, k is even and > 2 .

We look at the sequence (13) which breaks into

$$\begin{aligned} 0 \rightarrow \wedge^2\mathcal{P}_k(N) \rightarrow \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \rightarrow \mathcal{D} \rightarrow 0, \\ 0 \rightarrow \mathcal{D} \rightarrow \mathcal{P}_{n-k}(M) \otimes [\mathcal{P} \oplus \mathcal{F}] \rightarrow S^2\mathcal{P}_{n-k}(M) \rightarrow 0, \end{aligned} \tag{14}$$

$H_*^3(\wedge^2 \mathcal{P}_2(N)) \neq 0$. This yields $H_*^{2k-1}(\wedge^2 \mathcal{P}_k(N)) \neq 0$, since $n > 2k - 1$. On the other hand, $H_*^{2k-1}(\mathcal{P})$ and $H_*^{2k-1}(\wedge^2 \mathcal{P})$ are zero, since $k < 2k - 1 < n - k$ when $n \geq 4k$. Hence $H_*^{2k-2}(\mathcal{D}) \neq 0$ using the first short exact sequence in (14).

In the second sequence in (14), $H_*^{2k-2}(\mathcal{P}_{n-k}(M) \otimes \mathcal{F}) = 0$ since $2k - 2 \neq n - k$. $H_*^{2k-2}(\mathcal{P}_{n-k}(M) \otimes \mathcal{P})$ can be studied using a resolution for $\mathcal{P}_{n-k}(M)$ and tensoring with \mathcal{P} .

$$0 \rightarrow \mathcal{L}'_{n+1} \otimes \mathcal{P} \rightarrow \mathcal{L}'_n \otimes \mathcal{P} \rightarrow \dots \rightarrow \mathcal{L}'_{n-k+2} \otimes \mathcal{P} \rightarrow \mathcal{L}'_{n-k+1} \otimes \mathcal{P} \rightarrow \mathcal{P}_{n-k}(M) \otimes \mathcal{P} \rightarrow 0.$$

Then $H_*^{2k-2}(\mathcal{P}_{n-k}(M) \otimes \mathcal{P}) = 0$ provided $H_*^{2k-2}(\mathcal{P}), H_*^{2k-1}(\mathcal{P}), \dots, H_*^{3k-2}(\mathcal{P})$ are all zero. Since $n \geq 4k, n - k > 3k - 2$ and since $k > 2, k < 2k - 2$. Hence, these vanishings hold.

We compute $H_*^{2k-3}(S^2 \mathcal{P}_{n-k}(M))$, breaking up the resolution of \mathcal{P}_{n-k} (suppressing the letter M) into short exact sequences:

$$\begin{aligned} 0 \rightarrow \wedge^2 \mathcal{P}_{n-k+1} \rightarrow \wedge^2 \mathcal{L}'_{n-k+1} \rightarrow \mathcal{L}'_{n-k+1} \otimes \mathcal{P}_{n-k} \rightarrow S^2 \mathcal{P}_{n-k} \\ 0 \rightarrow S^2 \mathcal{P}_{n-k+2} \rightarrow \wedge^2 \mathcal{L}'_{n-k+2} \rightarrow \mathcal{L}'_{n-k+2} \otimes \mathcal{P}_{n-k+1} \rightarrow \wedge^2 \mathcal{P}_{n-k+1} \\ 0 \rightarrow \wedge^2 \mathcal{P}_{n-k+3} \rightarrow \wedge^2 \mathcal{L}'_{n-k+3} \rightarrow \mathcal{L}'_{n-k+3} \otimes \mathcal{P}_{n-k+2} \rightarrow S^2 \mathcal{P}_{n-k+2} \\ \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \\ 0 \rightarrow S^2 \mathcal{L}'_{n+1} \rightarrow S^2 \mathcal{L}'_n \rightarrow \mathcal{L}'_n \otimes \mathcal{P}_{n-1} \rightarrow \wedge^2 \mathcal{P}_{n-1}. \end{aligned} \tag{15}$$

$H_*^{2k-3}(S^2 \mathcal{P}_{n-k}(M))$ will vanish provided $H_*^{2k-3}(\mathcal{P}_{n-k}), H_*^{2k-1}(\mathcal{P}_{n-k+1}), \dots, H_*^{4k-5}(\mathcal{P}_{n-1})$ and $H_*^{4k-3}(S^2 \mathcal{L}'_{n+1})$ are all zero. $H_*^{4k-3}(S^2 \mathcal{L}'_{n+1}) = 0$ since $n > 4k - 3$. For the others, $H_*^{2k-3+2i}(\mathcal{P}_{n-k+i}) = 0$ since $n - k + i > 2k - 3 + 2i$ when $0 \leq i \leq k - 1$. We have concluded that $H_*^{2k-2}(\mathcal{D}) = 0$ from the second sequence, contradicting the earlier result of being nonzero.

Case 2 The case where N is noncyclic, $k > 2$ is even.

This is very similar to Case 1. We use the same sequence (13). Now $H_*^4(\wedge^2 \mathcal{P}_2(N)) \neq 0$. Hence, $H_*^{2k}(\wedge^2 \mathcal{P}_k(N)) \neq 0$, since $n > 2k$. $H_*^{2k}(\mathcal{P})$ and $H_*^{2k}(\wedge^2 \mathcal{P})$ are zero, since $k < 2k < n - k$, hence $H_*^{2k-1}(\mathcal{D}) \neq 0$.

Again, $H_*^{2k-1}(\mathcal{P}_{n-k}(M) \otimes \mathcal{F}) = 0$ since $2k - 1 \neq n - k$ and $H_*^{2k-1}(\mathcal{P}_{n-k}(M) \otimes \mathcal{P}) = 0$ since $n - k > 3k - 1$ and $k < 2k - 1$. Lastly, $H_*^{2k-2}(S^2 \mathcal{P}_{n-k}(M)) = 0$ since $n > 4k - 2$ and $n - k + i > 2k - 2 + 2i$ when $0 \leq i \leq k - 1$. Hence, $H_*^{2k-1}(\mathcal{D})$ is also equal to 0.

Case 3 The case where k is odd.

$H_*^{2k}(\wedge^2 \mathcal{P}_k(N)) \neq 0$ as in Case 2. We use sequence (13) and copy the proof in Case 2. □

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