ASYMPTOTIC OF THE DISCRETE VOLUME-PRESERVING FRACTIONAL MEAN CURVATURE FLOW VIA A NONLOCAL QUANTITATIVE ALEXANDROV THEOREM

DANIELE DE GENNARO, ANDREA KUBIN, AND ANNA KUBIN

Abstract. We characterize the long time behaviour of a discrete-in-time approximation of the volume preserving fractional mean curvature flow. In particular, we prove that the discrete flow starting from any bounded set of finite fractional perimeter converges exponentially fast to a single ball if the dimension $N \leq 7$ and the fractional esponent $s \approx 1$, or for any $s \in (0,1)$ when $N = 2$. As an intermediate result we establish a fractional quantitative Alexandrov type estimate for normal deformations of a ball. Finally, we provide existence for flat flows as limit points of the discrete flow when the time discretization parameter tends to zero. Furthermore, analogous results for the classical perimeter are obtained in the limit $s \to 1$.

Keywords: Geometric evolutions; fractional mean curvature; Alexandrov Theorem; minimizing movements; variational methods.

AMS subject classifications: 35R11; 53E10; 49M25; 49Q20.

Contents

Introduction

We consider the geometric evolution of sets called *the volume preserving fractional mean curvature flow*. The *classical mean curvature flow* is defined as a flow of sets $(E_t)_{0\leq t\leq T}$ in $\mathbb{R}^N$ following the motion law

$$v_t = -H_{E_t} \quad \text{on} \quad \partial E_t,$$

where v_t denotes the component of the velocity relative to the outer normal vector to ∂E_t and H_E is the mean curvature of the set E. In order to include the volume constraint, one can consider the following velocity

$$v_t = \bar{H}_{E_t} - H_{E_t} \quad \text{on} \quad \partial E_t$$

for all $t \in [0,T]$, where $\bar{H}_{E_t}$ denotes the average of H_{E_t} over ∂E_t. The defined geometric evolution is called *volume preserving mean curvature flow*, as one can observe that the volume of the evolving sets is constant.

In the fractional setting, the velocity of the flow is given by the *fractional mean curvature*, a geometric quantity introduced by Caffarelli, Roquejoffre and Savin in [5] and defined as the first variation of the fractional perimeter functional. The latter functional is defined on a measurable set $E \subset \mathbb{R}^N$ simply as

$$P^s(E) = \int_E \int_{E^c} \frac{1}{|x-y|^{N+s}} \, \mathrm{d}x \, \mathrm{d}y.$$

It turns out that its first variation on any C^2 set E is given by the formula

$$H^s_E(x) = \int_{\mathbb{R}^N} \frac{\chi_E(y) - \chi_{E^c}(y)}{|x-y|^{N+s}} \, \mathrm{d}y, \quad x \in \partial E.$$

In both the previous formulae, the integrals are intended in the principal value sense. In analogy with the classical case, the evolution law for the *volume preserving fractional mean curvature flow* is given by

$$v_t = \bar{H}^s_{E_t} - H^s_{E_t} \quad \text{on} \quad \partial E_t, \tag{0.1}$$

with the notations previously introduced.

Up to now, a satisfactory study of this type of evolution is still missing. While the evolution without the volume constraint is well-understood (see e.g. [10, 22]), the lack of a comparison principle in our case makes the study much harder. Moreover, the generated flow may present singularities of different kinds, as happens for the classical mean curvature flow: see [12] for some explicit examples of pinch-like singularities. In [23] short-time existence is proved for the smooth flow (0.1), while existence of the smooth flow starting from convex sets (under suitable assumptions) is provided in [11].

