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Projective Embedding of Dynamical Systems: *uniform mean field equations*

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Abstract

We study embeddings of continuous dynamical systems in larger dimensions via projector operators. We call this technique PEDS, projective embedding of dynamical systems, as the stable fixed point of the original system dynamics are recovered via projection from the higher dimensional space. In this paper we provide a general definition and prove that for a particular type of rank-1 projector operator, the uniform mean field projector, the equations of motion become a mean field approximation of the dynamical system. While in general the embedding depends on a specified variable ordering, the same is not true for the uniform mean field projector. We prove a variety of results on the relationship between the spectrum of the Jacobian for fixed points in the original and in the embedded system. Direct applications of PEDS can be non-convex optimization and machine learning.

Keywords: projective embedding, projector operators, dynamical systems, fixed points, PEDS

1. Introduction

The past decades witnessed an increased interest in physics- or neuro-inspired algorithms for the analysis of dynamical systems, with the main area of application being problems that can be mapped onto optimization ones, whether continuous or discrete [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. Among the most important neuro-inspired algorithms, we mention neural networks, which received a large amount of attention given their wide applicability and remarkable achievements: this is an active area of research falling at the boundary between complex systems, neuromorphic

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computing and nonlinear dynamics, dating back to Turing [15] at least. In the study of neural networks, one of the most important open problems is the acceleration of the training phase, a problem connected to the roughness of the energy landscape [16, 17]. Network training is one of the most difficult tasks, requiring in general huge computational power and a vast number of samples. Many algorithms attempt at modifying the energy function to reduce the time spent on saddle points [18, 19]. Changing the landscape is however challenging in general, as it somehow requires some *a priori* knowledge of what type of local extrema should be modified. Thus, finding valuable alternatives and/or generalizations of gradient descent has been a topic of intense study. In addition to this, analog models of computation is an active area of research [20] with several applications.

From the point of view of a dynamical system, however, there are not many strategies that one can attempt to employ. A possibility, incidentally the one we explore in this paper, is to increase the dimensionality of the system, by attempting to preserve some properties related to the original dynamical system, while aiming at a trade-off between convergence optimality and the curse of dimensionality. The basic rationale for this strategy is that increasing dimensions, there are more pathways that a system can take in order to reach a certain target point. As a simple example, consider a one dimensional barrier between two minima in a potential: following gradients, one could never move from one local minimum to the other, while in a higher dimension system, pathways around that confinement barrier are, at least in principle, possible.

The technique we propose here is inspired by recent results in the context of memristive circuits [21, 22, 23, 24, 25, 26]. In circuits, Kirchhoff laws are manifestations of the conservation of physical quantities such as charge or energy. Mathematically, these can be expressed via the introduction of projection operators, i.e. matrices $\mathbf{\Omega}$ satisfying the constraint $\mathbf{\Omega}^2 = \mathbf{\Omega}$, and directly connected to circuit topology. For instance, for a resistive circuit made of identical unitary resistances in series with impressed voltage generators, the Ohm's law for the network can be expressed as

$$\vec{i} = \mathbf{\Omega}\vec{v}, \quad (1)$$

where \vec{v} is the collection of voltage generators connected in series to each resistance, while \vec{i} contains the branch currents. The underlying assumption of (1) is that the voltage generators v_i 's are in series to the resistances i 's, while the circuit can be represented as graph with E edges. Given the branch currents and a certain orientation of the graph loops $1, \dots, L$, we can obtain the so called loop matrix of the circuit \mathbf{A} , of size $L \times E$, such that $\mathbf{\Omega} = \mathbf{A}^t(\mathbf{A}\mathbf{A}^t)^{-1}\mathbf{A}$, where t denotes the transpose. The details of the derivation of $\mathbf{\Omega}$ from the circuit topology are beyond the scope of this paper, where $\mathbf{\Omega}$ will be kept generic and unrelated to any underlying graph or conservation law.

We assume a continuous dynamical system, but the technique can in principle be extended to vector maps, and thus works also for numerical implementations of a dynamical system. Let us consider a dynamical system expressed in vector form as a first-order differential system

$$\frac{dx_i}{dt} = f_i(\vec{x}) \quad i = 1, \dots, m \quad (2)$$

where functions $f_i(\cdot)$ are assumed known, and analytic. We are in general interested in recovering the stable fixed points of (2), i.e. the values \vec{x}^* such that $f_i(\vec{x}^*) = 0$, if they exist.

To this aim we consider another dynamical system, of size mN , written in the form

$$\frac{d}{dt}\vec{X}_i = \mathbf{\Omega}\vec{F}_i(\vec{X}_1, \dots, \vec{X}_m) + \vec{G}_i(\vec{X}_i) \quad i = 1, \dots, m \quad (3)$$

where for each i value we define an augmented vector \vec{X}_i of size N . The question we aim at answering in this contribution is to ascertain whether functions \vec{F}_i and \vec{G}_i exist such that the dynamical system (2) is contained into the extended system (3).¹ The answer we provide in this paper is affirmative, as we will explicitly construct such system along with the technique to recover the original dynamical system.

From a mathematical perspective, these generalizations can be investigated by the study of the properties of fixed points in the embedded system in terms of the original ones, which is the strategy we use in this paper. A fixed point \vec{x}^* is a particular point of the phase space satisfying $\frac{d\vec{x}}{dt}|_{\vec{x}^*} = f(\vec{x}^*) = 0$. We dub the method developed in this paper *Projective Embedding of Dynamical Systems* (PEDS), as the technique involves the embedding of a target dynamical system of dimension m into one of dimension mN ; ultimately, we recover the fixed points of the original dynamical system by projecting back onto a chosen set (of size m) of observables. We will prove that the information of the fixed points of the original target system are related to the fixed points of the reduced observables. As we will see, the dynamical system in which the embedding is contained is a nontrivial and nonlinear extension of the original dynamical system which is obtained via a map between the original one and an extended one. Although the projection operator may be quite general, we prove most of the results here for a specific operator, that we call uniform mean-field projector, as in this simplified case mostly analytical proofs are available.

The structure of the paper is as follows. With the aim of introducing the main concepts and providing examples of application to highlight the usefulness of the proposed approach, we present in Section 2 a compendium of the main results and we show an example of application. The formal introduction of the PEDS procedure is discussed in Section 3, where we also provide various examples to intuitively grasp why these definitions make sense. In Section 4 we study the uniform mean field projector, and both for 1-dimensional and m -dimensional dynamical systems we prove exact results about the properties of the asymptotic stable fixed points and their Jacobians. Section 5 is devoted to some numerical examples of application. Finally, conclusions follow.

2. Summary of results and examples

2.1. Informal definition of PEDS and physical motivation

As discussed in the introduction, PEDS is conceived as an embedding of a specific continuous-time dynamical system into a higher dimensional one. More precisely, PEDS defines a map between two dynamical systems of different size. There are two

¹In the next section, we will make more precise the word “contained”.

key motivations for studying these embeddings. First, there is a class of physical systems, such as memristors and networks thereof, that can be described in terms of PEDS. Specifically, for a network of memristors, the differential equation governing the dynamics takes the form of coupled ODEs as follows [23]:

$$\frac{d}{dt}\vec{x} = \frac{1}{\beta}(\mathbf{I} - \chi\mathbf{\Omega X})^{-1}\mathbf{\Omega}\vec{S} - \alpha\vec{x}, \quad (4)$$

where \mathbf{X} is a diagonal matrix with elements $X_{ii} = x_i$. As we will see, the equation above can be cast into the form of a PEDS. Informally, a PEDS for a one-dimensional dynamical system is a mapping between an ODE of the form:

$$\frac{dx}{dt} = f(x) \quad (5)$$

to a higher dimensional (namely, size N) dynamical system as in (4), e.g.

$$\frac{d\vec{X}}{dt} = \mathbf{\Omega F}(\mathbf{\Omega X}) - \alpha(\mathbf{I} - \mathbf{\Omega})\vec{X} \quad (6)$$

where $\mathbf{\Omega}$ is a $N \times N$ dimensional projector operator. As a rule of thumb, the variables x_1, \dots, x_m of the original dynamical systems are associated to vectors \vec{X}_i and diagonal matrices \mathbf{X}_i , which are then introduced as “matrix variables” into the function \mathbf{F} . For technical reasons, we assume in the present paper that the only target dynamical system’s attractors are asymptotically stable equilibria. A more precise and general definition of PEDS is provided in Definition 3.2. The key requirement is that the dynamics of the original system (5) could be recovered from the dynamics of (6) through projection.

The reason why such mapping is interesting is twofold. First, it allows to interpret the asymptotic behavior of equations such as (4) in terms of a lower dimensional system such as (5). The exact mapping will be shown in Fig. 3. The reason why this is possible, is because of Lemma 4.4 (“Banality Lemma”), Proposition 4.5 and Corollary 4.6, which show that projecting the dynamics of the PEDS from a Nm dimensional space, down to a m dimensional space, provides a mapping between the asymptotic fixed points of (4) to those of the system of type (5). This is true for a large class of “decay functions” \vec{G}_i in (3), the ones we term $\mathbf{\Omega}$ -eligible, as we discuss in Definition 3.8. The function $-\alpha(\mathbf{I} - \mathbf{\Omega})\vec{X}$ is an especially simple case. In the Physics jargon, this is a type of dynamical mean field, as it allows to study the behavior of a high dimensional system making use of a simplified dynamics. In fact, this is true for a projected effective variable, to which we refer here loosely as $\tilde{x}(t) = \mathcal{P}\vec{x}$, which represents the “mean field”. Now, while the proofs are based on the fact that matrix $(\mathbf{\Omega}_1)_{ij} = 1/N$ is a projector, these proofs only exploit the fact that $(\mathbf{\Omega}_1)^2 = \mathbf{\Omega}_1$, and thus remain valid for arbitrary projectors. These generalizations, however, will be discussed in future papers. Thus, from a physical standpoint, the key motivation for studying these types of embeddings is to develop a mean field theory to understand the fixed points of the system. In this contribution we do not involve any approximation, the reason being that the focus, in most of our proofs, is on the “trivial” projector $\mathbf{\Omega}_1$, due to the fact that under this assumption exact calculations can be performed. Nonetheless, these results provide the intuition and a more formal background to the

recent paper [27], in which it was shown that a mean field theory exists to explain the behavior of (4).

For dynamical systems of size m , the “banality lemma” is extended in Corollary 4.8. Unfortunately, generalizing the previous statements requires a significant amount of extra work. First, the way in which the function $\mathbf{F}(\mathbf{\Omega}\mathbf{X})$ is defined is not obvious. The reason is that there are many possible definitions, due to the fact that matrices of the form $\mathbf{\Omega}\mathbf{X}$ not necessarily commute. In fact, in Section 4.3, we will introduce the notion of embedding map \mathcal{O} (see Definition 3.9), which is the map specifying the definition of function \mathbf{F} in (6). One of the advantages of using the projector $\mathbf{\Omega}_1$ is that the dynamics is effectively independent of the embedding map, as proven in Proposition 4.13 and Corollary 4.14.

Besides this physically motivated definition, there are also some interesting byproducts of this embedding. This brings us to the second reason why PEDS is, in our opinion, interesting. We take here a completely opposite point of view: given a set of equations like (5), can we modify the spectral properties of the fixed points via the embedding? Given that we have a map between the fixed points of the original (or “target”) system, and the fixed points of the embedding system, it is worth mentioning that the spectral properties of the two are rather different. For simplicity, let us assume that the target system has only fixed points whose Jacobians are full rank, e.g. that there are no zero eigenvalues or non-hyperbolic (the derivative changes sign across) directions. Section 4.4 is devoted to the study of the Jacobian of the fixed points of the PEDS, in relation to the assumed spectral properties of the Jacobian of the target system. For instance, for a one dimensional target system, the study of the spectral properties of the Jacobians is found in Proposition 4.9 and Corollary 4.10. We summarize here the result as

$$(1 \text{ dimension}) \quad \vec{X}^* \text{ is } \begin{cases} \text{stable} & \text{if } x^* \text{ is stable,} \\ \text{a saddle point} & \text{if } x^* \text{ is unstable,} \end{cases} \quad (7)$$

where x^* is a fixed point of the target system, while \vec{X}^* is the fixed point of the PEDS embedding. These results can also be generalized to higher dimensions. The spectral decomposition of the Jacobian for d -dimensional target systems is derived in Proposition 4.11, obtaining later in eqn. (98) that

$$(d > 1 \text{ dimensions}) \quad \vec{X}^* \text{ is } \begin{cases} \text{stable} & \text{if } x^* \text{ is stable,} \\ \text{a saddle point} & \text{if } x^* \text{ is a saddle point,} \\ \text{a saddle point} & \text{if } x^* \text{ is unstable,} \end{cases} \quad (8)$$

A visual interpretation of this is presented in Fig. 1, for the case of a gradient-based dynamical system, e.g. $f(x) = -dV(x)/dx$, which we can consider the main motivation of this work beyond memristive circuits. Barriers between minima typically prevent a system from exploring the local stable equilibria of the system, with possible practical applications for instance in the field of neural networks, specifically in the application of gradient descent, where the local minima of the function $V(x)$ are the stable asymptotic points. However, since barriers can be made circumventable thanks to the PEDS embedding, there can be trajectories that, starting from initial conditions with different velocities, can reach two different stable points. Another important

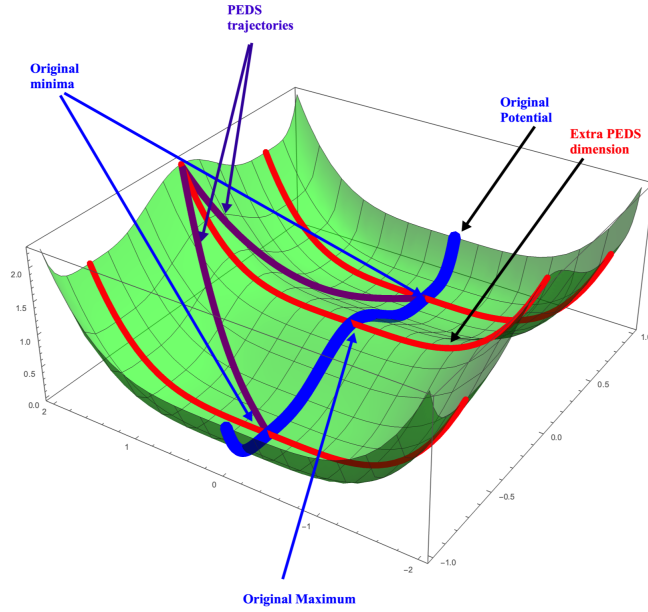


Figure 1: A simple fictitious example showing how PEDS modifies the spectral properties of the system. The blue curve represents the original system potential, while the red curves visually represent the “added dimension” in the PEDS. In such a situation, the spectral properties of the fixed points are modified. This is an intuitive representation of the possible advantages of the PEDS extension.

comment that it is worth making is that if the target system has hyperbolic fixed points, then also the PEDS’ fixed points will be hyperbolic. In the next Section, we anticipate an example to corroborate and better motivate the rest of the paper. As a last comment before our first example, it is also worth mentioning that the embedding does not preserve the gradient-like behavior of the dynamics: if the target system follows the gradient of a certain energy function $V(x)$, PEDS equations are not in fact typically following the gradients of an analogous function.

2.2. Numerical Example

We anticipate here one numerical example of application of the PEDS procedure for the uniform mean field projector $\mathbf{\Omega}_1$ to clarify how PEDS works.

We consider the example of a one dimensional nonlinear dynamical system, written as:

$$\frac{dx}{dt} = f(x) = -\frac{\partial V(x)}{\partial x}. \quad (9)$$

Let us thus analyze numerically the PEDS $\mathcal{O} = \{f(x), \mathbf{\Omega}_1, -\alpha(\mathbf{I} - \mathbf{\Omega}_1)\vec{X}, \vec{1}, N\}$, i.e. the dynamical system defined in (56), with a potential of the form

$$V(x) = a_0 + a_1x + \frac{a_2}{2}x^2 + \frac{a_3}{3}x^3 + \frac{a_4}{4}x^4 \quad (10)$$

for a set of parameters for which two minima are present, as shown in Fig. 2 (left).

The explicit form of the PEDS embedding is given by the differential system

$$\begin{aligned} \frac{d\vec{X}}{dt} &= \mathbf{F}(\vec{X})\vec{1} - \alpha(\mathbf{I} - \mathbf{\Omega}_1)\vec{X} \\ &= -\left(a_1\mathbf{I} + a_2(\mathbf{\Omega}_1\mathbf{X}) + a_3(\mathbf{\Omega}_1\mathbf{X})^2 + a_4(\mathbf{\Omega}_1\mathbf{X})^3\right)\vec{1} - \alpha(\mathbf{I} - \mathbf{\Omega}_1)\vec{1} \end{aligned} \quad (11)$$

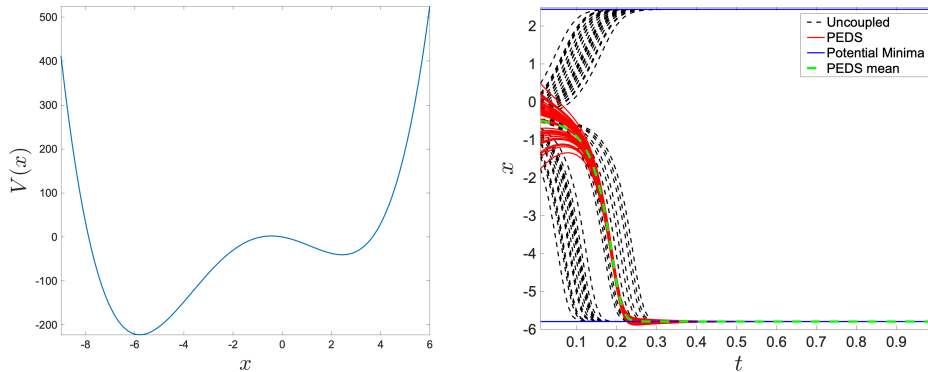


Figure 2: Left: Potential $V(x)$ in (10), for $a_0 = 0$, $a_1 = 9.85$, $a_2 = 10$, $a_3 = 2$ and $a_4 = -0.395$. The parameters are chosen to provide the potential minima. Right: Evolution of the PEDS $\{f(x), \mathbf{\Omega}_1, -\alpha(\mathbf{I} - \mathbf{\Omega}_1)\vec{X}, \vec{1}, N\}$ for random initial conditions around the maximum for random initial conditions for the non-commutative map. The blue solid lines are the target system minima. The black dashed curves are the target system trajectories ($\mathbf{\Omega} = \mathbf{I}$), that asymptotically reach both the two minima. The red curves refer to the coupled PEDS with the uniform mean field projector using the standard non-commutative map, leading to the global minimum, as it can be seen from the plot of the $\tilde{x}(t) = \mathcal{P}_{\mathbf{\Omega}_1}\vec{X}(t)$, the green dashed line.

The results of the numerical integration, using a simple Euler scheme for Gaussian-distributed initial conditions around the potential maximum, at $x^* = -0.51$, of the target system and of the PEDS embedding with the standard non-commutative map (48) are shown in Fig. 2 (right). The PEDS trajectories all reach the global minimum of the potential $V(x)$, while the uncoupled trajectories split between the two stable equilibria.

3. The PEDS procedure: key definitions and motivating examples

In order to clarify the techniques developed in this paper, we now construct the simplest example of embedding, before introducing the necessary definitions. Notation-wise, we will denote with \mathbf{I} the identity matrix, while $\vec{1}$ is a column vector with elements equal to 1.

Example 3.1. Exponential dynamics.

Let us consider the following one dimensional dynamical system:

$$\frac{dx}{dt} = a\tilde{x} \quad x(0) = x_0, \quad (12)$$

with $a \in \mathbb{R}$, whose analytical solution is given by

$$x(t) = e^{at}x_0. \quad (13)$$

Considering an $N \times N$ projector matrix $\mathbf{\Omega}$ such that $\mathbf{\Omega}^2 = \mathbf{\Omega}$ and, thus, $\mathbf{\Omega}(\mathbf{I} - \mathbf{\Omega}) = 0$, we define the following enlarged (size N) dynamical system

$$\frac{d\vec{X}}{dt} = a\mathbf{\Omega}\vec{X} - \alpha(\mathbf{I} - \mathbf{\Omega})\vec{X} \quad \vec{X}(0) = x_0\vec{b}, \quad (14)$$

where $\alpha > 0$, and \vec{b} is an arbitrary vector, satisfying the only requirement $\mathbf{\Omega}\vec{b} \neq \vec{0}$.

