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ON THE GREATEST COMMON DIVISOR OF n AND THE n TH FIBONACCI NUMBER, II

ABHISHEK JHA AND CARLO SANNA[†]

ABSTRACT. Let \mathcal{A} be the set of all integers of the form $\gcd(n, F_n)$, where n is a positive integer and F_n denotes the n th Fibonacci number. Leonetti and Sanna proved that \mathcal{A} has natural density equal to zero, and asked for a more precise upper bound. We prove that

$$\#(\mathcal{A} \cap [1, x]) \ll \frac{x \log \log \log x}{\log \log x}$$

for all sufficiently large x .

1. INTRODUCTION

Let (u_n) be a nondegenerate linear recurrence with integral values. Arithmetic relations between n and u_n have been studied by several authors. For example, the set of positive integers such that n divides u_n has been studied by Alba González, Luca, Pomerance, and Shparlinski [2], assuming that the characteristic polynomial of (u_n) is separable, and by André-Jeannin [3], Luca and Tron [11], Sanna [17], and Somer [21], when (u_n) is a Lucas sequence. Furthermore, Sanna [19] showed that the set of natural numbers n such that $\gcd(n, u_n) = 1$ has a natural density (see [13] for a generalization). Mastrostefano and Sanna [12, 18] studied the moments of $\log(\gcd(n, u_n))$ and $\gcd(n, u_n)$ when (u_n) is a Lucas sequence, and Jha and Nath [7] performed a similar study over shifted primes. (See also the survey of Tron [23] on greatest common divisors of terms of linear recurrences.)

Let (F_n) be the linear recurrence of Fibonacci numbers, which is defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for every positive integer n . Sanna and Tron [20] proved that, for each positive integer k , the set of positive integers n such that $\gcd(n, F_n) = k$ has a natural density, which is given by an infinite series. Kim [9] and Jha [6] obtained formally analogous results in cases of elliptic divisibility sequences and orbits of polynomial maps, respectively. Let \mathcal{A} be the set of numbers of the form $\gcd(n, F_n)$, for some positive integer n . Leonetti and Sanna [10] provided an effective method to enumerate the elements of \mathcal{A} in increasing order. In particular, the first elements of \mathcal{A} are

1, 2, 5, 7, 10, 12, 13, 17, 24, 25, 26, 29, 34, 35, 36, ...

see [1, A285058] for more terms. Then they proved that

$$(1) \quad \#\mathcal{A}(x) \gg \frac{x}{\log x}$$

for all $x \geq 2$. Their approach relied on a result of Cubre and Rouse [4], which in turn follows from Galois theory and the Chebotarev density theorem. Later, Jha and Sanna [8, Proposition 1.4] obtained an elementary proof as an application of related arithmetic problem over shifted primes. Leonetti and Sanna [10] also gave the upper bound $\#\mathcal{A}(x) = o(x)$ as $x \rightarrow +\infty$; and asked for a more precise estimate. We prove the following upper bound on $\#\mathcal{A}(x)$.

Theorem 1.1. *We have*

$$\#\mathcal{A}(x) \ll \frac{x \log \log \log x}{\log \log x}$$

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for all sufficiently large x .

In light of the gap between the upper bound of Theorem 1.1 and the lower bound (1) it is natural to wonder which is the true order of $\#\mathcal{A}(x)$. By performing some numerical experiments (see Section 4 later), we found that $\#\mathcal{A}(x)$ appears to be asymptotic to $x/(\log x)^c$, as $x \rightarrow +\infty$, for some constant $c \approx 0.63$, see Figure 1. Of course, these kind of experiments has to be taken with a grain of salt, since they cannot reveal slow-growing factors like $\log \log x$.