We will then follow the approach of the celebrated papers by Almgren, Taylor and Wang [2] and Luckhaus and Sturzenhecker [26] consisting in building a discrete-in-time approximation of the flow via a variational approach and then sending the time discretization parameter to zero. Our approach follows closely the work done by Mugnai, Seis and Spadaro in [28], where the variational problem studied incorporates a volume penalization to take into account the volume constraint. First of all we define a discrete-in-time approximation of the flow that will be called the *discrete flow*. Given any initial set E_0, with $|E_0| = m$, and a time-step $h > 0$ we define $E_0^{(h)} := E_0$ and, iteratively, for $n \geq 0$ we set

$$E^{(h)}_{n+1} \in \operatorname{argmin} \left\{ P^s(F) + \frac{1}{h} \int_F \mathrm{sd}_{E_n^{(h)}}(x) \, \mathrm{d}x + \frac{1}{h^{\frac{s}{1+s}}} ||F| - m| : F \subset \mathbb{R}^N \text{ measurable} \right\},$$

where $\mathrm{sd}_{E_n^{(h)}}$ is the signed distance function from the set $E_n^{(h)}$. We can define for every $t \geq 0$, the discrete flow by $E^{(h)}(t) := E^{(h)}_{[t/h]}$. We will prove that such a flow is well-defined. Any L^1_{loc}-limit point of this flow as the time-step h converges to zero will be called a *flat flow*. For the classical mean curvature flow, under the hypothesis of convergence of the perimeters, this approach produces global-in-time distributional solutions of the evolution law (0.1), as shown in [28]. In the fractional case, we fall short of this result, since we lack some regularity results needed to characterize the evolution law of a flat flow. Moreover, from a technical point of view, proving the uniform boundness of the discrete flow in the fractional setting is nontrivial. Nonetheless, we will prove in the Appendix the existence of flat flows, defined simply as $L^1_{loc}-$limit points of the discrete flow as $h \to 0$, and address some of its continuity properties. Moreover, under the additional hypothesis of uniform boundedness of the flow in some time interval, we will prove that the flat flow is volume-preserving.

In the recent years, the study of the long time behaviour of the volume preserving mean curvature flow has attracted more and more attention. In the local case, after some classical studies [18, 21], in a recent paper [27] the authors proved the asymptotic behaviour of the classical discrete flow by showing its convergence to unions of equal balls. Then, they improved their results in [24], proving uniform estimates with respects to the time parameter h in dimension $N = 2$, thus obtaining the same result for the flat flow. In this regard, we want to mention that it is not clear if the same result also holds for the fractional evolution. The proof of the aforementioned paper relies on a "global" quantitative Alexandrov theorem, holding for general sets (and not just normal deformations of a ball). It would be interesting to try to extend some of the local techniques used in the aforementioned paper to the nonlocal one, even at least in dimension $N = 2$. Even the situation in the periodic setting is quite studied, with results for the discrete flow in [14] and for the smooth flow in dimensions $N = 3, 4$ in [29]. In this case, it has been proved the asymptotic stability of the so-called *strictly stable sets,* namely periodic sets having constant mean curvature and strictly positive second variation of the perimeter (apart from a finite subspace of degenerate directions). In the aforementioned papers it is proved that if the flow considered starts suitably close to a strictly stable set then it will converge exponentially fast to a translate of the latter. In the fractional setting some recent results have been proved. For example, in [11] the authors prove that the smooth flow starting from a convex set converges to a ball, up to translations possibly depending on time and under the hypothesis of equiboundedness for the fractional curvatures along the flow.

In this paper the long-time convergence analysis for the discrete flow is developed in the fractional setting. The main result of the paper is Theorem 0.1 below. It provides a complete characterization of the long-time behaviour of the discrete fractional mean curvature flow starting from any bounded set of finite fractional perimeter, providing also an estimate on the convergence speed. We will assume that the dimension N is such that any $\Lambda-$minimizer of the fractional perimeter is a smooth set. Namely, we will assume that either:

- $N = 2$;
- $N \leq 7$ and $s \in (s_0, 1)$, where s_0 is the constant of Proposition 3.1, item *ii).*

