THE ROLE OF NOISE ON THE SYNCHRONIZATION OF OSCILLATORS

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THE ROLE OF NOISE ON THE SYNCHRONIZATION OF OSCILLATORS

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Abstract. Synchronization of coupled oscillators is a paradigm for complexity in many areas of science and engineering. Any realistic network model should include noise effects. For long time, noise has been considered a nuisance for synchronization, but recent developments, e.g. stochastic resonance, reveals that noise can play an active role to enhance self organization. Traditionally, phase noise has been described as a diffusion process, i.e. noise is responsible for phase diffusion, leaving the oscillation frequency unchanged. We show that phase noise in oscillator is best described as a convection diffusion problem, i.e. noise is responsible for both phase diffusion and frequency drift. We derive a simplified model to study the influence if noise on the oscillation frequency, and we discuss the implication to the synchronization of coupled and periodically driven oscillators.

Keywords. Synchronization, nonlinear oscillators, phase noise, stochastic differential equations, Itô calculus.

1 Introduction

Periodically driven oscillators and coupled oscillators are classical problems in nonlinear dynamics, with many relevant applications in physics, chemistry, biology and engineering [1, 2, 3]. To make the models more realistic, external inputs can be included, to represent the unavoidable random fluctuations that occur in real world systems, due to the physical properties of the oscillators or induced by the environment. Such disturbances can be modeled by stochastic forces applied to the oscillators, which are then described by stochastic differential equations [4].

Corrupting noise can dramatically affect the performance of oscillators. This is of particular relevance, for instance, in the field of modern electronic devices. Phase noise in oscillators can produce distortion or complete loss of incoming information in traditional receivers, and high bit error rates in phase modulated applications. Traditionally, the action of noise on electronic oscillators has been described as purely diffusive process [5, 6]. It is commonly assumed that the effect of white noise on the spectrum of an oscillator is to produce a broadening of the oscillator's spectrum without affecting the positions of the peaks. Recently, this assumption has been questioned by the analysis of some simple solvable models, and by the development of improved mathematical techniques [7, 8]. These works have shown that the phase noise problem is best described as a convection–diffusion process. That is, white noise may also be responsible for a shift in the oscillator's angular frequency.

On the one hand, that an external disturbance may modify the oscillation frequency should not come as a surprise. In fact, synchronization is commonly defined as a frequency adjustment in response to an external signal. On the other hand, it may sound surprising that a random perturbation can produce some kind of coherent modification to the oscillator's frequency. In fact one may expect that, as a result of their random nature, fluctuations have a null net effect and leave the oscillation frequency and amplitude unaffected. However we must keep in mind that we are dealing with systems that are not only stochastic, but also nonlinear in nature. Some directions are preferred to others, so that perturbations along some directions are amplified, while others are attenuated. The result is that coherent behavior can emerge from random excitations.

In this paper we present a novel derivation of the amplitude–phase equations for nonlinear oscillators driven by white Gaussian noise. The amplitude–phase equations are derived within the framework of Itô stochastic calculus, this allows a natural evaluation of the expected angular frequency which, in turn, is instrumental for synchronization analysis. The main advantage of this novel derivation is that the influence of noise on the oscillation frequency is made transparent and arises through Itô formula naturally. The relationship with Statonovich interpretation is also discussed.

2 Noisy oscillators

Noisy oscillators can be conveniently described by the stochastic differential equation (SDE) (see [4])

$$d\boldsymbol{X}(t) = \boldsymbol{a}(\boldsymbol{X}(t))\,dt + \varepsilon\,\boldsymbol{B}(\boldsymbol{X}(t))\,d\boldsymbol{W}(t) \qquad (1)$$

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where $\boldsymbol{X} : \mathbb{R} \mapsto \mathbb{R}^n$, $\boldsymbol{a} : \mathbb{R}^n \mapsto \mathbb{R}^n$, and $\boldsymbol{B} : \mathbb{R}^n \mapsto \mathbb{R}^{n,m}$. $\varepsilon \in \mathbb{R}$ is a parameter, non necessarily small, and $\boldsymbol{W} : \mathbb{R} \mapsto \mathbb{R}^m$ is a Brownian motion, i.e. a Gaussian distributed stochastic process characterized by zero expectation value, uncorrelated increments and representing the integral of a white noise [4]. The Brownian motion describes the unavoidable noise sources that are always present in real world systems.

