Equilibrium Propagation and (Memristor-based) Oscillatory Neural Networks

Equilibrium Propagation and (Memristor-based) Oscillatory Neural Networks

Gianluca Zoppo, Francesco Marrone, Michele Bonnin and Fernando Corinto
Department of Electronics and Telecommunications
Politecnico di Torino, Torino, Italy
Emails: gianluca.zoppo@polito.it, francesco.marrone@polito.it, michele.bonnin@polito.it, fernando.corinto@polito.it

***Abstract*—Weakly Connected Oscillatory Networks (WCONs) are bio-inspired models which exhibit associative memory properties and can be exploited for information processing. It has been shown that the nonlinear dynamics of WCONs can be reduced to equations for the phase variable if oscillators admit stable limit cycles with nearly identical periods. Moreover, if connections are symmetric, the phase deviation equation admits a gradient formulation establishing a one-to-one correspondence between phase equilibria, limit cycle of the WCON and minima of the system's potential function. The overall objective of this work is to provide a simulated WCON based on memristive connections and Van der Pol oscillators that exploits the device mem-conductance programmability to implement a novel local supervised learning algorithm for gradient models: Equilibrium Propagation (EP). Simulations of the phase dynamics of the WCON system trained with EP show that the retrieval accuracy of the proposed novel design outperforms the current state-of-the-art performance obtained with the Hebbian learning.**

***Index Terms*—Associative Memory, Equilibrium Propagation, Kuramoto, Memristor, Oscillatory Neural Networks.**

I. Introduction

Weakly Connected Oscillatory Networks (WCONs) have found application in many fields of Science and Engineering as modeling tools for complex phenomena [1]. Noticeably, the property of WCONs that has caught most of the researchers' attention is synchronization which has proven to be exploitable to carry out actual computation [2].

Recently, the unique switching features and signal storing capability of memristive devices have been exploited to implement synaptic connections of bio-plausible neural networks [3]. Their non-volatile resistance can be programmed to desired values within a conductance range enabling adjustable neural connections necessary in the training phase. Since memristors can implement only positive connections, different approaches can be used to enable the system to have zeros and negative weights [4], [5].

Nowadays, most training algorithms rely on weight updates that do not exclusively depend on the pre- and post- synaptic neurons. The novel learning framework known as Equilibrium Propagation [6] has gained popularity in the neuromorphic computing community thanks to the locality of its weight update rule which makes it, amongst many proposed bio-plausible learning techniques [7], the most suitable for VLSI implementation. In this contribution, we apply EP learning rule to train a network of weakly coupled oscillators to solve pattern reconstruction tasks.

Research works on the mammalian brain have paved the way to novel neuromorphic computing architectures made of coupled arrays of oscillators which show intriguing dynamic behaviors [5], [8]–[10]. In those studies, it has been shown that WCONs may work as Hopfield Neural Networks (HNNs) with limit cycles as attractors [8], [9].

Weakly Connected Oscillatory Networks are mathematically described by large systems of coupled nonlinear ordinary differential equations (ODEs) which may feature many attractors and/or bifurcation phenomena. Under the assumption of having all the oscillators' limit cycles stable and isofrequential, WCONs' nonlinear dynamics can be studied using the phase deviation equation [9], which describes the evolution of the phase deviations of all the oscillators in the network with respect to a single reference. This method enabled the investigation of a one-dimensional weakly connected network, composed by second order Van der Pol oscillators [11].

As a result of the biunivocal correspondence between the limit cycles of the WCON and equilibrium points of the phase deviation equation, oscillatory networks can work as models of associative memory [8], [12].

In Section II, we show that the equilibria of the phase deviation equation can be designed by means of Equilibrium Propagation learning rule in order to recover a given set of learnt pattern when presenting the network with their corrupted version. In Section III, we derive an analytic expression of the phase deviation equation of WCONs by applying the technique proposed in [13], [14]. This allows us to establish a biunivocal correspondence between the periodic limit cycles of the network and the equilibrium points of the phase deviation equation. The system performance is evaluated in Section IV. Section V concludes the paper.

