Pseudo-differential operators with isotropic symbols, Wick and anti-Wick operators, and hypoellipticity

PSEUDO-DIFFERENTIAL OPERATORS WITH ISOTROPIC SYMBOLS, WICK AND ANTI-WICK OPERATORS, AND HYPOELLIPTICITY

NENAD TEOFANOV, JOACHIM TOFT, AND PATRIK WAHLBERG

Abstract. We study the link between Ψdos and Wick operators via the Bargmann transform. We deduce a formula for the symbol of the Wick operator in terms of the short-time Fourier transform of the Weyl symbol. This gives characterizations of Wick symbols of Ψdos of Shubin type and of infinite order, and results on composition. We prove a series expansion of Wick operators in anti-Wick operators which leads to a sharp Gårding inequality and transition of hypoellipticity between Wick and and Shubin symbols. Finally we show continuity results for anti-Wick operators, and estimates for the Wick symbols of anti-Wick operators.

0. Introduction

In the paper we investigate conjugation with the Bargmann transformation of pseudo-differential and Toeplitz operators on $\mathbf{R}^d$ with isotropic symbols, and we explore relations between Wick and anti-Wick operators. Particularly we consider Shubin operators and operators of infinite order. This gives rise to analytic type pseudo-differential operators on $\mathbf{C}^d$ that are called Wick or Berezin operators because of the fundamental contributions by F. Berezin [6, 7], which in turns goes back to some ideas in [33] by G. C. Wick.

Let a be a suitable locally bounded function on $\mathbf{C}^{2d}$ such that $z \mapsto a(z,w)$ is analytic, $z, w \in \mathbf{C}^d$. Then the Wick operator $\operatorname{Op}_{\mathfrak{V}}(a)$ with symbol a is the operator which takes an appropriate entire function F on $\mathbf{C}^d$ into the entire function

$$\operatorname{Op}_{\mathfrak{V}}(a)F(z) = \pi^{-d} \int_{\mathbf{C}^d} a(z,w)F(w)e^{(z-w,w)}\, d\lambda(w), \tag{0.1}$$

where $d\lambda$ is the Lebesgue measure and $(\,\cdot\,,\,\cdot\,)$ is the scalar product on $\mathbf{C}^d$. (See [19] and Section 1 for notation.) Wick operators appear naturally in several problems in analysis and its applications, e. g. in quantum mechanics. For example, the harmonic oscillator, the creation and annihilation operators take the simple forms

$$F \mapsto \langle z, \nabla_z \rangle F + cF, \quad F \mapsto z_j F \quad \text{and} \quad F \mapsto \partial_{z_j} F,$$

2010 *Mathematics Subject Classification.* Primary: 32W25, 35S05, 32A17, 46F05, 42B35 Secondary: 32A25, 32A05.

Key words and phrases. pseudo-differential operators, Shubin symbols, isotropic symbols, operators of infinite order, Bargmann transform, Wick operators, anti-Wick operators, hypoellipticity, ellipticity, Gelfand-Shilov spaces.

respectively, for some constant c, in the Wick formulation (see [4]).

An advantage of the Wick calculus compared to corresponding operators on functions and distributions defined on $\mathbf{R}^d$ is that in almost all situations, the involved functions are entire, which admits the use of the powerful techniques of complex analysis. (A more general approach is studied in [30], where the Wick calculus is formulated in terms of spaces of formal power series expansions instead of spaces of entire functions.) The possible lack of analyticity of $a(z,w)$ in (0.1) with respect to the w variable is removable in the sense that for any Wick symbol a, there is a unique a_0 such that $(z,w) \mapsto a_0(z,\overline{w})$ is entire, and $\operatorname{Op}_{\mathfrak{V}}(a) = \operatorname{Op}_{\mathfrak{V}}(a_0)$. Consequently it is no restriction to assume that $a(z,w)$ in (0.1) is analytic in z and conjugate analytic in w, which we do in the introduction henceforth. Any linear and continuous operator from the Schwartz space, a Fourier invariant Gelfand-Shilov space or Pilipović space, to the corresponding distribution spaces, respectively, is in a unique way transformed into a Wick operator by the Bargmann transform (see [30]).

Several operators in quantum mechanics are so-called Shubin operators, i. e. pseudo-differential operators

$$\operatorname{Op}(\mathfrak{a})f(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} \mathfrak{a}(x,\xi)\widehat{f}(\xi)e^{i\langle x,\xi\rangle}\, d\xi, \qquad f \in \mathscr{S}(\mathbf{R}^d),$$

where the symbol $\mathfrak{a}$ belongs to the Shubin class $\operatorname{Sh}_\rho^{(\omega)}(\mathbf{R}^{2d})$, the set of all $\mathfrak{a} \in C^\infty(\mathbf{R}^{2d})$ such that

$$|\partial_x^\alpha \partial_\xi^\beta \mathfrak{a}(x,\xi)| \lesssim \omega(x,\xi)(1+|x|+|\xi|)^{-\rho|\alpha+\beta|}, \quad \alpha,\beta \in \mathbf{N}^d.$$

Here ω is a suitable weight function on $\mathbf{R}^{2d}$ and $0 \leqslant \rho \leqslant 1$. Partial differential operators with polynomial coefficients, e. g. the creation and annihilation operators or the harmonic oscillator mentioned above, are examples of Shubin operators. In Section 2 we prove that the Bargmann image of Shubin operators with symbols in $\operatorname{Sh}_\rho^{(\omega)}(\mathbf{R}^{2d})$ is the set of all Wick operators in (0.1) such that a belongs to $\widehat{\mathcal{A}}^{(\omega)}_{\operatorname{Sh},\rho}(\mathbf{C}^{2d})$. This means that $\mathbf{C}^{2d} \ni (z,w) \mapsto a(z,\overline{w})$ is an entire function that satisfies

$$|\partial_z^\beta \overline{\partial}_w^\gamma a(z,w)| \lesssim e^{\frac{1}{2}|z-w|^2}\omega(\sqrt{2}\overline{z})\langle z+w\rangle^{-\rho|\beta+\gamma|}\langle z-w\rangle^{-N} \tag{0.2}$$

for every $N \geqslant 0$.

An important subclass of Wick operators are the anti-Wick operators, which are Wick operators where the symbol $a(z,w)$ does not depend on z. That is, for an appropriate measurable function a_0 on $\mathbf{C}^d$, its anti-Wick operator is given by

$$\operatorname{Op}_{\mathfrak{V}}^{\mathrm{aw}}(a_0)F(z) = \pi^{-d}\int_{\mathbf{C}^d} a_0(w)F(w)e^{(z-w,w)}\, d\lambda(w). \tag{0.1$'$}$$

Again F is a suitable entire function on $\mathbf{C}^d$. The anti-Wick operators can also be described as the Bargmann image of Toeplitz operators on $\mathbf{R}^d$. (See e. g. [22, 27, 31] for the definition of Toeplitz operators.)

A feature of Toeplitz operators and anti-Wick operators, useful for energy estimates in quantum mechanics and time-frequency analysis, is that non-negative symbols give rise to non-negative operators. (Cf. e. g. [20–22].)

An operator $T = \operatorname{Op}_{\mathfrak{V}}(a)$ with a satisfying (0.2) for every $N \geqslant 0$, is called *positive* (*non-negative*), if there is a constant $C > 0$ ($C \geqslant 0$) such that

$$(TF, F)_{A^2} \geqslant C\|F\|_{A^2}^2,$$

for every analytic polynomial F on $\mathbf{C}^d$, where $(\,\cdot\,,\,\cdot\,)_{A^2}$ is the scalar product induced by the Hilbert norm

$$\|F\|_{A^2} = \pi^{-\frac{d}{2}} \left(\int_{\mathbf{C}^d} |F(z)|^2 e^{-|z|^2}\, d\lambda(z) \right)^{\frac{1}{2}}.$$

The implication from non-negative symbols to non-negative operators is not relevant for Wick operators in (0.1) when $a(z,w)$ is not constant with respect to z, since the analyticity of the map $z \mapsto a(z,w)$ implies that $a(z,w)$ is non-real almost everywhere. For such symbols it is instead natural to check whether positivity of the map $w \mapsto a(w,w)$ leads to positive operators (see e. g. [6, 7, 13]). By choosing

$$d = 1, \quad a(z,w) = 1 - 2z\overline{w} + 2z^2\overline{w}^2 \quad \text{and} \quad F(z) = z$$

we obtain

$$a(w,w) = (1 - |w|^2)^2 + |w|^4 > 0 \quad \text{but} \quad (\operatorname{Op}_{\mathfrak{V}}(a)F, F)_{A^2} = -1 < 0.$$

Consequently $\operatorname{Op}_{\mathfrak{V}}(a)$ may fail to be a non-negative operator even though $a(w,w)$ is positive.

On the other hand, for certain conditions on a, we deduce in Section 3 a weaker positivity result for Wick operators, which is equivalent to the sharp Gårding inequality in isotropic pseudo-differential calculus on $\mathbf{R}^d$ (see Theorem 18.6.7 and the proof of Theorem 18.6.8 in [19]). That is for $a \in \widehat{\mathcal{A}}_{\mathrm{Sh},\rho}^{(\omega)}(\mathbf{C}^{2d})$ with $\omega(z) = \langle z \rangle^{2\rho}$ and $\rho > 0$ we prove

$$\operatorname{Re}(\operatorname{Op}_{\mathfrak{V}}(a)F, F)_{A^2} \geqslant -C\|F\|_{A^2}^2 \tag{0.3}$$

and

$$|\operatorname{Im}(\operatorname{Op}_{\mathfrak{V}}(a)F, F)_{A^2}| \leqslant C\|F\|_{A^2}^2, \quad \text{when} \quad a(w,w) \geqslant 0 \tag{0.4}$$

(cf. Theorem 4.2). In particular we obtain energy estimates also for Wick operators with symbols that are non-negative on the diagonal.

The latter result is obtained by approximating Wick operators by anti-Wick operators, using for the Wick operator (0.1) with $a \in \widehat{\mathcal{A}}_{\mathrm{Sh},\rho}^{(\omega)}(\mathbf{C}^{2d})$ the remarkable identity

$$\operatorname{Op}_{\mathfrak{V}}(a) = \sum_{|\alpha|<N} \frac{(-1)^{|\alpha|}}{\alpha!} \operatorname{Op}_{\mathfrak{V}}^{\mathrm{aw}}(b_\alpha) + \operatorname{Op}_{\mathfrak{V}}(c_N) \quad \text{where} \quad b_\alpha(w) = \partial_z^\alpha \overline{\partial}_w^\alpha a(w,w), \tag{0.5}$$

for some $c_N \in \mathcal{A}_{\mathrm{Sh},\rho}^{(\omega_N)}(\mathbf{C}^{2d})$ with $\omega_N(z) = \omega(z)\langle z \rangle^{-2N\rho}$. Here we again assume $\rho > 0$. The decay conditions on b_α and c_N are, respectively,

$$|\partial_w^\beta \overline{\partial}_w^\gamma b_\alpha(w)| \lesssim \omega(\sqrt{2}\overline{w})\langle w \rangle^{-\rho|2\alpha+\beta+\gamma|}, \qquad \alpha, \beta, \gamma \in \mathbf{N}^d, \tag{0.6}$$

and

$$|\partial_z^\beta \overline{\partial}_w^\gamma c_N(z,w)| \lesssim e^{\frac{1}{2}|z-w|^2} \omega(\sqrt{2}\overline{z})\langle z \rangle^{-2N\rho} \langle z+w \rangle^{-\rho|\beta+\gamma|} \langle z-w \rangle^{-N}. \tag{0.7}$$

Consequently, several Wick operators can essentially be expressed as linear combinations of anti-Wick operators. The expansion (0.5) is deduced in Section 3 using Taylor expansion and integration by parts, see Proposition 3.1 and Remark 3.2.

The conditions on b_α are the same as the conditions on a (0.2), restricted to the diagonal $z = w$, and with improved decay. On the diagonal, the growth term $e^{\frac{1}{2}|z-w|^2}$ disappears, which dominates in (0.2) when $|z-w| \gtrsim |z|$ or $|z-w| \gtrsim |w|$. The right-hand side of (0.6) becomes as large as possible when $\alpha = \beta = \gamma = 0$, that is b_0 is the dominating term in the sum (0.5).

The conditions on c_N are the same as the estimates (0.2) again with improved decay due to the factor $\langle z\rangle^{-2N\rho}$.

For polynomial symbols, (0.5) agrees with the integral formula [6, Theorem 3] due to Berezin which carry over Wick operators into anti-Wick operators. For the general case, (0.5) is analogous to the approximation technique of pseudo-differential operators on $\mathbf{R}^d$ in terms of Toeplitz operators given in [27, Theorem 24.1] and its proof, by Shubin.

The anti-Wick symbols in (0.5) $b_\alpha(w) = \partial_z^\alpha \overline{\partial}_w^\alpha a(w,w)$ extend to have the property that $\partial_z^\alpha \overline{\partial}_w^\alpha a(z,w)$ is entire in z and conjugate entire in w. Note that restriction to the diagonal also appears in the positivity condition (0.3) on Wick symbols.

The sharp Gårding inequality (0.3) is reached by using the fact that $\operatorname{Op}^{\mathrm{aw}}_{\mathfrak{V}}(b_0)$ is non-negative, and that if T is either $\operatorname{Op}^{\mathrm{aw}}_{\mathfrak{V}}(b_\alpha)$ or $\operatorname{Op}_{\mathfrak{V}}(c_N)$ for $\alpha \neq 0$, then $\|TF\|_{A^2} \lesssim \|F\|_{A^2}$ when $F \in A(\mathbf{C}^d)$ is a polynomial.

In Section 5 we deduce links concerning ellipticity, hypoellipticity (in Shubin's sense) and weak ellipticity between Shubin and Wick symbols. The notion of hypoelliptic symbol resembles hypoelliptic symbols in Shubin's sense (see [27]). More specifically, we say that the symbol $\mathfrak{a} \in \operatorname{Sh}_\rho^{(\omega)}(\mathbf{R}^{2d})$ is hypoelliptic of order $\rho_0 \geqslant 0$, whenever there is an $R > 0$ such that

$$|\mathfrak{a}(x,\xi)| \gtrsim \omega(x,\xi)\langle(x,\xi)\rangle^{-\rho_0} \quad \text{and} \quad |\partial^\alpha \mathfrak{a}(x,\xi)| \lesssim |\mathfrak{a}(x,\xi)|\langle(x,\xi)\rangle^{-\rho|\alpha|}$$

when $|(x,\xi)| \geqslant R$.

A linear operator T from $\mathscr{S}'(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ is called globally hypoelliptic if

$$Tf = g,\ f \in \mathscr{S}'(\mathbf{R}^d),\ g \in \mathscr{S}(\mathbf{R}^d) \quad \Rightarrow \quad f \in \mathscr{S}(\mathbf{R}^d).$$

(See e. g. [12].) It can be proved that a pseudo-differential operator with hypoelliptic symbol in Shubin's sense is globally hypoelliptic as operator (see e. g. [27, Corollary 25.1]).

We show, similarly to our investigations of the sharp Gårding inequality and for expansion (0.5), that ellipticity, hypoellipticity and to some degree weak ellipticity for the Shubin symbol $\mathfrak{a}$ can be characterized by certain conditions for the corresponding Wick symbol $a(z,w)$ along the diagonal $z = w$. For example, let $\mathfrak{a}$ be a polynomial on $\mathbf{R}^d$ with principal symbol $\mathfrak{a}_p$, and let $a(z,w)$ be a polynomial in $z, \overline{w} \in \mathbf{C}^d$ with principal part a_p. Then $\mathfrak{a}$ is elliptic means that $\mathfrak{a}_p(x,\xi) \neq 0$ when $(x,\xi) \neq (0,0)$, and a is elliptic means that $a_p(z,z) \neq 0$ when $z \neq 0$. For such $\mathfrak{a}$ we prove

$$\mathfrak{a} \text{ is elliptic} \quad \Leftrightarrow \quad a \text{ is elliptic},$$

when $a(z,w)$ is the Wick symbol corresponding to $\mathfrak{a}$ (which must be a polynomial in z and $\overline{w}$).

Our investigations include the Bargmann transform of certain operators of infinite order, i. e. pseudo-differential operators with ultra-differentiable symbols that are permitted to grow faster than polynomially at infinity together with their derivatives. Particularly we consider Wick operators of infinite order, i. e. the Bargmann images $\operatorname{Op}_{\mathfrak{V}}(a)$ of operators $\operatorname{Op}(\mathfrak{a})$ of infinite order in [1], and characterize their images under the Bargmann transform (see Theorem 2.6). Then we deduce in Subsections 3.2 and 3.3 continuity results for anti-Wick operators which holds for the symbols b_α in (0.5) when $\operatorname{Op}_{\mathfrak{V}}(a)$ is the Bargmann image of an operator of infinite order.

In fact, in Subsection 3.2 we show that $\operatorname{Op}_{\mathfrak{V}}^{\mathrm{aw}}(b_\alpha)$ possess several other continuity properties than what is valid for $\operatorname{Op}_{\mathfrak{V}}(a)$ in the expansion (0.5) (see Propositions 3.6 and 3.9). In Subsection 3.3 we deduce estimates of the Wick symbol b_α^{aw} to the anti-Wick operator $\operatorname{Op}_{\mathfrak{V}}^{\mathrm{aw}}(b_\alpha)$, i. e. the unique element $b_\alpha^{\mathrm{aw}} \in \widehat{A}(\mathbf{C}^{2d})$ such that $\operatorname{Op}_{\mathfrak{V}}(b_\alpha^{\mathrm{aw}}) = \operatorname{Op}_{\mathfrak{V}}^{\mathrm{aw}}(b_\alpha)$. We show that usually, b_α^{aw} satisfies stronger conditions than a when $\operatorname{Op}_{\mathfrak{V}}(a)$ is a Wick operator of infinite order (see Theorems 3.11, 3.14 and 3.13).

The paper is organized as follows. In Section 1 we set the stage by providing necessary background notions and fixing the notation. It contains useful properties for weight functions, Gelfand-Shilov spaces, the Bargmann transform, pseudo-differential operators, Wick and anti-Wick operators. Thereafter we characterize in Section 2 Shubin operators and operators of infinite order in terms of appropriate classes of Wick operators on the Bargmann side. These considerations are based on a formula for the Wick symbol expressed in terms of a short-time Fourier transform of the Weyl symbol, and admits characterization of the Wick symbols corresponding to Shubin Weyl symbols and symbols for operators of infinite order (see Proposition 2.3).

In Section 2 we also study composition and show for example that the well-known closure under composition of Shubin operators and operators of infinite orders have simple and natural proofs on the Wick symbol side.

In Section 3 we deduce series expansions of Wick operators in terms of anti-Wick operators, and between Wick symbols and symbols to corresponding Shubin operators. We also consider anti-Wick operators, and show continuity results for them. We show that the upper bounds for the Wick symbols of anti-Wick operators are stricter than for general Wick symbols.

In Section 4 we discuss lower bounds for Wick operators and deduce the sharp Gårding's inequality. Section 5 concerns ellipticity, hypoellipticity and weak ellipticity.

Finally we observe in Section 6 that a polynomial bound of a Wick symbol implies that the symbol is a polynomial. For pseudo-differential operators this corresponds to partial differential operators with polynomial coefficients. This gives a characterization of such operators as those having polynomially bounded Wick symbols.

Various types of function spaces, distribution spaces, their Bargmann images, and symbol classes for pseudo-differential, Wick and anti-Wick operators appear frequently in the paper. For the reader's convenience we summarize several of these items in an Appendix.

Acknowledgement

N. Teofanov is partially supported by MNPTR Grant No. 451- 03-68/ 2022-14/200125. J. Toft is supported by Vetenskapsrådet (Swedish Science Council), within the project 2019-04890. P. Wahlberg is partially supported by the MIUR project "Dipartimenti di Eccellenza 2018-2022" (CUP E11G18000350001).

1. Preliminaries

In this section we recall some facts on function and distribution spaces as well as on pseudo-differential operators, Wick and anti-Wick operators. Subsection 1.1 concerns weight functions and Subsection 1.2 treats Gelfand-Shilov spaces. In Subsection 1.3 we introduce the Bargmann transform and topological spaces of entire functions on $\mathbf{C}^d$, and in Subsection 1.4 we recall the definitions and some facts on pseudo-differential operators on $\mathbf{R}^d$ as well as Wick and anti-Wick operators on $\mathbf{C}^d$. Subsection 1.5 defines certain symbol classes for pseudo-differential operators on $\mathbf{R}^d$.

1.1. Weight functions.

A *weight* on $\mathbf{R}^d$ is a positive function $\omega \in L^\infty_{loc}(\mathbf{R}^d)$ such that $1/\omega \in L^\infty_{loc}(\mathbf{R}^d)$. The weight ω is called *moderate* if there is a positive locally bounded function v such that

$$\omega(x+y) \leqslant C\omega(x)v(y), \quad x,y \in \mathbf{R}^d, \tag{1.1}$$

for some constant $C \geqslant 1$. If ω and v are weights such that (1.1) holds, then ω is also called v*-moderate*. The set of all moderate weights on $\mathbf{R}^d$ is denoted by $\mathscr{P}_E(\mathbf{R}^d)$. The set $\mathscr{P}(\mathbf{R}^d)$ consists of weights that are v-moderate for a polynomially bounded weight, that is a weight of the form $v(x) = \langle x\rangle^s$ where $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$ and $s \geqslant 0$. The bracket notation is also used for complex arguments as $\langle z\rangle = (1+|z|^2)^{\frac{1}{2}}$ when $z \in \mathbf{C}^d$. In particular, $\omega \in \mathscr{P}(\mathbf{R}^d)$, if and only if

$$\omega(x+y) \leqslant C\omega(x)\langle y\rangle^r, \quad x,y \in \mathbf{R}^d, \tag*{(1.1)$'$}$$

for some $r \geqslant 0$. If $s \in \mathbf{R}$ then $x \mapsto \langle x\rangle^s$ belongs to $\mathscr{P}(\mathbf{R}^d)$, due to Peetre's inequality [26, Lemma 2.1]

$$\langle x+y\rangle^s \leqslant \left(\frac{2}{\sqrt{3}}\right)^{|s|} \langle x\rangle^s \langle y\rangle^{|s|} \qquad x,y \in \mathbf{R}^d, \qquad s \in \mathbf{R}. \tag{1.2}$$

The weight v is called *submultiplicative* if it is even and (1.1) holds for $\omega = v$. If (1.1) holds and v is submultiplicative then

$$\begin{gathered}\frac{\omega(x)}{v(y)} \lesssim \omega(x+y) \lesssim \omega(x)v(y),\\ v(x+y) \lesssim v(x)v(y) \quad \text{and} \quad v(x) = v(-x), \quad x,y \in \mathbf{R}^d.\end{gathered} \tag{1.3}$$

The notation $A(\theta) \lesssim B(\theta)$, $\theta \in \Omega$, means that there is a constant $c > 0$ such that $A(\theta) \leqslant cB(\theta)$ for all $\theta \in \Omega$.

If ω is a moderate weight then by [31] there is a submultiplicative weight v such that (1.1) and (1.3) hold. If v is submultiplicative then

$$1 \lesssim v(x) \lesssim e^{r|x|} \tag{1.4}$$

for some constant $r > 0$ (cf. [16]). In particular, if ω is moderate, then

$$\omega(x+y) \lesssim \omega(x)e^{r|y|} \quad \text{and} \quad e^{-r|x|} \lesssim \omega(x) \lesssim e^{r|x|}, \quad x, y \in \mathbf{R}^d \tag{1.5}$$

for some $r > 0$. If not otherwise specified the symbol v always denote a submultiplicative weight.

1.2. **Gelfand-Shilov spaces.** Let $s, \sigma > 0$. The Gelfand-Shilov space $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ ($\Sigma_s^\sigma(\mathbf{R}^d)$) of Roumieu (Beurling) type consists of all $f \in C^\infty(\mathbf{R}^d)$ such that

$$\|f\|_{\mathcal{S}_{s,h}^\sigma} \equiv \sup \frac{|x^\alpha \partial^\beta f(x)|}{h^{|\alpha+\beta|}\alpha!^s\, \beta!^\sigma} \tag{1.6}$$

is finite for some (every) $h > 0$. The supremum refers to all $\alpha, \beta \in \mathbf{N}^d$ and $x \in \mathbf{R}^d$. The seminorms $\| \cdot \|_{\mathcal{S}_{s,h}^\sigma}$ induce an inductive limit topology for the space $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ and a projective limit topology for $\Sigma_s^\sigma(\mathbf{R}^d)$. The latter space is a Fréchet space under this topology. The space $\mathcal{S}_s^\sigma(\mathbf{R}^d) \neq \{0\}$ ($\Sigma_s^\sigma(\mathbf{R}^d) \neq \{0\}$), if and only if $s + \sigma \geqslant 1$ ($s + \sigma \geqslant 1$ and $(s, \sigma) \neq (\frac{1}{2}, \frac{1}{2})$). We write $\mathcal{S}_s(\mathbf{R}^d) = \mathcal{S}_s^s(\mathbf{R}^d)$ and $\Sigma_s(\mathbf{R}^d) = \Sigma_s^s(\mathbf{R}^d)$.

The *Gelfand-Shilov distribution spaces* $(\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$ and $(\Sigma_s^\sigma)'(\mathbf{R}^d)$ are the dual spaces of $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ and $\Sigma_s^\sigma(\mathbf{R}^d)$, respectively.

The embeddings

$$\mathcal{S}_{s_1}^{\sigma_1}(\mathbf{R}^d) \hookrightarrow \Sigma_{s_2}^{\sigma_2}(\mathbf{R}^d) \hookrightarrow \mathcal{S}_{s_2}^{\sigma_2}(\mathbf{R}^d) \hookrightarrow \mathscr{S}(\mathbf{R}^d)$$

$$\hookrightarrow \mathscr{S}'(\mathbf{R}^d) \hookrightarrow (\mathcal{S}_{s_2}^{\sigma_2})'(\mathbf{R}^d) \hookrightarrow (\Sigma_{s_2}^{\sigma_2})'(\mathbf{R}^d) \hookrightarrow (\mathcal{S}_{s_1}^{\sigma_1})'(\mathbf{R}^d),$$

$$s_1 + \sigma_1 \geqslant 1,\ s_1 < s_2,\ \sigma_1 < \sigma_2, \tag{1.7}$$

are dense. For topological spaces A and B, $A \hookrightarrow B$ means that the inclusion $A \subseteq B$ is continuous.

The spaces $\mathcal{S}_s$ and Σ_s, and their duals spaces, admit characterizations in terms of coefficients with respect to expansions with respect to the Hermite functions

$$h_\alpha(x) = \pi^{-\frac{d}{4}}(-1)^{|\alpha|}(2^{|\alpha|}\alpha!)^{-\frac{1}{2}}e^{\frac{|x|^2}{2}}(\partial^\alpha e^{-|x|^2}), \quad \alpha \in \mathbf{N}^d.$$

The set of Hermite functions on $\mathbf{R}^d$ is an orthonormal basis for $L^2(\mathbf{R}^d)$. We use $\mathcal{H}_0(\mathbf{R}^d)$ to denote the space of finite linear combinations of Hermite functions. Then $\mathcal{H}_0(\mathbf{R}^d)$ is dense in the Schwartz space $\mathscr{S}(\mathbf{R}^d)$, as well as in $\mathscr{S}'(\mathbf{R}^d)$, with respect to its weak* topology. The same conclusion is true for $\Sigma_s(\mathbf{R}^d)$ when $s > \frac{1}{2}$, $\mathcal{S}_s(\mathbf{R}^d)$ when $s \geqslant \frac{1}{2}$ and their distribution dual spaces $\Sigma_s'(\mathbf{R}^d)$ and $\mathcal{S}_s'(\mathbf{R}^d)$. An f in any of these spaces possess an expansion of the form

$$f = \sum_{\alpha \in \mathbf{N}^d} c(f, \alpha)h_\alpha, \quad c(f, \alpha) = (f, h_\alpha), \quad \alpha \in \mathbf{N}^d. \tag{1.8}$$

Here $(\cdot , \cdot)$ denotes the unique extensions of the L^2 form, which is linear in the first variable and conjugate linear in the second variable, from $\mathcal{H}_0(\mathbf{R}^d) \times$

$\mathcal{H}_0(\mathbf{R}^d)$ to $\mathcal{S}_s'(\mathbf{R}^d) \times \mathcal{S}_s(\mathbf{R}^d)$ or $\Sigma_s'(\mathbf{R}^d) \times \Sigma_s(\mathbf{R}^d)$. We recall that (cf. [25, Chapter V.3])

$$\begin{aligned} f \in \mathscr{S}(\mathbf{R}^d) \quad &\Leftrightarrow \quad |c(f,\alpha)| \lesssim \langle \alpha \rangle^{-N} \text{ for every } N \geqslant 0, \\ f \in \mathscr{S}'(\mathbf{R}^d) \quad &\Leftrightarrow \quad |c(f,\alpha)| \lesssim \langle \alpha \rangle^{N} \quad \text{for some } N \geqslant 0. \end{aligned} \tag{1.9}$$

The topology on $\mathscr{S}(\mathbf{R}^d)$ is equivalent to the Fréchet space topology defined by the sequence space seminorms

$$\mathscr{S}(\mathbf{R}^d) \ni f \mapsto \sum_{\alpha \in \mathbf{N}^d} \langle \alpha \rangle^{2N} |c(f,\alpha)|^2, \quad N \geqslant 0.$$

For $f \in \mathscr{S}'(\mathbf{R}^d)$ the sum in (1.8) converges in the weak* topology.

The Hermite functions are eigenfunctions to the harmonic oscillator $H = H_d \equiv |x|^2 - \Delta$ and to the Fourier transform $\mathscr{F}$, given by

$$\mathscr{F} f(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} f(x) e^{-i\langle x,\xi \rangle}\, dx, \quad \xi \in \mathbf{R}^d,$$

when $f \in L^1(\mathbf{R}^d)$. Here $\langle \,\cdot\,,\,\cdot\, \rangle$ denotes the scalar product on $\mathbf{R}^d$. In fact

$$H_d h_\alpha = (2|\alpha| + d) h_\alpha, \quad \alpha \in \mathbf{N}^d.$$

The Fourier transform $\mathscr{F}$ extends uniquely to homeomorphisms on $\mathscr{S}'(\mathbf{R}^d)$, from $(\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$ to $(\mathcal{S}_\sigma^s)'(\mathbf{R}^d)$ and from $(\Sigma_s^\sigma)'(\mathbf{R}^d)$ to $(\Sigma_\sigma^s)'(\mathbf{R}^d)$. It also restricts to homeomorphisms on $\mathscr{S}(\mathbf{R}^d)$, from $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ to $\mathcal{S}_\sigma^s(\mathbf{R}^d)$, from $\Sigma_s^\sigma(\mathbf{R}^d)$ to $\Sigma_\sigma^s(\mathbf{R}^d)$, and to a unitary operator on $L^2(\mathbf{R}^d)$. Similar facts hold true when the Fourier transform is replaced by a partial Fourier transform.