This is a technical hypothesis that could be dropped if, for example, we knew that the evolving sets were smooth. In particular, it is essential to characterize the possible long-time limit points for the discrete flow. In the local case such characterization has been proved in [15]. Before stating our main Theorem, We briefly give some definitions. Set B the ball of radius one and centered at the origin, and let $f : \partial B \to \mathbb{R}$ be a function with $\|f\|_{L^\infty(\partial B)}$ sufficiently small. The *normal deformation* B_f of the set B induced by f is defined as

$$\partial B_f := \{x(1 + f(x)) \ : \ x \in \partial B\}. \tag{0.2}$$

A normal deformation B_f is said to be of class C^k if $f \in C^k(\partial B)$.

Theorem 0.1. *Let m, $M > 0$ and let E_0 be an initial bounded set with $P^s(E_0) \leq M$, $|E_0| = m$. Then, for $h = h(s, M, m) > 0$ small enough the following holds: for any discrete flow $E_n^{(h)}$ starting from E_0, there exists $\xi \in \mathbb{R}^N$ such that*

$$E_n^{(h)} - \xi \to B^{(m)} \quad \text{as } n \to \infty \text{ in } C^k$$

for all $k \in \mathbb{N}$, where $B^{(m)}$ denotes the ball centered at the origin with volume equal to m. Moreover, the convergence is exponentially fast, meaning that there exist functions $f_n \in C^\infty(B^{(m)})$ such that $E_n^{(h)} - \xi = B_{f_n}^{(m)}$ and $\|f_n\|_{C^k(\partial B^{(m)})} \leq c_k e^{-c_k n}$, for some constants c_k depending on k, m and M.

We stress the difference between our result and the one holding in the classic setting, where the limit points of the discrete flow are in general unions of disjointed balls having the same radius and not necessarily only a single ball. This is a peculiar feature of the nonlocal perimeter considered, that penalizes non-connected components.

A crucial intermediate result consists in generalizing the Alexandrov-type estimate [27, Theorem 1.3] and [14, Theorem 1.3] (see also [25]) to the fractional setting. This result provides a stability inequality for normal deformations of balls which can be seen as a sharp Łojasiewicz-Simon inequality.

Theorem 0.2 (Theorem 1.1 in [27])**.** *There exist $\delta \in (0, 1/2)$ and $C > 0$ with the following property: for any $f \in C^1(\partial B) \cap H^2(\partial B)$ such that $\|f\|_{C^1(\partial B)} \leq \delta$, $|B_f| = \omega_N$ and* $\mathrm{bar}(B_f) =$

$\fint_{B_f} x\,\mathrm{d}x = 0$, *we have*

$$\|f\|_{H^1(\partial B)} \leq C\|H_{B_f} - \bar{H}_{B_f}\|_{L^2(\partial B)}.$$

We are able to extend the previous result to the fractional setting. Namely, we obtain the following.

Theorem 0.3. *There exist $\delta = \delta(N) > 0$ with the following property: for any $f \in C^2(\partial B)$ such that $\|f\|_{C^1(\partial B)} \leq \delta$, $|B_f| = \omega_N$ and $\mathrm{bar}(B_f) = \fint_{B_f} x\,\mathrm{d}x = 0$, and for any $s \in (0,1)$, there exists $C = C(N,s) > 0$ such that*

$$\|f\|_{H^{\frac{1+s}{2}}(\partial B)} \leq C\|H^s_{B_f} - \bar{H}^s_{B_f}\|_{L^2(\partial B)},$$

where we have set $\bar{H}^s_{B_f} := \fint_{\partial B} H^s_{B_f}(x + f(x)x)\,\mathrm{d}\mathcal{H}^{N-1}(x)$. Furthermore, there exists $s^* > 1$ such that for every $s \in (s^*, 1)$ it holds

$$(1-s)\|f\|^2_{H^{\frac{1+s}{2}}(\partial B)} \leq C(1-s)^2\|H^s_{B_f} - \bar{H}^s_{B_f}\|^2_{L^2(\partial B)}, \tag{0.3}$$

for a dimensional constant C.