Eq. (1) can be interpreted following two main schemes: Stratonovich or Itô [4]. Both interpretations are completely rigorous, and both have their own advantages and disadvantages. Roughly speaking, in Stratonovich view the Brownian motion is interpreted as the limit of a correlated process for correlation time approaching zero. The main advantage of Stratonovich interpretation is that traditional calculus rules applies. The drawback is that stochastic variables and noise increments are not statistically independent (a consequence of the "look in the future property" of Stratonovich stochastic integral), making the calculation of expectation values difficult. By contrast in Itô interpretation Brownian motion is a truly uncorrelated process. The pros are that Itô integrals are adapted processes, they do not suffer of the "look in the future property", and stochastic variables and noise increments are statistically independent, making calculation of expectation values easier. The cons is that a new set of calculus rules, known as Itô calculus, are required. However, any Stratonovich (respectively Itô) SDE can be transformed into an equivalent Itô (respectively Stratonovich) SDE. By equivalent we mean a different SDE, interpreted with different rules, but that has the same solution. This equivalence opens the possibility to switch from one interpretation to the other taking advantage of the pros of both the definitions.

In this paper we shall interpret (1) as an Itô SDE, to take advantage of the non anticipating nature of Itô stochastic integral, and we shall use the notation $\boldsymbol{B}(\boldsymbol{X}) \circ d\boldsymbol{W}$ to denote the Stratonovich stochastic integral used in Stratonovich SDE.

3 Amplitude and phase description of noisy oscillators

For $\varepsilon = 0$ the SDE (1) reduces to the ordinary differential equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{a}(\boldsymbol{x}(t)) \tag{2}$$

In absence of noise an oscillator exhibits a perfectly periodic solution. The periodic solution is represented by an asymptotically stable limit cycle $\boldsymbol{x}_0(t)$ in the oscillator's state space defined by

$$\begin{cases} \dot{\boldsymbol{x}}_0(t) &= \boldsymbol{a}(\boldsymbol{x}_0(t)) \\ \boldsymbol{x}_0(t) &= \boldsymbol{x}_0(t+T). \end{cases} \tag{3}$$

where T is the period od the oscillation. We denote by $\omega_0 = 2\pi/T$ the oscillator's free running frequency.

The synchronization of nonlinear oscillators is best studied by looking to the phase relations between the oscillators. To this scope, we introduce a phase function $\phi : \mathbb{R}^n \mapsto [0, T)$, $\mod T$, and an amplitude function $\boldsymbol{R} : \mathbb{R}^n \mapsto \mathbb{R}^{n-1}$, with $\phi, \boldsymbol{R} \in \mathcal{C}^m(\mathbb{R}^n)$, $m \geq 2$. The phase function is interpreted as an elapsed time from an initial reference point. Let us take a reference initial point $\boldsymbol{x}_0(0)$ on the limit cycle, and let us assign phase zero to this point, i.e. $\phi(\boldsymbol{x}_0(0)) = 0$. The phase of the point $\boldsymbol{x}_0(t)$ is $\phi(\boldsymbol{x}_0(t)) = t$. Thus, the phase represent a new parametrization of the limit cycle. The amplitude function $\boldsymbol{R}(\boldsymbol{x})$ is the Euclidean distance from the limit cycle. For our purposes we introduce the tangent unit vector

$$\boldsymbol{u}_1(t) = \frac{\boldsymbol{a}(\boldsymbol{x}_0(t))}{|\boldsymbol{a}(\boldsymbol{x}_0(t))|} \tag{4}$$

Together with $\boldsymbol{u}_1$ we consider other $n-1$ linear independent unit vectors $\boldsymbol{u}_2(t), \ldots, \boldsymbol{u}_n(t)$, such that the set $\{\boldsymbol{u}_1(t), \ldots, \boldsymbol{u}_n(t)\}$ is a basis for $\mathbb{R}^n$. We also consider another basis of $\mathbb{R}^n$, $\{\boldsymbol{v}_1(t), \ldots, \boldsymbol{v}_n(t)\}$ constructed as follows. Given the matrix $\boldsymbol{U}(t) = [\boldsymbol{u}_1(t), \ldots, \boldsymbol{u}_n(t)]$, we take its inverse

$$\boldsymbol{V}(t) = \boldsymbol{U}^{-1}(t) = \begin{bmatrix} \boldsymbol{v}_1^T(t) \\ \vdots \\ \boldsymbol{v}_n^T(t) \end{bmatrix} \tag{5}$$

It follows the bi–orthogonality condition $\boldsymbol{v}_i^T \boldsymbol{u}_j = \delta_{ij}$. To simplify notation we also introduce $r(\phi) = |\boldsymbol{a}(\boldsymbol{x}_0(\phi))|$.