Let $\phi \in \mathscr{S}(\mathbf{R}^d)\backslash 0$ be fixed. We use the transform

$$\begin{aligned} \mathcal{T}_\phi f(x,\xi) &= (2\pi)^{-\frac{d}{2}} e^{i\langle x,\xi \rangle} (f, e^{i\langle \,\cdot\,,\xi \rangle} \phi(\,\cdot\, - x)) \\ &= e^{i\langle x,\xi \rangle} \mathscr{F}(f \cdot \overline{\phi(\,\cdot\, - x)})(\xi) \\ &= \mathscr{F}(f(\,\cdot\, + x)\overline{\phi})(\xi), \quad x,\xi \in \mathbf{R}^d, \end{aligned} \tag{1.10}$$

where $f \in \mathscr{S}'(\mathbf{R}^d)$ and $\phi \in \mathscr{S}(\mathbf{R}^d)\backslash 0$ (cf. [9]). If $f,\phi \in \mathscr{S}(\mathbf{R}^d)$ then

$$\begin{aligned} \mathcal{T}_\phi f(x,\xi) &= (2\pi)^{-\frac{d}{2}} e^{i\langle x,\xi \rangle} \int_{\mathbf{R}^d} f(y) \overline{\phi(y-x)} e^{-i\langle y,\xi \rangle}\, dy \\ &= (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} f(y+x) \overline{\phi(y)} e^{-i\langle y,\xi \rangle}\, dy, \quad x,\xi \in \mathbf{R}^d. \end{aligned} \tag{1.10$'$}$$

We notice that the short-time Fourier transform $V_\phi f$ of f is given by

$$V_\phi f(x,\xi) = e^{-i\langle x,\xi \rangle} \mathcal{T}_\phi f(x,\xi). \tag{1.11}$$

That is, $\mathcal{T}_\phi$ is a *modulated* short-time Fourier transform. Thus by [31, Theorem 2.3] it follows that the definition of the map $(f,\phi) \mapsto \mathcal{T}_\phi f$ from $\mathscr{S}(\mathbf{R}^d) \times \mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}(\mathbf{R}^{2d})$ is uniquely extendable to a continuous map from $\mathcal{S}_s'(\mathbf{R}^d) \times \mathcal{S}_s'(\mathbf{R}^d)$ to $\mathcal{S}_s'(\mathbf{R}^{2d})$, and restricts to a continuous map from $\mathcal{S}_s(\mathbf{R}^d) \times \mathcal{S}_s(\mathbf{R}^d)$ to $\mathcal{S}_s(\mathbf{R}^{2d})$. The same conclusion holds with Σ_s in place of $\mathcal{S}_s$, at each place.

The adjoint $\mathcal{T}_\phi^*$ is given by

$$(\mathcal{T}_\phi^* F, g)_{L^2(\mathbf{R}^d)} = (F, \mathcal{T}_\phi g)_{L^2(\mathbf{R}^{2d})}$$

for $F\in \mathcal S_s'(\mathbf R^{2d})$ and $g\in \mathcal S_s(\mathbf R^d)$, and similarly with Σ_s or with $\mathscr S$ in place of $\mathcal S_s$ at each occurrence. When F is a polynomially bounded measurable function we write

$$\mathcal T_\phi^*F(y) = (2\pi)^{-\frac d2}\iint\limits_{\mathbf R^{2d}} F(x,\xi)\, e^{i\langle y-x,\xi\rangle}\phi(y-x)\, dxd\xi, \tag{1.12}$$

where the integral is defined weakly so that $(\mathcal T_\phi^*F,g)_{L^2(\mathbf R^d)} = (F,\mathcal T_\phi g)_{L^2(\mathbf R^{2d})}$ for $g\in \mathscr S(\mathbf R^d)$. The identity (1.12) is called Moyal's formula.

We have

$$(\mathcal T_\psi^*\circ \mathcal T_\phi)f = (\psi,\phi)f,\qquad f\in \mathcal S_s'(\mathbf R^d),\ \phi,\psi\in \mathcal S_s(\mathbf R^d), \tag{1.13}$$

and similarly with Σ_s or with $\mathscr S$ in place of $\mathcal S_s$ at each occurrence.

Two important features of $\mathcal T_\phi$ which distinguish it from the short-time Fourier transform are the differential identities

$$\partial_x^\alpha \mathcal T_\phi f(x,\xi) = \mathcal T_\phi(\partial^\alpha f)(x,\xi),\qquad \alpha\in \mathbf N^d \tag{1.14}$$

and

$$D_\xi^\beta \mathcal T_\phi f(x,\xi) = \mathcal T_{g_\beta} f(x,\xi),\qquad \beta\in \mathbf N^d,\qquad \phi_\beta(x) = (-x)^\beta\phi(x). \tag{1.15}$$

By (1.11) it follows that characterizations of Gelfand-Shilov spaces and their distribution spaces in terms of estimates of their short-time Fourier transforms carry over to estimates on $\mathcal T_\phi$ in place of V_ϕ. For example we have the following (see e. g. [17,28] for the proof of (1) and [32] for the proof of (2)). See also [11] for related results.

Proposition 1.1. *Let $s,\sigma>0$, $\phi\in \mathcal S_s^\sigma(\mathbf R^d)\backslash 0$ ($\phi\in \Sigma_s^\sigma(\mathbf R^d)\backslash 0$) and let $f\in (\mathcal S_s^\sigma)'(\mathbf R^d)$ ($f\in (\Sigma_s^\sigma)'(\mathbf R^d)$). Then the following is true:*

(1) *$f\in \mathcal S_s^\sigma(\mathbf R^d)$ ($f\in \Sigma_s^\sigma(\mathbf R^d)$) if and only if*

$$|\mathcal T_\phi f(x,\xi)|\lesssim e^{-r(|x|^{\frac 1s}+|\xi|^{\frac 1\sigma})},\quad x,\xi\in \mathbf R^d, \tag{1.16}$$

for some (every) $r>0$.

(2) *$f\in (\mathcal S_s^\sigma)'(\mathbf R^d)$ ($f\in (\Sigma_s^\sigma)'(\mathbf R^d)$) if and only if*

$$|\mathcal T_\phi f(x,\xi)|\lesssim e^{r(|x|^{\frac 1s}+|\xi|^{\frac 1\sigma})},\quad x,\xi\in \mathbf R^d, \tag{1.17}$$

for every (some) $r>0$.

1.3. **The Bargmann transform and spaces of analytic functions.** If $\Omega\subseteq \mathbf C^d$ is open then $A(\Omega)$ consists of all (complex-valued) analytic functions on Ω. Complex derivatives are denoted, with $z=x+iy\in\Omega$,

$$\partial_{z_j} = \frac 12\left(\partial_{x_j}-i\partial_{y_j}\right),\quad \overline\partial_{z_j} = \frac 12\left(\partial_{x_j}+i\partial_{y_j}\right)$$

for $1\leqslant j\leqslant d$, which admits the Cauchy-Riemann equations to be written as $\overline\partial_{z_j}f=0$, $1\leqslant j\leqslant d$.

The Bargmann kernel is defined by

$$\mathfrak A_d(z,y) = \pi^{-\frac d4}\exp\Big(-\frac 12(\langle z,z\rangle+|y|^2)+2^{1/2}\langle z,y\rangle\Big),\quad z\in \mathbf C^d,\quad y\in \mathbf R^d,$$

where

$$\langle z,w\rangle = \sum_{j=1}^{d} z_j w_j \quad \text{and} \quad (z,w) = \langle z,\overline{w}\rangle$$

when

$$z=(z_1,\dots,z_d)\in \mathbf{C}^d \quad \text{and} \quad w=(w_1,\dots,w_d)\in \mathbf{C}^d.$$

Sometimes $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the duality between a test function space and its dual. The context precludes confusion between its double use. The Bargmann transform $\mathfrak{V}_d f$ of $f\in \mathcal{S}_{1/2}'(\mathbf{R}^d)$ is the entire function

$$\mathfrak{V}_d f(z) = \langle f, \mathfrak{A}_d(z,\,\cdot\,)\rangle, \quad z\in \mathbf{C}^d. \tag{1.18}$$

The right-hand side is a well defined element in $A(\mathbf{C}^d)$, since $y\mapsto \mathfrak{A}_d(z,y)$ belongs to $\mathcal{S}_{1/2}(\mathbf{R}^d)$ for $z\in \mathbf{C}^d$ fixed, and $\mathfrak{A}_d(\,\cdot\,,y)$ is entire for all $y\in\mathbf{R}^d$. Let $p\in[1,\infty]$ and $\omega\in\mathscr{P}_E(\mathbf{R}^d)$. Then $L^p_{(\omega)}(\mathbf{R}^d)$ consists of all $f\in L^1_{loc}(\mathbf{R}^d)$ such that $\|f\|_{L^p_{(\omega)}}\equiv \|f\cdot\omega\|_{L^p}$ is finite. If $f\in L^p_{(\omega)}(\mathbf{R}^d)$, then

$$\begin{aligned}\mathfrak{V}_d f(z) &= \int_{\mathbf{R}^d} \mathfrak{A}_d(z,y)f(y)\,dy \\ &= \pi^{-\frac d4}\int_{\mathbf{R}^d}\exp\Big(-\frac12(\langle z,z\rangle+|y|^2)+2^{1/2}\langle z,y\rangle\Big)f(y)\,dy, \quad z\in\mathbf{C}^d. \end{aligned}\tag{1.19}$$

(Cf. [4, 5, 31, 32].)

For $p\in(0,\infty]$, $\omega\in\mathscr{P}_E(\mathbf{C}^d)$ and $\omega_0(z)=\omega(\sqrt2\overline{z})$, let $A^p_{(\omega)}(\mathbf{C}^d)$ be the set of all $F\in A(\mathbf{C}^d)$ such that

$$\|F\|_{A^p_{(\omega)}}\equiv \pi^{-\frac dp}\|F\cdot e^{-\frac12|\,\cdot\,|^2}\cdot\omega_0\|_{L^p},$$

and set $A^p=A^p_{(\omega)}$ when $\omega=1$. It was proved by Bargmann [4] that

$$\mathfrak{V}_d : L^2(\mathbf{R}^d)\to A^2(\mathbf{C}^d) \tag{1.20}$$

is bijective and isometric. The space $A^2(\mathbf{C}^d)$ is the Hilbert space of entire functions with scalar product

$$(F,G)_{A^2}\equiv\int_{\mathbf{C}^d}F(z)\overline{G(z)}\,d\mu(z), \quad F,G\in A^2(\mathbf{C}^d),$$

where $d\mu(z)=\pi^{-d}e^{-|z|^2}\,d\lambda(z)$ and $d\lambda(z)$ is the Lebesgue measure on $\mathbf{C}^d$. The space $A^2(\mathbf{C}^d)$ is known as the Fock or Segal-Bargmann space in quantum mechanics (see [13, 18]).

In [4] it was proved that the Bargmann transform maps the Hermite functions to monomials as

$$\mathfrak{V}_d h_\alpha = e_\alpha, \qquad e_\alpha(z)=\frac{z^\alpha}{\alpha!^{\frac12}}, \quad z\in\mathbf{C}^d, \quad \alpha\in\mathbf{N}^d. \tag{1.21}$$

The orthonormal basis $\{h_\alpha\}_{\alpha\in\mathbf{N}^d}\subseteq L^2(\mathbf{R}^d)$ is thus mapped to the orthonormal basis $\{e_\alpha\}_{\alpha\in\mathbf{N}^d}\subseteq A^2(\mathbf{C}^d)$. Bargmann also proved that there is a reproducing formula for $A^2(\mathbf{C}^d)$. Let Π_A be the operator from $L^2(d\mu)$ to $A(\mathbf{C}^d)$, given by

$$\Pi_A F(z)=\int_{\mathbf{C}^d}F(w)e^{(z,w)}\,d\mu(w), \quad z\in\mathbf{C}^d. \tag{1.22}$$

Then Π_A is the orthogonal projection from $L^2(d\mu)$ to $A^2(\mathbf{C}^d)$ (cf. [4]).

When we discuss extensions and restrictions of the Bargmann transform to Gelfand-Shilov spaces and their distribution spaces, we use

$$|z|_{s,\sigma} = |\operatorname{Re} z|^{\frac{1}{s}} + |\operatorname{Im} z|^{\frac{1}{\sigma}}, \qquad z \in \mathbf{C}^d, \tag{1.23}$$

and consider the seminorms

$$\|F\|_{\mathcal{A}_{\mathscr{S};r}} \equiv \|F \cdot e^{-\frac{1}{2}|\,\cdot\,|^2} \langle\,\cdot\,\rangle^r\|_{L^\infty}, \qquad \|F\|_{\mathcal{A}'_{\mathscr{S};r}} \equiv \|F \cdot e^{-\frac{1}{2}|\,\cdot\,|^2} \langle\,\cdot\,\rangle^{-r}\|_{L^\infty}$$

and

$$\|F\|_{\mathcal{A}_{\mathcal{S}^\sigma_{s;r}}} \equiv \|F \cdot e^{-\frac{1}{2}|\,\cdot\,|^2 + r|\,\cdot\,|_{s,\sigma}}\|_{L^\infty}, \quad \|F\|_{\mathcal{A}'_{\mathcal{S}^\sigma_{s;r}}} \equiv \|F \cdot e^{-\frac{1}{2}|\,\cdot\,|^2 - r|\,\cdot\,|_{s,\sigma}}\|_{L^\infty}$$

when $F \in A(\mathbf{C}^d)$, $r > 0$ and $s, \sigma \geqslant \frac{1}{2}$. Then $\mathcal{A}^\sigma_{0,s}(\mathbf{C}^d)$ for $s, \sigma > \frac{1}{2}$, $\mathcal{A}_{\mathscr{S}}(\mathbf{C}^d)$ and $(\mathcal{A}^\sigma_s)'(\mathbf{C}^d)$ for $s, \sigma \geqslant \frac{1}{2}$ are the sets of all $F \in A(\mathbf{C}^d)$ such that

$$\|F\|_{\mathcal{A}_{\mathcal{S}^\sigma_{s;r}}} < \infty, \quad \|F\|_{\mathcal{A}_{\mathscr{S};r}} < \infty \quad \text{and} \quad \|F\|_{\mathcal{A}'_{\mathcal{S}^\sigma_{s;r}}} < \infty, \tag{1.24}$$

respectively, for every $r > 0$. The spaces are equipped with the projective limit topology with respect to $r > 0$, defined by each class of seminorms, respectively.

In the same way we let $\mathcal{A}^\sigma_s(\mathbf{C}^d)$ for $s, \sigma \geqslant \frac{1}{2}$, $\mathcal{A}'_{\mathscr{S}}(\mathbf{C}^d)$ and $(\mathcal{A}^\sigma_{0,s})'(\mathbf{C}^d)$ for $s, \sigma > \frac{1}{2}$ be the sets of all $F \in A(\mathbf{C}^d)$ such that

$$\|F\|_{\mathcal{A}_{\mathcal{S}^\sigma_{s;r}}} < \infty, \quad \|F\|_{\mathcal{A}'_{\mathscr{S};r}} < \infty \quad \text{and} \quad \|F\|_{\mathcal{A}'_{\mathcal{S}^\sigma_{s;r}}} < \infty, \tag{1.25}$$

respectively, for some $r > 0$. Their topologies are the inductive limit topologies with respect to $r > 0$, defined by each class of seminorms, respectively. We also set

$$\mathcal{A}_{0,s} = \mathcal{A}^s_{0,s} \quad \text{and} \quad \mathcal{A}_s = \mathcal{A}^s_s.$$

Then

$$\mathfrak{V}_d : \mathscr{S}(\mathbf{R}^d) \to \mathcal{A}_{\mathscr{S}}(\mathbf{C}^d), \qquad \mathfrak{V}_d : \quad \mathscr{S}'(\mathbf{R}^d) \to \mathcal{A}'_{\mathscr{S}}(\mathbf{C}^d),$$

$$\mathfrak{V}_d : \mathcal{S}^\sigma_s(\mathbf{R}^d) \to \mathcal{A}^\sigma_s(\mathbf{C}^d), \qquad \mathfrak{V}_d : (\mathcal{S}^\sigma_s)'(\mathbf{R}^d) \to (\mathcal{A}^\sigma_s)'(\mathbf{C}^d) \qquad s, \sigma \geqslant \frac{1}{2}$$

and

$$\mathfrak{V}_d : \Sigma^\sigma_s(\mathbf{R}^d) \to \mathcal{A}^\sigma_{0,s}(\mathbf{C}^d), \qquad \mathfrak{V}_d : (\Sigma^\sigma_s)'(\mathbf{R}^d) \to (\mathcal{A}^\sigma_{0,s})'(\mathbf{C}^d), \quad s, \sigma > \frac{1}{2}$$

are homeomorphisms [32].

From these homeomorphisms, the fact that the map (1.20) is a homeomorphism and duality properties for Gelfand-Shilov spaces, it follows that $(\,\cdot\,,\,\cdot\,)_{A^2}$ on $\mathcal{A}_{1/2}(\mathbf{C}^d) \times \mathcal{A}_{1/2}(\mathbf{C}^d)$ is uniquely extendable to a continuous sesqui-linear form on $(\mathcal{A}^\sigma_s)'(\mathbf{C}^d) \times \mathcal{A}^\sigma_s(\mathbf{C}^d)$. The dual of $\mathcal{A}^\sigma_s(\mathbf{C}^d)$ can be identified with $(\mathcal{A}^\sigma_s)'(\mathbf{C}^d)$ through this form. Similar facts hold for $\mathcal{A}^\sigma_{0,s}$ in place of $\mathcal{A}^\sigma_s$ at each occurrence. (Cf. e. g. [31, 32].)

Finally let $\mathcal{A}_{\flat_1;r}(\mathbf{C}^d)$ and $\mathcal{A}_{\flat_\infty;r}(\mathbf{C}^d)$ for $r > 0$ be the Banach spaces which consist of all $F \in A(\mathbf{C}^d)$ such that

$$\|F\|_{\mathcal{A}_{\flat_1;r}} \equiv \|F \cdot e^{-r|\,\cdot\,|}\|_{L^\infty} \quad \text{respectively} \quad \|F\|_{\mathcal{A}_{\flat_\infty;r}} \equiv \|F \cdot e^{-r|\,\cdot\,|^2}\|_{L^\infty}$$

is finite, and let $\mathcal{A}_{\flat_1}(\mathbf{C}^d)$ be the inductive limit of $\mathcal{A}_{\flat_1;r}(\mathbf{C}^d)$ with respect to $r > 0$. Also let $\mathcal{A}_{0,\flat_\infty}(\mathbf{C}^d)$ and $\mathcal{A}'_{0,\flat_\infty}(\mathbf{C}^d)$ be the projective respectively inductive limit topologies of $\mathcal{A}_{\flat_\infty;r}(\mathbf{C}^d)$ with respect to $r > 0$.

It is evident that $\mathcal{A}_{\flat_1}(\mathbf{C}^d)$ is densely embedded in $\mathcal{A}_s^\sigma(\mathbf{C}^d)$ for every $s, \sigma \geqslant \frac{1}{2}$, as well as in $\mathcal{A}_{0,s}^\sigma(\mathbf{C}^d)$ for every $s, \sigma > \frac{1}{2}$. The form $(\,\cdot\,,\,\cdot\,)_{A^2}$ on $\mathcal{A}_{\flat_1}(\mathbf{C}^d) \times \mathcal{A}_{\flat_1}(\mathbf{C}^d)$ is uniquely extendable to a continuous sesqui-linear form on $A(\mathbf{C}^d) \times \mathcal{A}_{\flat_1}(\mathbf{C}^d)$ and the dual of $\mathcal{A}_{\flat_1}(\mathbf{C}^d)$ can be identified with $A(\mathbf{C}^d)$. The Fréchet space topology of $A(\mathbf{C}^d)$ can be defined by the seminorms

$$F \mapsto \sup_{|z| \leqslant N} |F(z)|, \qquad N = 1, 2, \dots .$$

(Cf. [32].)

Remark 1.2. The spaces $\mathcal{A}_{\flat_1}(\mathbf{C}^d)$ and $\mathcal{A}_{0,\flat_\infty}(\mathbf{C}^d)$ are examples of Bargmann images of special Pilipović spaces, a family of Fourier invariant topological vector spaces which are smaller than any Fourier invariant Gelfand-Shilov space, and which were introduced and investigated in [32]. For any $\sigma > 0$, the Bargmann image of the Pilipović spaces $\mathcal{H}_{\flat_\sigma}(\mathbf{R}^d)$ and $\mathcal{H}_{0,\flat_\sigma}(\mathbf{R}^d)$ are given by

$$\mathcal{A}_{\flat_\sigma}(\mathbf{C}^d) \equiv \{\, F \in A(\mathbf{C}^d)\,;\, |F(z)| \lesssim e^{r|z|^{\frac{2\sigma}{\sigma+1}}} \text{ for some } r > 0 \,\}$$

respectively

$$\mathcal{A}_{0,\flat_\sigma}(\mathbf{C}^d) \equiv \{\, F \in A(\mathbf{C}^d)\,;\, |F(z)| \lesssim e^{r|z|^{\frac{2\sigma}{\sigma+1}}} \text{ for every } r > 0 \,\}.$$

If $\sigma > 1$, then the (strong) duals of $\mathcal{A}_{\flat_\sigma}(\mathbf{C}^d)$ and $\mathcal{A}_{0,\flat_\sigma}(\mathbf{C}^d)$ are given by

$$\mathcal{A}'_{\flat_\sigma}(\mathbf{C}^d) \equiv \{\, F \in A(\mathbf{C}^d)\,;\, |F(z)| \lesssim e^{r|z|^{\frac{2\sigma}{\sigma-1}}} \text{ for every } r > 0 \,\}$$

respectively

$$\mathcal{A}'_{0,\flat_\sigma}(\mathbf{C}^d) \equiv \{\, F \in A(\mathbf{C}^d)\,;\, |F(z)| \lesssim e^{r|z|^{\frac{2\sigma}{\sigma-1}}} \text{ for some } r > 0 \,\}$$

through a unique extension of the A^2 scalar product on $\mathcal{A}_{\flat_1}(\mathbf{C}^d) \times \mathcal{A}_{\flat_1}(\mathbf{C}^d)$. In particular, if σ tends to ∞, it follows that some of these conditions tend to

$$\mathcal{A}_{0,\flat_\infty}(\mathbf{C}^d) \equiv \{\, F \in A(\mathbf{C}^d)\,;\, |F(z)| \lesssim e^{r|z|^2} \text{ for every } r > 0 \,\}$$

respectively

$$\mathcal{A}'_{0,\flat_\infty}(\mathbf{C}^d) \equiv \{\, F \in A(\mathbf{C}^d)\,;\, |F(z)| \lesssim e^{r|z|^2} \text{ for some } r > 0 \,\}.$$

Note that in [30,32], the set $\mathcal{A}_{0,\flat_\infty}(\mathbf{C}^d)$ is denoted by $\mathcal{A}_{0,\frac{1}{2}}(\mathbf{C}^d)$, and its dual $\mathcal{A}'_{0,\flat_\infty}(\mathbf{C}^d)$ is denoted by $\mathcal{A}'_{0,\frac{1}{2}}(\mathbf{C}^d)$.

At many places it will be crucial to use the Gaussian window

$$\phi(x) = \pi^{-\frac{d}{4}} e^{-\frac{1}{2}|x|^2}, \quad x \in \mathbf{R}^d, \tag{1.26}$$

in the transform $\mathcal{T}_\phi$. For this ϕ the relationship between the Bargmann transform and $\mathcal{T}_\phi$ is

$$\mathfrak{V}_d = U_{\mathfrak{V}} \circ \mathcal{T}_\phi, \quad \text{and} \quad U_{\mathfrak{V}}^{-1} \circ \mathfrak{V}_d = \mathcal{T}_\phi, \tag{1.27}$$

where $U_{\mathfrak{V}}$ is the linear, continuous and bijective operator on $\mathscr{D}'(\mathbf{R}^{2d}) \simeq \mathscr{D}'(\mathbf{C}^d)$, given by

$$U_{\mathfrak{V}} F(x+i\xi) = (2\pi)^{\frac{d}{2}} e^{\frac{1}{2}(|x|^2+|\xi|^2)} e^{i\langle x,\xi\rangle} F(\sqrt{2}\,x, -\sqrt{2}\,\xi), \quad x,\xi \in \mathbf{R}^d, \tag{1.28}$$

cf. [31] in combination with (1.11).

In analytic operator theory we need subspaces of

$$\widehat{A}(\mathbf{C}^{2d}) \equiv \left\{ \Theta K\, ;\, K \in A(\mathbf{C}^{2d}) \right\},$$

where the semi-conjugation operator is

$$(\Theta K)(z,w) = K(z,\overline{w}), \qquad z,w \in \mathbf{C}^d. \tag{1.29}$$

If T is a linear and continuous operator from $\mathcal{S}_{1/2}(\mathbf{R}^d)$ to $\mathcal{S}_{1/2}'(\mathbf{R}^d)$, then there is a unique $K \in \widehat{A}(\mathbf{C}^{2d})$ such that $\Theta K \in \mathcal{A}_{1/2}'(\mathbf{C}^{2d})$ and $\mathfrak{V}_d \circ T \circ \mathfrak{V}_d^{-1}$ is given by

$$F(z) \mapsto \int_{\mathbf{C}^d} K(z,w)F(w)\, d\mu(w). \tag{1.30}$$

(See e.g. [30].) For these reasons we let

$$\widehat{\mathcal{A}}_{0,s}(\mathbf{C}^{2d}), \quad \widehat{\mathcal{A}}_{s}(\mathbf{C}^{2d}), \quad \widehat{\mathcal{A}}_{\mathscr{S}}(\mathbf{C}^{2d}), \quad \widehat{\mathcal{A}}_{\mathscr{S}}'(\mathbf{C}^{2d}), \quad \widehat{\mathcal{A}}_{s}'(\mathbf{C}^{2d}) \quad \text{and} \quad \widehat{\mathcal{A}}_{0,s}'(\mathbf{C}^{2d})$$

be the images of

$$\mathcal{A}_{0,s}(\mathbf{C}^{2d}), \quad \mathcal{A}_{s}(\mathbf{C}^{2d}), \quad \mathcal{A}_{\mathscr{S}}(\mathbf{C}^{2d}), \quad \mathcal{A}_{\mathscr{S}}'(\mathbf{C}^{2d}), \quad \mathcal{A}_{s}'(\mathbf{C}^{2d}) \quad \text{and} \quad \mathcal{A}_{0,s}'(\mathbf{C}^{2d})$$

respectively, under the map Θ. We also let $\widehat{A}^p(\mathbf{C}^{2d})$ and $\widehat{\mathcal{A}}_{\flat_1}(\mathbf{C}^{2d})$ be the images of $A^p(\mathbf{C}^{2d})$ and $\mathcal{A}_{\flat_1}(\mathbf{C}^{2d})$, respectively, under the map Θ. The topologies of the former spaces are inherited from the corresponding latter spaces.

The semi-conjugated Bargmann (SCB) transform is defined as

$$\mathfrak{V}_{\Theta,d} = \Theta \circ \mathfrak{V}_{2d}.$$

All properties of the Bargmann transform carry over naturally to analogous properties for the SCB transform.

1.4. **Pseudo-differential operators.** Let A be a real $d \times d$ matrix. The *pseudo-differential operator* $\operatorname{Op}_A(\mathfrak{a})$ with *symbol* $\mathfrak{a} \in \mathcal{S}_{1/2}(\mathbf{R}^{2d})$ is the linear and continuous operator on $\mathcal{S}_{1/2}(\mathbf{R}^d)$ given by

$$\operatorname{Op}_A(\mathfrak{a})f(x) = (2\pi)^{-d} \iint\limits_{\mathbf{R}^{2d}} \mathfrak{a}(x-A(x-y),\xi) f(y) e^{i\langle x-y,\xi\rangle}\, dyd\xi, \quad x \in \mathbf{R}^d. \tag{1.31}$$

For $\mathfrak{a} \in \mathcal{S}_{1/2}'(\mathbf{R}^{2d})$ the pseudo-differential operator $\operatorname{Op}_A(\mathfrak{a})$ is defined as the continuous operator from $\mathcal{S}_{1/2}(\mathbf{R}^d)$ to $\mathcal{S}_{1/2}'(\mathbf{R}^d)$ with distribution kernel

$$K_{\mathfrak{a},A}(x,y) = (2\pi)^{-\frac{d}{2}} \mathscr{F}_2^{-1}\mathfrak{a}(x-A(x-y), x-y), \quad x,y \in \mathbf{R}^d, \tag{1.32}$$

where $\mathscr{F}_2 F$ is the partial Fourier transform of $F(x,y) \in \mathcal{S}_{1/2}'(\mathbf{R}^{2d})$ with respect to the y variable. This definition makes sense since the mappings

$$\mathscr{F}_2 \quad \text{and} \quad F(x,y) \mapsto F(x, x-y) \tag{1.33}$$

are homeomorphisms on $\mathcal{S}_{1/2}'(\mathbf{R}^{2d})$. The map $a \mapsto K_{\mathfrak{a},A}$ is hence a homeomorphism on $\mathcal{S}_{1/2}'(\mathbf{R}^{2d})$.

If A and B are real $d \times d$ matrices and $\mathfrak{a} \in \mathcal{S}_{1/2}'(\mathbf{R}^{2d})$, then there is a unique $\mathfrak{b} \in \mathcal{S}_{1/2}'(\mathbf{R}^{2d})$ such that $\operatorname{Op}_A(\mathfrak{a}) = \operatorname{Op}_B(\mathfrak{b})$, and that $\mathfrak{b}$ can be obtained by

$$\operatorname{Op}_A(\mathfrak{a}) = \operatorname{Op}_B(\mathfrak{b}) \quad \Leftrightarrow \quad e^{i\langle AD_\xi, D_x\rangle}\mathfrak{a}(x,\xi) = e^{i\langle BD_\xi, D_x\rangle}\mathfrak{b}(x,\xi) \tag{1.34}$$

(see [10, 19]).

Remark 1.3. By Fourier's inversion formula, (1.32) and the kernel theorem [23, Theorem 2.2], [29, Theorem 2.5] for operators from Gelfand-Shilov spaces to their duals, it follows that the map $\mathfrak{a} \mapsto \operatorname{Op}_A(\mathfrak{a})$ is bijective from $\mathcal{S}_{1/2}'(\mathbf{R}^{2d})$ to the set of all linear and continuous operators from $\mathcal{S}_{1/2}(\mathbf{R}^d)$ to $\mathcal{S}_{1/2}'(\mathbf{R}^{2d})$.