Since the system above is linear, we do know the analytical solution, which is given by

$$\vec{X}(t) = e^{[a\mathbf{\Omega} - \alpha(\mathbf{I} - \mathbf{\Omega})]t} \vec{X}(0) \approx e^{a\mathbf{\Omega}t} \vec{X}(0) \quad (15)$$

where the approximation holds for $t \rightarrow +\infty$, i.e. for $t \gg 1/\alpha$. As for any projector $\mathbf{\Omega}$ the following identity holds

$$e^{a\mathbf{\Omega}} = \mathbf{I} + (e^a - 1)\mathbf{\Omega} \quad (16)$$

the asymptotic solution reads

$$\vec{X}(t) \approx (\mathbf{I} - \mathbf{\Omega})\vec{X}(0) + e^{at}x_0\mathbf{\Omega}\vec{b} = (\mathbf{I} - \mathbf{\Omega})\vec{X}(0) + x(t)\mathbf{\Omega}\vec{b} \quad (17)$$

Therefore, projecting (17)

$$\mathbf{\Omega}\vec{X}(t) \approx x(t)\mathbf{\Omega}\vec{b}, \quad (18)$$

i.e., the asymptotic solution of (15) is contained as a common factor in all the modes of $\vec{X}(t)$, the “replicated” dynamics.

As a last comment, we recover the solution of the original dynamical system by averaging the elements of (18)

$$\frac{1}{N}\vec{1}^T\mathbf{\Omega}\vec{X}(t) \approx x(t)\frac{1}{N}\vec{1}^T\mathbf{\Omega}\vec{b}, \quad (19)$$

where T represents the transpose. Therefore, choosing vector \vec{b} such that

$$\frac{1}{N}\vec{1}^T\mathbf{\Omega}\vec{b} = \frac{1}{N}\sum_{i,j=1}^N \Omega_{ij}b_j = 1 \quad (20)$$

we find

$$x(t) = \frac{1}{N}\vec{1}^T\mathbf{\Omega}\vec{X}(t) = \frac{1}{N}\sum_{i,j=1}^N \Omega_{ij}X_j(t) \quad (21)$$

i.e., the projected dynamics recovers the original one dimensional system.

The main goal of this paper is to extend the results of Example 3.1 to arbitrary dynamical systems. Let us now identify the key steps of the procedure. First, we begin with a dynamical system in the standard form.

Definition 3.2. *Embedding procedure: PEDS.* We explicitly define here the steps involved in developing the PEDS procedure.

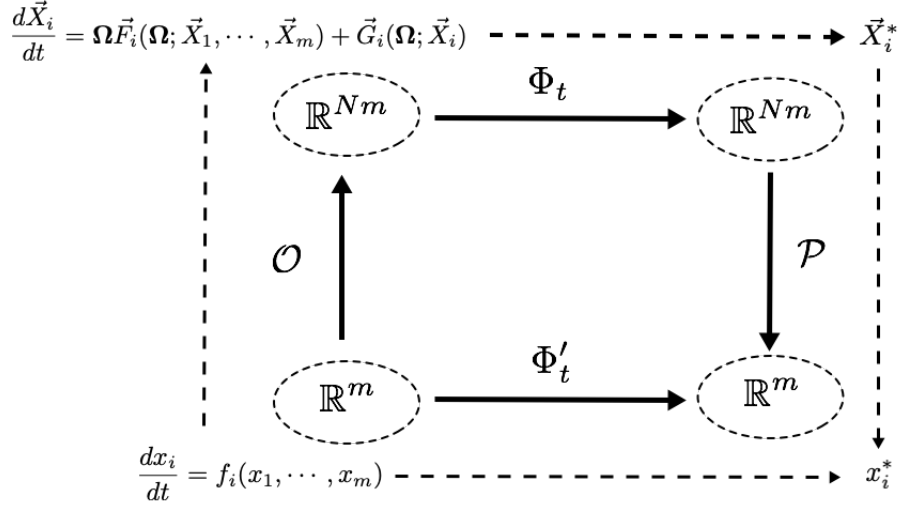


Figure 3: Graphical representation of the PEDS procedure and of the associated maps. The horizontal arrows represent the time evolution maps Φ_t and Φ'_t , while the vertical arrows represent the embedding \mathcal{O} and the projection \mathcal{P} map, respectively.

1. We begin with a tuple $(\{f_1(\vec{x}), \dots, f_m(\vec{x})\}, \mathbf{\Omega}, \{\vec{G}_1, \dots, \vec{G}_m\}, \mathcal{S}, \{\vec{b}_1, \dots, \vec{b}_m\}, N)$, where $\mathbf{\Omega}$ is a size N projector operator. We call $\{f_1(\vec{x}), \dots, f_m(\vec{x})\}$ the *target dynamical system*, while x_i represent the *target variables*. We assume that in the target system the only attractors are asymptotically stable equilibria. \mathcal{S} represents an ordering, relevant for the case of a multi-dimensional target system if the embedding is non-commutative. Vector \vec{b} is constant and such that $\mathbf{\Omega}\vec{b} \neq \vec{0}$. Finally, N is the dimension of the embedding for each scalar variable. As such, it can also be interpreted as the number of dimensions in which each scalar variable is expanded into.
2. Given the target dynamical system of dimension m , we build an extended dynamical system of size mN , represented by a set of canonical equations of the form

$$\frac{d}{dt} \vec{X}_i = \mathbf{\Omega} \mathbf{F}_i(\vec{X}_1, \dots, \vec{X}_m) \vec{b}_i + \vec{G}_i(\mathbf{\Omega}; \vec{X}_i) \quad i = 1, \dots, m \quad (22)$$

This step is represented by the arrow \mathcal{O} in Fig. 3, being it a mapping between each scalar functions f_i to the vector function $\vec{F}_i = \mathbf{F}_i \vec{b}_i$. Thus, for each dimension of the original dynamical system, we obtain an extended N dimensional subspace in the Nm dynamical system, so that

$$\begin{aligned} (\{f_1(\vec{x}), \dots, f_m(\vec{x})\}) &\xrightarrow{\mathcal{O}} (\{\vec{F}_1(\vec{X}_i), \dots, \vec{F}_m(\vec{X}_i)\}, \{\vec{G}_1(\mathbf{\Omega}; \vec{X}_i), \dots, \vec{G}_m(\mathbf{\Omega}; \vec{X}_i)\}) \\ &\equiv (F, G) \end{aligned} \quad (23)$$

We call the specific map \mathcal{O} the *embedding*, while (F, G) is the *extended system* and \vec{X}_i the *extended variable* (i.e., a set of N scalar variables in the extended system). We also dub the set \vec{G}_i , the *decay functions*. In each extended subspace,

\vec{X}_i is a vector of components $X_{i,j}$, while diagonal matrix \mathbf{X}_i is made of elements $X_{i,jk} = X_{i,j}\delta_{jk}$, where δ_{jk} represents Kronecker symbol. The original vector can be easily recovered from the diagonal matrix as $\vec{X}_i = \mathbf{X}_i\vec{1}$. We stress that in principle \mathbf{F}_i can be a non-trivial function of Ω , as we shall discuss later on.

3. We consider the time evolution of both the original and the extended system, represented by maps Φ'_t and Φ_t , respectively, in Fig. 3. In Example 3.1, the two maps were analytically expressed, thanks to the simplicity of the target system.
4. Arrow \mathcal{P} in Fig. 3, finally, projects the extended dynamical system from size Nm to an m dimensional system, that is required to coincide with the trajectory of the target dynamical system. For each variable, the projection is derived from the projector operator as, given a certain extended variable \vec{X}_i , we obtain $\bar{x}_i = \frac{1}{N} \sum_{j,k=1}^N \Omega_{jk} X_{i,k}$.

3.1. Extended variable ordering

Before delving into the details of the construction of (22), let us clarify what we mean by ordering. During the development of the PEDS procedure, commuting variable products such as x_1x_2 will be mapped onto matrix products of the form $(\Omega\mathbf{X}_1)(\Omega\mathbf{X}_2)$. As matrix products do not commute, the ordering of the variables will have a role.

Definition 3.3. *Ordering.* Within the context of PEDS, an ordering \mathcal{S} is a map between commuting monomials of the form $x_1^{i_1}x_2^{i_2}\dots x_m^{i_m}$ and non-commuting matrix monomials of the form $(\Omega\mathbf{X}_1)^{i_1}(\Omega\mathbf{X}_2)^{i_2}\dots(\Omega\mathbf{X}_m)^{i_m}$.

In general, an ordering can be written in terms of a certain set of coefficients. We will use the following notation

$$\{(\Omega\mathbf{X}_1)^{i_1}\dots(\Omega\mathbf{X}_m)^{i_m}\}_{\mathcal{S}} = \sum_{\sigma \in \mathcal{S}(m)} o_{\sigma(1)\dots\sigma(m)} (\Omega\mathbf{X}_{\sigma(1)})^{i_{\sigma(1)}} \dots (\Omega\mathbf{X}_{\sigma(m)})^{i_{\sigma(m)}} \quad (24)$$

where σ is an element of the permutation group $\mathcal{S}(m)$ over m variables, the coefficients $o_{\sigma(1)\dots\sigma(m)}$ are zero if at least two indices are equal, and

$$\sum_{\sigma \in \mathcal{S}(m)} o_{\sigma(1)\dots\sigma(m)} = 1. \quad (25)$$

Definition 3.4. Given the monomial $x_1^{i_1}x_2^{i_2}\dots x_m^{i_m}$, the *standard* ordering is given by $(\Omega\mathbf{X}_1)^{i_1}(\Omega\mathbf{X}_2)^{i_2}\dots(\Omega\mathbf{X}_m)^{i_m}$, i.e. a matrix monomial where the matrix products strictly follow the same sequence as in the scalar case.

Definition 3.5. Given the monomial $x_1^{i_1}x_2^{i_2}\dots x_m^{i_m}$, the *balanced* ordering is given by

$$\frac{1}{m!} \sum_{\sigma \in \mathcal{S}(m)} (\Omega\mathbf{X}_{\sigma(1)})^{i_{\sigma(1)}} (\Omega\mathbf{X}_{\sigma(2)})^{i_{\sigma(2)}} \dots (\Omega\mathbf{X}_{\sigma(m)})^{i_{\sigma(m)}}$$

Notice that

$$\frac{1}{m!} \sum_{\sigma \in \mathcal{S}(m)} 1 = 1.$$

and that, given an order-independent function M , i.e. a function satisfying

$$M(a_1, \dots, a_m) = M(a_{\sigma(1)}, \dots, a_{\sigma(m)}), \quad (26)$$

for any permutation $\sigma \in \mathcal{S}(m)$, then

$$\begin{aligned} \sum_{\sigma \in \mathcal{S}(m)} o_{\sigma(1), \dots, \sigma(m)} M(a_{\sigma(1)}, \dots, a_{\sigma(m)}) &= \sum_{\sigma \in \mathcal{S}(m)} o_{\sigma(1), \dots, \sigma(m)} M(a_1, \dots, a_m) \\ &= M(a_1, \dots, a_m) \sum_{\sigma \in \mathcal{S}(m)} o_{\sigma(1), \dots, \sigma(m)} \\ &= M(a_1, \dots, a_m) \end{aligned} \quad (27)$$

Example 3.6.

The standard ordering is characterized by

$$o_{\sigma(1)\dots\sigma(m)} = \delta_{\sigma(1)1} \cdots \delta_{\sigma(m)m}, \quad (28)$$

while for the balanced ordering, $o_{\sigma(1)\dots\sigma(m)} = 1/m!$.

Considering the case of two scalar variables (i.e., $m = 2$), choosing $o_{12} = 1$ and $o_{21} = o_{11} = o_{22} = 0$, we get $\{(\mathbf{\Omega}X_1)^{i_1}(\mathbf{\Omega}X_2)^{i_2}\}_S = (\mathbf{\Omega}X_1)^{i_1}(\mathbf{\Omega}X_2)^{i_2}$. Another possible choice is $o_{12} = a, o_{21} = b, o_{11} = o_{22} = 0$, where $0 \leq a, b \leq 1$ and $a + b = 1$, so that

$$\{(\mathbf{\Omega}X_1)^{i_1}(\mathbf{\Omega}X_2)^{i_2}\}_S = a(\mathbf{\Omega}X_1)^{i_1}(\mathbf{\Omega}X_2)^{i_2} + b(\mathbf{\Omega}X_2)^{i_2}(\mathbf{\Omega}X_1)^{i_1} \quad (29)$$

3.2. Decay functions

We discuss now the decay functions $\vec{G}_i(\mathbf{\Omega}; \vec{X}_i)$. The choice made in Example 3.1 was

$$\vec{G}(\mathbf{\Omega}; \vec{X}) = -\alpha(\mathbf{I} - \mathbf{\Omega})\vec{X} \quad (30)$$

where $\alpha \geq 0$. This particular choice corresponds to a precise definition:

Definition 3.7. *Standard decay function.* The decay function in (30) is called *standard decay function*.

As seen in Example 3.1, the standard decay function allowed to recover the target dynamical system dynamics, that in turn was reconstructed in the $\text{Span}(\mathbf{\Omega})$. The role played by the decay functions is to enforce that in each extended subspace, the modal components associated to the $\text{Ker}(\mathbf{\Omega})$ are asymptotically vanishing.

Definition 3.8. A decay function $\vec{G}_i(\mathbf{\Omega}; \vec{X}_i)$ is $\mathbf{\Omega}$ -eligible if

$$\lim_{t \rightarrow +\infty} \mathbf{\Omega} \vec{G}_i(\mathbf{\Omega}; \vec{X}_i(t)) = \vec{0} \quad (31)$$

and if the solution of the dynamical system obtained projecting (22) onto the $\text{Ker}(\mathbf{\Omega})$ (i.e., projecting the extended equation via $(\mathbf{I} - \mathbf{\Omega})$ and defining $\vec{X}_{ci}(t) = (\mathbf{I} - \mathbf{\Omega})\vec{X}_i(t)$)

$$\frac{d\vec{X}_{ci}}{dt} = (\mathbf{I} - \mathbf{\Omega})\vec{G}_i(\mathbf{\Omega}; \vec{X}_i) \quad (32)$$

is decaying, i.e. if $\lim_{t \rightarrow \infty} \vec{X}_{ci}(t) = \vec{0}$.

Obviously, the standard decay functions are $\mathbf{\Omega}$ -eligible.

3.3. Embedding map \mathcal{O}

We are now ready to state the exact definition of the \mathcal{O} map. However, this step requires to express the nonlinear scalar functions defining the target dynamical system as a power series. From this standpoint, it is convenient to formulate the Taylor expansion of an m variable, scalar analytic function $f(\vec{x})$ as a superposition of monomials exploiting Kronecker symbol:

$$f(\vec{x}) = \sum_{j=0}^{\infty} \sum_{i_1, \dots, i_j=0}^j \delta_{j, \sum_{k=1}^m i_k} b_{j; i_1 \dots i_j} x_1^{i_1} \cdots x_m^{i_m} = \sum_{j=0}^{\infty} \sum_{i_1, \dots, i_j=0}^j a_{j; i_1 \dots i_j} x_1^{i_1} \cdots x_m^{i_m}, \quad (33)$$

where $a_{j; i_1 \dots i_m} = \delta_{j, \sum_{k=1}^m i_k} b_{j; i_1 \dots i_j}$.

Definition 3.9. Matrix map.

Given a scalar analytic function $f(\vec{x})$ with Taylor expansion as in (33), we call a *matrix map* for f the following construction

$$\mathbf{F}(\vec{X}_1, \dots, \vec{X}_m) = \sum_{j=0}^{\infty} \sum_{i_1, \dots, i_j=0}^j a_{j; i_1 \dots i_j} \{(\mathbf{\Omega}_1 \mathbf{X}_1)^{i_1} \cdots (\mathbf{\Omega}_1 \mathbf{X}_m)^{i_m}\}_S \quad (34)$$

where S is a properly defined ordering.

Definition 3.10. Embedding map.

The embedding map \mathcal{O} is defined as the tuple $\mathcal{O} = (\{f_i\}, \mathbf{\Omega}, \{\vec{G}_i\}, S, \{\vec{b}_i\}, N)$, where \vec{G}_i represents decay functions, and \vec{b}_i is a set of constant vectors satisfying condition $\mathbf{\Omega} \vec{b}_i \neq \vec{0}$. The *embedding map* \mathcal{O} of (2) is given by

$$\frac{dx_i}{dt} = f_i(\vec{x}) \xrightarrow{\mathcal{O}} \frac{d\vec{X}_i}{dt} = \mathbf{\Omega} \mathbf{F}_i(\vec{X}_1, \dots, \vec{X}_m) \vec{b}_i + \vec{G}_i(\mathbf{\Omega}; \vec{X}_i) \quad i = 1, \dots, m \quad (35)$$

where \mathbf{F}_i is the matrix map associated to f_i according to Definition 3.9.

Let us now provide three examples of matrices \mathbf{F}_i which will be used in the following. Each target function is analytical, with series representation as in (33):

$$f_i(x_1, \dots, x_m) = \sum_{k=0}^{\infty} \sum_{j_1, \dots, j_m}^k a_{i, k; j_1, \dots, j_m} x_1^{j_1} \cdots x_m^{j_m}$$

Definition 3.11. We define the following three possible matrix embeddings:

- the *standard commutative map* is

$$\mathbf{F}_i^{(c)}(\mathbf{X}_1, \dots, \mathbf{X}_m) = \sum_{k=0}^{\infty} \sum_{j_1, \dots, j_m}^k a_{i, k; j_1, \dots, j_m} \mathbf{X}_1^{j_1} \cdots \mathbf{X}_m^{j_m}, \quad (36)$$

- the *mixed commutative map* is

$$\mathbf{F}_i^{(mc)}(\mathbf{\Omega}; \mathbf{X}_1, \dots, \mathbf{X}_m) = a_{i,0} \mathbf{I} + \sum_{k=1}^{\infty} \sum_{j_1, \dots, j_m}^k a_{i, k; j_1, \dots, j_m} (\mathbf{\Omega}(\mathbf{X}_1^{j_1} \cdots \mathbf{X}_m^{j_m})^{1/k})^k, \quad (37)$$

where $a_{i,0}$ denotes the constant term of the series expansion for function f_i

- the *standard non-commutative map* is

$$\mathbf{F}_i^{(nc)}(\mathbf{\Omega}; \mathbf{X}_1, \dots, \mathbf{X}_m) = \sum_{k=0}^{\infty} \sum_{j_1, \dots, j_m}^k a_{i,k;j_1, \dots, j_m} \{(\mathbf{\Omega}\mathbf{X}_1)^{j_1} \dots (\mathbf{\Omega}\mathbf{X}_m)^{j_m}\}_S \quad (38)$$

where S is the chosen ordering.

Clearly, since diagonal matrices \mathbf{X}_i commute, defining an ordering for the standard and the mixed commutative maps is unnecessary. As we will see, such difference is important for embeddings of vector dynamical systems in the case of the mixed commutative map, but not for a scalar system. Notice also that

- the standard commutative map is a linear mix of the dynamical systems functions $f_i(\vec{x})$, since a direct calculation shows

$$\mathbf{F}_i^{(c)}(\mathbf{X}_1, \dots, \mathbf{X}_m) = \text{diag}(f_i(X_{1,1}, \dots, X_{m,1}), \dots, f_i(X_{1,N}, \dots, X_{m,N})). \quad (39)$$

which simplifies drastically the evaluation

- for scalar dynamical systems, the mixed commutative map and the standard non-commutative map reduce to the same quantity
- in the case of vector dynamical systems, the mixed commutative map preserves the commutativity of the target variables, since diagonal matrices \mathbf{X}_i commute among themselves.

