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# ON THE GREATEST COMMON DIVISOR OF $n$ AND THE $n$ TH FIBONACCI NUMBER, II

ABHISHEK JHA AND CARLO SANNA<sup>†</sup>

ABSTRACT. Let  $\mathcal{A}$  be the set of all integers of the form  $\gcd(n, F_n)$ , where  $n$  is a positive integer and  $F_n$  denotes the  $n$ th Fibonacci number. Leonetti and Sanna proved that  $\mathcal{A}$  has natural density equal to zero, and asked for a more precise upper bound. We prove that

$$\#(\mathcal{A} \cap [1, x]) \ll \frac{x \log \log \log x}{\log \log x}$$

for all sufficiently large  $x$ .

## 1. INTRODUCTION

Let  $(u_n)$  be a nondegenerate linear recurrence with integral values. Arithmetic relations between  $n$  and  $u_n$  have been studied by several authors. For example, the set of positive integers such that  $n$  divides  $u_n$  has been studied by Alba González, Luca, Pomerance, and Shparlinski [2], assuming that the characteristic polynomial of  $(u_n)$  is separable, and by André-Jeannin [3], Luca and Tron [11], Sanna [17], and Somer [21], when  $(u_n)$  is a Lucas sequence. Furthermore, Sanna [19] showed that the set of natural numbers  $n$  such that  $\gcd(n, u_n) = 1$  has a natural density (see [13] for a generalization). Mastrostefano and Sanna [12, 18] studied the moments of  $\log(\gcd(n, u_n))$  and  $\gcd(n, u_n)$  when  $(u_n)$  is a Lucas sequence, and Jha and Nath [7] performed a similar study over shifted primes. (See also the survey of Tron [23] on greatest common divisors of terms of linear recurrences.)

Let  $(F_n)$  be the linear recurrence of Fibonacci numbers, which is defined by  $F_1 = F_2 = 1$  and  $F_{n+2} = F_{n+1} + F_n$  for every positive integer  $n$ . Sanna and Tron [20] proved that, for each positive integer  $k$ , the set of positive integers  $n$  such that  $\gcd(n, F_n) = k$  has a natural density, which is given by an infinite series. Kim [9] and Jha [6] obtained formally analogous results in cases of elliptic divisibility sequences and orbits of polynomial maps, respectively. Let  $\mathcal{A}$  be the set of numbers of the form  $\gcd(n, F_n)$ , for some positive integer  $n$ . Leonetti and Sanna [10] provided an effective method to enumerate the elements of  $\mathcal{A}$  in increasing order. In particular, the first elements of  $\mathcal{A}$  are

1, 2, 5, 7, 10, 12, 13, 17, 24, 25, 26, 29, 34, 35, 36, ...

see [1, A285058] for more terms. Then they proved that

$$(1) \quad \#\mathcal{A}(x) \gg \frac{x}{\log x}$$

for all  $x \geq 2$ . Their approach relied on a result of Cubre and Rouse [4], which in turn follows from Galois theory and the Chebotarev density theorem. Later, Jha and Sanna [8, Proposition 1.4] obtained an elementary proof as an application of related arithmetic problem over shifted primes. Leonetti and Sanna [10] also gave the upper bound  $\#\mathcal{A}(x) = o(x)$  as  $x \rightarrow +\infty$ ; and asked for a more precise estimate. We prove the following upper bound on  $\#\mathcal{A}(x)$ .

**Theorem 1.1.** *We have*

$$\#\mathcal{A}(x) \ll \frac{x \log \log \log x}{\log \log x}$$

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<sup>†</sup> C. Sanna is a member of GNSAGA of INdAM and of CryptTO, the group of Cryptography and Number Theory of Politecnico di Torino.

for all sufficiently large  $x$ .

In light of the gap between the upper bound of Theorem 1.1 and the lower bound (1) it is natural to wonder which is the true order of  $\#\mathcal{A}(x)$ . By performing some numerical experiments (see Section 4 later), we found that  $\#\mathcal{A}(x)$  appears to be asymptotic to  $x/(\log x)^c$ , as  $x \rightarrow +\infty$ , for some constant  $c \approx 0.63$ , see Figure 1. Of course, these kind of experiments has to be taken with a grain of salt, since they cannot reveal slow-growing factors like  $\log \log x$ .