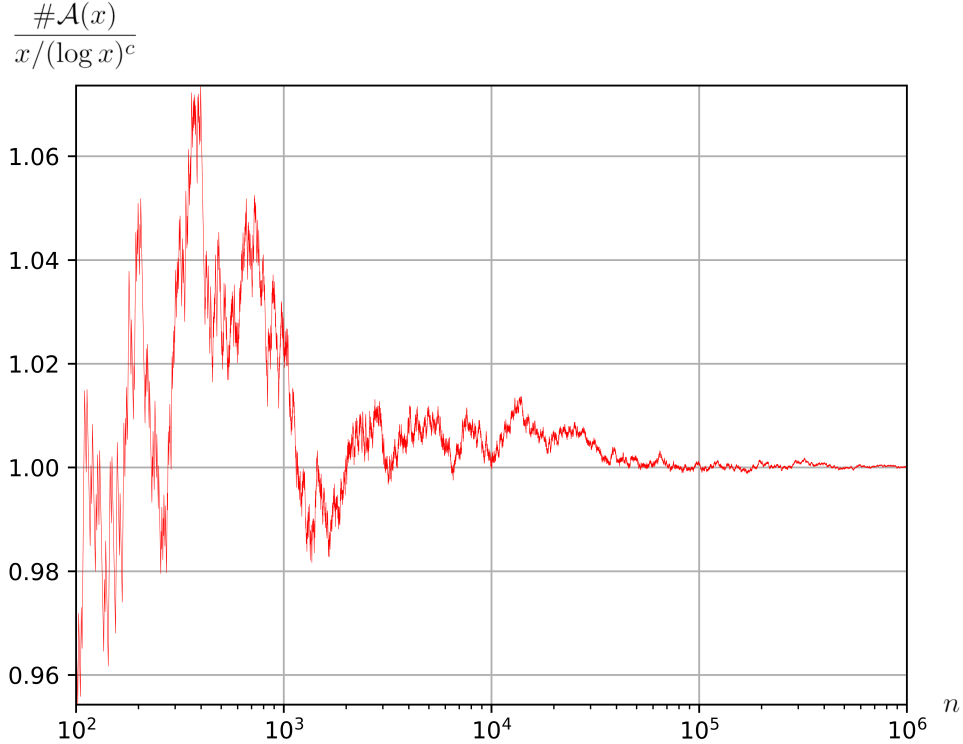


FIGURE 1. A plot of $\#\mathcal{A}(x)/(x/(\log x)^c)$ for x up to 10^6 .

Notation. For every set of positive integers \mathcal{S} and for every $x > 0$, we define $\mathcal{S}(x) := \mathcal{S} \cap [1, x]$. We employ the Landau–Bachmann “Big Oh” and “little oh” notation O and o , as well as the associated Vinogradov symbols \ll and \gg . In particular, all of the implied constants are intended to be absolute. We let $\text{Li}(x) := \int_2^x (\log t)^{-1} dt$ denote the integral logarithm.

2. PRELIMINARIES

For each positive integer n , let $z(n)$ be the *rank of appearance* of n , that is, $z(n)$ is the smallest positive integer k such that n divides F_k . It is well known that $z(n)$ exists. Moreover, put $\ell(n) := \text{lcm}(n, z(n))$ and $g(n) := \text{gcd}(n, F_n)$. The next lemma collects some elementary properties of z , ℓ , and g .

Lemma 2.1. *For all positive integer m, n and all prime numbers p , we have:*

- (i) $z(m) \mid z(n)$ whenever $m \mid n$.
- (ii) $n \mid g(m)$ if and only if $\ell(n) \mid m$.
- (iii) $n \in \mathcal{A}$ if and only if $n = g(\ell(n))$.
- (iv) $m \mid n$ whenever $\ell(m) \mid \ell(n)$ and $n \in \mathcal{A}$.
- (v) $z(p) \mid p - (p/5)$ where $(p/5)$ is a Legendre symbol.

- (vi) $z(p^n) = p^{\max(n-e(p), 0)} z(p)$, where $e(p) := \nu_p(F_{z(p)}) \geq 1$ and ν_p is the usual p -adic valuation.
- (vii) $\ell(p^n) = p^n z(p)$ if $p \neq 5$, and $\ell(5^n) = 5^n$.