If $A = 0$ then $\operatorname{Op}_A(\mathfrak{a}) = \operatorname{Op}_0(\mathfrak{a}) = \operatorname{Op}(\mathfrak{a}) = \mathfrak{a}(x,D)$ is the Kohn-Nirenberg or standard representation. If $A = \frac{1}{2}I_d$ where I_d is the $d \times d$ identity matrix then $\operatorname{Op}_A(\mathfrak{a}) = \operatorname{Op}^w(\mathfrak{a})$ is the Weyl quantization. In this paper we use mainly the Weyl quantization and we put

$$K_{\mathfrak{a}}^w = K_{\mathfrak{a},I_d/2}\ .$$

The Weyl product $\mathfrak{a}\#\mathfrak{b}$ of two Weyl symbols $\mathfrak{a}, \mathfrak{b} \in \mathcal{S}_{1/2}(\mathbf{R}^{2d})$ is defined as the product of symbols corresponding to operator composition. Thus

$$\operatorname{Op}^w(\mathfrak{a}\#\mathfrak{b}) = \operatorname{Op}^w(\mathfrak{a}) \circ \operatorname{Op}^w(\mathfrak{b})$$

and the Weyl product can be extended to larger spaces as long as composition is well defined.

Next we recall the definition of Wick operators. Suppose that $a \in \widehat{A}(\mathbf{C}^{2d})$ satisfies

$$w \mapsto a(z,w)e^{r|w|-|w|^2} \in L^1(\mathbf{C}^d) \tag{1.35}$$

locally uniformly with respect to $z \in \mathbf{C}^d$ for every $r > 0$. Then the *analytic pseudo-differential operator*, or *Wick operator* $\operatorname{Op}_{\mathfrak{V}}(a)$ with symbol a and acting on $F \in \mathcal{A}_{\flat_1}(\mathbf{C}^d)$, is defined by

$$\operatorname{Op}_{\mathfrak{V}}(a)F(z) = \int_{\mathbf{C}^d} a(z,w)F(w)e^{(z,w)}\,d\mu(w), \quad z \in \mathbf{C}^d. \tag{1.36}$$

(Cf. e.g. [6, 13, 30–32].) The condition (1.35) and $F \in \mathcal{A}_{\flat_1}(\mathbf{C}^d)$ imply that the integrand on the right-hand side of (1.36) is well defined. The locally uniform condition (1.35) with respect to $z \in \mathbf{C}^d$ implies that $\operatorname{Op}_{\mathfrak{V}}(a)F \in A(\mathbf{C}^d)$.

In [30] several extensions and restrictions of $\operatorname{Op}_{\mathfrak{V}}(a)$ are given. The following result follows from [30, Theorems 2.7 and 2.8]. Here $\mathcal{L}(\mathcal{A}_{\flat_1}(\mathbf{C}^d), A(\mathbf{C}^d))$ is the space of all linear and continuous operators from $\mathcal{A}_{\flat_1}(\mathbf{C}^d)$ to $A(\mathbf{C}^d)$.

Proposition 1.4. *The map $a \mapsto \operatorname{Op}_{\mathfrak{V}}(a)$ from $\widehat{\mathcal{A}}_{\flat_1}(\mathbf{C}^{2d})$ to $\mathcal{L}(\mathcal{A}_{\flat_1}(\mathbf{C}^d), A(\mathbf{C}^d))$ is uniquely extendable to a bijective map from $\widehat{A}(\mathbf{C}^{2d})$ to $\mathcal{L}(\mathcal{A}_{\flat_1}(\mathbf{C}^d), A(\mathbf{C}^d))$.*

Let $L_A(\mathbf{C}^{2d})$ be the set of all $a \in L^1_{\mathrm{loc}}(\mathbf{C}^{2d})$ such that $z \mapsto a(z,w)$ is entire for almost every $w \in \mathbf{C}^d$ and

$$w \mapsto \sup_{\alpha \in \mathbf{N}^d} \left| \frac{\partial_z^\alpha a(z,w) \cdot e^{r|w|-|w|^2}}{h^{|\alpha|}\alpha!} \right| \in L^1(\mathbf{C}^d) \tag{1.37}$$

for every $h, r > 0$ and $z \in \mathbf{C}^d$. If $a \in \widehat{A}(\mathbf{C}^{2d})$ satisfies (1.35) then $a \in L_A(\mathbf{C}^{2d})$ as a consequence of Cauchy's integral formula. Thus $L_A(\mathbf{C}^{2d})$ is a relaxation of the former condition.

If $a \in L_A(\mathbf{C}^{2d})$ then $\operatorname{Op}_{\mathfrak{V}}(a) : \mathcal{A}_{\flat_1}(\mathbf{C}^d) \to \mathcal{A}'_{\flat_1}(\mathbf{C}^d) = A(\mathbf{C}^d)$ is continuous. Hence the following result is a straight-forward consequence of Proposition 1.4 and the fact that $\widehat{\mathcal{A}}'_{\flat_1}(\mathbf{C}^{2d}) = \widehat{A}(\mathbf{C}^{2d})$.

Proposition 1.5. *Let $a \in L_A(\mathbf{C}^{2d})$. Then there is a unique $a_0 \in \widehat{A}(\mathbf{C}^{2d})$ such that $\operatorname{Op}_{\mathfrak{V}}(a) = \operatorname{Op}_{\mathfrak{V}}(a_0)$ as mappings from $\mathcal{A}_{\flat_1}(\mathbf{C}^d)$ to $\mathcal{A}'_{\flat_1}(\mathbf{C}^d)$. It holds*

$$\operatorname{Op}_{\mathfrak{V}}(a) = \operatorname{Op}_{\mathfrak{V}}(a_0)$$

$$\textit{where} \quad a_0(z,w) = \pi^{-d} \int_{\mathbf{C}^d} a(z,w_1) e^{-(z-w_1, w-w_1)} \, d\lambda(w_1). \tag{1.38}$$

Proof. The operator Π_A defined in (1.22) is the orthogonal projection from $L^2(d\mu)$ to $A^2(\mathbf{C}^d)$ which is uniquely extendable to a continuous map from

$$L_{0,A}(\mathbf{C}^d) \equiv \{\, a_0 \in L^1_{\mathrm{loc}}(\mathbf{C}^d) \, ; \, w \mapsto a_0(w) e^{r|w|-|w|^2} \in L^1(\mathbf{C}^d) \text{ for every } r > 0 \,\} \tag{1.39}$$

to $A(\mathbf{C}^d)$ (see e. g. [31]). Hence, if $F, G \in \mathcal{A}_{\flat_1}(\mathbf{C}^d)$ and a_0 is given by (1.38) then

$$\begin{aligned}
(\operatorname{Op}_{\mathfrak{V}}(a)F, G)_{A^2} &= ((\operatorname{Op}_{\mathfrak{V}}(a) \circ \Pi_A)F, G)_{A^2} \\
&= \left(\int_{\mathbf{C}^d} \left(\int_{\mathbf{C}^d} a(\,\cdot\,, w_1) e^{(\,\cdot\,, w_1)} e^{(w_1, w)} \, d\mu(w_1) \right) F(w) \, d\mu(w), G \right)_{A^2} \\
&= \left(\int_{\mathbf{C}^d} a_0(\,\cdot\,, w) e^{(\,\cdot\,, w)} F(w) \, d\mu(w), G \right)_{A^2} = (\operatorname{Op}_{\mathfrak{V}}(a_0)F, G)_{A^2},
\end{aligned}$$

and thus $\operatorname{Op}_{\mathfrak{V}}(a) = \operatorname{Op}_{\mathfrak{V}}(a_0)$ follows. The assertion now follows from Proposition 1.4 and the fact that a_0 in the integral formula of (1.38) defines an element in $\widehat{A}(\mathbf{C}^{2d})$. $\square$

We will also consider *anti-Wick operators* [6, 7, 13] defined by

$$\operatorname{Op}^{\mathrm{aw}}_{\mathfrak{V}}(a_0)F(z) = \int_{\mathbf{C}^d} a_0(w) F(w) e^{(z,w)} \, d\mu(w), \quad z \in \mathbf{C}^d, \tag{1.40}$$

when $a_0 \in L_{0,A}(\mathbf{C}^d)$ and F belongs to $\mathcal{A}_0(\mathbf{C}^d)$, the space of analytic polynomials on $\mathbf{C}^d$. Then $a_0 \in L_{0,A}(\mathbf{C}^d)$ if and only if $a(z,w) \equiv a_0(w)$ belongs to $L_A(\mathbf{C}^{2d})$, and then $\operatorname{Op}^{\mathrm{aw}}_{\mathfrak{V}}(a_0) = \operatorname{Op}_{\mathfrak{V}}(a)$. Consequently, all results for Wick operators with symbols in $L_A(\mathbf{C}^{2d})$ hold for anti-Wick operators. In particular, if $a_0 \in L_{0,A}(\mathbf{C}^d)$, then $\operatorname{Op}^{\mathrm{aw}}_{\mathfrak{V}}(a_0) : \mathcal{A}_{\flat_1}(\mathbf{C}^d) \to A(\mathbf{C}^d)$ is continuous. We denote the Wick symbol of the anti-Wick operator $\operatorname{Op}^{\mathrm{aw}}_{\mathfrak{V}}(a_0)$ by a_0^{aw}. Then (1.38) takes the form

$$\operatorname{Op}^{\mathrm{aw}}_{\mathfrak{V}}(a_0) = \operatorname{Op}_{\mathfrak{V}}(a_0^{\mathrm{aw}})$$

$$\text{where} \quad a_0^{\mathrm{aw}}(z,w) = \pi^{-d} \int_{\mathbf{C}^d} a_0(w_1) e^{-(z-w_1, w-w_1)}\, d\lambda(w_1). \qquad (1.38)'$$

Pseudo-differential operators on $\mathbf{R}^d$ may be transferred to Wick operators on $\mathbf{C}^d$ by means of the Bargmann transform.

Definition 1.6. Let $\mathfrak{a} \in \mathcal{S}_{1/2}'(\mathbf{R}^{2d})$.

(1) the *Bargmann assignment* $\mathsf{S}_{\mathfrak{V}}\mathfrak{a}$ of $\mathfrak{a}$ is the unique element $a \in \widehat{A}(\mathbf{C}^{2d})$ which fulfills

$$\operatorname{Op}_{\mathfrak{V}}(a) = \mathfrak{V}_d \circ \operatorname{Op}^w(\mathfrak{a}) \circ \mathfrak{V}_d^* \quad \Leftrightarrow \quad a = \mathsf{S}_{\mathfrak{V}}\mathfrak{a}; \qquad (1.41)$$

(2) the *Bargmann kernel assignment* $K_{\mathfrak{V},\mathfrak{a}}$ of $\mathfrak{a}$ is the unique element $K \in \widehat{A}(\mathbf{C}^{2d})$, which is the kernel of the map $\mathfrak{V}_d \circ \operatorname{Op}^w(\mathfrak{a}) \circ \mathfrak{V}_d^*$ with respect to the sesquilinear A^2 form.

By the definitions we have

$$K_{\mathfrak{V},\mathfrak{a}}(z,w) = e^{(z,w)} \mathsf{S}_{\mathfrak{V}}\mathfrak{a}(z,w). \qquad (1.42)$$

Example 1.7. The creation and annihilation operators

$$2^{-\frac{1}{2}}(x_j - \partial_{x_j}) \quad \text{and} \quad 2^{-\frac{1}{2}}(x_j + \partial_{x_j}),$$

are transfered to the operators

$$F \mapsto z_j F \quad \text{and} \quad F \mapsto \partial_{z_j} F, \qquad (1.43)$$

by the Bargmann transform (see [4]). The Wick symbols of the operators in (1.43) are z_j and $\overline{w}_j$, respectively [6, 31]. By combining these identities with the fact that the Weyl symbol of $i^{-1}\partial_{x_j}$ equals ξ_j we get

$$\begin{aligned} &\mathsf{S}_{\mathfrak{V}}(2^{-\frac{1}{2}}(x_j - i\xi_j)) = z_j, \qquad \mathsf{S}_{\mathfrak{V}}(2^{-\frac{1}{2}}(x_j + i\xi_j)) = \overline{w}_j, \\ &\mathsf{S}_{\mathfrak{V}}(x_j) = 2^{-\frac{1}{2}}(z_j + \overline{w}_j) \qquad \text{and} \qquad \mathsf{S}_{\mathfrak{V}}(\xi_j) = 2^{-\frac{1}{2}} i(z_j - \overline{w}_j). \end{aligned} \qquad (1.44)$$

We need to compare $K_{\mathfrak{a}}^w$ and $K_{\mathfrak{V},\mathfrak{a}}$. On the one hand we have for $f, g \in \mathscr{S}(\mathbf{R}^d)$

$$(\operatorname{Op}^w(\mathfrak{a})f, g)_{L^2(\mathbf{R}^d)} = (K_{\mathfrak{a}}^w, g \otimes \overline{f})_{L^2(\mathbf{R}^{2d})} = (\mathfrak{V}_{2d} K_{\mathfrak{a}}^w, \mathfrak{V}_{2d}(g \otimes \overline{f})_{A^2(\mathbf{C}^{2d})}$$

and on the other hand

$$\begin{aligned} (\operatorname{Op}^w(\mathfrak{a})f, g)_{L^2(\mathbf{R}^d)} &= (\operatorname{Op}_{\mathfrak{V}}(a)\mathfrak{V}_d f, \mathfrak{V}_d g)_{A^2(\mathbf{C}^d)} \\ &= (K_{\mathfrak{V},\mathfrak{a}}, \mathfrak{V}_d g \otimes \overline{\mathfrak{V}_d f})_{A^2(\mathbf{C}^{2d})} \\ &= (\Theta K_{\mathfrak{V},\mathfrak{a}}, \Theta(\mathfrak{V}_d g \otimes \overline{\mathfrak{V}_d f}))_{A^2(\mathbf{C}^{2d})}. \end{aligned}$$

Since

$$\Theta(\mathfrak{V}_d g \otimes \overline{\mathfrak{V}_d f})(z,w) = \mathfrak{V}_d g(z) \overline{\mathfrak{V}_d f(\overline{w})} = \mathfrak{V}_{2d}(g \otimes \overline{f})(z,w)$$

we obtain

$$K_{\mathfrak{V},\mathfrak{a}} = \Theta \mathfrak{V}_{2d} K_{\mathfrak{a}}^w = \mathfrak{V}_{\Theta,d} K_{\mathfrak{a}}^w. \qquad (1.45)$$

1.5. **Symbol classes for pseudo-differential operators on $\mathbf{R}^d$.** In order to define a generalized family of Shubin symbol classes [27], we need to add a restriction of the involved weights. Let $\rho \in [0,1]$, and let $\mathscr{P}_{\mathrm{Sh},\rho}(\mathbf{R}^d)$ be the set of all $\omega \in \mathscr{P}(\mathbf{R}^d) \cap C^\infty(\mathbf{R}^d)$ such that for every multi-index $\alpha \in \mathbf{N}^d$,

$$|\partial^\alpha \omega(x)| \lesssim \omega(x)\langle x\rangle^{-\rho|\alpha|}, \quad x \in \mathbf{R}^d. \tag{1.46}$$

For $\omega \in \mathscr{P}_{\mathrm{Sh},\rho}(\mathbf{R}^d)$ the Shubin symbol class $\mathrm{Sh}_\rho^{(\omega)}(\mathbf{R}^d)$ is the set of all $f \in C^\infty(\mathbf{R}^d)$ such that for every $\alpha \in \mathbf{N}^{2d}$,

$$|\partial^\alpha f(x)| \lesssim \omega(x)\langle x\rangle^{-\rho|\alpha|}, \qquad x \in \mathbf{R}^d. \tag{1.47}$$

Let $\rho \in [0,1]$, $\omega \in \mathscr{P}_{\mathrm{Sh},\rho}(\mathbf{R}^{2d})$ and A be a real $d\times d$ matrix. Then it follows from [27] or [19, Section 18.5] that $e^{i\langle AD_\xi,D_x\rangle}$ is a homeomorphism on $\mathrm{Sh}_\rho^{(\omega)}(\mathbf{R}^{2d})$, which implies that the set

$$\{\, \mathrm{Op}_A(\mathfrak{a})\,;\, \mathfrak{a} \in \mathrm{Sh}_\rho^{(\omega)}(\mathbf{R}^{2d}) \,\}$$

is independent of the choice of A, in view of (1.34). If B is another real $d\times d$ matrix and $\mathfrak{a},\mathfrak{b} \in \mathrm{Sh}_\rho^{(\omega)}(\mathbf{R}^{2d})$ satisfy (1.34), then it follows from [19, Section 18.5] that

$$\mathfrak{a} - \mathfrak{b} \in \mathrm{Sh}_\rho^{(\omega_\rho)}(\mathbf{R}^{2d}), \quad \text{where} \quad \omega_\rho(x,\xi) = \omega(x,\xi)\langle (x,\xi)\rangle^{-2\rho}. \tag{1.48}$$

In particular

$$|\mathfrak{a}(x,\xi) - \mathfrak{b}(x,\xi)| \lesssim \omega(x,\xi)\langle (x,\xi)\rangle^{-2\rho}. \tag{1.49}$$

We also need the symbol classes defined in [1, Definition 1.8] with symbols satisfying estimates of the form

$$|\partial_x^\alpha \partial_\xi^\beta \mathfrak{a}(x,\xi)| \lesssim h^{|\alpha+\beta|}\alpha!^\sigma \beta!^s e^{r(|x|^{\frac{1}{s}}+|\xi|^{\frac{1}{\sigma}})}, \qquad x,\xi \in \mathbf{R}^d. \tag{1.50}$$

(See also [10] for the restricted case when $s=\sigma$.)

Definition 1.8. Let $s,\sigma > 0$. Then

(1) $\Gamma_{s,\sigma}^{\sigma,s;0}(\mathbf{R}^{2d})$ consists of all $\mathfrak{a} \in C^\infty(\mathbf{R}^d)$ such that for some $r > 0$, (1.50) holds for every $h > 0$;

(2) $\Gamma_{s,\sigma;0}^{\sigma,s}(\mathbf{R}^{2d})$ consists of all $\mathfrak{a} \in C^\infty(\mathbf{R}^d)$ such that for some $h > 0$, (1.50) holds for every $r > 0$;

(3) $\Gamma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$ consists of all $\mathfrak{a} \in C^\infty(\mathbf{R}^d)$ such that (1.50) holds for some $h > 0$ and some $r > 0$.

Remark 1.9. The symbol classes $\mathrm{Sh}_\rho^{(\omega)}(\mathbf{R}^{2d})$ have isotropic behaviour with respect to phase space $T^*\mathbf{R}^d \simeq \mathbf{R}^{2d}$, and the same holds for the symbol classes in Definition 1.8 when $\sigma = s$. See also [10] for the restricted case when $s=\sigma$, and [2] for a bilinear extension. Important classes similar to those given by Definition 1.8 are considered in [24].

Pseudo-differential operators with symbols in the classes in Definition 1.8 are examples of so called operators of infinite order. These operators are continuous on appropriate Gelfand-Shilov (distribution) spaces [1, 10]. The next result characterizes the symbol classes in Definition 1.8 by means of estimates of form

$$|\mathcal{T}_\psi \mathfrak{a}(x,\xi,\eta,y)| \lesssim e^{r_1(|x|^{\frac{1}{s}}+|\xi|^{\frac{1}{\sigma}})-r_2(|\eta|^{\frac{1}{\sigma}}+|y|^{\frac{1}{s}})}, \quad x,\xi,y,\eta \in \mathbf{R}^d. \tag{1.51}$$

We omit the proof since the result is a special case of [1, Proposition 2.1′]. We refer to [1, Subsection 1.1] for the definition of the Gelfand-Shilov spaces $\mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$, $\Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$ and their distribution spaces.

Proposition 1.10. *Let $s,\sigma > 0$ and let $\mathfrak{a} \in C^\infty(\mathbf{R}^{2d})$. Then the following is true:*

(1) *if $\psi \in \mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})\backslash 0$, then $\mathfrak{a} \in \Gamma_{s,\sigma;0}^{\sigma,s}(\mathbf{R}^{2d})$ if and only if* (1.51) *holds for some $r_2 > 0$ and every $r_1 > 0$;*

(2) *if $\psi \in \Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})\backslash 0$, then $\mathfrak{a} \in \Gamma_{s,\sigma}^{\sigma,s;0}(\mathbf{R}^{2d})$ if and only if* (1.51) *holds for some $r_1 > 0$ and all $r_2 > 0$;*

(3) *if $\psi \in \Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})\backslash 0$, then $\mathfrak{a} \in \Gamma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$ if and only if* (1.51) *holds for some $r_1 > 0$ and some $r_2 > 0$.*

1.6. Elliptic, weakly elliptic and hypoelliptic elements in $\mathrm{Sh}_\rho^{(\omega)}(\mathbf{R}^d)$.

Let $\rho \geqslant 0$ and $\omega \in \mathscr{P}_{\mathrm{Sh},\rho}(\mathbf{R}^d)$. Then $f \in \mathrm{Sh}_\rho^{(\omega)}(\mathbf{R}^d)$ is called *weakly elliptic* of order $\rho_0 \geqslant 0$, (in $\mathrm{Sh}_\rho^{(\omega)}(\mathbf{R}^d)$), or ρ_0*-weakly elliptic*, if there is an $R > 0$ such that

$$|f(x)| \gtrsim \langle x\rangle^{-\rho_0}\omega(x), \qquad |x| \geqslant R.$$

A weakly elliptic function of order 0 is called *elliptic*.

Let A and B be real $d \times d$ matrices, $\rho > 0$, $\rho_0 \in [0,2\rho)$, $\omega \in \mathscr{P}_{\mathrm{Sh},\rho}(\mathbf{R}^{2d})$ and suppose that $\mathfrak{a},\mathfrak{b} \in \mathrm{Sh}_\rho^{(\omega)}(\mathbf{R}^{2d})$ satisfy (1.34). It follows from (1.48) that $\mathfrak{a}$ is weakly elliptic of order ρ_0, if and only if $\mathfrak{b}$ is weakly elliptic of order ρ_0. In particular, $\mathfrak{a}$ is elliptic, if and only if $\mathfrak{b}$ is elliptic.

Next we define Shubin hypoelliptic symbols (cf. Definitions 5.1 and 25.1 in [27]).

Definition 1.11. Let $\rho > 0$, $\rho_0 \geqslant 0$, $\omega_0 \in \mathscr{P}_{\mathrm{Sh},\rho}(\mathbf{R}^d)$ and $f \in \mathrm{Sh}_\rho^{(\omega_0)}(\mathbf{R}^d)$. Then f is called *hypoelliptic* (in *Shubin's sense* in $\mathrm{Sh}_\rho^{(\omega_0)}(\mathbf{R}^d)$) of order ρ_0, if there is an $R > 0$ such that for every $\alpha \in \mathbf{N}^d$, it holds

$$|\partial^\alpha f(x)| \lesssim |f(x)|\langle x\rangle^{-\rho|\alpha|}, \qquad |x| \geqslant R,$$

and

$$|f(x)| \gtrsim \omega_0(x)\langle x\rangle^{-\rho_0}, \qquad |x| \geqslant R.$$

Elliptic and hypoelliptic symbols are important since they give rise to parametrices. For ρ, ω as above and $\mathfrak{a} \in \mathrm{Sh}_\rho^{(\omega)}(\mathbf{R}^{2d})$ elliptic, there is an elliptic symbol $\mathfrak{b} \in \mathrm{Sh}_\rho^{(1/\omega)}(\mathbf{R}^{2d})$ such that

$$\mathrm{Op}_A(\mathfrak{a}) \circ \mathrm{Op}_A(\mathfrak{b}) = I + \mathrm{Op}_A(\mathfrak{c}_1) \quad \text{and} \quad \mathrm{Op}_A(\mathfrak{b}) \circ \mathrm{Op}_A(\mathfrak{a}) = I + \mathrm{Op}_A(\mathfrak{c}_2)$$

for some $\mathfrak{c}_1,\mathfrak{c}_2 \in \mathscr{S}(\mathbf{R}^{2d})$. An operator $\mathrm{Op}(\mathfrak{c})$ with $c \in \mathscr{S}(\mathbf{R}^{2d})$ is regularizing in the sense that $\mathrm{Op}(\mathfrak{c})$ is continuous from $\mathscr{S}'(\mathbf{R}^d)$ to $\mathscr{S}(\mathbf{R}^d)$. (Cf. e.g. [8,27].)

2. Reformulation of pseudo-differential calculus using the Bargmann transform

In this section we characterize the Bargmann assignment of pseudo-differential operator symbols from Subsection 1.5, using estimates of complex derivatives. In Subsection 2.1 we show how pseudo-differential operators on $\mathbf{R}^d$ with Shubin symbols are transformed to Wick operators by the Bargmann transform. In Subsection 2.3 we deduce similar links between pseudo-differential operators of infinite order, given in the second part of Subsection 1.5, and suitable classes of Wick operators. Subsection 2.4 treats composition formulae for symbols of Wick operators, which leads to algebraic properties for operators in Subsection 2.1 and 2.3. As an application we obtain short proofs of composition results for pseudo-differential operators on $\mathbf{R}^d$ from Subsection 1.5.

2.1. Wick symbols of Shubin pseudo-differential operators.

The following proposition is essential in the characterization of Shubin type pseudo-differential operators on $\mathbf{R}^d$ by means of the corresponding Wick symbols. The Shubin classes can be characterized using the transform $\mathcal{T}_\phi$ by means of estimates of the form

$$|\partial_x^\alpha \partial_\xi^\beta \mathcal{T}_\phi f(x,\xi)| \lesssim \omega(x)\langle x\rangle^{-\rho|\alpha|}\langle \xi\rangle^{-N}, \tag{2.1}$$

$$|\partial_x^\alpha \mathcal{T}_\phi f(x,\xi)| \lesssim \omega(x)\langle x\rangle^{-\rho|\alpha|}\langle \xi\rangle^{-N} \tag{2.2}$$

and

$$|\mathcal{T}_\phi f(x,\xi)| \lesssim \omega(x)\langle \xi\rangle^{-N}. \tag{2.3}$$

The proof of the following result is similar to the proof of [9, Proposition 3.2].

Proposition 2.1. *Let* $0 \leqslant \rho \leqslant 1$, *let* $\omega \in \mathscr{P}_{\mathrm{Sh},\rho}(\mathbf{R}^d)$, *and suppose* $f \in \mathscr{S}'(\mathbf{R}^d)$ *and* $\phi \in \mathscr{S}(\mathbf{R}^d)\backslash 0$. *The following conditions are equivalent:*

(1) $f \in \mathrm{Sh}_\rho^{(\omega)}(\mathbf{R}^d)$,

(2) (2.1) *holds true for any* $N \geqslant 0$ *and* $\alpha, \beta \in \mathbf{N}^d$,

(3) (2.2) *holds true for any* $N \geqslant 0$ *and* $\alpha \in \mathbf{N}^d$,

and the following conditions are equivalent:

(1)′ $f \in \mathrm{Sh}_0^{(\omega)}(\mathbf{R}^d)$,

(2)′ (2.3) *holds true for any* $N \geqslant 0$.

Proof. First we prove that (1) implies (2). Suppose $f \in \mathrm{Sh}_\rho^{(\omega)}(\mathbf{R}^d)$ and let $\alpha, \beta, \gamma \in \mathbf{N}^d$ be arbitrary. We will show

$$|\xi^\gamma \partial_x^\alpha \partial_\xi^\beta \mathcal{T}_\phi f(x,\xi)| \lesssim \omega(x)\langle x\rangle^{-\rho|\alpha|}.$$

By (1.14), (1.15) and integration by parts we get

$$
\begin{aligned}
|\xi^\gamma \partial_x^\alpha \partial_\xi^\beta \mathcal{T}_\phi f(x,\xi)| &= \left|\xi^\gamma \mathcal{T}_{\phi_\beta}(\partial^\alpha f)(x,\xi)\right| \\
&= (2\pi)^{-\frac{d}{2}} \left| \int_{\mathbf{R}^d} \left((i\partial_y)^\gamma e^{-i\langle \xi, y\rangle} \right) \overline{\phi_\beta(y)}\, \partial^\alpha f(x+y)\, dy \right| \\
&\lesssim \int_{\mathbf{R}^d} \left| \partial_y^\gamma \left[\overline{\phi_\beta(y)}\, \partial^\alpha f(x+y) \right] \right| dy \\
&= \int_{\mathbf{R}^d} \left| \sum_{\kappa \leqslant \gamma} \binom{\gamma}{\kappa} \partial^{\gamma-\kappa} \overline{\phi_\beta(y)}\, \partial^{\alpha+\kappa} f(x+y) \right| dy \\
&\lesssim \sum_{\kappa \leqslant \gamma} \binom{\gamma}{\kappa} \int_{\mathbf{R}^d} \left| \partial^{\gamma-\kappa} \phi_\beta(y) \right| \omega(x+y) \langle x+y\rangle^{-\rho|\alpha+\kappa|}\, dy.
\end{aligned}
$$

Since ω is polynomially moderate, Peetre's inequality (1.2) and the fact that $\phi \in \mathscr{S}$ give

$$
\begin{aligned}
&|\xi^\gamma \partial_x^\alpha \partial_\xi^\beta \mathcal{T}_\phi f(x,\xi)| \\
&\quad \lesssim \omega(x)\langle x\rangle^{-\rho|\alpha|} \sum_{\kappa \leqslant \gamma} \binom{\gamma}{\kappa} \int_{\mathbf{R}^d} \left| \partial^{\gamma-\kappa} \phi_\beta(y) \right| \omega(y)\, \langle y\rangle^{|m|+\rho|\alpha+\kappa|}\, dy \\
&\qquad \asymp \omega(x)\langle x\rangle^{-\rho|\alpha|}.
\end{aligned}
$$

Thus $f \in \mathrm{Sh}_\rho^{(\omega)}(\mathbf{R}^d)$ implies implies (2.1), and as a special case (2.2), and $f \in \mathrm{Sh}_0^{(\omega)}(\mathbf{R}^d)$ implies (2.3). We have proved that (1) implies (2) which in turn implies (3), and that (1)$'$ implies (2)$'$.