II. Equilibrium Propagation

Inspired by the novel algorithm described in [6] for training gradient-based models, the goal of this section is to generalize Equilibrium Propagation to Oscillatory Neural Networks that admit a gradient formulation. Let $\boldsymbol{\psi} \in \mathbb{R}^N$ be the state variable vector, $\mathbf{W} \in \mathbb{R}^{N \times N}$ the coupling weight matrix and the phase dynamics defined by the system:

$$\dot{\psi}_i = \sum_{j=1}^{N} w_{ij} \sin\left(\psi_j - \psi_i\right) \quad \forall i = 1, \ldots, N \tag{1}$$

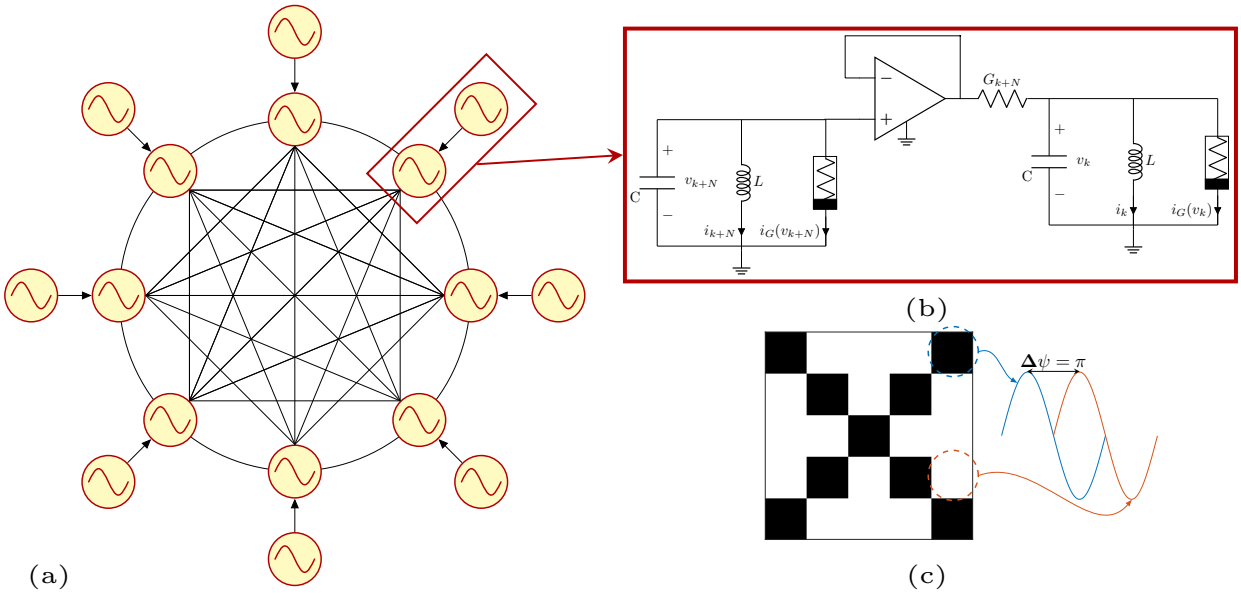

Figure 1: a) Illustration of a weakly connected oscillatory neural network. b) Master-slave configuration of the k-th Van der Pol oscillator with the corresponding $(k+N)$-th driving unit. The connection between the two oscillators is one-directional. c) A black square corresponds to the phase difference between each oscillator and a reference unit equal to $\Delta\psi = 0$, and a white square corresponds to the phase difference equal to $\Delta\psi = \pi$.

It is easy to verify that the system (1) can be expressed in the form of a gradient system $\dot{\boldsymbol{\psi}} = -\nabla_{\boldsymbol{\psi}} E(\boldsymbol{\psi}, \mathbf{W})$ with potential function defined as

$$E(\boldsymbol{\psi}, \mathbf{W}) = -\frac{1}{2} \sum_{i,j=1}^{N} w_{ij} \cos(\psi_i - \psi_j) \tag{2}$$

with symmetric interconnections $w_{ij} = w_{ji}$ and $w_{ii} = 0$.