As a consequence, for scalar dynamical system we will study only the standard commutative and non-commutative maps, while the result will follow also for the mixed commutative map from the non-commutative one. However, we will have to be more careful in the vector case.

3.4. Projection operator $\mathbf{\Omega}$

We provide here a few definitions on the projection operators of size N we will consider in the following.

Definition 3.12. A projector $\mathbf{\Omega}$ is called *trivial* if $\text{rank}(\mathbf{\Omega}) = N$, or, equivalently, if $\text{Span}(\mathbf{\Omega}) = \mathbb{R}^N$.

A simple proof shows that the only trivial projector is the identity matrix \mathbf{I} .

Definition 3.13. The *uniform mean-field projector* $\mathbf{\Omega}_1$ is defined as the square matrix with elements

$$\Omega_{1,ij} = \frac{1}{N}$$

Let us consider \mathbf{X} to be a diagonal matrix, as in the PEDS embedding procedure. Projection using the uniform mean-field operator yields

$$\mathbf{\Omega}_1 \mathbf{X} = \frac{1}{N} \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} X_1 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & X_N \end{pmatrix} = \frac{1}{N} \begin{pmatrix} X_1 & X_2 & \dots & X_N \\ \vdots & \vdots & \vdots & \vdots \\ X_1 & X_2 & \dots & X_N \end{pmatrix} \quad (40)$$

therefore, the powers of $\mathbf{\Omega}\mathbf{X}$ appearing in the PEDS procedure, are neither trivial expressions nor sparse matrices, and indeed contain non linear components in the X_i variables.

Example 3.14.

For $N = 2$ we have

$$\mathbf{\Omega}_1 \mathbf{X} = \frac{1}{2} \begin{pmatrix} X_1 & X_2 \\ X_1 & X_2 \end{pmatrix}, \quad (\mathbf{\Omega}_1 \mathbf{X})^2 = \frac{1}{4} \begin{pmatrix} X_1(X_1 + X_2) & X_2(X_1 + X_2) \\ X_1(X_1 + X_2) & X_2(X_1 + X_2) \end{pmatrix} = \langle X \rangle \mathbf{\Omega}_1 \mathbf{X} \quad (41)$$

where $\langle X \rangle = \frac{1}{N} \sum_{j=1}^N X_j$.

The previous example can easily be generalized to size N , showing that $(\mathbf{\Omega}_1 \mathbf{X})^k = \langle X \rangle^{k-1} \mathbf{\Omega}_1 \mathbf{X}$, thus justifying the definition of $\mathbf{\Omega}_1$ as the uniform mean-field projector.

A similar property applies to vectors, as $\mathbf{\Omega}_1 \vec{X} = \langle X \rangle \vec{1}$.

Example 3.15.

Consider again the Example 3.1. We can write the embedded system as

$$\frac{d\vec{X}}{dt} = a\mathbf{\Omega}\vec{X} - \alpha(\mathbf{I} - \mathbf{\Omega})\vec{X} = \mathbf{\Omega}(a\mathbf{\Omega}\mathbf{X})\vec{1} - \alpha(\mathbf{I} - \mathbf{\Omega})\vec{X} \quad (42)$$

which is in the form of a PEDS (3), with $\mathbf{F} = a\mathbf{\Omega}\mathbf{X}$, $\vec{b} = \vec{1}$ and $\vec{G} = -\alpha(\mathbf{I} - \mathbf{\Omega})\vec{X}$.

We would like to stress the fact that the PEDS mapping is, in general, highly non-trivial, at least as far as the projection operator is not the trivial one: this condition is required because in this case the matrix powers of the form $(\mathbf{\Omega}\mathbf{X}_i)^k$ couple all the subspace variables in a nonlinear way.

On the other hand, for the trivial projector, the standard decay function is identically zero, and the extended system as well as any extended monomial are ordering independent. In fact, as the diagonal matrices \mathbf{X}_j commute among themselves, we have that

$$\{(\mathbf{\Omega}\mathbf{X}_1)^{j_1} \dots (\mathbf{\Omega}\mathbf{X}_m)^{j_m}\}_S = (\mathbf{X}_1)^{j_1} \dots (\mathbf{X}_m)^{j_m} \quad (43)$$

if and only if $\mathbf{\Omega} = \mathbf{I}$. As a consequence, (35) becomes

$$\frac{d\vec{X}_i}{dt} = \mathbf{\Omega}\mathbf{F}_i(\vec{X}_1, \dots, \vec{X}_m)\vec{b}_i = \begin{pmatrix} f_i(X_{i,1}, \dots, X_{i,m})b_{i,1} \\ \vdots \\ f_i(X_{i,N}, \dots, X_{i,m})b_{i,N} \end{pmatrix} \quad i = 1, \dots, m \quad (44)$$

thus showing that the PEDS procedure for the trivial projector decouples into N identical copies of the original system.

4. Embedding via the uniform mean field projector

We derive here in a more rigorous way the key results presented above. We focus on the uniform mean field projector $\mathbf{\Omega}_1$, as the proofs are easier to be carried out. Nevertheless, several results are actually valid even for a more general projection operator $\mathbf{\Omega}$: these will be explicitly denoted by using the general projector $\mathbf{\Omega}$ in place of the uniform mean field operator.

4.1. Simple case: Scalar target system, embedding without decay function

We start from the case of a one dimensional target dynamical system

$$\frac{dx}{dt} = f(x) \quad (45)$$

where $f(x)$ is analytic, so that

$$f(x) = \sum_{i=0}^{\infty} a_i x^i. \quad (46)$$

Following the PEDS procedure, we introduce the projector operator $\mathbf{\Omega}$. The extended variable \vec{X} is thus an N -dimensional vector with components X_i , and the matrix map associated to f takes either the standard commutative form (36) so that

$$\vec{F}(\vec{X}) = \sum_{i=0}^{\infty} a_i X^i \vec{1} \quad (47)$$

where we have chosen $\vec{b} = \vec{1}$, or the standard non-commutative form (38) (we remind that for scalar target systems, the mixed commutative and the standard non-commutative forms coincide)

$$\vec{F}(\vec{X}) = \sum_{i=0}^{\infty} a_i \mathbf{\Omega} X^i \vec{1} \quad (48)$$

Before we begin our discussion on the embedding, it is worth giving a definition of what we mean when we say that a dynamical system is *contained* in another one. Taking Fig. 3 as a reference, let our target system be described by the evolution map (the solution of the dynamical system) $\phi'_t : \mathbb{R} \rightarrow \mathbb{R}$, while the PEDS evolution is instead a map $\phi_t : \mathbb{R}^N \rightarrow \mathbb{R}^N$.

Definition 4.1. *Embedding.* A dynamical system A of size N_A is *contained* in a dynamical system B of dimensions $N_B > N_A$ if a linear operator $\mathcal{P} : \mathbb{R}^{N_B} \rightarrow \mathbb{R}^{N_A}$, and a set of initial conditions exists such that, for $\vec{X} \in \mathbb{R}^{N_B}$

$$\mathcal{P} \phi_t(\vec{X}) = \phi'_t(\vec{x}), \quad (49)$$

where \vec{x} has size N_A .

In this paper, we are in particular interested in the fixed points of the dynamics. Fixed points are invariant with respect with the flow of a dynamical system, e.g. given the flow operator $\phi_t(\cdot) : \mathbb{R}^N \rightarrow \mathbb{R}^N$, the union of all fixed points $M \in \mathbb{R}^N$ satisfy

$$\phi_t(M) = M. \quad (50)$$

Definition 4.2. *Weak embedding.* The fixed points of a dynamical system A of size N_A are *weakly contained* in the fixed points dynamical system B of dimensions $N_B > N_A$ if a linear operator $\mathcal{P} : \mathbb{R}^{N_B} \rightarrow \mathbb{R}^{N_A}$ exists such that, for $\vec{X} \in \mathbb{R}^{N_B}$

$$\mathcal{P} \phi_t(M) = \phi'_t(M'), \quad (51)$$

and moreover

$$\phi_t(M) = M, \quad \phi_t(M') = M'. \quad (52)$$

It is easy to see that if both dynamical systems contain fixed points, then if a dynamical system is contained into another, a portion of their fixed points will be contained too. This follows easily from sending $t \rightarrow \infty$ in eqn. (49).

Given the definition above, we can now prove the following

Proposition 4.3. Banality of mean value. *Let $\mathcal{O} = (f(x), \mathbf{\Omega}, 0, \vec{\mathbf{1}}, N)$ be a PEDS tuple of a target dynamical system as in (45), where the matrix map can take either the standard commuting or standard non-commuting forms. Then, the extended dynamical system (47) or (48) contains the dynamics of (46) for a generic projection operator $\mathbf{\Omega}$ satisfying $\mathbf{\Omega}\vec{\mathbf{1}} \neq \vec{\mathbf{0}}$.*

Proof. Let us consider an extended variable \vec{X} subject to the condition $\vec{X} = x\vec{\mathbf{1}}$. Then, as $\mathbf{X} = x\vec{\mathbf{1}}$ and $\mathbf{\Omega}^i = \mathbf{\Omega}$ $i > 0$, both definitions of the standard commuting and non-commuting maps reduce to the same equation, and we have:

$$\frac{d}{dt}\vec{X} = \frac{dx}{dt}\vec{\mathbf{1}} = \sum_{i=0}^{\infty} a_i x^i \mathbf{\Omega}\vec{\mathbf{1}} = a_0 \mathbf{\Omega}\vec{\mathbf{1}} + \sum_{i=1}^{\infty} a_i x^i \mathbf{\Omega}\vec{\mathbf{1}} = f(x) \mathbf{\Omega}\vec{\mathbf{1}} \quad (53)$$

Therefore, projecting the previous relation onto the span of $\mathbf{\Omega}$, i.e. evaluating $\mathbf{\Omega} \frac{d}{dt} \vec{X} = \mathbf{\Omega} \vec{F}(\mathbf{X})$, we obtain

$$\left(\frac{dx}{dt} - f(x) \right) \mathbf{\Omega}\vec{\mathbf{1}} = 0. \quad (54)$$

It follows that as $\mathbf{\Omega}\vec{\mathbf{1}} \neq \vec{\mathbf{0}}$, then $\frac{dx}{dt} - f(x) = 0$. In order to prove that the dynamical system is contained, we can project on any component i , obtaining

$$\omega_i \left(\frac{dx}{dt} - f(x) \right) = 0 \quad (55)$$

if $\omega_i = (\mathbf{\Omega}\vec{\mathbf{1}})_i \neq 0$, we have then proven that an initial condition exists for which (49) applies. \square

Proposition 4.3 is a warm up for the type of proofs that will follow. It shows that if the initial condition for the variables \vec{X} are chosen homogeneously, then the extended dynamical system will follow the one dimensional dynamics of (45). However, condition $\vec{X} = x\vec{\mathbf{1}}$ is a strong requirement for the dynamical system. In principle, a dynamically obtained convergence towards a state of the form $\vec{X} = x\vec{\mathbf{1}}$ would be a much better demand. To this aim, we introduce the decay functions.

4.2. Scalar target system: Enforcing the convergence to the mean via decay functions

We consider now the following form for the extended system (22) based on the uniform mean field projector:

$$\frac{d\vec{X}}{dt} = \mathbf{\Omega}_1 \vec{F}(\vec{X}) - \alpha (\vec{X} - \langle X \rangle \vec{\mathbf{1}}) = \mathbf{\Omega}_1 \vec{F}(\vec{X}) - \alpha (\mathbf{I} - \mathbf{\Omega}_1) \vec{X} \quad (56)$$

where $\langle X \rangle = (1/N) \sum_{i=1}^N X_i$ and $\alpha > 0$. Notice that $\vec{X} - \langle X \rangle \vec{\mathbf{1}} = (\mathbf{I} - \mathbf{\Omega}_1) \vec{X}$ because of the properties of $\mathbf{\Omega}_1$ discussed in Sec. 3.4. The second term on the right hand side of (56) is an ‘‘elastic’’ force compelling the extended trajectories to remain close to the mean. The relative strength of the two addends determines the behavior of the system.

Proposition 4.4. Convergence to the mean. *The dynamics of (56) is characterized by the same fixed points, if they exist, as for the target system (45) both for the standard commuting and non-commuting matrix maps.*

Proof. Projecting (56) through $\mathbf{\Omega}_1$ and using (47) we find

$$\mathbf{\Omega}_1 \frac{d\vec{X}}{dt} = \begin{cases} \sum_{i=0}^{\infty} a_i \mathbf{\Omega}_1 X^i \vec{1} - \alpha \mathbf{\Omega}_1 (\mathbf{I} - \mathbf{\Omega}_1) \vec{X} & \text{standard commuting map} \\ \sum_{i=0}^{\infty} a_i \mathbf{\Omega}_1 (\mathbf{\Omega}_1 X)^i \vec{1} - \alpha \mathbf{\Omega}_1 (\mathbf{I} - \mathbf{\Omega}_1) \vec{X} & \text{standard noncommuting map} \end{cases} \quad (57)$$

which reduces to the banality lemma enforcing $\vec{X} = x\vec{1}$.

Considering the complementary projection, we have

$$(\mathbf{I} - \mathbf{\Omega}_1) \frac{d\vec{X}}{dt} = (\mathbf{I} - \mathbf{\Omega}_1) \mathbf{\Omega}_1 \vec{F}(\vec{X}) - \alpha ((\mathbf{I} - \mathbf{\Omega}_1) \vec{X} - \langle X \rangle (\mathbf{I} - \mathbf{\Omega}_1) \vec{1}) \quad (58)$$

or, defining $\vec{X}_c = (\mathbf{I} - \mathbf{\Omega}_1) \vec{X}$,

$$\frac{d}{dt} \vec{X}_c = -\alpha \vec{X}_c. \quad (59)$$

Equation (59) represents the dynamics of the $N - 1$ modes that make \vec{X} non-uniform. The above implies that any non-uniform mode of \vec{X} decays exponentially, and thus $\vec{X} - \langle X \rangle \vec{1} \rightarrow 0$ in a time $t \gg \tau = 1/\alpha$. This concludes the proof. \square

In conclusion, using the uniform mean field projector $\mathbf{\Omega}_1$ and the standard decay functions as in (56), $\vec{X}(t)$ converges to the right mean, and thus to the same fixed points as the target system. Let us now provide some technical results to support the idea that the decay functions project back on the subspace of our interest. The result is in fact not strictly limited to the standard decay functions. We now prove the decay of the modes in $\text{Ker}(\mathbf{\Omega}_1)$ for a generalized set of decay functions. Let us consider

$$\frac{d\vec{X}}{dt} = \mathbf{\Omega}_1 \vec{F} - \begin{cases} \mathbf{D}(\mathbf{I} - \mathbf{\Omega}_1) \vec{X} & \text{generalization A} \\ (\mathbf{I} - \mathbf{\Omega}_1) \mathbf{D}(\mathbf{I} - \mathbf{\Omega}_1) \vec{X} & \text{generalization B} \end{cases} \quad (60)$$

where \mathbf{D} is a positive diagonal matrix with diagonal elements $\alpha_1, \dots, \alpha_n > 0$. If $\alpha = \alpha_1 = \dots = \alpha_n$, both generalizations reduce to the standard decay function. In the general case, their difference becomes evident projecting via $\mathbf{\Omega}_1$

$$\mathbf{\Omega}_1 \frac{d\vec{X}}{dt} = \mathbf{\Omega}_1 \vec{F} - \begin{cases} \mathbf{\Omega}_1 \mathbf{D}(\mathbf{I} - \mathbf{\Omega}_1) \vec{X} & \text{generalization A} \\ \vec{0} & \text{generalization B} \end{cases} \quad (61)$$

i.e., the PEDS embeddings

$$\mathcal{O}_A = (f(x), \mathbf{\Omega}_1, -\mathbf{D}(\mathbf{I} - \mathbf{\Omega}_1) \vec{X}, \vec{1}, N)$$

and

$$\mathcal{O}_B = (f(x), \mathbf{\Omega}_1, -(\mathbf{I} - \mathbf{\Omega}_1) \mathbf{D}(\mathbf{I} - \mathbf{\Omega}_1) \vec{X}, \vec{1}, N).$$

Clearly, the first part of the proof of the banality lemma remains valid also in these cases. On the other hand, projecting via $(\mathbf{I} - \mathbf{\Omega}_1)$, we obtain for both generalizations the following governing equation for the non-uniform modes

$$\frac{d\vec{X}_c}{dt} = -(\mathbf{I} - \mathbf{\Omega}_1)\mathbf{D}\vec{X}_c \quad (62)$$

whose solution is

$$\vec{X}_c(t) = e^{-(\mathbf{I}-\mathbf{\Omega})\mathbf{D}t} \vec{X}_c(0) \Big|_{\mathbf{\Omega}=\mathbf{\Omega}_1}. \quad (63)$$

We show now that the two generalizations A and B are $\mathbf{\Omega}$ -eligible, i.e. that \vec{X}_c asymptotically approaches the zero vector. We prove the following proposition for general projectors:

Proposition 4.5. *Given the governing equation (62) written for a general projector $\mathbf{\Omega}$, assuming $\vec{X}_c(0) \in \text{Span}(\mathbf{I} - \mathbf{\Omega})$ then $\vec{X}_c(t) \in \text{Span}(\mathbf{I} - \mathbf{\Omega}) \forall t$.*

Proof. See App. Appendix A. □

As a result of the proposition above, vector $\vec{X}_c(t)$ is contained in the subspace spanned by $\mathbf{I} - \mathbf{\Omega}$ at all times and for any projector, and thus also for $\mathbf{\Omega}_1$.

Corollary 4.6. *, Equation (62) implies that, if $\vec{X}_c(0) \in \text{Span}(\mathbf{I} - \mathbf{\Omega}_1)$, then in (61) one has $\lim_{t \rightarrow \infty} \vec{X}(t) = \langle X \rangle \vec{\mathbf{1}}$.*

Proof. See App. Appendix B. □

Propositions 4.3, 4.4 and 4.5, along with Corollary 4.6 imply that for a one-dimensional dynamical system, the PEDS $\mathcal{O} = (f(x), \mathbf{\Omega}_1, -(\mathbf{I} - \mathbf{\Omega}_1)\vec{X}, \vec{\mathbf{1}}, N)$ contains the fixed points of the original dynamical system. This implies a weak embedding, using the definition of Def. 4.2. In particular, Corollary 4.6 implies that the extended system converges to an asymptotic state of the form $\vec{X}(t) = x(t)\vec{\mathbf{1}}$. Therefore, through the banality of the mean value Lemma, the PEDS embedding will contain the original dynamical system.

For practical purposes, it is sufficient to consider the observable $\tilde{x}(t) = \langle X \rangle = \frac{1}{N} \sum_{i,j=1}^N \mathbf{\Omega}_{1,ij} X_j(t)$ in order to recover the location of the fixed points. Clearly, this example applies only to a one-dimensional dynamical system. However, the result can be extended to the vector case following similar considerations.

Example 4.7.

As an example of a one dimensional dynamical system embedded in $N = 2$ dimensions, let us consider the dynamical system

$$\frac{dx}{dt} = x - x^2 \quad (64)$$

whose stable fixed point is given by $x^* = 1$. A PEDS embedding $\mathcal{O} = (x - x^2, \mathbf{\Omega}_1, -\alpha(\mathbf{I} - \mathbf{\Omega}_1)\vec{X}, \vec{\mathbf{1}}, 2)$ is given by the two coupled differential equations:

$$\frac{d\vec{X}}{dt} = \mathbf{\Omega}_1 \left(\mathbf{X} - \mathbf{X}^2 \right) \vec{\mathbf{1}} - \alpha(\mathbf{I} - \mathbf{\Omega}_1)\vec{X} \quad (65)$$

whose components can be made explicit introducing $\langle X \rangle = \frac{1}{2}(X_1 + X_2)$ and evaluating the matrix expressions. The result is

$$\frac{dX_1}{dt} = \langle X \rangle - \langle X \rangle^2 - \alpha(X_1 - \langle X \rangle) \quad (66)$$

$$\frac{dX_2}{dt} = \langle X \rangle - \langle X \rangle^2 - \alpha(X_2 - \langle X \rangle) \quad (67)$$

whose stable fixed point is easily seen to be $X_1^* = X_2^* = 1$.