Proof. For (i), (ii), and (iii), see [10, Lemma 2.1 and 2.2]. Fact (iv) follows easily from (ii) and (iii). Facts (vi) and (v) are well known (cf. [11, Lemma 1]). Fact (vii) follows quickly from (vi) and (v). \square

For each positive integer d , let \mathcal{P}_d be the set of prime numbers p such that d divides $z(p)$. Cubre and Rouse [4] proved that $\#\mathcal{P}_d(x) \sim \delta(d) \text{Li}(x)$, as $x \rightarrow +\infty$, where

$$\delta(d) := \frac{1}{d} \prod_{p|d} \left(1 - \frac{1}{p^2}\right)^{-1} \begin{cases} 1 & \text{if } 10 \nmid d; \\ 5/4 & \text{if } d \equiv 10 \pmod{20}; \\ 1/2 & \text{if } 20 \mid d. \end{cases}$$

Sanna [16] extended this result to Lucas sequences (under some mild restrictions) and provided also an error term. In particular, as a consequence of [16, Theorem 1.1], we have the following asymptotic formula.

Lemma 2.2. *There exists an absolute constant $B > 0$ such that*

$$(2) \quad \#\mathcal{P}_d(x) = \delta(d) \text{Li}(x) + O\left(\frac{x}{(\log x)^{12/11}}\right),$$

for all odd positive integers d and for all $x \geq \exp(Bd^{40})$.

Proof. From [16, Theorem 1.1] we have that there exists an absolute constant $B > 0$ such that

$$\#\mathcal{P}_d(x) = \delta(d) \text{Li}(x) + O\left(\frac{d}{\varphi(d)} \cdot \frac{x (\log \log x)^{\omega(d)}}{(\log x)^{9/8}}\right),$$

for all odd positive integers d and for all $x \geq \exp(Bd^{40})$, where $\varphi(d)$ and $\omega(d)$ are the Euler totient function and the number of prime factors of d , respectively. Note that we can assume that B (and consequently x) is sufficiently large. In particular, we have that $d \leq (\log x)^{1/40}$. Put $\varepsilon := 1/330$. By the classic lower bound for $\varphi(d)$ (see, e.g., [22, Ch. I.5, Theorem 5.6]) we have that

$$\frac{d}{\varphi(d)} \ll \log \log d \ll \log \log \log x \leq (\log x)^\varepsilon.$$

Recall that $\omega(d) \leq (1 + o(1)) \log d / \log \log d$ as $d \rightarrow +\infty$ (see, e.g., [22, Ch. I.5, Theorem 5.5]). Therefore, there exists an absolute constant $C > 0$ such that if $d > C$ then

$$\omega(d) \leq (1 + \varepsilon) \frac{\log d}{\log \log d} \leq \left(\frac{1}{40} + 2\varepsilon\right) \frac{\log \log x}{\log \log \log x},$$

and consequently $(\log \log x)^{\omega(d)} \leq (\log x)^{\frac{1}{40} + 2\varepsilon}$. Also, if $d \leq C$ then $(\log \log x)^{\omega(d)} \leq (\log x)^\varepsilon$. The claim follows. \square

Remark 2.1. In Lemma 2.2 the exponent $12/11$ can be replaced by $11/10 + \varepsilon$, for every $\varepsilon > 0$, assuming that x is sufficiently large depending on ε .

We also need an upper bound for $\#\mathcal{P}_d(x)$.

Lemma 2.3. *We have*

$$\#\mathcal{P}_d(x) \ll \frac{x}{\varphi(d) \log(x/d)}$$

for all positive integers d and for all $x > d$.

Proof. By Lemma 2.1(v), we have that

$$\#\mathcal{P}_d(x) \leq 1 + \#\{p \leq x : p \equiv \pm 1 \pmod{d}\} \ll \frac{x}{\varphi(d) \log(x/d)},$$

where we applied the Brun–Titchmarsh inequality [22, Ch. I.4, Theorem 4.16]. \square

Now we give an upper bound for the sum of reciprocals of primes in \mathcal{P}_d .