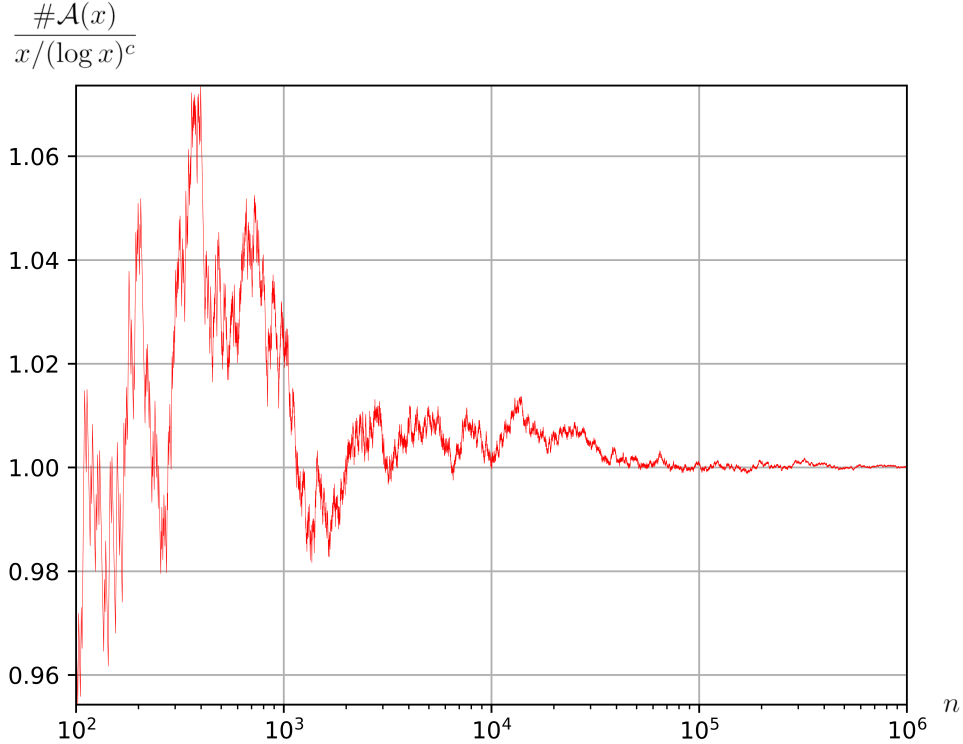


FIGURE 1. A plot of  $\#\mathcal{A}(x)/(x/(\log x)^c)$  for  $x$  up to  $10^6$ .

**Notation.** For every set of positive integers  $\mathcal{S}$  and for every  $x > 0$ , we define  $\mathcal{S}(x) := \mathcal{S} \cap [1, x]$ . We employ the Landau–Bachmann “Big Oh” and “little oh” notation  $O$  and  $o$ , as well as the associated Vinogradov symbols  $\ll$  and  $\gg$ . In particular, all of the implied constants are intended to be absolute. We let  $\text{Li}(x) := \int_2^x (\log t)^{-1} dt$  denote the integral logarithm.

## 2. PRELIMINARIES

For each positive integer  $n$ , let  $z(n)$  be the *rank of appearance* of  $n$ , that is,  $z(n)$  is the smallest positive integer  $k$  such that  $n$  divides  $F_k$ . It is well known that  $z(n)$  exists. Moreover, put  $\ell(n) := \text{lcm}(n, z(n))$  and  $g(n) := \text{gcd}(n, F_n)$ . The next lemma collects some elementary properties of  $z$ ,  $\ell$ , and  $g$ .

**Lemma 2.1.** *For all positive integer  $m, n$  and all prime numbers  $p$ , we have:*

- (i)  $z(m) \mid z(n)$  whenever  $m \mid n$ .
- (ii)  $n \mid g(m)$  if and only if  $\ell(n) \mid m$ .
- (iii)  $n \in \mathcal{A}$  if and only if  $n = g(\ell(n))$ .
- (iv)  $m \mid n$  whenever  $\ell(m) \mid \ell(n)$  and  $n \in \mathcal{A}$ .
- (v)  $z(p) \mid p - (p/5)$  where  $(p/5)$  is a Legendre symbol.