Conversely, suppose (3), that is $f \in \mathscr{S}'(\mathbf{R}^d)$ and (2.2) holds for all $N \geqslant 0$ and all $\alpha \in \mathbf{N}^d$, which is a weaker assumption than (2). We obtain from (1.13)

$$
\begin{aligned}
f(x) &= \|\phi\|_{L^2}^{-2}\, \mathcal{T}_\phi^* \mathcal{T}_\phi f(x) \\
&= \|\phi\|_{L^2}^{-2}\, (2\pi)^{-\frac{d}{2}} \iint_{\mathbf{R}^{2d}} \mathcal{T}_\phi f(y,\xi)\, e^{i\langle \xi, x-y\rangle}\, \phi(x-y)\, dy d\xi,
\end{aligned}
$$

which is an absolutely convergent integral due to (2.2) and the fact that $\phi \in \mathscr{S}(\mathbf{R}^d)$. We may differentiate under the integral, so integration by parts, (2.2) and Peetre's inequality give for some $N_0 \geqslant 0$, any $\alpha \in \mathbf{N}^d$ and any $x \in \mathbf{R}^d$

$$
\begin{aligned}
|\partial^\alpha f(x)| &= \|\phi\|_{L^2}^{-2} (2\pi)^{-\frac{d}{2}} \left| \iint_{\mathbf{R}^{2d}} \mathcal{T}_\phi f(y,\xi)\, \partial_y^\alpha \left(e^{i\langle \xi, x-y\rangle} \phi(x-y) \right) dy d\xi \right| \\
&= \|\phi\|_{L^2}^{-2} (2\pi)^{-\frac{d}{2}} \left| \iint_{\mathbf{R}^{2d}} \partial_y^\alpha \mathcal{T}_\phi f(y,\xi)\, e^{i\langle \xi, x-y\rangle} \phi(x-y)\, dy d\xi \right| \\
&= \|\phi\|_{L^2}^{-2} (2\pi)^{-\frac{d}{2}} \left| \iint_{\mathbf{R}^{2d}} \partial_y^\alpha \mathcal{T}_\phi f(x-y,\xi)\, e^{i\langle \xi, y\rangle} \phi(y)\, dy d\xi \right| \\
&\lesssim \iint_{\mathbf{R}^{2d}} \omega(x-y) \langle x-y\rangle^{-\rho|\alpha|} \langle \xi \rangle^{-d-1} |\phi(y)|\, dy d\xi \\
&\lesssim \omega(x)\langle x\rangle^{-\rho|\alpha|} \iint_{\mathbf{R}^{2d}} \langle \xi \rangle^{-d-1} \langle y \rangle^{N_0+\rho|\alpha|} |\phi(y)|\, dy d\xi \\
&\asymp \omega(x)\langle x\rangle^{-\rho|\alpha|}.
\end{aligned}
$$

Thus $f \in \mathrm{Sh}_\rho^{(\omega)}(\mathbf{R}^d)$ and we have proved the equivalence of (1), (2) and (3).

It remains to show that $(2)'$ implies $(1)'$, that is (2.3) for all $N \geqslant 0$ implies $f \in \mathrm{Sh}_0^{(\omega)}(\mathbf{R}^d)$. We have for some $N_0 \geqslant 0$, any $\alpha \in \mathbf{N}^d$, $x \in \mathbf{R}^d$ and $N \geqslant 0$,

$$
\begin{aligned}
|\partial^\alpha f(x)| &= \|\phi\|_{L^2}^{-2} (2\pi)^{-\frac{d}{2}} \left| \iint_{\mathbf{R}^{2d}} \mathcal{T}_\phi f(y,\xi)\, \partial_x^\alpha \left(e^{i\langle \xi, x-y\rangle} \phi(x-y) \right) dy d\xi \right| \\
&\lesssim \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \iint_{\mathbf{R}^{2d}} |\mathcal{T}_\phi f(y,\xi)| \langle \xi \rangle^{|\beta|} \left| \partial^{\alpha-\beta} \phi(x-y) \right| dy d\xi \\
&\lesssim \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \iint_{\mathbf{R}^{2d}} \omega(y) \langle \xi \rangle^{|\alpha|-N} \left| \partial^{\alpha-\beta} \phi(x-y) \right| dy d\xi \\
&\lesssim \omega(x) \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \iint_{\mathbf{R}^{2d}} \langle \xi \rangle^{|\alpha|-N} \langle x-y \rangle^{N_0} \left| \partial^{\alpha-\beta} \phi(x-y) \right| dy d\xi \\
&\lesssim \omega(x)
\end{aligned}
$$

provided N is sufficiently large, since $\phi \in \mathscr{S}$. This shows that $f \in \mathrm{Sh}_0^{(\omega)}(\mathbf{R}^d)$. □

We may now characterize the Shubin classes $\mathrm{Sh}_\rho^{(\omega)}(\mathbf{R}^{2d})$ by estimates on their Bargmann (kernel) assignments of the forms

$$\begin{aligned}\big|(\partial_z+\overline{\partial}_w)^\alpha(\partial_z-\overline{\partial}_w)^\beta \mathsf{S}_{\mathfrak{V}}\mathfrak{a}(z,w)\big|&\\ \lesssim e^{\frac12|z-w|^2}\omega(\sqrt2\,\overline{z})\langle z+w\rangle^{-\rho|\alpha+\beta|}\langle z-w\rangle^{-N},&\qquad (2.4)\end{aligned}$$

$$\left|\partial_z^\alpha\overline{\partial}_w^\beta \mathsf{S}_{\mathfrak{V}}\mathfrak{a}(z,w)\right| \lesssim e^{\frac12|z-w|^2}\omega(\sqrt2\,\overline{z})\langle z+w\rangle^{-\rho|\alpha+\beta|}\langle z-w\rangle^{-N}, \qquad (2.5)$$

$$|\mathsf{S}_{\mathfrak{V}}\mathfrak{a}(z,w)| \lesssim e^{\frac12|z-w|^2}\omega(\sqrt2\,\overline{z})\langle z-w\rangle^{-N} \qquad (2.6)$$

and

$$|K_{\mathfrak{V},\mathfrak{a}}(z,w)| \lesssim \omega(\sqrt2\,\overline{z})\langle z-w\rangle^{-N}e^{\frac12(|z|^2+|w|^2)}. \qquad (2.7)$$

Theorem 2.2. *Let $0\leqslant\rho\leqslant1$, $\omega\in\mathscr{P}_{\mathrm{Sh},\rho}(\mathbf{R}^{2d})$ and $\mathfrak{a}\in\mathscr{S}'(\mathbf{R}^{2d})$. The following conditions are equivalent:*

(1) $\mathfrak{a}\in\mathrm{Sh}_\rho^{(\omega)}(\mathbf{R}^{2d})$,

(2) (2.4) *holds true for every* $N\geqslant0$, $z,w\in\mathbf{C}^d$ *and* $\alpha,\beta\in\mathbf{N}^d$,

(3) (2.5) *holds true for every* $N\geqslant0$, $z,w\in\mathbf{C}^d$ *and* $\alpha,\beta\in\mathbf{N}^d$,

and the following conditions are equivalent:

$(1)'$ $\mathfrak{a}\in\mathrm{Sh}_0^{(\omega)}(\mathbf{R}^{2d})$,

$(2)'$ (2.6) *holds true for any* $N\in\mathbf{N}$ *and* $z,w\in\mathbf{C}^d$,

$(3)'$ (2.7) *holds true for any* $N\in\mathbf{N}$ *and* $z,w\in\mathbf{C}^d$.

For the proof we need the following proposition of independent interest. Here we recall that $\mathsf{S}_{\mathfrak{V}}$ is bijective from $\mathcal{S}_{1/2}'(\mathbf{R}^{2d})$ to the set

$$\{\,a\in\widehat{A}(\mathbf{C}^{2d})\,;\,|a(z,w)|\lesssim e^{(\frac12+r)|z-w|^2}\text{ for every } r>0\,\}. \qquad (2.8)$$

Proposition 2.3. *Let $\psi(x,\xi)=(\frac{2}{\pi})^{\frac d2}e^{-(|x|^2+|\xi|^2)}$, $x,\xi\in\mathbf{R}^d$, $\mathfrak{a}\in\mathcal{S}_{1/2}'(\mathbf{R}^{2d})$ and a belongs to the set in* (2.8). *Then*

$$\mathsf{S}_{\mathfrak{V}}\mathfrak{a}(z,w)=(2\pi)^{\frac d2}e^{\frac12|z-w|^2}\mathcal{T}_\psi\mathfrak{a}\left(\frac{x+y}{\sqrt2},-\frac{\xi+\eta}{\sqrt2},\sqrt2(\eta-\xi),\sqrt2(y-x)\right), \qquad (2.9)$$

and

$$(\mathsf{S}_{\mathfrak{V}}^{-1}a)(x,-\xi)=\left(\frac{2}{\pi}\right)^d\int_{\mathbf{C}^d}a\left(\frac{z}{\sqrt2}-w,\frac{z}{\sqrt2}+w\right)e^{-2|w|^2}\,d\lambda(w), \qquad (2.10)$$

with $z=x+i\xi$, $w=y+i\eta$ and $x,y,\xi,\eta\in\mathbf{R}^d$.

Proof. Let $\phi(x,y)=\pi^{-\frac d2}e^{-\frac12(|x|^2+|y|^2)}$ for $x,y\in\mathbf{R}^d$, and let $K_{\mathfrak{a}}^w$ be the kernel of $\mathrm{Op}^w(\mathfrak{a})$. Then $\psi=\mathscr{F}_2(\phi\circ\kappa)$, where $\kappa(x,y)=(x+y/2,x-y/2)$. By

(1.27) (or [30, Eq. (1.35)]) and [9, Lemma 4.1] we have

$$\mathfrak{V}_{\Theta,d}K_{\mathfrak{a}}^w(z,w) = \mathfrak{V}_{2d}K_{\mathfrak{a}}^w(z,\overline{w}) = \mathfrak{V}_{2d}K_{\mathfrak{a}}^w((x,y)+i(\xi,-\eta))$$

$$= (2\pi)^d e^{\frac{1}{2}\left(|z|^2+|w|^2\right)+i(\langle x,\xi\rangle - \langle y,\eta\rangle)}\mathcal{T}_\phi K_{\mathfrak{a}}^w\left(\sqrt{2}(x,y), -\sqrt{2}(\xi,-\eta)\right)$$

$$= (2\pi)^{\frac{d}{2}} e^{\frac{1}{2}\left(|z|^2+|w|^2\right)+i(\langle y,\xi\rangle - \langle x,\eta\rangle)}\mathcal{T}_\psi \mathfrak{a}\left(\frac{x+y}{\sqrt{2}}, -\frac{\xi+\eta}{\sqrt{2}}, \sqrt{2}(\eta-\xi), \sqrt{2}(y-x)\right),$$

$$= (2\pi)^{\frac{d}{2}} e^{\frac{1}{2}\left(|z|^2+|w|^2\right)+i\,\mathrm{Im}(z,w)}\mathcal{T}_\psi \mathfrak{a}\left(\frac{x+y}{\sqrt{2}}, -\frac{\xi+\eta}{\sqrt{2}}, \sqrt{2}(\eta-\xi), \sqrt{2}(y-x)\right).$$

Together with the identity

$$|z|^2 + |w|^2 + 2i\,\mathrm{Im}(z,w) = |z-w|^2 + 2(z,w)$$

this gives

$$\mathfrak{V}_{\Theta,d}K_{\mathfrak{a}}^w(z,w)$$
$$= (2\pi)^{\frac{d}{2}} e^{\frac{1}{2}|z-w|^2+(z,w)}\mathcal{T}_\psi \mathfrak{a}\left(\frac{x+y}{\sqrt{2}}, -\frac{\xi+\eta}{\sqrt{2}}, \sqrt{2}(\eta-\xi), \sqrt{2}(y-x)\right). \quad (2.11)$$

A combination of this identity with (1.42) and (1.45) gives (2.9).

In order to prove (2.10), we use Moyal's formula (1.12), (1.13) and the fact that $\|\psi\|_{L^2} = 1$. This implies that the inverse of $\mathcal{T}_\psi$ is given by

$$(\mathcal{T}_\psi^{-1}F)(x,\xi) = (\mathcal{T}_\psi^* F)(x,\xi)$$
$$= (2\pi)^{-d} \iiiint_{\mathbf{R}^{4d}} F(x_1,\xi_1,\eta_1,y_1)\psi(x-x_1,\xi-\xi_1)e^{i(\langle x-x_1,\eta_1\rangle + \langle y_1,\xi-\xi_1\rangle}\,dx_1d\xi_1d\eta_1dy_1.$$

Writing

$$G(z,w) = F(x,\xi,\eta,y), \qquad z = x+i\xi,\ w = y+i\eta,$$

we obtain

$$\mathcal{T}_\psi^* F(x,\xi) = 2^d(2\pi)^{-\frac{3d}{2}} \iint\limits_{\mathbf{C}^{2d}} G(w_1,w_2)e^{-|z-w_1|^2}e^{i\mathrm{Im}\langle z-w_1,w_2\rangle}\,d\lambda(w_1)d\lambda(w_2). \quad (2.12)$$

If $\mathfrak{a} = \mathcal{T}_\psi^* F$ and $a = \mathsf{S}_{\mathfrak{V}}\mathfrak{a}$, then (2.9) shows that

$$a(z,w) = (2\pi)^{\frac{d}{2}} e^{\frac{1}{2}|z-w|^2} G\left(\frac{\overline{z}+\overline{w}}{\sqrt{2}}, \sqrt{2}(w-z)\right)$$

which gives

$$G(z,w) = (2\pi)^{-\frac{d}{2}} e^{-\frac{1}{4}|w|^2} a\left(\frac{2\overline{z}-w}{2\sqrt{2}}, \frac{2\overline{z}+w}{2\sqrt{2}}\right).$$

Inserting this into (2.12) we get

$$\mathcal{T}_\psi^* F(x,-\xi)$$
$$= \frac{1}{2^d\pi^{2d}} \iint\limits_{\mathbf{C}^{2d}} a\left(\frac{2\overline{w}_1-w_2}{2\sqrt{2}}, \frac{2\overline{w}_1+w_2}{2\sqrt{2}}\right) e^{-|\overline{z}-w_1|^2}e^{-\frac{1}{4}|w_2|^2}e^{i\mathrm{Im}\langle \overline{z}-w_1,w_2\rangle}\,d\lambda(w_1)d\lambda(w_2),$$

and by taking

$$\frac{2\overline{w}_1 - w_2}{2\sqrt{2}} - \frac{z}{\sqrt{2}} \quad \text{and} \quad \frac{2\overline{w}_1 + w_2}{2\sqrt{2}} - \frac{z}{\sqrt{2}}$$

as new variables of integration, we obtain using (1.22)

$$\begin{aligned}
&\mathcal{T}_\psi^* F(x, -\xi) \\
&= \frac{2^d}{\pi^{2d}} \iint\limits_{\mathbf{C}^{2d}} a\left(w_1 + \frac{z}{\sqrt{2}}, w_2 + \frac{z}{\sqrt{2}}\right) e^{-(|w_1|^2+|w_2|^2)} e^{2i\mathrm{Im}(w_1,w_2)} \, d\lambda(w_1) d\lambda(w_2) \\
&= 2^d \iint\limits_{\mathbf{C}^{2d}} a\left(w_1 + \frac{z}{\sqrt{2}}, w_2 + \frac{z}{\sqrt{2}}\right) e^{2i\mathrm{Im}(w_1,w_2)} \, d\mu(w_1) d\mu(w_2) \\
&= 2^d \int_{\mathbf{C}^d} \left(\int_{\mathbf{C}^d} a\left(w_1 + \frac{z}{\sqrt{2}}, w_2 + \frac{z}{\sqrt{2}}\right) e^{(w_1,w_2)} \, e^{(-w_2,w_1)} \, d\mu(w_1) \right) d\mu(w_2) \\
&= 2^d \int_{\mathbf{C}^d} a\left(-w_2 + \frac{z}{\sqrt{2}}, w_2 + \frac{z}{\sqrt{2}}\right) e^{-|w_2|^2} \, d\mu(w_2) \\
&= \left(\frac{2}{\pi}\right)^d \int_{\mathbf{C}^d} a\left(\frac{z}{\sqrt{2}} - w, \frac{z}{\sqrt{2}} + w\right) e^{-2|w|^2} \, d\lambda(w).
\end{aligned}$$

□

Proof of Theorem 2.2. Combining Propositions 2.1 and 2.3, writing $z + w = 2z + w - z$, we obtain that $\mathfrak{a} \in \mathrm{Sh}_\rho^{(\omega)}(\mathbf{R}^{2d})$ if and only if for all $\alpha, \beta \in \mathbf{N}^d$ and $N \in \mathbf{N}$ we have

$$\begin{aligned}
&\left| (\partial_x + \partial_y)^\alpha (\partial_\xi + \partial_\eta)^\beta \left(e^{-\frac{1}{2}|z-w|^2} \mathsf{S}_{\mathfrak{V}} \mathfrak{a}(z, w) \right) \right| \\
&\qquad \lesssim \omega\left(\frac{\overline{z + w}}{\sqrt{2}} \right) \langle z + w \rangle^{-\rho|\alpha+\beta|} \langle z - w \rangle^{-N} \\
&\qquad\qquad \lesssim \omega(\sqrt{2}\,\overline{z}) \langle z + w \rangle^{-\rho|\alpha+\beta|} \langle z - w \rangle^{-N+k}
\end{aligned}$$

for some $k \in \mathbf{N}$ that can be absorbed into N.

Note that multi-index powers of the differential operators $\partial_x + \partial_y$ and $\partial_\xi + \partial_\eta$ acting on the factor $e^{-\frac{1}{2}|z-w|^2} = e^{-\frac{1}{2}\left(|x-y|^2 + |\xi-\eta|^2\right)}$ are zero. Thus we obtain the equivalent condition

$$\begin{aligned}
&\left| (\partial_x + \partial_y)^\alpha (\partial_\xi + \partial_\eta)^\beta \mathsf{S}_{\mathfrak{V}} \mathfrak{a}(z, w) \right| \\
&\qquad\qquad \lesssim \omega(\sqrt{2}\,\overline{z}) \langle z + w \rangle^{-\rho|\alpha+\beta|} \langle z - w \rangle^{-N} e^{\frac{1}{2}|z-w|^2}.
\end{aligned}$$

Using the (conjugate) analyticity of $\mathsf{S}_{\mathfrak{V}} \mathfrak{a}(z, w)$ with respect to $z \in \mathbf{C}^d$ ($w \in \mathbf{C}^d$) we can formulate this as (2.4). We have now shown the equivalence between (1) and (2).

The equivalence between (2) and (3) follows from the binomial formulae

$$(\partial_z + t\overline{\partial}_w)^\alpha = \sum_{\gamma\leqslant\alpha} \binom{\alpha}{\gamma} t^{|\gamma|}\partial_z^{\alpha-\gamma}\overline{\partial}_w^\gamma, \qquad t \in \{-1,1\},$$

$$\partial_z^\alpha = 2^{-|\alpha|} \sum_{\gamma\leqslant\alpha} \binom{\alpha}{\gamma} (\partial_z + \overline{\partial}_w)^{\alpha-\gamma}(\partial_z - \overline{\partial}_w)^\gamma$$

and

$$\overline{\partial}_w^\beta = 2^{-|\beta|} \sum_{\gamma\leqslant\beta} \binom{\beta}{\gamma} (-1)^{|\gamma|}(\partial_z + \overline{\partial}_w)^{\beta-\gamma}(\partial_z - \overline{\partial}_w)^\gamma .$$

It remains to consider the case $\rho = 0$. We obtain from Propositions 2.1 and 2.3 that $\mathfrak{a} \in \mathrm{Sh}_0^{(\omega)}(\mathbf{R}^{2d})$ if and only if for all $N \in \mathbf{N}$ we have

$$|\mathsf{S}_{\mathfrak{V}}\mathfrak{a}(z,w)| \lesssim \omega(\sqrt{2}\,\overline{z})\langle z-w\rangle^{-N} e^{\frac{1}{2}|z-w|^2}, \quad z,\zeta \in \mathbf{C}^d .$$

This shows the equivalence between $(1)'$ and $(2)'$.

Finally the equivalence of $(2)'$ and $(3)'$ is an immediate consequence of (1.42) and

$$|e^{(|z|^2+|w|^2)/2}e^{-(z,w)}| = e^{(|z|^2-2\,\mathrm{Re}(z,w)+|w|^2)/2} = e^{|z-w|^2/2}. \qquad \square$$

Let $\widehat{\mathcal{A}}_{\mathrm{Sh},\rho}^{(\omega)}(\mathbf{C}^{2d})$, be the set of all $a \in \widehat{A}(\mathbf{C}^{2d})$ such that

$$\left|\partial_z^\alpha\overline{\partial}_w^\beta a(z,w)\right| \leqslant Ce^{\frac{1}{2}|z-w|^2}\omega(\sqrt{2}\,\overline{z})\langle z+w\rangle^{-\rho|\alpha+\beta|}\langle z-w\rangle^{-N}, \quad N \geqslant 0. \tag{2.13}$$

The smallest constant $C \geqslant 0$ defines a semi-norm parameterized by α, β and N, and we equip $\widehat{\mathcal{A}}_{\mathrm{Sh},\rho}^{(\omega)}(\mathbf{C}^{2d})$ with the Fréchet space topology defined by these semi-norms. The following result is an immediate consequence of Theorem 2.2 and its proof.

Proposition 2.4. *Let $0 \leqslant \rho \leqslant 1$ and $\omega \in \mathscr{P}_{\mathrm{Sh},\rho}(\mathbf{R}^{2d})$. Then $\mathsf{S}_{\mathfrak{V}}$ is a homeomorphism from $\mathrm{Sh}_\rho^{(\omega)}(\mathbf{R}^{2d})$ to $\widehat{\mathcal{A}}_{\mathrm{Sh},\rho}^{(\omega)}(\mathbf{C}^{2d})$.*

2.2. **Extensions and variations.** There are several extensions and variations of Theorem 2.2. First we observe that by playing with N in (2.4) and (2.5) and using Peetre's inequality, it follows that $\langle z+w\rangle$ in (2.4) and (2.5) can be replaced by Ψ, where

$$\Psi(z,w) \in \{\langle z+w\rangle, \langle z\rangle, \langle w\rangle, \max(\langle z\rangle,\langle w\rangle), \min(\langle z\rangle,\langle w\rangle)\}\,. \tag{2.14}$$

In particular (2.5) in Theorem 2.2 can be replaced by

$$\left|\partial_z^\alpha\overline{\partial}_w^\beta \mathsf{S}_{\mathfrak{V}}\mathfrak{a}(z,w)\right| \lesssim e^{\frac{1}{2}|z-w|^2}\omega(\sqrt{2}\,\overline{z})\Psi(z,w)^{-\rho|\alpha+\beta|}\langle z-w\rangle^{-N}, \tag{2.5$'$}$$

where Ψ is given by (2.14).

Secondly, let

$$\Omega_{k,M} = \{\,(\alpha_1,\dots,\alpha_k) \in \mathbf{N}^d \times \cdots \times \mathbf{N}^d \simeq \mathbf{N}^{kd}\,;\, |\alpha_1 + \cdots + \alpha_k| = M\,\},.$$

where $k \geqslant 1$ and $M \geqslant 0$ are integers.

If $a \in \widehat{A}(\mathbf{C}^{2d})$ and $\alpha \in \Omega_{4,M}$, then

$$\begin{gathered}\partial_x^{\alpha_1}\partial_\xi^{\alpha_3}\partial_y^{\alpha_2}\partial_\eta^{\alpha_4} a = i^{|\alpha_3|-|\alpha_4|}\partial_z^{\alpha_1+\alpha_3}\overline{\partial}_w^{\alpha_2+\alpha_4} a,\\ z = x+i\xi,\ w = y+i\eta,\end{gathered} \tag{2.15}$$

because of the analyticity with respect to z and conjugate analyticity with respect to w for $a(z,w)$. In particular, $(2.5)'$ implies

$$\begin{aligned}&\left|\partial_x^{\alpha_1}\partial_\xi^{\alpha_3}\partial_y^{\alpha_2}\partial_\eta^{\alpha_4}\mathsf{S}_{\mathfrak{V}}\mathfrak{a}(z,w)\right|\\ &\qquad\lesssim e^{\frac{1}{2}|z-w|^2}\omega(\sqrt{2}\,\overline{z})\Psi(z,w)^{-\rho M}\langle z-w\rangle^{-N}, \quad \alpha\in\Omega_{4,M}. \end{aligned} \tag{2.5$''$}$$

On the other hand, if we let $\alpha_3 = \alpha_4 = 0$ in (2.15), $(2.5)''$ implies $(2.5)'$. Hence $(2.5)'$ and $(2.5)''$ are equivalent.

Let $M \geqslant 0$ be an integer and let T be the operator

$$T = \sum_{\alpha\in\Omega_{2,M}} c(\alpha)\partial_z^{\alpha_1}\overline{\partial}_w^{\alpha_2}, \qquad c(\alpha)\in\mathbf{C},\ \alpha=(\alpha_1,\alpha_2)\in\Omega_{2,M}. \tag{2.16}$$

Then $(2.5)'$ implies that

$$|T(\mathsf{S}_{\mathfrak{V}}\mathfrak{a})(z,w)| \lesssim e^{\frac{1}{2}|z-w|^2}\omega(\sqrt{2}\,\overline{z})\Psi(z,w)^{-\rho M}\langle z-w\rangle^{-N}, \tag{2.5$'''$}$$

holds for every $M \geqslant 0$ and and every operator T of the form (2.16). On the other hand, the operators $\partial_z^\alpha\overline{\partial}_w^\beta$ in $(2.5)'$ are special cases of the operators T in $(2.5)'''$. This shows that $\partial_z^\alpha\overline{\partial}_w^\beta$ in $(2.5)'$ can be replaced by operators T in (2.16).

In the same way it follows that $(2.5)''$ is equivalent to $(2.5)'''$, after T in (2.16) is replaced by

$$\begin{aligned}T &= \sum_{\alpha\in\Omega_{4,M}} C(\alpha)\partial_x^{\alpha_1}\partial_\xi^{\alpha_3}\partial_y^{\alpha_2}\partial_\eta^{\alpha_4}, &&z=x+i\xi,\ w=y+i\eta,\\ C(\alpha)&\in\mathbf{C}, &&\alpha=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\in\Omega_{4,M}.\end{aligned} \tag{2.17}$$

Finally we observe that we may replace the set of operators in (2.16) by the set of operators

$$\begin{aligned}T &= \sum_{\alpha\in\Omega_{4,M}} C(\alpha)\partial_z^{\alpha_1}\overline{\partial}_z^{\alpha_3}\overline{\partial}_w^{\alpha_2}\partial_w^{\alpha_4}, &&z=x+i\xi,\ w=y+i\eta,\\ C(\alpha)&\in\mathbf{C}, &&\alpha=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\in\Omega_{4,M},\end{aligned} \tag{2.18}$$

in the estimate $(2.5)'''$. In fact, obviously the operators of form (2.18) contains the operators of form (2.16). Hence if $(2.5)'''$ holds true for operators of form (2.18), it also holds for operators of form (2.16). On the other hand, if $\alpha_3 \neq 0$ or $\alpha_4 \neq 0$ in (2.18), then

$$\partial_z^{\alpha_1}\overline{\partial}_z^{\alpha_3}\overline{\partial}_w^{\alpha_2}\partial_w^{\alpha_4} a = 0$$

because of the analyticity in z and conjugate analyticity in w for $a(z,w)$. Consequently, it suffices to consider operators in (2.18) where all $C(\alpha) = 0$ when $\alpha_3 \neq 0$ or $\alpha_4 \neq 0$, when investigating the condition $(2.5)'''$. This set of operators is exactly the set of operators in (2.16). This implies that the set of operators in (2.16) can be replaced by the set of operators in (2.18) when checking the condition $(2.5)'''$.

From these observations Theorem 2.2 gives the following conclusion.

Theorem 2.5. *Suppose that* $\rho \geqslant 0$*,* $\omega \in \mathscr{P}_{\mathrm{Sh},\rho}(\mathbf{C}^d)$*,* $a \in \widehat{A}(\mathbf{C}^{2d})$ *and* Ψ *is given by* (2.14)*. Then the following conditions are equivalent:*

(1) $\mathfrak{a} \in \mathrm{Sh}_\rho^{(\omega)}(\mathbf{R}^{2d})$,

(2) $(2.5)'$ *holds true for every* $N \geqslant 0$*,* $\alpha, \beta \in \mathbf{N}^d$ *and* $z, w \in \mathbf{C}^d$*;*

(3) *for every* $M \geqslant 0$*,* $(2.5)''$ *holds true for every* $N \geqslant 0$*, and* $z, w \in \mathbf{C}^d$*;*

(4) *for every* $M \geqslant 0$*,* $(2.5)'''$ *holds true for every* $N \geqslant 0$*,* T *in* (2.16) *and* $z, w \in \mathbf{C}^d$*;*

(5) *for every* $M \geqslant 0$*,* $(2.5)'''$ *holds true for every* $N \geqslant 0$*,* T *in* (2.17) *and* $z, w \in \mathbf{C}^d$*;*

(6) *for every* $M \geqslant 0$*,* $(2.5)'''$ *holds true for every* $N \geqslant 0$*,* T *in* (2.18) *and* $z, w \in \mathbf{C}^d$*.*

2.3. **Wick operators corresponding to Gevrey type pseudo-differential operators.** Using (2.9) and (1.23) we obtain the following theorem expressed with estimates of the form

$$|a(z,w)| \lesssim \exp\left(\frac{1}{2}|z-w|^2 + r_1|z+w|_{s,\sigma} - r_2|z-w|_{s,\sigma}\right) \tag{2.19}$$

(cf. Definition 1.8). The verification is left for the reader.

Theorem 2.6. *The following is true:*

(1) *if* $s,\sigma \geqslant \frac{1}{2}$*, then* $\mathsf{S}_{\mathfrak{V}}$ *is homeomorphic from* $\Gamma^{\sigma,s}_{s,\sigma;0}(\mathbf{R}^{2d})$ *to the set of all* $a \in \widehat{A}(\mathbf{C}^{2d})$ *such that for some* $r_2 > 0$*,* (2.19) *holds for every* $r_1 > 0$*;*

(2) *if* $s,\sigma > \frac{1}{2}$*, then* $\mathsf{S}_{\mathfrak{V}}$ *is homeomorphic from* $\Gamma^{\sigma,s;0}_{s,\sigma}(\mathbf{R}^{2d})$ *to the set of all* $a \in \widehat{A}(\mathbf{C}^{2d})$ *such that for some* $r_1 > 0$*,* (2.19) *holds for every* $r_2 > 0$*;*

(3) *if* $s,\sigma > \frac{1}{2}$*, then* $\mathsf{S}_{\mathfrak{V}}$ *is homeomorphic from* $\Gamma^{\sigma,s}_{s,\sigma}(\mathbf{R}^{2d})$ *to the set of all* $a \in \widehat{A}(\mathbf{C}^{2d})$ *such that* (2.19) *holds for some* $r_1 > 0$ *and some* $r_2 > 0$*.*

Remark 2.7. The restrictions on s and σ in Theorem 2.6 are needed since we must choose ψ in (1.51) as the Gauss function in Proposition 2.3. According to the proof of Theorem 2.2 this is necessary for the use of the formula (1.27) that relates $\mathcal{T}_\phi K^w_{\mathfrak{a}}$ and the Bargmann transform $\mathfrak{V}_{2d}K^w_{\mathfrak{a}}$. For this ψ we have $\psi \in \mathcal{S}^\sigma_s(\mathbf{R}^d)$ ($\psi \in \Sigma^\sigma_s(\mathbf{R}^d)$), if and only if $s,\sigma \geqslant \frac{1}{2}$ ($s,\sigma > \frac{1}{2}$).