The following theorem establishes the amplitude–phase model corresponding to the SDE (1).

Theorem 3.1. *Consider the Itô diffusion* (1)*, and a coordinate transformation* $\boldsymbol{x} = \boldsymbol{h}(\phi, \boldsymbol{R})$. *Let* $\boldsymbol{h}$ *be invertible, at least locally in a neighborhood of the limit cycle* $I(\boldsymbol{x}_0)$*, and let the inverse* $\boldsymbol{h}^{-1} \in \mathcal{C}^m(I(\boldsymbol{x}_0))$*,* $m \geq 2$*. Let* $\boldsymbol{Y}(\phi) = [\boldsymbol{u}_2(\phi), \ldots, \boldsymbol{u}_n(\phi)]$ *and* $\boldsymbol{Z}(\phi) = [\boldsymbol{v}_2(\phi, \ldots, \boldsymbol{v}_n(\phi)]$. *Then the amplitude and phase are Itô processes given by*

$$d\phi = \big(1 + a_1(\phi, \boldsymbol{R})\big)dt + \varepsilon \boldsymbol{B}_1(\phi, \boldsymbol{R})\, d\boldsymbol{W}_t \tag{6}$$

$$d\boldsymbol{R} = \big(\boldsymbol{A}(\phi) + \boldsymbol{a}_2(\phi, \boldsymbol{R})\big)dt + \varepsilon \boldsymbol{B}_2(\phi, \boldsymbol{R})\, d\boldsymbol{W}_t \tag{7}$$

where

$$\begin{aligned} a_1 &= \left(r + \boldsymbol{v}_1^T \frac{\partial \boldsymbol{Y}}{\partial \phi}\boldsymbol{R}\right)^{-1} \boldsymbol{v}_1^T \Bigg(\boldsymbol{a}(\boldsymbol{x}_0 + \boldsymbol{Y}\boldsymbol{R}) - \boldsymbol{a}(\boldsymbol{x}_0) - \frac{\partial \boldsymbol{Y}}{\partial \phi}\boldsymbol{R} \\ &\quad - \frac{\varepsilon^2}{2}\left(\frac{\partial \boldsymbol{a}(\boldsymbol{x}_0)}{\partial \phi} + \frac{\partial^2 \boldsymbol{Y}}{\partial \phi^2}\boldsymbol{R}\right)\boldsymbol{B}_1^T\boldsymbol{B}_1 - \varepsilon^2 \frac{\partial \boldsymbol{Y}}{\partial \phi}\boldsymbol{B}_2^T\boldsymbol{B}_1\Bigg) \end{aligned} \tag{8}$$

$$\boldsymbol{B}_1 = \left(r + \boldsymbol{v}_1^T \frac{\partial \boldsymbol{Y}}{\partial \phi}\boldsymbol{R}\right)^{-1} \boldsymbol{v}_1^T \boldsymbol{B}(\boldsymbol{x}_0 + \boldsymbol{Y}\boldsymbol{R}) \tag{9}$$

$$\boldsymbol{A} = -\boldsymbol{Z}^T \frac{\partial \boldsymbol{Y}}{\partial \phi} \tag{10}$$

$$\boldsymbol{a}_2 = \boldsymbol{Z}^T\Bigg(- \frac{\partial \boldsymbol{Y}}{\partial \phi}\boldsymbol{R}\,\boldsymbol{a}_1 + \boldsymbol{a}(\boldsymbol{x}_0 + \boldsymbol{Y}\boldsymbol{R}) \tag{11}$$

$$- \frac{\varepsilon^2}{2}\left(\frac{\partial \boldsymbol{a}(\boldsymbol{x}_0)}{\partial \phi} + \frac{\partial^2 \boldsymbol{Y}}{\partial \phi^2}\boldsymbol{R}\right)\boldsymbol{B}_1^T\boldsymbol{B}_1 - \varepsilon^2 \frac{\partial \boldsymbol{Y}}{\partial \phi}\boldsymbol{B}_2^T\boldsymbol{B}_1\Bigg) \tag{12}$$

$$\boldsymbol{B}_2 = \boldsymbol{Z}^T \boldsymbol{B}(\boldsymbol{x}_0 + \boldsymbol{Y}\boldsymbol{R}) - \boldsymbol{Z}^T \frac{\partial \boldsymbol{Y}}{\partial \phi}\boldsymbol{R}\boldsymbol{B}_1(\boldsymbol{x}_0 + \boldsymbol{Y}\boldsymbol{R}) \tag{13}$$

