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Research Article

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Abstract: We consider the non-symmetric coupling of finite and boundary elements to solve second-order nonlinear partial differential equations defined in unbounded domains. We present a novel condition that ensures that the associated semi-linear form induces a strongly monotone operator, keeping track of the dependence on the linear combination of the interior domain equation with the boundary integral one. We show that an optimal ellipticity condition, relating the nonlinear operator to the contraction constant of the shifted double-layer integral operator, is guaranteed by choosing a particular linear combination. These results generalize those obtained by Of and Steinbach [Is the one-equation coupling of finite and boundary element methods always stable?, *ZAMM Z. Angew. Math. Mech.* **93** (2013), no. 6–7, 476–484] and [On the ellipticity of coupled finite element and one-equation boundary element methods for boundary value problems, *Numer. Math.* **127** (2014), no. 3, 567–593], and by Steinbach [A note on the stable one-equation coupling of finite and boundary elements, *SIAM J. Numer. Anal.* **49** (2011), no. 4, 1521–1531], where the simple sum of the two coupling equations has been considered. Numerical examples confirm the theoretical results on the sharpness of the presented estimates.

Keywords: Finite Elements, Boundary Elements, Non-symmetric Coupling

MSC 2010: 65N30, 65N12, 65N38

1 Introduction

The coupling of finite and boundary element methods is well established in many applications, in particular when considering partial differential equations defined in unbounded domains, with non-constant coefficients restricted to a bounded region.

At least two types of coupling have been proposed and extensively studied: the symmetric (or Costabel–Han) coupling [4, 9] and the non-symmetric (or Johnson–Nédélec) one [3, 11]. The former approach, relying on the symmetric formulation of the exterior Steklov–Poincaré operator, yields a symmetric and non-positive definite scheme, providing stability and a satisfying error analysis. However, involving a boundary integral operator of hypersingular type, it turns out to be quite onerous from the computational point of view and, even if there are efficient implementations available, its use is still not very popular in more advanced applications. On the contrary, the non-symmetric coupling, relying only on the use of single- and double-layer boundary integral operators, turns out to be cheaper and easier to implement and, consequently, more appealing from the engineering point of view.

In [11], Johnson and Nédélec have proved that this coupling is a well-posed and stable procedure, assuming the compactness of the double-layer boundary operator and, hence, provided that smooth interfaces boundaries are considered. In [15], Sayas proved the stability of the non-symmetric coupling in case of arbitrary interfaces for free-space transmission problems. Successively, in [17], the ellipticity of the related bilinear form was

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obtained. These results were further extended and improved in [12, 13] where a sufficient and necessary condition, relating the diffusion matrix and the contraction constant of the shifted double-layer integral operator, was given. The necessity of conditions weaker than those present in literature, even for problems with constant coefficients, is of particular interest when dealing with the coupling of a generalized Galerkin method, such as, e.g., the recent virtual element method, with a boundary one. Indeed, the ellipticity constants of the approximated bilinear forms may not be explicitly known, or they could be badly dependent on the order of the method. For these couplings, the stability on arbitrary interfaces is still an open issue (see [6, 7]).

In this paper, we generalize the results given previously in [12, 13, 17]. Moreover, we adapt the approaches of [1, 2] to deal with (possible) nonlinearities. The novelty of the paper consists in studying the dependence of the ellipticity constant on the linear combination coefficients of the interior domain equation with the boundary integral one. We show that there exists an optimal computable choice of these coefficients which ensures the weakest ellipticity condition in terms of the ellipticity constant of the nonlinear operator. The proof relies on a generalization of the contraction property of the shifted double-layer integral operator; see [19]. We mainly consider exterior boundary value problems, and in the last section, we present a similar result for transmission interface problems. The theoretical analysis presented here can be extended to the solution of other boundary value problems, as done in [13].

The paper is organized as follows. In the next section, we present the model problem and its variational formulation for the one-equation coupling of finite with boundary elements. In Section 3, we present a generalization of the contraction property proved in [19], in a general framework involving arbitrary Hilbert spaces. In Section 4, the main result for an exterior boundary value problem on the ellipticity of the semi-linear form related to the non-symmetric coupling is proved. In Section 5, we present two numerical tests which confirm the sharpness of the theoretical results. Finally, in the last section, we show how to adapt the main ideas to transmission interface problems.

2 Model Problem and Variational Formulation

Let $\Omega_0 \subset \mathbb{R}^n$ $(n = 2, 3)$ be a bounded domain with a Lipschitz boundary $\Gamma_0 = \partial\Omega_0$ having positive Hausdorff measure. Its exterior region is decomposed in two non-overlapping subdomains Ω and Ω_∞ with interface Γ, i.e.

$$\mathbb{R}^n \setminus \Omega_0 = \Omega_\infty \cup \Gamma \cup \Omega.$$

We consider an exterior boundary value problem in the unknown u, u_∞,

$$\begin{cases} -\operatorname{div}(\mathcal{U}\nabla u(\mathbf{x})) = f(\mathbf{x}), & \mathbf{x} \in \Omega, \\ -\Delta u_\infty(\mathbf{x}) = 0, & \mathbf{x} \in \Omega_\infty, \end{cases} \tag{2.1}$$

with boundary, transmission and radiation conditions

$$\begin{cases} u(\mathbf{x}) = 0, & \mathbf{x} \in \Gamma_0, \\ u(\mathbf{x}) = u_\infty(\mathbf{x}), \quad \mathbf{n}(\mathbf{x}) \cdot (\mathcal{U}\nabla u(\mathbf{x})) = -\mathbf{n}_\infty(\mathbf{x}) \cdot \nabla u_\infty(\mathbf{x}), & \mathbf{x} \in \Gamma, \\ u_\infty(\mathbf{x}) = \gamma + O\Big(\dfrac{1}{\|\mathbf{x}\|}\Big), & \|\mathbf{x}\| \to \infty. \end{cases} \tag{2.2}$$

We assume $f \in L^2(\Omega)$, the asymptotic behavior of u_∞ at infinity $\gamma \in \mathbb{R}$, and $\mathcal{U}\colon \mathbb{R}^n \to \mathbb{R}^n$ a coefficient function Lipschitz continuous and strongly monotone, i.e. there exists $c_{\mathrm{ell}}(\mathcal{U}) > 0$ such that

$$(\mathcal{U}\nabla v - \mathcal{U}\nabla w, \nabla v - \nabla w)_{L^2(\Omega)} \geq c_{\mathrm{ell}}(\mathcal{U})|v - w|^2_{H^1(\Omega)} \tag{2.3}$$

for all $v, w \in H^1(\Omega)$. Here, $(\cdot\,,\cdot)_{L^2(\Omega)}$ denotes the $L^2(\Omega)$-scalar product, and $|\cdot|_{H^1(\Omega)}$ the $H^1(\Omega)$-seminorm.

Note that $\mathbf{n}$, $\mathbf{n}_\infty$ denote the exterior normal vectors on Γ with respect to the domains Ω and Ω_∞, respectively. Introducing $\lambda(\mathbf{x}) := \mathbf{n}(\mathbf{x}) \cdot (A(\mathbf{x})\nabla u(\mathbf{x}))$ for $\mathbf{x} \in \Gamma$, to ensure the correct radiation condition for u_∞, we observe that $\lambda \in H_0^{-\frac{1}{2}}(\Gamma) := \{\lambda \in H^{-\frac{1}{2}}(\Gamma) : \langle 1, \lambda\rangle_\Gamma = 0\}$, where $\langle\cdot\,,\cdot\rangle_\Gamma$ represents the duality pairing between $H^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$.

By considering the Neumann transmission condition in (2.2), the variational formulation of the interior Poisson equation in (2.1) is to find $u \in H^1_{0,\Gamma_0}(\Omega) := \{u \in H^1(\Omega) : u(\mathbf{x}) = 0,\ \mathbf{x} \in \Gamma_0\}$ such that

$$(\mathcal{U}\nabla u, \nabla v)_{L^2(\Omega)} - \langle v, \lambda\rangle_\Gamma = (f, v)_{L^2(\Omega)} \tag{2.4}$$

is satisfied for all $v \in H^1_{0,\Gamma_0}(\Omega)$. Moreover, the weak formulation of the boundary integral equation related to the Laplace equation in Ω_∞ reads as follows: find $\lambda \in H_0^{-\frac{1}{2}}(\Gamma)$ such that

$$\langle \mathrm{V}\lambda, \mu\rangle_\Gamma + \Big\langle\Big(\frac{1}{2}\mathrm{I} - \mathrm{K}\Big)u, \mu\Big\rangle_\Gamma = 0 \tag{2.5}$$

for all $\mu \in H_0^{-\frac{1}{2}}(\Gamma)$, where V and K represent, respectively, the single- and double-layer integral operators, defined by

$$\mathrm{V}\chi(\mathbf{x}) := \int_\Gamma G(\mathbf{x}, \mathbf{y})\chi(\mathbf{y})\, d\Gamma_\mathbf{y} \quad \mathrm{K}w(\mathbf{x}) := \int_\Gamma w(\mathbf{y})\mathbf{n}(\mathbf{y}) \cdot \nabla_\mathbf{y} G(\mathbf{x}, \mathbf{y})\, d\Gamma_\mathbf{y}$$

for $\chi \in H^{-\frac{1}{2}}(\Gamma)$, $w \in H^{\frac{1}{2}}(\Gamma)$ and $\mathbf{x} \in \Gamma$, with

$$G(\mathbf{x}, \mathbf{y}) := \begin{cases} -\frac{1}{2\pi} \log\|\mathbf{x} - \mathbf{y}\| & \text{for } n = 2, \\ \frac{1}{4\pi} \frac{1}{\|\mathbf{x}-\mathbf{y}\|} & \text{for } n = 3 \end{cases}$$

the fundamental solution of the Laplace operator.

