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RESEARCH ARTICLE

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Non-maturing deposits modelling in a Ornstein-Uhlenbeck framework

Marina Marena[1] | **Andrea Romeo**[1] | **Patrizia Semeraro**[2]

[1]Department of Economics and Statistics, Università di Torino, Torino, Italy

[2]Department of Mathematical Sciences G. Lagrange, Politecnico di Torino, Torino, Italy

Correspondence
Patrizia Semeraro, Department of Mathematical Sciences G. Lagrange, Politecnico di Torino, Torino, Italy.
Email: patrizia.semeraro@polito.it

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Abstract

This paper builds a multivariate Lévy-driven Ornstein-Uhlenbeck process for the management of non-maturing deposits, that are a major source of funding for banks. The contribution of the paper is both theoretical and operational. On the theoretical side, the novelty of this model is to include three independent sources of randomness in a Lévy framework: market interest rates, deposit rates and deposit volumes. The choice of a Lévy background driving process allows us to model rare but severe events. On the operational side, we propose a procedure to include severe volume outflows with positive probability in future scenarios simulation, explaining its implementation with an illustrative example using Italian banking sector data.

KEYWORDS

Lévy processes, multivariate Ornstein-Uhlenbeck processes, normal inverse Gaussian, operational research in non-maturing deposits

1 | INTRODUCTION

Non-maturing deposits (NMDs) are a major source of funding for financial institutions. Many definitions for NMDs can be provided. According to the Basel Committee on Banking Supervision,[1] NMDs are liabilities of the banks where there is no contractual maturity and consequently where depositors are free to withdraw them at any time, partially or entirely. On top of the possibility for depositors to withdraw their money at any time, another intrinsic characteristic of NMDs is the faculty for banks to reprice these products at any time. Several types of products issued by banks can fall under this definition, including (in a non-exhaustive list): non-maturing liability checking accounts, savings accounts and short time deposits. NMDs are important for banks, due to their stability and cheapness on the liability side. These characteristics become increasingly important in periods when market turmoil threatens to preclude other sources of funding. Considering that NMDs entail these two features, managing them is anything but a trivial task for banks. In addition, during the few past years (mostly after the financial crisis), a number of credit institutions experienced or almost experienced unexpected shortages—the so-called "bank run". A number of factors can produce this phenomenon, mainly involving a lack of confidence in the creditworthiness and accountability of the bank. Consequently there is a clear need for a modeling approach for NMDs which is appropriate, effective and prudent. Although the topic has not been extensively investigated in the literature, different approaches have been proposed to tackle this issue, for example, References 2-4 and 5 to cite just some of them.

This work has both theoretical and operational contributions. The theoretical contribution of the present work is to introduce a multivariate Ornstein-Uhlenbeck (OU) process to model the interactions among market interest rates, deposit

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rates and deposit volumes. The novelty of our approach is to explicitly model three independent sources of randomness, each one affecting one or more of the one-dimensional marginal driving processes within our multivariate framework. These three explicit sources of randomness are assumed to be the implicit options held by the bank and the customers, alongside market interest rates. The idea is that, in order to adequately capture the possibility for a single bank of facing stressed situations (irrespective of the general economic environment), both the deposit rates and the volumes of deposits should depend on idiosyncratic components specific to the bank and its customers. Given the economic relationships among the modelled variables, a triangular specification of the multivariate background driving process is assumed. Driving processes belonging to the Lévy class are considered to be fit for our purpose. This class of processes are in our view particularly suitable for the purpose of NMDs modelling, for their ability to accommodate skewness and excess kurtosis.[6,7] Indeed, both skewness and excess kurtosis play a key role in modelling rare events that can entail a significant risk in terms of liquidity management for a bank.

The estimation procedure is based on maximum likelihood (ML) methods applied in our multidimensional setting. The operational contribution is to introduce an *ad-hoc* procedure to simulate future cash flows in stressed scenarios which encompass rare but severe volume outflows. This procedure can be used for liquidity risk management purposes.

In what follows, Section 2 presents the stochastic model and Section 3 describes the estimation procedure. Then, in Section 4, the parameters of the model are estimated. The application to liquidity risk management for banks is in Section 5. Section 6 concludes.

2 | THE MODEL: LÉVY-DRIVEN OU PROCESS

Let $\boldsymbol{L}(t)$ be a Lévy process on $\mathbb{R}^n$ starting from the origin with independent components. Let $\boldsymbol{K}$ be a real $n \times n$ matrix such that its eigenvalues have positive real part, $\boldsymbol{\Sigma}$ be a real $n \times n$ matrix and $\boldsymbol{\theta} \in \mathbb{R}^n$. Consider the following OU process

$$d\boldsymbol{X}(t) = -\boldsymbol{K}\left(\boldsymbol{X}(t) - \boldsymbol{\theta}\right) dt + \boldsymbol{\Sigma} d\boldsymbol{L}(t), \tag{2.1}$$

where $\boldsymbol{X}(0)$ is supposed to be independent of $\boldsymbol{L}(t)$. Before going into details of our model we discuss the mathematical framework. The general n-dimensional OU process with background driving Lévy process (BDLP) is discussed both in Reference 6, where the solution to (2.1) is related to self decomposable distributions, and in Reference 7. The solution to (2.1) is expressed as

$$\boldsymbol{X}(t) = \left(I - e^{-t\boldsymbol{K}}\right)\boldsymbol{\theta} + e^{-t\boldsymbol{K}}\boldsymbol{X}(\boldsymbol{0}) + \int_0^t e^{-(t-u)\boldsymbol{K}}\boldsymbol{\Sigma} d\boldsymbol{L}(u), \quad t \geq 0. \tag{2.2}$$

A probability distribution η is a stationary distribution for (2.1) if $\mathcal{L}(\boldsymbol{X}(0)) = \eta$ implies that $\mathcal{L}(\boldsymbol{X}(t)) = \eta$ for every t. If the stationary solution exists then it is unique. It exists a stationary solution if and only if

$$\int_{\mathbb{R}^n} \log^+ |\boldsymbol{x}| \nu_0(d\boldsymbol{x}) < \infty, \tag{2.3}$$

where ν_0 is the Lévy measure of $\boldsymbol{L}(t)$. An equivalent condition is

$$\int_{\mathbb{R}^n} \log^+ |\boldsymbol{x}| \eta_0(d\boldsymbol{x}) < \infty, \tag{2.4}$$

where $\eta_0 = \mathcal{L}(\boldsymbol{L}(1))$.