Example 4.7 highlights an important property of the uniform mean field embedding. We gain some intuition on how the uniform mean field projector works by exploiting a direct evaluation of the matrix powers. For instance, for the PEDS $\mathcal{O} = (f(x), \mathbf{\Omega}_1, -\alpha(\mathbf{I} - \mathbf{\Omega}_1)\vec{X}, \vec{1}, N)$, then

$$\frac{d\vec{X}}{dt} = \mathbf{\Omega}_1 \vec{F}(\vec{X}) - \alpha(\mathbf{I} - \mathbf{\Omega}_1)\vec{X}. \quad (68)$$

Using the properties

$$(\mathbf{\Omega}_1 \mathbf{X})^k = \langle X \rangle^{k-1} \mathbf{\Omega}_1 \mathbf{X} \quad k \geq 1, \quad \mathbf{\Omega}_1 \mathbf{X} \vec{1} = \mathbf{\Omega}_1 \vec{X} = \langle X \rangle \vec{1}, \quad \mathbf{\Omega}_1 \vec{1} = \vec{1}$$

we obtain the equivalent form for (68)

$$\frac{d\vec{X}}{dt} = f(\langle x \rangle) \vec{1} - \alpha(\vec{X} - \langle x \rangle \vec{1}). \quad (69)$$

Multiplying on the left times $\mathbf{\Omega}_1$ and times $\mathbf{I} - \mathbf{\Omega}_1$, yields

$$\frac{d}{dt} (\langle X \rangle \vec{1}) = f(\langle X \rangle) \vec{1}, \quad (70)$$

$$\frac{d}{dt} (\vec{X} - \langle X \rangle \vec{1}) = -\alpha(\vec{X} - \langle X \rangle \vec{1}). \quad (71)$$

Therefore, the mean value $\langle X \rangle$ in (70) follows exactly the target system dynamics, while (71) asymptotically determines $\vec{X} \rightarrow \langle X \rangle \vec{1}$ for $t \gg 1/\alpha$. It follows that for scalar target systems, because of the identity $(\mathbf{\Omega}_1 \mathbf{X})^k = \langle X \rangle^{k-1} \mathbf{\Omega}_1 \mathbf{X}$, the standard commutative and non-commutative maps are identical. This is no longer true for vector target systems, as we discuss below.

4.3. General case: Vector target system

We will now focus on the generalization of the previous results to the case of a vector target dynamical system of arbitrary dimension. Such generalization is involved, basically because the ordering \mathcal{S} becomes important (at least for the standard non-commutative map) and this depends on the fact that matrix terms such as $(\mathbf{\Omega} \mathbf{X})^i$ and $(\mathbf{\Omega} \mathbf{X})^j$ do not commute. Nevertheless, exploiting the properties of the uniform mean field projector $\mathbf{\Omega}_1$ certain exact results can be obtained.

We consider an m dimensional target system as in (2), defined by the functions $f_i(x_1, \dots, x_m)$. These can both be defined in terms of (33) after choosing the ordering

S or in analytical form. Following the PEDS procedure, we construct as a first instance the embedding map $\mathcal{O} = (\{f_i\}, \mathbf{\Omega}_1, 0, S, \vec{\mathbf{1}}, N)$ in the absence of decay functions. Given the extended variables \vec{X}_s ($s = 1, \dots, m$) of size N , we write the embedding maps (36)–(38) as

$$\frac{d\vec{X}_s}{dt} = \mathbf{\Omega}_1 \sum_{k=0}^{\infty} \sum_{i_1, \dots, i_m}^k a_{s,k;i_1 \dots i_m} \mathbf{X}_1^{i_1} \dots \mathbf{X}_m^{i_m} \vec{\mathbf{1}}$$

standard commuting map (72a)

$$\frac{d\vec{X}_s}{dt} = a_{s,0} \mathbf{\Omega}_1 + \mathbf{\Omega}_1 \sum_{k=1}^{\infty} \sum_{i_1, \dots, i_m}^k a_{s,k;i_1, \dots, i_m} (\mathbf{\Omega}_1 (\mathbf{X}_1^{i_1} \dots \mathbf{X}_m^{i_m})^{1/k})^k \vec{\mathbf{1}}$$

mixed commuting map (72b)

$$\frac{d\vec{X}_s}{dt} = \mathbf{\Omega}_1 \sum_{k=0}^{\infty} \sum_{i_1, \dots, i_m}^k a_{s,k;i_1 \dots i_m} \{(\mathbf{\Omega}_1 \mathbf{X}_1)^{i_1} \dots (\mathbf{\Omega}_1 \mathbf{X}_m)^{i_m}\}_S \vec{\mathbf{1}}$$

standard noncommuting map (72c)

where S is the chosen ordering of the variables. The question is whether also in this case the banality of the mean value Proposition 4.3 still holds.

An easy proof shows that assuming a solution $\vec{X}_s = x_s \vec{\mathbf{1}}$, the *banality of the mean value* lemma applies also in the vector case, irrespective of the chosen extended variable ordering.

Corollary 4.8. Multivariate banality of the mean value. *Let $\mathbf{\Omega}_1$ be the uniform mean field projector, and $\mathcal{O} = (\{f_i\}, \mathbf{\Omega}_1, \{-\alpha(\mathbf{I} - \mathbf{\Omega}_1)\vec{X}_i\}, S, \vec{\mathbf{1}}, N)$ the embedding with S an arbitrary ordering. Following the PEDS procedure, the dynamics of the m variables \vec{X}_i is determined by*

$$\frac{d\vec{X}_i}{dt} = \mathbf{\Omega}_1 \vec{F}_i(\vec{X}_1, \dots, \vec{X}_m) - \alpha(\mathbf{I} - \mathbf{\Omega}_1) \vec{X}_i$$

(73)

where $\vec{F}_i = \mathbf{F}_i(\vec{X}_1, \dots, \vec{X}_m) \vec{\mathbf{1}}$ is the vector constructed following the PEDS procedure defined in (72). We define the projection operator $\mathcal{P}_{\mathbf{\Omega}_1} = \frac{1}{N} \vec{\mathbf{1}} \vec{\mathbf{1}}^T$, and the projected variables $\langle X_i(t) \rangle = \mathcal{P}_{\mathbf{\Omega}_1} \vec{X}_i(t)$. Then, the following two statements hold true

$$(a) \quad \frac{d}{dt} \langle X_i \rangle = 0 \implies \frac{d}{dt} x_i = 0,$$

(74)

and (b) if the extended system approaches a fixed point \vec{X}_i^* for times $t \gg 1/\alpha$, then the projection $\mathcal{P}_{\mathbf{\Omega}_1} \vec{X}_i^*$ is a fixed point of the target system.

Proof. See App. Appendix C. □

It follows from Corollary (4.8) that, since the proofs hold at $t \rightarrow \infty$, PEDS with mean field projectors guarantee fixed points with a weak embedding. The results of this section show that at least one type of PEDS exists which preserves the fixed points

of the target system, thus justifying the entire construction of the PEDS embedding. We now focus the attention on how the PEDS procedure modifies the properties of the fixed points, by analyzing their stability. Therefore, we need to look at the spectral properties of the Jacobian at the embedding fixed points.

4.4. Properties of the Jacobian and fixed points

Let us now investigate the properties of the Jacobian.

We begin with the one dimensional case

$$\frac{d\vec{X}}{dt} = \underbrace{\mathbf{\Omega}_1 \mathbf{F} \vec{1}}_1 - \underbrace{\alpha(\mathbf{I} - \mathbf{\Omega}_1) \vec{X}}_2 \quad (75)$$

For the sake of generality, we consider also the generalized decay functions in (60)

$$\frac{d\vec{X}}{dt} = \underbrace{\mathbf{\Omega}_1 \mathbf{F} \vec{1}}_1 - \begin{cases} \underbrace{\mathbf{D}(\mathbf{I} - \mathbf{\Omega}_1) \vec{X}}_{2A} & \text{generalization A} \\ \underbrace{(\mathbf{I} - \mathbf{\Omega}_1) \mathbf{D}(\mathbf{I} - \mathbf{\Omega}_1) \vec{X}}_{2B} & \text{generalization B} \end{cases} \quad (76)$$

As usual, we shall derive the results for a general projector $\mathbf{\Omega}$ whenever possible.

4.4.1. Simple case: scalar target system

We will focus first on the PEDS of a one dimensional target system, as in this case the ordering is immaterial and proofs are easier to carry out. Initially we consider the PEDS map $\mathcal{O} = (f(x), \mathbf{\Omega}_1, -\alpha(\mathbf{I} - \mathbf{\Omega}_1) \vec{X}, \vec{b}, N)$, i.e. the standard decay function. The embedding thus takes the form

$$\frac{d}{dt} X_i = (\mathbf{\Omega}_1 \mathbf{F} \vec{b} - \alpha(\mathbf{I} - \mathbf{\Omega}_1) \vec{X})_i = M_i. \quad (77)$$

We prove the following:

Proposition 4.9. One dimensional PEDS Jacobian

For a scalar target system, a PEDS of the form $\mathcal{O} = (f(x), \mathbf{\Omega}, -\alpha(\mathbf{I} - \mathbf{\Omega}) \vec{X}, \vec{b}, N)$ is characterized by the following functional form of the Jacobian

- for the standard non-commuting map

$$J_{ir}^{(nc)} = \sum_{j=1}^N \sum_{l=1}^N \Omega_{il} \sum_{z=1}^{\infty} a_z \sum_{k=0}^{z-1} \sum_{s=1}^N (\mathbf{\Omega X})_{ls}^k \Omega_{sr} (\mathbf{\Omega X})_{rj}^{z-1-k} b_j - \alpha(\mathbf{I} - \mathbf{\Omega})_{ir}, \quad (78)$$

- for the standard commuting map

$$J_{ir}^{(c)} = \Omega_{ir} f'(X_r) b_r - \alpha(\mathbf{I} - \mathbf{\Omega})_{ir}. \quad (79)$$

Proof. See App. Appendix D. □

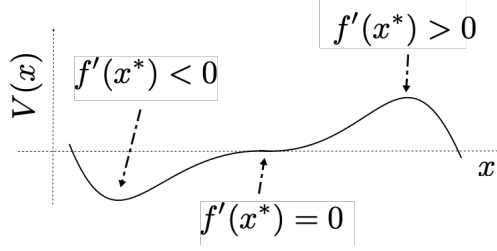


Figure 4: Interpreting the one dimensional dynamical system as the gradient of a potential, via $f(x) = -\partial V(x)/\partial x$.

As a direct Corollary of Proposition 4.9, we obtain that for $\mathbf{\Omega} = \mathbf{\Omega}_1$ and $\vec{b} = \vec{1}$, the Jacobian takes a simpler form.

Corollary 4.10. *Consider the uniform mean field PEDS with standard decay function $\mathcal{O} = (f(x), \mathbf{\Omega}_1, -\alpha(\mathbf{I} - \mathbf{\Omega}_1)\vec{X}, \vec{1}, N)$ of a scalar dynamical system characterized by a fixed point x^* . The Jacobian of the PEDS in its fixed point $\vec{X}^* = x^*\vec{1}$ is given by*

$$J_{ir}(\vec{X}^*) = -\alpha\delta_{ir} + \frac{1}{N}(f'(x^*) + \alpha) \iff \mathbf{J}(\vec{X}^*) = -\alpha\mathbf{I} + (f'(x^*) + \alpha)\mathbf{\Omega}_1, \quad (80)$$

both for the standard commutative and non-commutative maps.

Proof. See App. Appendix E. □

At this point we can start to draw some partial conclusions. In fact, we can use the properties of the fixed points of the target system to understand how these are transformed by the embedding procedure. The target system equilibrium is unstable if $f'(x^*) > 0$, while it is stable for $f'(x^*) < 0$. Finally, $f'(x^*) = 0$ corresponds to a saddle. Since any scalar dynamical system is conservative, it can be expressed as

$$\frac{dx}{dt} = f(x) = -\frac{\partial V(x)}{\partial x}, \quad (81)$$

where V is a potential function. The extrema x^* of $V(x)$ correspond to minima, maxima and non-hyperbolic points with $f'(x^*) = 0$, as we show in Fig. 4.

For the PEDS $\mathcal{O} = (f(x), \mathbf{\Omega}_1, -\alpha(\mathbf{I} - \mathbf{\Omega}_1)\vec{X}, \vec{1}, N)$, Corollary 4.10 implies that the Jacobian spectrum follows from the spectral properties of $\mathbf{\Omega}_1$. In fact, $\mathbf{\Omega}_1$ has one eigenvalue equal to 1, and $N - 1$ identical, null eigenvalues. Thus, the spectrum Λ of the Jacobian at $\vec{X}^* = x^*\vec{1}$ is given by

$$\Lambda(\mathbf{J}(x^*)) = \{-\alpha\}_{N-1}, \{f'(x^*)\}_1, \quad (82)$$

i.e. $-\alpha$ with multiplicity $N - 1$, and $f'(x^*)$ with multiplicity 1.

As a particular case of interest, if $f'(x^*) \neq 0$, we can therefore carry out a stability analysis of the PEDS fixed point \vec{X}^* as follows:

$$\vec{X}^* \text{ is } \begin{cases} \text{stable} & \text{if } x^* \text{ is stable,} \\ \text{non-hyperbolic} & \text{if } x^* \text{ a saddle point,} \\ \text{a saddle point} & \text{if } x^* \text{ is unstable.} \end{cases} \quad (83)$$

We thus see what are the benefits of the PEDS from the point of view of the target system. While in the scalar case “barriers” can be present, these can be made to disappear via the PEDS. Although this specific feature is peculiar to scalar target systems, this result will later be useful also for vector target systems. A graphical representation is shown in Fig. 5.

We can conclude that the PEDS procedure $\mathcal{O} = (f(x), \mathbf{\Omega}_1, -\alpha(\mathbf{I} - \mathbf{\Omega}_1)\vec{X}, \vec{1}, N)$ preserves stable and saddle fixed points of the target dynamics, while it turns unstable fixed points into saddle points.

Let us now consider the Jacobian properties in presence of the generalized decay functions in (76). A simple generalization of (D.7) yields

$$J_{ir}^{(2)}(\vec{X}^*) = \begin{cases} -(\mathbf{D}(\mathbf{I} - \mathbf{\Omega}))_{ir} & \text{generalization A} \\ -((\mathbf{I} - \mathbf{\Omega})\mathbf{D}(\mathbf{I} - \mathbf{\Omega}))_{ir} & \text{generalization B.} \end{cases} \quad (84)$$

Therefore, the full Jacobian for $\mathbf{\Omega} = \mathbf{\Omega}_1$ and $\vec{b} = \vec{1}$, becomes

$$J(\vec{X}^*) = f'(x^*)\mathbf{\Omega}_1 - \begin{cases} \mathbf{D}(\mathbf{I} - \mathbf{\Omega}_1) & \text{generalization A} \\ (\mathbf{I} - \mathbf{\Omega}_1)\mathbf{D}(\mathbf{I} - \mathbf{\Omega}_1) & \text{generalization B.} \end{cases} \quad (85)$$

We wish, now, to determine the spectrum $\Lambda(J(\vec{X}^*))$. We consider the two generalizations separately:

- Generalization B. As a consequence of the identities $\mathbf{\Omega}(\mathbf{I} - \mathbf{\Omega}) = (\mathbf{I} - \mathbf{\Omega})\mathbf{\Omega} = \mathbf{0}$, we can deduce $[J^{(1)}, J^{(2)}] = \mathbf{0} \forall \mathbf{D}$. Thus, both Jacobian components can be diagonalized in the same basis assembled in matrix \mathbf{T} , such that $\mathbf{T}\mathbf{\Omega}\mathbf{T}^{-1} = \mathbf{D}_1$ and $\mathbf{T}(\mathbf{I} - \mathbf{\Omega})\mathbf{D}(\mathbf{I} - \mathbf{\Omega})\mathbf{T}^{-1} = \mathbf{D}_2$. Then the eigenvalues are given by the elements of the diagonal matrix $\mathbf{D}_1 + \mathbf{D}_2$. Since it is not hard to see that $\text{Span}(\mathbf{\Omega}) = \text{Ker}((\mathbf{I} - \mathbf{\Omega})\mathbf{D}(\mathbf{I} - \mathbf{\Omega}))$ and $\text{Span}(\mathbf{\Omega}) \cup \text{Span}((\mathbf{I} - \mathbf{\Omega})\mathbf{D}(\mathbf{I} - \mathbf{\Omega})) = \mathbb{R}^N$ with $\text{Span}(\mathbf{\Omega}) \cap \text{Span}((\mathbf{I} - \mathbf{\Omega})\mathbf{D}(\mathbf{I} - \mathbf{\Omega})) = \emptyset$, we can focus on the eigenvalues of the two addends of \mathbf{J} . For $\mathbf{J}^{(1)}$, there are M eigenvalues equal to 0 and $N - M$ identical eigenvalues $\lambda = f'(x^*)$. For $\mathbf{J}^{(2B)} = (\mathbf{I} - \mathbf{\Omega})\mathbf{D}(\mathbf{I} - \mathbf{\Omega})$, there are $N - M$ null eigenvalues, while the remaining M eigenvalues satisfy $0 < \lambda \leq \max\{D_{ii}\}$. Clearly, M is determined by the cardinality of $\text{Span}(\mathbf{\Omega})$, equal to 1 for $\mathbf{\Omega}_1$.
- Generalization A. This case is slightly more complicated, since the Jacobian is not symmetric and, thus, its eigenvalues can be complex. We can provide some results exploiting Gerschgorin’s theorem

$$|\lambda - J_{ii}| \leq \sum_{k \neq i} |J_{ki}|. \quad (86)$$

Since $\mathbf{\Omega} = \mathbf{\Omega}_1$ and $\vec{b} = \vec{1}$, we have

$$J_{ii} = \frac{1}{N}f'(x^*) - \frac{N-1}{N}D_{ii} \quad \sum_{k \neq i} |J_{ki}| = \frac{N-1}{N}|f'(x^*) + D_{ii}| = R_i$$

Let $\bar{D} = \max_i |D_{ii}|$ and $\bar{R} = \frac{N-1}{N}|f'(x^*)|$. It follows that the eigenvalues $\lambda \in \Lambda(\mathbf{J})$ must be enclosed, in the complex plane, in circles of radius \bar{R} and center $z_i = \frac{1}{N}f'(x^*) - \frac{N-1}{N}D_{ii}$.

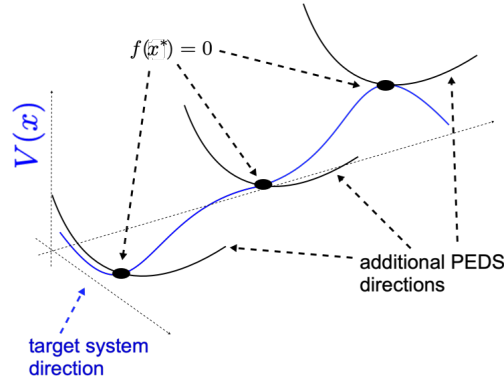


Figure 5: Interpreting the N dimensional PEDS dynamical system fixed points, from the point of view of the potential in Fig. 4 (blue curve). Notice that while the PEDS procedure preserves the extrema of potential $V(x)$, the embedded dynamical system cannot, in general, be expressed as the gradient of a generalized potential.

4.4.2. General case: Vector target system

The analysis of the PEDS embedding for scalar target systems carried out in the previous section showed that exploiting the uniform mean field projector and the standard or generalized decay functions, the embedding Jacobian can be fully characterized on the basis of the features of the target system fixed points (stable, unstable, or non-hyperbolic or saddle). We have obtained this result both for the standard commutative map, in which essentially one “mixes” linearly the dynamical systems, and in the case of the standard non-commuting map, in which the mix is non-trivial. The difference between the two is, thus, essentially contained in the embedding intermediate dynamics.