Proof: That ϕ and $\boldsymbol{R}$ are Itô processes is a direct consequence of the hypothesis that $\boldsymbol{h}$ is invertible with inverse of class at least $\mathcal{C}^2(I(\boldsymbol{x}_0))$. Then we can find $\phi = \phi(\boldsymbol{x})$ and $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{x})$. It follows from Itô formula that ϕ and $\boldsymbol{R}$ are Itô processes. If $\boldsymbol{X}(t)$ is a solution of (1), then using Itô formula ϕ and $\boldsymbol{R}$ satisfy equations of type

$$\begin{aligned} d\phi &= \alpha\, dt + \boldsymbol{\beta}\, d\boldsymbol{W} && (14)\\ d\boldsymbol{R} &= \gamma\, dt + \boldsymbol{\sigma}\, d\boldsymbol{W} && (15)\end{aligned}$$

that using Itô lemma gives

$$\begin{aligned} d\phi^2 &= \boldsymbol{\beta}\boldsymbol{\beta}^T\, dt && (16)\\ d\phi\, d\boldsymbol{R} &= \boldsymbol{\sigma}\boldsymbol{\beta}^T\, dt && (17)\end{aligned}$$

From $\boldsymbol{x} = \boldsymbol{h}(\phi, \boldsymbol{R})$ we have, using Itô formula and (1)

$$\frac{\partial \boldsymbol{h}}{\partial \phi}\, d\phi + \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{R}}\, d\boldsymbol{R} + \frac{1}{2}\frac{\partial^2 \boldsymbol{h}}{\partial \phi^2} d\phi^2 + \frac{1}{2}\frac{\partial^2 \boldsymbol{h}}{\partial \boldsymbol{R}\partial\phi} d\boldsymbol{R} d\phi + \frac{1}{2} d\boldsymbol{R}^T \frac{\partial^2 \boldsymbol{h}}{\partial \boldsymbol{R}^2} d\boldsymbol{R} = \boldsymbol{a}(\boldsymbol{h}(\phi,\boldsymbol{R}))dt + \varepsilon \boldsymbol{B}(\boldsymbol{h}(\phi,\boldsymbol{R}))d\boldsymbol{W} \quad (18)$$

We look for a change of coordinates in the form

$$\boldsymbol{h}(\phi, \boldsymbol{R}) = \boldsymbol{x}_0(\phi) + \boldsymbol{Y}(\phi)\boldsymbol{R}(t)$$

Introducing this ansatz in (18) yields

$$\left(\boldsymbol{a}(\boldsymbol{x}_0) + \frac{\partial \boldsymbol{Y}}{\partial \phi}\boldsymbol{R}\right) d\phi + \boldsymbol{Y} d\boldsymbol{R} + \frac{1}{2}\left(\frac{\partial \boldsymbol{a}(\boldsymbol{x}_0)}{\partial \phi} + \frac{\partial^2 \boldsymbol{Y}}{\partial \phi^2}\boldsymbol{R}\right) d\phi^2 + \frac{\partial \boldsymbol{Y}}{\partial \phi}\, d\phi\, d\boldsymbol{R} = \boldsymbol{a}(\boldsymbol{x}_0 + \boldsymbol{Y}\boldsymbol{R})dt + \varepsilon \boldsymbol{B}(\boldsymbol{x}_0 + \boldsymbol{Y}\boldsymbol{R}) \quad (19)$$

Multiplying to the left by $\boldsymbol{v}_1^T$ and using the bi–orthogonality condition we get

$$\left(r + \boldsymbol{v}_1^T \frac{\partial \boldsymbol{Y}}{\partial \phi}\boldsymbol{R}\right) d\phi + \frac{1}{2}\boldsymbol{v}_1^T\left(\frac{\partial \boldsymbol{a}(\boldsymbol{x}_0)}{\partial \phi} + \frac{\partial^2 \boldsymbol{Y}}{\partial \phi^2}\boldsymbol{R}\right) d\phi^2 + \boldsymbol{v}_1^T \frac{\partial \boldsymbol{Y}}{\partial \phi}\, d\phi\, d\boldsymbol{R} = \boldsymbol{v}_1^T \boldsymbol{a}(\boldsymbol{x}_0 + \boldsymbol{Y}\boldsymbol{R})dt + \varepsilon \boldsymbol{v}_1^T \boldsymbol{B}(\boldsymbol{x}_0 + \boldsymbol{Y}\boldsymbol{R})d\boldsymbol{W} \quad (20)$$