Combining (2.4) and (2.5), the weak formulation of the exterior boundary value problem (2.1)–(2.2) associated to the one-equation coupling reads as follows: find $(u, \lambda) \in H^1_{0,\Gamma_0}(\Omega) \times H_0^{-\frac{1}{2}}(\Gamma)$ such that

$$\begin{cases} (\mathcal{U}\nabla u, \nabla v)_{L^2(\Omega)} - \langle v, \lambda\rangle_\Gamma = (f, v)_{L^2(\Omega)} & \text{for all } v \in H^1_{0,\Gamma_0}(\Omega), \\ \Big\langle\Big(\frac{1}{2}\mathrm{I} - \mathrm{K}\Big)u, \mu\Big\rangle_\Gamma + \langle \mathrm{V}\lambda, \mu\rangle_\Gamma = 0 & \text{for all } \mu \in H_0^{-\frac{1}{2}}(\Gamma). \end{cases} \tag{2.6}$$

To illustrate the main result, we need to list several properties of boundary integral operators and related norm equivalences. We recall that the single-layer integral operator $\mathrm{V}\colon H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ is $H_0^{-\frac{1}{2}}(\Gamma)$-elliptic (see [10]), i.e. there exists a constant $c_\mathrm{V} > 0$ such that

$$\langle \mathrm{V}\mu, \mu\rangle_\Gamma \geq c_\mathrm{V}\|\mu\|^2_{H^{-\frac{1}{2}}(\Gamma)} \quad \text{for all } \mu \in H_0^{-\frac{1}{2}}(\Gamma). \tag{2.7}$$

Let us define the subspace of $H^{\frac{1}{2}}(\Gamma)$,

$$H_0^{\frac{1}{2}}(\Gamma) := \{v \in H^{\frac{1}{2}}(\Gamma) : \langle v, \mu_{\mathrm{eq}}\rangle_\Gamma = 0\},$$

where μ_{eq} is the natural density, $\mu_{\mathrm{eq}} := \mathrm{V}^{-1}1$. Since $\mathrm{V}\colon H_0^{-\frac{1}{2}}(\Gamma) \to H_0^{\frac{1}{2}}(\Gamma)$ is positive definite and bounded, $\|\cdot\|_\mathrm{V} := \sqrt{\langle \mathrm{V}\cdot, \cdot\rangle_\Gamma}$ defines an equivalent norm in $H_0^{-\frac{1}{2}}(\Gamma)$ (see, e.g., [16, Theorem 2.6]), and correspondingly, $\|\cdot\|_{\mathrm{V}^{-1}} := \sqrt{\langle \cdot, \mathrm{V}^{-1}\cdot\rangle_\Gamma}$ defines an equivalent norm in $H_0^{\frac{1}{2}}(\Gamma)$.

The ellipticity results obtained in [12] are mainly based on the contraction property of the double-layer integral operator (first obtained in [19])

$$(1 - c_\mathrm{K})\|v\|_{\mathrm{V}^{-1}} \leq \Big\|\Big(\frac{1}{2}\mathrm{I} + \mathrm{K}\Big)v\Big\|_{\mathrm{V}^{-1}} \leq c_\mathrm{K}\|v\|_{\mathrm{V}^{-1}} \quad \text{for all } v \in H_0^{\frac{1}{2}}(\Gamma), \tag{2.8}$$

with the positive constant

$$c_\mathrm{K} := \frac{1}{2} + \sqrt{\frac{1}{4} - c_\mathrm{V} c_\mathrm{D}}. \tag{2.9}$$

Here, c_V is defined in (2.7), and $c_\mathrm{D} > 0$ is the ellipticity constant of the hypersingular boundary integral operator

$$\mathrm{D}v(\mathbf{x}) := -\mathbf{n}(\mathbf{x}) \cdot \nabla_\mathbf{x}\Big(\int_\Gamma \mathbf{n}(\mathbf{y}) \cdot \nabla_\mathbf{y} G(\mathbf{x}, \mathbf{y})v(\mathbf{y})\, d\Gamma_\mathbf{y}\Big), \quad \mathbf{x} \in \Gamma,$$

in the Hilbert space $H_0^{\frac{1}{2}}(\Gamma)$, i.e.

$$\langle v, \mathrm{D}v\rangle_\Gamma \geq c_\mathrm{D}\|v\|^2_{H^{\frac{1}{2}}(\Gamma)} \quad \text{for all } v \in H_0^{\frac{1}{2}}(\Gamma). \tag{2.10}$$

We define the interior Steklov–Poincaré operator $\mathrm{S}\colon H_0^{\frac{1}{2}}(\Gamma) \to H_0^{-\frac{1}{2}}(\Gamma)$ as

$$\mathrm{S}v(\mathbf{x}) := \mathbf{n}(\mathbf{x}) \cdot \nabla u(\mathbf{x}) \quad \mathbf{x} \in \Gamma, \tag{2.11}$$

where $u \in H^1(\Omega \cup \Gamma_0 \cup \Omega_0)$ solves the Dirichlet–Laplace problem

$$\begin{cases} \Delta u(\mathbf{x}) = 0, & \mathbf{x} \in \Omega \cup \Gamma_0 \cup \Omega_0, \\ u(\mathbf{x}) = v(\mathbf{x}), & \mathbf{x} \in \Gamma. \end{cases}$$

It is worth to point out that this operator can be characterized also by

$$\mathrm{S} = \mathrm{V}^{-1}\Big(\frac{1}{2}\mathrm{I} + \mathrm{K}\Big). \tag{2.12}$$

A strictly related operator more suitable for exterior problems like the one we are considering is $\mathrm{S}_{\mathrm{int}}$, the interior Steklov–Poincaré operator associated to the following Dirichlet–Laplace boundary value problem

$$\begin{cases} \Delta u(\mathbf{x}) = 0, & \mathbf{x} \in \Omega, \\ u(\mathbf{x}) = v(\mathbf{x}), & \mathbf{x} \in \Gamma, \\ u(\mathbf{x}) = 0, & \mathbf{x} \in \Gamma_0. \end{cases}$$

As in (2.11), the definition of $\mathrm{S}_{\mathrm{int}}$ is

$$\mathrm{S}_{\mathrm{int}} v(\mathbf{x}) := \mathbf{n}(\mathbf{x}) \cdot \nabla u(\mathbf{x}), \quad \mathbf{x} \in \Gamma. \tag{2.13}$$

The operators S and $\mathrm{S}_{\mathrm{int}}$ are related by the spectral equivalence inequality

$$\mu_{\min} \langle v, \mathrm{S}v\rangle_\Gamma \le \langle v, \mathrm{S}_{\mathrm{int}} v\rangle_\Gamma \quad \text{for all } v \in H_0^{\frac{1}{2}}(\Gamma), \tag{2.14}$$

for a positive constant $\mu_{\min} > 0$. Now we are able to state the main result of the paper.

In order to analyze the stability of the coupled system (2.6), we introduce the semi-linear forms $\mathcal{A}_\alpha^\beta$, multiplying and summing (2.4) and (2.5) by two coefficients $\alpha, \beta \in \mathbb{R} \setminus \{0\}$. For $\hat{u} = (u, \lambda)$, $\hat{v} = (v, \mu) \in H^1_{0,\Gamma_0}(\Omega) \times H_0^{-\frac{1}{2}}(\Gamma)$, we then obtain

$$\mathcal{A}_\alpha^\beta(\hat{u}, \hat{v}) := \alpha[(\mathfrak{U}\nabla u, \nabla v)_{L^2(\Omega)} - \langle v, \lambda\rangle_\Gamma] + \beta\Big[\langle \mathrm{V}\lambda, \mu\rangle_\Gamma + \Big\langle \Big(\frac{1}{2}\mathrm{I} - \mathrm{K}\Big)u, \mu\Big\rangle_\Gamma\Big]. \tag{2.15}$$

The coupling of equations (2.4) and (2.5) is equivalent to finding $\hat{u} \in H^1_{0,\Gamma_0}(\Omega) \times H_0^{-\frac{1}{2}}(\Gamma)$ such that

$$\mathcal{A}_\alpha^\beta(\hat{u}, \hat{v}) = \alpha(f, v)_{L^2(\Omega)}$$

for all $\hat{v} = (v, \mu) \in H^1_{0,\Gamma_0}(\Omega) \times H_0^{-\frac{1}{2}}(\Gamma)$. By varying α and β, all these problems are equivalent, but the ellipticity of the semi-linear forms $\mathcal{A}_\alpha^\beta$ clearly depends on the choice of α and β. By scaling, it suffices to consider $\alpha = 1$, although, in the next section, we will show some properties of the inherited operator $(\alpha - \frac{1}{2}\beta)\mathrm{I} + \beta\mathrm{K}$ for general α and β.

Johnson and Nédélec in [11] have shown (for $\mathfrak{U} = \mathrm{I}$) that $\mathcal{A}_1^2$ is a Fredholm operator when we assume that the double-layer integral operator $\mathrm{K}\colon H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ is compact, but the compactness of K allows only the consideration of Γ being Lyapunov regular, i.e. of class $C^{1,\theta}$ for some $\theta > 0$. Steinbach in [17], and successively the same author with Of in [12, 13], has shown that $\mathcal{A}_1^1$ is elliptic, also for Lipschitz interfaces, when $\mathfrak{U}$ is a coefficient matrix, whose minimal eigenvalue is bigger than $\frac{c_{\mathrm{K}}}{(4\mu_{\min})}$. We recall that $c_{\mathrm{K}} \in [\frac{1}{2}, 1)$ is the contraction constant defined in (2.9), depending only on Γ, and $\mu_{\min} > 0$ is defined in (2.14), depending both on Γ and Γ_0. The analysis in [17] was extended to nonlinear operators by Aurada et al. in [1] by showing the strong monotonicity of the associated semi-linear forms. In the next sections, we will show that, for each fixed Γ, we can choose β^* such that $\mathcal{A}_1^{\beta^*}$ is a strongly monotone semi-linear form for a larger range than that obtained in [12]. Precisely, we will obtain that if

$$c_{\mathrm{ell}}(\mathfrak{U}) > \frac{1 - 2\sqrt{c_{\mathrm{K}}(1 - c_{\mathrm{K}})}}{4\mu_{\min}(1 - \sqrt{c_{\mathrm{K}}(1 - c_{\mathrm{K}})})},$$

then $\mathcal{A}_1^{\beta^*}$ is strongly monotone. In particular, if Γ is a circle, and so $c_{\mathrm{K}} = \frac{1}{2}$, then $\mathcal{A}_1^2$ induces always a strongly monotone operator.

Remark 1. We point out that the computation of c_K is not needed in the applications. Indeed, the choice of a particular scaling parameter β is only useful in the theory to prove the stability since solving the scaled problem is equivalent to solving the coupling with coefficient $\beta = 1$.

3 Generalized Contraction Properties

In order to improve the results in [12], we first need a generalized contraction property for $(\alpha - \frac{1}{2}\beta)\mathrm{I} + \beta\mathrm{K}$. Costabel, in [5], observed that the contraction property (2.8) of $\frac{1}{2}\mathrm{I} + \mathrm{K}$ is essentially based on the following idea: "if a number is bigger than its square, then it must lie between 0 and 1". For operators, this idea can be stated as follows.