In particular, we recover Theorem 0.2 as a corollary of our result, see Remark 2.4. The proof of the previous theorem follows closely the proof of the quantitative Alexandrov type estimate obtained in the flat torus and contained in [14]. In particular, the approach is based on some Taylor approximations of the factor $\bar{H}^s_{B_f} - H^s_{B_f}(x)$ combined with the coercivity of the second variation of the fractional perimeter, proved in [19]. The additional regularity assumption $f \in C^2$ is technical and needed to properly define $H^s_{B_f}$.

After this work was completed, we were informed that a similar quantitative Alexandrov type estimate has been independently proved in [9]. In this paper the authors use this result to prove the stability of spheres for the smooth evolution (0.1).

Notation

We work in the Euclidian space $\mathbb{R}^N$, with $N \geq 2$. We denote with $|\cdot|$ the standard Lebesgue measure in $\mathbb{R}^N$, $\mathrm{M}(\mathbb{R}^N)$ is the family of measurable set of $\mathbb{R}^N$. We denote with E^c the complement of a set $E \subset \mathbb{R}^N$. We denote by $\mathcal{H}^{N-1}$ the Hausdorff measure, and sometimes we denote $\mathrm{d}\mathcal{H}^{N-1}_x := \mathrm{d}\mathcal{H}^{N-1}(x)$. If E is a set with C^1 boundary the outer normal to E at a point x in ∂E is denoted by $\nu = \nu_E(x)$. We denote the ball of radius r and center x both as $B(x,r)$ and $B_r(x)$, and we set $B = B(0,1)$. Also, with $B^{(m)}$ we denote the ball centered at zero and having volume $|B^{(m)}| = m$. Let f be a real valued function, with $O(f)$ we will denote the family of all function g such that $|g| \leq C|f|$. Finally, we denote by $C(*,\cdots,*)$ a constant that depends on $*,\cdots,*$; such a constant may change from line to line.

1. Preliminaries

Let $s \in (0,1)$ we define the s-fractional perimeter as the following function

$$P^s : \mathrm{M}(\mathbb{R}^N) \to [0,+\infty], \quad P^s(E) := \int_E \int_{E^c} \frac{1}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2}[\chi_E]^2_{H^{\frac{s}{2}}}.$$

More in general, for every $E, F \in \mathrm{M}(\mathbb{R}^N)$ we set

$$\mathcal{L}_s(E,F) := \int_E \int_F \frac{1}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y$$

and, for any bounded set Ω, we define the fractional perimeter of E relative to Ω as

$$P^s(E;\Omega) := \mathcal{L}_s(E\cap\Omega, E^c\cap\Omega) + \mathcal{L}_s(E\cap\Omega, E^c\setminus\Omega) + \mathcal{L}_s(E\setminus\Omega, E^c\cap\Omega).$$

Let $E \in \mathrm{M}(\mathbb{R}^N)$ be a set of class C^2. Given a vector field $X \in C^1_c(\mathbb{R}^N;\mathbb{R}^N)$, let

$$\Phi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N, \quad \Phi(t,x) = x + tX(x).$$

We recall that the first variation of the s-fractional perimeter of E in the direction of X is given by

$$\partial P^s(E)[X] \coloneqq \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P^s(\Phi(t,E)) = \int_{\partial E} H^s_E(x) X(x) \cdot \nu_E(x)\, \mathrm{d}\mathcal{H}^{N-1}_x,$$

where $H^s_E(x)$ is the s-fractional mean curvature of E evaluated at $x \in \partial E$, that is

$$H^s_E(x) \coloneqq \int_{\mathbb{R}^N} \frac{\chi_E(y) - \chi_{E^c}(y)}{|x-y|^{N+s}}\, \mathrm{d}y,$$

where the integral has to be intended in the principal value sense. Applying the divergence theorem in the above formula, with $\operatorname{div}(-\frac{1}{s}\frac{\xi}{|\xi|^{N+s}}) = \frac{1}{|\xi|^{N+s}}$, the fractional curvature can be written as

$$H^s_E(p) = \frac{1}{s} \int_{\partial E} \frac{(x-p)\cdot \nu_E(x)}{|x-p|^{N+s}}\, \mathrm{d}\mathcal{H}^{N-1}(x) \quad \forall\, p \in \partial E.$$