While in higher dimensions the nature of the questions to be answered is quite similar, the derivations are technically more challenging. The reason lies in the ordering, that, as mentioned earlier, does play a role in how the PEDS is defined. However, the case $\mathbf{\Omega} = \mathbf{\Omega}_1$ still makes it possible to carry out an almost entirely analytical derivation, even if at least some results can be proved for a more general projector structure.

Let us focus on the following PEDS for an m -dimensional target system

$$\mathcal{O} = \{\{f_1(\vec{x}), \dots, f_m(\vec{x})\}, \mathbf{\Omega}_1, -\{\mathcal{Q}_1(\mathbf{\Omega}_1)\vec{X}, \dots, \mathcal{Q}_m(\mathbf{\Omega}_1)\vec{X}\}, \vec{1}, N\}.$$

Therefore we consider an extended system as in (22) characterized by the mean field projector and any eligible decay functions.

Similarly to the scalar case, we also consider here the two generalizations \mathcal{Q} to the standard decay functions defined in (60). The corresponding PEDS take the form $\mathcal{O}_A = (\{f_i(x)\}, \mathbf{\Omega}_1, \{-\mathbf{D}(\mathbf{I} - \mathbf{\Omega}_1)\vec{X}_i\}, S, \vec{1}, N)$ and $\mathcal{O}_B = (\{f_i(x)\}, \mathbf{\Omega}_1, \{-(\mathbf{I} - \mathbf{\Omega}_1)\mathbf{D}(\mathbf{I} - \mathbf{\Omega}_1)\vec{X}_i\}, S, \vec{1}, N)$, which in terms of PEDS equations, read

$$\frac{d\vec{X}_i}{dt} = \mathbf{\Omega}_1 \mathbf{F}_i(\vec{X}_1, \dots, \vec{X}_m) \vec{1} - \mathcal{Q}_i(\mathbf{\Omega}_1) \vec{X}_i \quad (87)$$

where

$$\mathcal{Q}_i(\mathbf{\Omega}_1) = \begin{cases} \mathbf{D}_i(\mathbf{I} - \mathbf{\Omega}_1) & \text{generalization A} \\ (\mathbf{I} - \mathbf{\Omega}_1)\mathbf{D}_i(\mathbf{I} - \mathbf{\Omega}_1) & \text{generalization B} \end{cases} \quad (88)$$

being \mathbf{D}_i positive, diagonal matrices that make the corresponding decay function $\mathbf{\Omega}_1$ -eligible.

For any eligible decay function, we can obtain the vector target system Jacobian for the PEDS system as follows. Let us consider first the standard commuting map. We use the representation in (39), so that

$$F_{i,k}^{(c)}(\vec{X}_1, \dots, \vec{X}_m) = f_i(X_{1,k}, \dots, X_{m,k})b_k = f_i(X_{1,k}, \dots, X_{m,k}) \quad (89)$$

as $\vec{b} = \vec{1}$. We evaluate the Jacobian in blocks, starting from component 1 as in the scalar case

$$(\mathbf{J}_{ij}^{(c,1)}(\{\vec{X}_i\}))_{ks} = \sum_{r=1}^N \mathbf{\Omega}_{1,kr} \frac{\partial f_i(X_{1,r}, \dots, X_{m,r})}{\partial X_{j,s}} = \mathbf{\Omega}_{1,ks} \frac{\partial f_i(X_{1,s}, \dots, X_{m,s})}{\partial X_{j,s}} \quad (90)$$

thus, in the fixed point $\vec{X}_i^* = x_i^* \vec{1}$ (see the multivariate banality of the mean Corollary 4.8) we find

$$\mathbf{J}_{ij}^{(c)}(\{\vec{X}_i^*\}) = \mathbf{\Omega}_1 f'_{i,x_j}(\vec{x}^*) + \delta_{ij} \mathbf{Q}_i(\mathbf{\Omega}_1). \quad (91)$$

where $f'_{i,x_j} = \partial f_i(\vec{x}) / \partial x_j$. In other words, the Jacobian in the equilibria can be built as

$$\begin{pmatrix} f'_{1,x_1}(\vec{x}^*)\mathbf{\Omega}_1 + \mathbf{Q}_1(\mathbf{\Omega}_1) & f'_{1,x_2}(\vec{x}^*)\mathbf{\Omega}_1 & \cdots & f'_{1,x_n}(\vec{x}^*)\mathbf{\Omega}_1 \\ f'_{2,x_1}(\vec{x}^*)\mathbf{\Omega}_1 & f'_{2,x_2}(\vec{x}^*)\mathbf{\Omega}_1 + \mathbf{Q}_2(\mathbf{\Omega}_1) & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ f'_{m,x_1}(\vec{x}^*)\mathbf{\Omega}_1 & \cdots & \cdots & f'_{m,x_m}(\vec{x}^*)\mathbf{\Omega}_1 + \mathbf{Q}_m(\mathbf{\Omega}_1) \end{pmatrix} \quad (92)$$

where each block is of size $N \times N$.

Surprisingly, the same result holds also for the mixed and standard non-commutative maps. We start by proving this in the mixed commuting case, where ordering is immaterial.

Proposition 4.11. *Let $\mathcal{O} = (\{f_1(\vec{x}), \dots, f_m(\vec{x})\}, \mathbf{\Omega}_1, \{\mathbf{Q}_i(\mathbf{\Omega}_1)\vec{X}_i\}, S, \vec{1}, N)$ be the PEDS built on a mixed commutative map, considering the decay functions $\mathbf{Q}_i(\mathbf{\Omega}_1)\vec{X}_i$ assumed to be $\mathbf{\Omega}_1$ -eligible. Then, for any ordering S , the Jacobian matrix, evaluated at the equilibrium $\vec{X}_i^* = x_i^* \vec{1}$ being \vec{x} an equilibrium of the target system, is given by*

$$\mathbf{J}_{ij}^{(mc)}(\{\vec{X}_i^*\}) = \mathbf{\Omega}_1 f'_{i,x_j}(\vec{x}^*) + \delta_{ij} \mathbf{Q}_i(\mathbf{\Omega}_1). \quad (93)$$

Proof. See App. Appendix F. □

Finally, let us turn to the non-commutative map Jacobian, for which we have the following result.

Proposition 4.12. *Let $\mathcal{O} = (\{f_1(\vec{x}), \dots, f_m(\vec{x})\}, \mathbf{\Omega}_1, \{\mathbf{Q}_i(\mathbf{\Omega}_1)\vec{X}_i\}, S, \vec{1}, N)$ be the PEDS built on a non-commutative map, considering the decay functions $\mathbf{Q}_i(\mathbf{\Omega}_1)\vec{X}_i$ assumed to be $\mathbf{\Omega}_1$ -eligible. Then, for any ordering S , the Jacobian matrix, evaluated at the equilibrium $\vec{X}_i^* = x_i^* \vec{1}$, is given by*

$$\mathbf{J}_{ij}^{(nc)}(\{\vec{X}_i^*\}) = \mathbf{\Omega}_1 f'_{i,x_j}(\vec{x}^*) + \delta_{ij} \mathbf{Q}_i(\mathbf{\Omega}_1). \quad (94)$$

Proof. See App. Appendix G. □

We are now ready to discuss the Jacobian spectral properties irrespective of the chosen map, as the matrix is the same for all of the three maps that we consider. For the sake of simplicity, we limit the discussion to the standard decay functions $\mathcal{Q}_i(\mathbf{\Omega}_1) = -\alpha_i(\mathbf{I} - \mathbf{\Omega}_1)$, so that

$$\mathbf{J}(\{\vec{X}_i^*\}) = \begin{pmatrix} f'_{1,x_1}(\vec{x}^*)\mathbf{\Omega}_1 & f'_{1,x_2}(\vec{x}^*)\mathbf{\Omega}_1 & \cdots & \cdots & f'_{1,x_n}(\vec{x}^*)\mathbf{\Omega}_1 \\ f'_{2,x_1}(\vec{x}^*)\mathbf{\Omega}_1 & f'_{2,x_2}(\vec{x}^*)\mathbf{\Omega}_1 & f'_{2,x_3}(\vec{x}^*)\mathbf{\Omega}_1 & \cdots & f'_{2,x_n}(\vec{x}^*)\mathbf{\Omega}_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f'_{n,x_1}(\vec{x}^*)\mathbf{\Omega}_1 & \cdots & \cdots & \cdots & f'_{n,x_n}(\vec{x}^*)\mathbf{\Omega}_1 \\ \alpha_1(\mathbf{I} - \mathbf{\Omega}_1) & 0 & \cdots & \cdots & 0 \\ 0 & \alpha_2(\mathbf{I} - \mathbf{\Omega}_1) & \cdots & \vdots & \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & \cdots & 0 & \alpha_m(\mathbf{I} - \mathbf{\Omega}_1) \end{pmatrix} \quad (95)$$

that can be cast in the following form

$$\mathbf{J}(\{\vec{X}_i^*\}) = (\mathbf{J}_m(\vec{x}^*) + \mathbf{D}_\alpha^1) \otimes \mathbf{\Omega}_1 - \mathbf{D}_\alpha^N \quad (96)$$

where $\mathbf{J}_m(\vec{x}^*)$ is the Jacobian of the target system functions $\{f_i(\vec{x})\}$ evaluated at the target system equilibrium \vec{x}^* , \otimes denotes matrix Kronecker product², and

$$\mathbf{D}_\alpha^k = \text{diag}(\underbrace{\alpha_1, \dots, \alpha_1}_{k \text{ times}}, \underbrace{\alpha_2, \dots, \alpha_2}_{k \text{ times}}, \dots, \underbrace{\alpha_m, \dots, \alpha_m}_{k \text{ times}}).$$

The Jacobian (96) is a generalization of the scalar case (see Proposition 4.9). We are interested in assessing the properties of its eigenvalues.

For the time being, we discuss the simpler case $\alpha_i \equiv \alpha$, and since it will be useful later, let us think of this Jacobian for a general $\mathbf{\Omega}$, only to then consider $\mathbf{\Omega}_1$ as a special case.³ Let us therefore discuss the spectrum of

$$\mathbf{J}(\{\vec{X}_i^*\}) = (\mathbf{J}_m(\vec{x}^*) + \alpha \mathbf{I}_m) \otimes \mathbf{\Omega} - \alpha \mathbf{I}_{Nm}. \quad (97)$$

where \mathbf{I}_q is the identity matrix of size $q \times q$. Being λ_i the eigenvalues of $\mathbf{J}_m(\vec{x}^*)$, the eigenvalues of $\mathbf{J}_m(\vec{x}^*) + \alpha \mathbf{I}_m$ are given by $\lambda_i + \alpha$, and $-\alpha$ are the eigenvalues of $-\alpha \mathbf{I}_{Nm}$ for the whole matrix. Let us assume that $\mathbf{\Omega}$ has k unitary eigenvalues ($k = 1$ for $\mathbf{\Omega} = \mathbf{\Omega}_1$), while the remaining $N - k$ eigenvalues are equal to 0. Then matrix $\mathbf{J}(\{\vec{X}_i^*\})$ has eigenvalues [28]

- λ_i , $1 \leq i \leq m$ with multiplicity k
- $-\alpha$ with multiplicity $m(N - k)$.

²According to the definition, the i, j block of the matrix Kronecker product $A \otimes B$ is $a_{ij}B$.

³To motivate this generalized discussion, we briefly anticipate the result of an upcoming paper, in which we show that (97) is in fact the first term of the representation obtained for the Jacobian of a general projector $\mathbf{\Omega}$. This general case is, however, beyond the scope of this paper.

As a consequence, if $\lambda_i < 0 \forall i$, then a stable equilibrium point for the target system is still stable in the PEDS embedding. Similarly, if the equilibrium point is unstable, or if at least some i values exist for which $\lambda_i > 0$, then it becomes a saddle point for the extended system, being characterized by $m(N - k)$ negative eigenvalues and mk positive eigenvalues. Thus, the following classification holds

$$\{\vec{X}_i^*\} = \begin{cases} \text{stable} & \text{if } \vec{x}^* \text{ is stable,} \\ \text{saddle point} & \text{if } \vec{x}^* \text{ is a saddle point,} \\ \text{saddle point} & \text{if } \vec{x}^* \text{ is unstable.} \end{cases} \quad (98)$$

This analysis suggests that the presence of “barriers” in the target system, characterized by unstable equilibria, can (in principle) be overcome in the PEDS embedding via their transformation into saddle points in the extended system.

4.5. Dynamical ordering-equivalence for the uniform mean field projector

The uniform mean field projector has various properties that are interesting *per se*. In particular, we wish to show here that not only the fixed points, but also the embedding dynamics is ordering independent.

For this purpose, consider a PEDS of the form

$$O_r = (\{f_i(x_1, \dots, x_m)\}, \mathbf{\Omega}_1, \{\mathbf{G}_i(\mathbf{\Omega}_1)\vec{X}_i\}, S_r, \vec{1}, N),$$

and for arbitrary $\mathbf{\Omega}_1$ -eligible decay functions. Given the PEDS above, the standard non-commutative matrix embedding is given by

$$\vec{F}_s = \sum_{k=0}^{\infty} \sum_{i_1, \dots, i_j}^k a_{s,k;i_1 \dots i_m} \{(\mathbf{\Omega}_1 \mathbf{X}_1)^{i_1} \dots (\mathbf{\Omega}_1 \mathbf{X}_m)^{i_m}\}_{S_r} \vec{1} \quad (99)$$

We prove the following

Proposition 4.13. *The quantity*

$$\{(\mathbf{\Omega}_1 \mathbf{X}_1)^{i_1} \dots (\mathbf{\Omega}_1 \mathbf{X}_m)^{i_m}\}_{S_r} \vec{1}$$

is independent of the ordering S_r , for any i_1, \dots, i_m .

Proof. See App. Appendix H. □

Proposition 4.13 is important because it implies the following

Corollary 4.14. Dynamical ordering independence for the uniform mean field projector. *For any analytic functions f_i , we have*

$$\begin{aligned} O_r &= (\{f_i(x_1, \dots, x_m)\}, \mathbf{\Omega}_1, \{\mathbf{Q}_i(\mathbf{\Omega}_1)\vec{X}_i\}, S_r, \vec{1}, N) \\ &= (\{f_i(x_1, \dots, x_m)\}, \mathbf{\Omega}_1, \{\mathbf{Q}_i(\mathbf{\Omega}_1)\vec{X}_i\}, \vec{1}, N), \end{aligned} \quad (100)$$

or, alternatively, the uniform mean field PEDS are ordering independent.

Proof. The proof follows directly from the fact that any analytic function $f_i(x_1, \dots, x_m)$ can be written in the form of a series expansion as in Proposition 4.13. □

This implies essentially that for any dynamical system, we can write the dynamics with the most convenient ordering, without affecting the dynamics.

We would like to stress here that analyticity is a technical requirement, due to the fact that PEDS is easily defined in terms of monomials, e.g. $x^k \rightarrow (\mathbf{\Omega X})^k$. However, we could in principle define it for non-analytic functions as well. In principle we could define PEDS for the following ODE:

$$\dot{x} = ae^{-\frac{1}{x}}. \quad (101)$$

Above, the function on the right hand side is not analytical, but we can define the PEDS formally:

$$\frac{d}{dt}\vec{X} = -\alpha(I - \mathbf{\Omega})\vec{X} + a \cdot \exp(-(\mathbf{\Omega X})^{-1})\vec{b}, \quad (102)$$

where the exponential and the inverses above are intended as matrix exponential and matrix inverses. We will use these expressions explicitly in the following.

5. Examples of application

We now provide several examples to show applications of the theory developed in this paper, for the case of the uniform mean field projector.

5.1. Uniform mean field projector

Non-commutative example. In Section 2.2 we anticipated the results of the embedding of a one dimensional system with potential as in Fig. 2 (left) embedded exploiting the uniform mean field projector. The results were shown in Fig. 2 (right). However, while the fixed points do not depend on the map, the intermediate dynamics does. For this reason, we compare the standard commutative (47) and non-commutative (48) maps, for the potential (10). The difference is shown in Fig. 6 for identical initial conditions. As we can see, the two dynamics reach two different fixed points, showing that indeed the basins of attraction can depend on the type of dynamics used in the embedding.

Vector target system. As an example of vector target system, we consider a two-dimensional dynamical system embedded with the standard non-commutative map. The target system is:

$$\frac{dx}{dt} = -\frac{\partial V(x, y)}{\partial x} \quad (103)$$

$$\frac{dy}{dt} = -\frac{\partial V(x, y)}{\partial y} \quad (104)$$

where

$$V(x, y) = \exp\left(\frac{x^2}{2} - \frac{y^2}{2} + \frac{y^4}{4}\right) \quad (105)$$

which is characterized by two local minima, $(x^* = 0, y^* = \pm 1)$. The equations of motion define the gradient descent dynamics

$$\frac{dx}{dt} = f_x(x, y) = -xV(x, y) \quad (106)$$

$$\frac{dy}{dt} = f_y(x, y) = -(y - y^3)V(x, y). \quad (107)$$

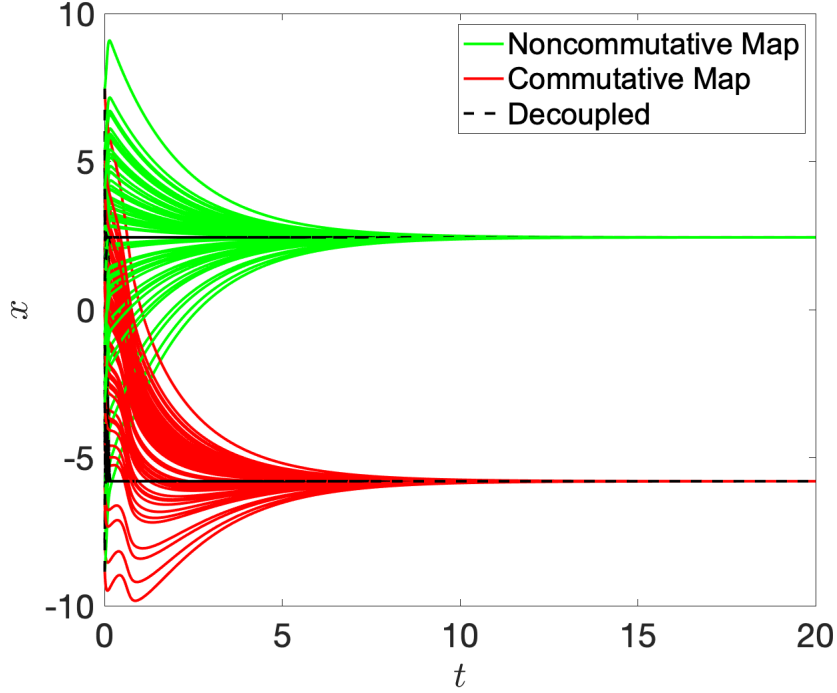


Figure 6: Comparison between the standard commutative and non-commutative maps for the 1D potential (10) and the target system.