Lemma 1. *Let A, B_1 be bounded operators on a Hilbert space $\mathcal{H}$, with B_1 self-adjoint. If it holds*

$$B_1 = B_1^2 + A \tag{3.1}$$

and there exists $c_A > 0$ such that $(Av, v)_{\mathcal{H}} \geq c_A \|v\|^2_{\mathcal{H}}$ for all $v \in \mathcal{H}$, then $c_A \leq \frac{1}{4}$ and B_1 is a contraction satisfying

$$(1 - c^{\mathrm{up}})\|v\|_{\mathcal{H}} \leq \|B_1 v\|_{\mathcal{H}} \leq c^{\mathrm{up}}\|v\|_{\mathcal{H}} \quad \text{for all } v \in \mathcal{H},$$

with

$$c^{\mathrm{up}} := \frac{1}{2} + \sqrt{\frac{1}{4} - c_A}. \tag{3.2}$$

This lemma can be used with $A = \mathrm{V}\,\mathrm{D}$ and $B_1 = \frac{1}{2}\mathrm{I} + \mathrm{K}$, in the Hilbert space $\mathcal{H} = H_0^{\frac{1}{2}}(\Gamma)$ equipped the norm $\|\cdot\|^2_{\mathrm{V}^{-1}}$. Property (3.1) reduces to

$$\frac{1}{2}\mathrm{I} + \mathrm{K} = \left(\frac{1}{2}\mathrm{I} + \mathrm{K}\right)^2 + \mathrm{V}\,\mathrm{D},$$

but this is exactly the well-known relation (see, e.g., [10])

$$\left(\frac{1}{2}\mathrm{I} + \mathrm{K}\right)\left(\frac{1}{2}\mathrm{I} - \mathrm{K}\right) = \mathrm{V}\,\mathrm{D}.$$

The symmetry relation $\mathrm{K}\,\mathrm{V} = \mathrm{V}\,\mathrm{K}'$ (see, e.g., [16]) with the adjoint double-layer operator integral operator K', implies that $B_1 = \frac{1}{2}\mathrm{I} + \mathrm{K}$ is self-adjoint in the $(\cdot,\cdot)_{\mathcal{H}}$ scalar product. In fact, we can write, for $v, w \in H_0^{\frac{1}{2}}(\Gamma)$,

$$(B_1 v, w)_{\mathcal{H}} = \left\langle w, \mathrm{V}^{-1}\left(\frac{1}{2}\mathrm{I} + \mathrm{K}\right)v\right\rangle_\Gamma = \left\langle v, \left(\frac{1}{2}\mathrm{I} + \mathrm{K}'\right)\mathrm{V}^{-1}w\right\rangle_\Gamma = \left\langle\left(\frac{1}{2}\mathrm{I} + \mathrm{K}\right)w, \mathrm{V}^{-1}v\right\rangle_\Gamma = (v, B_1 w)_{\mathcal{H}}.$$

Finally, since the operator $\mathrm{V}\,\mathrm{D}$ is positive semi-definite in the inner product $\langle\cdot, \mathrm{V}^{-1}\cdot\rangle_\Gamma$ with ellipticity constant $c_A = c_\mathrm{V} c_\mathrm{D}$, with c_V and c_D defined in (2.7) and (2.10), respectively, the contraction property (2.8) for $\frac{1}{2}\mathrm{I} + \mathrm{K}$ easily follows from Lemma 1.

We want to apply a similar result to the operators $\mathrm{V}\,\mathrm{D}$ and $(\alpha - \frac{1}{2}\beta)\mathrm{I} + \beta\mathrm{K}$. Since condition (3.1) is not, in general, satisfied by these operators, we extend the previous lemma to the following situation.

Lemma 2. *Let A, B_2 be bounded operators on a Hilbert space $\mathcal{H}$, with B_2 self-adjoint. If it holds*

$$\gamma_1 B_2 = B_2^2 + \gamma_2 A + \gamma_3 \mathrm{I} \tag{3.3}$$

for non-negative numbers γ_1, γ_2 and $\gamma_3 \in \mathbb{R}$, and there exists $c_A > 0$ such that $(Av, v)_{\mathcal{H}} \geq c_A\|v\|^2_{\mathcal{H}}$ for all $v \in \mathcal{H}$, then $\gamma_2 c_A \leq \frac{\gamma_1^2}{4} - \gamma_3$ and B_2 is a contraction satisfying

$$(\gamma_1 - c^{\mathrm{up}}_{\gamma_1,\gamma_2,\gamma_3})\|v\|_{\mathcal{H}} \leq \|B_2 v\|_{\mathcal{H}} \leq c^{\mathrm{up}}_{\gamma_1,\gamma_2,\gamma_3}\|v\|_{\mathcal{H}} \quad \text{for all } v \in \mathcal{H},$$

with

$$c^{\mathrm{up}}_{\gamma_1,\gamma_2,\gamma_3} := \frac{\gamma_1}{2} + \sqrt{\frac{\gamma_1^2}{4} - (\gamma_2 c_A + \gamma_3)}. \tag{3.4}$$

Proof. Using the self-adjointness of B_2, assumption (3.3), the ellipticity of A and Cauchy–Schwarz, we obtain

$$\begin{aligned}\|B_2 v\|^2_{\mathcal{H}} &= (B_2 v, B_2 v)_{\mathcal{H}} = (B_2^2 v, v)_{\mathcal{H}} = ((\gamma_1 B_2 - \gamma_2 A - \gamma_3 \mathrm{I})v, v)_{\mathcal{H}} \\ &\le \gamma_1 (B_2 v, v)_{\mathcal{H}} - c_A \gamma_2 \|v\|^2_{\mathcal{H}} - \gamma_3 \|v\|^2_{\mathcal{H}} \le \gamma_1 \|B_2 v\|_{\mathcal{H}} \|v\|_{\mathcal{H}} - (c_A \gamma_2 + \gamma_3)\|v\|^2_{\mathcal{H}}.\end{aligned}$$

Defining $t := \frac{\|B_2 v\|_{\mathcal{H}}}{\|v\|_{\mathcal{H}}}$, the latter inequality is equivalent to $t^2 - \gamma_1 t + c_A \gamma_2 + \gamma_3 \le 0$, which implies

$$\gamma_1 - c^{\mathrm{up}}_{\gamma_1,\gamma_2,\gamma_3} \le t \le c^{\mathrm{up}}_{\gamma_1,\gamma_2,\gamma_3},$$

with $c^{\mathrm{up}}_{\gamma_1,\gamma_2,\gamma_3}$ defined in (3.4). □

To show that the choice $B_2 = (\alpha - \frac{1}{2}\beta)\mathrm{I} + \beta \mathrm{K}$ satisfies condition (3.3) with $A = \mathrm{V}\,\mathrm{D}$ for some γ_1, γ_2 and γ_3, we prove an auxiliary result: if an operator B_2 is a linear combination of an operator B_1, satisfying property (3.1), and the identity, i.e. it holds

$$B_2 = \omega_1 B_1 + \omega_2 \mathrm{I} \quad \text{and} \quad B_1 = B_1^2 + A \tag{3.5}$$

for some $\omega_1, \omega_2 \in \mathbb{R}$, then condition (3.3) is fulfilled with

$$\gamma_1 = \omega_1 + 2\omega_2, \quad \gamma_2 = \omega_1^2 \quad \text{and} \quad \gamma_3 = \omega_1\omega_2 + \omega_2^2. \tag{3.6}$$

Indeed, this is an easy calculation:

$$\begin{aligned}B_2^2 &= (\omega_1 B_1 + \omega_2 \mathrm{I})^2 = \omega_1^2 B_1^2 + \omega_2^2 \mathrm{I} + 2\omega_1\omega_2 B_1 = \omega_1^2 B_1 - \omega_1^2 A + \omega_2^2 \mathrm{I} + 2\omega_1\omega_2 B_1 \\ &= \omega_1 B_2 - \omega_1\omega_2 \mathrm{I} - \omega_1^2 A + \omega_2^2 \mathrm{I} + 2\omega_2 B_2 - 2\omega_2^2 \mathrm{I} = (\omega_1 + 2\omega_2) B_2 - \omega_1^2 A - (\omega_1\omega_2 + \omega_2^2)\mathrm{I}.\end{aligned}$$

The latter remark turns out to be useful in our case since it holds

$$\Big(\alpha - \frac{1}{2}\beta\Big)\mathrm{I} + \beta \mathrm{K} = \beta\Big(\frac{1}{2}\mathrm{I} + \mathrm{K}\Big) + (\alpha - \beta)\mathrm{I}. \tag{3.7}$$

Summarizing the previous results, we obtain the following contraction property.

Theorem 1. *Let $\alpha, \beta \in \mathbb{R}$, satisfying $2\alpha - \beta \ge 0$. Then it holds*

$$(2\alpha - \beta - c_{\mathrm{K}}^{\alpha,\beta})\|v\|_{\mathrm{V}^{-1}} \le \Big\|\Big(\Big(\alpha - \frac{1}{2}\beta\Big)\mathrm{I} + \beta \mathrm{K}\Big)v\Big\|_{\mathrm{V}^{-1}} \le c_{\mathrm{K}}^{\alpha,\beta}\|v\|_{\mathrm{V}^{-1}}$$

for all $v \in H_0^{\frac{1}{2}}(\Gamma)$, with

$$c_{\mathrm{K}}^{\alpha,\beta} := \frac{2\alpha - \beta}{2} + |\beta|\sqrt{\frac{1}{4} - c_{\mathrm{V}} c_{\mathrm{D}}}. \tag{3.8}$$

Proof. We take $A = \mathrm{V}\,\mathrm{D}$, and $B_2 = (\alpha - \frac{1}{2}\beta)\mathrm{I} + \beta \mathrm{K}$ in Lemma 2. These operators satisfy (3.5) with $B_1 = \frac{1}{2}\mathrm{I} + \mathrm{K}$, $\omega_1 = \beta$ and $\omega_2 = \alpha - \beta$ by virtue of (3.7). Then condition (3.3) is satisfied with, recalling (3.6),

$$\gamma_1 = 2\alpha - \beta, \quad \gamma_2 = \beta^2 \quad \text{and} \quad \gamma_3 = \alpha(\alpha - \beta).$$

Easy calculations prove the assertion, observing that the assumption $2\alpha - \beta \ge 0$ is equivalent to $\gamma_1 \ge 0$. □

To collect all the tools useful to prove our main result, it remains to show a generalization of the bound that relates $(\alpha - \frac{1}{2}\beta)\mathrm{I} + \beta \mathrm{K}$ with the interior Steklov–Poincaré operator S defined in (2.11). In [12], in fact, was proved that, for all $v \in H_0^{\frac{1}{2}}(\Gamma)$, there holds the inequality

$$\Big\|\Big(\frac{1}{2}\mathrm{I} + \mathrm{K}\Big)v\Big\|^2_{\mathrm{V}^{-1}} \le c_{\mathrm{K}} \langle v, \mathrm{S}v\rangle_\Gamma. \tag{3.9}$$

Observing that, using the characterization (2.12), we can write, for $v \in \mathcal{H} = H_0^{\frac{1}{2}}(\Gamma)$ endowed with the norm $\|\cdot\|_{\mathrm{V}^{-1}}$,

$$\langle v, \mathrm{S}v\rangle_\Gamma = (B_1 v, v)_{\mathcal{H}},$$

where $B_1 = \frac{1}{2}\mathrm{I} + \mathrm{K}$, we see that (3.9) is exactly

$$\|B_1 v\|^2_{\mathcal{H}} \le c^{\mathrm{up}} (B_1 v, v)_{\mathcal{H}}.$$

As we did before for the contraction property, now we derive a generalization of inequality (3.9) for arbitrary operators.