The stationary solution is $\boldsymbol{K}$-self decomposable (see Reference 6). Furthermore, given a measure η on $\mathbb{R}^n$ it exists a Lévy process such that η is the unique stationary solution of (2.1) if η is $\boldsymbol{K}$-self decomposable. Therefore, there is a one to one relationship between the $\boldsymbol{K}$-self decomposable measures η on $\mathbb{R}^n$ and the measures η_0 satisfying (2.4).

Usually the model specification of a OU process with BDLP is performed by giving the stationary $\boldsymbol{K}$-self decomposable distribution η. The law of η is chosen to belong to a family of self decomposable distributions and η_0 is derived. An important example in the financial framework of model specification for the one-dimensional case is given in Reference 8, where the stationary distribution is specified to be a normal inverse Gaussian (NIG) distribution–which is self-decomposable. The corresponding process is called NIG-OU process. The multivariate extension of NIG-OU processes is considered in Reference 7, where the stationary distribution is chosen to be a multivariate NIG process. Since Reference 9 proved that

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a multivariate normal distribution subordinated by a generalized gamma distribution is self-decomposable if the multivariate normal distribution has zero drift, this assumption is made in Reference 7. This restriction leads to symmetric stationary distributions. Because of that, and in order to specify the dependence structure of the driving process, we choose a different approach. Instead of specifying the stationary distribution we go the other way round and construct the model by specifying the BDLP $\boldsymbol{L}(t)$. As a result, the only constraint we have on the choice of $\boldsymbol{L}(t)$ is given by (2.3).

2.1 | Model specification

Let now consider the OU process $\boldsymbol{X}(t)$ in (2.1) and assume that $n = 3$, $\boldsymbol{K}$ is lower triangular, $\boldsymbol{\Sigma}$ is lower triangular with unit diagonal elements and $\boldsymbol{L}(t)$ is a Lévy process on $\mathbb{R}^3$ with independent zero-mean components. In our economic setting, the three components $X_1(t)$, $X_2(t)$ and $X_3(t)$ represent the market interest rates, the deposit rates and the deposit volume, respectively. Since $\boldsymbol{K}$ is triangular, so it is $e^{-t\boldsymbol{K}}$. Since $\boldsymbol{\Sigma}$ is lower triangular with unit diagonal elements, we have

$$\boldsymbol{\Sigma} d\boldsymbol{L}(t) = \begin{pmatrix} dL_1(t) \\ \sigma_{21} dL_1(t) + dL_2(t) \\ \sigma_{31} dL_1(t) + \sigma_{32} dL_2(t) + dL_3(t) \end{pmatrix} \tag{2.5}$$

This specification of the transformations $\boldsymbol{K}$ and $\boldsymbol{\Sigma}$ captures the economic relationship among the interest rates, the level of deposit rates and the level of NMDs. The latter two are built to include components which are specific to the bank and its customers, together with a systemic factor. In particular, $L_1(t)$ is responsible for systemic shocks in the market interest rates which have an impact on both the implicit options held by the bank and the customers. The factor $L_2(t)$ captures the idiosyncratic component of deposit rates, which has also an impact on the level of NMDs. Finally $L_3(t)$ represents the idiosyncratic component specific to the level of NMDs only.

We consider the case of $\boldsymbol{L}(t)$ being a pure jump Lévy process, to accommodate possible skewness and kurtosis. In addition, the Gaussian case, which leads to the standard Ornstein-Uhlenbeck process, is used as a benchmark to assess the performance of our model. The Lévy driving process $\boldsymbol{L}(t)$ is specified to have one dimensional independent normal inverse Gaussian margins (NIG)–the univariate NIG process is recalled in Appendix A. We choose the NIG specification because it has a good fit on financial data, it is parsimonious in the number of parameters and with this choice $\boldsymbol{L}(t)$ satisfies (2.3).

The process $\boldsymbol{Y}(t) = \boldsymbol{\Sigma}\boldsymbol{L}(t)$ is a triangular linear transformation of $\boldsymbol{L}(t)$. As a consequence, it is still a Lévy process, but its marginal one-dimensional distributions at time $t = 1$ do not belong in general to the NIG family. In particular, $Y_1(1)$ is NIG distributed, while the distributions of $Y_2(1)$ and $Y_3(1)$ are convolutions of NIG distributions. The constraint for closure under convolution of a NIG distribution (see Equation (A1)) are in general not compatible with our model. Indeed, enforcing convolution conditions means that we should assume that $L_1(t)$, $L_2(t)$ and $L_3(t)$ have the same asymmetry and tail parameters. Assuming that those parameters should be equal for interest rates, deposit rates and deposit volumes seems to be unrealistic when we apply the model to real data, as we can see in Section 4 from the analysis of the empirical distributions of residuals (see Figure 2).

3 | ESTIMATION OF THE MODEL

We observe the multivariate process $\boldsymbol{X}(t)$ at fixed times $0 = t_0 < t_1 < \ldots < t_n = T$, where $\Delta = t_{k+1} - t_k$ is constant. We have

$$\boldsymbol{X}(t_{k+1}) = \left(I - e^{-\boldsymbol{K}\Delta}\right)\boldsymbol{\theta} + e^{-\Delta\boldsymbol{K}}\boldsymbol{X}(t_k) + \boldsymbol{u}(t_k), \tag{3.1}$$

with

$$\boldsymbol{u}(t_k) = \int_{t_k}^{t_{k+1}} e^{-(t_{k+1}-u)\boldsymbol{K}}\boldsymbol{\Sigma} d\boldsymbol{L}(u). \tag{3.2}$$

The solution (3.1) is a vector autoregressive process of order one *VAR* (1). Letting

$$\boldsymbol{a} = \left(I - e^{-\boldsymbol{K}\Delta}\right)\boldsymbol{\theta}, \quad \boldsymbol{B} = e^{-\boldsymbol{K}\Delta}, \tag{3.3}$$

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consider

$$\boldsymbol{X}(t_{k+1}) = \boldsymbol{a} + \boldsymbol{B}\boldsymbol{X}(\boldsymbol{t}_k) + \boldsymbol{u}(t_k). \tag{3.4}$$