We recall some useful results concerning sets of finite fractional perimeter. The proofs of the following results can be found, respectively, in [5, Proposition 3.1], [17, Theorem 7.1] and [16, Lemma 2.5].

Proposition 1.1 (Lower semi-continuity)**.** *Let $\{E_n\}_{n\in\mathbb{N}} \subset \mathrm{M}(\mathbb{R}^N)$ such that $\chi_{E_n} \to \chi_E$ in L^1_{loc}, as $n \to +\infty$, for some $E \in \mathrm{M}(\mathbb{R}^N)$. Then, for all $s \in (0,1)$, we have*

$$P^s(E) \leq \liminf_{n\to+\infty} P^s(E_n).$$

Theorem 1.2 (Compactness)**.** *If $R > 0$ and $\{E_n\}_{n\in\mathbb{N}} \subset \mathrm{M}(\mathbb{R}^N)$, with*

$$E_n \subset B(0,R) \quad \forall n \in \mathbb{N} \quad \text{and} \quad \sup_{n\in\mathbb{N}} P^s(E_n) < +\infty,$$

then, up to a subsequence, $E_n \to E$ in $L^1(\mathbb{R}^N)$, where $E \subset B(0,R)$ and $P^s(E) < +\infty$.

Theorem 1.3. *(Relative isoperimetric inequality) Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz continuous boundary and let $E \in \mathrm{M}(\mathbb{R}^N)$. Then there exists a constant $C = C(s,N,\Omega) > 0$ such that*

$$P^s(E,\Omega) \geq \mathcal{L}_s(E\cap\Omega, E^c\cap\Omega) \geq C\min\left\{|E\cap\Omega|^{\frac{N-s}{N}}, |E\setminus\Omega|^{\frac{N-s}{N}}\right\}.$$

We recall the following convergence theorems. The first one concerns the convergence of the fractional perimeter to the classical one and its proof can be found in [6, Theorem 1].

Theorem 1.4. *Let E be a bounded set of class $C^{1,\alpha}$ for $\alpha \in (0,1)$. Then,*

$$\lim_{s\to1^-}(1-s)P^s(E) = \omega_{N-1}P(E).$$

The second one relates to the convergence of the fractional curvatures. It was proved in a more general setting in [1, 6, 7].

Theorem 1.5. *Let E be a bounded set of class C^2. Then,*

$$\lim_{s\to1^-}(1-s)H^s_E = \omega_{N-1}H_E$$

uniformly on ∂E.

Finally, we recall the pointwise convergence of the fractional Gagliardo seminorms to the Sobolev one. The classical proof is contained in [3, Corollary 2], see also [20, Proposition 3.7] for the same result in a more general setting. Here and in the following with ∇ we denote the tangential gradient on a hypersurface.

Theorem 1.6. *Assume $f \in H^s(\partial B)$. Then*

$$\lim_{s\to 1^-}(1-s)[f]^2_{H^{\frac{1+s}{2}}(\partial B)} = C\|\nabla f\|^2_{L^2(\partial B)},$$

where $C>0$ is a constant that depends only on N.