Lemma 3. *Let A, B_1, B_2 be bounded operators on a Hilbert space $\mathcal{H}$, with B_1 self-adjoint. Assume the following hold:*

$$B_1 = B_1^2 + A, \quad B_2 = \omega_1 B_1 + \omega_2 \mathrm{I} \tag{3.10}$$

for some $\omega_1, \omega_2 \in \mathbb{R}$ satisfying $\omega_1 + 2\omega_2 > 0$, $\omega_1 > 0$, and there exists $c_A > 0$ such that $(Av, v)_{\mathcal{H}} \geq c_A \|v\|^2_{\mathcal{H}}$ for all $v \in \mathcal{H}$. Moreover, suppose that $\omega_1^2 c_A - \omega_2^2 \geq 0$. Then it holds

$$\|B_2 v\|^2_{\mathcal{H}} \leq \frac{(c^{\mathrm{up}}_{\omega_1,\omega_2})^2}{c^{\mathrm{up}}} (B_1 v, v)_{\mathcal{H}} \quad \textit{for all } v \in \mathcal{H},$$

where

$$c^{\mathrm{up}}_{\omega_1,\omega_2} := \frac{\omega_1 + 2\omega_2}{2} + |\omega_1| \sqrt{\frac{1}{4} - c_A}. \tag{3.11}$$

Proof. We recall that, by virtue of assumptions (3.10), there exist $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$ such that

$$\gamma_1 B_2 = B_2^2 + \gamma_2 A + \gamma_3 \mathrm{I}, \tag{3.12}$$

precisely those defined in (3.6). Multiplying the second equation of (3.10) by γ_1 and using (3.12), we can deduce

$$\gamma_1 \omega_1 B_1 = \gamma_1 B_2 - \gamma_1 \omega_2 \mathrm{I} = B_2^2 + \gamma_2 A + (\gamma_3 - \gamma_1 \omega_2)\mathrm{I}.$$

Testing with $v \in \mathcal{H}$, using the ellipticity of A and the self-adjointness of B_2, we obtain

$$\begin{aligned}\gamma_1 \omega_1 (B_1 v, v)_{\mathcal{H}} &= (B_2^2 v, v)_{\mathcal{H}} + \gamma_2 (Av, v)_{\mathcal{H}} + (\gamma_3 - \gamma_1 \omega_2)\|v\|^2_{\mathcal{H}} \\ &\geq \|B_2 v\|^2_{\mathcal{H}} + (\gamma_2 c_A + \gamma_3 - \gamma_1 \omega_2)\|v\|^2_{\mathcal{H}}.\end{aligned}$$

Assuming that $\gamma_2 c_A + \gamma_3 - \gamma_1 \omega_2 \geq 0$, i.e. $\omega_1^2 c_A - \omega_2^2 \geq 0$, the latter term can be bounded using Lemma 2, provided that $\gamma_1 \geq 0$, i.e. $\omega_1 + 2\omega_2 \geq 0$, as

$$\gamma_1 \omega_1 (B_1 v, v)_{\mathcal{H}} \geq \|B_2 v\|^2_{\mathcal{H}} + \frac{\gamma_2 c_A + \gamma_3 - \gamma_1 \omega_2}{(c^{\mathrm{up}}_{\gamma_1,\gamma_2,\gamma_3})^2} \|B_2 v\|^2_{\mathcal{H}}.$$

By using (3.6), we rewrite the above inequality only in terms of ω_1 and ω_2 as follows:

$$(\omega_1 + 2\omega_2)\omega_1 (B_1 v, v)_{\mathcal{H}} \geq \left(1 + \frac{\omega_1^2 c_A - \omega_2^2}{(c^{\mathrm{up}}_{\omega_1,\omega_2})^2}\right) \|B_2 v\|^2_{\mathcal{H}},$$

where $c^{\mathrm{up}}_{\omega_1,\omega_2}$ is defined in (3.11). Since it easy to verify that

$$1 + \frac{\omega_1^2 c_A - \omega_2^2}{(c^{\mathrm{up}}_{\omega_1,\omega_2})^2} = \omega_1(\omega_1 + 2\omega_2) \frac{c^{\mathrm{up}}}{(c^{\mathrm{up}}_{\omega_1,\omega_2})^2},$$

with c^{up} as in (3.2), we conclude that

$$\omega_1(\omega_1 + 2\omega_2)(B_1 v, v)_{\mathcal{H}} \geq \omega_1(\omega_1 + 2\omega_2) \frac{c^{\mathrm{up}}}{(c^{\mathrm{up}}_{\omega_1,\omega_2})^2} \|B_2 v\|^2_{\mathcal{H}},$$

and this prove the assertion. □

In the boundary operators setting, the previous lemma translates in the following.

Corollary 1. *Let $\alpha, \beta \in \mathbb{R}$, with $\beta > 0$, satisfying $2\alpha - \beta > 0$ and*

$$\frac{\alpha - |\alpha|\sqrt{c_{\mathrm{V}} c_{\mathrm{D}}}}{1 - c_{\mathrm{V}} c_{\mathrm{D}}} \leq \beta \leq \frac{\alpha + |\alpha|\sqrt{c_{\mathrm{V}} c_{\mathrm{D}}}}{1 - c_{\mathrm{V}} c_{\mathrm{D}}}.$$

Then it holds

$$\left\|\left(\left(\alpha - \frac{1}{2}\beta\right)\mathrm{I} + \beta \mathrm{K}\right)v\right\|^2_{\mathrm{V}^{-1}} \leq \frac{(c_{\mathrm{K}}^{\alpha,\beta})^2}{c_{\mathrm{K}}} \langle v, \mathrm{S}v\rangle_\Gamma \quad \textit{for all } v \in H_0^{\frac{1}{2}}(\Gamma). \tag{3.13}$$

Remark 2. We remark that, when $c_{\mathrm{K}} = \frac{1}{2}$, choosing $\alpha = 1$ and $\beta = 2$ in Theorem 1 gives us directly that $\|\mathrm{K}v\|_{\mathrm{V}^{-1}} = 0$ for all $v \in H_0^{\frac{1}{2}}(\Gamma)$. Therefore, even if not included in the hypotheses of Corollary 1, when $c_{\mathrm{K}} = \frac{1}{2}$, the result (3.13) holds as well with the choice $\alpha = 1$ and $\beta = 2$ since $c_{\mathrm{K}}^{1,2} = 0$.

4 The Main Result

By virtue of the results obtained in the previous section, we are able to state our main theorem on the stability of the non-symmetric coupling of the finite element and the boundary element methods.

Theorem 2. *Let $\beta > 0$ be such that*

$$\frac{1-\sqrt{c_{\mathrm{K}}(1-c_{\mathrm{K}})}}{1-c_{\mathrm{K}}(1-c_{\mathrm{K}})} \le \beta \le \frac{1+\sqrt{c_{\mathrm{K}}(1-c_{\mathrm{K}})}}{1-c_{\mathrm{K}}(1-c_{\mathrm{K}})}.$$

Let also

$$c_{\mathrm{ell}}(\mathcal{U}) > \frac{(c_{\mathrm{K}}^{1,\beta})^2}{4\beta\mu_{\min}c_{\mathrm{K}}}$$

be satisfied, with $\mu_{\min}$ defined in (2.14), *and $c_{\mathrm{K}}^{1,\beta}$ defined in* (3.8). *Then the semi-linear form $\mathcal{A}_1^\beta$, as defined in* (2.15), *is $H^1_{0,\Gamma_0}(\Omega) \times H_0^{-\frac{1}{2}}(\Gamma)$-strongly monotone satisfying*

$$\mathcal{A}_1^\beta(\hat{p}-\hat{v},\hat{p}-\hat{v}) \ge C_{\mathrm{stab}}^\beta[|p-v|^2_{H^1(\Omega)} + \|\delta-\mu\|^2_{\mathrm{V}}] \tag{4.1}$$

for all $\hat{p} = (p,\delta)$, $\hat{v} = (v,\mu) \in H^1_{0,\Gamma_0}(\Omega) \times H_0^{-\frac{1}{2}}(\Gamma)$, where

$$C_{\mathrm{stab}}^\beta := \min\biggl\{c_{\mathrm{ell}}(\mathcal{U}), \frac{1}{2}\biggl[c_{\mathrm{ell}}(\mathcal{U}) + \beta - \sqrt{(c_{\mathrm{ell}}(\mathcal{U})-\beta)^2 + \frac{(c_{\mathrm{K}}^{1,\beta})^2}{\mu_{\min}c_{\mathrm{K}}}}\biggr]\biggr\}. \tag{4.2}$$

Proof. As in the proofs given in [17, Theorem 2.2], [12, Theorem 3.1] and [13, Section 5.2], we start using (2.3), the strong monotonicity of $\mathcal{U}$,

$$\begin{aligned}\mathcal{A}_1^\beta(\hat{p}-\hat{v},\hat{p}-\hat{v}) &= (\mathcal{U}\nabla p - \mathcal{U}\nabla v, \nabla w)_{L^2(\Omega)} + \beta\langle \mathrm{V}\chi,\chi\rangle_\Gamma - \Bigl\langle\Bigl(\Bigl(1-\frac{1}{2}\beta\Bigr)\mathrm{I} + \beta\mathrm{K}\Bigr)w,\chi\Bigr\rangle_\Gamma \\ &\ge c_{\mathrm{ell}}(\mathcal{U})|w|^2_{H^1(\Omega)} + \beta\|\chi\|^2_{\mathrm{V}} - \Bigl\langle\Bigl(\Bigl(1-\frac{1}{2}\beta\Bigr)\mathrm{I} + \beta\mathrm{K}\Bigr)w,\chi\Bigr\rangle_\Gamma,\end{aligned}$$

where we have denoted $\hat{p}-\hat{v} = \hat{w} = (w,\chi)$. We split $w = w_1 + w_2$, w_1 and w_2 being the solutions of the Poisson problems

$$\begin{cases}\Delta w_1(\mathbf{x}) = \Delta w(\mathbf{x}), & \mathbf{x}\in\Omega,\\ w_1(\mathbf{x}) = 0, & \mathbf{x}\in\Gamma,\\ w_1(\mathbf{x}) = -\dfrac{\langle w,\mu_{\mathrm{eq}}\rangle_\Gamma}{\langle 1,\mu_{\mathrm{eq}}\rangle_\Gamma}, & \mathbf{x}\in\Gamma_0,\end{cases} \qquad \begin{cases}\Delta w_2(\mathbf{x}) = 0, & \mathbf{x}\in\Omega,\\ w_2(\mathbf{x}) = w(\mathbf{x}), & \mathbf{x}\in\Gamma,\\ w_2(\mathbf{x}) = \dfrac{\langle w,\mu_{\mathrm{eq}}\rangle_\Gamma}{\langle 1,\mu_{\mathrm{eq}}\rangle_\Gamma}, & \mathbf{x}\in\Gamma_0,\end{cases} \tag{4.3}$$

where μ_{eq} is the natural density $\mu_{\mathrm{eq}} = \mathrm{V}^{-1}1$. Observing that

$$|w|^2_{H^1(\Omega)} = |w_1|^2_{H^1(\Omega)} + 2\langle w_1, \mathbf{n}_{\Gamma_0}\cdot\nabla w_2\rangle_{\Gamma_0} + |w_2|^2_{H^1(\Omega)}, \tag{4.4}$$

where $\mathbf{n}_{\Gamma_0}$ is the exterior normal from Ω_0, we deduce

$$\mathcal{A}_1^\beta(\hat{w},\hat{w}) \ge c_{\mathrm{ell}}(\mathcal{U})[|w_1|^2_{H^1(\Omega)} + 2\langle w_1,\mathbf{n}_{\Gamma_0}\cdot\nabla w_2\rangle_{\Gamma_0} + |w_2|^2_{H^1(\Omega)}] + \beta\|\chi\|^2_{\mathrm{V}} - \Bigl\langle\Bigl(\Bigl(1-\frac{1}{2}\beta\Bigr)\mathrm{I}+\beta\mathrm{K}\Bigr)w_2,\chi\Bigr\rangle_\Gamma. \tag{4.5}$$