- (vi)  $z(p^n) = p^{\max(n-e(p), 0)} z(p)$ , where  $e(p) := \nu_p(F_{z(p)}) \geq 1$  and  $\nu_p$  is the usual  $p$ -adic valuation.
- (vii)  $\ell(p^n) = p^n z(p)$  if  $p \neq 5$ , and  $\ell(5^n) = 5^n$ .

*Proof.* For (i), (ii), and (iii), see [10, Lemma 2.1 and 2.2]. Fact (iv) follows easily from (ii) and (iii). Facts (vi) and (v) are well known (cf. [11, Lemma 1]). Fact (vii) follows quickly from (vi) and (v).  $\square$

For each positive integer  $d$ , let  $\mathcal{P}_d$  be the set of prime numbers  $p$  such that  $d$  divides  $z(p)$ . Cubre and Rouse [4] proved that  $\#\mathcal{P}_d(x) \sim \delta(d) \text{Li}(x)$ , as  $x \rightarrow +\infty$ , where

$$\delta(d) := \frac{1}{d} \prod_{p|d} \left(1 - \frac{1}{p^2}\right)^{-1} \begin{cases} 1 & \text{if } 10 \nmid d; \\ 5/4 & \text{if } d \equiv 10 \pmod{20}; \\ 1/2 & \text{if } 20 \mid d. \end{cases}$$

Sanna [16] extended this result to Lucas sequences (under some mild restrictions) and provided also an error term. In particular, as a consequence of [16, Theorem 1.1], we have the following asymptotic formula.

**Lemma 2.2.** *There exists an absolute constant  $B > 0$  such that*

$$(2) \quad \#\mathcal{P}_d(x) = \delta(d) \text{Li}(x) + O\left(\frac{x}{(\log x)^{12/11}}\right),$$

for all odd positive integers  $d$  and for all  $x \geq \exp(Bd^{40})$ .

*Proof.* From [16, Theorem 1.1] we have that there exists an absolute constant  $B > 0$  such that

$$\#\mathcal{P}_d(x) = \delta(d) \text{Li}(x) + O\left(\frac{d}{\varphi(d)} \cdot \frac{x (\log \log x)^{\omega(d)}}{(\log x)^{9/8}}\right),$$

for all odd positive integers  $d$  and for all  $x \geq \exp(Bd^{40})$ , where  $\varphi(d)$  and  $\omega(d)$  are the Euler totient function and the number of prime factors of  $d$ , respectively. Note that we can assume that  $B$  (and consequently  $x$ ) is sufficiently large. In particular, we have that  $d \leq (\log x)^{1/40}$ . Put  $\varepsilon := 1/330$ . By the classic lower bound for  $\varphi(d)$  (see, e.g., [22, Ch. I.5, Theorem 5.6]) we have that

$$\frac{d}{\varphi(d)} \ll \log \log d \ll \log \log \log x \leq (\log x)^\varepsilon.$$

Recall that  $\omega(d) \leq (1 + o(1)) \log d / \log \log d$  as  $d \rightarrow +\infty$  (see, e.g., [22, Ch. I.5, Theorem 5.5]). Therefore, there exists an absolute constant  $C > 0$  such that if  $d > C$  then

$$\omega(d) \leq (1 + \varepsilon) \frac{\log d}{\log \log d} \leq \left(\frac{1}{40} + 2\varepsilon\right) \frac{\log \log x}{\log \log \log x},$$

and consequently  $(\log \log x)^{\omega(d)} \leq (\log x)^{\frac{1}{40} + 2\varepsilon}$ . Also, if  $d \leq C$  then  $(\log \log x)^{\omega(d)} \leq (\log x)^\varepsilon$ . The claim follows.  $\square$

*Remark 2.1.* In Lemma 2.2 the exponent  $12/11$  can be replaced by  $11/10 + \varepsilon$ , for every  $\varepsilon > 0$ , assuming that  $x$  is sufficiently large depending on  $\varepsilon$ .

We also need an upper bound for  $\#\mathcal{P}_d(x)$ .

**Lemma 2.3.** *We have*

$$\#\mathcal{P}_d(x) \ll \frac{x}{\varphi(d) \log(x/d)}$$

for all positive integers  $d$  and for all  $x > d$ .