Theorem 2.6 can be combined with continuity results in [1] to deduce continuity of Wick operators acting on the Bargmann images of $\Sigma^\sigma_s(\mathbf{R}^d)$, $\mathcal{S}^\sigma_s(\mathbf{R}^d)$, $(\mathcal{S}^\sigma_s)'(\mathbf{R}^d)$ and $(\Sigma^\sigma_s)'(\mathbf{R}^d)$, respectively. The following result follows by a straight-forward combination of Theorems 3.8, 3.15 and 3.16 in [1], (1.41) and Theorem 2.6.

Proposition 2.8. *Let* $a \in \widehat{A}(\mathbf{C}^{2d})$*. Then the following is true:*

(1) *if* $s,\sigma \geqslant \frac{1}{2}$ *and some* $r_2 > 0$*,* (2.19) *holds for every* $r_1 > 0$*, then* $\mathrm{Op}_{\mathfrak{V}}(a)$ *is continuous on* $\mathcal{A}^\sigma_s(\mathbf{C}^d)$ *and on* $(\mathcal{A}^\sigma_s)'(\mathbf{C}^d)$*;*

(2) *if* $s, \sigma > \frac{1}{2}$ *and for some* $r_1 > 0$, (2.19) *holds for every* $r_2 > 0$, *then* $\operatorname{Op}_{\mathfrak{V}}(a)$ *is continuous on* $\mathcal{A}_{0,s}^{\sigma}(\mathbf{C}^d)$ *and on* $(\mathcal{A}_{0,s}^{\sigma})'(\mathbf{C}^d)$*;*

(3) *if* $s, \sigma > \frac{1}{2}$ *and* (2.19) *holds for some* $r_1 > 0$ *and some* $r_2 > 0$, *then* $\operatorname{Op}_{\mathfrak{V}}(a)$ *is continuous from* $\mathcal{A}_{0,s}^{\sigma}(\mathbf{C}^d)$ *to* $\mathcal{A}_{s}^{\sigma}(\mathbf{C}^d)$, *and from* $(\mathcal{A}_{s}^{\sigma})'(\mathbf{C}^d)$ *to* $(\mathcal{A}_{0,s}^{\sigma})'(\mathbf{C}^d)$.

2.4. **Composition of Wick operators.** Let $a_1, a_2 \in \widehat{A}(\mathbf{C}^{2d})$. If composition is well defined then the complex twisted product $a_1 \#_{\mathfrak{V}} a_2$ is defined by

$$\operatorname{Op}_{\mathfrak{V}}(a_1) \circ \operatorname{Op}_{\mathfrak{V}}(a_2) = \operatorname{Op}_{\mathfrak{V}}(a_1 \#_{\mathfrak{V}} a_2).$$

By straight-forward computations it follows that the product $\#_{\mathfrak{V}}$ is given by

$$a_1 \#_{\mathfrak{V}} a_2(z,w) = \pi^{-d} \int_{\mathbf{C}^d} a_1(z,u) a_2(u,w) e^{-(z-u,w-u)} \, d\lambda(u), \quad z, w \in \mathbf{C}^d, \tag{2.20}$$

provided the integral is well defined. Inserting derivatives, (2.20) takes the form

$$(\partial_z^{\alpha_1} \overline{\partial}_w^{\beta_1} a_1) \#_{\mathfrak{V}} (\partial_z^{\alpha_2} \overline{\partial}_w^{\beta_2} a_2)(z,w)$$
$$= \pi^{-d} \int_{\mathbf{C}^d} (\partial_z^{\alpha_1} \overline{\partial}_u^{\beta_1} a_1)(z,u) (\partial_u^{\alpha_2} \overline{\partial}_w^{\beta_2} a_2)(u,w) e^{-(z-u,w-u)} \, d\lambda(u), \quad z, w \in \mathbf{C}^d. \tag{2.20$'$}$$

The following lemma is a product rule for the complex twisted product.

Lemma 2.9. *Let* $a_1, a_2 \in \widehat{A}(\mathbf{C}^{2d})$ *and suppose the integral in* (2.20)$'$ *is well defined for all* $z, w \in \mathbf{C}^d$ *and all* $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbf{N}^d$ *such that*

$$|\alpha_1 + \alpha_2 + \beta_1 + \beta_2| \leqslant 1.$$

Suppose also that the integrand in (2.20) *is zero at infinity. Then*

$$\partial_{z_j}(a_1 \#_{\mathfrak{V}} a_2) = (\partial_{z_j} a_1) \#_{\mathfrak{V}} a_2 + a_1 \#_{\mathfrak{V}} (\partial_{z_j} a_2), \quad j = 1, \dots, d \tag{2.21}$$

and

$$\overline{\partial}_{w_j}(a_1 \#_{\mathfrak{V}} a_2) = (\overline{\partial}_{w_j} a_1) \#_{\mathfrak{V}} a_2 + a_1 \#_{\mathfrak{V}} (\overline{\partial}_{w_j} a_2), \quad j = 1, \dots, d. \tag{2.22}$$

Proof. If

$$F_{a_1,a_2}(z,w,u) = a_1(z,u) a_2(u,w) e^{(z,u-w)+(u,w)}$$

then

$$\pi^d (a_1 \#_{\mathfrak{V}} a_2)(z,w) = \int_{\mathbf{C}^d} F_{a_1,a_2}(z,w,u) e^{-|u|^2} \, d\lambda(u).$$

This gives

$$\pi^d \partial_{z_j}(a_1 \#_{\mathfrak{V}} a_2)(z,w) = b_1(z,w) + b_2(z,w) - b_3(z,w),$$

where

$$b_1(z,w) = \int_{\mathbf{C}^d} F_{\partial_{z_j} a_1, a_2}(z,w,u) e^{-|u|^2} \, d\lambda(u),$$

$$b_2(z,w) = \int_{\mathbf{C}^d} F_{a_1,a_2}(z,w,u) \overline{u}_j e^{-|u|^2} \, d\lambda(u)$$

and

$$\begin{aligned}b_3(z,w) &= \overline{w}_j \int_{\mathbf{C}^d} F_{a_1,a_2}(z,w,u) e^{-|u|^2}\, d\lambda(u)\\ &= \overline{w}_j \pi^d (a_1 \#_{\mathfrak{V}} a_2)(z,w).\end{aligned}$$

The conjugate analyticity of $u \mapsto a_1(z,u)$ and $u \mapsto e^{(z,u-w)}$ implies $\partial_{u_j} a_1(z,u) = \partial_{u_j} e^{(z,u-w)} = 0$ which gives

$$\begin{aligned}&\partial_{u_j} F_{a_1,a_2}(z,w,u)\\ &\qquad = \big(a_1(z,u)\partial_{u_j} a_2(u,w) + \overline{w}_j a_1(z,u) a_2(u,w)\big)\, e^{(z,u-w)+(u,w)}\\ &\qquad\qquad = F_{a_1,\partial_{z_j} a_2}(z,w,u) + \overline{w}_j F_{a_1,a_2}(z,w,u).\end{aligned}$$

Consider $b_2(z,w)$. Integration by parts gives

$$\begin{aligned}b_2(z,w) &= \int_{\mathbf{C}^d} F_{a_1,a_2}(z,w,u) \overline{u}_j e^{-|u|^2}\, d\lambda(u)\\ &= - \int_{\mathbf{C}^d} F_{a_1,a_2}(z,w,u)\, \partial_{u_j} e^{-|u|^2}\, d\lambda(u)\\ &= \int_{\mathbf{C}^d} \partial_{u_j} F_{a_1,a_2}(z,w,u) e^{-|u|^2}\, d\lambda(u)\\ &= \int_{\mathbf{C}^d} F_{a_1,\partial_{z_j} a_2}(z,w,u) e^{-|u|^2}\, d\lambda(u) + \overline{w}_j \int_{\mathbf{C}^d} F_{a_1,a_2}(z,w,u) e^{-|u|^2}\, d\lambda(u)\\ &= \int_{\mathbf{C}^d} F_{a_1,\partial_{z_j} a_2}(z,w,u) e^{-|u|^2}\, d\lambda(u) + b_3(z,w).\end{aligned}$$

A combination of these identities now gives

$$\begin{aligned}&\pi^d \partial_{z_j} (a_1 \#_{\mathfrak{V}} a_2)(z,w)\\ &\qquad = \int_{\mathbf{C}^d} (F_{\partial_{z_j} a_1,a_2}(z,w,u) + F_{a_1,\partial_{z_j} a_2}(z,w,u)) e^{-|u|^2}\, d\lambda(u)\\ &\qquad\qquad = \pi^d (\partial_{z_j} a_1) \#_{\mathfrak{V}} a_2(z,w) + \pi^d a_1 \#_{\mathfrak{V}} (\partial_{z_j} a_2)(z,w),\end{aligned}$$

and (2.21) follows.

The assertion (2.22) is proved by similar arguments. □

The characterization in Theorem 2.2 (3) can be applied to prove the following composition result, which is a generalization of [27, Theorem 23.6] to include the case when $\rho = 0$.

Proposition 2.10. *Let $0 \leqslant \rho \leqslant 1$ and $\omega_j \in \mathscr{P}_{\mathrm{Sh},\rho}(\mathbf{R}^{2d})$ for $j = 1,2$. If $\mathfrak{a}_j \in \mathrm{Sh}_\rho^{(\omega_j)}(\mathbf{R}^{2d})$ for $j = 1,2$, then $\mathfrak{a}_1 \# \mathfrak{a}_2 \in \mathrm{Sh}_\rho^{(\omega_1\omega_2)}(\mathbf{R}^{2d})$.*

Proof. If $\mathfrak{a}_0 = \mathfrak{a}_1 \# \mathfrak{a}_2$ and $a_j = \mathsf{S}_{\mathfrak{V}} \mathfrak{a}_j$, $j = 0, 1, 2$, then $a_0 = a_1 \#_{\mathfrak{V}} a_2$. From Lemma 2.9 and (2.20) we obtain for $\alpha, \beta \in \mathbf{N}^d$,

$$
\begin{aligned}
&\partial_z^\alpha \overline{\partial}_w^\beta a_0(z,w) \\
&= \sum_{\gamma \leqslant \alpha} \sum_{\kappa \leqslant \beta} \binom{\alpha}{\gamma} \binom{\beta}{\kappa} \Big((\partial_z^{\alpha-\gamma} \overline{\partial}_w^{\beta-\kappa} a_1) \#_{\mathfrak{V}} (\partial_z^\gamma \overline{\partial}_w^\kappa a_2) \Big) (z,w) \\
&= \pi^{-d} \sum_{\gamma \leqslant \alpha} \sum_{\kappa \leqslant \beta} \binom{\alpha}{\gamma} \binom{\beta}{\kappa} \int_{\mathbf{C}^d} \partial_z^{\alpha-\gamma} \overline{\partial}_u^{\beta-\kappa} a_1(z,u) \partial_u^\gamma \overline{\partial}_w^\kappa a_2(u,w) e^{(z,u-w)+(u,w)} \, d\mu(u).
\end{aligned}
$$

Since $\omega_2 \in \mathscr{P}(\mathbf{R}^{2d}) \simeq \mathscr{P}(\mathbf{C}^d)$ is moderate, Theorem 2.2 gives for some $N_0 \geqslant 0$ and any $N_1, N_2 \geqslant 0$

$$
|\partial_u^{\alpha-\gamma} \overline{\partial}_w^{\beta-\kappa} a_1(z,u)| \lesssim \omega_1(\sqrt{2}\,\overline{z}) \langle z+u \rangle^{-\rho|\alpha+\beta-\gamma-\kappa|} \langle z-u \rangle^{-N_1} e^{\frac{1}{2}|z-u|^2}
$$

and

$$
|\partial_u^\gamma \overline{\partial}_w^\kappa a_2(u,w)| \lesssim \omega_2(\sqrt{2}\,\overline{z}) \langle z-u \rangle^{N_0} \langle u+w \rangle^{-\rho|\gamma+\kappa|} \langle u-w \rangle^{-N_2} e^{\frac{1}{2}|u-w|^2}.
$$

This gives

$$
\begin{aligned}
&\left| \partial_z^\alpha \overline{\partial}_w^\beta a_0(z,w) \right| \\
&\qquad \lesssim \omega_1(\sqrt{2}\,\overline{z}) \omega_2(\sqrt{2}\,\overline{z}) e^{\frac{1}{2}|z-w|^2} \int_{\mathbf{C}^d} F(z,w,u) e^{\Phi(z,w,u)} \, d\lambda(u) \qquad (2.23)
\end{aligned}
$$

where for any $N_1 \geqslant 0$

$$
F(z,w,u) = \langle z+u \rangle^{-\rho|\alpha+\beta-\gamma-\kappa|} \langle z-u \rangle^{N_0-N_1} \langle u+w \rangle^{-\rho|\gamma+\kappa|} \langle u-w \rangle^{-N_2}
$$

and

$$
\begin{aligned}
\Phi(z,w,u) = -\frac{1}{2}|z-w|^2 + \frac{1}{2}|z-u|^2 + \frac{1}{2}|u-w|^2 - |u|^2 \\
+ \operatorname{Re}(z,u-w) + \operatorname{Re}(u,w) = 0.
\end{aligned}
$$

By Peetre's inequality and the facts that $\gamma \leqslant \alpha$ and $\kappa \leqslant \beta$ we get

$$
\begin{aligned}
\langle z+u \rangle^{\rho|\gamma+\kappa|} \langle u+w \rangle^{-\rho|\gamma+\kappa|} &\lesssim \langle z-w \rangle^{\rho|\gamma+\kappa|} \\
&\lesssim \langle z-u \rangle^{\rho|\gamma+\kappa|} \langle u-w \rangle^{\rho|\gamma+\kappa|} \\
&\leqslant \langle z-u \rangle^{\rho|\alpha+\beta|} \langle u-w \rangle^{\rho|\alpha+\beta|}
\end{aligned}
$$

and

$$
\langle z+u \rangle^{-\rho|\alpha+\beta|} \lesssim \langle z+w \rangle^{-\rho|\alpha+\beta|} \langle u-w \rangle^{\rho|\alpha+\beta|}
$$

wherefrom

$$
F(z,w,u) \leqslant \langle z+w \rangle^{-\rho|\alpha+\beta|} \langle z-u \rangle^{\rho|\alpha+\beta|+N_0-N_1} \langle u-w \rangle^{2\rho|\alpha+\beta|-N_2}. \qquad (2.24)
$$

Hence a combination of (2.23) and (2.24) gives for any $N \geqslant 0$

$$
\begin{aligned}
&(\omega_1(\sqrt{2}\,\overline{z})\omega_2(\sqrt{2}\,\overline{z}))^{-1}\langle z+w\rangle^{\rho|\alpha+\beta|}\left|\partial_z^\alpha\overline{\partial}_w^\beta a_0(z,w)\right| \\
&\lesssim e^{\frac{1}{2}|z-w|^2}\int_{\mathbf{C}^d}\langle z-u\rangle^{\rho|\alpha+\beta|+N_0-N_1}\langle u-w\rangle^{2\rho|\alpha+\beta|-N_2}\,d\lambda(u) \\
&\lesssim \langle z-w\rangle^{-N}e^{\frac{1}{2}|z-w|^2}\int_{\mathbf{C}^d}\langle z-u\rangle^{\rho|\alpha+\beta|+N_0+N-N_1}\langle u-w\rangle^{2\rho|\alpha+\beta|+N-N_2}\,d\lambda(u).
\end{aligned}
$$

By letting

$$
N_1 \geqslant \rho|\alpha+\beta|+N_0+N \quad \text{and} \quad N_2 > 2\rho|\alpha+\beta|+N+2d
$$

we obtain

$$
\left|\partial_z^\alpha\overline{\partial}_w^\beta a_0(z,w)\right| \lesssim \omega_1(\sqrt{2}\,\overline{z})\omega_2(\sqrt{2}\,\overline{z})\langle z+w\rangle^{-\rho|\alpha+\beta|}\langle z-w\rangle^{-N}e^{\frac{1}{2}|z-w|^2}.
$$

According to Theorem 2.2 (3) this estimate implies that $\mathfrak{a}_0 \in \operatorname{Sh}_\rho^{(\omega_1\omega_2)}(\mathbf{R}^{2d})$. □

Remark 2.11. Eq. (2.20) combined with Theorem 2.6 can be used to show composition results for pseudo-differential operators with symbols in $\Gamma_{s,\sigma;0}^{\sigma,s}(\mathbf{R}^{2d})$. In fact we may use an argument similar to the proof of Proposition 2.10, but simpler since derivatives can be avoided. We obtain

$$
\mathfrak{a}_1\#\mathfrak{a}_2 \in \Gamma_{s,\sigma;0}^{\sigma,s}(\mathbf{R}^{2d}) \quad \text{when} \quad \mathfrak{a}_1,\mathfrak{a}_2 \in \Gamma_{s,\sigma;0}^{\sigma,s}(\mathbf{R}^{2d}), \quad s,\sigma \geqslant \frac{1}{2},
$$

and similarly with $\Gamma_{s,\sigma}^{\sigma,s;0}$ in place of $\Gamma_{s,\sigma;0}^{\sigma,s}$, provided $\sigma > \frac{1}{2}$. Thereby we regain parts of [1, Theorem 3.18] for certain restrictions on s and σ.

3. Relations and estimates for Wick and anti-Wick operators

In this section we first show how to approximate a Wick operator by means of a sum of anti-Wick operators. Then we prove continuity results for anti-Wick operators with symbols having exponential type bounds. Finally we deduce estimates for the Wick symbol of these anti-Wick operators.

3.1. **Expansion of Shubin type Wick operators with respect to anti-Wick operators.**

The first result can be stated for semi-conjugate analytic symbols on $\mathbf{C}^{2d}$.

Proposition 3.1. *Suppose $s \geqslant \frac{1}{2}$, $a \in \widehat{\mathcal{A}}_s'(\mathbf{C}^{2d})$, let $N \geqslant 0$ be an integer, and let*

$$
a_\alpha(w) = \partial_z^\alpha\overline{\partial}_w^\alpha a(w,w), \quad \alpha \in \mathbf{N}^d,
$$

and

$$
b_\alpha(z,w) = |\alpha|\int_0^1 (1-t)^{|\alpha|-1}\partial_z^\alpha\overline{\partial}_w^\alpha a(w+t(z-w),w)\,dt, \quad \alpha \in \mathbf{N}^d\backslash 0.
$$

Then

$$
\operatorname{Op}_{\mathfrak{V}}(a) = \sum_{|\alpha|\leqslant N}\frac{(-1)^{|\alpha|}\operatorname{Op}_{\mathfrak{V}}^{\mathrm{aw}}(a_\alpha)}{\alpha!} + \sum_{|\alpha|=N+1}\frac{(-1)^{|\alpha|}\operatorname{Op}_{\mathfrak{V}}(b_\alpha)}{\alpha!}. \tag{3.1}
$$

Proof. Taylor expansion gives

$$a(z,w) = \sum_{|\alpha|\leqslant N} \frac{(-1)^{|\alpha|} c_\alpha(z,w)}{\alpha!} + \sum_{|\alpha|=N+1} \frac{(-1)^{|\alpha|} c_{0,\alpha}(z,w)}{\alpha!},$$

where

$$c_\alpha(z,w) = (-1)^{|\alpha|}(z-w)^\alpha \partial_z^\alpha a(w,w)$$

and

$$c_{0,\alpha}(z,w) = (-1)^{|\alpha|}|\alpha|(z-w)^\alpha \int_0^1 (1-t)^{|\alpha|-1}\partial_z^\alpha a(w+t(z-w),w)\,dt.$$

Hence

$$\operatorname{Op}_{\mathfrak{V}}(a) = \sum_{|\alpha|\leqslant N} \frac{(-1)^{|\alpha|}\operatorname{Op}_{\mathfrak{V}}(c_\alpha)}{\alpha!} + \sum_{|\alpha|=N+1} \frac{(-1)^{|\alpha|}\operatorname{Op}_{\mathfrak{V}}(c_{0,\alpha})}{\alpha!},$$

and the result follows if we prove

$$\operatorname{Op}_{\mathfrak{V}}(c_\alpha) = \operatorname{Op}_{\mathfrak{V}}^{\mathrm{aw}}(a_\alpha) \quad \text{and} \quad \operatorname{Op}_{\mathfrak{V}}(c_{0,\alpha}) = \operatorname{Op}_{\mathfrak{V}}(b_\alpha). \tag{3.2}$$

It follows from (1.38) that

$$\operatorname{Op}_{\mathfrak{V}}(b_\alpha) = \operatorname{Op}_{\mathfrak{V}}(c_{1,\alpha}) \quad \text{and} \quad \operatorname{Op}_{\mathfrak{V}}(c_{0,\alpha}) = \operatorname{Op}_{\mathfrak{V}}(c_{2,\alpha})$$

where

$$c_{j,\alpha}(z,w) = (-1)^{|\alpha|}\pi^{-d}|\alpha| \int_0^1 (1-t)^{|\alpha|-1} h_{j,\alpha}(a;t,z,w)\,dt, \tag{3.3}$$

$j = 1,2$, with

$$h_{1,\alpha}(a;t,z,w) = (-1)^{|\alpha|}\int_{\mathbf{C}^d} \partial_z^\alpha \overline{\partial}_w^\alpha a(w_1+t(z-w_1),w_1)e^{-(z-w_1,w-w_1)}\,d\lambda(w_1) \tag{3.4}$$

and

$$h_{2,\alpha}(a;t,z,w) = \int_{\mathbf{C}^d} (z-w_1)^\alpha \partial_z^\alpha a(w_1+t(z-w_1),w_1)e^{-(z-w_1,w-w_1)}\,d\lambda(w_1).$$

Since

$$(z-w_1)^\alpha e^{-(z-w_1,w-w_1)} = \overline{\partial}_{w_1}^\alpha e^{-(z-w_1,w-w_1)}$$

integration by parts yields

$$h_{2,\alpha}(a;t,z,w) = \int_{\mathbf{C}^d} \partial_z^\alpha a(w_1+t(z-w_1),w_1)\overline{\partial}_{w_1}^\alpha e^{-(z-w_1,w-w_1)}\,d\lambda(w_1)$$

$$= (-1)^{|\alpha|}\int_{\mathbf{C}^d} \partial_z^\alpha \overline{\partial}_w^\alpha a(w_1+t(z-w_1),w_1)e^{-(z-w_1,w-w_1)}\,d\lambda(w_1) = h_{1,\alpha}(a;t,z,w),$$

and the second equality in (3.2) follows. The first equality in (3.2) follows by similar arguments. The details are left for the reader. □

Remark 3.2. Proposition 3.1 and its proof show that

$$\operatorname{Op}_{\mathfrak{V}}(a) = \sum_{|\alpha|\leqslant N} \frac{(-1)^{|\alpha|}\operatorname{Op}^{\mathrm{aw}}_{\mathfrak{V}}(a_\alpha)}{\alpha!} + \sum_{|\alpha|=N+1} \frac{(-1)^{|\alpha|}\operatorname{Op}_{\mathfrak{V}}(c_{1,\alpha})}{\alpha!} \tag{3.1$'$}$$

where $c_{1,\alpha}$ is defined by (3.3) and (3.4).

In the following result we estimate a_α in Proposition 3.1 and $c_{1,\alpha}$ in (3.3) when $a = \mathsf{S}_{\mathfrak{V}}\mathfrak{a}$ satisfies (2.5) for every $N \geqslant 0$ and $\alpha, \beta \in \mathbf{N}^d$. By Theorem 2.2 this means that $\operatorname{Op}_{\mathfrak{V}}(a)$ is the Bargmann transform of a Shubin type operator.

Proposition 3.3. *Let $0 \leqslant \rho \leqslant 1$, $\omega \in \mathscr{P}_{\mathrm{Sh},\rho}(\mathbf{R}^{2d})$, $a \in \widehat{\mathcal{A}}^{(\omega)}_{\mathrm{Sh},\rho}(\mathbf{C}^{2d})$, and let a_α and b_α be as in Proposition 3.1 for $\alpha \in \mathbf{N}^d$. Then $\operatorname{Op}_{\mathfrak{V}}(b_\alpha) = \operatorname{Op}_{\mathfrak{V}}(c_{1,\alpha})$ for a unique $c_{1,\alpha} \in \widehat{A}(\mathbf{C}^{2d})$,*

$$|\partial^\beta_w \overline{\partial}^\gamma_w a_\alpha(w)| \lesssim \omega(\sqrt{2}\overline{w})\langle w\rangle^{-\rho(2|\alpha|+|\beta+\gamma|)}, \quad \alpha,\beta,\gamma \in \mathbf{N}^d, \tag{3.5}$$

and

$$|\partial^\beta_z \overline{\partial}^\gamma_w c_{1,\alpha}(z,w)| \lesssim e^{\frac{1}{2}|z-w|^2}\omega(\sqrt{2}\overline{z})\langle z+w\rangle^{-\rho(2|\alpha|+|\beta+\gamma|)}\langle z-w\rangle^{-N}, \quad \alpha,\beta,\gamma \in \mathbf{N}^d. \tag{3.6}$$

Remark 3.4. The Wick symbol $c_{1,\alpha}$ in Proposition 3.3 is uniquely defined and given by (3.3) in view of Proposition 1.5, when $h_{1,\alpha}$ is defined by (3.4). The conditions in Proposition 3.3 imply that $c_{1,\alpha} \in \widehat{\mathcal{A}}^{(\omega_\alpha)}_{\mathrm{Sh},\rho}(\mathbf{C}^{2d})$ where $\omega_\alpha = \langle\,\cdot\,\rangle^{-2\rho|\alpha|}\cdot\omega$.

Proof of Proposition 3.3. The estimate (3.5) is an immediate consequence of

$$\partial^\beta_w \overline{\partial}^\gamma_w a_\alpha(w) = \partial^{\alpha+\beta}_w \overline{\partial}^{\alpha+\gamma}_w a(w,w)$$

and (2.5).

In order to prove (3.6) we first note that the uniqueness assertion for $c_{1,\alpha}$ is a consequence of Remark 3.4. Let $h_{1,\alpha}(a;z,w)$ be the same as in the proof of Proposition 3.1. Integration by parts gives

$$\partial^\beta_z \overline{\partial}^\gamma_w h_{1,\alpha}(a;t,z,w) = h_{1,\alpha}(\partial^\beta_z \overline{\partial}^\gamma_w a;t,z,w),$$

which reduce the problem to prove that (3.6) holds for $\beta = \gamma = 0$.

The assumption $a \in \widehat{\mathcal{A}}^{(\omega)}_{\mathrm{Sh},\rho}(\mathbf{C}^{2d})$ combined with ω and $\langle\,\cdot\,\rangle^{-|\alpha|}$ being moderate imply

$$|\partial^\alpha_z \overline{\partial}^\beta_w a(z,w)| \lesssim e^{\frac{1}{2}|z-w|^2}\omega(\sqrt{2}\,\overline{w})\langle w\rangle^{-\rho|\alpha+\beta|}\langle z-w\rangle^{-N}$$

for every $N \geqslant 0$. This gives

$$e^{\operatorname{Re}(z,w)}|h_{1,\alpha}(a;t,z,w)|$$
$$\lesssim \int_{\mathbf{C}^d} \omega(\sqrt{2}\overline{w_1})e^{\frac{t^2}{2}|z-w_1|^2}\langle w_1\rangle^{-2\rho|\alpha|}\langle t(z-w_1)\rangle^{-N}e^{\operatorname{Re}(z+w-w_1,w_1)}\,d\lambda(w_1),$$

that is

$$e^{-\frac{1}{4}|z-w|^2}|h_{1,\alpha}(a;t,z,w)|$$

$$\lesssim \int_{\mathbf{C}^d} \omega(\sqrt{2}\overline{w_1})e^{\frac{t^2}{2}|z-w_1|^2}\langle w_1\rangle^{-2\rho|\alpha|}\langle t(z-w_1)\rangle^{-N}e^{-|w_1-z_2|^2}\,d\lambda(w_1)$$

$$= \int_{\mathbf{C}^d} \omega(\sqrt{2}(\overline{z_2+w_1}))e^{\frac{t^2}{2}|z_1-w_1|^2}\langle z_2+w_1\rangle^{-2\rho|\alpha|}\langle t(z_1-w_1)\rangle^{-N}e^{-|w_1|^2}\,d\lambda(w_1) \tag{3.7}$$

for every $N \geqslant 0$, where $z_1 = \frac{1}{2}(z-w)$ and $z_2 = \frac{1}{2}(z+w)$.

If $t \in [0, \frac{1}{2}]$, then the last estimate together with the moderateness of ω give

$$e^{-|z_1|^2}|h_{1,\alpha}(a;t,z,w)| \lesssim \omega(\sqrt{2}\overline{z_2})\langle z_2\rangle^{-2\rho|\alpha|}\int_{\mathbf{C}^d} e^{\frac{1}{8}|w_1|^2}e^{\frac{1}{8}|z_1-w_1|^2}e^{-|w_1|^2}\,d\lambda(w_1)$$

$$\lesssim \omega(\sqrt{2}\overline{z_2})\langle z_2\rangle^{-2\rho|\alpha|}e^{\frac{1}{4}|z_1|^2}\int_{\mathbf{C}^d} e^{\frac{1}{4}|w_1|^2}e^{-\frac{7}{8}|w_1|^2}\,d\lambda(w_1)$$

$$\lesssim \omega(\sqrt{2}\overline{z_2})\langle z_2\rangle^{-2\rho|\alpha|}e^{\frac{1}{2}|z_1|^2}\langle z_1\rangle^{-N},$$

for every $N \geqslant 0$. The moderateness of ω again gives

$$|h_{1,\alpha}(a;t,z,w)| \lesssim e^{\frac{1}{2}|z-w|^2}\omega(\sqrt{2}\overline{z})\langle z+w\rangle^{-2\rho|\alpha|}\langle z-w\rangle^{-N} \tag{3.8}$$

or every $N \geqslant 0$, when $t \in [0, \frac{1}{2}]$.

Suppose instead $t \in [\frac{1}{2}, 1]$. Then $\langle t(z_1-w_1)\rangle^{-N} \asymp \langle z_1-w_1\rangle^{-N}$. Moderateness again gives

$$\omega(\sqrt{2}(\overline{z_2+w_1}))\langle z_2+w_1\rangle^{-2\rho|\alpha|}\langle z_1-w_1\rangle^{-N_0} \lesssim \omega(\sqrt{2}\overline{z})\langle z\rangle^{-2\rho|\alpha|}$$

for some N_0. Hence (3.7) gives

$$e^{-|z_1|^2}\omega(\sqrt{2}\overline{z})^{-1}\langle z\rangle^{2\rho|\alpha|}|h_{1,\alpha}(a;t,z,w)|$$

$$\lesssim \int_{\mathbf{C}^d} e^{\frac{1}{2}|z_1-w_1|^2}\langle z_1-w_1\rangle^{-N}e^{-|w_1|^2}\,d\lambda(w_1)$$

$$= e^{|z_1|^2}\int_{\mathbf{C}^d}\langle z_1-w_1\rangle^{-N}e^{-\frac{1}{2}|w_1+z_1|^2}\,d\lambda(w_1) \asymp e^{|z_1|^2}\langle z_1\rangle^{-N}$$

for every $N \geqslant 0$. This gives (3.8) also for $t \in [\frac{1}{2}, 1]$.