Since all the properties we have shown in the last section hold in $H_0^{\frac{1}{2}}(\Gamma)$, now we also split

$$w_2 = w_2^* + \frac{\langle w,\mu_{\mathrm{eq}}\rangle_\Gamma}{\langle 1,\mu_{\mathrm{eq}}\rangle_\Gamma},$$

where $w_2^* \in H_0^{\frac{1}{2}}(\Gamma)$. Integration by parts yields

$$|w_2|^2_{H^1(\Omega)} = \biggl|w_2 - \frac{\langle w,\mu_{\mathrm{eq}}\rangle_\Gamma}{\langle 1,\mu_{\mathrm{eq}}\rangle_\Gamma}\biggr|^2_{H^1(\Omega)} = \langle w_2^*, \mathrm{S}_{\mathrm{int}}w_2^*\rangle_\Gamma, \tag{4.6}$$

where $\mathrm{S}_{\mathrm{int}}$ is the interior Steklov–Poincaré operator defined in (2.13). Combining (4.5) and (4.6), we obtain

$$\begin{aligned}\mathcal{A}_1^\beta(\hat{w},\hat{w}) \ge c_{\mathrm{ell}}(\mathcal{U})[|w_1|^2_{H^1(\Omega)} &+ 2\langle w_1,\mathbf{n}_{\Gamma_0}\cdot\nabla w_2\rangle_{\Gamma_0} + \langle w_2^*,\mathrm{S}_{\mathrm{int}}w_2^*\rangle_\Gamma] \\ &+ \beta\|\chi\|^2_{\mathrm{V}} - \Bigl\langle\Bigl(\Bigl(1-\frac{1}{2}\beta\Bigr)\mathrm{I}+\beta\mathrm{K}\Bigr)w_2^*,\chi\Bigr\rangle_\Gamma,\end{aligned}$$

where we also used that

$$\left\langle\left(\left(1-\frac{1}{2}\beta\right)\mathrm{I}+\beta\mathrm{K}\right)w_2,\chi\right\rangle_\Gamma=\left\langle\left(\left(1-\frac{1}{2}\beta\right)\mathrm{I}+\beta\mathrm{K}\right)w_2^*,\chi\right\rangle_\Gamma,$$

due to the fact $(\frac{1}{2}\mathrm{I}-\mathrm{K})1=1$ and that $\chi\in H_0^{-\frac{1}{2}}(\Gamma)$.

From the equivalence inequalities (3.13), we continue estimating

$$\begin{aligned}\mathcal{A}_1^\beta(\hat{w},\hat{w})&\geq c_{\mathrm{ell}}(\mathcal{U})[|w_1|^2_{H^1(\Omega)}+2\langle w_1,\mathbf{n}_{\Gamma_0}\cdot\nabla w_2\rangle_{\Gamma_0}+\langle w_2^*,\mathrm{S}_{\mathrm{int}}w_2^*\rangle_\Gamma]\\&\quad+\beta\|\chi\|^2_{\mathrm{V}}-\left\|\left(\left(1-\frac{1}{2}\beta\right)\mathrm{I}+\beta\mathrm{K}\right)w_2^*\right\|_{\mathrm{V}^{-1}}\|\chi\|_{\mathrm{V}}\\&\geq c_{\mathrm{ell}}(\mathcal{U})[|w_1|^2_{H^1(\Omega)}+2\langle w_1,\mathbf{n}_{\Gamma_0}\cdot\nabla w_2\rangle_{\Gamma_0}+\langle w_2^*,\mathrm{S}_{\mathrm{int}}w_2^*\rangle_\Gamma]\\&\quad+\beta\|\chi\|^2_{\mathrm{V}}-\frac{c_{\mathrm{K}}^{1,\beta}}{\sqrt{c_{\mathrm{K}}}}\sqrt{\langle w_2^*,\mathrm{S}w_2^*\rangle_\Gamma}\|\chi\|_{\mathrm{V}}.\end{aligned}$$

Finally, using the spectral equivalence inequality (2.14), we obtain

$$\begin{aligned}\mathcal{A}_1^\beta(\hat{w},\hat{w})&\geq c_{\mathrm{ell}}(\mathcal{U})[|w_1|^2_{H^1(\Omega)}+2\langle w_1,\mathbf{n}_{\Gamma_0}\cdot\nabla w_2\rangle_{\Gamma_0}+\langle w_2^*,\mathrm{S}_{\mathrm{int}}w_2^*\rangle_\Gamma]\\&\quad+\beta\|\chi\|^2_{\mathrm{V}}-\frac{c_{\mathrm{K}}^{1,\beta}}{\sqrt{\mu_{\min}c_{\mathrm{K}}}}\sqrt{\langle w_2^*,\mathrm{S}_{\mathrm{int}}w_2^*\rangle_\Gamma}\|\chi\|_{\mathrm{V}}\\&=c_{\mathrm{ell}}(\mathcal{U})[|w_1|^2_{H^1(\Omega)}+2\langle w_1,\mathbf{n}_{\Gamma_0}\cdot\nabla w_2\rangle_{\Gamma_0}]\\&\quad+\begin{pmatrix}\sqrt{\langle w_2^*,\mathrm{S}_{\mathrm{int}}w_2^*\rangle_\Gamma}\\\|\chi\|_{\mathrm{V}}\end{pmatrix}^T\begin{pmatrix}c_{\mathrm{ell}}(\mathcal{U})&-\frac{c_{\mathrm{K}}^{1,\beta}}{2\sqrt{\mu_{\min}c_{\mathrm{K}}}}\\-\frac{c_{\mathrm{K}}^{1,\beta}}{2\sqrt{\mu_{\min}c_{\mathrm{K}}}}&\beta\end{pmatrix}\begin{pmatrix}\sqrt{\langle w_2^*,\mathrm{S}_{\mathrm{int}}w_2^*\rangle_\Gamma}\\\|\chi\|_{\mathrm{V}}\end{pmatrix}.\end{aligned}$$

Since $c_{\mathrm{ell}}(\mathcal{U})+\beta>0$, the quadratic form in the right-hand side of the above estimate is positive definite if and only if

$$\begin{vmatrix}c_{\mathrm{ell}}(\mathcal{U})&-\frac{c_{\mathrm{K}}^{1,\beta}}{2\sqrt{\mu_{\min}c_{\mathrm{K}}}}\\-\frac{c_{\mathrm{K}}^{1,\beta}}{2\sqrt{\mu_{\min}c_{\mathrm{K}}}}&\beta\end{vmatrix}=\beta c_{\mathrm{ell}}(\mathcal{U})-\frac{(c_{\mathrm{K}}^{1,\beta})^2}{4\mu_{\min}c_{\mathrm{K}}}>0.$$

Calculating the smallest eigenvalue of the matrix above, we can bound

$$\mathcal{A}_1^\beta(\hat{w},\hat{w})\geq C_{\mathrm{stab}}^\beta[|w_1|^2_{H^1(\Omega)}+2\langle w_1,\mathbf{n}_{\Gamma_0}\cdot\nabla w_2\rangle_{\Gamma_0}+\langle w_2^*,\mathrm{S}_{\mathrm{int}}w_2^*\rangle_\Gamma+\|\chi\|^2_{\mathrm{V}}],$$

which proves the assertion by virtue of (4.6) and (4.4). □

To obtain the optimal condition on $c_{\mathrm{ell}}(\mathcal{U})$ that guarantees the stability of the coupling, we look for β^* such that $\frac{(c_{\mathrm{K}}^{1,\beta})^2}{\beta}$ is minimized. Using definition (3.8), we equivalently write

$$\frac{(c_{\mathrm{K}}^{1,\beta})^2}{\beta}=\frac{(1-\beta)^2}{\beta}+\beta c_{\mathrm{K}}^2+2(1-\beta)c_{\mathrm{K}}$$

and easily obtain that, since $c_{\mathrm{K}}\in[\frac{1}{2},1)$, the optimal choice is

$$\beta^*=\frac{1+\sqrt{c_{\mathrm{K}}(1-c_{\mathrm{K}})}}{1-c_{\mathrm{K}}(1-c_{\mathrm{K}})}. \tag{4.7}$$

With the latter choice, we deduce a stability condition of the form

$$c_{\mathrm{ell}}(\mathcal{U})>\frac{1-2\sqrt{c_{\mathrm{K}}(1-c_{\mathrm{K}})}}{4\mu_{\min}(1-\sqrt{c_{\mathrm{K}}(1-c_{\mathrm{K}})})}. \tag{4.8}$$

To concretely observe what we have earned in condition (4.8) with respect to the one obtained in [12, 13], which were essentially

$$c_{\mathrm{ell}}(\mathcal{U})>\frac{c_{\mathrm{K}}}{4\mu_{\min}}, \tag{4.9}$$

we compare the functions $y=x$ and $y=\frac{1-2\sqrt{x(1-x)}}{1-\sqrt{x(1-x)}}$ for $x\in[\frac{1}{2},1)$ in Figure 1.

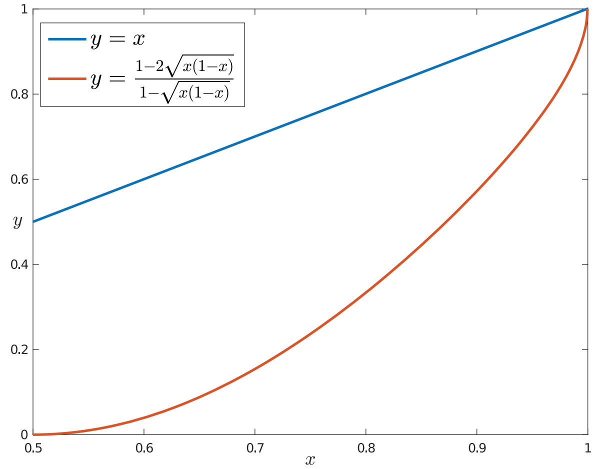

Figure 1: Comparison between conditions (4.8) and (4.9).

5 Numerical Results

In this section we provide two 2D numerical examples to support the results of Theorem 2. We first consider a ring domain, where we are able to test the sharpness on the ellipticity estimate (4.1); then we deal with an elliptic domain with a circular inclusion to test the optimality of β^* as in (4.7).