2. A fractional quantitative Alexandrov type estimate

In this section, we are going to prove the quantitative Alexandrov inequality Theorem 0.3 in the nonlocal setting of the fractional perimeter. From now on we set

$$[f]^2_{\frac{1+s}{2}} \coloneqq [f]^2_{H^{\frac{1+s}{2}}(\partial B)} = \int_{\partial B}\int_{\partial B} \frac{|f(x)-f(y)|^2}{|x-y|^{N+s}}\,\mathrm{d}\mathcal{H}^{N-1}_x\,\mathrm{d}\mathcal{H}^{N-1}_y.$$

We start by recalling representation formulas for the s-fractional perimeter and its first variation on smooth sets. As usual, given a vector field $X \in C^1_c(\mathbb{R}^N;\mathbb{R}^N)$, we define the first variation of the fractional perimeter of a C^2 set E with respect to X as

$$\partial P^s(E)[X] \coloneqq \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P^s(\Phi(t,E)),$$

where $\Phi : \mathbb{R}^N \times (-1,1) \to \mathbb{R}^N$ is the flow defined by $\Phi(x,t) \coloneqq x + tX$. Analogously, we define the second variation of the fractional perimeter of a C^2 set E with respect to X as

$$\partial^2 P^s(E)[X] \coloneqq \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} P^s(\Phi(t,E)).$$

For a normal deformation B_f of B induced by a function $f \in C^1(\partial B)$, see (0.2), and for every function $\psi \in C^1(\partial B)$, with a slight abuse of notation, we set

$$\partial P^s(B_f)[\psi] \coloneqq \partial P^s(B_f)[X] = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P^s(B_{f+t\psi}),$$

where the field X is defined by

$$X(x) = \psi\left(\frac{x}{|x|}\right)\frac{x}{|x|}, \qquad x \in \mathbb{R}^N \setminus \{0\}.$$

Lemma 2.1. *The following equalities hold true:*

(1) *If $f \in C^2(\partial B)$ with $\|f\|_\infty$ sufficiently small, then*

$$\begin{aligned} P^s(B_f) = {} & \frac{P^s(B)}{P(B)}\int_{\partial B}(1+f)^{N-s}\,\mathrm{d}\mathcal{H}^{N-1} + \\ & + \frac{1}{2}\int_{\partial B}\int_{\partial B}\int_{1+f(y)}^{1+f(x)}\int_{1+f(y)}^{1+f(x)} F_{|x-y|}(r,\rho)\,\mathrm{d}r\,\mathrm{d}\rho\,\mathrm{d}\mathcal{H}^{N-1}_x\,\mathrm{d}\mathcal{H}^{N-1}_y, \end{aligned} \tag{2.1}$$

where, for every $\theta, r, \rho \in (0,+\infty)$, we have set

$$F_\theta(r,\rho) := \frac{r^{N-1}\rho^{N-1}}{((r-\rho)^2 + r\rho\theta^2)^{\frac{N+s}{2}}}.$$

(2) *If $f \in C^2(\partial B)$ with $\|f\|_\infty$ sufficiently small, then, for every $\psi \in C^1(\partial B)$, we have*

$$\begin{aligned} \partial P^s(B_f)[\psi] = {} & (N-s)\frac{P^s(B)}{P(B)}\int_{\partial B}(1+f)^{N-s-1}\psi\,\mathrm{d}\mathcal{H}^{N-1} \\ & + \int_{\partial B}\int_{\partial B}\int_{f(y)}^{f(x)}\left(\psi(x)F_{|x-y|}(1+f(x),1+\rho) - \psi(y)F_{|x-y|}(1+f(y),1+\rho)\right). \end{aligned} \tag{2.2}$$