Let us define the cost function $\mathcal{C}$ that measures the phase difference between a target memory $\mathbf{T} \in \mathcal{T} = \{\mathbf{T}^1, \dots, \mathbf{T}^m\}$ and the state of the system $\boldsymbol{\psi}$:

$$\mathcal{C}(\mathbf{T}, \boldsymbol{\psi}) = N - \sum_{i=1}^{N} \cos(T_i - \hat{\psi}_i) \tag{3}$$

where $C(\mathbf{T}, \boldsymbol{\psi}) \geq 0$. The learning process consists in minimizing the distance between the target and the stable equilibrium point of the system $\hat{\boldsymbol{\psi}}$. This process can be recast into the following optimization problem [6], [15]:

$$\min_{\mathbf{W}} \ \mathcal{C}(\mathbf{T}, \boldsymbol{\psi}) \quad \text{subject to} \quad \nabla_{\boldsymbol{\psi}} E(\boldsymbol{\psi}, \mathbf{W}) = 0. \tag{4}$$

In order to solve system (4), let us introduce the Lagrange multipliers $\boldsymbol{\lambda} \in \mathbb{R}^N$ and consider the Lagrangian function

$$\mathcal{L}(\boldsymbol{\psi}, \boldsymbol{\lambda}, \mathbf{W}, \mathbf{T}) = \mathcal{C}(\mathbf{T}, \boldsymbol{\psi}) + \boldsymbol{\lambda}^T \nabla_{\boldsymbol{\psi}} E(\boldsymbol{\psi}, \mathbf{W}). \tag{5}$$

The critical points of the Lagrangian function correspond to the stationary values of the original function that satisfy the constraint. Thus, for solving the optimization problem, one has to find the minima of the augmented objective function. This problem does not have a closed form solution for $\mathbf{W}$. For this reason, let us first keep constant the entries of the weight matrix and find $\hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\psi}}$, solutions of the system:

$$\begin{cases} \nabla_{\boldsymbol{\lambda}} \mathcal{L}(\hat{\boldsymbol{\psi}}, \hat{\boldsymbol{\lambda}}, \mathbf{W}, \mathbf{T}) = \nabla_{\boldsymbol{\psi}} E(\hat{\boldsymbol{\psi}}, \mathbf{W}) = 0 \\ \nabla_{\boldsymbol{\psi}} \mathcal{L}(\hat{\boldsymbol{\psi}}, \hat{\boldsymbol{\lambda}}, \mathbf{W}, \mathbf{T}) = \nabla_{\boldsymbol{\psi}} \mathcal{C}(\mathbf{T}, \hat{\boldsymbol{\psi}}) + \mathbf{H}_E(\hat{\boldsymbol{\psi}}, \mathbf{W}) \hat{\boldsymbol{\lambda}} = 0 \end{cases} \tag{6}$$

where $\mathbf{H}_E$ is the Hessian of the function E.

Now, in order to update the remaining system parameters, let us perform a stochastic descent on the total cost

$$\begin{aligned} \Delta W_{ij} &= -\eta \frac{\partial \mathcal{L}}{\partial W_{ij}}(\hat{\boldsymbol{\psi}}, \hat{\boldsymbol{\lambda}}, \mathbf{W}, \mathbf{T}) = \\ &= -\eta \hat{\boldsymbol{\lambda}}^T \frac{\partial^2 E}{\partial \mathbf{W} \partial \boldsymbol{\psi}}(\hat{\boldsymbol{\psi}}, \mathbf{W}) \end{aligned} \tag{7}$$

where $\eta > 0$ is the learning rate. Equation (7) shows that the direction that minimizes the cost function depends on $\hat{\boldsymbol{\lambda}}$. The authors in [6] proposed to find the Lagrange multipliers by defining the following augmented potential function

$$F(\boldsymbol{\psi}, \mathbf{T}, \mathbf{W}, \beta) = E(\boldsymbol{\psi}, \mathbf{W}) + \beta \mathcal{C}(\mathbf{T}, \boldsymbol{\psi}) \tag{8}$$

where $\beta > 0$ is the forcing parameter, and considering the corresponding gradient system

$$\begin{aligned} \dot{\psi}_i &= -\frac{\partial F}{\partial \psi_i}(\boldsymbol{\psi}, \mathbf{T}, \mathbf{W}, \beta) = \\ &= \sum_{j=1}^{N} w_{ij} \sin(\psi_j - \psi_i) + \beta \sin(T_i - \psi_i) \end{aligned} \tag{9}$$

for $\forall i = 1, \dots, N$. A fixed point $\hat{\boldsymbol{\psi}}^{\beta}$ of (9) is given by

$$\nabla F(\hat{\boldsymbol{\psi}}^{\beta}, \mathbf{T}, \mathbf{W}, \beta) = 0 \tag{10}$$

which corresponds to a local minimum of the augmented potential function.