Lemma 2.4. *We have*

$$\sum_{p \in \mathcal{P}_d(x)} \frac{1}{p} = \delta(d) \log \log x + O\left(\frac{\log(2d)}{\varphi(d)}\right)$$

for all odd positive integers d and for all $x \geq 3$.

Proof. First, suppose that $x < \exp(Bd^{40})$, where B is the constant of Lemma 2.2. Hence, we have that

$$\delta(d) \log \log x \ll \frac{\log \log x}{d} \ll \frac{\log(2d)}{d}.$$

Moreover, by [15, Theorem 1 and Remark 1], we have that

$$\sum_{\substack{p \leq x \\ p \equiv \pm 1 \pmod{d}}} \frac{1}{p} = \frac{2 \log \log x}{\varphi(d)} + O\left(\frac{\log(2d)}{\varphi(d)}\right).$$

This together with Lemma 2.1(v) yields that

$$\sum_{p \in \mathcal{P}_d(x)} \frac{1}{p} \leq \frac{1}{d} + \sum_{\substack{p \leq x \\ p \equiv \pm 1 \pmod{d}}} \frac{1}{p} \ll \frac{1}{d} + \frac{\log(2d)}{\varphi(d)}.$$

Hence, the claim follows. Now suppose that $x \geq \exp(Bd^{40})$. By partial summation, we have that

$$\sum_{p \in \mathcal{P}_d(x)} \frac{1}{p} = \frac{\#\mathcal{P}_d(x)}{x} + \int_1^x \frac{\#\mathcal{P}_d(t)}{t^2} dt.$$

Obviously, $\#\mathcal{P}_d(x)/x \ll 1/d$ by the trivial inequality. Thus it remains to bound the integral. By Lemma 2.1(v), we have that

$$\int_1^{2d} \frac{\#\mathcal{P}_d(t)}{t^2} dt \leq \frac{1}{d^2} \int_1^{2d-2} 5 dt \ll \frac{1}{d},$$

after noticing that $\#\mathcal{P}_d(t) > 0$ only if $t \geq d-1$. By Lemma 2.3, we have that

$$\int_{2d}^{\exp(Bd^{40})} \frac{\#\mathcal{P}_d(t)}{t^2} dt \ll \int_{2d}^{\exp(Bd^{40})} \frac{dt}{\varphi(d) t \log(t/d)} = \left[\frac{\log \log(t/d)}{\varphi(d)} \right]_{t=2d}^{\exp(Bd^{40})} \ll \frac{\log d}{\varphi(d)}.$$

By Lemma 2.2, we have that

$$\begin{aligned} \int_{\exp(Bd^{40})}^x \frac{\#\mathcal{P}_d(t)}{t^2} dt &= \int_{\exp(Bd^{40})}^x \frac{\delta(d) \operatorname{Li}(t)}{t^2} dt + O\left(\int_{\exp(Bd^{40})}^x \frac{dt}{t(\log t)^{12/11}}\right) \\ &= \delta(d) \left[\log \log t - \frac{\operatorname{Li}(t)}{t} \right]_{t=\exp(Bd^{40})}^x + O\left(\frac{1}{d^{40/11}}\right) \\ &= \delta(d) (\log \log x + O(\log d)) + O\left(\frac{1}{d^{40/11}}\right) \\ &= \delta(d) \log \log x + O\left(\frac{\log d}{d}\right). \end{aligned}$$

Putting these together, the claim follows. \square

The following sieve result is a special case of [5, Theorem 7.2] (cf. [14, Lemma 2.2]).

Lemma 2.5. *We have*

$$\#\{n \leq x : p \mid n \Rightarrow p \notin \mathcal{P}\} \ll x \prod_{p \in \mathcal{P}(x)} \left(1 - \frac{1}{p}\right),$$

for all $x \geq 2$ and for all sets of prime numbers \mathcal{P} .