Proof. By explicit computations one can obtain equation (2.1), see for example the calculations in the proof of [19, Theorem 2.1]. To prove (2.2), we take the derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P^s(B_{f+t\psi})$$

in formula (2.1) and, recalling that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\alpha(t)}^{\beta(t)}\int_{\alpha(t)}^{\beta(t)} f(r,\rho)\,\mathrm{d}\rho\,\mathrm{d}r\right] &= \int_{\alpha(t)}^{\beta(t)} (f(\beta(t),\rho)\beta'(t) - f(\alpha(t),\rho)\alpha'(t))\,\mathrm{d}\rho \\ &\quad + \int_{\alpha(t)}^{\beta(t)} (f(r,\beta(t))\beta'(t) - f(r,\alpha(t))\alpha'(t))\,\mathrm{d}r\end{aligned}$$

for every function α, $\beta : \mathbb{R} \to \mathbb{R}$ of class C^1 and $f \in L^1_{loc}(\mathbb{R}\times\mathbb{R})$, we conclude

$$\begin{aligned}\partial P^s(B_f)[\psi] = &\int_{\partial B}\int_{\partial B}\int_{1+f(y)}^{1+f(x)} \big(\psi(x)F_{|x-y|}(1+f(x),\rho) - \psi(y)F_{|x-y|}(1+f(y),\rho)\big)\,\mathrm{d}\rho \\ &+ (N-s)\frac{P^s(B)}{P(B)}\int_{\partial B}(1+f)^{N-s-1}\psi\,\mathrm{d}\mathcal{H}^{N-1}.\end{aligned}$$

A simple change of coordinates then yields the thesis. $\square$

Lemma 2.2. *If $f \in C^2(\partial B)$ with $\|f\|_{C^1(\partial B)} \le \delta$ sufficiently small, then we have*

$$\partial P^s(B_f)[1] = (N-s)\frac{P^s(B)}{P(B)}\int_{\partial B}\big(1+(N-s-1)f+O(f^2))\big)\,\mathrm{d}\mathcal{H}^{N-1} + O([f]^2_{\frac{1+s}{2}}), \tag{2.3}$$

$$\begin{aligned}\partial P^s(B_f)[f] =&(N-s)\frac{P^s(B)}{P(B)}\int_{\partial B}\big(1+(N-s-1)f+O(f^2))\big)\,f\,\mathrm{d}\mathcal{H}^{N-1} \\ &+\int_{\partial B}\int_{\partial B}\frac{(f(x)-f(y))^2}{|x-y|^{N+s}} + O([f]^2_{\frac{1+s}{2}})\|f\|_{C^1}.\end{aligned} \tag{2.4}$$

Proof. Let $\psi \in C^1(\partial B)$, we remark that, by expanding the first term in (2.2), we obtain

$$\begin{aligned}\partial P^s(B_f)[\psi] =&(N-s)\frac{P^s(B)}{P(B)}\int_{\partial B}(1+(N-s-1)f+O(f^2))\psi\,\mathrm{d}\mathcal{H}^{N-1} \\ &+\int_{\partial B}\int_{\partial B}\int_{f(y)}^{f(x)}\big(\psi(x)F_{|x-y|}(1+f(x),1+\rho) - \psi(y)F_{|x-y|}(1+f(y),1+\rho)\big)\,\mathrm{d}\rho.\end{aligned}$$

By simmetry, using a change of variables in the formula above, we get

$$\begin{aligned}\partial P^s(B_f)[\psi] =&(N-s)\frac{P^s(B)}{P(B)}\int_{\partial B}(1+(N-s-1)f+O(f^2))\psi\,\mathrm{d}\mathcal{H}^{N-1} \\ &+2\int_{\partial B}\int_{\partial B}\int_{f(y)}^{f(x)}\psi(x)F_{|x-y|}(1+f(x),1+\rho)\,\mathrm{d}\rho\,\mathrm{d}\mathcal{H}^{N-1}_x\,\mathrm{d}\mathcal{H}^{N-1}_y.\end{aligned} \tag{2.5}$$