Since (10) is constant for all β, we can compute the total derivative with respect to β and evaluate the result in $\beta = 0$ and get

$$\nabla_{\boldsymbol{\psi}} \mathcal{C}(\mathbf{T}, \hat{\boldsymbol{\psi}}) + \mathbf{H}_E(\hat{\boldsymbol{\psi}}, \mathbf{W}) \frac{\partial \boldsymbol{\psi}^\beta}{\partial \beta}\Big|_{\beta=0} = 0 \tag{11}$$

Comparing this result with Eq. (6), we can identify $\hat{\boldsymbol{\lambda}} = \frac{\partial \boldsymbol{\psi}^\beta}{\partial \beta}\Big|_{\beta=0}$, obtaining

$$\begin{aligned} \Delta W_{ij} &\propto -\left(\frac{\partial \boldsymbol{\psi}^\beta}{\partial \beta}\Big|_{\beta=0}\right)^T \frac{\partial^2 E}{\partial W_{ij} \partial \boldsymbol{\psi}}(\hat{\boldsymbol{\psi}}, \mathbf{W}) = \\ &= -\frac{d}{d\beta}\left[\frac{\partial E}{\partial W_{ij}}(\hat{\boldsymbol{\psi}}^\beta, \mathbf{W})\right]_{\beta=0} = \\ &= \frac{d}{d\beta}\left[\cos\left(\hat{\psi}_i^\beta - \hat{\psi}_j^\beta\right)\right]_{\beta=0} = \\ &= \lim_{\beta \to 0} \frac{\cos\left(\hat{\psi}_i^\beta - \hat{\psi}_j^\beta\right) - \cos\left(\hat{\psi}_i - \hat{\psi}_j\right)}{\beta} \end{aligned} \tag{12}$$

where we used the symmetry of second derivatives of the potential function E. Thus, the learning rule can be approximated by

$$\Delta W_{ij} \propto \frac{\cos\left(\hat{\psi}_i^\beta - \hat{\psi}_j^\beta\right) - \cos\left(\hat{\psi}_i - \hat{\psi}_j\right)}{\beta}, \quad \beta \simeq 0. \tag{13}$$

The training process can be summarized as follows:

1) First, we set $\psi^0 = \mathbf{T}$ and let the network follow the free dynamics defined by the gradient system (1) relaxing to the free fixed point $\hat{\boldsymbol{\psi}}$, and $\cos\left(\hat{\psi}_i - \hat{\psi}_j\right)$ is computed;
2) Secondly, a small perturbation is introduced to the system (1) allowing the network (9) to relax to a new fixed point $\hat{\boldsymbol{\psi}}^\beta$ where $\cos\left(\hat{\psi}_i^\beta - \hat{\psi}_j^\beta\right)$ is measured;
3) Lastly, the weights of the matrix $\mathbf{W}$ are changed according to (13).

III. Phase dynamics of WCON

The existence of synchronized oscillations in the brain has prompted research to use oscillatory network as models for associative memory based on temporal coding of information. Traditionally, these models consist of coupled oscillators interacting with each other in accordance with a given topology, and the information is coded as phase-locked oscillations. As a result of a recently developed description of WCONs dynamics in terms of amplitude and phase variables [14], [16], it is possible to show that the phase dynamics of the system coincides locally in the neighborhood of the limit cycle with the asymptotic phase defined by Kuramoto model [1]. This attractive result motivates this work to consider WCONs as models able to implement the learning rule described in the previous section.

Consider a network of N weakly connected nonlinear oscillators whose states are individually controllable through the coupling with an additional driving oscillator unit as shown in Fig. 1(a). The driving unit oscillators have two different important roles:

- to set the phase of each oscillator as equal to the perturbed/target pattern ($\beta \gg 0$);
- to model the teaching signal described in the second phase of EP (see Eq. (9)).