*Proof.* By Lemma 2.1(v), we have that

$$\#\mathcal{P}_d(x) \leq 1 + \#\{p \leq x : p \equiv \pm 1 \pmod{d}\} \ll \frac{x}{\varphi(d) \log(x/d)},$$

where we applied the Brun–Titchmarsh inequality [22, Ch. I.4, Theorem 4.16].  $\square$

Now we give an upper bound for the sum of reciprocals of primes in  $\mathcal{P}_d$ .

**Lemma 2.4.** *We have*

$$\sum_{p \in \mathcal{P}_d(x)} \frac{1}{p} = \delta(d) \log \log x + O\left(\frac{\log(2d)}{\varphi(d)}\right)$$

for all odd positive integers  $d$  and for all  $x \geq 3$ .

*Proof.* First, suppose that  $x < \exp(Bd^{40})$ , where  $B$  is the constant of Lemma 2.2. Hence, we have that

$$\delta(d) \log \log x \ll \frac{\log \log x}{d} \ll \frac{\log(2d)}{d}.$$

Moreover, by [15, Theorem 1 and Remark 1], we have that

$$\sum_{\substack{p \leq x \\ p \equiv \pm 1 \pmod{d}}} \frac{1}{p} = \frac{2 \log \log x}{\varphi(d)} + O\left(\frac{\log(2d)}{\varphi(d)}\right).$$

This together with Lemma 2.1(v) yields that

$$\sum_{p \in \mathcal{P}_d(x)} \frac{1}{p} \leq \frac{1}{d} + \sum_{\substack{p \leq x \\ p \equiv \pm 1 \pmod{d}}} \frac{1}{p} \ll \frac{1}{d} + \frac{\log(2d)}{\varphi(d)}.$$

Hence, the claim follows. Now suppose that  $x \geq \exp(Bd^{40})$ . By partial summation, we have that

$$\sum_{p \in \mathcal{P}_d(x)} \frac{1}{p} = \frac{\#\mathcal{P}_d(x)}{x} + \int_1^x \frac{\#\mathcal{P}_d(t)}{t^2} dt.$$

Obviously,  $\#\mathcal{P}_d(x)/x \ll 1/d$  by the trivial inequality. Thus it remains to bound the integral. By Lemma 2.1(v), we have that

$$\int_1^{2d} \frac{\#\mathcal{P}_d(t)}{t^2} dt \leq \frac{1}{d^2} \int_1^{2d-2} 5 dt \ll \frac{1}{d},$$

after noticing that  $\#\mathcal{P}_d(t) > 0$  only if  $t \geq d-1$ . By Lemma 2.3, we have that

$$\int_{2d}^{\exp(Bd^{40})} \frac{\#\mathcal{P}_d(t)}{t^2} dt \ll \int_{2d}^{\exp(Bd^{40})} \frac{dt}{\varphi(d) t \log(t/d)} = \left[ \frac{\log \log(t/d)}{\varphi(d)} \right]_{t=2d}^{\exp(Bd^{40})} \ll \frac{\log d}{\varphi(d)}.$$

By Lemma 2.2, we have that

$$\begin{aligned} \int_{\exp(Bd^{40})}^x \frac{\#\mathcal{P}_d(t)}{t^2} dt &= \int_{\exp(Bd^{40})}^x \frac{\delta(d) \operatorname{Li}(t)}{t^2} dt + O\left(\int_{\exp(Bd^{40})}^x \frac{dt}{t(\log t)^{12/11}}\right) \\ &= \delta(d) \left[ \log \log t - \frac{\operatorname{Li}(t)}{t} \right]_{t=\exp(Bd^{40})}^x + O\left(\frac{1}{d^{40/11}}\right) \\ &= \delta(d) (\log \log x + O(\log d)) + O\left(\frac{1}{d^{40/11}}\right) \\ &= \delta(d) \log \log x + O\left(\frac{\log d}{d}\right). \end{aligned}$$

Putting these together, the claim follows.  $\square$

The following sieve result is a special case of [5, Theorem 7.2] (cf. [14, Lemma 2.2]).

**Lemma 2.5.** *We have*

$$\#\{n \leq x : p \mid n \Rightarrow p \notin \mathcal{P}\} \ll x \prod_{p \in \mathcal{P}(x)} \left(1 - \frac{1}{p}\right),$$

for all  $x \geq 2$  and for all sets of prime numbers  $\mathcal{P}$ .