Multiplying (17) to the left by $\boldsymbol{Z}^T$ we get

$$\boldsymbol{Z}^T \frac{\partial \boldsymbol{Y}}{\partial \phi}\boldsymbol{R}\, d\phi + d\boldsymbol{R} + \frac{1}{2}\boldsymbol{Z}^T\left(\frac{\partial \boldsymbol{a}(\boldsymbol{x}_0)}{\partial \phi} + \frac{\partial^2 \boldsymbol{Y}}{\partial \phi^2}\boldsymbol{R}\right) d\phi^2 + \boldsymbol{Z}^T \frac{\partial \boldsymbol{Y}}{\partial \phi}\, d\phi\, d\boldsymbol{R} = \boldsymbol{Z}^T \boldsymbol{a}(\boldsymbol{x}_0 + \boldsymbol{Y}\boldsymbol{R})dt + \varepsilon \boldsymbol{Z}^T \boldsymbol{B}(\boldsymbol{x}_0 + \boldsymbol{Y}\boldsymbol{R})d\boldsymbol{W} \quad (21)$$

Introducing (14)–(17) into (20)–(21) and equating term in $d\boldsymbol{W}$ yields

$$\begin{aligned} \boldsymbol{\beta} &= \varepsilon\left(r + \boldsymbol{v}_1^T \frac{\partial \boldsymbol{Y}}{\partial \phi}\boldsymbol{R}\right)^{-1} \boldsymbol{v}_1^T \boldsymbol{B}(\boldsymbol{x}_0 + \boldsymbol{Y}\boldsymbol{R}) && (22)\\ \boldsymbol{\sigma} &= \varepsilon \boldsymbol{Z}^T \boldsymbol{B}(\boldsymbol{x}_0 + \boldsymbol{Y}\boldsymbol{R}) - \boldsymbol{Z}^T \frac{\partial \boldsymbol{Y}}{\partial \phi}\boldsymbol{R}\boldsymbol{\beta} && (23)\end{aligned}$$

Finally introducing (16)–(17) and (22)–(23) into (20)–(21) and rearranging the terms we get the thesis. □

The amplitude–phase equations (6)–(7) crucially depends on the choice of the basis vectors $\boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$. Two special sets of vectors look particularly suitable. The first is an orthonormal set. This choice allows to take advantage of Frenet formulas for moving orthonormal coordinate systems. The second choice is related to Floquet's basis [6, 7, 9].

Corollary 3.1. *If the basis vectors $\{\boldsymbol{u}_2(\phi), \ldots, \boldsymbol{u}_n(\phi)\}$ are such that*

$$\frac{\partial \boldsymbol{Y}}{\partial \phi} = \frac{\partial \boldsymbol{a}(\boldsymbol{x}_0)}{\partial \boldsymbol{x}}\boldsymbol{Y} \quad (24)$$

then, up to the first perturbative order, the Itô processes for the phase and amplitude reduce to

$$\begin{aligned} d\phi &= \big(1 + \varepsilon^2 \hat{a}_1(\phi, \boldsymbol{R})\big)dt + \varepsilon \boldsymbol{B}_1(\phi, \boldsymbol{R})\, d\boldsymbol{W}_t && (25)\\ d\boldsymbol{R} &= \varepsilon^2 \hat{\boldsymbol{a}}_2(\phi, \boldsymbol{R})dt + \varepsilon \boldsymbol{B}_2(\phi, \boldsymbol{R})\, d\boldsymbol{W}_t && (26)\end{aligned}$$

where

$$\hat{a}_1 = -\left(r + \boldsymbol{v}_1^T \frac{\partial \boldsymbol{Y}}{\partial \phi}\boldsymbol{R}\right)^{-1} \boldsymbol{v}_1^T \left(\frac{1}{2}\left(\frac{\partial \boldsymbol{a}(\boldsymbol{x}_0)}{\partial \phi} + \frac{\partial^2 \boldsymbol{Y}}{\partial \phi^2}\boldsymbol{R}\right)\boldsymbol{B}_1^T\boldsymbol{B}_1 + \frac{\partial \boldsymbol{Y}}{\partial \phi}\boldsymbol{B}_2^T\boldsymbol{B}_1\right) \quad (27)$$

$$\hat{\boldsymbol{a}}_2 = -\boldsymbol{Z}^T\left(\frac{\partial \boldsymbol{Y}}{\partial \phi}\boldsymbol{R}\hat{a}_1 + \frac{1}{2}\left(\frac{\partial \boldsymbol{a}(\boldsymbol{x}_0)}{\partial \phi} + \frac{\partial^2 \boldsymbol{Y}}{\partial \phi^2}\boldsymbol{R}\right)\boldsymbol{B}_1^T\boldsymbol{B}_1 + \frac{\partial \boldsymbol{Y}}{\partial \phi}\boldsymbol{B}_2^T\boldsymbol{B}_1\right) \quad (28)$$