The interest in this examples lies in the fact that the PEDS equations of motion depend on the ordering prescription considered. We discuss here the two cases defined below:

$$S_1 \rightarrow \begin{cases} \mathbf{F}_x^{(1)}(\vec{X}, \vec{Y}) &= -\boldsymbol{\Omega}_1 \mathbf{X} V_1(\mathbf{X}, \mathbf{Y}) \\ \mathbf{F}_y^{(1)}(\vec{X}, \vec{Y}) &= -(\boldsymbol{\Omega}_1 \mathbf{Y})(\mathbf{I} - (\boldsymbol{\Omega}_1 \mathbf{Y})^2) V_1(\mathbf{X}, \mathbf{Y}) \end{cases} \quad (108)$$

$$S_2 \rightarrow \begin{cases} \mathbf{F}_x^{(2)}(\vec{X}, \vec{Y}) &= -\boldsymbol{\Omega}_1 \mathbf{X} V_2(\mathbf{X}, \mathbf{Y}) \\ \mathbf{F}_y^{(2)}(\vec{X}, \vec{Y}) &= -(\boldsymbol{\Omega}_1 \mathbf{Y})(\mathbf{I} - (\boldsymbol{\Omega}_1 \mathbf{Y})^2) V_2(\mathbf{X}, \mathbf{Y}) \end{cases} \quad (109)$$

where

$$V_1(\mathbf{X}, \mathbf{Y}) = \exp\left(\frac{1}{2}(\boldsymbol{\Omega}_1 \mathbf{X})^2 + (\boldsymbol{\Omega}_1 \mathbf{Y})^2 \left(\frac{1}{2}\mathbf{I} - \frac{1}{4}(\boldsymbol{\Omega}_1 \mathbf{Y})^2\right)\right) \quad (110)$$

$$V_2(\mathbf{X}, \mathbf{Y}) = \exp\left(\frac{1}{2}(\boldsymbol{\Omega}_1 \mathbf{X})^2\right) \exp\left((\boldsymbol{\Omega}_1 \mathbf{Y})^2 \left(\frac{1}{2}\mathbf{I} - \frac{1}{4}(\boldsymbol{\Omega}_1 \mathbf{Y})^2\right)\right) \quad (111)$$

where $V_1 \neq V_2$ since $\exp(\boldsymbol{\Omega}_1 \mathbf{X} \boldsymbol{\Omega}_1 \mathbf{Y}) \neq \exp(\boldsymbol{\Omega}_1 \mathbf{X}) \exp(\boldsymbol{\Omega}_1 \mathbf{Y})$. Thus, choosing one versus the other is equivalent to a different ordering choice.

We embed this system of equations via $\mathcal{O}_1 = (\{f_x, f_y\}, \boldsymbol{\Omega}_1, \{-\alpha(\mathbf{I} - \boldsymbol{\Omega}_1)\vec{X}, -\alpha(\mathbf{I} - \boldsymbol{\Omega}_1)\vec{Y}\}, S_1, \vec{1}, N)$ and $\mathcal{O}_2 = (\{f_x, f_y\}, \boldsymbol{\Omega}_1, \{-\alpha(\mathbf{I} - \boldsymbol{\Omega}_1)\vec{X}, -\alpha(\mathbf{I} - \boldsymbol{\Omega}_1)\vec{Y}\}, S_2, \vec{1}, N)$, obtaining

$$\frac{d\vec{X}}{dt} = \boldsymbol{\Omega}_1 \mathbf{F}_x^{(i)} \vec{1} - \alpha(\mathbf{I} - \boldsymbol{\Omega}_1) \vec{X}, \quad (112)$$

$$\frac{d\vec{Y}}{dt} = \boldsymbol{\Omega}_1 \mathbf{F}_y^{(i)} \vec{1} - \alpha(\mathbf{I} - \boldsymbol{\Omega}_1) \vec{Y}, \quad (113)$$

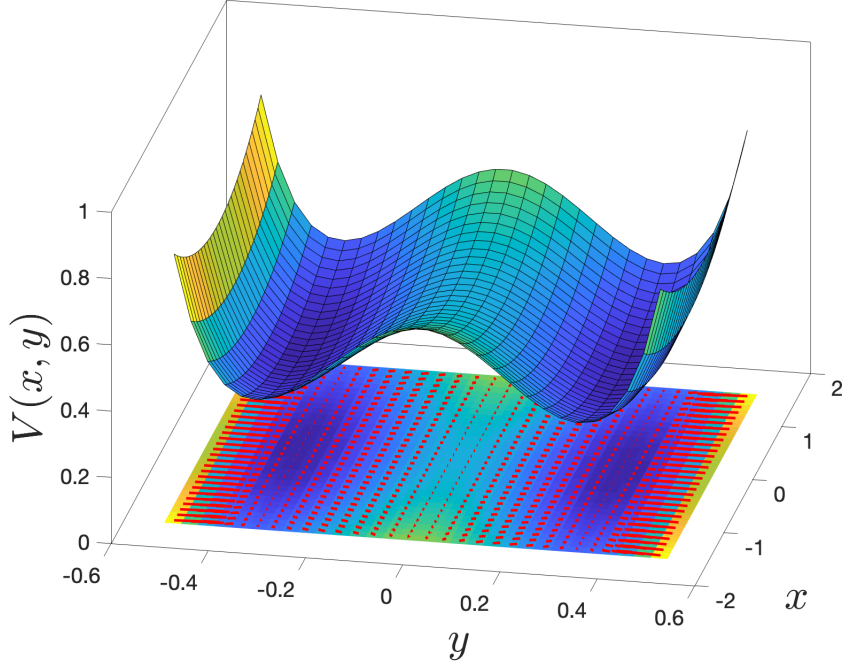


Figure 7: The potential $V(x, y)$ in (105), the projection with the two minima.

where $i = 1, 2$ is the label for S_1, S_2 . The numerical solutions are shown in Fig. 8 using $\alpha = 0.1$ and $N = 50$. The two dynamic behaviours are essentially identical, as per Corollary 4.14. Interestingly, even if in general $[\mathbf{\Omega}_1 \mathbf{X}, \mathbf{\Omega}_1 \mathbf{Y}] \neq 0$, for this example we can work out the full details leading to the independence on the ordering.

First, let us note that $\mathbf{\Omega}_1 \mathbf{X} \mathbf{\Omega}_1 = \langle X \rangle \mathbf{\Omega}_1$. Using this expression, we can write

$$f(\mathbf{\Omega}_1 \mathbf{X}) = \mathbf{I} + \frac{f(\langle X \rangle) - 1}{\langle X \rangle} \mathbf{\Omega}_1 \mathbf{X}$$

therefore

$$\begin{aligned} \exp(f(\mathbf{\Omega}_1 \mathbf{X}) + g(\mathbf{\Omega}_1 \mathbf{Y})) &= \exp\left(2\mathbf{I} + \frac{f(\langle X \rangle) - 1}{\langle X \rangle} \mathbf{\Omega}_1 \mathbf{X} + \frac{g(\langle Y \rangle) - 1}{\langle Y \rangle} \mathbf{\Omega}_1 \mathbf{Y}\right) \\ &= \exp(2) \exp\left[\mathbf{\Omega}_1 \left(\frac{f(\langle X \rangle) - 1}{\langle X \rangle} \mathbf{X} + \frac{g(\langle Y \rangle) - 1}{\langle Y \rangle} \mathbf{Y}\right)\right] \end{aligned} \quad (114)$$

We can now apply again the same formula

$$\begin{aligned} &\exp\left[\mathbf{\Omega}_1 \left(\frac{f(\langle X \rangle) - 1}{\langle X \rangle} \mathbf{X} + \frac{g(\langle Y \rangle) - 1}{\langle Y \rangle} \mathbf{Y}\right)\right] \\ &= \exp\left[\mathbf{I} + \frac{f(\langle X \rangle) + g(\langle Y \rangle) - 2 - 1}{f(\langle X \rangle) + g(\langle Y \rangle) - 2} \mathbf{\Omega}_1 \left(\frac{f(\langle X \rangle) - 1}{\langle X \rangle} \mathbf{X} + \frac{g(\langle Y \rangle) - 1}{\langle Y \rangle} \mathbf{Y}\right)\right] \end{aligned} \quad (115)$$

yielding, after taking into account the multiplication times $\vec{\mathbf{1}}$

$$\begin{aligned} &\exp(2) \exp\left[\mathbf{I} + \frac{f(\langle X \rangle) + g(\langle Y \rangle) - 2 - 1}{f(\langle X \rangle) + g(\langle Y \rangle) - 2} \mathbf{\Omega}_1 \left(\frac{f(\langle X \rangle) - 1}{\langle X \rangle} \mathbf{X} + \frac{g(\langle Y \rangle) - 1}{\langle Y \rangle} \mathbf{Y}\right)\right] \vec{\mathbf{1}} \\ &= \exp(f(\langle X \rangle) + g(\langle Y \rangle)) \vec{\mathbf{1}}. \end{aligned} \quad (116)$$

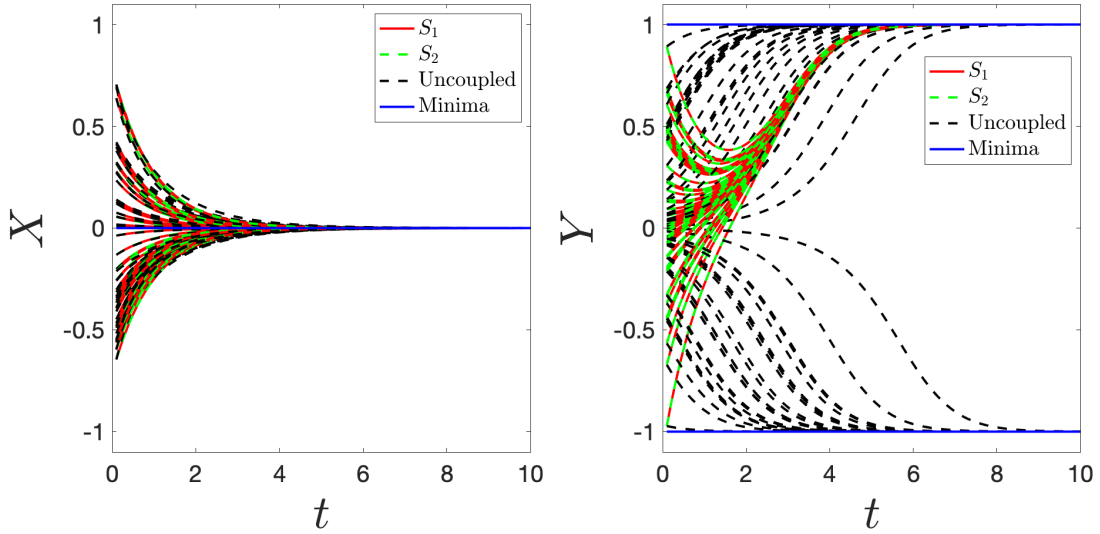


Figure 8: The two PEDS dynamics \mathcal{O}_1 and \mathcal{O}_2 for the x and y variables (left and right) for $N = 50$ and integrated using an Euler scheme with step size $dt = 0.1$. We consider here the noncommutative map. We can see that the dynamics of S_1 and S_2 are identical. Note that both S_1 , S_2 and the uncoupled systems have the identical initial conditions.

In other words, we have shown that

$$\exp(f(\mathbf{\Omega}_1 \mathbf{X}) + g(\mathbf{\Omega}_1 \mathbf{Y})) \vec{\mathbb{1}} = \exp(f(\mathbf{\Omega}_1 \mathbf{X})) \exp(g(\mathbf{\Omega}_1 \mathbf{Y})) \vec{\mathbb{1}}, \quad (117)$$

i.e. the equivalence of the dynamics of S_1 and S_2 .

Hamiltonian equations with dissipation. As a third example, let us consider another two-dimensional vector target system: the description of a dissipative Hamiltonian system for a single particle of mass m . The target system reads

$$\frac{dx}{dt} = \frac{p}{m} \quad (118)$$

$$\frac{dp}{dt} = -\frac{\partial V}{\partial x} - \chi \frac{p}{m} \quad (119)$$

where χ denotes the dissipation and we define the force $f(x, p) = -\partial V/\partial x$. Following the prescription of the previous sections, we write the PEDS

$$\mathcal{O} = (\{\frac{p}{m}, f(x, p) - \chi p/m\}, \mathbf{\Omega}_1, \{-\alpha_x(\mathbf{I} - \mathbf{\Omega}_1)\vec{X}, -\alpha_p(\mathbf{I} - \mathbf{\Omega}_1)\vec{P}\}, \vec{\mathbb{1}}, N).$$

The extended system equations are given by

$$\frac{d\vec{X}}{dt} = \frac{1}{m} \mathbf{\Omega}_1 \mathbf{P} \vec{\mathbb{1}} - \alpha_x (\mathbf{I} - \mathbf{\Omega}_1) \vec{X} \quad (120)$$

$$\frac{d\vec{P}}{dt} = \mathbf{\Omega}_1 \left(\mathbf{F}(\mathbf{X}) - \chi \frac{\mathbf{P}}{m} \right) \vec{\mathbb{1}} - \alpha_p (\mathbf{I} - \mathbf{\Omega}_1) \vec{P} \quad (121)$$

which is thus a $2N$ set of equations. Let us focus on the potential $V(x, p)$ defined in (10). The results are shown in Fig. 9.

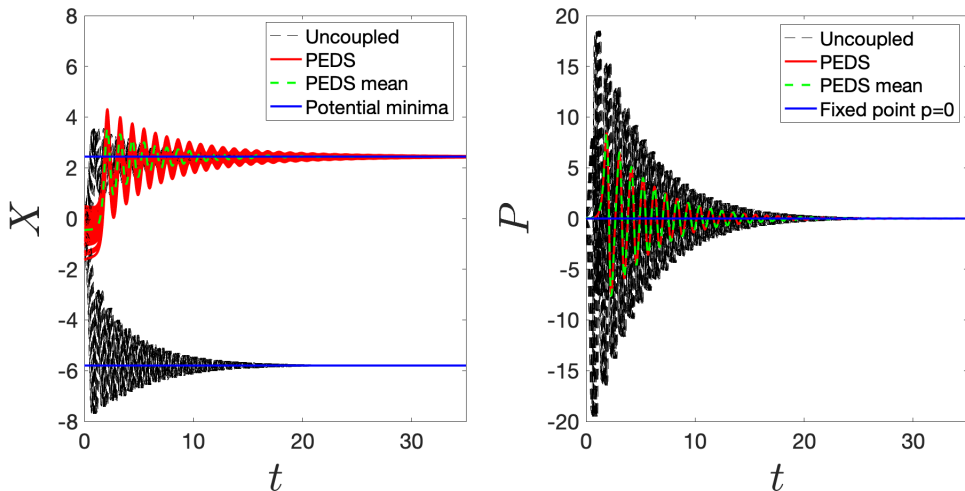


Figure 9: PEDS dynamics for the Hamiltonian system with one particle, integrated using an Euler scheme with step size $dt = 0.001$. The extended system has $N = 50$. Also in this case, the dynamics approaches the fixed points of the potential for the x variable, reaching a local minimum, while the p variables tends to the asymptotic value $p^* = 0$.

5.2. Beyond the uniform mean field projector

This paper is focused mostly on the uniform mean field projection $\mathbf{\Omega}_1$. Before concluding, we wish to numerically simulate also the case of a PEDS with a different projection operator. Let us consider a PEDS where $\mathbf{\Omega} = \mathbf{B}^t(\mathbf{B}\mathbf{B}^t)^{-1}\mathbf{B}$, where \mathbf{B} is a random square matrix (uniformly distributed on $[0, 1]$) of size $N \times K$. The scalar target dynamical system we are interested in is again

$$\frac{dx}{dt} = f(x) = -\frac{\partial V(x)}{\partial x}. \quad (122)$$

with the potential in (10), choosing parameters $a_4 = a_0 = 0$, $a_3 = -2$, $a_2 = -10$ and $a_1 = 9.85$ that guarantee a single potential minimum, as shown Fig. 10 (left). We then consider the PEDS $\mathcal{O} = (f(x), \mathbf{\Omega}, -\alpha(\mathbf{I} - \mathbf{\Omega})\vec{X}, \mathbf{\Omega}\vec{1}, N)$, with $N = 50$, and follow the observable $\tilde{x} = \mathcal{P}_{\mathbf{\Omega}}\vec{X}$. The results are shown in Fig. 10 (right). The PEDS embedding converges also in this case to the potential absolute minimum, thus confirming that a generalization of the results of this paper to arbitrary projectors is possible.

As a last comment, one of the main motivations for this study is that in circuits, conservation laws can be expressed in terms of projector operators. An example is the volatile but (almost) ideal memristor. A resistor with memory can be described, at the lowest level of approximation for a current controlled device, by an effective dynamical resistance depending on an internal parameter x . In this sense, memristors are approximately described by the functional form $R(x) = R_{\text{off}}(1 - x) + xR_{\text{on}}$, where $R_{\text{on}} < R_{\text{off}}$ are the boundary resistances, and $x \in [0, 1]$. We assume that the internal memory parameter x evolves according to a simple equation of the form $dx/dt = R_{\text{on}}I/\beta - \alpha x$. The parameters α and β are the decay constant and the effective activation voltage per unit time, respectively. For a recent paper which inspired this study, consider [27], where transitions between effective minima of a lower dimensional potential were observed. Using Ohm's law, we define voltage $V = R(x)I$, so as to obtain a normalized

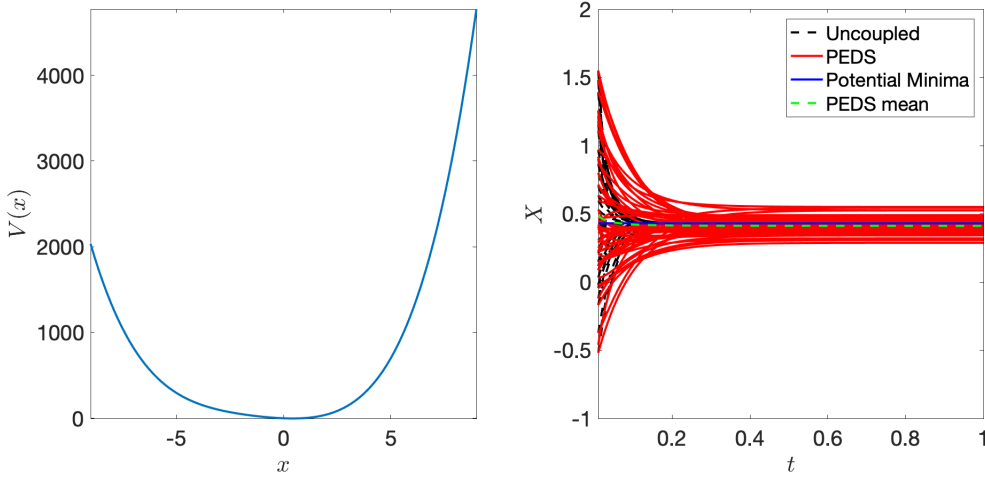


Figure 10: PEDS dynamics for the target system (122) with a random projection operator. We can see that the results we established for this paper in the case $\mathbf{\Omega}_1$ could be possibly extended to more general projectors.

equation for $x(t)$

$$\frac{dx}{dt} = \frac{V}{\beta} \frac{1}{1 - \chi x} - \alpha x = -\alpha \frac{\partial V(x, s)}{\partial x} \quad (123)$$

where $\chi = (R_{\text{off}} - R_{\text{on}})/R_{\text{off}}$ and $s = \frac{S}{\alpha\beta}$, with $0 \leq \chi \leq 1$ in the physically relevant cases, and $V(x, s)$ as an effective potential, where S is the voltage applied to the circuit, and s is a normalized quantity with units of inverse time.

The dynamics of a single memristor (123) is fully characterized by the gradient following the dynamics of the effective potential

$$V(x, s) = \frac{1}{2}x^2 + \frac{s}{\chi} \log(1 - \chi x), \quad (124)$$

with $s = \frac{S}{\alpha\beta}$; the constant α also acts as the learning rate in (123).

Let us now discuss in detail the case of the memristor equation (4). The matrix $\mathbf{\Omega}$ is the projection operator on the vector space of cycles of \mathcal{G} , the graph representing the circuit [23], and, as discussed in the Introduction, it is a mathematical consequence of Kirchhoff's conservation laws. Now, we note that we can write (4) as:

$$\frac{d}{dt}\vec{x} = \mathbf{\Omega} \left(\frac{1}{\beta} (\mathbf{I} - \chi \mathbf{\Omega} \mathbf{X})^{-1} \mathbf{\Omega} \vec{S} - \alpha \vec{x} \right) - \alpha (\mathbf{I} - \mathbf{\Omega}) \vec{x}, \quad (125)$$

which is exactly in the form of a PEDS, with a standard decay function. Thus, the results of [27] can be interpreted as the relaxation of the system towards the minima defined by the embedding function. If $\mathbf{\Omega} = \mathbf{\Omega}_1$, then using the results of this paper we know that the potential (124) determines the effective minima of the system. However, in order to justify the presence of the rumbling transitions shown in [27], a deeper understanding of the PEDS properties for a general projector $\mathbf{\Omega}$ is required.