The result now follows by using (3.8) when estimating $|c_{1,\alpha}(z,w)|$ in (3.3) and evaluating the arising integral. □

The next result, analogous to Proposition 3.3, will be useful in Section 5 when we discuss hypoellipticity for Shubin operators in the Wick setting.

Proposition 3.5. *Let $\rho \geqslant 0$, $\omega \in \mathscr{P}_{\mathrm{Sh},\rho}(\mathbf{C}^d)$, $\omega_t = \omega\cdot\langle\,\cdot\,\rangle^{-2\rho t}$ when $t \geqslant 0$, $a \in \widehat{\mathcal{A}}^{(\omega)}_{\mathrm{Sh},\rho}(\mathbf{C}^{2d})$, $\mathfrak{a} = \mathsf{S}_{\mathfrak{V}}^{-1}a$ and $N \geqslant 0$ be an integer. Then*

$$\mathfrak{a}(x,-\xi) = \sum_{|\alpha|\leqslant N}\frac{(-1)^{|\alpha|}(\partial_z^\alpha\overline{\partial}_w^\alpha a)(2^{-\frac{1}{2}}z, 2^{-\frac{1}{2}}z)}{2^{|\alpha|}\alpha!} + c_N(z), \quad z = x+i\xi, \tag{3.9}$$

where

$$\partial_z^\alpha \overline{\partial}_w^\alpha a \in \widehat{\mathcal{A}}_{\mathrm{Sh},\rho}^{(\omega_{|\alpha|})}(\mathbf{C}^{2d}) \quad \textit{and} \quad (x,\xi) \mapsto c_N(x-i\xi) \in \mathrm{Sh}_\rho^{(\omega_{N+1})}(\mathbf{R}^{2d}). \tag{3.10}$$

Proof. The first claim in (3.10) $\partial_z^\alpha \overline{\partial}_w^\alpha a \in \widehat{\mathcal{A}}_{\mathrm{Sh},\rho}^{(\omega_{|\alpha|})}(\mathbf{C}^{2d})$ is an immediate consequence of the definition (2.13) and Peetre's inequality.

By Taylor expanding the right-hand side of (2.10) we obtain

$$\mathfrak{a}(x,-\xi) = \sum_{|\alpha+\beta|\leqslant 2N+1} \frac{(2/\pi)^d I_{\alpha,\beta} \cdot (\partial_z^\alpha \overline{\partial}_w^\beta a)(2^{-\frac{1}{2}}z, 2^{-\frac{1}{2}}z)}{\alpha!\beta!} + c_N(z), \tag{3.11}$$

where

$$I_{\alpha,\beta} = \int_{\mathbf{C}^d} (-w)^\alpha \overline{w}^\beta e^{-2|w|^2}\, d\lambda(w),$$

and

$$c_N(z) = 2(N+1) \sum_{|\alpha+\beta|=2N+2} \frac{(-1)^{|\beta|}}{\alpha!\beta!} \int_0^1 (1-\theta)^{2N+1} H_{\alpha,\beta}(z,\theta)\, d\theta \tag{3.12}$$

with

$$H_{\alpha,\beta}(z,\theta) = \left(\frac{2}{\pi}\right)^d \int_{\mathbf{C}^d} (\partial_z^\alpha \overline{\partial}_w^\beta a)\left(\frac{z}{\sqrt{2}} - \theta w, \frac{z}{\sqrt{2}} + \theta w\right) w^\alpha \overline{w}^\beta e^{-2|w|^2}\, d\lambda(w). \tag{3.13}$$

The orthonormality of $\{e_\alpha\}_{\alpha\in\mathbf{N}^d} \subseteq A^2(\mathbf{C}^d)$ (cf. (1.21)) yields $I_{\alpha,\beta} = 0$ if $\alpha \neq \beta$ and

$$\begin{aligned} I_{\alpha,\alpha} &= \int_{\mathbf{C}^d} (-w)^\alpha \overline{w}^\alpha e^{-2|w|^2}\, d\lambda(w) \\ &= (-1)^{|\alpha|} 2^{-d-|\alpha|} \alpha! \pi^d \int_{\mathbf{C}^d} |e_\alpha(w)|^2\, d\mu(w) \\ &= (-1)^{|\alpha|} 2^{-d-|\alpha|} \alpha! \pi^d. \end{aligned}$$

Comparing (3.11) with (3.9) we see that the sum in the latter formula has been proven correct.

It remains to study the remainder c_N. We need to prove that $\mathfrak{c}(x,\xi) = c_N(x-i\xi)$ belongs to $\mathrm{Sh}_\rho^{(\omega_{N+1})}(\mathbf{R}^{2d})$. If

$$h_{\alpha,\beta}(z,w,\theta) = (\partial_z^\alpha \overline{\partial}_w^\beta a)\left(\frac{z}{\sqrt{2}} - \theta w, \frac{z}{\sqrt{2}} + \theta w\right) w^\alpha \overline{w}^\beta e^{-2|w|^2}$$

then

$$H_{\alpha,\beta}(z,\theta) = \left(\frac{2}{\pi}\right)^d \int_{\mathbf{C}^d} h_{\alpha,\beta}(z,w,\theta)\, d\lambda(w).$$

First we notice that

$$\partial_z^\alpha \overline{\partial}_z^\beta c_N(z) = 2(N+1) \sum_{|\gamma+\delta|=2N+2} \frac{(-1)^{|\delta|}}{\gamma!\delta!} \int_0^1 (1-\theta)^{2N+1} \partial_z^\alpha \overline{\partial}_z^\beta H_{\gamma,\delta}(z,\theta)\, d\theta,$$

$$\partial_z^\alpha \overline{\partial}_z^\beta H_{\gamma,\delta}(z,\theta) = \left(\frac{2}{\pi}\right)^d \int_{\mathbf{C}^d} \partial_z^\alpha \overline{\partial}_z^\beta h_{\gamma,\delta}(z,w,\theta)\, d\lambda(w)$$

and

$$\partial_z^\alpha \overline{\partial}_z^\beta h_{\gamma,\delta}(z,w,\theta) = 2^{-\frac{|\alpha+\beta|}{2}} (\partial_z^{\alpha+\gamma}\overline{\partial}_z^{\beta+\delta} a)\left(\frac{z}{\sqrt{2}} - \theta w, \frac{z}{\sqrt{2}} + \theta w\right) w^\gamma \overline{w}^\delta e^{-2|w|^2}.$$

From the definition (2.13) this implies that for every $M \geqslant 0$ and some $M_0 \geqslant 0$ we have

$$\begin{aligned} &|\partial_z^\alpha \overline{\partial}_z^\beta h_{\gamma,\delta}(z,w,\theta)| \\ &\quad\lesssim e^{-2(1-\theta^2)|w|^2} \omega(\overline{z} - \sqrt{2}\theta\overline{w}) \langle z\rangle^{-\rho(|\alpha+\beta|+2N+2)} \langle \theta w\rangle^{-M-M_0} |w|^{2N+2} \\ &\qquad\qquad \lesssim e^{-2(1-\theta)|w|^2} \omega(\overline{z}) \langle z\rangle^{-\rho(|\alpha+\beta|+2N+2)} \langle \theta w\rangle^{-M} |w|^{2N+2}. \end{aligned}$$

This gives

$$|\partial_z^\alpha \overline{\partial}_z^\beta H_{\gamma,\delta}(z,\theta)| \lesssim \omega(\overline{z}) \langle z\rangle^{-\rho(|\alpha+\beta|+2N+2)} \cdot J(\theta),$$

where

$$J(\theta) = \int_{\mathbf{C}^d} e^{-2(1-\theta)|w|^2} \langle \theta w\rangle^{-M} |w|^{2N+2} \, d\lambda(w).$$

For $\theta \in [0, \frac{1}{2}]$ we get

$$J(\theta) = \int_{\mathbf{C}^d} e^{-|w|^2} |w|^{2N+2} \, d\lambda(w),$$

which is finite and independent of θ. If instead $\theta \in [\frac{1}{2}, 1]$, and choosing $M > 2d + 2N + 2$, then

$$J(\theta) \leqslant \int_{\mathbf{C}^d} \langle \theta w\rangle^{-M} |w|^{2N+2} \, d\lambda(w),$$

which is again finite and independent of θ.

A combination of these estimates give

$$|\partial_z^\alpha \overline{\partial}_z^\beta H_{\gamma,\delta}(z,\theta)| \lesssim \omega(\overline{z}) \langle z\rangle^{-\rho(|\alpha+\beta|+2N+2)},$$

which in turn implies

$$|\partial_z^\alpha \overline{\partial}_z^\beta c_N(z)| \lesssim \omega(\overline{z}) \langle z\rangle^{-\rho(|\alpha+\beta|+2N+2)}.$$

This means that $\mathfrak{c} \in \operatorname{Sh}_\rho^{(\omega_{N+1})}(\mathbf{R}^{2d})$. □

3.2. **Continuity of anti-Wick operators with exponentially bounded symbols.** Next we consider anti-Wick symbols that satisfy exponential bounds of the form

$$|a_0(w)| \lesssim e^{-r_0|w|^{\frac{1}{s}}}, \tag{3.14}$$

or

$$|a_0(w)| \lesssim e^{r_0|w|^{\frac{1}{s}}}. \tag{3.15}$$

In order to formulate our results we introduce new spaces of entire functions. Let $s > \frac{1}{2}$, $t_0, r > 0$, and let $\mathcal{A}_{s,t_0,r}(\mathbf{C}^d)$ be the Banach space of all $F \in A(\mathbf{C}^d)$ such that

$$\|F\|_{\mathcal{A}_{s,t_0,r}} \equiv \|F \cdot e^{-t_0|\,\cdot\,|^2 + r|\,\cdot\,|^{\frac{1}{s}}}\|_{L^\infty} < \infty.$$

Set

$$\mathcal{A}_{0,(s,t_0)}(\mathbf{C}^d) = \bigcap_{r>0} \mathcal{A}_{s,t_0,r}(\mathbf{C}^d) \quad \text{and} \quad \mathcal{A}'_{(s,t_0)}(\mathbf{C}^d) = \bigcap_{r>0} \mathcal{A}_{s,t_0,-r}(\mathbf{C}^d)$$

equipped with the projective limit topology. Likewise we set

$$\mathcal{A}_{(s,t_0)}(\mathbf{C}^d) = \bigcup_{r>0} \mathcal{A}_{s,t_0,r}(\mathbf{C}^d) \quad \text{and} \quad \mathcal{A}'_{0,(s,t_0)}(\mathbf{C}^d) = \bigcup_{r>0} \mathcal{A}_{s,t_0,-r}(\mathbf{C}^d)$$

equipped with the inductive limit topology.

Referring to Section 1.3 it is clear that the spaces $\mathcal{A}_{0,(s,t_0)}(\mathbf{C}^d)$, $\mathcal{A}_{(s,t_0)}(\mathbf{C}^d)$, $\mathcal{A}'_{(s,t_0)}(\mathbf{C}^d)$ and $\mathcal{A}'_{0,(s,t_0)}(\mathbf{C}^d)$ are generalizations of

$$\mathcal{A}_{0,(s,\frac{1}{2})}(\mathbf{C}^d) = \mathfrak{V}_d(\Sigma_s(\mathbf{R}^d)) = \mathcal{A}_{0,s}(\mathbf{C}^d)$$
$$\mathcal{A}_{(s,\frac{1}{2})}(\mathbf{C}^d) = \mathfrak{V}_d(\mathcal{S}_s(\mathbf{R}^d)) = \mathcal{A}_s(\mathbf{C}^d)$$
$$\mathcal{A}'_{(s,\frac{1}{2})}(\mathbf{C}^d) = \mathfrak{V}_d(\mathcal{S}'_s(\mathbf{R}^d)) = \mathcal{A}'_s(\mathbf{C}^d)$$

and

$$\mathcal{A}'_{0,(s,\frac{1}{2})}(\mathbf{C}^d) = \mathfrak{V}_d(\Sigma'_s(\mathbf{R}^d)) = \mathcal{A}'_{0,s}(\mathbf{C}^d),$$

respectively.

Proposition 3.6. *Let $a_0 \in L^\infty_{loc}(\mathbf{C}^d)$, $s > \frac{1}{2}$, $0 < t_0 < 1$ and*

$$t_1 = \frac{1}{4(1-t_0)}. \tag{3.16}$$

Then the following is true:

(1) *if* (3.15) *holds for some $r_0 > 0$ then*

$$\begin{aligned} \operatorname{Op}^{\mathrm{aw}}_{\mathfrak{V}}(a_0) &: \mathcal{A}_{0,(s,t_0)}(\mathbf{C}^d) \to \mathcal{A}_{0,(s,t_1)}(\mathbf{C}^d), \\ \operatorname{Op}^{\mathrm{aw}}_{\mathfrak{V}}(a_0) &: \mathcal{A}'_{0,(s,t_0)}(\mathbf{C}^d) \to \mathcal{A}'_{0,(s,t_1)}(\mathbf{C}^d) \end{aligned} \tag{3.17}$$

are continuous;

(2) *if* (3.15) *holds for every $r_0 > 0$ then*

$$\begin{aligned} \operatorname{Op}^{\mathrm{aw}}_{\mathfrak{V}}(a_0) &: \mathcal{A}_{(s,t_0)}(\mathbf{C}^d) \to \mathcal{A}_{(s,t_1)}(\mathbf{C}^d), \\ \operatorname{Op}^{\mathrm{aw}}_{\mathfrak{V}}(a_0) &: \mathcal{A}'_{(s,t_0)}(\mathbf{C}^d) \to \mathcal{A}'_{(s,t_1)}(\mathbf{C}^d) \end{aligned} \tag{3.18}$$

are continuous.

Proof. We only prove that the first map in (3.17) is continuous. The other continuity assertions follow by similar arguments and are left for the reader.

Let $r_2 > 0$ be given, $r_1 > r_0$ and $F \in \mathcal{A}_{0,(s,t_0)}(\mathbf{C}^d)$. We have for $z \in \mathbf{C}^d$

$$
\begin{aligned}
&| \operatorname{Op}_{\mathfrak{V}}^{\mathrm{aw}}(a_0)F(z)|e^{-t_1|z|^2+r_2|z|^{\frac{1}{s}}} \\
&\lesssim e^{-t_1|z|^2+r_2|z|^{\frac{1}{s}}} \int_{\mathbf{C}^d} |a_0(w)|\,|F(w)|\,e^{\operatorname{Re}(z,w)-|w|^2}\,d\lambda(w) \\
&\lesssim e^{-t_1|z|^2+r_2|z|^{\frac{1}{s}}} \|F\|_{\mathcal{A}_{s,t_0,r_1}} \int_{\mathbf{C}^d} e^{r_0|w|^{\frac{1}{s}}+t_0|w|^2-r_1|w|^{\frac{1}{s}}+\operatorname{Re}(z,w)-|w|^2}\,d\lambda(w) \\
&= e^{r_2|z|^{\frac{1}{s}}} \|F\|_{\mathcal{A}_{s,t_0,r_1}} \int_{\mathbf{C}^d} e^{-(r_1-r_0)|w|^{\frac{1}{s}}-(1-t_0)|w|^2+\operatorname{Re}(z,w)-t_1|z|^2}\,d\lambda(w) \\
&= e^{r_2|z|^{\frac{1}{s}}} \|F\|_{\mathcal{A}_{s,t_0,r_1}} \int_{\mathbf{C}^d} e^{-(r_1-r_0)|w|^{\frac{1}{s}}-\left|\sqrt{1-t_0}w-\frac{1}{2\sqrt{1-t_0}}z\right|^2}\,d\lambda(w) \\
&= e^{r_2|z|^{\frac{1}{s}}} \|F\|_{\mathcal{A}_{s,t_0,r_1}} \int_{\mathbf{C}^d} e^{-(r_1-r_0)\left|w+\frac{1}{2(1-t_0)}z\right|^{\frac{1}{s}}-(1-t_0)|w|^2}\,d\lambda(w) \\
&\leqslant e^{(r_2-c_1(r_1-r_0))|z|^{\frac{1}{s}}} \|F\|_{\mathcal{A}_{s,t_0,r_1}} \int_{\mathbf{C}^d} e^{c_2(r_1-r_0)|w|^{\frac{1}{s}}-(1-t_0)|w|^2}\,d\lambda(w) \\
&\asymp \|F\|_{\mathcal{A}_{s,t_0,r_1}} e^{(r_2-c_1(r_1-r_0))|z|^{\frac{1}{s}}}
\end{aligned}
$$

for some constants $c_1, c_2 > 0$. By choosing r_1 sufficiently large we get

$$
\| \operatorname{Op}_{\mathfrak{V}}^{\mathrm{aw}}(a_0)F\|_{\mathcal{A}_{s,t_1,r_2}} \lesssim \|F\|_{\mathcal{A}_{s,t_0,r_1}}.
$$

The estimates and (1.40) imply $\operatorname{Op}_{\mathfrak{V}}^{\mathrm{aw}}(a_0)F \in A(\mathbf{C}^d)$. □

Remark 3.7. Note that (3.16) implies $t_1 > \frac{1}{4}$ and $t_0 \leqslant t_1$ with equality if and only if $t_0 = \frac{1}{2}$. Hence $\mathcal{A}_{0,(s,t_0)}(\mathbf{C}^d) \subseteq \mathcal{A}_{0,(s,t_1)}(\mathbf{C}^d)$, and similarly for the other spaces.

The particular case $t_0 = \frac{1}{2}$ gives

Corollary 3.8. *Let $a_0 \in L^\infty_{loc}(\mathbf{C}^d)$ and $s > \frac{1}{2}$. If* (3.15) *holds for some (every) $r_0 > 0$ then $\operatorname{Op}_{\mathfrak{V}}^{\mathrm{aw}}(a_0)$ is continuous on $\mathcal{A}_{0,s}(\mathbf{C}^d)$ (on $\mathcal{A}_s(\mathbf{C}^d)$).*

With a technique similar to the proof of Proposition 3.6 one shows the following result.

Proposition 3.9. *Let $a_0 \in L^\infty_{loc}(\mathbf{C}^d)$, $s > \frac{1}{2}$, $0 < t_0 < 1$ and suppose* (3.16) *holds. Then the following is true:*

(1) *if* (3.14) *holds for all $r_0 > 0$ then*
$$
\operatorname{Op}_{\mathfrak{V}}^{\mathrm{aw}}(a_0) : \mathcal{A}'_{0,(s,t_0)}(\mathbf{C}^d) \to \mathcal{A}_{0,(s,t_1)}(\mathbf{C}^d) \tag{3.19}
$$
is continuous;

(2) *if* (3.14) *holds for some $r_0 > 0$ then*
$$
\operatorname{Op}_{\mathfrak{V}}^{\mathrm{aw}}(a_0) : \mathcal{A}'_{(s,t_0)}(\mathbf{C}^d) \to \mathcal{A}_{(s,t_1)}(\mathbf{C}^d) \tag{3.20}
$$
is continuous.

Again the particular case $t_0 = \frac{1}{2}$ gives

Corollary 3.10. *Let $a_0 \in L^\infty_{loc}(\mathbf{C}^d)$ and $s > \frac{1}{2}$. Then the following is true:*

(1) *if* (3.14) *holds for all $r_0 > 0$ then*

$$\operatorname{Op}^{\mathrm{aw}}_{\mathfrak{V}}(a_0) : \mathcal{A}'_{0,s}(\mathbf{C}^d) \to \mathcal{A}_{0,s}(\mathbf{C}^d)$$

is continuous;

(2) *if* (3.14) *holds for some $r_0 > 0$ then*

$$\operatorname{Op}^{\mathrm{aw}}_{\mathfrak{V}}(a_0) : \mathcal{A}'_{s}(\mathbf{C}^d) \to \mathcal{A}_{s}(\mathbf{C}^d)$$

is continuous.

3.3. **Estimates of Wick symbols of anti-Wick operators with exponentially bounded symbols.** For anti-Wick operators in [13, Eq. (2.94)] we have the following result.

Theorem 3.11. *If $a_0 \in L^\infty_{loc}(\mathbf{C}^d)$ satisfies*

$$|a_0(w)| \lesssim e^{r|w|^2}, \quad w \in \mathbf{C}^d, \quad \text{for some} \quad r < 1, \tag{3.21}$$

then $a_0 \in L_{0,A}(\mathbf{C}^d)$ and $(1.38)'$ holds for some $a_0^{\mathrm{aw}} \in \widehat{A}(\mathbf{C}^{2d})$ with

$$|a_0^{\mathrm{aw}}(z,w)| \lesssim e^{r_0|z+w|^2 - \operatorname{Re}(z,w)}, \qquad r_0 = 4^{-1}(1-r)^{-1}.$$

Proof. The claim $a_0 \in L_{0,A}(\mathbf{C}^d)$ is an immediate consequence of the assumption (3.21) and the definition (1.39). The integral in $(1.38)'$ can be estimated as

$$
\begin{aligned}
&\left| \int_{\mathbf{C}^d} a_0(w_1) e^{-(z-w_1,w-w_1)}\, d\lambda(w_1) \right| \\
&\qquad \lesssim \int_{\mathbf{C}^d} e^{r|w_1|^2} \left| e^{-(z-w_1,w-w_1)} \right| \, d\lambda(w_1) \\
&\qquad = e^{-\operatorname{Re}(z,w)} \int_{\mathbf{C}^d} e^{-(1-r)|w_1|^2} e^{\operatorname{Re}(z+w,w_1)}\, d\lambda(w_1) \\
&\qquad = e^{\frac{1}{4(1-r)}|z+w|^2 - \operatorname{Re}(z,w)} \int_{\mathbf{C}^d} e^{-(1-r)|w_1-(z+w)/(2(1-r))|^2}\, d\lambda(w_1) \\
&\qquad \asymp e^{r_0|z+w|^2 - \operatorname{Re}(z,w)}. \qquad \square
\end{aligned}
$$

Remark 3.12. The condition on a_0^{aw} in Theorem 3.11 implies that a_0^{aw} belongs to $\widehat{\mathcal{A}}'_{0,\frac{1}{2}}(\mathbf{C}^{2d})$ (see [30]). In particular it follows that $\operatorname{Op}^{\mathrm{aw}}_{\mathfrak{V}}(a_0) = \operatorname{Op}_{\mathfrak{V}}(a_0^{\mathrm{aw}})$ is continuous from $\mathcal{A}_{0,\frac{1}{2}}(\mathbf{C}^d)$ to $\mathcal{A}'_{0,\frac{1}{2}}(\mathbf{C}^d)$ (cf. [30, Theorem 2.10] and Remark 1.2).

The following result concerns exponentially moderate weight functions.

Theorem 3.13. *Let $a_0 \in L_{0,A}(\mathbf{C}^d)$, $a_0^{\mathrm{aw}} \in \widehat{A}(\mathbf{C}^{2d})$ is given by $(1.38)'$ and $\omega \in \mathscr{P}_E(\mathbf{C}^d)$. If*

$$|a_0(w)| \lesssim \omega(2w), \quad w \in \mathbf{C}^d,$$

then

$$|a_0^{\mathrm{aw}}(z,w)| \lesssim e^{\frac{1}{4}|z-w|^2}\omega(z+w), \quad z,w \in \mathbf{C}^d.$$

Proof. Let $r \geqslant 0$ be chosen such that $\omega(z+w) \lesssim \omega(z)e^{r|w|}$, $z, w \in \mathbf{C}^d$. From $(1.38)'$ we get

$$
\begin{aligned}
|a_0^{\mathrm{aw}}(z,w)| &\lesssim \int_{\mathbf{C}^d} \omega(2w_1) e^{-\operatorname{Re}(z-w_1, w-w_1)}\, d\lambda(w_1) \\
&= e^{-\operatorname{Re}(z,w)} \int_{\mathbf{C}^d} \omega(2w_1) e^{\operatorname{Re}(z+w,w_1)-|w_1|^2}\, d\lambda(w_1) \\
&= e^{-\operatorname{Re}(z,w)+\frac{1}{4}|z+w|^2} \int_{\mathbf{C}^d} \omega(2w_1) e^{-|w_1-(z+w)/2|^2}\, d\lambda(w_1) \\
&= e^{\frac{1}{4}|z-w|^2} \int_{\mathbf{C}^d} \omega(2w_1+z+w) e^{-|w_1|^2}\, d\lambda(w_1) \\
&\lesssim e^{\frac{1}{4}|z-w|^2}\omega(z+w) \int_{\mathbf{C}^d} e^{2r|w_1|-|w_1|^2}\, d\lambda(w_1) \asymp e^{\frac{1}{4}|z-w|^2}\omega(z+w). \qquad \square
\end{aligned}
$$

The anti-Wick operators in Propositions 3.6 and 3.9 can also be described as Wick operators with symbols that have smaller growth bounds than $\widehat{\mathcal{A}}_s(\mathbf{C}^{2d})$ and its dual. The following result extends Theorem 3.13 for weights of the form $e^{c|z|^{\frac{1}{s}}}$ with $c \in \mathbf{R}$ from $s \geqslant 1$ to $s \geqslant \frac{1}{2}$.

Theorem 3.14. *Let $s \geqslant \frac{1}{2}$ ($s > \frac{1}{2}$), $a_0 \in L_{0,A}(\mathbf{C}^d)$ and let a_0^{aw} be given by $(1.38)'$. Then the following is true:*

(1) *if (3.14) holds for some (every) $r_0 > 0$ then*

$$
|a_0^{\mathrm{aw}}(z,w)| \lesssim e^{\frac{1}{4}|z-w|^2 - r|z+w|^{\frac{1}{s}}} \tag{3.22}
$$

for some (every) $r > 0$;

(2) *if (3.15) holds for every (some) $r_0 > 0$ then*

$$
|a_0^{\mathrm{aw}}(z,w)| \lesssim e^{\frac{1}{4}|z-w|^2 + r|z+w|^{\frac{1}{s}}} \tag{3.23}
$$

for every (some) $r > 0$.

Remark 3.15. Thanks to the parameter $\frac{1}{4}$ in the factor $e^{\frac{1}{4}|z-w|^2}$ rather than $\frac{1}{2}$, the estimates (3.23) are much stronger than the estimates (2.19) with $\sigma = s$. Corollary 3.8 can thus be seen as a consequence of Theorems 2.6 and 3.14, and [10, Definition 2.4, and Theorems 4.10 and 4.11].

Remark 3.16. The estimates for a_0^{aw} in Theorem 3.14 may seem weak since the dominating factor $e^{\frac{1}{4}|z-w|^2}$ is present in (3.22) and (3.23) but absent in the original estimates (3.14) and (3.15) for a_0.

On the other hand, Wick symbols for operators with continuity involving the spaces $\mathcal{A}_s(\mathbf{C}^d)$ and $\mathcal{A}_s'(\mathbf{C}^d)$, as well as $\mathcal{A}_{0,s}(\mathbf{C}^d)$ and $\mathcal{A}_{0,s}'(\mathbf{C}^d)$, usually satisfies conditions of the form

$$
|a(z,w)| \lesssim e^{\frac{1}{2}|z-w|^2 \pm r_1|z+w|^{\frac{1}{2}} \pm |z-w|^{\frac{1}{s}}}
$$

in view of [30, Theorems 2.9 and 2.10], and Theorem 2.6. Here the dominating factor is $e^{\frac{1}{2}|z-w|^2}$, which is larger than the factor $e^{\frac{1}{4}|z-w|^2}$ in Theorem 3.14.

This factor has a large impact on functions on $\mathbf{R}^d$ that are transformed back by the inverse of the Bargmann transform. For instance, if $\varepsilon > 0$, then the Bargmann image of any non-trivial Gelfand-Shilov space and its distribution space contain

$$\{ F \in A(\mathbf{C}^d) ; |F(z)| \lesssim e^{(\frac{1}{2}-\varepsilon)|z|^2} \}$$

and are contained in

$$\{ F \in A(\mathbf{C}^d) ; |F(z)| \lesssim e^{(\frac{1}{2}+\varepsilon)|z|^2} \}.$$

The same holds true for the Bargmann images of $\mathscr{S}(\mathbf{R}^d)$ and $\mathscr{S}'(\mathbf{R}^d)$.

Theorem 3.14 is a straight-forward consequence of the following two propositions, which give more details on the relationships between r and r_0 in (3.14), (3.15), (3.22) and (3.23).

Proposition 3.17. *Let $s \geqslant \frac{1}{2}$ and let $r_0, r \in (0, \infty)$ be such that*

$$r_0 \in (0, \infty) \quad \text{and} \quad r < \frac{r_0}{4(1 + r_0)}, \qquad \text{when} \quad s = \tfrac{1}{2}, \tag{3.24}$$

and

$$r_0 \in (0, \infty) \quad \text{and} \quad r \leqslant 2^{-\frac{1}{s}} r_0, \qquad \text{when} \quad s \in (\tfrac{1}{2}, \infty), \tag{3.25}$$

with strict inequality in (3.25) *when $s < 1$. If $a_0 \in L^\infty_{loc}(\mathbf{C}^d)$ satisfies* (3.14) *and $a_0^{\mathrm{aw}} \in \widehat{A}(\mathbf{C}^{2d})$ is given by* $(1.38)'$, *then* (3.22) *holds.*

Proposition 3.18. *Let $s \geqslant \frac{1}{2}$ and $r_0, r \in (0, \infty)$ be such that*

$$r_0 \in (0, 1) \quad \text{and} \quad r > \frac{r_0}{4(1 - r_0)}, \qquad \text{when} \quad s = \tfrac{1}{2}, \tag{3.24$'$}$$

and

$$r_0 \in (0, \infty) \quad \text{and} \quad r \geqslant 2^{-\frac{1}{s}} r_0, \qquad \text{when} \quad s \in (\tfrac{1}{2}, \infty), \tag{3.25$'$}$$

with strict inequality in $(3.25)'$ *when $s < 1$. If $a_0 \in L^\infty_{loc}(\mathbf{C}^d)$ satisfies* (3.15) *and $a_0^{\mathrm{aw}} \in \widehat{A}(\mathbf{C}^{2d})$ is given by* $(1.38)'$, *then* (3.23) *holds.*

For the proofs of Propositions 3.17 and 3.18 we use the inequalities

$$|z|^\theta - |w|^\theta \leqslant |z + w|^\theta \leqslant |z|^\theta + |w|^\theta, \qquad \theta \in (0, 1],\ z, w \in \mathbf{C}^d \tag{3.26}$$

$$|z + w|^\theta \leqslant (1 + \varepsilon)|z|^\theta + (1 + \varepsilon^{-1})|w|^\theta, \qquad \theta \in [1, 2],\ z, w \in \mathbf{C}^d, \tag{3.27}$$

and

$$|z + w|^\theta \geqslant (1 - \varepsilon)|z|^\theta + (1 - \varepsilon^{-1})|w|^\theta, \qquad \theta \in [1, 2],\ z, w \in \mathbf{C}^d, \tag{3.28}$$

for every $\varepsilon > 0$.