For the discretization of the coupled variational formulation (2.4)–(2.5), we use a globally quasi-uniform triangular finite element mesh in Ω with piecewise linear continuous basis functions $\{\Phi_i\}_{i=1}^{N_\mathrm{F}}$, and the inherited boundary element mesh on Γ with piecewise linear basis functions $\{\varphi_j\}_{j=1}^{N_\mathrm{B}}$, where N_F and N_B are the numbers of degrees of freedom related to the finite and boundary elements, respectively. In particular, we consider the case of $\mathcal{U} = s\mathrm{I}$ for $s \in (0,1]$. We remark that, when $\mathcal{U}$ is a coefficient matrix, $c_{\mathrm{ell}}(\mathcal{U})$ is the minimal eigenvalue of $\mathcal{U}(\mathbf{x})$ by varying $\mathbf{x} \in \Omega$, and the strongly monotone semi-linear form $\mathcal{A}_1^\beta$ is indeed an elliptic bilinear form with the ellipticity constant C_{stab}^β.

We compute the discrete approximations of the ellipticity constant by the Rayleigh quotient

$$C_{\mathrm{stab}}^\beta = \min_{\substack{\hat{v}=(v,\mu)\in H^1_{0,\Gamma_0}(\Omega)\times H_0^{-\frac{1}{2}}(\Gamma)\\ \hat{v}\neq(0,0)}} \frac{\mathcal{A}_1^\beta(\hat{v},\hat{v})}{|v|^2_{H^1(\Omega)} + \|\mu\|^2_{\mathrm{V}}} = \min_{\substack{\hat{v}=(v,\mu)\in H^1_{0,\Gamma_0}(\Omega)\times H_0^{-\frac{1}{2}}(\Gamma)\\ \hat{v}\neq(0,0)}} \frac{\mathcal{A}_1^{\mathrm{Sym},\beta}(\hat{v},\hat{v})}{|v|^2_{H^1(\Omega)} + \|\mu\|^2_{\mathrm{V}}}$$

with the symmetrized bilinear form, for $\hat{w} = (w,\chi)$, $\hat{v} = (v,\mu) \in H^1_{0,\Gamma_0}(\Omega) \times H_0^{-\frac{1}{2}}(\Gamma)$,

$$\mathcal{A}_1^{\mathrm{Sym},\beta}(\hat{w},\hat{v}) := s(\nabla w, \nabla v)_{L^2(\Omega)} + \beta\langle \mathrm{V}\chi, \mu\rangle_\Gamma - \frac{1}{2}\Big\langle\Big(\Big(1-\frac{1}{2}\beta\Big)\mathrm{I} + \beta\mathrm{K}\Big)w, \mu\Big\rangle_\Gamma - \frac{1}{2}\Big\langle v, \Big(\Big(1-\frac{1}{2}\beta\Big)\mathrm{I} + \beta\mathrm{K}'\Big)\chi\Big\rangle_\Gamma.$$

An approximation of the ellipticity constant C_{stab}^β is now given by the minimal eigenvalue of the algebraic eigenvalue problem

$$\begin{pmatrix} s\mathbb{A} & -\frac{1}{2}(1-\frac{1}{2}\beta)\mathbb{Q}^T - \frac{1}{2}\beta\mathbb{K}^T \\ -\frac{1}{2}(1-\frac{1}{2}\beta)\mathbb{Q} - \frac{1}{2}\beta\mathbb{K} & \mathbb{V} \end{pmatrix}\begin{pmatrix}\mathbf{w}\\ \chi\end{pmatrix} = \sigma\begin{pmatrix}\mathbb{A} & \mathbb{O}\\ \mathbb{O} & \mathbb{V}\end{pmatrix}\begin{pmatrix}\mathbf{w}\\ \chi\end{pmatrix}, \tag{5.1}$$

where the entries of the single blocks are given by

$$\mathbb{Q}_{ij} := \langle \Phi_{i|_\Gamma}, \varphi_j\rangle_\Gamma, \quad \mathbb{A}_{il} := (\nabla\Phi_i, \nabla\Phi_l)_{L^2(\Omega)}, \quad \mathbb{K}_{ij} := \langle \mathrm{K}\Phi_{i|_\Gamma}, \varphi_j\rangle_\Gamma, \quad \mathbb{V}_{kj} := \langle \mathrm{V}\varphi_k, \varphi_j\rangle_\Gamma$$

for $i, l = 1, \ldots, N_\mathrm{F}$ and $j, k, = 1, \ldots, N_\mathrm{B}$.

We compute the minimal eigenvalue of problem (5.1) for a sequence of coefficients $s_i := \frac{i}{100}$, $i = 1, \ldots, 100$, by using the Matlab eigenvalue solver *eigs*.

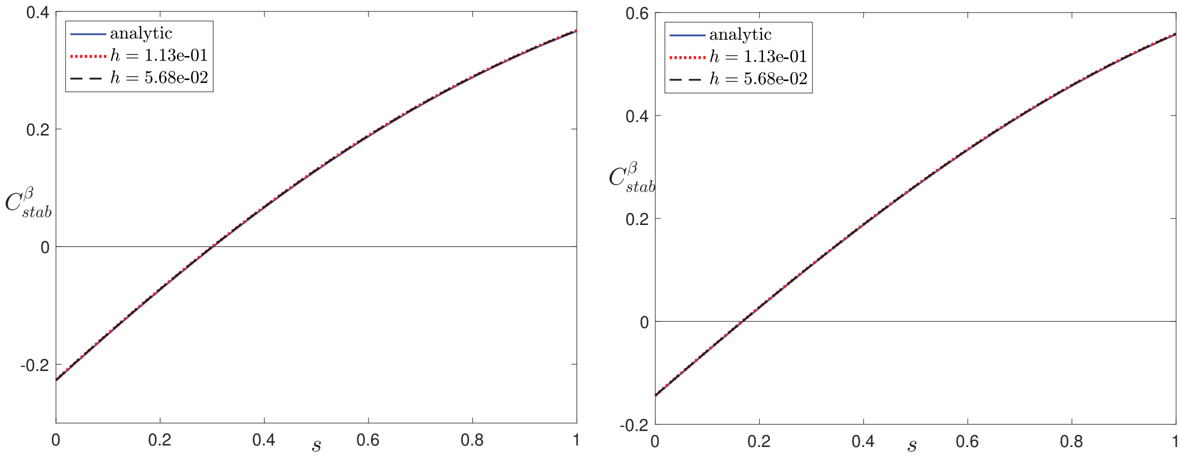

Figure 2: (Ring domain). Minimal eigenvalues of (5.1) for $s \in (0, 1]$ and $\beta = 0.7$ (left plot), $\beta = 0.9$ (right plot).

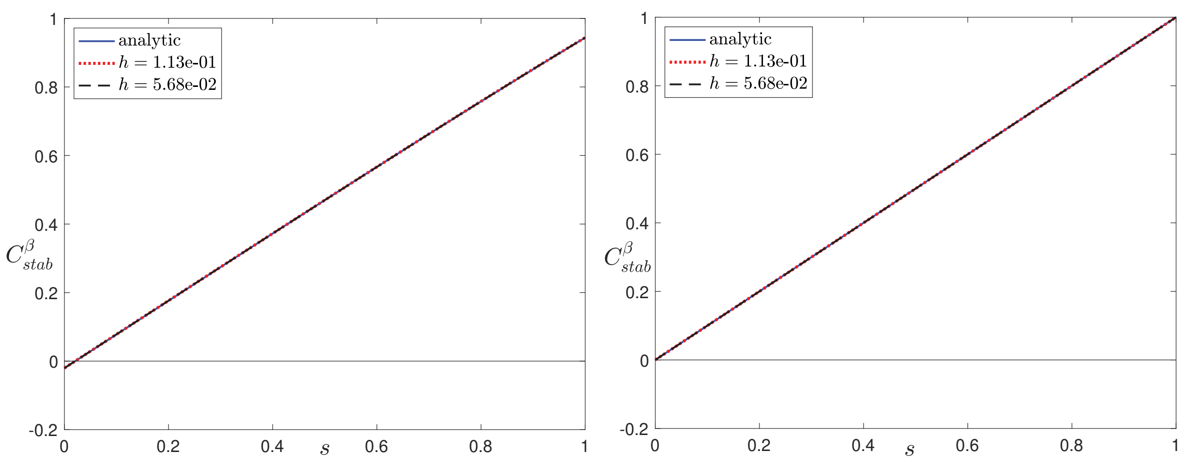

Figure 3: (Ring domain). Minimal eigenvalues of (5.1) for $s \in (0, 1]$ and $\beta = 1.5$ (left plot), and $\beta = \beta^* = 2$ (right plot).

For particular domains, like circles, ellipses or rectangles, the contraction constant c_{K}, as given in (2.9), is explicitly known. However, in general, c_{K} is unknown and, proceeding as in [18, Section 3.2], one can compute an approximation of it by using the characterization via the Rayleigh quotient

$$c_{\mathrm{K}}^2 = \max_{v \in H_0^{\frac{1}{2}}(\Gamma)} \frac{\|(\frac{1}{2}\mathrm{I} + \mathrm{K})v\|^2_{\mathrm{V}^{-1}}}{\|v\|^2_{\mathrm{V}^{-1}}}.$$

As a first example, we consider a ring domain with internal radius 1 and external radius 2. In Figures 2 and 3, the computed minimal eigenvalues for two refinement levels are compared with the behavior of the constant in the ellipticity estimate (4.1) as a function of the variable s, for some choices of β. The first refinement level is associated with a mesh of diameter $h = 1.13\mathrm{e}{-01}$, while the second level of diameter $h = 5.68\mathrm{e}{-02}$, having $h = \max_{i,j}\|\mathbf{x}_i - \mathbf{y}_j\|$ by varying all the vertices of the meshes $\mathbf{x}_i$ and $\mathbf{y}_j$. The lines of both refinement levels are on top of the expected ones given by (4.2). We remark that, for a circle, we know $c_{\mathrm{K}} = \frac{1}{2}$, and so the optimal scaling parameter is $\beta^* = 2$. As we can see in Figure 3 (right plot), the bilinear form $\mathcal{A}_1^2$ is elliptic for all s.

As a second example, we consider the case when Γ is an ellipse of semiaxes $a = 1.5$ and $b = 5$, and Γ_0 is again a circle of radius 1. In this case, the contraction constant c_{K} has been analytically derived in [14], precisely $c_{\mathrm{K}} = \frac{b}{(a+b)}$, which in our case reduces to $c_{\mathrm{K}} = \frac{10}{13}$. In Figure 4, we observe that the theoretical optimal value $\beta^* \approx 1.73$, obtained with formula (4.7), is confirmed in the practice by varying β and fixing a level of refinement with mesh parameter $h = 3.36\mathrm{e}{-01}$.

