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Objectification processes in engineering freshmen while jointly learning eigentheory

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In this paper we present the first results of an ongoing PhD study which investigates eigentheory teaching and learning processes. Drawing on a sociocultural theory, namely the theory of objectification, we study students' collective meaning-making processes. A specifically designed activity, aimed at supporting these objectification processes, is described. University engineering freshmen, working in small groups, are prompted to jointly reconceptualize eigentheory notions and rules and to solve some problems. Then a few excerpts of one small group's work are presented and analysed with a focus on students' use of different semiotic resources, their mutual relationship and evolution.

Keywords: teaching and learning of specific topics in university mathematics, teaching and learning of linear and abstract algebra, eigentheory, objectification, embodiment.

INTRODUCTION

Linear algebra is widely recognised to be a major obstacle for university freshmen. A growing body of literature has investigated the sources of these difficulties and the way students comprehend linear algebra concepts. Nevertheless, only a small number of studies has focused on eigentheory teaching and learning processes, despite its importance in different applications in STEM subjects. This paper describes the first results of an ongoing PhD project, concerning the didactics of this specific topic.

As described by Stewart & Thomas (2006), when eigenvector and eigenvalue concepts are introduced to students, the focus is turned too soon to the manipulation of algebraic representations. In a standard instructional sequence, the formula to compute eigenvalues, i.e. $det(A - \lambda I)x = 0$, follows their formal definition almost without delay. Immediately after, the algorithm to compute the eigenvectors associated to each eigenvalue is given. We agree that in this way students are provided with a trusty procedure and do not feel the need to elaborate further these concepts' definitions. As a result, "the strong visual, or embodied metaphorical, image of eigenvectors is obscured by the strength of this formal and symbolic thrust" (p.185). Most of the few studies concerning this topic, agree on the fact that consequently students prefer to rely on the standard algebraic procedure rather than draw on conceptual understanding to solve exercises and problems (Bouhjar et al., 2018; Salgado & Trigueros, 2015). Nevertheless, some of these researches bring evidence on how students' understanding of eigentheory could be enhanced by the use of dynamic-geometry software (Gol Tabaghi & Sinclair, 2013), inquiry-oriented instruction (Bouhjar et al., 2018; Wawro et al., 2019) or modelling activities (Salgado & Trigueros, 2015). However, the comprehension of how students develop and coordinate the interpretations needed for

a deep conceptual understanding of eigentheory is not so clear and deserves further investigation (Bouhjar et al., 2018).

This research tries to fill this gap, analyzing how students collectively reinterpret an introductory standard frontal lecture on eigentheory, in order to construct a robust meaning for the presented concepts. We build on a sociocultural theory on mathematics teaching and learning, namely the Theory of Objectification (Radford, 2021). Hence, we are particularly interested in collective forms of knowledge production, with a focus on their multimodal features (Arzarello, 2006; Radford, 2014).

THEORETICAL FRAMEWORK

Radford (2010, 2021), defines the process of objectification as "the process through which cultural knowledge (Objekt) is progressively transformed into an object of consciousness" (Radford, 2021, p. 99). Students must engage in suitable activities in order to be able to transform cultural knowledge into *knowing* (p. 49). Through this activity, the student has the chance to encounter and attend mathematics as a cultural-historical system of thinking. This encounter does not happen all of a sudden but must be considered as a process; a process which is highly determined by the student's effort to attend the object of knowledge. The word *Activity* in the theory of objectification "refers to a dynamic system where individuals interact collectively in a strong social sense" (p. 29). To distinguish this specific formulation from activity as merely meaning "doing something", the notion of *joint labour* has been introduced (Radford, 2021). In joint labour, the acts of teaching and learning are not distinguished from each other any longer. In particular, students do not passively receive the knowledge in an "alienated" form of learning, but actively take part, through collective work, to the production of cultural social knowledge. Joint labour not only includes language as a mean for collective activity, rather encompasses the agency of body, matter, movement, rhythm, passion and sensations. Indeed, in order to become objects of consciousness, concepts must be actualised through material, sensuous activities (Radford, 2014). During the objectification process, students and teachers resort to multiple semiotic resources: written symbols, uttered and written words, diagrams, gestures, etc. These, together with object and tools, are intentionally used in social meaning-making activities in order to carry out actions aimed at fulfilling the goal of such activity: in Radford's theory (2001) they are called *semiotic means of objectification*. Since we are interested in analysing how these different signs jointly contribute to the process of knowledge objectification, they must be looked at in an integrated and systemic way, with attention to relationships and dynamics between them (Radford & Sabena, 2013). For this reason, methodologically speaking, it is important to analyse *semiotic nodes*, namely those segments of students' activities, in which different kind of semiotic resources intertwine and play a key role. In this investigation we emphasise the importance of the genesis of new signs, their evolution and the evolution of their mutual relationships in the process of objectification. Hence, we adopt the notion of *semiotic bundle*