If $\boldsymbol{L}(t)$ is a multivariate Wiener process, then the conditional distribution of $\boldsymbol{X}(t_{k+1})|\boldsymbol{X}(t_k)$ is normal with mean $\left(I - e^{-\boldsymbol{K}\Delta}\right)\boldsymbol{\theta} + e^{-\Delta\boldsymbol{K}}\boldsymbol{X}(t_k)$ and covariance matrix given by

$$\mathrm{vec}(\boldsymbol{\Sigma}_u) = (\boldsymbol{K} \oplus \boldsymbol{K})^{-1}[\boldsymbol{I} - \exp(-(\boldsymbol{K} \oplus \boldsymbol{K})\Delta)]\mathrm{vec}(\boldsymbol{\Sigma}\boldsymbol{\Sigma}')$$

where vec denotes the stack operator and $\oplus$ denote the Kronecker sum, since $\boldsymbol{u}(t_k) \sim N(0, \boldsymbol{\Sigma}_u)$ (see Reference 10). Then we can retrieve the original model parameters from (3.3)

$$\boldsymbol{K} = -\frac{\log(\boldsymbol{B})}{\Delta}, \quad \boldsymbol{\theta} = \left(I - e^{-\boldsymbol{K}\Delta}\right)^{-1}\boldsymbol{a}.$$

In the NIG case, the use of ML estimation requires the inversion of the multivariate characteristic function, which is challenging from a numerical point of view. From (3.1) and (3.2), the error term $\boldsymbol{u}(t_k)$ has a triangular structure

$$\boldsymbol{u}(t_k) = \begin{pmatrix} u_{11}(t_k) \\ u_{21}(t_k) + u_{22}(t_k) \\ u_{31}(t_k) + u_{32}(t_k) + u_{33}(t_k) \end{pmatrix}, \tag{3.5}$$

where

$$u_{ij}(t_k) = \int_{t_k}^{t_{k+1}} C_{ij}^k(u)\, dL_j(u),$$

and $C_{ij}^k(u)$ is a function of $\boldsymbol{K}$ and $\boldsymbol{\Sigma}$. If Δ is small enough, we can assume that $u_{ij}(t_k)$ has a NIG distribution with zero mean and variance provided by Ito's isometry. We can write

$$Var\left(u_{ij}(t_k)\right) = s_{ij}^2 Var\left(L_j(1)\right), \quad i \geq j,$$

where $L_j(1)$ has zero mean and variance σ_j^2. Therefore we estimate the process

$$\boldsymbol{X}(t_{k+1}) = \boldsymbol{a} + \boldsymbol{B}\boldsymbol{X}(t_k) + \boldsymbol{S}\boldsymbol{\varepsilon}(t_k), \tag{3.6}$$

where S is a lower triangular matrix with unit diagonal elements

$$\boldsymbol{S}\boldsymbol{\varepsilon}(t_k) = \begin{pmatrix} \varepsilon_1(t_k) \\ s_{21}\varepsilon_1(t_k) + \varepsilon_2(t_k) \\ s_{31}\varepsilon_1(t_k) + s_{32}\varepsilon_2(t_k) + \varepsilon_3(t_k) \end{pmatrix}, \tag{3.7}$$

and $\varepsilon_i(t_k)$ are independent identically distributed $L_i(1) \sim \mathrm{NIG}(\alpha_i, \beta_i, \delta_i, \mu_i)$, with zero-mean and variance σ_i^2 for all k.

The model can be estimated by maximum likelihood (see Reference 11). The log-likelihood function of the sequence $\boldsymbol{\varepsilon}(t_k) = \boldsymbol{S}^{-1}(\boldsymbol{X}_{k+1} - \boldsymbol{a} - \boldsymbol{B}\boldsymbol{X}_k)$, $k = 0 \ldots, n-1$, is given by

$$L(\boldsymbol{\theta}) = \sum_{k=0}^{n-1}\sum_{i=1}^{3} \log f_i\left(\left(\boldsymbol{e}^i\right)'\boldsymbol{S}^{-1}(\boldsymbol{X}_{k+1} - \boldsymbol{a} - \boldsymbol{B}\boldsymbol{X}_k)\right), \tag{3.8}$$

where $\boldsymbol{e}^i$ is the i-th unit vector and f_i denotes the log-likelihood function of the sequence of noises $\varepsilon_i(t_k)$. In the Gaussian case, $\varepsilon_i(t_k) \sim N\left(0, \sigma_i^2\right)$, while in the NIG case, we have that $\varepsilon_i(t_k) \sim \mathrm{NIG}(\alpha_i, \beta_i, \delta_i, \mu_i)$, $k = 0 \ldots, n-1$ and f_i is given by

$$f_i(x; \alpha_i, \beta_i, \delta_i, \mu_i) = n\log\left(\frac{\alpha_i}{\pi}\right) + n\delta_i\sqrt{\alpha_i^2 - \beta_i^2} + \sum_{k=0}^{n-1}\left[\beta_i\delta_i\tau_i(x) - \log c_{i,k} + \log H_1\left(\alpha_i\delta_i c_{i,k}\right)\right], \tag{3.9}$$

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being H_1 the modified Bessel function of the third kind, $\tau_i(x) = \frac{x-\mu_i}{\delta_i}$ and $c_{i,k} = \sqrt{1+\tau_{i,k}^2}$, for any $i = 1, 2, 3$, and $k = 0 \ldots, n-1$.

In the NIG error specification, we need to estimate the parameter set

$$\{\boldsymbol{a}, \boldsymbol{B}, \boldsymbol{S}, \boldsymbol{\sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}, \boldsymbol{\mu}\}.$$

Let $\gamma_i = \sqrt{\alpha_i^2 - \beta_i^2}$. By assumption, ϵ_i has zero mean and variance σ_i^2. We set

$$\mu_i + \frac{\delta_i \beta_i}{\gamma_i} = 0, \quad \frac{\delta_i \alpha_i^2}{\gamma_i^3} = \sigma_i^2. \tag{3.10}$$

In fact, the constraints will leave us with two free NIG parameters only, for each i. The NIG parameters that enters the log-likelihood optimization are $\left(\log \gamma_i, \beta_i\right)$, to directly enforce the positivity of γ_i and the constraints in (3.10). Summing up, the parameter set to be estimated is

$$\theta = \left\{\boldsymbol{a}, \boldsymbol{B}, \boldsymbol{S}, \boldsymbol{\sigma}, \log(\boldsymbol{\gamma}), \boldsymbol{\beta}\right\},$$

while $\boldsymbol{\alpha}, \boldsymbol{\delta}$ and $\boldsymbol{\mu}$ are derived as

$$\alpha_i = \sqrt{\gamma_i^2 + \beta_i^2}, \quad \delta_i = \frac{\sigma_i^2 \gamma_i^3}{\alpha_i^2}, \quad \mu_i = -\frac{\delta_i \beta_i}{\gamma_i}, \quad i = 1, 2, 3.$$

ML estimators can be computationally demanding in a multidimensional setting. Initial conditions are derived by a two-step procedure, generalizing the approach in Reference 12 to a multidimensional setting.