Proof: It is sufficient to substitute in the previous equations the truncated Taylor expansion

$$\boldsymbol{a}(\boldsymbol{x}_0 + \boldsymbol{Y}\boldsymbol{R}) = \boldsymbol{a}(\boldsymbol{x}_0) + \frac{\partial \boldsymbol{a}(\boldsymbol{x}_0)}{\partial \boldsymbol{x}}\boldsymbol{Y}$$

and consider that $\boldsymbol{Z}^T\boldsymbol{a}(\boldsymbol{x}_0) = 0$. □

It can be shown that $\boldsymbol{v}_1$ is locally tangent on the limit cycle to the manifold on which the phase is most sensitive to perturbations. By contrast $\boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ span the direction tangent to the manifold where the phase is insensitive to perturbations [7, 9]. The projection of the noise along these linear spaces allows a partial decoupling of the amplitude and phase dynamics. In absence of noise ($\varepsilon = 0$) the system evolves with constant amplitude and constant angular frequency, similarly to the popular action-angle variables formalism of classical mechanics. In the limit $\varepsilon \ll 1$, the amplitude dynamics is one order of magnitude slower than the phase dynamics. This observation suggests the ida to neglect the amplitude fluctuations, and substitute $\boldsymbol{R} = 0$ in (25). This approximation leads to the phase reduced model

$$d\phi = \big(1 + \varepsilon^2 \hat{a}_1(\phi)\big)dt + \varepsilon \boldsymbol{B}_1(\phi)\, d\boldsymbol{W} \quad (29)$$

We remark that the assumptions leading to (25)–(26) and (29) are often made more for mathematical convenience than being physically plausible. In fact, (25)–(26) rely on linear approximation of manifolds, and nonlinear effects will become stronger the further we move away from the limit cycle. Moreover, (29) is based on the assumption that any perturbation is instantaneously adsorbed and trajectories immediately relaxes back to the limit cycle. However, the presence of nearby invariant structures such as equilibrium points and invariant manifolds may result in trajectories spending long periods of time away from the limit cycle.

4 Influence of noise on the frequency of an oscillator

The phase reduced model (29) shows that the drift effect due to $\varepsilon^2 \hat{a}_1$ becomes negligible in the limit of vanishing small noise ($\varepsilon \to 0$). The phase model in this limit has been extensively studied both at the single oscillator and network level [5, 6, 10]. However, for small but finite values of ε, the drift effect may become significant if $\hat{a}_1$ becomes large enough. That the drift effect should not be neglected even for small values of ε can also be seen as follows. Let $h(\phi)$ be an arbitrary function of the phase, and let $u(t, \phi) = E[h(\phi)]$ be the expected value of this function, with $u(0, \phi) = h(\phi)$ then the time evolution of $u(t, \phi)$ is governed by the Kolmogorov backward equation [4]

$$\frac{\partial u}{\partial t} = Au \tag{30}$$

where A is the generator of the Itô diffusion

$$Ah(\phi) = \big(\omega_0 + \varepsilon^2 a_1(\phi)\big) \frac{\partial h(\phi)}{\partial \phi} + \frac{\varepsilon^2}{2} \left(\boldsymbol{B}_1(\phi) \boldsymbol{B} 1(\phi)^T\right) \frac{\partial^2 h(\phi)}{\partial \phi^2} \tag{31}$$

Equations (30) and (31) show that both the $\mathcal{O}(\varepsilon^2)$ drift coefficient and the $\mathcal{O}(\varepsilon)$ diffusion coefficient contribute for ε^2 terms to the evolution of expected quantities, and therefore we are not allowed to neglect one term with respect to the other.