(Arzarello, 2006), in order to perform an analysis of students' sign production and their evolution in time. It allows to have a more precise view on the way objectification is occurring. A semiotic bundle has been defined as:

> a system of signs […] that is produced by one or more interacting subjects and that evolves in time. Typically, a semiotic bundle is made of the signs that are produced by a student or by a group of students while solving a problem and/or discussing a mathematical question. Possibly the teacher too participates to this production and so the semiotic bundle may include also the signs produced by the teacher. (Arzarello et al. 2009, p.100).

In this study, specifically, we will use two theoretical constructs originating in the field of gestures study, namely those of *growth point* and *catchment* (McNeill, 2005*)*, to show how the evolution of the relationship between gestures and other semiotic resources can provide information about students' cognitive processes. A growth point is a cognitive mechanism that integrates linguistic and imagistic components (McNeill, 2005) and in a discourse is identified as "the starting point for the emergence of noteworthy information prior to its full articulation" (Arzarello et al., 2015, p. 22). The information condensed in a growth point could be progressively unpacked through a catchment, defined as an observable sequence of recurring gestural imagery (McNeill, 2015). Arzarello and colleagues (2015) have shown how catchments are produced by students in meaning-making processes of a new mathematical concept (Arzarello et al., 2015).

RESEARCH AIM AND METHODOLOGY

The investigation here presented has been conducted in an Italian public university in the fall term of 2021. The aim of the study was to analyse if/how students can objectify the concepts of eigenvector and eigenvalue, while engaged on joint labour in a specifically designed activity. Data were collected in three different linear algebra and geometry courses offered to first-year engineering students; in the Italian curriculum this is the unique linear algebra course offered to students in their first year of engineering studies, and covers standard vector-space theory (approximately: vector spaces, matrix algebra, linear systems, eigentheory, euclidean spaces). In total 64 students attended the activity and they worked divided in small groups of three, or in a few cases four students each. Sheets of paper used were collected for all the groups, while eight of them were video-taped during the whole activity. This last kind of data was necessary to collect, considering the theoretical framework that we have outlined. Indeed, from a methodological point of view, "the identification of the semiotic nodes and the semiotic means of objectification mobilised by the teacher and the students provides a kind of window to the investigation of objectification processes" (Radford, 2021, p. 106). We made sure that the recordings would capture not only the whole discussions, but also gestures and gazes produced by the students.

Activity design

As previously emphasised, activity is a key component of the objectification process. Even more, it is a key component of the investigation of this process, meaning that the

design of an appropriate activity not only can support the process of meaning-making, but can also provide the observer with important information about how this process occurs and develops (Radford and Sabena, 2015). Another key component of the theory of objectification is classroom interaction, and this is why we shaped our activity as a small-groups work.