Let us assume that each single nonlinear oscillator admits an asymptotically stable, T periodic limit cycle. In this work we consider a network of van der Pol oscillators, but the generalization to other nonlinear oscillators is straightforward. Let $\varepsilon \ll 1$ be the interaction strength, by applying Kirchhoff laws to the network in Figs. 1(a)-(b) we obtain the equations

$$\begin{cases} C\dfrac{dv_k}{dt} = -i_k - i_G(v_k) - \varepsilon f(v_1, \ldots, v_{2N}) \\ L\dfrac{di_k}{dt} = v_k \\ C\dfrac{dv_{k+N}}{dt} = -i_{k+N} - i_G(v_{k+N}) \\ L\dfrac{di_{k+N}}{dt} = v_{k+N} \end{cases} \tag{14}$$

where

$$f(v_1, \ldots, v_{2N}) = \sum_{j=1}^{N} G_{kj}(v_k - v_j) + G_{k+N}(v_k - v_{k+N})$$

is the coupling function describing the interactions. We shall assume $i_G(v_k) = -g_a v_k + g_b v_k^3$. Introducing the adimensional time $\tau = \frac{t}{LG}$, and state variables $x_k = \frac{v_k}{V_0}$, $y_k = \frac{i_k}{GV_0}$, where G is a generic reference conductance and V_0 is a reference voltage, the state equations can be rewritten in the adimensional form

$$\begin{cases} \dfrac{dx_k}{d\tau} = -\alpha y_k + \delta x_k - \gamma x_k^3 - \varepsilon \tilde{f}(x_1, \ldots, x_{2N}) \\ \dfrac{dy_k}{d\tau} = x_k \\ \dfrac{dx_{k+N}}{d\tau} = -\alpha y_{k+N} + \delta x_{k+N} - \gamma x_{k+N}^3 \\ \dfrac{dy_{k+N}}{d\tau} = x_{k+N} \end{cases} \tag{15}$$

where

$$\tilde{f}(x_1, \ldots, x_{2N}) = \sum_{j=1}^{N} \Gamma_{kj}(x_k - x_j) + \Gamma_k(x_k - x_{k+N})$$

is the normalized coupling function and $\alpha = \frac{LG^2}{C}$, $\delta = g_a \frac{LG}{C}$, $\gamma = g_b V_0^2 \frac{LG}{C}$, $\Gamma_{kj} = G_{kj}\frac{LG}{C}$ and $\Gamma_k = G_{k+N}\frac{LG}{C}$.

Applying the procedure described in [13] (Section 3), we obtain the phase deviation equation for the k-th state:

$$\begin{cases} \dfrac{d\psi_k}{dt} = \tilde{\varepsilon}\left[\sum_{j=1}^{N} \Gamma_{kj} \sin(\psi_j - \psi_k) + \Gamma_k \sin(\psi_{k+N} - \psi_i)\right] \\ \dfrac{d\psi_{k+N}}{dt} = 0 \Rightarrow \psi_{k+N}(t) = \psi_{k+N}(0) \end{cases} \tag{16}$$

where $\tilde{\varepsilon} = \varepsilon \dfrac{\alpha}{2\omega}$.

It is trivial to verify that, assuming $w_{kj} = \tilde{\varepsilon}\Gamma_{kj}$, $\beta = \delta_k = \tilde{\varepsilon}\Gamma_k$ and $\psi_{i+N}(0) = T_i$, Eq. (16) is equivalent to the Kuramoto model (9).