3. PROOF OF THEOREM 1.1

Suppose that $x > 0$ is sufficiently large, and put

$$k := \left\lfloor \frac{1}{\log 5} \log \left(\frac{25}{24 \log 5} \cdot \frac{\log \log x}{\log \log \log x} \right) \right\rfloor$$

and $d := 5^k$. Note that $\delta(d) = 5^{-k} \cdot 25/24$. Hence, we get that

$$\log \left(\frac{\log d}{\delta(d)} \right) = k \log 5 + \log k + \log \left(\frac{24 \log 5}{25} \right) \leq \log \log \log x.$$

Therefore, we have that $(\log d)/\delta(d) \leq \log \log x$ and

$$(3) \quad (\log x)^{\delta(d)} \geq d \gg \frac{\log \log x}{\log \log \log x}.$$

We split \mathcal{A} into two subsets: \mathcal{A}_1 is the subset of \mathcal{A} consisting of integers without prime factors in \mathcal{P}_d , and $\mathcal{A}_2 := \mathcal{A} \setminus \mathcal{A}_1$.

First, we give an upper bound on $\#\mathcal{A}_1(x)$. By Lemma 2.5 and Lemma 2.4, we get that

$$(4) \quad \#\mathcal{A}_1(x) \ll x \prod_{p \in \mathcal{P}_d(x)} \left(1 - \frac{1}{p} \right) \ll x \exp \left(- \sum_{p \in \mathcal{P}_d(x)} \frac{1}{p} \right) \ll \frac{x}{(\log x)^{\delta(d)}},$$

where we also used the inequality $1 - x \leq \exp(-x)$, which holds for $x \geq 0$.

Now we give an upper bound on $\#\mathcal{A}_2(x)$. If $n \in \mathcal{A}_2$ then n has a prime factor $p \in \mathcal{P}_d$. Hence, we have that $p \mid n$ and $d \mid z(p)$. Thus, by Lemma 2.1(i), we get that $z(p) \mid z(n)$ and so $d \mid \ell(n)$. Recalling that $d = 5^k$, by Lemma 2.1(vii) we have that $\ell(d) = d$. Hence, we get that $\ell(d) \mid \ell(n)$ and, by Lemma 2.1(iv), it follows that $d \mid n$. Thus all the elements of \mathcal{A}_2 are multiples of d . Consequently, we have that

$$(5) \quad \#\mathcal{A}_2(x) \leq \frac{x}{d}.$$

Therefore, putting together (4) and (5), and using (3), we obtain that

$$\#\mathcal{A}(x) = \#\mathcal{A}_1(x) + \#\mathcal{A}_2(x) \ll \frac{x}{(\log x)^{\delta(d)}} + \frac{x}{d} \ll \frac{x \log \log \log x}{\log \log x},$$

as desired. The proof is complete.

4. NUMERICAL COMPUTATIONS

We computed the elements of $\mathcal{A} \cap [1, 10^6]$ by using a program written in \mathbb{C} that employs Lemma 2.1(iii). Note that computing $g(\ell(n))$ directly as $\gcd(\ell(n), F_{\ell(n)})$ would be prohibitive, in light of the exponential growth of Fibonacci numbers. Instead, we used the fact that

$$g(\ell(n)) = \gcd(\ell(n), F_{\ell(n)} \bmod \ell(n)),$$

and we computed Fibonacci numbers modulo an integer by efficient matrix exponentiation.