Because of institutional constraints - among others, the deeply-rooted habit in Italian engineering firs year courses of performing traditional blackboard frontal lectures and the extremely high number of attending students (around 200) per course - we had to accommodate the planning of our activity to the standard schedule of the linear algebra courses involved in the research and were not able to plan the activity as a first introduction to the topic. Consequently, we decided to perform a pilot study after the teachers would have conducted their frontal lecture of introduction to eigentheory. Because of this, we designed the first part of the activity as a collective review of the lecture to be performed during a two-hours tutoring class, which occurred a few days after the teacher's introductory lectures on the topic. The activity was guided and attended by the course tutor and/or the researcher author of this paper. We prepared guidelines that could direct the small groups in the meaning-making process. These guidelines comprised very open questions such as "How would you explain the concept of eigenvector to someone who has never heard of that before?". Students were not specifically asked to answer the question in a written or oral form, but could freely benefit from trying to answer to these questions in order to jointly making sense of eigenvalue and eigenvector concepts. They were free to use any tool and encouraged to use other resources that they had encountered, besides the book or notes taken during the lessons. In fact, the teachers of all the three courses had shown or suggested to use a GeoGebra applet to explore eigenvectors in two-dimensional space and to watch some videos about this topic retrieved from the web.

For the second part of the activity, we prepared a set of five problems. In this paper we focus on student's engagement in the first part of the activity, while students' solution strategies to the problems are left for future works. For this reason we will not further elaborate here on the design of the problems.

Research questions

Considering the outlined theoretical framework, we can phrase our research questions as follows:

1. Can our designed activity trigger and support first year university students' objectification process of eigenvector and eigenvalue concepts, and if so, how?
2. What information can the analysis of the evolution in time of the semiotic means of objectification mobilized by students give about these objectification processes?

For space reasons, we will limit to the description and analysis of one small-group's work, which we consider as illustrative of a trajectory for the objectification process towards eigenvectors and eigenvalues: we refer to it as Group 1. We will present three particularly significant extracts from their first part of activity and describe key semiotic nodes in their objectification process.

Tackling obstacles with the definition of eigenvalue and the formula $Ax = \lambda x$

The three students start from the guiding question "What are eigenvalues and eigenvectors and how would you explain these concepts to someone who is following a linear algebra course but still has not encountered this topic?"

They decide to write the answers on a sheet of paper and one student, that we will call A, takes on the task of writing. They glance at their lecture notes and start focusing on the term "eigenvalue". At the beginning, they seem to focus on writing a correct definition of the term, without really trying to make sense of the concept or to look for specific and possibly clear examples.

A:	So I would say, starting from eigenvalues, that eigenvalues are values that can represent a linear transformation with a number.
B:	Yes
A:	Via a value …
B:	Yes, at the end, if you think about it, if I'm not wrong, it is like multiplying the matrix of the associated function …

Student B, immediately starts focusing on procedures to find eigenvalues and A stops him and goes back to trying to find a definition. They keep looking for a reasonable definition until B's intervention leads them to facing another conflict:

B:	because λ can be a 2x1 matrix
A:	[thinks about it some seconds] No, λ is just a number
B:	eh!
A:	λI is the matrix
B:	yes, ok, but you can think about λ also as a matrix, can't you?

The two students discuss about this conflict, each persuaded by his own idea. After a while, B understands that he is not able to make A understand his point with verbal language only. He starts writing formulas on his tablet. This is a first significant semiotic node to be analyzed in the group's activity. He insists on the fact that when finding the image of a vector, $f(v)$, a matrix that he calls M must be multiplied by that vector. He links then this idea to the formula used by the teacher and the textbook to define eigenvalues, namely $f(v) = \lambda v$. He correctly deduces the equality $Mv = \lambda v$, but interprets it as if λ must be a matrix as well, for the equality to stand. Stewart and Thomas (2006) have indeed described how the use of this formula can be a source of difficulty for students:

One serious problem with Ax = λx for students is that the two sides of the equation are quite different processes, but they have to be encapsulated to give the same mathematical

object. In the first case the left hand side is the process of multiplying (on the left) a vector by a matrix; the right hand side is the process of multiplying a vector by a scalar. Yet in each case the final object is a vector that has to be interpreted as the product of the eigenvalue and its eigenvector. (p. 186)

B's explanation of why he thinks that λ could be a matrix, shows how he has encountered this misconception. A seems to understand the reason of B's error, and tries to solve the conflict by rewriting the equality as $Mv = \lambda Iv$, so to make clear that λ is a scalar, while λI is a matrix. He keeps using this formulation from that moment on. We cannot say from the analysis of the rest of this segment of activity if B has understood his error; surely, as argued in Stewart and Thomas' work, reasons behind and ways to avoid this misconception need to be better studied.