## 3. PROOF OF THEOREM 1.1

Suppose that  $x > 0$  is sufficiently large, and put

$$k := \left\lfloor \frac{1}{\log 5} \log \left( \frac{25}{24 \log 5} \cdot \frac{\log \log x}{\log \log \log x} \right) \right\rfloor$$

and  $d := 5^k$ . Note that  $\delta(d) = 5^{-k} \cdot 25/24$ . Hence, we get that

$$\log \left( \frac{\log d}{\delta(d)} \right) = k \log 5 + \log k + \log \left( \frac{24 \log 5}{25} \right) \leq \log \log \log x.$$

Therefore, we have that  $(\log d)/\delta(d) \leq \log \log x$  and

$$(3) \quad (\log x)^{\delta(d)} \geq d \gg \frac{\log \log x}{\log \log \log x}.$$

We split  $\mathcal{A}$  into two subsets:  $\mathcal{A}_1$  is the subset of  $\mathcal{A}$  consisting of integers without prime factors in  $\mathcal{P}_d$ , and  $\mathcal{A}_2 := \mathcal{A} \setminus \mathcal{A}_1$ .

First, we give an upper bound on  $\#\mathcal{A}_1(x)$ . By Lemma 2.5 and Lemma 2.4, we get that

$$(4) \quad \#\mathcal{A}_1(x) \ll x \prod_{p \in \mathcal{P}_d(x)} \left( 1 - \frac{1}{p} \right) \ll x \exp \left( - \sum_{p \in \mathcal{P}_d(x)} \frac{1}{p} \right) \ll \frac{x}{(\log x)^{\delta(d)}},$$

where we also used the inequality  $1 - x \leq \exp(-x)$ , which holds for  $x \geq 0$ .

Now we give an upper bound on  $\#\mathcal{A}_2(x)$ . If  $n \in \mathcal{A}_2$  then  $n$  has a prime factor  $p \in \mathcal{P}_d$ . Hence, we have that  $p \mid n$  and  $d \mid z(p)$ . Thus, by Lemma 2.1(i), we get that  $z(p) \mid z(n)$  and so  $d \mid \ell(n)$ . Recalling that  $d = 5^k$ , by Lemma 2.1(vii) we have that  $\ell(d) = d$ . Hence, we get that  $\ell(d) \mid \ell(n)$  and, by Lemma 2.1(iv), it follows that  $d \mid n$ . Thus all the elements of  $\mathcal{A}_2$  are multiples of  $d$ . Consequently, we have that

$$(5) \quad \#\mathcal{A}_2(x) \leq \frac{x}{d}.$$

Therefore, putting together (4) and (5), and using (3), we obtain that

$$\#\mathcal{A}(x) = \#\mathcal{A}_1(x) + \#\mathcal{A}_2(x) \ll \frac{x}{(\log x)^{\delta(d)}} + \frac{x}{d} \ll \frac{x \log \log \log x}{\log \log x},$$

as desired. The proof is complete.

## 4. NUMERICAL COMPUTATIONS

We computed the elements of  $\mathcal{A} \cap [1, 10^6]$  by using a program written in  $\mathbb{C}$  that employs Lemma 2.1(iii). Note that computing  $g(\ell(n))$  directly as  $\gcd(\ell(n), F_{\ell(n)})$  would be prohibitive, in light of the exponential growth of Fibonacci numbers. Instead, we used the fact that

$$g(\ell(n)) = \gcd(\ell(n), F_{\ell(n)} \bmod \ell(n)),$$

and we computed Fibonacci numbers modulo an integer by efficient matrix exponentiation.

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INDRAPRATHA INSTITUTE OF INFORMATION TECHNOLOGY,  
 OKHLA INDUSTRIAL ESTATE, PHASE-3, NEW DELHI, INDIA  
*Email address:* [abhishek20553@iiitd.ac.in](mailto:abhishek20553@iiitd.ac.in)

DEPARTMENT OF MATHEMATICAL SCIENCES, POLITECNICO DI TORINO  
 CORSO DUCA DEGLI ABRUZZI 24, 10129 TORINO, ITALY  
*Email address:* [carlo.sanna.dev@gmail.com](mailto:carlo.sanna.dev@gmail.com)