6. Conclusions and perspective

In the present paper we presented and studied a map between dynamical systems of size m and dynamical systems in a higher number of variables. This is the first of a series of papers formally investigating the projective embeddings of dynamical systems (PEDS) paradigm that we defined here. The purpose of this work was to formally show the properties of this type of embeddings, within the context of a particular projector matrix. As we have seen, their structure is such that for long times, the asymptotic equilibria of the target dynamical system can be recovered.

We have discussed in particular the case of the uniform mean field projector operator $\Omega_{1,j} = \frac{1}{N}$. For this choice, we have been able to prove analytically that the asymptotic equilibria are strictly connected to those of the original system. Aside from establishing the formalism, this paper also established some exact results about how the embedding changes the properties of the dynamics critical, including the cases of unstable equilibria and saddle points.

Specifically, we have studied the embedding of m dimensional dynamical systems in Nm -dimensional systems. More precisely, we studied the weak embedding, e.g. that after projector fixed points can be recovered. The purpose of such embedding is to modify the nature of the fixed points of the dynamics, i.e. those satisfying $\{\vec{x}^* \text{ s.t. } d\vec{x}/dt|_{\vec{x}^*} = 0\}$. In particular, we have shown that stable and saddle type fixed points retain their properties, while unstable fixed points become saddles. This observation justify future works in this direction, in particular exploiting different types of decay functions, matrix embeddings and projectors with respect to this contribution. It is worth to mention that a follow up of this work is in preparation, in which we discuss the behavior of PEDS for general projectors; many of the results on the Jacobian obtain in this paper do actually apply also in the general case [29]. It is important to stress that the initial conditions still play an important role in the ability to reach fixed points. While our proofs focused on the spectral properties of the fixed points, it is unclear from this work how basins of attractions are modified, an aspect to be investigated in the future.

An important aspect of interest of future works will be to focus on how to further modify the spectral properties of the fixed points, i.e. the nature of the Jacobian once evaluated at \vec{x}^* . What we have shown in the present paper is that, for the uniform projector, the PEDS Jacobian is always symmetric, and thus characterized by real eigenvalues, that in particular are negative if the corresponding fixed point of the target system is stable. This implies that the dynamics near stable fixed points is always laminar, e.g. slowly decaying towards the fixed point. As we will see in future works, this is not the case for general projectors, for which approximate but special techniques will have to be employed.

As discussed, the spectral signature is in part inherited by the original, target dynamical system, but modified through the extended number of dimensions. The idea of generalizing the space of solutions to higher dimensions is not new. In a way, the PEDS technique is in spirit close to both Markov Chain Monte Carlo methods [30] and the notion of lifts in convex optimization [31], but is specifically developed for the fixed points of dynamical systems.

In particular, in [27] it was observed that memristive circuits have an effective lower dimensional representation in terms of an effective potential, and that they can exhibit

a “rumbling” transition, i.e. a transient chaotic tunneling between local minima of a properly defined potential. As it turns out, such dynamics is only a particular case of the PEDS introduced here, in which the projector operator was given by random circuit connections.

The rumbling transition in [27] was pinpointed numerically to be due to an effective “Lyapunov force”, shown to be present in connection with the rumbling transition phenomenon. Such force was defined essentially as a deviation from a mean field theory, and we provided evidence of an athermal and novel mechanism in which barrier escapes emerge in the effective description of a multi-particle system. This paper is a continuation of that work, attempting at generalizing those findings to general systems, although focusing specifically on a particular type of projector: in this case, these “Lyapunov” forces are not present. Similar yet different types of behavior were also observed previously within the context of memory-based computing (memcomputing) solutions [32, 33, 34, 35, 36, 37].

The main focus of this paper represents a first step towards a clarification of the general reasons why the introduction of hidden variables in a dynamical system can lead to transitions between local and global minima of the effective description via instabilities in the full system. Since maxima can be turned into saddle points, generically there cannot be no “barriers” when the target system is a gradient descent. However, as we will show formally in future works, in order to obtain the rumbling transitions, one has to go beyond the uniform mean field approximation and study a more general type of projector.

Clearly, the projective embedding studied in this paper can be employed in a variety of dynamical systems, including all sort of gradient-based dynamics, with applicability to machine learning and neural networks. These applications will also be the subject of future studies. In particular, we hope that the introduction of “hidden variables” in dynamical systems [38] can be further investigated for the purpose of machine learning and optimization applications [39]. In general, the study of transient chaos in dynamical systems and optimization [40, 41] is an interesting area of research with possible applications also in memristor-based algorithms [42].

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Appendix A. Proof of Prop. 4.5

Proof. The solution of (62) takes the form

$$\vec{X}_c(t) = e^{\mathbf{A}t} \vec{X}_c(0) \quad (\text{A.1})$$

where $\mathbf{A} = (\mathbf{I} - \mathbf{\Omega})\mathbf{D}$. Expanding the exponential, we get

$$\vec{X}_c(t) = \vec{X}_c(0) + \sum_{k=1}^{+\infty} \frac{t^k}{k!} ((\mathbf{I} - \mathbf{\Omega})\mathbf{D})^k \vec{X}_c(0)$$

that, projecting through $\mathbf{I} - \mathbf{\Omega}$, becomes

$$(\mathbf{I} - \mathbf{\Omega})\vec{X}_c(t) = (\mathbf{I} - \mathbf{\Omega})\vec{X}_c(0) + \sum_{k=1}^{+\infty} \frac{t^k}{k!} ((\mathbf{I} - \mathbf{\Omega})\mathbf{D})^k \vec{X}_c(0). \quad (\text{A.2})$$

We notice that if $\vec{X}_c(0) \in \text{Span}(\mathbf{I} - \mathbf{\Omega})$, then we can express $\vec{X}_c(0) = \sum_j a_j \vec{v}_j$ where \vec{v}_j are eigenvectors associated to the eigenvalue equal to 1 of $\mathbf{I} - \mathbf{\Omega}$. Thus, $(\mathbf{I} - \mathbf{\Omega})\vec{X}_c(0) = \sum_j a_j (\mathbf{I} - \mathbf{\Omega})\vec{v}_j = \sum_j a_j \vec{v}_j = \vec{X}_c(0)$. This implies that

$$(\mathbf{I} - \mathbf{\Omega})\vec{X}_c(t) = \vec{X}_c(0) + \sum_{k=1}^{+\infty} \frac{t^k}{k!} ((\mathbf{I} - \mathbf{\Omega})\mathbf{D})^k \vec{X}_c(0) = e^{(\mathbf{I} - \mathbf{\Omega})\mathbf{D}t} \vec{X}_c(0) = \vec{X}_c(t) \quad (\text{A.3})$$

Thus, we have $(\mathbf{I} - \mathbf{\Omega})\vec{X}_c(t) = \vec{X}_c(t)$, i.e. $\vec{X}_c(t) \in \text{Span}(\mathbf{I} - \mathbf{\Omega})$. \square

Appendix B. Proof of Prop. 4.6

Proof. We consider the dynamics for the modes $\vec{X}_c = (\mathbf{I} - \mathbf{\Omega}_1)\vec{X}(t)$ from (62), and we use a Lyapunov stability argument. Let us consider the following functional: $V(\vec{X}_c) = \vec{X}_c \cdot \vec{X}_c \geq 0$. Then,

$$\begin{aligned} \frac{d}{dt}V &= 2 \left(\frac{d}{dt} \vec{X}_c(t) \right) \cdot \vec{X}_c(t) \\ &= -2((\mathbf{I} - \mathbf{\Omega}_1)\mathbf{D} \vec{X}_c(t)) \cdot \vec{X}_c(t) \\ &= -2(\sqrt{\mathbf{D}} \vec{X}_c(t)) \cdot \sqrt{\mathbf{D}}(\mathbf{I} - \mathbf{\Omega}_1)^T \vec{X}_c(t) \\ &= -2(\sqrt{\mathbf{D}} \vec{X}_c(t)) \cdot \sqrt{\mathbf{D}}(\mathbf{I} - \mathbf{\Omega}_1) \vec{X}_c(t). \end{aligned} \quad (\text{B.1})$$

Using the fact that $\vec{X}_c(t) \in \text{Span}(\mathbf{I} - \mathbf{\Omega}_1)$ from the previous Proposition, we obtain that

$$\frac{d}{dt}V = -2\|\sqrt{\mathbf{D}}\vec{X}_c(t)\|^2 \leq 0. \quad (\text{B.2})$$

Since the only minimum of $V(\vec{X})$ is $\vec{X} = \vec{0}$, then $\vec{X}_c(t) \rightarrow \vec{0}$ for $t \rightarrow \infty$. This proves that $\vec{X}_c(0) \rightarrow \vec{0}$, and thus $\vec{X} \rightarrow \langle X \rangle \vec{1}$, for $t \rightarrow \infty$. \square

Appendix C. Proof of Corollary 4.8

Proof. Let us first prove statement (a), which is a corollary of the banality of the mean value Proposition 4.3. We set $\vec{X}_i(t) = \langle X_i(t) \rangle \vec{1}$. In all cases of the standard commutative, mixed commutative and non-commutative maps, the standard decay function is identically zero, and it is not hard to see that, for an arbitrary orderings S , we have in all cases the same expression for each term of the expansion:

$$\left. \begin{aligned} & \Omega_1(X_1)^{i_1} \dots (X_m)^{i_m} \vec{1} \\ & \left(\Omega_1(X_1)^{\frac{i_1}{\Sigma_k i_k}} \dots (X_m)^{\frac{i_m}{\Sigma_k i_k}} \right)^{\Sigma_k i_k} \vec{1} \\ & \{(\Omega_1 X_1)^{i_1} \dots (\Omega_1 X_m)^{i_m}\}_S \vec{1} \end{aligned} \right\} = \langle X_1(t) \rangle^{i_1} \dots \langle X_m(t) \rangle^{i_m} \Omega_1 \vec{1} \\ = \langle X_1(t) \rangle^{i_1} \dots \langle X_m(t) \rangle^{i_m} \vec{1} \quad (\text{C.1})$$

As a consequence, a relatively simple calculation shows

$$\frac{d}{dt} \vec{X}_i = \frac{d}{dt} \langle X_i(t) \rangle \vec{1} = f_i(\langle X_1(t) \rangle, \dots, \langle X_m(t) \rangle) \Omega_1 \vec{1} = f_i(\langle X_1(t) \rangle, \dots, \langle X_m(t) \rangle) \vec{1}, \quad (\text{C.2})$$

or

$$\left(\frac{d}{dt} \langle X_i(t) \rangle - f_i(\langle X_1(t) \rangle, \dots, \langle X_m(t) \rangle) \right) \vec{1} = 0. \quad (\text{C.3})$$

Replacing $\langle X_i(t) \rangle$ with $x_i(t)$, we obtain that the fixed points of the extended system under the assumption $\vec{X}_i(t) = \langle X_i(t) \rangle \vec{1}$ must be the same as for the target dynamical system.

We now turn to statement (b). The initial condition $\vec{X}_i(t=0)$ is now arbitrary. Multiplying on the left (73) times $(\mathbf{I} - \Omega_1)$, we get

$$(\mathbf{I} - \Omega_1) \frac{d\vec{X}_i}{dt} = -\alpha(\mathbf{I} - \Omega_1) \vec{X}_i \quad (\text{C.4})$$

where we define $\vec{X}_{i,c} = (\mathbf{I} - \Omega_1) \vec{X}_i$. As $\text{Span}(\mathbf{I} - \Omega_1) = \text{Ker}(\Omega_1)$, $\vec{X}_{i,c}$ can be interpreted as a deviation from the average, since $\vec{X}_i = \langle X_i \rangle \vec{1} + \vec{X}_{i,c}$. Following almost the same steps as in the proof of the convergence of the mean for the one dimensional system, we arrive at

$$\begin{aligned} & \frac{d\vec{X}_{i,c}}{dt} = -\alpha \vec{X}_{i,c} \implies \\ & (\langle X_i(t) \rangle - f_i(\langle X_1(t) \rangle, \dots, \langle X_m(t) \rangle)) \vec{1} \\ & = \left(\mathcal{P}_{\Omega_1} \vec{X}_i(t) - f_i(\mathcal{P}_{\Omega_1} \vec{X}_1(t), \dots, \mathcal{P}_{\Omega_1} \vec{X}_m(t)) \right) \vec{1} \rightarrow \vec{0}, \text{ for } t \gg \frac{1}{\alpha}. \end{aligned} \quad (\text{C.5})$$

where the second expression follows from the first being $\alpha > 0$, so that $\vec{X}_{i,c}(t) \rightarrow \vec{0}$ for $t \gg 1/\alpha$. Essentially, this implies that the extended system fixed points are those of the target dynamical system: for long enough times the system converges exponentially to the mean in each variable, for which the banality lemma applies. Thus, if the projected PEDS given by \mathcal{O} approaches a fixed point, it has to be a fixed point of the target system. Alternatively, the system must not converge. \square

Appendix D. Proof of Jacobian formula, Prop. 4.9

Proof. The Jacobian elements are defined as

$$J_{ir} = \frac{\partial M_i}{\partial X_r} = \frac{\partial}{\partial X_r} \sum_{j=1}^N (\mathbf{\Omega F})_{ij} b_j - \alpha \frac{\partial}{\partial X_r} ((\mathbf{I} - \mathbf{\Omega}) \vec{X})_i. \quad (\text{D.1})$$

As we need to evaluate the derivatives of the matrix maps, we consider first the derivatives of the matrix quantities depending on \mathbf{X} . We have

$$\begin{aligned} \frac{\partial}{\partial X_r} (\mathbf{\Omega X})_{ij}^m &= \sum_{k=0}^{m-1} \sum_{s=1}^N (\mathbf{\Omega X})_{is}^k \sum_{l,t=1}^N \frac{\partial (\mathbf{\Omega}_{sl} X_l \delta_{lt})}{\partial X_r} (\mathbf{\Omega X})_{tj}^{m-1-k} \\ &= \sum_{k=0}^{m-1} \sum_{s=1}^N (\mathbf{\Omega X})_{is}^k \mathbf{\Omega}_{sr} (\mathbf{\Omega X})_{rj}^{m-1-k}, \end{aligned} \quad (\text{D.2})$$

where a matrix to zero power coincides with the identity matrix.

Taking into account definition (38) we find

$$\begin{aligned} \frac{\partial F_{ij}^{(nc)}}{\partial X_r} &= \sum_{l=1}^N \mathbf{\Omega}_{il} \frac{\partial}{\partial X_r} \sum_{z=1}^{\infty} a_z (\mathbf{\Omega X})_{lj}^z \\ &= \sum_{l=1}^N \mathbf{\Omega}_{il} \sum_{z=1}^{\infty} a_z \sum_{k=0}^{z-1} \sum_{s=1}^N (\mathbf{\Omega X})_{ls}^k \mathbf{\Omega}_{sr} (\mathbf{\Omega X})_{rj}^{z-1-k} \end{aligned} \quad (\text{D.3})$$

Thus, the first term of the Jacobian is simply given by

$$J_{ir}^{(nc,1)} = \sum_{j=1}^N \sum_{l=1}^N \mathbf{\Omega}_{il} \sum_{z=1}^{\infty} a_z \sum_{k=0}^{z-1} \sum_{s=1}^N (\mathbf{\Omega X})_{ls}^k \mathbf{\Omega}_{sr} (\mathbf{\Omega X})_{rj}^{z-1-k} b_j. \quad (\text{D.4})$$

In the case of the standard commutative map, the result can be easily derived exploiting (39) if function $f(x)$ is known in closed form. On the other hand, making use of the power expansion of the function, we can directly calculate the derivatives noticing that

$$\frac{\partial}{\partial X_r} X_{ij}^k = \frac{\partial}{\partial X_r} (\text{diag}(X_1^k, \dots, X_m^k))_{ij} = (\text{diag}(0, \dots, k X_r^{k-1}, \dots, 0))_{ij} = k \delta_{ij} \delta_{ir} X_r^{k-1} \quad (\text{D.5})$$

so that

$$\begin{aligned} J_{ir}^{(c,1)} &= \sum_{j=1}^N \frac{\partial (\mathbf{\Omega F}^{(c)})_{ij}}{\partial X_r} b_j \\ &= \sum_{j=1}^N \sum_{l=1}^N \mathbf{\Omega}_{il} \sum_{z=0}^{\infty} a_z \frac{\partial X_{lj}^z}{\partial X_r} b_j \\ &= \sum_{j=1}^N \sum_{l=1}^N \mathbf{\Omega}_{il} \sum_{z=1}^{\infty} z a_z \delta_{lj} \delta_{lr} X_r^{z-1} b_j \\ &= \mathbf{\Omega}_{ir} f'(X_r) b_r \end{aligned} \quad (\text{D.6})$$

where $f'(x)$ denotes the derivative of $f(x)$.

Similarly, for the standard decay function $\mathbf{D} = \alpha \mathbf{I}$, it is not hard to see that the second term of the Jacobian is the same irrespective of the chosen matrix embedding

$$\begin{aligned} J_{ir}^{(2)} &= -\alpha \sum_{j=1}^N \frac{\partial}{\partial x_r} \left((\mathbf{I} - \mathbf{\Omega}) \mathbf{X} \right)_{ij} \\ &= -\alpha \sum_{j=1}^N (\mathbf{I} - \mathbf{\Omega})_{ir} \delta_{rj} = -\alpha (\mathbf{I} - \mathbf{\Omega})_{ir} \end{aligned} \quad (\text{D.7})$$

Summing $\mathbf{J}^{(1)}$ and $\mathbf{J}^{(2)}$, we find the expression to be proven. \square

Appendix E. Proof of Corollary 4.10

Proof. We exploit Proposition 4.9, considering $\mathbf{\Omega} = \mathbf{\Omega}_1$ and $\vec{\mathbf{b}} = \vec{\mathbf{1}}$, from which, because of Proposition 4.3, we have $\mathbf{X}^* = x^* \mathbf{I}$. Substituting into (D.3), we obtain

$$\begin{aligned} J_{ir}^{(nc,1)}(\vec{\mathbf{X}}^*) &= \sum_{j=1}^N \sum_{l=1}^N \mathbf{\Omega}_{1,il} \sum_{z=1}^{\infty} a_z(x^*)^{z-1} \sum_{k=0}^{z-1} \sum_{s=1}^N \mathbf{\Omega}_{1,ls}^k \mathbf{\Omega}_{1,sr} \mathbf{\Omega}_{1,rj}^{z-1-k} \\ &= \sum_{j=1}^N \sum_{z=1}^{\infty} a_z(x^*)^{z-1} \sum_{k=0}^{z-1} \sum_{s=1}^N \mathbf{\Omega}_{1,is} \mathbf{\Omega}_{1,sr} \sum_{j=1}^N \mathbf{\Omega}_{1,rj}^{z-1-k} \end{aligned} \quad (\text{E.1})$$

Since $\mathbf{I} \vec{\mathbf{1}} = \mathbf{\Omega}_1 \vec{\mathbf{1}} = \vec{\mathbf{1}}$, we find

$$\begin{aligned} J_{ir}^{(nc,1)}(\vec{\mathbf{X}}^*) &= \sum_{z=1}^{\infty} a_z(x^*)^{z-1} \sum_{k=0}^{z-1} \mathbf{\Omega}_{1,ir} \\ &= \sum_{z=1}^{\infty} z a_z(x^*)^{z-1} \mathbf{\Omega}_{1,ir} = f'(x^*) \mathbf{\Omega}_{1,ir} \end{aligned} \quad (\text{E.2})$$

therefore, in these conditions the first part of the Jacobian takes the same expression as for the standard commutative map. Summing the second term (D.7), finally yields for both maps

$$\begin{aligned} J_{ir}(\vec{\mathbf{X}}^*) &= f'(x^*) \mathbf{\Omega}_{1,ir} - \alpha (\mathbf{I} - \mathbf{\Omega}_1)_{ir} = -\alpha \delta_{ir} + (f'(x^*) + \alpha) \mathbf{\Omega}_{1,ir} \\ &= -\alpha \delta_{ir} + \frac{1}{N} (f'(x^*) + \alpha). \end{aligned} \quad (\text{E.3})$$