Firstly, we estimate the subset of parameters $\{\boldsymbol{a}, \boldsymbol{B}, \boldsymbol{S}, \boldsymbol{\sigma}\}$ using the least squares approach. In particular, in the first step, we estimate $\boldsymbol{a}$ and $\boldsymbol{B}$ by multivariate LS estimation of (3.4), which is equivalent to OLS estimation of the three equations separately. The white noise covariance matrix estimator of $\boldsymbol{\Sigma}_u$ of the error term $\boldsymbol{u}$ is computed from the LS residuals. From $\boldsymbol{u} = \boldsymbol{S}\boldsymbol{\epsilon}$, the covariance matrix $\boldsymbol{\Sigma}_u$ is equal to $\boldsymbol{S}\boldsymbol{D}\boldsymbol{S}'$, being $\boldsymbol{D}$ the covariance matrix of the error term $\boldsymbol{\epsilon}$ with independent components. By assumption, $\boldsymbol{D}$ is a diagonal matrix with diagonal elements σ_i^2, $i = 1, 2, 3$. Given the triangular structure of the matrix $\boldsymbol{S}$, we can derive $\boldsymbol{S}$ and $\boldsymbol{\sigma}$ by equating component-wise the six distinct elements of the two matrices $\boldsymbol{\Sigma}_u$ and $\boldsymbol{S}\boldsymbol{D}\boldsymbol{S}'$.

Secondly, we estimate the remaining subset of parameters $\{\log(\boldsymbol{\gamma}), \boldsymbol{\beta}\}$ related to the error terms $\boldsymbol{\epsilon}(t_k)$ using a ML approach, keeping $\{\boldsymbol{a}, \boldsymbol{B}, \boldsymbol{S}, \boldsymbol{\sigma}\}$ fixed.

The parameters estimated via the two-step procedure are then used as initial points in the numerical ML procedure.

4 | MODELING NON-MATURING DEPOSITS

In the present Section, we estimate the model on real data and discuss the fit of the model, the role of its single components and how those components interact. The proposed case study is performed using publicly available aggregate data. Consequently the results obtained are discussed with the only purpose of exhibiting the main functioning and features of the model. In fact, it should be stressed that the estimates obtained are meaningful only where data relevant for a specific bank are used to calibrate the model. Indeed, the use of less aggregate data enhances the ability of the model to capture specific features of types of customers or types of products, reflecting particular market situations such as the relative positioning of a specific bank within the competitive environment. Therefore, it would be advisable to have at least a segmentation of the bank' customers into retail and wholesale categories,[1] but the bank is free to have further refinements in its classification of customers for internal purposes. In addition, the bank can further discriminate among types of deposits, identifying, for example, transactional accounts or non-interest bearing deposits.

Once the scope of the model has been defined and the model has been calibrated to the relevant data, it will be able to provide concrete answers to the questions faced by banks in their asset and liability management (ALM).

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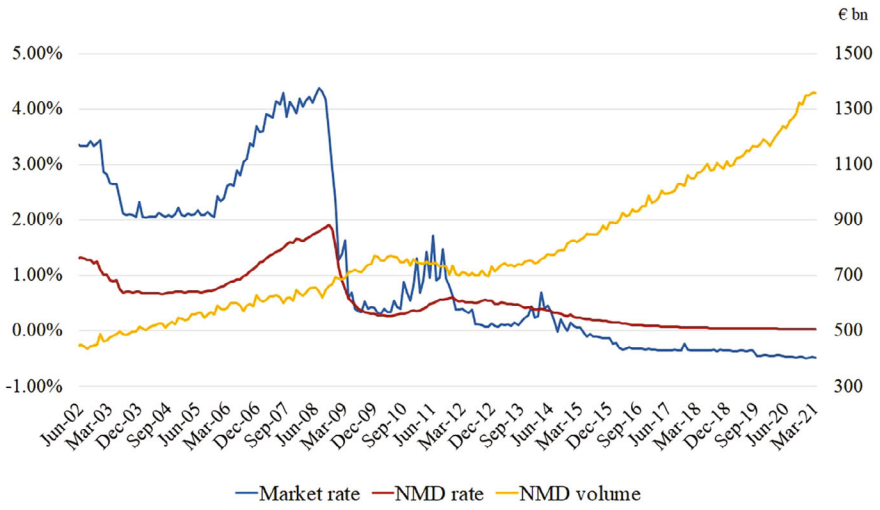

FIGURE 1 Time series of market rates, deposit rates, and deposit volume used for the calibration of the model.

4.1 | Market rates, deposit rates, and volume dynamics

As discussed in the previous section, the three components $X_1(t)$, $X_2(t)$ and $X_3(t)$ represent the interest rate, the level of deposit rates and the level of NMDs, respectively. Some further specifications are needed. The dynamics of interest rates is modelled by the component $X_1(t)$ itself, since negative interest rates are possible and observed. This means that the dynamics of interest rate follows a traditional Vasicek model. In recent years, deposit rates, represented by the second component of the model, have turned negative on corporate deposits of several banks of the Euro area.[13] Therefore, $X_2(t)$ could directly represent their dynamics. Nevertheless, retail deposit rates are usually positive and floored at zero. Those floored deposits are hugely relevant for liquidity risk management. In order to encompass the modeling of floored deposits within our setting, we set out an alternative specification where $X_2(t)$ is the deposit log-rates. This means to adopt a dynamics for deposit rates which is of exponential Vasicek type (see Reference 14). On the contrary, deposit volumes can never be negative and cannot be modelled by $X_3(t)$ directly. Thus, $X_3(t)$ is the dynamics of log-volumes.

4.2 | Dataset

For illustrative purposes three different types of monthly data for market rates, deposit rates and deposit volumes are used in what follows. Data for EONIA rates are used as a proxy for market interest rates, and are collected from Bloomberg. The Statistical Database of Banca d'Italia provides publicly available time series from the Italian bank system on a number of topics. We select the total deposit rates* and volumes†. The data are shown in Figure 1.