REFERENCES

- [1] OEIS Foundation Inc. (2022), The On-Line Encyclopedia of Integer Sequences, Published electronically at <https://oeis.org>.
- [2] J. J. Alba González, F. Luca, C. Pomerance, and I. E. Shparlinski, *On numbers n dividing the n th term of a linear recurrence*, Proc. Edinb. Math. Soc. (2) **55** (2012), no. 2, 271–289.
- [3] R. André-Jeannin, *Divisibility of generalized Fibonacci and Lucas numbers by their subscripts*, Fibonacci Quart. **29** (1991), no. 4, 364–366.
- [4] P. Cubre and J. Rouse, *Divisibility properties of the Fibonacci entry point*, Proc. Amer. Math. Soc. **142** (2014), no. 11, 3771–3785.
- [5] H. Halberstam and H.-E. Richert, *Sieve methods*, London Mathematical Society Monographs, No. 4, Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1974.
- [6] A. Jha, *On terms in a dynamical divisibility sequence having a fixed G.C.D. with their index*, <https://arxiv.org/abs/2105.06190>.

- [7] A. Jha and A. Nath, *The distribution of G.C.D.s of shifted primes and Lucas sequences*, <https://arxiv.org/abs/2207.00825>.
- [8] A. Jha and C. Sanna, *Greatest common divisors of shifted primes and Fibonacci numbers*, <https://arxiv.org/abs/2204.05161>.
- [9] S. Kim, *The density of the terms in an elliptic divisibility sequence having a fixed G.C.D. with their indices*, *J. Number Theory* **207** (2020), 22–41, With an appendix by M. Ram Murty.
- [10] P. Leonetti and C. Sanna, *On the greatest common divisor of n and the n th Fibonacci number*, *Rocky Mountain J. Math.* **48** (2018), no. 4, 1191–1199.
- [11] F. Luca and E. Tron, *The distribution of self-Fibonacci divisors*, *Advances in the theory of numbers*, Fields Inst. Commun., vol. 77, Fields Inst. Res. Math. Sci., Toronto, ON, 2015, pp. 149–158.
- [12] D. Mastrostefano, *An upper bound for the moments of a gcd related to Lucas sequences*, *Rocky Mountain J. Math.* **49** (2019), no. 3, 887–902.
- [13] D. Mastrostefano and C. Sanna, *On numbers n with polynomial image coprime with the n th term of a linear recurrence*, *Bull. Aust. Math. Soc.* **99** (2019), no. 1, 23–33.
- [14] P. Pollack, *Numbers which are orders only of cyclic groups*, *Proc. Amer. Math. Soc.* **150** (2022), no. 2, 515–524.
- [15] C. Pomerance, *On the distribution of amicable numbers*, *J. Reine Angew. Math.* **293(294)** (1977), 217–222.
- [16] C. Sanna, *On the divisibility of the rank of appearance of a Lucas sequence*, *Int. J. Number Theory*, online ready, <https://doi.org/10.1142/S1793042122501093>.
- [17] C. Sanna, *On numbers n dividing the n th term of a Lucas sequence*, *Int. J. Number Theory* **13** (2017), no. 3, 725–734.
- [18] C. Sanna, *The moments of the logarithm of a G.C.D. related to Lucas sequences*, *J. Number Theory* **191** (2018), 305–315.
- [19] C. Sanna, *On numbers n relatively prime to the n th term of a linear recurrence*, *Bull. Malays. Math. Sci. Soc.* **42** (2019), no. 2, 827–833.
- [20] C. Sanna and E. Tron, *The density of numbers n having a prescribed G.C.D. with the n th Fibonacci number*, *Indag. Math. (N.S.)* **29** (2018), no. 3, 972–980.
- [21] L. Somer, *Divisibility of terms in Lucas sequences by their subscripts*, *Applications of Fibonacci numbers*, Vol. 5 (St. Andrews, 1992), Kluwer Acad. Publ., Dordrecht, 1993, pp. 515–525.
- [22] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, third ed., Graduate Studies in Mathematics, vol. 163, American Mathematical Society, Providence, RI, 2015, Translated from the 2008 French edition by Patrick D. F. Ion.
- [23] E. Tron, *The greatest common divisor of linear recurrences*, *Rend. Semin. Mat. Univ. Politec. Torino* **78** (2020), no. 1, 103–124.

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