It is possible to determine the expected mean angular frequency directly from the reduced phase model (28) without solving the Kolmogorov backward equation. The mean angular frequency can be defined as $\omega = 1/T \int_0^T d\phi$. The expected mean angular frequency is computed using the following property of Itô stochastic integral: for any non anticipating function (adapted process) $h(\boldsymbol{X}_t)$

$$E\left[\int_{t_0}^{t} h(\boldsymbol{X}_t)\, d\boldsymbol{W}_t\right] = 0 \tag{32}$$

as a consequence of the fact that $E[\boldsymbol{W}_t] = 0$. Thus from (28) we have

$$E[\omega] = 1 + \frac{\varepsilon^2}{T} E\left[\int_0^T \hat{a}_1(\phi)\, dt\right] \tag{33}$$

One may argue that the drift term is an artifact due to Itô interpretation. However, it turns out that the frequency drift is also present if Stratonovich interpretation is used [7]. To clarify the point, consider the Stratonovich SDE

$$d\boldsymbol{X} = \boldsymbol{a}^{(S)}(\boldsymbol{X})\, dt + \varepsilon \boldsymbol{B}(\boldsymbol{X}) \circ d\boldsymbol{W} \tag{34}$$

where the index (S) means "Stratonovich". Taking into account that in Stratonovich interpretation traditional calculus rules apply, repeating the procedure used in the previous section we arrive to the reduced phase model

$$d\phi = dt + \varepsilon \boldsymbol{B}_1(\phi) \circ d\boldsymbol{W} \tag{35}$$

However, deriving the expected mean angular frequency from (35) is not trivial, because in Stratonovich interpretation $E[\int_{t_0}^t h(\boldsymbol{X}) \circ d\boldsymbol{W}] \neq 0$. This is a consequence of the anticipating nature or "look in the future" property of Stratonovich stochastic integral. Because of its anticipating nature, in Stratonovich view stochastic processes and noise increments are correlated. To resolve the statistical dependence, a Stratonovich SDE has to be transformed into its equivalent Itô SDE by the addition of the drift correction term [4, 11]. Here is where the drift coefficient, that arises naturally from the quadratic terms in Itô formula, comes into play.

5 Discussion

Synchronization of oscillators is commonly defined as an adjustment of frequencies in response to an external stimulus. Usually, the external stimulus is taken in the form of either a coherent signal, e.g. a periodic function, or as couplings among oscillators. The synchronization is the result of different competing mechanisms. On the one hand, the application of a periodic input, and/or couplings between the oscillators play a constructive role, and favor the formation of common rhythms. On the other hand, differences between the oscillators natural frequencies are destructive to synchronization. In the traditional picture, noise is added as a nuisance to synchronization. The negative effect produced by phase diffusion and the resulting occurrence of phase slips has been extensively studied and is rather well understood [1, 2]. However it has been recently discovered that noise can play a constructive role in information transmission and processing, and in the emergence of coherent structures in complex systems [12, 13].

Here we limit ourselves to discuss the role of noise in synchronization through its action on the frequency of an oscillator. It is well known that an oscillator will synchronize with a periodic external signal, provided that the strength of the signal exceeds a thresholds determined by the frequency mismatch between the oscillator and the input. The situation is analogous for coupled oscillators. Because of the dependence of the frequency on the noise intensity, noise can both favor synchronization if it reduces the frequency mismatch, or it can contrast the formation of rhythms if the mismatch is increased. It is worth noting that this mechanism is different from stochastic resonance, although they bear some resemblance. In stochastic resonance the role of noise is to enhance the signal level at a certain frequency. By contrast, the present mechanism is based on the modification of the oscillator frequency.

In figure 1 we show the expected angular frequency $E[\omega]$ versus the noise intensity for a van der Pol oscillator with additive noise described by the SDE

$$\begin{aligned} dx &= y\,dt + \varepsilon\, dB_1 \\ dy &= \big(-x + \mu(1-x^2)y\big)dt + \varepsilon\, dB_2 \end{aligned} \tag{36}$$

The expected angular frequency has been obtained through Monte Carlo simulations. For each value of the noise intensity we have run simulations for one thousand different realizations of the noise process. For each realization, the mean frequency has been evaluated as $\omega = (\phi(t_2) - \phi(t_1))/(t_2 - t_1)$ for $t_2 \gg t_1$. The expected values has been computed as the mean of the ω values. Since the noise is additive, there is no difference between Itô and Stratonovich interpretation. The quadratic dependence of the frequency drift on the noise intensity is well reproduced by numerical data.