Both the data used for deposit rates and NMDs volume have monthly frequency, starting from January 2002 up to March 2021, for a total of 231 observations.

4.3 | Estimates of the model parameters

We now estimate the parameters of Equation (3.6), where $X_1(t)$ represents the market rate dynamics, $X_2(t)$ represents the deposit log-rate dynamics and $X_3(t)$ represents the log-volume dynamics. In the optimation of the log-likelihood function (3.8) we enforce the following sign constraints

$$b_{21} > 0, \quad b_{31} < 0, \quad b_{32} > 0, \quad s_{21} > 0, \quad s_{31} < 0, \quad s_{32} > 0, \tag{4.1}$$

*Code BAM_MIR.M.1300010.MIR5421.3.950.1000.SBI78.EUR.101.997.

†Code BAM_BSIB.M.1070001.52000102.9.101.IT.S1O.1000.997.

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TABLE 1 Estimated parameters $\boldsymbol{a}$, $\boldsymbol{B}$, $\boldsymbol{S}$, $\boldsymbol{\sigma}$ and corresponding $\boldsymbol{\theta}$ and $\boldsymbol{K}$.

	$\boldsymbol{a}$	$\boldsymbol{B}$		
Gaussian	−0.000039	0.988688		
	−0.114547	1.734262	0.986268	
	0.047423	−0.060261	0.000000	0.996912
NIG	−0.000112	0.996328		
	−0.074274	1.130800	0.992096	
	0.062410	−0.147520	0.000000	0.995876
	$\boldsymbol{\theta}$	$\boldsymbol{K}$		
Gaussian	−0.003478	0.136515		
	−8.780658	−21.075061	0.165929	
	15.424397	0.728377	−0.000001	0.037114
NIG	−0.030580	0.044145		
	−13.772694	−13.648619	0.095220	
	16.223972	1.777170	−0.000005	0.049591
	$\boldsymbol{\sigma}$	$\boldsymbol{S}$		
Gaussian	0.002045	1.000000		
	0.055157	10.072156	1.000000	
	0.019052	−0.000031	0.000004	1.000000
NIG	0.002729	1.000000		
	0.059975	5.859505	1.000000	
	0.019063	−0.000246	0.007663	1.000000

TABLE 2 Estimated parameters of the idiosyncratic components $\boldsymbol{L}(1)$ for the NIG specification (second step).

	α	β	δ	μ	**Skewness**	**Kurtosis**
$L_1(1)$	52.52986	−9.29901	0.00037	0.00007	−1.91	46.77
$L_2(1)$	17.09158	−9.14173	0.03709	0.02348	−1.10	6.00
$L_3(1)$	71.33072	12.01585	0.02483	−0.00424	0.19	3.48

Note: The last two columns show the annual skewness and kurtosis.

given the financial interpretation of the parameters in our application. In particular, deposit volumes are expected to decrease with market rates and increase with deposit rates.

Table 1 provides the estimates of the parameters of Equation (3.6) obtained by ML estimation, together with their translation into the parameters of Equation (3.1).

Table 2 provides the estimated parameters for the NIG specification. The first row refers to the idiosyncratic component of the market interest rate noise, the second row refers to the idiosyncratic component $L_2(1)$ of the deposit log-rates noise, and the third row refers to the idiosyncratic component $L_3(1)$ of the deposit log-volumes. The last two columns of Table 2 show the estimated annual skewness and kurtosis of the three scaled idiosyncratic components representing the marginal idiosyncratic components of the error. Notice that in our dataset market rates show high negative skewness and high excess kurtosis that cannot be captured in the Gaussian case.

Figure 2 shows how the NIG assumption critically improves the fit of the empirical distribution of the idiosyncratic errors. This is particularly evident for the market rates and deposit log-rates.

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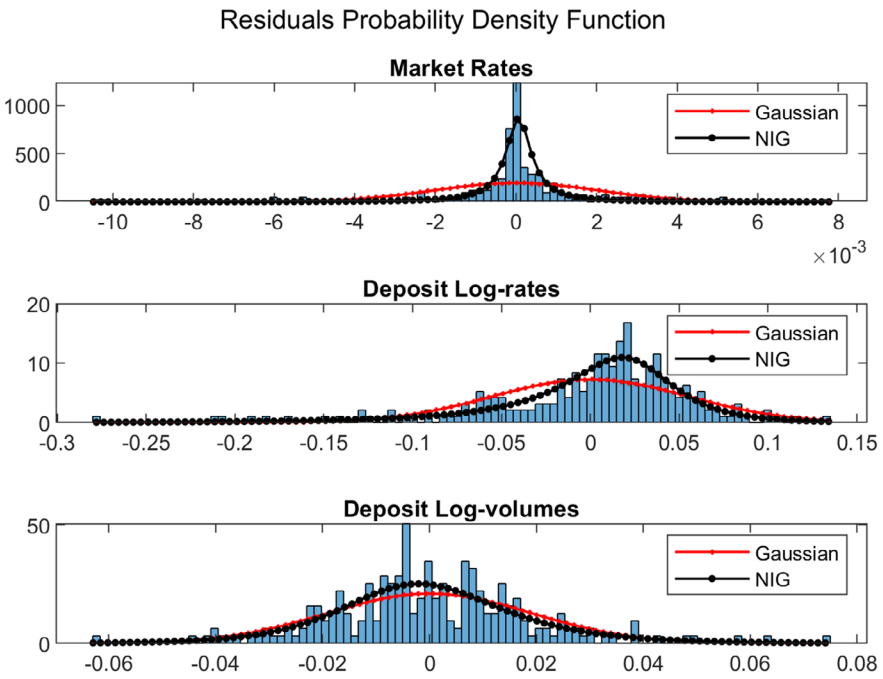

FIGURE 2 Histogram of the empirical residuals and the corresponding Gaussian and NIG fitted pdf.

5 | APPLICATION: LIQUIDITY RISK MANAGEMENT

In this section we make use of our estimated multivariate Lévy-driven OU process as a tool for liquidity risk management. We recall that the three components of the process model interest rates $X_1(t)$, deposit log-rates $X_2(t)$ and log-volumes $X_3(t)$ of NMDs.