As we have shown in this subsection, students struggle in finding a suitable verbal definition of Eigenvalue. In our opinion, their difficulty might be due to the fact that, ontologically speaking, it is challenging to think of an eigenvalue before even considering the existence of a linear transformation and of eigenvectors. In the following subsection, we will see how the comprehension of what an eigenvalue is can be supported by a geometric context. In fact, in it, we can define a linear transformation, and what happens to different vectors under its effect becomes more tangible.

Picturing a geometric example and gesturing as a meaning-making tool

An important shift in the advance of the activity, occurs when B suggests to use an example. In particular he suggests to consider an example offered by the teacher during the lecture. He refers to the teacher using a GeoGebra applet to explore and show the students a possible representation of eigenvectors in the two-dimensional space. Student C, who had not particularly got involved in the first part of the discussion, suddenly appears interested. He tries to recall the way eigenvectors could be identified in the applet, by gesturing with his two index fingers: first he moves them towards each other and then overlaps them (Fig. 1). These gestures allow a shift in the focus of the discussion: it moves from trying to define eigenvalues, to attempt to understand what eigenvectors are. After different efforts to verbally describing the situation, finally A states:

A: […] It is possible to find an eigenvalue associated with an eigenvector when the image of the linear application coincides…[B and C look baffled]
B: How to say it? Can we say "overlapping"?
A: When the eigenvector and its image overlap.

The three of them seem happy with this definition, but C, again with the help of gestures to make himself understood, shows that the words "coincides" and "overlapping" are not satisfactory because

C: With this definition it means that the vectors reach the same point (Fig. 1)

A then refines his definition with:

A: When they have the same direction.

C: Same direction and same sense.

At this point B steps in and, he too gesturing (Fig. 2a and 2b), shows that actually the eigenvector and its image can have opposite senses. The so refined definition satisfies the whole group.

Fig. 1   **Fig. 2a**   **Fig. 2b**

Objectifying eigenspaces

One last episode deserves being mentioned. Later in the discussion, the doubt about the number of eigenvalues that can exist for a same direction, triggers the need to bring eigenspaces into play. Talking about eigenvectors laying on the same line, B asks:

B: There are different values for λ, aren't there?

A: No

C: Why not?

A: No, because if a linear application let's say multiplies an eigenvector times 3, if you multiply the eigenvector times 3, its image is time 3, then times 9 with respect to the first one.

Providing this answer, A performs a gesture (Fig. 3) that is the first one of a series of repeated and very similar gestures that will have a key role in the development of the discussion. Apparently he starts gesturing – he almost hadn't done that yet during the activity – in order to align with his group mates' discourses. Obviously this is just our interpretation. In order to convince B and C that all vectors lining on an eigenvector's direction are associated to the same eigenvalue, he starts with this embodied idea of stretching different vectors in the span of (1,1) by the same factor 3:

A: if you, the vector (1,1)..(3,3), the vector (3,3) goes into (9,9). On the other way round if you take (-1,-1) (Fig. 4a) it goes into (-3,-3) (Fig. 4b)

The interesting part of this excerpt is the way the semiotic bundle evolves: from a gesture used to convey an embodied conceptualization of this property, A progressively moves to the use of written diagrams and then to symbolic formulas. Firstly, he converts the idea shown with his gestures into a diagram and then from this he shows to B and C how this idea can be formalized with symbols (Fig. 5) and to provide an

almost correct proof of the fact that each vector laying on the same direction of an eigenvector is an eigenvector as well, associated to the same eigenvalue.