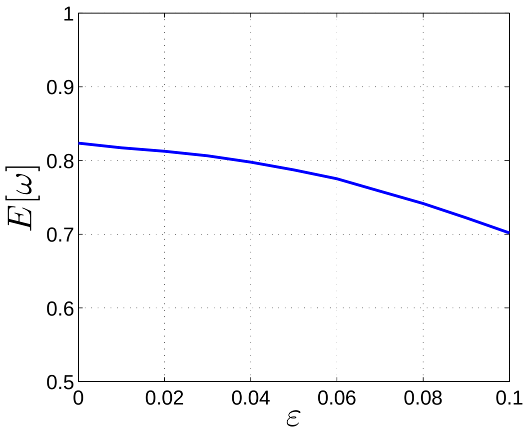

Figure 1: Expected mean angular frequency vs noise intensity for a van der Pol oscillator with additive noise. The parameter is set to $\mu = 2$.

Figure 2 shows the time evolution of the phase difference between two uncoupled Stuart–Landau oscillators. In polar coordinates the Stuart–Landau oscillator is described by the SDE

$$\begin{aligned} dr &= \big(r - r^3\big)dt + \varepsilon dB_r \\ d\theta &= \big(\alpha - \beta r^2\big)\,dt + \varepsilon dB_\theta \end{aligned} \tag{37}$$

It is easy to see that the oscillator admits an asymptotically stable limit cycle of amplitude $r = 1$ and angular frequency $\omega_0 = \alpha - \beta$. This example is instructive because, introducing the new phase function $\phi = \theta - \beta \log r$, we obtain the Itô SDE

$$\begin{aligned} dr &= \big(r - r^3\big)dt + \varepsilon dB_r \\ d\phi &= \left(\omega_0 + \frac{\varepsilon^2}{2}\frac{\beta}{r^2}\right)dt + \varepsilon\left(dB_\theta - \frac{\beta}{r}dB_r\right) \end{aligned} \tag{38}$$

Thus, in absence of noise, the system admits a solution representing oscillations of constant amplitude and constant angular frequency, in complete analogy with (25)–(26). If amplitude fluctuations are neglected, we have the expected angular frequency $E[\omega] = \alpha - \beta + \varepsilon^2\beta/2$. Figure 2 shows the result for two oscillators with free running angular frequencies $\omega_1 = 4$ and $\omega_2 = 3.995$, respectively. The two oscillators are expected to have the same mean angular frequency for $\varepsilon \simeq 0.25$. The figure shows that, in absence of noise the phase difference would grow linearly in time, while for the proper noise intensity it remains bounded, as expected. The large fluctuations in the phase difference are due to the relatively high noise intensity required to achieve the same mean angularfrequency.

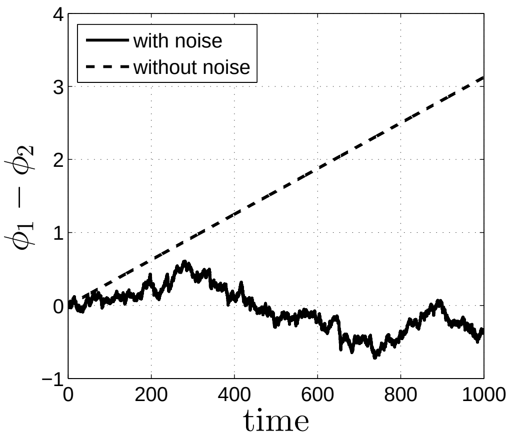

Figure 2: Phase difference for two uncoupled Stuart–Landau oscillator with slightly different frequencies both in presence of noise (solid line) and without noise. Noise intensity is $\varepsilon = 0.25$

6 Conclusions

We have discussed the role of noise on nonlinear oscillators subject to white Gaussian noise. Noisy oscillators can be conveniently described by stochastic differential equations. Using projection techniques and Itô calculus a set of stochastic differential equations describing the evolution of the phase and the amplitude of the oscillator can be derived.

The resulting amplitude–phase equations describe the phase noise problem as a convection–diffusion process. White noise is responsible for both phase diffusion and a drift in the frequency of oscillations. We have discussed the condition under which an approximate phase reduced model can be derived. The phase reduced model is a simplified model, that describe the behavior of the noisy oscillator in terms of the sole phase variable. We have shown how the frequency drift emerges if the Stratonovich interpretation is used instead of Itô view.

We discussed the implication of the frequency drift to the synchronization of coupled and forced oscillators. Oscillators adjust their frequency in response to noise intensity, and as a consequence noise can actively contribute to the synchronization by decreasing the frequency mismatch between an oscillator and a periodic driving signal. Thus noise can favor the emergence of coherent behavior, through a mechanism similar to stochastic resonance.

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