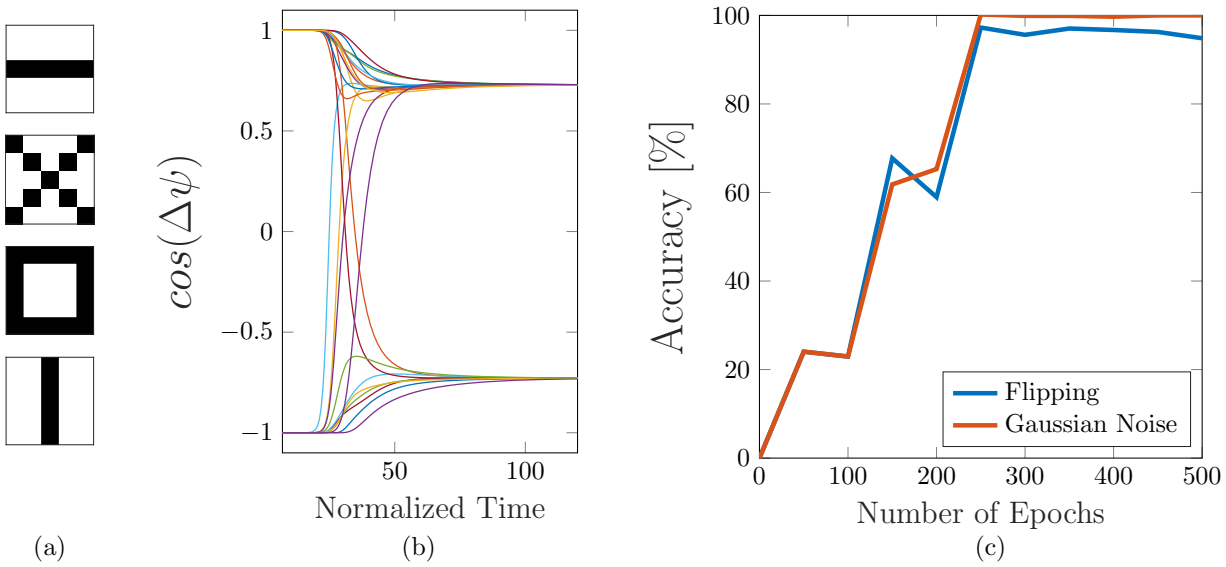

Figure 2: (a) Training patterns of the network. (b) Evolution of the oscillators in terms of the cosine of the phase difference. (c) Accuracy of the network over 1000 different corrupted patterns during the training phase. In blue the accuracy of the WCON for reconstructing patterns with pixels flipped with probability 0.1. In orange the accuracy of the WCON for reconstructing patterns corrupted with an additive Gaussian Noise ($\sigma = 0.5$).

IV. Simulations

The network architecture consists of a fully connected WCON with $N = 25$ oscillators and symmetric weights as described in Fig. 1(a). Weights are randomly initialized by sampling from a uniform distribution $\mathcal{U}(-0.1, 0.1)$. The adjustable connections are implemented by using memristors as programmable resistors coupled in a differential pair configuration. The update of the weights is performed by averaging the back propagated errors over the total number of training images shown in Fig. 2(a). This approach allows us to lower the amount of total updates of the weight matrix. The learning rate is $\eta = 0.01$ and it decreases during the iterations using a step decay schedule. The forcing parameter is set to $\beta = 1$ and the training process ends whenever a prefixed accuracy is reached. The phase dynamics is simulated using the built-in `MatLAB` routine `ode15s` and time spans are chosen in order to guarantee the convergence of the state variables. Since all target patterns composed by 0 and/or π are equilibria of the system defined in Eq. (1), a small perturbation ε is added to the phase initial conditions of the oscillators in order to let the system escape from a constant evolution of the phase dynamics.

In order to evaluate the effectiveness of EP for training oscillatory networks in associative memory's tasks, the novel algorithm is compared with the unsupervised Hebbian learning rule. Patterns are corrupted using either a uniform flipping of the pixels with probability $p = 0.1$ or an additive Gaussian noise with standard deviation $\sigma = 0.5$. As can be observed in Fig. 2(b) phase trajectories do not converge to multiple values of 0 and π. This is probably due to the cosinusoidal combination in the potential function that allows the system to have many possible equilibria. After convergence, the cosine of the output phase differences is computed and results are saturated to the closest values -1 or 1. A pattern is recognized as correctly reconstructed if the Hamming difference between the reconstruction and the target image is zero. As shown in Fig. 2(c), results provide evidence that the WCON trained with EP is perfectly able to reconstruct the corrupted patterns. In contrast, the network with hebbian connections gets stuck in some local minima reaching 45% accuracy. However, the accuracy degrades as a function of the flipping probability p and the standard deviation σ. For example, with either $p = 0.3$ or $\sigma = 1.5$, the network is able to reconstruct only the 50% of the corrupted patterns.