Proof of Proposition 3.17. Suppose that a_0 satisfies (3.14) for some $r_0 > 0$. First we consider the case $s > \frac{1}{2}$. If $s < 1$ let $\varepsilon_1 > 0$ and $\varepsilon_2 = \varepsilon_1^{-1}$, and if

$s \geqslant 1$ let $\varepsilon_1 = 0$ and $\varepsilon_2 = 2$, and let $c = 2^{-\frac{1}{s}}$. Then (1.38)$'$, (3.26) and (3.28) give

$$
\begin{aligned}
|a_0^{\mathrm{aw}}(z,w)| &\lesssim \int_{\mathbf{C}^d} e^{-r_0|w_1|^{\frac{1}{s}}} e^{-\operatorname{Re}(z-w_1,w-w_1)}\, d\lambda(w_1) \\
&= e^{\frac{1}{4}|z+w|^2-\operatorname{Re}(z,w)} \int_{\mathbf{C}^d} e^{-r_0|w_1|^{\frac{1}{s}}-|w_1-(z+w)/2|^2}\, d\lambda(w_1) \\
&= e^{\frac{1}{4}|z-w|^2} \int_{\mathbf{C}^d} e^{-r_0|w_1+(z+w)/2|^{\frac{1}{s}}-|w_1|^2}\, d\lambda(w_1) \\
&\leqslant e^{\frac{1}{4}|z-w|^2} e^{-cr_0(1-\varepsilon_1)|z+w|^{\frac{1}{s}}} \int_{\mathbf{C}^d} e^{-r_0(1-\varepsilon_2)|w_1|^{\frac{1}{s}}-|w_1|^2}\, d\lambda(w_1) \\
&\asymp e^{\frac{1}{4}|z-w|^2} e^{-cr_0(1-\varepsilon_1)|z+w|^{\frac{1}{s}}}. \qquad (3.29)
\end{aligned}
$$

If $s \geqslant 1$, then $\varepsilon_1 = 0$ and $\varepsilon_2 = 2$, and the result follows from (3.29). If instead $s < 1$, then the result follows by choosing $\varepsilon_1 > 0$ small enough, and we have proved the result in the case $s > \frac{1}{2}$.

Next suppose that $s = \frac{1}{2}$. For $\varepsilon_1 > 0$ and $\varepsilon_2 = \varepsilon_1^{-1}$ (3.29) gives

$$
|a_0^{\mathrm{aw}}(z,w)| \lesssim e^{\frac{1}{4}|z-w|^2} e^{-\frac{1}{4}r_0(1-\varepsilon_1)|z+w|^2} \int_{\mathbf{C}^d} e^{-(r_0(1-\varepsilon_2)+1)|w_1|^2}\, d\lambda(w_1).
$$

For any $\varepsilon_2 < \frac{1+r_0}{r_0}$ it follows that the integral converges, and

$$
1-\varepsilon_1 = 1-\varepsilon_2^{-1} < (1+r_0)^{-1}.
$$

By the assumptions there is $\delta > 0$ such that

$$
r = \frac{r_0(1-\delta)}{4(1+r_0)}.
$$

Since

$$
1-\varepsilon_1 \nearrow (1+r_0)^{-1} \quad \text{as} \quad \varepsilon_2 \nearrow \frac{1+r_0}{r_0}
$$

we may pick $0 < \varepsilon_2 < \frac{1+r_0}{r_0}$ such that

$$
\frac{1-\delta}{1+r_0} \leqslant 1-\varepsilon_1
$$

and the result follows in the case $s = \frac{1}{2}$. □

Proof of Proposition 3.18. First we consider the case when $s > \frac{1}{2}$. Suppose that a_0 satisfies (3.15) for some $r_0 > 0$, let $\varepsilon_1, \varepsilon_2 \geqslant 0$ be such that $\varepsilon_1 = \varepsilon_2 = 0$ when $s \geqslant 1$ and $\varepsilon_1\varepsilon_2 = 1$ when $s < 1$, and let $c = 2^{-\frac{1}{s}}$. Then (1.38)$'$, (3.26) and (3.27) give

$$
\begin{aligned}
|a_0^{\mathrm{aw}}(z,w)| &\lesssim \int_{\mathbf{C}^d} e^{r_0|w_1|^{\frac{1}{s}}} e^{-\operatorname{Re}(z-w_1,w-w_1)}\, d\lambda(w_1)\\
&= e^{\frac{1}{4}|z+w|^2-\operatorname{Re}(z,w)} \int_{\mathbf{C}^d} e^{r_0|w_1|^{\frac{1}{s}}-|w_1-(z+w)/2|^2}\, d\lambda(w_1)\\
&= e^{\frac{1}{4}|z-w|^2} \int_{\mathbf{C}^d} e^{r_0|w_1+(z+w)/2|^{\frac{1}{s}}-|w_1|^2}\, d\lambda(w_1)\\
&\leqslant e^{\frac{1}{4}|z-w|^2} e^{cr_0(1+\varepsilon_1)|z+w|^{\frac{1}{s}}} \int_{\mathbf{C}^d} e^{r_0(1+\varepsilon_2)|w_1|^{\frac{1}{s}}-|w_1|^2}\, d\lambda(w_1)\\
&\asymp e^{\frac{1}{4}|z-w|^2} e^{cr_0(1+\varepsilon_1)|z+w|^{\frac{1}{s}}}. \qquad (3.30)
\end{aligned}
$$

If $s \geqslant 1$, then $\varepsilon_1 = \varepsilon_2 = 0$, and the result follows from (3.30). If instead $s < 1$, then the result follows by choosing $\varepsilon_1 > 0$ small enough, and the result follows in the case $s > \frac{1}{2}$.

Next suppose that $s = \frac{1}{2}$. Then (3.30) gives

$$
|a_0^{\mathrm{aw}}(z,w)| \lesssim e^{\frac{1}{4}|z-w|^2} e^{\frac{1}{4}r_0(1+\varepsilon_1)|z+w|^2} \int_{\mathbf{C}^d} e^{r_0(1+\varepsilon_2)|w_1|^2-|w_1|^2}\, d\lambda(w_1).
$$

For any $\varepsilon_2 < \frac{1-r_0}{r_0}$ the integral converges, and

$$
1+\varepsilon_1 = 1+\varepsilon_2^{-1} > (1-r_0)^{-1}.
$$

Since

$$
1+\varepsilon_1 \searrow (1-r_0)^{-1} \quad \text{as} \quad \varepsilon_2 \nearrow \frac{1-r_0}{r_0},
$$

the result follows in the case $s = \frac{1}{2}$ by letting $r = \frac{r_0(1+\varepsilon_1)}{4}$. □

4. A lower bound for Wick operators

In this section we apply the asymptotic expansions in the previous section for Shubin-Wick operators to deduce a sharp Gårding inequality.

First we have the following result. We put $\widehat{\mathcal{A}}_{\mathrm{Sh},\rho}(\mathbf{C}^{2d}) = \widehat{\mathcal{A}}^{(\omega)}_{\mathrm{Sh},\rho}(\mathbf{C}^{2d})$ when $\omega = 1$.

Proposition 4.1. *Let $\omega \in \mathscr{P}(\mathbf{C}^d)$, $p \in [1,\infty]$, $a \in \widehat{\mathcal{A}}_{\mathrm{Sh},0}(\mathbf{C}^{2d})$ and $a_0 \in L^\infty(\mathbf{C}^d)$. Then $\operatorname{Op}_{\mathfrak{V}}(a)$ and $\operatorname{Op}^{\mathrm{aw}}_{\mathfrak{V}}(a_0)$ are both continuous on $A^p_{(\omega)}(\mathbf{C}^d)$.*

The claimed continuity of $\operatorname{Op}_{\mathfrak{V}}(a)$ is a straight-forward consequence of [30, Theorem 3.3], in combination with Proposition 2.1 and the relationship $K(z,w) = a(z,w)e^{(z,w)}$ between the kernel and symbol of a Wick operator (cf. (0.1)). In order to be self-contained we include an alternative and shorter proof.

Proof. Let $F \in A^p_{(\omega)}(\mathbf{C}^d)$, $G(z) = e^{-\frac{1}{2}|z|^2}|F(z)\omega(\sqrt{2}\overline{z})|$,

$$
\begin{aligned}
H_1(z) &= e^{-\frac{1}{2}|z|^2}|\operatorname{Op}_{\mathfrak{V}}(a)F(z)\omega(\sqrt{2}\overline{z})| \quad \text{and}\\
H_2(z) &= e^{-\frac{1}{2}|z|^2}|\operatorname{Op}^{\mathrm{aw}}_{\mathfrak{V}}(a_0)F(z)\omega(\sqrt{2}\overline{z})|.
\end{aligned}
$$

We have

$$\omega(\sqrt{2}\overline{z}) \lesssim \omega(\sqrt{2}\overline{w})\langle z-w\rangle^{N_0}$$

for some $N_0 \geqslant 0$. By Theorem 2.2 and (2.6) we get

$$H_1(z) \lesssim e^{-\frac{1}{2}|z|^2} \int_{\mathbf{C}^d} e^{\frac{1}{2}|z-w|^2} \langle z-w\rangle^{-N} |F(w)\omega(\sqrt{2}\overline{z})| e^{\operatorname{Re}(z,w)-|w|^2}\, d\lambda(w)$$

$$= (\langle\,\cdot\,\rangle^{N_0-N} * G)(z),$$

for every $N \geqslant 0$. By choosing $N > 2d + N_0$ and using Young's inequality we get $\|H_1\|_{L^p} \lesssim \|G\|_{L^p}$ which means $\|\operatorname{Op}_{\mathfrak{V}}(a)F\|_{A^p_{(\omega)}} \lesssim \|F\|_{A^p_{(\omega)}}$, and the asserted continuity for $\operatorname{Op}_{\mathfrak{V}}(a)$ follows.

In the same way we get

$$H_2(z) \lesssim \|a_0\|_{L^\infty} e^{-\frac{1}{2}|z|^2} \int_{\mathbf{C}^d} |F(w)\omega(\sqrt{2}\overline{w})| \langle z-w\rangle^{N_0} e^{\operatorname{Re}(z,w)-|w|^2}\, d\lambda(w)$$

$$\asymp ((\langle\,\cdot\,\rangle^{N_0} e^{-\frac{1}{2}|\,\cdot\,|^2}) * G)(z),$$

and another application of Young's inequality shows that $\|H_2\|_{L^p_{(\omega)}} \lesssim \|G\|_{L^p_{(\omega)}}$ that is $\|\operatorname{Op}^{\mathrm{aw}}_{\mathfrak{V}}(a_0)F\|_{A^p_{(\omega)}} \lesssim \|F\|_{A^p_{(\omega)}}$. □

We have finally a version of the sharp Gårding inequality.

Theorem 4.2. *Let $\rho > 0$, $\omega(z) = \langle z\rangle^{2\rho}$ and let $a \in \widehat{\mathcal{A}}^{(\omega)}_{\mathrm{Sh},\rho}(\mathbf{C}^{2d})$ be such that $a(w,w) \geqslant -C_0$ for all $w \in \mathbf{C}^d$, for some constant $C_0 \geqslant 0$. Then*

$$\operatorname{Re}\big((\operatorname{Op}_{\mathfrak{V}}(a)F,F)_{A^2}\big) \geqslant -C\|F\|^2_{A^2}, \qquad F \in \mathcal{A}_{\mathscr{S}}(\mathbf{C}^d) \tag{4.1}$$

and

$$\big|\operatorname{Im}\big((\operatorname{Op}_{\mathfrak{V}}(a)F,F)_{A^2}\big)\big| \leqslant C\|F\|^2_{A^2}, \qquad F \in \mathcal{A}_{\mathscr{S}}(\mathbf{C}^d) \tag{4.2}$$

for some constant $C \geqslant 0$.

Proof. Let $b_0(w) = a(w,w)$. Then $\operatorname{Op}_{\mathfrak{V}}(a) = \operatorname{Op}^{\mathrm{aw}}_{\mathfrak{V}}(b_0) + \operatorname{Op}_{\mathfrak{V}}(a_1)$ for some $a_1 \in \widehat{\mathcal{A}}_{\mathrm{Sh},\rho}(\mathbf{C}^{2d}) \subseteq \widehat{\mathcal{A}}_{\mathrm{Sh},0}(\mathbf{C}^{2d})$, in view of Proposition 3.3. Since $\Pi_A F = F$ for $F \in A^2(\mathbf{C}^d)$ (cf. (1.22)), the assumption $b_0 \geqslant -C_0$ implies $(\operatorname{Op}^{\mathrm{aw}}_{\mathfrak{V}}(b_0)F,F)_{A^2} \geqslant -C_0\|F\|^2_{A^2}$ for every $F \in \mathcal{A}_{\mathscr{S}}(\mathbf{C}^d)$. The operator $\operatorname{Op}_{\mathfrak{V}}(a_1)$ is continuous on $A^2(\mathbf{C}^d)$ in view of Proposition 4.1. A combination of these facts gives the result. □

5. Ellipticity and hypoellipticity for Shubin and Wick operators

In this section we show that the Bargmann assignment $\mathsf{S}_{\mathfrak{V}}$ maps the sets of hypoelliptic symbols and weakly elliptic symbols in the Shubin class $\mathrm{Sh}^{(\omega)}_\rho(\mathbf{R}^{2d})$ bijectively into the sets of hypoelliptic symbols and weakly elliptic Wick symbols in $\widehat{\mathcal{A}}^{(\omega)}_{\mathrm{Sh},\rho}(\mathbf{C}^{2d})$, respectively. Then we explain some consequences for polynomial symbols.

5.1. **Transition of weakly elliptic symbols.** For symbols in $\widehat{\mathcal{A}}^{(\omega)}_{\mathrm{Sh},\rho}(\mathbf{C}^{2d})$ we define ellipticity and weak ellipticity as follows.

Definition 5.1. Let $\rho > 0$, $\omega \in \mathscr{P}_{\mathrm{Sh},\rho}(\mathbf{C}^d)$ and $a \in \widehat{\mathcal{A}}^{(\omega)}_{\mathrm{Sh},\rho}(\mathbf{C}^{2d})$. Then a is called *weakly elliptic* of order $\rho_0 \geqslant 0$, or ρ_0-*weakly elliptic*, if for some $R > 0$

$$|a(z,z)| \gtrsim \langle z\rangle^{-\rho_0}\omega(\sqrt{2}\overline{z}), \quad |z| \geqslant R.$$

If a is weakly elliptic of order 0 then a is called *elliptic*.

Theorem 5.2. *Let $\omega \in \mathscr{P}(\mathbf{R}^{2d}) \simeq \mathscr{P}(\mathbf{C}^d)$, $\rho > 0$ and $\mathfrak{a} \in \mathrm{Sh}^{(\omega)}_\rho(\mathbf{R}^{2d})$. Then the following is true:*

(1) *if $z = x + i\xi$, $x, \xi \in \mathbf{R}^d$, then*

$$|\mathsf{S}_{\mathfrak{V}}\mathfrak{a}(z,z) - \mathfrak{a}(\sqrt{2}\,x, -\sqrt{2}\,\xi)| \lesssim \omega(\sqrt{2}\,\overline{z})\langle z\rangle^{-2\rho}; \tag{5.1}$$

(2) *if $\rho_0 \in [0, 2\rho)$, then $\mathsf{S}_{\mathfrak{V}}$ is bijective from the set of weakly elliptic symbols in $\mathrm{Sh}^{(\omega)}_\rho(\mathbf{R}^{2d})$ of order ρ_0 to the set of weakly elliptic symbols in $\widehat{A}^{(\omega)}_{\mathrm{Sh},\rho}(\mathbf{C}^{2d})$ of order ρ_0.*

As a consequence of (2) in the previous theorem we get the following.

Corollary 5.3. *Let $\mathfrak{a}$ be as in Theorem 5.2. Then the following is true:*

(1) *if $\rho_0 \in [0, 2\rho)$, then $\mathfrak{a} \in \mathrm{Sh}^{(\omega)}_\rho(\mathbf{R}^{2d})$ is weakly elliptic of order ρ_0, if and only if $\mathsf{S}_{\mathfrak{V}}\mathfrak{a} \in \widehat{A}^{(\omega)}_{\mathrm{Sh},\rho}(\mathbf{C}^{2d})$ is weakly elliptic of order ρ_0;*

(2) *$\mathfrak{a} \in \mathrm{Sh}^{(\omega)}_\rho(\mathbf{R}^{2d})$ is elliptic if and only if $\mathsf{S}_{\mathfrak{V}}\mathfrak{a} \in \widehat{A}^{(\omega)}_{\mathrm{Sh},\rho}(\mathbf{C}^{2d})$ is elliptic.*

For the proof of Theorem 5.2 we need the following proposition, related to Propositions 3.1 and 3.5.

Proposition 5.4. *Let $N \geqslant 0$ be an integer, $\rho \geqslant 0$, $\omega \in \mathscr{P}_{\mathrm{Sh},\rho}(\mathbf{R}^{2d}) \simeq \mathscr{P}_{\mathrm{Sh},\rho}(\mathbf{C}^d)$, $\omega_k(x,\xi) = \omega(x,\xi)\langle(x,\xi)\rangle^{-2\rho k}$ and $\mathfrak{a} \in \mathrm{Sh}^{(\omega)}_\rho(\mathbf{R}^{2d})$. Then for some $\mathfrak{c}_N \in \mathrm{Sh}^{(\omega_{N+1})}_\rho(\mathbf{R}^{2d})$ and constants $\{c_\alpha\}_{|\alpha|\leqslant 2N}$ with $c_0 = 1$, it holds*

$$\mathsf{S}_{\mathfrak{V}}\mathfrak{a}(2^{-\frac{1}{2}}z, 2^{-\frac{1}{2}}z) = \sum_{k=0}^{N}\mathfrak{a}_k(x,-\xi) + \mathfrak{c}_N(x,-\xi), \qquad \mathfrak{a}_k = \sum_{|\alpha|=2k} c_\alpha \partial^\alpha \mathfrak{a}. \tag{5.2}$$

Proof. Let ψ be as in Proposition 2.3. If we put $z = w$, then (2.9) and Taylor's formula give

$$\begin{aligned}
&(2\pi)^d \mathsf{S}_{\mathfrak{V}}\mathfrak{a}(2^{-\frac{1}{2}}z, 2^{-\frac{1}{2}}z) = (2\pi)^{\frac{3d}{2}}\mathcal{T}_\psi\mathfrak{a}(x,-\xi,0,0)\\
&= 2^d \iint_{\mathbf{R}^{2d}} \mathfrak{a}(t+x, \tau-\xi)e^{-(|t|^2+|\tau|^2)}\,dtd\tau = \sum_{k=0}^{2N+1}\mathfrak{b}_k(x,-\xi) + \mathfrak{c}(x,-\xi)
\end{aligned} \tag{5.3}$$

where

$$\mathfrak{b}_k(x,\xi) = \frac{2^d}{k!}\iint_{\mathbf{R}^{2d}} \langle \mathfrak{a}^{(k)}(x,\xi); (t,\tau), \dots, (t,\tau)\rangle e^{-(|t|^2+|\tau|^2)}\,dtd\tau$$

and

$$\mathfrak{c}(x,\xi) = \frac{1}{(2N+1)!}\int_0^1 (1-\theta)^{2N+1}\mathfrak{c}_\theta(x,\xi)\,d\theta,$$

with

$$\mathfrak{c}_\theta(x,\xi) = 2^d \iint_{\mathbf{R}^{2d}} \langle \mathfrak{a}^{(2N+2)}(x+\theta t,\xi+\theta\tau);(t,\tau),\dots,(t,\tau)\rangle e^{-(|t|^2+|\tau|^2)}\,dtd\tau.$$

If k is odd, then

$$(t,\tau)\mapsto \langle \mathfrak{a}^{(k)}(x,\xi);(t,\tau),\dots,(t,\tau)\rangle e^{-(|t|^2+|\tau|^2)}$$

is odd which implies that the integral is zero. Hence $\mathfrak{b}_k(x,\xi)=0$ when k is odd. For $k=0$ we observe that the integral for $\mathfrak{b}_0$ becomes

$$2^d\iint_{\mathbf{R}^{2d}} e^{-(|t|^2+|\tau|^2)}\,dtd\tau = (2\pi)^d,$$

and it follows from these relations that

$$(2\pi)^{-d}\sum_{k=0}^{2N+1}\mathfrak{b}_k = \sum_{k=0}^{N}\mathfrak{a}_k,$$

with $\mathfrak{a}_k$ as in (5.2) and $c_0=1$. Hence the result follows if we prove that the last term in (5.3) satisfies $\mathfrak{c}_N \in \operatorname{Sh}_\rho^{(\omega_{N+1})}(\mathbf{R}^{2d})$.

For $\theta\in[0,1]$ and $\alpha\in\mathbf{N}^{2d}$ we have

$$|\partial^\alpha \mathfrak{c}_\theta(x,\xi)| \lesssim \iint_{\mathbf{R}^{2d}} |\partial^\alpha\mathfrak{a}^{(2N+2)}(x+\theta t,\xi+\theta\tau)|\langle (t,\tau)\rangle^{2N+2}e^{-(|t|^2+|\tau|^2)}\,dtd\tau$$

$$\lesssim \iint_{\mathbf{R}^{2d}} \omega(x+\theta t,\xi+\theta\tau)\langle (x+\theta t,\xi+\theta\tau)\rangle^{-(2N+2+|\alpha|)\rho}\langle (t,\tau)\rangle^{2N+2}e^{-(|t|^2+|\tau|^2)}\,dtd\tau$$

$$\lesssim \omega(x,\xi)\langle (x,\xi)\rangle^{-(2N+2+|\alpha|)\rho}\iint_{\mathbf{R}^{2d}}\langle (t,\tau)\rangle^{N_0}e^{-(|t|^2+|\tau|^2)}\,dtd\tau$$

$$\asymp \omega(x,\xi)\langle (x,\xi)\rangle^{-(2N+2+|\alpha|)\rho}$$

for some $N_0>0$. In the last inequality we have used the fact that ω is polynomially moderate.

This implies

$$|\partial^\alpha\mathfrak{c}(x,\xi)| \lesssim \int_0^1 |\partial^\alpha \mathfrak{c}_\theta(x,\xi)|\,d\theta \lesssim \omega(x,\xi)\langle (x,\xi)\rangle^{-(2N+2+|\alpha|)\rho},$$

which shows that $\mathfrak{c},\mathfrak{c}_N\in\operatorname{Sh}_\rho^{(\omega_{N+1})}(\mathbf{R}^{2d})$. □

Proof of Theorem 5.2. Let ψ be as in Proposition 2.3 and $N=0$ in Proposition 5.4. Then

$$|\mathsf{S}_{\mathfrak{V}}\mathfrak{a}(2^{-\frac12}z,2^{-\frac12}z)-\mathfrak{a}(x,-\xi)| \lesssim \omega(x,-\xi)\langle (x,-\xi)\rangle^{-2\rho}, \qquad (5.1)'$$

and (1) follows.

Suppose $\rho_0 \in [0, 2\rho)$. Then it follows from the latter inequality that

$$|\mathsf{S}_{\mathfrak{V}}\mathfrak{a}(z,z)| \gtrsim \langle z\rangle^{-\rho_0}\omega(\sqrt{2}\,\overline{z}), \qquad |z| \geqslant R$$

for some $R > 0$, if and only if

$$|\mathfrak{a}(x,\xi)| \gtrsim \langle (x,\xi)\rangle^{-\rho_0}\omega(x,\xi), \qquad |z| \geqslant R$$

for some $R > 0$, and the asserted equivalence in (2) follows. □

5.2. **Shubin hypoellipticity in a Wick setting.**

Definition 5.5. Let $\rho > 0$, $\rho_0 \geqslant 0$, $\omega \in \mathscr{P}_{\mathrm{Sh},\rho}(\mathbf{C}^d)$ and $a \in \widehat{\mathcal{A}}^{(\omega)}_{\mathrm{Sh},\rho}(\mathbf{C}^{2d})$. Then a is called *hypoelliptic* (in the *Shubin-Wick sense* in $\widehat{\mathcal{A}}^{(\omega)}_{\mathrm{Sh},\rho}(\mathbf{C}^{2d})$) of order ρ_0, if there is an $R > 0$ such that for every $\alpha, \beta \in \mathbf{N}^d$, it holds

$$|\partial_z^\alpha \overline{\partial}_w^\beta a(z,z)| \lesssim |a(z,z)|\langle z\rangle^{-\rho|\alpha+\beta|}, \qquad |z| \geqslant R. \tag{5.4}$$

and

$$|a(z,z)| \gtrsim \omega_0(\sqrt{2}\overline{z})\langle z\rangle^{-\rho_0}, \qquad |z| \geqslant R. \tag{5.5}$$

According to Definition 1.11, if ω, ρ and ρ_0 are as in the definition, then $\mathfrak{a} \in \mathrm{Sh}^{(\omega)}_\rho(\mathbf{R}^{2d})$ is hypoelliptic of order ρ_0 means that there is an $R > 0$ such that for every $\alpha \in \mathbf{N}^{2d}$, it holds

$$|\partial^\alpha \mathfrak{a}(x,\xi)| \lesssim |\mathfrak{a}(x,\xi)|\langle (x,\xi)\rangle^{-\rho|\alpha|}, \qquad |(x,\xi)| \geqslant R. \tag{5.6}$$

and

$$|\mathfrak{a}(x,\xi)| \gtrsim \omega(x,\xi)\langle (x,\xi)\rangle^{-\rho_0}, \qquad |(x,\xi)| \geqslant R. \tag{5.7}$$

Similar to Theorem 5.2 we have the following.

Theorem 5.6. *Let $\rho > 0$, $\rho_0 \geqslant 0$, $\omega \in \mathscr{P}_{\mathrm{Sh},\rho}(\mathbf{R}^{2d}) \simeq \mathscr{P}_{\mathrm{Sh},\rho}(\mathbf{C}^d)$, $\mathfrak{a} \in \mathrm{Sh}^{(\omega)}_\rho(\mathbf{R}^{2d})$ and $a = \mathsf{S}_{\mathfrak{V}}\mathfrak{a}$. Then $\mathfrak{a}$ is hypoelliptic of order ρ_0 in $\mathrm{Sh}^{(\omega)}_\rho(\mathbf{R}^{2d})$, if and only if a is* hypoelliptic *of order ρ_0 in $\widehat{\mathcal{A}}^{(\omega)}_{\mathrm{Sh},\rho}(\mathbf{C}^{2d})$.*

Proof. Suppose that $\mathfrak{a} \in \mathrm{Sh}^{(\omega)}_\rho(\mathbf{R}^{2d})$ is hypoelliptic of order ρ_0, and choose $N \geqslant 0$ such that $2N\rho > \rho_0$. Suppose that $R > 0$ is chosen such that (5.6) and (5.7) are fulfilled. Then Proposition 5.4 gives for $z = x + i\xi$ with $|z| \geqslant R$ where $R > 0$ is sufficiently large

$$\begin{aligned}
|a(2^{-\frac{1}{2}}z, 2^{-\frac{1}{2}}z)| &\gtrsim |\mathfrak{a}(x,-\xi)| - \sum_{k=1}^{N}\sum_{|\alpha|=2k}(|\partial^\alpha\mathfrak{a}(x,-\xi)| + |\mathfrak{c}(x,-\xi)|) \\
&\gtrsim |\mathfrak{a}(x,-\xi)| - |\mathfrak{a}(x,-\xi)|\langle (x,-\xi)\rangle^{-2\rho} - \omega(x,-\xi)\langle (x,-\xi)\rangle^{-\rho(2N+2)} \\
&\gtrsim |\mathfrak{a}(x,-\xi)| - |\mathfrak{a}(x,-\xi)|\langle (x,-\xi)\rangle^{-2\rho} \\
&\gtrsim |\mathfrak{a}(x,-\xi)| \gtrsim \omega(x,-\xi)\langle (x,-\xi)\rangle^{-\rho_0},
\end{aligned}$$

and (5.5) follows. In particular it follows from the previous estimates that

$$|a(2^{-\frac{1}{2}}z, 2^{-\frac{1}{2}}z)| \gtrsim |\mathfrak{a}(x,-\xi)|, \qquad |z| \geqslant R. \tag{5.8}$$

For fixed $\alpha,\beta\in\mathbf{N}^d$, let Ω_k be the set of all $(\gamma,\delta)\in\mathbf{N}^{2d}\times\mathbf{N}^{2d}$ such that $|\gamma|=2k$ and $|\delta|=|\alpha+\beta|$. By Proposition 5.4 and (5.8) we have for some R large enough and $|z|\geqslant R$,

$$|(\partial_z^\alpha\overline{\partial}_w^\beta a)(2^{-\frac{1}{2}}z,2^{-\frac{1}{2}}z)|\lesssim\sum_{k=0}^N\sum_{(\gamma,\delta)\in\Omega_k}(|\partial^{\gamma+\delta}\mathfrak{a}(x,-\xi)|+|\partial^\delta\mathfrak{c}(x,-\xi)|)$$

$$\lesssim\sum_{k=0}^N\sum_{(\gamma,\delta)\in\Omega_k}\big(|\mathfrak{a}(x,-\xi)|\langle(x,-\xi)\rangle^{-\rho(2k+|\alpha+\beta|)}+\omega(x,-\xi)\langle(x,-\xi)\rangle^{-\rho(2N+|\alpha+\beta|)}\big)$$

$$\asymp|\mathfrak{a}(x,-\xi)|\langle(x,-\xi)\rangle^{-\rho|\alpha+\beta|}+\omega(x,-\xi)\langle(x,-\xi)\rangle^{-\rho(2N+|\alpha+\beta|)}$$

$$\lesssim|\mathfrak{a}(x,-\xi)|\langle(x,-\xi)\rangle^{-\rho|\alpha+\beta|}+|\mathfrak{a}(x,-\xi)|\langle(x,-\xi)\rangle^{\rho_0-\rho(2N+|\alpha+\beta|)}$$

$$\asymp|\mathfrak{a}(x,-\xi)|\langle(x,-\xi)\rangle^{-\rho|\alpha+\beta|}\lesssim|a(2^{-\frac{1}{2}}z,2^{-\frac{1}{2}}z)|\langle(x,-\xi)\rangle^{-\rho|\alpha+\beta|},$$

which implies that (5.4) holds.

This shows that a is hypoelliptic of order ρ_0 in $\widehat{\mathcal{A}}^{(\omega)}_{\mathrm{Sh},\rho}(\mathbf{C}^{2d})$ when $\mathfrak{a}$ is hypoelliptic of order ρ_0 in $\mathrm{Sh}^{(\omega)}_\rho(\mathbf{R}^{2d})$.