\square

Appendix F. Proof of Prop. 4.11

Proof. We aim at evaluating the Jacobian of⁴

$$\mathbf{F}_i^{(mc)}(\mathbf{X}_1, \dots, \mathbf{X}_m) = a_{i,0} \mathbf{I} + \sum_{k=1}^{\infty} \sum_{j_1, \dots, j_m}^k a_{k,i;j_1, \dots, j_m} (\mathbf{\Omega}(\mathbf{X}_1^{j_1} \dots \mathbf{X}_m^{j_m})^{1/k})^k. \quad (\text{F.1})$$

⁴As usual, we use a general projection operator wherever possible.

as it is needed for the estimation of the first Jacobian component

$$\begin{aligned}
(\mathbf{J}_{ij}^{(mc,1)}(\{\vec{X}_i\}))_{st} &= \frac{\partial(\mathbf{F}_i^{(mc)}(\mathbf{X}_1, \dots, \mathbf{X}_m)\vec{\mathbf{1}})_s}{\partial X_{j,t}} \\
&= \sum_{k=1}^{\infty} \sum_{j_1, \dots, j_m}^k a_{k,i;j_1, \dots, j_m} \left(\frac{\partial(\boldsymbol{\Omega}(\mathbf{X}_1^{j_1} \dots \mathbf{X}_m^{j_m})^{1/k})^k}{\partial X_{j,t}} \vec{\mathbf{1}} \right)_s. \quad (\text{F.2})
\end{aligned}$$

Therefore, we need to evaluate

$$\frac{\partial(\boldsymbol{\Omega}(\mathbf{X}_1^{j_1} \dots \mathbf{X}_m^{j_m})^{1/k})^k}{\partial X_{j,t}} \quad (\text{F.3})$$

We exploit the identity

$$\frac{\partial \mathbf{B}^k}{\partial X} = \sum_{z=0}^{k-1} \mathbf{B}^{k-1-z} \frac{\partial \mathbf{B}}{\partial X} \mathbf{B}^z \quad k \geq 1 \quad (\text{F.4})$$

valid for any matrix \mathbf{B} , obtaining

$$\begin{aligned}
\frac{\partial(\boldsymbol{\Omega}(\mathbf{X}_1^{j_1} \dots \mathbf{X}_m^{j_m})^{1/k})^k}{\partial X_{j,t}} &= \sum_{z=0}^{k-1} \left(\boldsymbol{\Omega}(\mathbf{X}_1^{j_1} \dots \mathbf{X}_m^{j_m})^{1/k} \right)^{k-1-z} \left(\boldsymbol{\Omega} \frac{\partial((\mathbf{X}_1^{j_1} \dots \mathbf{X}_m^{j_m})^{1/k})}{\partial X_{j,t}} \right) \\
&\quad \times \left(\boldsymbol{\Omega}(\mathbf{X}_1^{j_1} \dots \mathbf{X}_m^{j_m})^{1/k} \right)^z \quad (\text{F.5})
\end{aligned}$$

where, since \mathbf{X}_i are diagonal, we have

$$\left(\frac{\partial((\mathbf{X}_1^{j_1} \dots \mathbf{X}_m^{j_m})^{1/k})}{\partial X_{j,t}} \right)_{pq} = \delta_{pq} \delta_{pt} \left(\prod_{r \neq j} X_{r,t}^{j_r/k} \right) \frac{j_j}{k} X_{j,t}^{j_j/k-1}.$$

Now, taking into account that $\boldsymbol{\Omega} = \boldsymbol{\Omega}_1$ and that the fixed points are defined by $\vec{X}_i^* = x_i^* \vec{\mathbf{1}}$ (because of the multivariate banality of the mean Corollary 4.8), we find

$$\left((\boldsymbol{\Omega}_1(\mathbf{X}_1^{\alpha_1} \dots \mathbf{X}_m^{\alpha_m})^{1/\eta})^\beta \right)_{ij} \Big|_{\{\vec{X}_i^*\}} = \boldsymbol{\Omega}_{1,ij} \prod_{i=1}^m (x_i^*)^{\frac{\beta}{\eta} \alpha_i} \quad (\text{F.6})$$

$$\delta_{pq} \delta_{pt} \left(\prod_{r \neq j} X_{r,t}^{j_r/k} \right) \frac{j_j}{k} X_{j,t}^{j_j/k-1} \Big|_{\{\vec{X}_i^*\}} = \delta_{pq} \delta_{pt} \left(\prod_{r \neq j} (x_r^*)^{j_r/k} \right) \frac{j_j}{k} (x_j^*)^{j_j/k-1} \quad (\text{F.7})$$

thus, substituting into (F.5)

$$\begin{aligned}
& \left(\frac{\partial (\mathbf{\Omega}_1 (\mathbf{X}_1^{j_1} \dots \mathbf{X}_m^{j_m})^{1/k})^k}{\partial X_{j,t}} \right)_{pq} \Big|_{\{\vec{X}_i^*\}} = \sum_{b,c,d=1}^N \sum_{z=0}^{k-1} \left((\mathbf{\Omega}_1 (\mathbf{X}_1^{j_1} \dots \mathbf{X}_m^{j_m})^{1/k})^{k-1-z} \right)_{pb} \Big|_{\{\vec{X}_i^*\}} \\
& \times \left(\mathbf{\Omega}_{1,bc} \frac{\partial \left((\mathbf{\Omega}_1 (\mathbf{X}_1^{j_1} \dots \mathbf{X}_m^{j_m})^{1/k} \right)_{cd}}{\partial X_{j,t}} \right) \Big|_{\{\vec{X}_i^*\}} \left((\mathbf{\Omega}_1 (\mathbf{X}_1^{j_1} \dots \mathbf{X}_m^{j_m})^{1/k})^z \right)_{dq} \Big|_{\{\vec{X}_i^*\}} \\
& = j_j (x_j^*)^{j_j-1} \prod_{r \neq j} (x_r^*)^{j_r} \sum_{b,c,d=1}^N \mathbf{\Omega}_{1,pb} \mathbf{\Omega}_{1,bc} \delta_{cd} \delta_{ct} \mathbf{\Omega}_{1,dq} \\
& = j_j (x_j^*)^{j_j-1} \prod_{r \neq j} (x_r^*)^{j_r} \mathbf{\Omega}_{1,pt} \mathbf{\Omega}_{1,tq} \tag{F.8}
\end{aligned}$$

where each element of $\mathbf{\Omega}_1$ is equal to $1/N$. Substituting into (F.2) we recognize the series expansion of $f'_{i,x_j}(\vec{x}^*)$. This leads to the final expression

$$(\mathbf{J}_{ij}^{(mc,1)}(\{\vec{X}_i^*\}))_{st} = f'_{i,x_j}(\vec{x}^*) \mathbf{\Omega}_{1,st}. \tag{F.9}$$

Taking into account the second Jacobian component, i.e. the Jacobian of the decay functions, finally yields the result to be proved

$$\mathbf{J}_{ij}^{(mc)}(\{\vec{X}_i^*\}) = f'_{i,x_j}(\vec{x}^*) \mathbf{\Omega}_1 + \delta_{ij} \mathbf{Q}_i(\mathbf{\Omega}_1). \tag{F.10}$$

□

Appendix G. Proof of Prop. 4.12

Proof. As we aim at demonstrating the independence of the Jacobian from the ordering S (at the fixed points), we consider the formalism introduced in Section 3.1. Given the matrix map definition (34), we use the general form

$$\mathbf{F}_s(\vec{X}) = \sum_{k=0}^{\infty} \sum_{i_1, \dots, i_j=0}^j a_{s,k;i_1, \dots, i_m} \{ (\mathbf{\Omega} \mathbf{X}_1)^{i_1} \dots (\mathbf{\Omega} \mathbf{X}_m)^{i_m} \}_S \tag{G.1}$$

that forms the basis for the evaluation of the first Jacobian component. Writing, to simplify the notation, $o_{\sigma(s_1)\dots\sigma(s_m)} = o_{\vec{\sigma}}$, we compute the derivatives

$$\begin{aligned}
& \frac{\partial}{\partial X_{a,b}} \{(\mathbf{\Omega} \mathbf{X}_1)^{i_1} \dots (\mathbf{\Omega} \mathbf{X}_m)^{i_m}\}_S \\
&= \sum_{\sigma \in \mathcal{S}(m)} o_{\vec{\sigma}} \frac{\partial}{\partial X_{a,b}} ((\mathbf{\Omega} \mathbf{X}_{\sigma(s_1)})^{i_1} \dots (\mathbf{\Omega} \mathbf{X}_{\sigma(s_m)})^{i_m}) \\
&= \sum_{\sigma \in \mathcal{S}(m)} o_{\vec{\sigma}} \left(\frac{\partial}{\partial X_{a,b}} (\mathbf{\Omega} \mathbf{X}_{\sigma(s_1)})^{i_1} \right) \dots (\mathbf{\Omega} \mathbf{X}_{\sigma(s_m)})^{i_m} + \dots \\
&\quad + \sum_{\sigma \in \mathcal{S}(m)} o_{\vec{\sigma}} (\mathbf{\Omega} \mathbf{X}_{\sigma(s_1)})^{i_1} \dots \left(\frac{\partial}{\partial X_{a,b}} (\mathbf{\Omega} \mathbf{X}_{\sigma(s_m)})^{i_m} \right) \\
&= \sum_{\sigma \in \mathcal{S}(m)} o_{\vec{\sigma}} \delta_{a,\sigma(s_1)} \left(\frac{\partial}{\partial X_{\sigma(s_1),b}} (\mathbf{\Omega} \mathbf{X}_{\sigma(s_1)})^{i_1} \right) \dots (\mathbf{\Omega} \mathbf{X}_{\sigma(s_m)})^{i_m} + \dots \\
&\quad + \sum_{\sigma \in \mathcal{S}(m)} o_{\vec{\sigma}} \delta_{a,\sigma(s_m)} (\mathbf{\Omega} \mathbf{X}_{\sigma(s_1)})^{i_1} \dots \left(\frac{\partial}{\partial X_{\sigma(s_m),b}} (\mathbf{\Omega} \mathbf{X}_{\sigma(s_m)})^{i_m} \right) \quad (\text{G.2})
\end{aligned}$$

Applying identity (F.4) to matrix $(\mathbf{\Omega} \mathbf{X}_{s_k})^{i_k}$ we find

$$\frac{\partial}{\partial X_{s_k,b}} ((\mathbf{\Omega} \mathbf{X}_{s_k})^{i_k})_{ij} = \sum_{t=0}^{i_k-1} \sum_{l=1}^N ((\mathbf{\Omega} \mathbf{X}_{s_k})^t)_{il} \Omega_{lb} \left((\mathbf{\Omega} \mathbf{X}_{s_k})^{i_k-1-t} \right)_{bj}, \quad (\text{G.3})$$

At this point, the multivariate banality of the mean Corollary 4.8 proves that the PEDS fixed points are given by $\mathbf{X}_{s_k}^* = x_{s_k}^* \mathbf{I}$, therefore for the mean field projector $\mathbf{\Omega} = \mathbf{\Omega}_1$ we find

$$\begin{aligned}
\frac{\partial}{\partial X_{s_k,b}} ((\mathbf{\Omega}_1 \mathbf{X}_{s_k})^{i_k})_{ij} \Big|_{\{\vec{X}_i^*\}} &= \sum_{t=0}^{i_k-1} (x_{s_k}^*)^{i_k-1} \sum_{s=1}^N \Omega_{1,is}^t \Omega_{1,sb} \Omega_{1,bj}^{i_k-1-t} \\
&= i_k (x_{s_k}^*)^{i_k-1} \Omega_{1,ib} \Omega_{1,bj} \quad (\text{G.4})
\end{aligned}$$

Substituting this expression into the derivative of the ordered product, yields

$$\begin{aligned}
& \frac{\partial}{\partial X_{a,b}} \{(\mathbf{\Omega}_1 \mathbf{X}_1)^{i_1} \dots (\mathbf{\Omega}_1 \mathbf{X}_m)^{i_m}\}_S \Big|_{\{\vec{X}_i^*\}} \\
&= \sum_{\sigma \in \mathcal{S}(m)} o_{\vec{\sigma}} \delta_{a,\sigma(s_1)} \left(i_1 (x_{\sigma(s_1)}^*)^{i_1-1} (\mathbf{\Omega}_1)_{:b} (\mathbf{\Omega}_1)_{b:} \right) \\
&\quad \times (x_{\sigma(s_2)}^*)^{i_2} \mathbf{\Omega}_1^{i_2} \dots (x_{\sigma(s_m)}^*)^{i_m} \mathbf{\Omega}_1^{i_m} + \dots \\
&\quad + \sum_{\sigma \in \mathcal{S}(m)} o_{\vec{\sigma}} \delta_{a,\sigma(s_m)} (x_{\sigma(s_1)}^*)^{i_1} \mathbf{\Omega}_1^{i_1} (x_{\sigma(s_2)}^*)^{i_2} \mathbf{\Omega}_1^{i_2} \dots \\
&\quad \times (x_{\sigma(s_{m-1})}^*)^{i_{m-1}} \mathbf{\Omega}_1^{i_{m-1}} \left(i_m (x_{\sigma(s_m)}^*)^{i_m-1} (\mathbf{\Omega}_1)_{:b} (\mathbf{\Omega}_1)_{b:} \right) \\
&= \sum_{\sigma \in \mathcal{S}(m)} o_{\vec{\sigma}} \delta_{a,\sigma(s_1)} i_1 (x_{\sigma(s_1)}^*)^{i_1-1} (x_{\sigma(s_2)}^*)^{i_2} \dots (x_{\sigma(s_m)}^*)^{i_m} (\mathbf{\Omega}_1)_{:b} (\mathbf{\Omega}_1)_{b:} + \dots \\
&\quad + \sum_{\sigma \in \mathcal{S}(m)} o_{\vec{\sigma}} \delta_{a,\sigma(s_m)} i_m (x_{\sigma(s_1)}^*)^{i_1} (x_{\sigma(s_2)}^*)^{i_2} \dots (x_{\sigma(s_m)}^*)^{i_m-1} (\mathbf{\Omega}_1)_{:b} (\mathbf{\Omega}_1)_{b:}. \quad (\text{G.5})
\end{aligned}$$

where $(\mathbf{\Omega}_1)_{:b}$ and $(\mathbf{\Omega}_1)_b$ are vectors made of the b -th column and row of $\mathbf{\Omega}_1$, respectively. As the matrix products all collapse into the same quantity, the previous expression is independent of the ordering \mathcal{S} . Therefore, we can write

$$\frac{\partial}{\partial X_{a,b}} \{(\mathbf{\Omega}_1 \mathbf{X}_1)^{i_1} \cdots (\mathbf{\Omega}_1 \mathbf{X}_m)^{i_m}\}_{\mathcal{S}} \Big|_{\{\vec{X}_i^*\}} = \left(\frac{\partial}{\partial x_a} \prod_{j=1}^m (x_j^*)^{i_j} \right) (\mathbf{\Omega}_1)_{:b} (\mathbf{\Omega}_1)_b. \quad (\text{G.6})$$

This means that, for any ordering \mathcal{S} , at the fixed point the sum of the terms for the derivative with respect to the elements of each extended variable \vec{X}_a leads, once taking into account the factor $\vec{\mathbb{1}}$ in (87), to a scalar factor corresponding to $f'_{i,x_a}(\vec{x}^*)$, i.e. the corresponding element of the Jacobian of the target system multiplied times matrix $\mathbf{\Omega}_1$.

Concerning the second part of the Jacobian, i.e. the derivatives of $\mathbf{Q}_i(\mathbf{\Omega}_1)\vec{X}_i$, the result is a block diagonal matrix of the type $\text{diag}\{\mathbf{Q}_i(\mathbf{\Omega}_1)\}$.

In summary, even in this case the full Jacobian at the fixed points follows the block structure claimed in the proposition. \square

Appendix H. Proof of Prop. 4.13

Proof. The proof relies on the following observation. We have in general that

$$\mathbf{\Omega}_1 \mathbf{X}_s \mathbf{\Omega}_1 = \langle X_s \rangle \mathbf{\Omega}_1. \quad (\text{H.1})$$

with $\langle X_s \rangle = \frac{1}{N} \sum_{j=1}^N X_s^j$. The previous result can be easily shown as follows

$$\begin{aligned} (\mathbf{\Omega}_1 \mathbf{X}_s \mathbf{\Omega}_1)_{ij} &= \sum_{k_1, k_2=1}^N \Omega_{1,ik_1} X_{s,k_1 k_2} \Omega_{1,k_2 j} \\ &= \frac{1}{N^2} \sum_{k_1, k_2=1}^N X_{s,k_1} \delta_{k_1 k_2} \\ &= \frac{1}{N} \sum_{k_1=1}^N X_{s,k_1} \frac{1}{N} = \langle X_s \rangle \Omega_{1,ij} \end{aligned} \quad (\text{H.2})$$

Because of (H.1), we can always write the following

$$(\mathbf{\Omega}_1 \mathbf{X}_{\sigma(1)})^{\sigma(i_1)} \cdots (\mathbf{\Omega}_1 \mathbf{X}_{\sigma(m)})^{\sigma(i_m)} = f_{\sigma(i_1), \dots, \sigma(i_m)}(\langle X_{\sigma(1)} \rangle, \dots, \langle X_{\sigma(m)} \rangle) \mathbf{\Omega}_1 \mathbf{X}_{\sigma(m)}. \quad (\text{H.3})$$

where function f is scalar. To gain an intuition about the scalar f , consider for instance $(\mathbf{\Omega}_1 \mathbf{X}_1)^a (\mathbf{\Omega}_1 \mathbf{X}_2)^b$. Using (H.1), the previous expression can be written as

$$(\mathbf{\Omega}_1 \mathbf{X}_1)^a (\mathbf{\Omega}_1 \mathbf{X}_2)^b = \langle X_1 \rangle^{a-1} \langle X_2 \rangle^{b-1} \mathbf{\Omega}_1 \mathbf{X}_1 \mathbf{\Omega}_1 \mathbf{X}_2 = \langle X_1 \rangle^a \langle X_2 \rangle^{b-1} \mathbf{\Omega}_1 \mathbf{X}_2, \quad (\text{H.4})$$

so that $f_{12} = \langle X_1 \rangle^a \langle X_2 \rangle^{b-1}$. At this point we have

$$\begin{aligned} \{(\mathbf{\Omega}_1 \mathbf{X}_1)^{i_1} \cdots (\mathbf{\Omega}_1 \mathbf{X}_m)^{i_m}\}_{\mathcal{S}_r} \vec{\mathbb{1}} &= \sum_{\sigma \in \mathcal{S}_m} o_{\vec{\sigma}} f_{\sigma(i_1), \dots, \sigma(i_m)}(\langle X_{\sigma(1)} \rangle, \dots, \langle X_{\sigma(m)} \rangle) \mathbf{\Omega}_1 \mathbf{X}_{\sigma(m)} \vec{\mathbb{1}} \\ &= \sum_{\sigma \in \mathcal{S}_m} o_{\vec{\sigma}} f_{\sigma(i_1), \dots, \sigma(i_m)}(\langle X_{\sigma(1)} \rangle, \dots, \langle X_{\sigma(m)} \rangle) \langle X_{\sigma(m)} \rangle \vec{\mathbb{1}} \\ &= \sum_{\sigma \in \mathcal{S}_m} o_{\vec{\sigma}} f_{\sigma(i_1), \dots, \sigma(i_m)}(\langle X_{\sigma(1)} \rangle, \dots, \langle X_{\sigma(m)} \rangle) \langle X_{\sigma(m)} \rangle \vec{\mathbb{1}} \\ &= f_{i_1, \dots, i_m}(\langle X_{\sigma(1)} \rangle, \dots, \langle X_{\sigma(m)} \rangle) \langle X_m \rangle \vec{\mathbb{1}} \end{aligned} \quad (\text{H.5})$$

which follows from the fact that the scalar variables $\langle X_j \rangle$ do commute.

□