Fig. 3
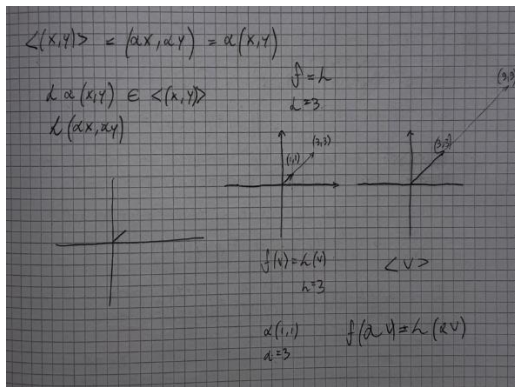

Fig. 4a

Fig. 4b

Fig. 5

We can identify A's recurring gesture as a catchment. He keeps replicating it, or a slightly modified versions of it, throughout his whole process of development of thought: from the example grounded in embodied reality, to the more formal formulation. The repeated gesture appears as the element of cohesion between these different levels of conceptualization, and that allows the other members of the group to follow and comprehend this development.

Moreover, the catchment generates from that first gesture (Fig. 3), accompanied by language. The idea guiding the described process seems to arise from this language-gesture integration that we have indeed identified as a growth point.

CONCLUSION

From the presented results we can outline some, however partial, conclusions. We can assert that the designed activity was suitable to make students engage in an objectification process. Firstly, students' management of time is a relevant indicator. As already stated, the whole activity lasted two hours. We had not recommended a partition of the whole available time, but were expecting students not to engage in the first part for longer than 25 minutes and that they would have hurried to start solving the problems. Unexpectedly, all the small groups engaged for at least 40 minutes in the first part of the task, before moving to the second one. We interpret this fact as an indication of the fact that students felt the need to really grasp the meaning of the concepts at stake. As we could notice from the recordings, students never settled for just repeating the definitions seen during the lecture. Rather, they tried with conviction to build strong meanings for those concepts and to pinpoint connections with other linear algebra concepts. As well, they tried to ensure that all the members of their group grasped the same meaning. Secondly, the analysis of students' means of objectification

and their evolution and mutual relationships actually allowed us to study their collective meaning-making process. Thanks to the use of the semiotic bundle as an analytical tool, we could detect semiotic nodes in which the emerging and evolving relationships between signs help accomplish the objectification process (Radford & Sabena , 2015). It is particularly interesting to notice how students preferably appealed to different semiotic registers. Student A from the beginning privileged the use of oral or written verbal discourse, and this, despite his evident confidence on the topics, represented an obstacle for the objectification of eigenvalues. B apparently was more confident with symbolic manipulation and resorted different times to this kind of representations in order to connect to A's discourse. The role of C was relevant, even if from the beginning he seems to be the least confident on the subject. In the first part of the activity, he struggles in following the conversation and easily gets distracted. When they switch to a geometric example, he is able to actively engage in the dialogue using gestures, with which he is able to convey the intended meanings. In this case, it is clear how gestures, as also highlighted by previous researches (e.g., Arzarello et al., 2015), are not only a means for communication, but can be productive resources that help constitute thought. They are indeed key actors in the objectification process. Even more, it is the combination of these different semiotic resources in the bundle and conflicts arising between them, that allowed objectification to occur. "In fact, the activity through which knowledge is actualized is an activity of conflicting significations" (Radford & Sabena, 2015, p. 164). The intertwining of means of objectification activated by different students was possible only thanks to their joint labour. One last remarkable aspect is the fact that the observed group, despite required to deal with eigenvalues and eigenvectors, autonomously felt the need to deeply investigate the concept of eigenspace, in order to really understand them. This is a quite informative result, considering also the fact that research concerning eigenspaces teaching and learning is really limited (cf. Wawro et al., 2019), and will be more deeply described in future works.

To conclude, it is important to remark the fact that in this activity the course's teacher was almost absent. The issue of considering teachers' lectures and students' reflections in two separated moments poses a relevant question which requires further research also because of still scarce consideration in the literature. How can the teacher's role be integrated with a students' joint activity as that described? In future stages of our research, we are planning to move the focus to this aspect, whose investigation might provide further insights and perspectives to the same process of objectification.

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