V. Conclusions

In this work, the dynamics of WCONs has been analyzed as a model of associative memory. The network was trained by adapting Equilibrium Propagation to the phase model defined by Kuramoto. Such two-phase local learning rule enables the memristor–based neural network to sequentially update and adjust the synaptic weights by simply computing a symmetric change that follows the cost function's gradient descent. Simulations showed compelling results that the method has significant capabilities comparable to the one reached by the current learning rules. However, further studies are needed to evaluate the performance of the system considering systematic and random errors arising from real implementations.

VI. Acknowledgements

This work was supported by the Ministero dell'Istruzione, dell'Università e della Ricerca (MIUR) under contract 2017LSCR4K-003.

REFERENCES

[1] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence*. Springer, Berlin, Heidelberg, 1984.

[2] G. Csaba and W. Porod, "Coupled oscillators for computing: A review and perspective," *Applied Physics Reviews*, vol. 7, no. 1, p. 011302, 2020.

[3] Q. Xia and J. J. Yang, "Memristive crossbar arrays for brain-inspired computing," *Nature materials*, vol. 18, no. 4, pp. 309–323, 2019.

[4] C. Li, Y. Li, H. Jiang, W. Song, P. Lin, Z. Wang, J. J. Yang, Q. Xia, M. Hu, E. Montgomery *et al.*, "Large memristor crossbars for analog computing," in *2018 IEEE International Symposium on Circuits and Systems (ISCAS)*. IEEE, 2018, pp. 1–4.

[5] J. Shamsi, M. J. Avedillo, B. Linares-Barranco, and T. Serrano-Gotarredona, "Hardware implementation of differential oscillatory neural networks using vo 2-based oscillators and memristor-bridge circuits," *Frontiers in Neuroscience*, vol. 15, p. 848, 2021. [Online]. Available: https://www.frontiersin.org/article/10.3389/fnins.2021.674567

[6] B. Scellier and Y. Bengio, "Equilibrium propagation: Bridging the gap between energy-based models and backpropagation," *Frontiers in computational neuroscience*, vol. 11, p. 24, 2017.

[7] J. C. Whittington and R. Bogacz, "Theories of error back-propagation in the brain," *Trends in Cognitive Sciences*, vol. 23, no. 03, pp. 235–250, 2019.

[8] F. C. Hoppensteadt and E. M. Izhikevich, "Oscillatory neurocomputers with dynamic connectivity," *Phys. Rev. Lett.*, vol. 82, pp. 2983–2986, Apr 1999.

[9] F. C. Hoppensteadt and E. M. Izhikevich, *Weakly connected neural networks*. Springer Science & Business Media, 2012, vol. 126.

[10] C. Delacour and A. Todri-Sanial, "Mapping Hebbian learning rules to coupling resistances for oscillatory neural networks," *Frontiers in Neuroscience*, vol. 15, p. 1489, 2021. [Online]. Available: https://www.frontiersin.org/article/10.3389/fnins.2021.694549

[11] M. Gilli, M. Bonnin, and F. Corinto, "On global dynamic behavior of weakly connected oscillatory networks," *International Journal of Bifurcation and Chaos*, vol. 15, no. 04, pp. 1377–1393, 2005.

[12] M. Itoh and L. O. Chua, "Star cellular neural networks for associative and dyncamic memories," *International Journal of Bifurcation and Chaos*, vol. 14, no. 05, pp. 1725–1772, 2004.

[13] M. Bonnin, F. L. Traversa, and F. Bonani, "Coupled oscillator networks for von Neumann and non-von Neumann computing," in *Advances in Artificial Intelligence-based Technologies*. Springer, 2022, pp. 179–207.

[14] M. Bonnin, "Phase oscillator model for noisy oscillators," *The European Physical Journal Special Topics*, vol. 226, no. 15, pp. 3227–3237, 2017.

[15] T. Mesnard, W. Gerstner, and J. Brea, "Towards deep learning with spiking neurons in energy based models with contrastive Hebbian plasticity," *arXiv preprint arXiv:1612.03214*, 2016.

[16] M. Bonnin, "Amplitude and phase dynamics of noisy oscillators," *International Journal of Circuit Theory and Applications*, vol. 45, no. 5, pp. 636–659, 2017.