According to Reference 15, banks should have a sound process for identifying, measuring, monitoring and controlling liquidity risk, including a robust framework for projecting cash flows arising from assets, liabilities and off-balance sheet items over appropriate time horizons. It is important, in particular, to correctly assess the "stickiness" of funding sources, that is, the tendency of these sources not to run off quickly under stress. Among the factors that influence the "stickiness" of deposits, the Basel standards cite the interest-rate sensitivity, the size of the deposit, the geographical location of depositors and the deposit channel. The first of these factors is explicitly modelled in our framework, while the others, as previously mentioned, can be captured by means of calibration to historical data for specific types of deposits or depositors.

We proceed by Monte Carlo simulation, generating 100,000 paths of the joint dynamics of the risk factors on a time grid with step size of one month and time horizon up to 10 years. We focus on the lowest quantiles of the projected deposit volume distribution. We define the Value-at-Risk at time t of the deposit volumes at the confidence level α as the quantile of order $1-\alpha$ of the distribution of deposit volume at any given time t

$$VaR_\alpha\left(D\left(t\right)\right)=\inf\left\{d\in(0,+\infty):\mathbb{P}\left(D\left(t\right)\leq d\right)\geq 1-\alpha\right\}.$$

The left-hand side of Figure 3 shows the historical evolution of the deposit volumes in the past 19 years, corresponding to our sample period, and their projected evolution over the next 10 years. The projected evolution is summarized by its expected value, value-at-risk levels and expected shortfall. Results are presented both for the NIG specification of the BDLP and the Gaussian one, with the latter serving as a benchmark for the first. From the plot, one can appreciate differences and similarities of the two specifications. While similar results are obtained in terms of the $VaR_{95\%}$ of the evolution of deposits, differences can be observed in terms of $VaR_{99\%}$. The phenomenon is a clear consequence of the "fat tails" embedded in the NIG distribution.

However in many practical cases, banks prefer to look at different metrics, which could be proven to be more meaningful than the distribution at time t of the NMDs. Indeed, very often banks are interested in identifying the minimum

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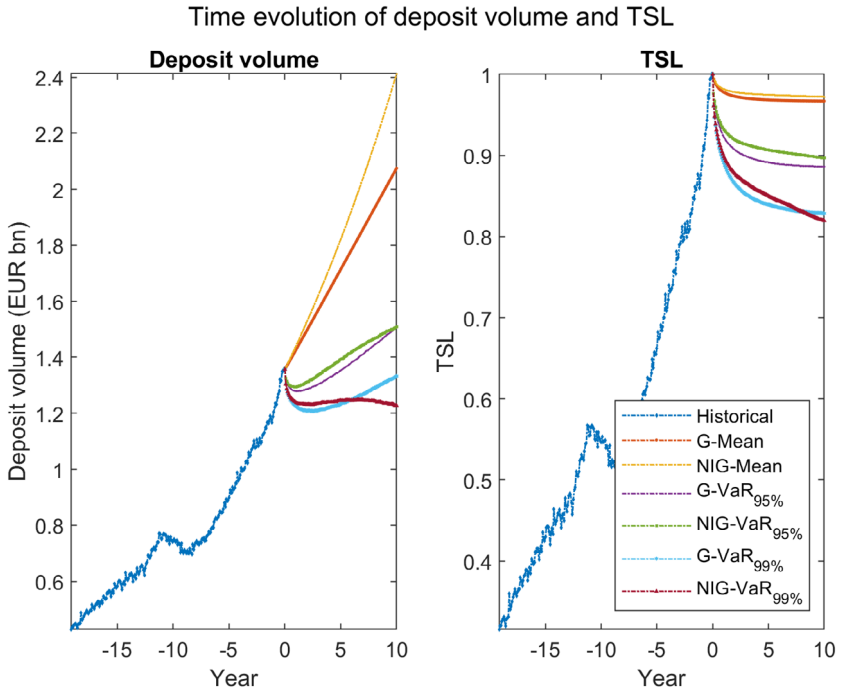

FIGURE 3 Time evolution of the deposit volume (EUR bn) and *TSL* at different confidence levels for the Gaussian and NIG specifications. The initial deposit volume is equal to 1356 EUR bn.

level of deposit volumes up to any given time t, with a given confidence level. This would be the actual amount of funds available for reinvestment on the horizon $[0, t]$. We define the lowest level of deposit volumes up to time t as

$$M(t) = \min_{0 \le s \le t} D(s).$$

We normalize the quantiles of $M(t)$ by the initial value of the deposit volume (see e.g., References 3 and 5). We obtain the so-called Term Structure of Liquidity with confidence level α, used in the following as reference metrics

$$TSL_\alpha(t) = \frac{VaR_\alpha(M(t))}{D(0)}, \quad t \ge 0.$$

By construction, $TSL_\alpha(t)$ is a non-increasing function of t. The normalization with respect to the initial value $D(0)$ eases the comparison among different clusters of depositors when a financial institution implements the model to internal data. Table 3 shows $TSL_\alpha(t)$ for our selected grid of confidence levels and time horizons, while the right-hand side of Figure 3 displays the projected evolution of the TSL_α. From the projection of the TSL_α, one can appreciate the stability of the deposit volume over time. In our illustrative example, 90% and 82% of the NMDs are expected to be available after 10 years with 95% and 99% confidence level, respectively (89% and 83% in the Gaussian specification of the BDLP, similarly to the NIG specification), proving to be very stable. Analysing the expected shortfall at 97.5%, the results are slightly more pronounced for the NIG specification (81%) than for the Gaussian one (83%), signalling the ability of the NIG specification to capture a larger part of the risks embedded in the historical data.

5.1 | Stressed parameters and inclusion of bank runs

Reference 15 stresses the importance of the assumptions used in projecting future cash flows. These assumptions should be adjusted according to market conditions or bank-specific circumstances, therefore they should include all the events that could entail a significant risk in terms of liquidity management for a bank.

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TABLE 3 VaR and expected shortfall of the term structure of liquidity at different time horizons and confidence levels.