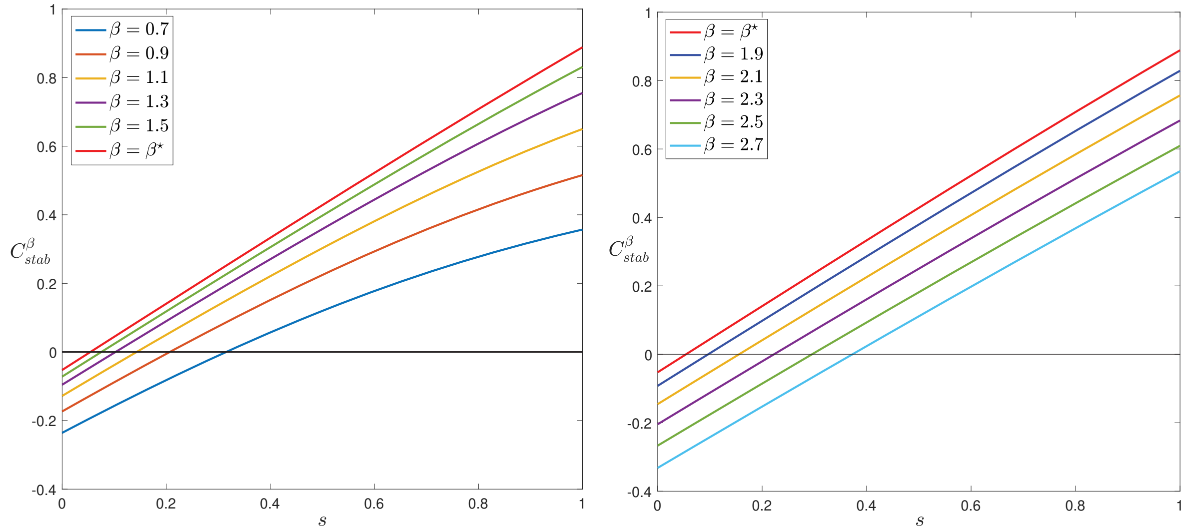

Figure 4: (Elliptic domain with a circular inclusion). Minimal eigenvalues of (5.1) for $s \in (0, 1]$ and various choices of β to test the optimality of $\beta^* \approx 1.73$.

6 Transmission Interface Problems

The analysis developed in Section 4 can be applied also to the coupling of finite and boundary elements to solve nonlinear interface problems. In this situation, $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ is a bounded domain with Lipschitz boundary Γ, and $\Omega^{\text{ext}} := \mathbb{R}^n \setminus \Omega$ with exterior normal vectors $\mathbf{n}, \mathbf{n}_{\text{ext}}$, respectively. We consider the following transmission interface problem:

$$\begin{cases} -\operatorname{div}(\mathcal{U}\nabla u(\mathbf{x})) = f(\mathbf{x}), & \mathbf{x} \in \Omega, \\ -\Delta u^{\text{ext}}(\mathbf{x}) = 0, & \mathbf{x} \in \Omega^{\text{ext}}, \\ u(\mathbf{x}) - u^{\text{ext}}(\mathbf{x}) = u_0(\mathbf{x}), & \mathbf{x} \in \Gamma, \\ \mathbf{n}(\mathbf{x}) \cdot (\mathcal{U}\nabla u(\mathbf{x})) + \mathbf{n}_{\text{ext}}(\mathbf{x}) \cdot \nabla u^{\text{ext}}(\mathbf{x}) = \lambda_0(\mathbf{x}), & \mathbf{x} \in \Gamma, \\ u^{\text{ext}}(\mathbf{x}) = O\Big(\frac{1}{\|\mathbf{x}\|}\Big), & \|\mathbf{x}\| \to \infty. \end{cases} \tag{6.1}$$

The given data $(f, u_0, \lambda_0) \in L^2(\Omega) \times H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ satisfy, when $n = 2$, the compatibility condition

$$(f, 1)_{L^2(\Omega)} + \langle \lambda_0, 1 \rangle_\Gamma = 0$$

to guarantee the radiation condition at infinity of u_{ext}.

When applying the weak formulation of the one-equation coupling for problem (6.1), one does not need, a priori, to restrict the density λ to $H_0^{-\frac{1}{2}}(\Gamma)$, but it is possible to allow for the bigger space $H^{-\frac{1}{2}}(\Gamma)$. This is due to the compatibility condition on f and λ_0, and the absence of an interior Dirichlet boundary condition. However, to guarantee the ellipticity of the single-layer operator V in $H^{-\frac{1}{2}}(\Gamma)$, when $n = 2$, we need to assume that $\operatorname{diam}(\Omega) < 1$. The semi-linear form associated to the weak formulation of (6.1) is the following:

$$\mathcal{B}^\beta(\hat{u}, \hat{v}) := (\mathcal{U}\nabla u, \nabla v)_{L^2(\Omega)} + c_{\text{ell}}(\mathcal{U})\langle u, \mu_{\text{eq}}\rangle_\Gamma \langle v, \mu_{\text{eq}}\rangle_\Gamma - \langle v, \lambda\rangle_\Gamma + \beta\Big[\langle \mathrm{V}\lambda, \mu\rangle_\Gamma + \Big\langle \Big(\frac{1}{2}\mathrm{I} - \mathrm{K}\Big)u, \mu\Big\rangle_\Gamma\Big] \tag{6.2}$$

for $\hat{u} = (u, \lambda)$, $\hat{v} = (v, \mu) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\Gamma)$, and $\beta \in \mathbb{R} \setminus \{0\}$. As in [17], we have added the stabilizing term $c_{\text{ell}}(\mathcal{U})\langle \cdot, \mu_{\text{eq}}\rangle_\Gamma \langle \cdot, \mu_{\text{eq}}\rangle_\Gamma$. The key idea is that (see [16, Theorem 2.6])

$$\|v\|^2_{H^1(\Omega),\Gamma} := \int_\Omega |\nabla v(\mathbf{x})|^2 \, d\mathbf{x} + [\langle v, \mu_{\text{eq}}\rangle_\Gamma]^2 \tag{6.3}$$

defines an equivalent norm in $H^1(\Omega)$.

Remark 3. It is worth noting that, in [1] (see also [8]), it has been shown that, to guarantee the stability of the weak formulation of problem (6.1), the stabilization term $c_{\mathrm{ell}}(\mathcal{U})\langle \cdot, \mu_{\mathrm{eq}}\rangle_\Gamma \langle \cdot, \mu_{\mathrm{eq}}\rangle_\Gamma$ is not needed. The authors have shown that there exists a semi-linear form, associated to an equivalent problem, which can be proved to be strongly monotone. We point out that our analysis does not directly apply to the artificial semi-linear form introduced in [1]. However, it is worth asking if similar results to those obtained in this paper can be obtained as well, with applications of Theorem 1, and this might be the subject of future investigations.

Proceeding along the lines of the proof of Theorem 2, we can show the following results regarding the strong monotonicity of the semi-linear form $\mathcal{B}^\beta$ with respect two different Sobolev spaces. In the first case, we consider $\mathcal{B}^\beta$ defined in $H^1(\Omega) \times H_0^{-\frac{1}{2}}(\Gamma)$, and the result we report is, essentially, the analogue of Theorem 2. However, although both variational formulations are equivalent to each other, the ellipticity results for the semi-linear form $\mathcal{B}^\beta$ are rather different when considering the space $H^1(\Omega) \times H^{-\frac{1}{2}}(\Gamma)$. Indeed, the final ellipticity condition on $c_{\mathrm{ell}}(\mathcal{U})$ is in general stronger with respect to the one in $H^1(\Omega) \times H_0^{-\frac{1}{2}}(\Gamma)$ since we had to introduce an additional splitting for the density function in $H^{-\frac{1}{2}}(\Gamma)$.

Theorem 3. *Let $\beta > 0$ be such that*

$$\frac{1 - \sqrt{c_K(1 - c_K)}}{1 - c_K(1 - c_K)} \le \beta \le \frac{1 + \sqrt{c_K(1 - c_K)}}{1 - c_K(1 - c_K)}.$$

(1) The case $H_0^{-\frac{1}{2}}(\Gamma)$. *Let*

$$c_{\mathrm{ell}}(\mathcal{U}) > \frac{(c_K^{1,\beta})^2}{4\beta c_K}$$

be satisfied, with $c_K^{1,\beta}$ defined in (3.8)*. Then the semi-linear form $\mathcal{B}^\beta$, as defined in* (6.2)*, is $H^1(\Omega) \times H_0^{-\frac{1}{2}}(\Gamma)$-strongly monotone satisfying*

$$\mathcal{B}^\beta(\hat{p} - \hat{v}, \hat{p} - \hat{v}) \ge D^\beta_{\mathrm{stab}}[\|p - v\|^2_{H^1(\Omega),\Gamma} + \|\delta - \mu\|^2_V]$$

for all $\hat{p} = (p, \delta)$, $\hat{v} = (v, \mu) \in H^1(\Omega) \times H_0^{-\frac{1}{2}}(\Gamma)$, where

$$D^\beta_{\mathrm{stab}} := \min\left\{c_{\mathrm{ell}}(\mathcal{U}), \frac{1}{2}\left[c_{\mathrm{ell}}(\mathcal{U}) + \beta - \sqrt{(c_{\mathrm{ell}}(\mathcal{U}) - \beta)^2 + \frac{(c_K^{1,\beta})^2}{c_K}}\right]\right\}.$$

(2) The case $H^{-\frac{1}{2}}(\Gamma)$. *Let*

$$c_{\mathrm{ell}}(\mathcal{U}) > \max\left\{\frac{(c_K^{1,\beta})^2}{4\beta c_K}, \frac{(1 - \beta)^2}{4\beta\langle 1, \mu_{\mathrm{eq}}\rangle_\Gamma}\right\} \tag{6.4}$$

be satisfied, with $c_K^{1,\beta}$ defined in (3.8)*, and $\mu_{\mathrm{eq}} := V^{-1}1$. If $n = 2$, suppose* diam$(\Omega) < 1$*. Then the semi-linear form $\mathcal{B}^\beta$, as defined in* (6.2)*, is $H^1(\Omega) \times H^{-\frac{1}{2}}(\Gamma)$-strongly monotone satisfying*

$$\mathcal{B}^\beta(\hat{p} - \hat{v}, \hat{p} - \hat{v}) \ge E^\beta_{\mathrm{stab}}[\|p - v\|^2_{H^1(\Omega),\Gamma} + \|\delta - \mu\|^2_V]$$

for all $\hat{p} = (p, \delta)$, $\hat{v} = (v, \mu) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\Gamma)$, where

$$E^\beta_{\mathrm{stab}} := \min\left\{c_{\mathrm{ell}}(\mathcal{U}), \frac{1}{2}\left[c_{\mathrm{ell}}(\mathcal{U}) + \beta - \sqrt{(c_{\mathrm{ell}}(\mathcal{U}) - \beta)^2 + \frac{(c_K^{1,\beta})^2}{c_K}}\right],\right.$$
$$\left.\frac{1}{2}\left[c_{\mathrm{ell}}(\mathcal{U}) + \beta - \sqrt{(c_{\mathrm{ell}}(\mathcal{U}) - \beta)^2 + \frac{(1 - \beta)^2}{\langle 1, \mu_{\mathrm{eq}}\rangle_\Gamma}}\right]\right\}.$$

Proof. In the first case, the proof is very similar to the proof of Theorem 2, without the spectral equivalence (2.14). Moreover, instead of the splitting (4.3), we use (in the same spirit of [17]) the simpler one, $w = \tilde{w} + w_\Gamma$, w_Γ being the harmonic extension of $w_{|\Gamma}$ and $\tilde{w} \in H_0^1(\Omega)$.