Suppose instead that a is hypoelliptic of order ρ_0 in $\widehat{\mathcal{A}}^{(\omega)}_{\mathrm{Sh},\rho}(\mathbf{C}^{2d})$. By using Proposition 3.5, (3.12) and (3.13) instead of Proposition 5.4, similar computations as in the first part of the proof shows that (5.6) and (5.7) hold for some $R>0$. This shows that $\mathfrak{a}$ is hypoelliptic of order ρ_0 in $\mathrm{Sh}^{(\omega)}_\rho(\mathbf{R}^{2d})$ when a is hypoelliptic of order ρ_0 in $\widehat{\mathcal{A}}^{(\omega)}_{\mathrm{Sh},\rho}(\mathbf{C}^{2d})$, and the result follows. □

5.3. **Ellipticity in the case of polynomial symbols.** Next we discuss ellipticity for polynomial symbols, i. e.

$$\mathfrak{a}(x,\xi)=\sum_{|\alpha+\beta|\leqslant N}c(\alpha,\beta)x^\alpha\xi^\beta,\qquad x,\xi\in\mathbf{R}^d,\tag{5.9}$$

and

$$a(z,w)=\sum_{|\alpha+\beta|\leqslant N}c(\alpha,\beta)z^\alpha\overline{w}^\beta,\quad z,w\in\mathbf{C}^d.\tag{5.10}$$

The corresponding principal symbols are

$$\mathfrak{a}_p(x,\xi)=\sum_{|\alpha+\beta|=N}c(\alpha,\beta)x^\alpha\xi^\beta,\qquad x,\xi\in\mathbf{R}^d,\tag{5.11}$$

and

$$a_p(z,w)=\sum_{|\alpha+\beta|=N}c(\alpha,\beta)z^\alpha\overline{w}^\beta,\quad z,w\in\mathbf{C}^d,\tag{5.12}$$

respectively.

First we relate polynomials on $\mathbf{R}^{2d}$ to Shubin classes.

Proposition 5.7. *Let $\mathfrak{a}$ and $\mathfrak{a}_p$ be as in* (5.9) *and* (5.11) *for some $c(\alpha,\beta)\in\mathbf{C}$, $\alpha,\beta\in\mathbf{N}^d$ and $N\geqslant 0$, and let $\omega_N(x,\xi)=\langle(x,\xi)\rangle^N$, $x,\xi\in\mathbf{R}^d$. Then the following is true:*

(1) $\mathfrak{a} \in \operatorname{Sh}_1^{(\omega_N)}(\mathbf{R}^{2d})$;

(2) $\mathfrak{a}$ *is elliptic with respect to* ω_N*, if and only if* $\mathfrak{a}_p(x,\xi) \neq 0$ *when* $(x,\xi) \neq 0$.

The result can be considered folklore. In order to be self-contained we present the arguments.

Proof. First we prove (1). Let $t = \max(|x_1|,\dots,|x_d|,|\xi_1|,\dots,|\xi_d|)$ when $x = (x_1,\dots,x_d) \in \mathbf{R}^d$ and $\xi = (\xi_1,\dots,\xi_d) \in \mathbf{R}^d$. Then

$$|\mathfrak{a}(x,\xi)| \leqslant \sum_{|\alpha+\beta|\leqslant N_0} |c(\alpha,\beta)| t^{|\alpha+\beta|} \lesssim 1 + t^N \leqslant \langle (x,\xi)\rangle^N,$$

which gives the desired bound for $|\mathfrak{a}(x,\xi)|$. Since the degree of a polynomial is lowered by at least one for every differentiation we get

$$|\partial^\alpha \mathfrak{a}(x,\xi)| \lesssim \langle (x,\xi)\rangle^{N-|\alpha|}$$

for every $\alpha \in \mathbf{N}^{2d}$, which gives (1).

In order to prove (2) we let $\mathfrak{a}_p$ be as in (5.11). First suppose that $\mathfrak{a}_p(x,\xi) \neq 0$ when $(x,\xi) \neq (0,0)$, and let g be the continuous function on $\mathbf{R}^{2d}\backslash 0$ given by

$$g(x,\xi) = \frac{|\mathfrak{a}_p(x,\xi)|}{|(x,\xi)|^N}, \qquad (x,\xi) \neq (0,0).$$

Since g is continuous and positive, and the sphere

$$\mathbf{S}^{2d-1} = \{\, (x,\xi) \in \mathbf{R}^{2d}\, ;\, |x|^2 + |\xi|^2 = 1 \,\}$$

is compact, it follows that there are constants $c_1, c_2 > 0$ such that

$$c_1 \leqslant g(x,\xi) \leqslant c_2, \qquad (x,\xi) \in \mathbf{S}^{2d-1}.$$

By homogeneity it now follows

$$c_1 |(x,\xi)|^N \leqslant |\mathfrak{a}_p(x,\xi)| \leqslant c_2 |(x,\xi)|^N, \qquad x,\xi \in \mathbf{R}^d.$$

Hence, if

$$\mathfrak{b}(x,\xi) = \mathfrak{a}(x,\xi) - \mathfrak{a}_p(x,\xi) = \sum_{|\alpha+\beta|\leqslant N-1} c(\alpha,\beta) x^\alpha \xi^\beta,$$

then the first part of the proof implies that for some constants $C > 0$ and $R > 0$ we have

$$|\mathfrak{a}(x,\xi)| \geqslant |\mathfrak{a}_p(x,\xi)| - |\mathfrak{b}(x,\xi)| \geqslant c_1 |(x,\xi)|^{N_0} - C\langle (x,\xi)\rangle^{N-1} \gtrsim \langle (x,\xi)\rangle^N$$

when $|(x,\xi)| \geqslant R$. Hence $\mathfrak{a}$ is elliptic with respect to ω_{N_0}.

Suppose instead $\mathfrak{a}_p(x_0,\xi_0) = 0$ for some $(x_0,\xi_0) \neq (0,0)$. For any $(x,\xi) = (tx_0, t\xi_0)$ we have

$$\begin{aligned} |\mathfrak{a}(x,\xi)| \leqslant |\mathfrak{a}_p(x,\xi)| + |\mathfrak{b}(x,\xi)| &= |t^N \mathfrak{a}_p(x_0,\xi_0)| + |\mathfrak{b}(x,\xi)| \\ &= |\mathfrak{b}(x,\xi)| \lesssim \langle (x,\xi)\rangle^{N-1}, \end{aligned}$$

giving that $|\mathfrak{a}(x,\xi)| \gtrsim \langle (x,\xi)\rangle^N$, $|(x,\xi)| \geqslant R$, cannot hold for any $R > 0$. □

By Theorems 5.2, 5.6 and Proposition 5.7 we get the following. The details are left for the reader.

Proposition 5.8. *Let a and a_p be as in* (5.10) *and* (5.12) *for some $c(\alpha,\beta)\in \mathbf{C}$, $\alpha,\beta\in\mathbf{N}^d$ and $N\geqslant 0$, and let $\omega_N(x,\xi)=\langle(x,\xi)\rangle^N$, $x,\xi\in\mathbf{R}^d$. Then the following is true:*

(1) *$a\in\mathcal{A}^{(\omega_N)}_{\mathrm{Sh},1}(\mathbf{C}^{2d})$;*

(2) *a is elliptic in $\mathcal{A}^{(\omega_N)}_{\mathrm{Sh},1}(\mathbf{C}^{2d})$ if and only if $a_p(z,z)\neq 0$ when $z\neq 0$.*

Remark 5.9. Let $\mathfrak{a}$, $\mathfrak{a}_p$, a and a_p be as in (5.9)–(5.12). Then it follows from Propositions 5.7 and Proposition 5.8 that $\mathfrak{a}$ is elliptic, if and only if $\mathfrak{a}_p$ is elliptic, and that a is elliptic, if and only if a_p is elliptic.

We have now the following.

Theorem 5.10. *Let $\mathfrak{a}\in\mathrm{Sh}_1^{(\omega_N)}(\mathbf{R}^{2d})$ and $\mathfrak{a}_p$ be as in* (5.9) *and* (5.11) *for some $c(\alpha,\beta)\in\mathbf{C}$, $\alpha,\beta\in\mathbf{N}^d$ and $N\geqslant 0$. Then the following is true:*

(1) *the principal symbol $a_p(z,w)$ of $\mathsf{S}_{\mathfrak{V}}\mathfrak{a}$ is given by*

$$a_p(z,w)=2^{-\frac{N}{2}}\sum_{|\alpha+\beta|=N}c(\alpha,\beta)i^{|\beta|}(z+\overline{w})^\alpha(z-\overline{w})^\beta; \tag{5.13}$$

(2) *$\mathfrak{a}$ is elliptic in $\mathrm{Sh}_1^{(\omega_N)}(\mathbf{R}^{2d})$ if and only if a_p is elliptic in $\mathcal{A}^{(\omega_N)}_{\mathrm{Sh},1}(\mathbf{C}^{2d})$;*

(3) *$\mathfrak{a}_p(x,\xi)>0$ for every $(x,\xi)\neq(0,0)$, if and only if $a_p(z,z)>0$ for every $z\neq 0$.*

Proof. Let $z=x+i\xi$, $x,\xi\in\mathbf{R}^d$, i. e. $x=\frac{1}{2}(z+\overline{z})$ and $\xi=\frac{1}{2i}(z-\overline{z})$. By Theorem 5.2 we get

$$a_p(2^{-\frac{1}{2}}z,2^{-\frac{1}{2}}z)=\mathfrak{a}_p(x,-\xi). \tag{5.14}$$

This implies

$$\begin{aligned}a_p(z,z)&=2^{\frac{N}{2}}\sum_{|\alpha+\beta|=N}c(\alpha,\beta)x^\alpha(-\xi)^\beta\\ &=2^{\frac{N}{2}}\sum_{|\alpha+\beta|=N}c(\alpha,\beta)2^{-|\alpha|}(z+\overline{z})^\alpha(2i)^{-|\beta|}(-(z-\overline{z}))^\beta,\end{aligned}$$

which gives

$$a_p(z,z)=2^{-\frac{N}{2}}\sum_{|\alpha+\beta|=N}c(\alpha,\beta)i^{|\beta|}(z+\overline{z})^\alpha(z-\overline{z})^\beta. \tag{5.13$'$}$$

The formula (5.13) now follows from (5.13)$'$ and analytic continuation, using the fact that $a_p(z,w)$ is analytic in z and conjugate analytic in w.

The assertion (2) follows by a combination of Corollary 5.3, Propositions 5.7 and 5.8, and the assertion (3) is a direct consequence of (5.14). □

6. A necessary condition for polynomially bounded Wick symbols

In [13, Section 2.7] Folland shows that polynomial symbols for pseudo-differential operators correspond to polynomial Wick and anti-Wick symbols. Thus partial differential operators with polynomial coefficients corresponds to polynomial Wick symbols.

Here we show that a Wick symbol that is polynomially bounded must be a polynomial. This gives a characterization of Wick symbols corresponding to polynomial symbols for pseudo-differential operators.

Cauchy's integral formula implies that an entire function which is polynomially bounded must be a polynomial:

Proposition 6.1. *Let $F \in A(\mathbf{C}^d)$ have Maclaurin series*

$$F(z) = \sum_{\alpha \in \mathbf{N}^d} c(\alpha) e_\alpha(z), \quad z \in \mathbf{C}^d.$$

Suppose that for some $j \in \{1, \dots, d\}$, $C > 0$, $N \geqslant 0$, and an open neighbourhood $I \subseteq \mathbf{C}$ of the origin we have

$$|F(z)| \leqslant C \langle z_j \rangle^N, \quad z_j \in \mathbf{C},$$

provided $z_k \in I$, $k \in \{1, \dots, d\} \setminus \{j\}$. Then $c(\alpha) = 0$ when $\alpha_j > N$.

Proof. By interchanging the variables, we may assume that $j = d$. Let $R \geqslant 1$ and $\varepsilon > 0$ be chosen such that

$$D_\varepsilon \equiv \{ z_0 \in \mathbf{C} \, ; \, |z_0| \leqslant \varepsilon \} \subseteq I.$$

Take $\alpha \in \mathbf{N}^d$ such that $\alpha_d > N$, let $\beta = (\alpha_1 + 1, \dots, \alpha_d + 1) \in \mathbf{N}^d$ and $\gamma_\varepsilon \subseteq \mathbf{C}$ be the boundary circle of D_ε. Then Cauchy's integral formula gives

$$\frac{|c(\alpha)|}{\alpha!^{\frac{1}{2}}} = \left| \frac{\partial^\alpha F(0)}{\alpha!} \right| = (2\pi)^{-d} \left| \int \cdots \int_{\gamma_\varepsilon^{d-1}} \left(\int_{|z_d| = R} \frac{F(z)}{z^\beta} \, dz_d \right) dz_1 \cdots dz_{d-1} \right|$$

$$\leqslant (2\pi)^{-d} \int \cdots \int_{\gamma_\varepsilon^{d-1}} \left(\int_{|z_d| = R} \frac{|F(z)|}{|z^\beta|} \, |dz_d| \right) |dz_1| \cdots |dz_{d-1}|$$

$$\lesssim R^{-\alpha_d} \langle R \rangle^N \varepsilon^{-(\alpha_1 + \cdots + \alpha_{d-1})} \to 0$$

as $R \to \infty$. □

Corollary 6.2. *Let $a \in \widehat{A}(\mathbf{C}^{2d})$ and suppose*

$$|a(z, w)| \lesssim \langle (z, w) \rangle^N \tag{6.1}$$

for some $N \geqslant 0$. Then a is a polynomial in $z \in \mathbf{C}^d$ and $\overline{w} \in \mathbf{C}^d$ of degree at most N.

Proof. By Proposition 6.1 it follows that a is a polynomial of degree at most $2dN$. We need to prove that the degree is at most N. In order to do this we may assume that a has degree at least one.

For some integer $M \geqslant 1$ we have

$$a(z, w) = a_M(z, w) + a_{M-1}(z, w),$$

where

$$a_M(z, w) = \sum_{|\alpha + \beta| = M} c(\alpha, \beta) z^\alpha \overline{w}^\beta$$

is non-trivial and

$$a_{M-1}(z, w) = \sum_{|\alpha + \beta| \leqslant M-1} c(\alpha, \beta) z^\alpha \overline{w}^\beta.$$

Since a_M is non-trivial, there are $z_0, w_0 \in \mathbf{C}^d$ such that $|z_0|^2 + |w_0|^2 = 1$ and $|a_M(z_0, w_0)| = c_0 \neq 0$. By homogeneity we get

$$|a_M(tz_0, tw_0)| = c_0|t|^M, \qquad t \in \mathbf{R}.$$

In the same way we get

$$|a_{M-1}(tz_0, tw_0)| \leqslant C(1 + |t|)^{M-1}, \qquad t \in \mathbf{R}$$

for some constant C which is independent of t.

Suppose contrary to the assumption that $M > N$. For $t \in \mathbf{R}$ with $|t| \geqslant 1$ we have

$$\left| \frac{a(tz_0, tw_0)}{\langle (tz_0, tw_0) \rangle^N} \right| \gtrsim |t|^{-N} \left(|a_M(tz_0, tw_0)| - |a_{M-1}(tz_0, tw_0)| \right)$$

$$\geqslant |t|^{-N} \left(c_0|t|^M - C(1 + |t|)^{M-1} \right) \to \infty \quad \text{as} \quad |t| \to \infty.$$

This contradicts (6.1), and the hence our assumption that $M > N$ must be false. □

Appendix

In this appendix we present some tables on weights, operators, spaces of entire functions on $\mathbf{C}^d$ and Wick symbol classes.

In the first two tables we review weight classes, transforms, and operators. Recall

$$\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}, \quad d\mu(w) = \pi^{-d} e^{-|w|^2} \, d\lambda(w),$$

where $d\lambda(w)$ is the Lebesgue measure on $\mathbf{C}^d$, and let $đy = (2\pi)^{-d} dy$.

Weight class	Features	Eq. ref.
$\mathscr{P}_E(\mathbf{R}^d)$	$\omega \in L^\infty_{loc}(\mathbf{R}^d; \mathbf{R}_+)$, $\omega(x + y) \lesssim \omega(x) e^{r\|y\|}$	(1.5)
$\mathscr{P}(\mathbf{R}^d)$	$\omega \in L^\infty_{loc}(\mathbf{R}^d; \mathbf{R}_+)$, $\omega(x + y) \lesssim \omega(x) \langle y \rangle^N$	(1.1)′
$\mathscr{P}_{\mathrm{Sh},\rho}(\mathbf{R}^d)$	$\omega \in \mathscr{P}(\mathbf{R}^d) \ \wedge \ \|\partial^\alpha \omega(x)\| \lesssim \omega(x) \langle x \rangle^{-\rho\|\alpha\|}$	(1.46)

Table 1: Weight classes.

Operator	Notation	Features	Eq. ref.
Modul. STFT	$\mathcal{T}$	$f \mapsto \int f(y + x) \overline{\phi(y)} e^{-i\langle y, \xi \rangle} đy$	(1.10)′
Bargm. transf.	$\mathfrak{V}_d$	$f \mapsto \pi^{-\frac{d}{4}} \int e^{-\frac{1}{2}(\langle z,z \rangle + \|y\|^2) + 2^{1/2}\langle z,y \rangle} f(y) \, dy$	(1.19)
Semi-conj. op.	Θ	$K(z, w) \mapsto K(z, \overline{w})$	(1.29)
Pseudo-diff. op.	$\mathrm{Op}_A(\mathfrak{a})$	$f \mapsto \iint \mathfrak{a}(x - A(x - y), \xi) f(y) e^{i\langle x-y, \xi \rangle} đy d\xi$	(1.31)
Wick op.	$\mathrm{Op}_{\mathfrak{V}}(a)$	$F \mapsto \int_{\mathbf{C}^d} a(z, w) F(w) e^{(z,w)} \, d\mu(w)$	(1.36)
Anti-Wick op.	$\mathrm{Op}^{\mathrm{aw}}_{\mathfrak{V}}(a)$	$F \mapsto \int_{\mathbf{C}^d} a(w) F(w) e^{(z,w)} \, d\mu(w)$	(1.40)
Bargm. assignm.	$\mathsf{S}_{\mathfrak{V}}$	$\mathrm{Op}_{\mathfrak{V}}(\mathsf{S}_{\mathfrak{V}} \mathfrak{a}) = \mathfrak{V}_d \circ \mathrm{Op}_A(\mathfrak{a}) \circ \mathfrak{V}_d^*, \quad A = \frac{1}{2} I$	(1.41)

Table 2: Operators and transforms.

The next two tables deal with properties of the Bargmann images of Gelfand-Shilov function spaces, the Schwartz space, and their distribution spaces. Recall

$$|z|_{s,\sigma} = |\operatorname{Re} z|^{\frac{1}{s}} + |\operatorname{Im} z|^{\frac{1}{\sigma}}, \qquad z \in \mathbf{C}^d.$$

Function space	Bargmann image	$\lvert\mathfrak{V}_d f(z)\rvert \lesssim$	Eq. ref.
$\mathcal{S}_s^\sigma(\mathbf{R}^d), s, \sigma \geqslant \frac{1}{2}$	$\mathcal{A}_s^\sigma(\mathbf{C}^d)$	$e^{\frac{\lvert z\rvert^2}{2} - r\lvert z\rvert_{s,\sigma}}, \exists r > 0$	(1.6), (1.24), (1.25)
$\Sigma_s^\sigma(\mathbf{R}^d), s, \sigma > \frac{1}{2}$	$\mathcal{A}_{0,s}^\sigma(\mathbf{C}^d)$	$e^{\frac{\lvert z\rvert^2}{2} - r\lvert z\rvert_{s,\sigma}}, \forall r > 0$	(1.6), (1.24), (1.25)
$\mathscr{S}(\mathbf{R}^d)$	$\mathcal{A}_{\mathscr{S}}(\mathbf{C}^d)$	$e^{\frac{\lvert z\rvert^2}{2}} \langle z\rangle^{-N}, \forall N \geqslant 0$	(1.6), (1.24), (1.25)

Table 3: The Bargmann images of test function spaces.

Distribution space	Bargmann image	$\lvert\mathfrak{V}_d f(z)\rvert \lesssim$	Eq. ref.
$(\mathcal{S}_s^\sigma)'(\mathbf{R}^d), s, \sigma \geqslant \frac{1}{2}$	$(\mathcal{A}_s^\sigma)'(\mathbf{C}^d)$	$e^{\frac{\lvert z\rvert^2}{2} + r\lvert z\rvert_{s,\sigma}}, \forall r > 0$	(1.6), (1.24), (1.25)
$(\Sigma_s^\sigma)'(\mathbf{R}^d), s, \sigma > \frac{1}{2}$	$(\mathcal{A}_{0,s}^\sigma)'(\mathbf{C}^d)$	$e^{\frac{\lvert z\rvert^2}{2} + r\lvert z\rvert_{s,\sigma}}, \exists r > 0$	(1.6), (1.24), (1.25)
$\mathscr{S}'(\mathbf{R}^d)$	$\mathcal{A}'_{\mathscr{S}}(\mathbf{C}^d)$	$e^{\frac{\lvert z\rvert^2}{2}} \langle z\rangle^{N}, \exists N \geqslant 0$	(1.6), (1.24), (1.25)

Table 4: The Bargmann images of distribution spaces.

For the links between the Shubin class $\mathrm{Sh}_\rho^{(\omega)}(\mathbf{R}^{2d})$ (see (1.47)), and the symbol classes $\Gamma_{s,\sigma}^{\sigma,s;0}(\mathbf{R}^{2d})$, $\Gamma_{s,\sigma;0}^{\sigma,s}(\mathbf{R}^{2d})$ and $\Gamma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$ (see Definition 1.8) we have the following table. Here

$$\partial^\alpha a = \partial_x^{\alpha_1} \partial_\xi^{\alpha_3} \partial_y^{\alpha_2} \partial_\eta^{\alpha_4} a,$$

$$z = x + i\xi,\ w = y + i\eta,\ \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbf{N}^{2d},$$

when $a \in \widehat{A}(\mathbf{C}^{2d})$ (see (2.15)), and $\omega_r(z) = \omega(z)\langle z\rangle^{-r}$ when $\omega \in\in \mathscr{P}_{\mathrm{Sh},\rho}(\mathbf{C}^d)$ and $r \in \mathbf{R}$.

Wick class	$\lvert\partial^\alpha a(z,w)\rvert \lesssim$	Ref.
$\mathsf{S}_{\mathfrak{V}}(\mathrm{Sh}_0^{(\omega)}(\mathbf{R}^{2d}))$	$e^{\frac{1}{2}\lvert z-w\rvert^2} \omega(\sqrt{2}\,\overline{z}) \langle z - w\rangle^{-N}\ \forall N \geqslant 0,\ \alpha = 0$	Thm. 2.2
$\mathsf{S}_{\mathfrak{V}}(\mathrm{Sh}_\rho^{(\omega)}(\mathbf{R}^{2d}))$	$e^{\frac{1}{2}\lvert z-w\rvert^2} \omega_{\rho\lvert\alpha\rvert}(\sqrt{2}\,\overline{z}) \langle z - w\rangle^{-N},\ \alpha \in \mathbf{N}^{4d}$	Thm. 2.5
$\mathsf{S}_{\mathfrak{V}}(\Gamma_{s,\sigma;0}^{\sigma,s}(\mathbf{R}^{2d}))$	$e^{\frac{1}{2}\lvert z-w\rvert^2 + r_1\lvert z+w\rvert_{s,\sigma} - r_2\lvert z-w\rvert_{s,\sigma}}, \exists r_2 > 0, \forall r_1 > 0,\ \alpha = 0$	Thm. 2.6
$\mathsf{S}_{\mathfrak{V}}(\Gamma_{s,\sigma}^{\sigma,s;0}(\mathbf{R}^{2d}))$	$e^{\frac{1}{2}\lvert z-w\rvert^2 + r_1\lvert z+w\rvert_{s,\sigma} - r_2\lvert z-w\rvert_{s,\sigma}}, \exists r_1 > 0, \forall r_2 > 0,\ \alpha = 0$	Thm. 2.6
$\mathsf{S}_{\mathfrak{V}}(\Gamma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d}))$	$e^{\frac{1}{2}\lvert z-w\rvert^2 + r_1\lvert z+w\rvert_{s,\sigma} - r_2\lvert z-w\rvert_{s,\sigma}}, \exists r_1, r_2 > 0,\ \alpha = 0$	Thm. 2.6

Table 5: Estimates for Wick symbol classes

In Table 5, recall that $\widehat{\mathcal{A}}^{(\omega)}_{\mathrm{Sh},\rho} = \mathsf{S}_{\mathfrak{V}}(\mathrm{Sh}^{(\omega)}_{\rho}(\mathbf{R}^{2d}))$. For $\mathsf{S}_{\mathfrak{V}}(\Gamma^{\sigma,s}_{s,\sigma;0}(\mathbf{R}^{2d}))$ it is assumed that $s,\sigma \geqslant \frac{1}{2}$, while for $\mathsf{S}_{\mathfrak{V}}(\Gamma^{\sigma,s;0}_{s,\sigma}(\mathbf{R}^{2d}))$ and $\mathsf{S}_{\mathfrak{V}}(\Gamma^{\sigma,s}_{s,\sigma}(\mathbf{R}^{2d}))$ it is assumed that $s,\sigma > \frac{1}{2}$.

References

[1] A. Abdeljawad, M. Cappiello, J. Toft, *Pseudo-differential calculus in anisotropic Gelfand-Shilov setting*, Integr. Equ. Oper. Theory **91** (2019), 91–26.

[2] A. Abdeljawad, S. Coriasco, N. Teofanov, *Bilinear Pseudo-differential Operators with Gevrey–Hörmander Symbols*, Mediterr. J. Math. **17** 120 (2020).

[3] A. Abdeljawad, C. Fernandez, A. Galbis, J. Toft, R Üster, *Characterizations of Pilipović spaces by powers of harmonic oscillator*, RACSAM **114** (2020), 131.

[4] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform*, Comm. Pure Appl. Math. **14** (1961), 187–214.

[5] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform. Part II. A family of related function spaces. Application to distribution theory*, Comm. Pure Appl. Math. **20** (1967), 1–101.

[6] F. A. Berezin, *Wick and anti-Wick symbols of operators*, Mat. Sb. (N.S.) **86** (1971), 578–610.

[7] F. A. Berezin, *Covariant and contravariant symbols of operators*, Izv. Akad. Nauk SSSR **36** (1972), 1134–1167.

[8] J. M. Bony, J. Y. Chemin *Espaces Functionnels Associés au Calcul de Weyl-Hörmander*, Bull. Soc. math. France **122** (1994), 77–118.

[9] M. Cappiello, R. Schulz, P. Wahlberg, *Conormal distributions in the Shubin calculus of pseudodifferential operators*, J. Math. Phys. **59** (2018), 021502.

[10] M. Cappiello, J. Toft, *Pseudo-differential operators in a Gelfand–Shilov setting*, Math. Nachr. **290** (5-6) (2017), 738–755.

[11] E. Cordero, S. Pilipović, L. Rodino, N. Teofanov, *Quasianalytic Gelfand-Shilov spaces with applications to localization operators*, Rocky Mountain. J. Math. **40** (2010), 1123-1147.

[12] P. Boggiatto, E. Buzano, L. Rodino *Global Hypoellipticity and Spectral Theory*, Akademie Verlag, Berlin, 1996.

[13] G. B. Folland, *Harmonic analysis in phase space*, Princeton University Press, Princeton, 1989.

[14] I. M. Gelfand, G. E. Shilov, *Generalized functions, II-III*, Academic Press, NewYork London, 1968.

[15] K. Gröchenig, *Foundations of time-frequency analysis*, Birkhäuser, Boston, 2001.

[16] K. Gröchenig, *Weight functions in time-frequency analysis,* in: L. Rodino, M. W. Wong (Eds.) Pseudodifferential Operators: Partial Differential Equations and Time-Frequency Analysis, Fields Institute Comm., **52** (2007), pp. 343–366.

[17] K. Gröchenig, G. Zimmermann, *Spaces of test functions via the STFT* J. Funct. Spaces Appl. **2** (2004), 25–53.

[18] B. C. Hall, *Quantum Theory for Mathematicians*, Springer, New York, 2013.

[19] L. Hörmander, *The analysis of linear partial differential operators*, vol I–III, Springer-Verlag, Berlin Heidelberg NewYork Tokyo, 1983, 1985.

[20] E. H. Lieb, *Integral bounds for radar ambiguity functions and Wigner distributions*, J. Math. Phys. **31** (1990), 594–599.

[21] E. H. Lieb, *The stability of matter from atoms to stars*, Bull. Amer. Math. Soc. **22** (1990), 1–49.

[22] E. H. Lieb, J. P. Solovej, *Quantum coherent operators: a generalization of coherent states*, Lett. Math. Phys. **22** (1991), 145–154.

[23] Z. Lozanov Crvenković, D. Perišić, *Hermite expansions of elements of Gelfand Shilov spaces in quasianalytic and non quasianalytic case*, Novi Sad J. Math. **37** (2007), 129–147.

[24] B. Prangoski, *Pseudodifferential operators of infinite order in spaces of tempered ultradistributions*, J. Pseudo-Differ. Oper. Appl. **4** (2013), 495–549.

[25] M. Reed and B. Simon, *Methods of modern mathematical physics*, Academic Press, London New York, 1979.
[26] L. Rodino, P. Wahlberg, *Microlocal analysis of Gelfand–Shilov spaces*, arXiv:2202.05543 [math.AP] (2022).
[27] M. A. Shubin, *Pseudodifferential operators and spectral theory*, Springer, 2001.
[28] N. Teofanov, *Ultradistributions and time-frequency analysis.* In "Pseudo-differential operators and related topics", 173–192, Oper. Theory Adv. Appl., **164**, Birkhäuser, Basel, 2006.
[29] N. Teofanov, *Gelfand-Shilov spaces and localization operators*, Funct. Anal. Approx. Comput. **7** (2015), 135–158.
[30] N. Teofanov, J. Toft, *Pseudo-differential calculus in a Bargmann setting*, Ann. Acad. Sci. Fenn. Math. **45** (2020), 227–257.
[31] J. Toft, *The Bargmann transform on modulation and Gelfand-Shilov spaces, with applications to Toeplitz and pseudo-differential operators*, J. Pseudo-Differ. Oper. Appl. **3** (2012), 145–227.
[32] J. Toft, *Images of function and distribution spaces under the Bargmann transform*, J. Pseudo-Differ. Oper. Appl. **8** (2017), 83–139.
[33] G. C. Wick, *The evaluation of the collision matrix*, Phys. Rev. **80** (1950), 268–272.

Department of Mathematics and Informatics, Faculty of Sciences University of Novi Sad, Novi Sad, Serbia
Email address: nenad.teofanov@dmi.uns.ac.rs

Department of Mathematics, Linnæus University, Växjö, Sweden
Email address: joachim.toft@lnu.se

Dipartimento di Scienze Matematiche, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy
Email address: patrik.wahlberg@polito.it