	TSL-Gaussian			TSL-NIG			Stressed TSL-NIG		
	$\text{Var}_{95\%}$	$\text{Var}_{99\%}$	$\text{ES}_{97.5\%}$	$\text{Var}_{95\%}$	$\text{Var}_{99\%}$	$\text{ES}_{97.5\%}$	$\text{Var}_{95\%}$	$\text{Var}_{99\%}$	$\text{ES}_{97.5\%}$
1 Year	92%	89%	89%	93%	90%	90%	90%	82%	82%
3 Years	90%	85%	85%	91%	87%	87%	87%	77%	77%
5 Years	89%	84%	84%	91%	85%	85%	86%	76%	75%
10 Years	89%	83%	83%	90%	82%	81%	84%	73%	73%

Bank runs are rare events and as such one natural way to model them is by means of Poisson models. Lévy processes include a compound Poisson component which accounts for big jumps,[16] allowing their marginal (time t) distributions to model heavy tails driven by skewness and kurtosis parameters. Therefore, we think that the multivariate NIG-driven OU model presented in this paper is a good candidate for being applied to this setting and for the inclusion of bank run events, that is, events occurring when a large number of customers withdraw their deposits in a relatively short period of time. These events can be triggered by a number of different causes, but as common characteristics they are typically unpredictable and idiosyncratic to one or few banks. They can have a severe impact on a bank, as the more people withdraw their deposits, the more the stability of a bank is undermined, potentially encouraging further withdrawals. Although our model is well-suited for capturing rare but severe events, these events are quite infrequent and it is difficult to observe them in the time series used for the calibration of the model. Therefore, we propose here an alternative strategy for capturing such events within our model, outside from the calibration to historical data. As a starting point of our strategy, we take a real case of bank run. In 2019, Metro Bank PLC, one of the major banks in UK, experienced a severe outflow of deposits in a relative short period of time. In the "Half Year 2019 Results",[‡] the bank reported a 25% volume outflow of deposits from commercial customers in the period from 31 December 2018 to 30 June 2019. According to what reported by the bank, the total deposits' decrease (−13%) in that reported 6-month period was "*driven by a limited number of commercial customers withdrawing deposits during intense speculation in February and May*". As mentioned above, on the one hand, that event can be incorporated in the model via calibration, but that is possible only if the model is calibrated to the specific Metro Bank PLC historical data, given that that event has been recorded in those historical data only. On the other hand, an alternative strategy can be to calibrate the model to the available data and then to stress the calibrated parameters in order to enable the model to reproduce that stress event with a predetermined confidence level. Several approaches are possible and what we propose here is one of the many. Firstly, we define the relative amount of deposit volume outflow with confidence level α at time $t+h$, given the deposits' level at time t,

$$RDO_\alpha(t,h) = 1 - VaR_\alpha\left(\frac{D(t+h)}{D(t)}\middle|\mathcal{F}_t\right),$$

where $\mathcal{F}_t$ is the natural filtration of process $\boldsymbol{X}(t)$ in (3.6). Secondly, given that we want the stress event (i.e., 25% deposit outflow in 6 months) to be reproduced by the model from t to T (the time horizon of our simulation), we need to calibrate the parameters of the model so to reproduce, at least on average, that effect from t to T. Considering that we observe the multivariate process $\boldsymbol{X}(t)$ at fixed times $t = t_0 < t_1 < \ldots < t_n = T$, with $\Delta = t_{k+1} - t_k$ constant, and that for each t_k we can compute the relative amount of deposit outflow with confidence level α and time horizon $t_k + h$, where $h > \Delta$, the following measure is defined

$$\overline{RDO}_\alpha(t,h) = \frac{1}{m+1}\sum_{k=0}^{m} RDO_\alpha(t_k,h),$$

where $m = n - \frac{h}{\Delta}$. Thirdly, the parameters of the model are "stressed" (i.e., partially re-calibrated) in order to enable the model to produce on average the stress event (i.e., an deposit volumes outflow of at least 25%) at any time $t_k + h$, given the level of NMDs at time t_k, with a predetermined level of probability $1-\alpha$ (arbitrarily set at 0.1%). Given that the event to be captured by the model is idiosyncratic and affects the level of NMDs only, we focus on the parameters governing

[‡]https://www.metrobankonline.co.uk/globalassets/documents/investor_documents/trading-announcement-h1-2019.pdf.

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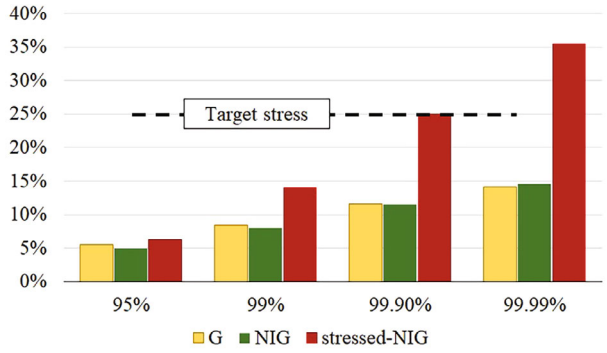

FIGURE 4 Calibration of $\overline{RDO}_{\alpha}(t, 6 \text{ months})$ to a target stress event (outflow of deposit volume of at least 25%).

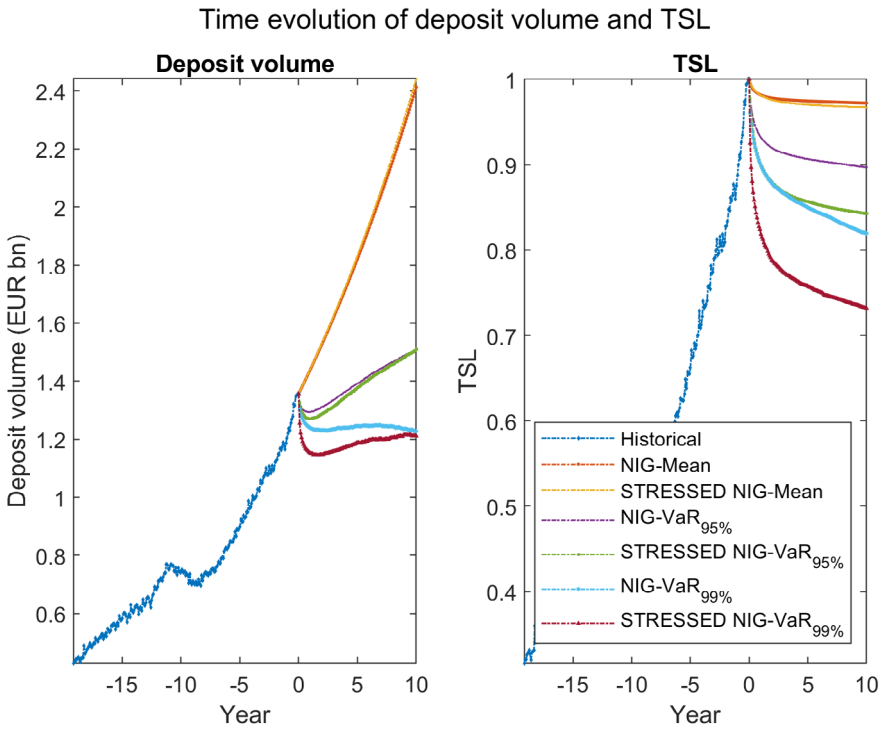

FIGURE 5 Time evolution of the deposit volume (EUR bn) and *TSL* at different confidence levels for the NIG and stressed-NIG specification. The initial deposit volume is equal to 1356 EUR bn.