In the second case, for arbitrary $\hat{p}, \hat{v} \in H^1(\Omega) \times H^{-\frac{1}{2}}(\Gamma)$, we consider

$$\begin{aligned}\mathcal{B}^\beta(\hat{p} - \hat{v}, \hat{p} - \hat{v}) &= (\mathcal{U}\nabla p - \mathcal{U}\nabla v, \nabla w)_{L^2(\Omega)} + c_{\mathrm{ell}}(\mathcal{U})[\langle w, \mu_{\mathrm{eq}}\rangle_\Gamma]^2 + \beta\langle V\chi, \chi\rangle_\Gamma \\ &\quad - \left\langle\left(\left(1 - \frac{1}{2}\beta\right)I + \beta K\right)w, \chi\right\rangle_\Gamma \\ &\ge c_{\mathrm{ell}}(\mathcal{U})[|\tilde{w}|^2_{H^1(\Omega)} + |w_\Gamma|^2_{H^1(\Omega)} + [\langle w_\Gamma, \mu_{\mathrm{eq}}\rangle_\Gamma]^2] + \beta\|\chi\|^2_V \\ &\quad - \left\langle\left(\left(1 - \frac{1}{2}\beta\right)I + \beta K\right)w_\Gamma, \chi\right\rangle_\Gamma,\end{aligned} \tag{6.5}$$

where we have denoted $\hat{p} - \hat{v} = \hat{w} = (w, \chi)$ and we have used, like in the first case, the splitting $w = \tilde{w} + w_\Gamma$, w_Γ being the harmonic extension of $w_{|_\Gamma}$ and $\tilde{w} \in H_0^1(\Omega)$. We introduce the splitting

$$w_\Gamma = w_\Gamma^* + \frac{\langle w_\Gamma, \mu_{eq}\rangle_\Gamma}{\langle 1, \mu_{eq}\rangle_\Gamma},$$

where $w_\Gamma^* \in H_0^{\frac{1}{2}}(\Gamma)$. By applying Green's first formula, we obtain

$$|w_\Gamma|^2_{H^1(\Omega)} = \langle w_\Gamma^*, \mathrm{S} w_\Gamma^*\rangle_\Gamma, \tag{6.6}$$

where S is the interior Steklov–Poincaré operator defined in (2.11). We introduce a last splitting for χ, precisely

$$\chi = \chi^* + \frac{\langle 1, \chi\rangle_\Gamma}{\langle 1, \mu_{eq}\rangle_\Gamma}\mu_{eq},$$

where $\chi^* \in H_0^{-\frac{1}{2}}(\Gamma)$. One can readily deduce that

$$\|\chi\|_\mathrm{V}^2 = \|\chi^*\|_\mathrm{V}^2 + \frac{[\langle 1, \chi\rangle_\Gamma]^2}{\langle 1, \mu_{eq}\rangle_\Gamma} \tag{6.7}$$

and that

$$\Big\langle\Big(\Big(1 - \frac{1}{2}\beta\Big)\mathrm{I} + \beta\mathrm{K}\Big)w_\Gamma, \chi\Big\rangle_\Gamma = \Big\langle\Big(\Big(1 - \frac{1}{2}\beta\Big)\mathrm{I} + \beta\mathrm{K}\Big)w_\Gamma^*, \chi^*\Big\rangle_\Gamma + (1-\beta)\frac{\langle w_\Gamma, \mu_{eq}\rangle_\Gamma\langle 1, \chi\rangle_\Gamma}{\langle 1, \mu_{eq}\rangle_\Gamma}, \tag{6.8}$$

using $(\frac{1}{2}\mathrm{I} - \mathrm{K})1 = 1$, $w_\Gamma^* \in H_0^{\frac{1}{2}}(\Gamma)$ and $\chi^* \in H_0^{-\frac{1}{2}}(\Gamma)$.

Combining (6.5) with (6.7), (6.6) and (6.8), we obtain

$$\begin{aligned}\mathcal{B}^\beta(\hat{w}, \hat{w}) \geq{}& c_{ell}(\mathcal{U})[|\tilde{w}|^2_{H^1(\Omega)} + \langle w_\Gamma^*, \mathrm{S}w_\Gamma^*\rangle_\Gamma + [\langle w_\Gamma, \mu_{eq}\rangle_\Gamma]^2] + \beta\Big[\|\chi^*\|_\mathrm{V}^2 + \frac{[\langle 1, \chi\rangle_\Gamma]^2}{\langle 1, \mu_{eq}\rangle_\Gamma}\Big] \\ & - \Big\|\Big(\Big(1 - \frac{1}{2}\beta\Big)\mathrm{I} + \beta\mathrm{K}\Big)w_\Gamma^*\Big\|_{\mathrm{V}^{-1}}\|\chi^*\|_\mathrm{V} + (1-\beta)\frac{\langle w_\Gamma, \mu_{eq}\rangle_\Gamma\langle 1, \chi\rangle_\Gamma}{\langle 1, \mu_{eq}\rangle_\Gamma}.\end{aligned}$$

From the equivalence inequalities (3.13), we finally estimate

$$\begin{aligned}\mathcal{B}^\beta(\hat{w}, \hat{w}) \geq{}& c_{ell}(\mathcal{U})[|\tilde{w}|^2_{H^1(\Omega)} + \langle w_\Gamma^*, \mathrm{S}w_\Gamma^*\rangle_\Gamma + [\langle w_\Gamma, \mu_{eq}\rangle_\Gamma]^2] + \beta\Big[\|\chi^*\|_\mathrm{V}^2 + \frac{[\langle 1, \chi\rangle_\Gamma]^2}{\langle 1, \mu_{eq}\rangle_\Gamma}\Big] \\ & - \frac{c_\mathrm{K}^{1,\beta}}{\sqrt{c_\mathrm{K}}}\sqrt{\langle w_\Gamma^*, \mathrm{S}w_\Gamma^*\rangle_\Gamma}\|\chi^*\|_\mathrm{V} + (1-\beta)\frac{\langle w_\Gamma, \mu_{eq}\rangle_\Gamma\langle 1, \chi\rangle_\Gamma}{\langle 1, \mu_{eq}\rangle_\Gamma} \\ ={}& c_{ell}(\mathcal{U})|\tilde{w}|^2_{H^1(\Omega)} \\ & + \begin{pmatrix}\sqrt{\langle w_\Gamma^*, \mathrm{S}w_\Gamma^*\rangle_\Gamma} \\ \|\chi^*\|_\mathrm{V} \\ \langle w_\Gamma, \mu_{eq}\rangle_\Gamma \\ \frac{\langle 1, \chi\rangle_\Gamma}{\sqrt{\langle 1, \mu_{eq}\rangle_\Gamma}}\end{pmatrix}^T \begin{pmatrix} c_{ell}(\mathcal{U}) & -\frac{c_\mathrm{K}^{1,\beta}}{2\sqrt{c_\mathrm{K}}} & 0 & 0 \\ -\frac{c_\mathrm{K}^{1,\beta}}{2\sqrt{c_\mathrm{K}}} & \beta & 0 & 0 \\ 0 & 0 & c_{ell}(\mathcal{U}) & \frac{1-\beta}{2\sqrt{\langle 1, \mu_{eq}\rangle_\Gamma}} \\ 0 & 0 & \frac{1-\beta}{2\sqrt{\langle 1, \mu_{eq}\rangle_\Gamma}} & \beta\end{pmatrix} \begin{pmatrix}\sqrt{\langle w_\Gamma^*, \mathrm{S}w_\Gamma^*\rangle_\Gamma} \\ \|\chi^*\|_\mathrm{V} \\ \langle w_\Gamma, \mu_{eq}\rangle_\Gamma \\ \frac{\langle 1, \chi\rangle_\Gamma}{\sqrt{\langle 1, \mu_{eq}\rangle_\Gamma}}\end{pmatrix}.\end{aligned}$$

Since $c_{ell}(\mathcal{U}) + \beta > 0$, the quadratic form is positive definite if and only if

$$c_{ell}(\mathcal{U})\beta - \frac{(c_\mathrm{K}^{1,\beta})^2}{4c_\mathrm{K}} > 0 \quad \text{and} \quad c_{ell}(\mathcal{U})\beta - \frac{(1-\beta)^2}{4\langle 1, \mu_{eq}\rangle_\Gamma} > 0.$$

Calculating the smallest eigenvalue of the matrix above, we can bound

$$\mathcal{B}^\beta(\hat{w}, \hat{w}) \geq E_{stab}^\beta\Big[|\tilde{w}|^2_{H^1(\Omega)} + \langle w_\Gamma^*, \mathrm{S}w_\Gamma^*\rangle_\Gamma + [\langle w_\Gamma, \mu_{eq}\rangle_\Gamma]^2 + \|\chi^*\|_\mathrm{V}^2 + \frac{[\langle 1, \chi\rangle_\Gamma]^2}{\langle 1, \mu_{eq}\rangle_\Gamma}\Big],$$

which proves the assertion thanks to (6.3), (6.6) and (6.7). □

We remark that assumption (6.4) is, again, a generalization of that proposed in [12]; in fact, for $\beta = 1$, this condition reduces to that one. The quantity $\langle 1, \mu_{\mathrm{eq}} \rangle_\Gamma$ is also called *capacity* of Γ, and for a circle of radius R, this can be computed analytically (see [13, Section 4]),

$$\langle 1, \mu_{\mathrm{eq}} \rangle_\Gamma = -\frac{\log R}{2\pi}.$$

Therefore, recalling that, in this case, $c_{\mathrm{K}} = \frac{1}{2}$, for a circle of radius R, condition (6.4) reduces to

$$c_{\mathrm{ell}}(\mathcal{U}) > \max\left\{\frac{(2-\beta)}{4\beta}, -\frac{(1-\beta)^2 \log R}{8\pi\beta}\right\}. \tag{6.9}$$

For $R \in (0,1)$, one can show that the optimal $\beta^* \in [\frac{2}{3}, 2]$ that minimizes the right-hand side of (6.9) is

$$\beta^* = \frac{\log R + \pi - \sqrt{\pi^2 - 2\pi \log R}}{\log R},$$

and the condition attained is

$$c_{\mathrm{ell}}(\mathcal{U}) > \frac{\log R - \pi + \sqrt{\pi^2 - 2\pi \log R}}{4(\log R + \pi - \sqrt{\pi^2 - 2\pi \log R})}.$$

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