the probability distribution of the noise ε_3 in the dynamics of the deposit volume $X_3(t)$ in (3.6). In order to do so, we look for the combination of the NIG parameters of the deposit volumes idiosyncratic component which results in an $\overline{RDO}_{99.9\%}(t, 6 \text{ months}) = 25\%$, where $\Delta = 1$ month and $T = t + 10$ years, that is, we calibrate the parameters such that the mean of all 6-month relative deposit outflows over a 10-year period is 25%. Therefore, a 6-month rolling horizon is considered for 10 years, where the first 6-month horizon goes from month 0 to 6, the second horizon goes from month 1 to 7 and so on, reaching the last horizon which goes from month 114 to 120. Considering that the event we want to reproduce belongs to the negative tail of the distribution of the deposit volumes, when we iteratively look for the set of "stressed" parameters, the mean and variance of the error term are kept at the level calibrated to historical data, while both the skewness and kurtosis are allowed to change.

The outcome of the calibration of the parameters to an $\overline{RDO}_{99.9\%}(t, 6 \text{ months}) = 25\%$ is represented in Figure 4. In particular, one can appreciate that the stressed parameters imply a significant change in the left tail of the distribution only. The observed average relative amount of deposits outflow in 6 months, that is, $\overline{RDO}_{95\%}(t, 6 \text{ months})$ represented in the first three columns of Figure 4, is mainly driven by the variance of the distribution and consequently the observed values of

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that measure are close for the three cases at hand (Gaussian, NIG and stressed NIG). Setting $\overline{RDO}_{99.9\%}(t, 6 \text{ months}) = 25\%$ (see ninth column of Figure 4) translates in an annual skewness of the idiosyncratic component of the deposit volume equal to −1.74 and an annual kurtosis equal to 5.17. By setting $\alpha = 269.4450$, $\beta = -256.7294$, $\delta = 0.0027$, and $\mu = 0.0086$, we obtain the stressed evolution of deposit volume and *TSL* , as shown in Table 3 and Figure 5.

In Figure 5 one can appreciate the difference between the calibrated NIG specification and the stressed-NIG, which is increasingly pronounced as one moves to the right end of the distribution and can be already very relevant from the $VaR_{99\%}$ of both the evolution of deposit volume and the TSL. As expected, the inclusion of the possibility of bank run events in the model generates an increasing of the risk measures. These measures depend on the likelihood of that event, as well as on its magnitude.

6 | CONCLUSION AND FURTHER APPLICATIONS

The contribution of the paper to the operational research literature relates to the management of NMDs for banks. We build a multivariate OU process to model the interactions among market interest rates, deposit rates and deposit volumes. By specifying the driving process to be a Lévy process, we are able to incorporate rare but significant events in the liquidity risk management of a bank. This paper also clarifies how the proposed model can be estimated via an ML approach.

We also propose an operational procedure to stress the calibrated parameters in order to enable the model to reproduce rare but significant events (e.g., bank runs) with a predetermined confidence level. As a starting point of our strategy, we take a real case of bank run. We show that the stressed parameters produce significantly increased measures of risk. Moreover, our risk factor model can also be used to perform scenario analysis. Focusing on scenarios for the market rates variable, the analysis can be based on internally selected interest rate shock scenarios, historical or hypothetical interest rate stress scenarios, Basel-prescribed interest rate shock scenarios or any additional interest rate shock scenarios required by supervisors (see Reference 1).

In addition to applications in the liquidity risk management, our multivariate Lévy-driven OU model can be used for the purpose of interest rate risk (in the Banking Book–IRRBB) management. As underlined in Reference 1, IRRBB is a material risk faced by banks, with this materiality expected to be more pronounced when interest rates may normalise from the current low levels. When interest rates change, the economic value of the NMDs changes, as well as the bank's earning capacity, measured by its net interest income (NII). In particular, the model can be used in the construction of a bond portfolio with fixed maturities, replicating the price and delta profile of the NMDs. This portfolio can be used then in the computation of the aggregated economic value and earning measures.

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DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

ORCID

Patrizia Semeraro https://orcid.org/0000-0003-2075-8196

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APPENDIX A. NORMAL INVERSE GAUSSIAN PROCESS

The univariate NIG process has been defined by Reference 17. A NIG process with parameters $\alpha > 0,\ -\alpha < \beta < \alpha,\ \delta > 0$ is a Lévy process $\{L_{NIG}(t), t \geq 0\}$ with characteristic function at time 1

$$\psi_{NIG}(u) = \exp\left(-\delta\left(\sqrt{\alpha^2 - (\beta + iu)^2} - \sqrt{\alpha^2 - \beta^2}\right)\right).$$

The NIG process has been chosen for its ability to accommodate skewness and kurtosis and for its analytical tractability. We recall below the mean m, the variance v, the skewness s and the kurtosis k of the NIG distribution:

$$m = \frac{\delta\beta}{\sqrt{\alpha^2 - \beta^2}},$$
$$v = \alpha^2\delta(\alpha^2 - \beta^2)^{-\frac{3}{2}},$$
$$s = 3\beta\alpha^{-1}\delta^{-\frac{1}{2}}(\alpha^2 - \beta^2)^{-\frac{1}{4}},$$
$$k = 3\left(1 + \frac{\alpha^2 + 4\beta^2}{\delta\alpha^2\sqrt{\alpha^2 - \beta^2}}\right).$$

The NIG class is closed under convolution provided the parameters α and β are fixed, that is,

$$NIG(\alpha, \beta, \delta_1) * NIG(\alpha, \beta, \delta_2) = NIG(\alpha, \beta, \delta_1 + \delta_2), \tag{A1}$$

α and β are the skewness and tail parameters.

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