We remark that, fixed $x, y \in \partial B$ and $x \neq y$, if $\|f\|_{C^1} \le \delta$ is sufficiently small, and if ρ varies between the values $f(y)$ and $f(x)$, then we have $|f(x)-\rho| \le \|\nabla f\|_\infty|x-y| \le \delta|x-y|$. From this observation we can expand the denominator of $F_{|x-y|}(1+f(x),1+\rho)$ and get

$$\begin{aligned}&|(f(x)-\rho)^2 + (1+f(x))(1+\rho)|x-y|^2|^{-\frac{N+s}{2}} \\ &= \frac{1}{|x-y|^{N+s}}((f(x)-\rho)^2/|x-y|^2 + f(x)+\rho+f(x)\rho+1)^{-\frac{N+s}{2}} \\ &= \frac{1}{|x-y|^{N+s}}\left(1+O(\|f\|_{C^1})\right).\end{aligned} \tag{2.6}$$

Plugging formula (2.6) into the second addend of (2.5) and by symmetry again, we obtain

$$\begin{aligned}
&2\int_{\partial B}\int_{\partial B}\int_{f(y)}^{f(x)} \psi(x)F_{|x-y|}(1+f(x),1+\rho)\,\mathrm{d}\rho\,\mathrm{d}\mathcal{H}^{N-1}_x\,\mathrm{d}\mathcal{H}^{N-1}_y\\
&=2\int_{\partial B\times\partial B}\frac{\psi(x)}{N}\frac{(1+f(x))^{N-1}}{|x-y|^{N+s}}\left((1+f(x))^N-(1+f(y))^N\right)(1+O(\|f\|_{C^1}))\,\mathrm{d}\mathcal{H}^{N-1}_x\,\mathrm{d}\mathcal{H}^{N-1}_y\\
&=\int\frac{\left(\psi(x)(1+f(x))^{N-1}-\psi(y)(1+f(y))^{N-1}\right)\left((1+f(x))^N-(1+f(y))^N\right)}{N|x-y|^{N+s}}(1+O(\|f\|_{C^1})).
\end{aligned}$$

Now, if $\psi=1$ by a simple Taylor expansion we conclude

$$2\int_{\partial B\times\partial B}\int_{f(y)}^{f(x)} F_{|x-y|}(1+f(x),1+\rho)=(N-1)\int_{\partial B\times\partial B}\frac{(f(x)-f(y))^2}{|x-y|^{N+s}}(1+O(\|f\|_{C^1}))=O([f]^2_{\frac{1+s}{2}}),$$

while the choice $\psi=f$ yields

$$2\int_{\partial B\times\partial B}\int_{f(y)}^{f(x)} f(x)F_{|x-y|}(1+f(x),1+\rho)=\int_{\partial B\times\partial B}\frac{(f(x)-f(y))^2}{|x-y|^{N+s}}(1+O(\|f\|_{C^1})).$$

□

In order to prove Theorem 0.3, we need the following lemma, which states the coercivity of the second variation of the fractional perimeter of a ball with respect to normal deformations. Its proof is contained in [19, Theorem 8.1]. We start by defining

$$\lambda_1^s := s(N-s)\frac{P^s(B)}{P(B)}. \tag{2.7}$$

Lemma 2.3. *There exists $\delta>0$ small such that, if $f\in C^2(\partial B)$ with $\|f\|_{C^1(\partial B)}\le\delta$, $|B_f|=\omega_N$ and $\mathrm{bar}(B_f)=0$, then we have*

$$\begin{aligned}
\partial^2P^s(B)[f]&=\int_{\partial B}\int_{\partial B}\frac{(f(x)-f(y))^2}{|x-y|^{N+s}}\,\mathrm{d}\mathcal{H}^{N-1}_x\,\mathrm{d}\mathcal{H}^{N-1}_y-\lambda_1^s\int_{\partial B}|f|^2\,\mathrm{d}\mathcal{H}^{N-1}\\
&\ge\frac14\Big([f]^2_{\frac{1+s}{2}}+\lambda_1^s\|f\|^2_{L^2(\partial B)}\Big).
\end{aligned}$$

We are now in position to prove Theorem 0.3.

Proof of Theorem 0.3. Without loss of generality, we assume that $\|H^s_{B_f}-\bar H^s_{B_f}\|_{L^2}\le 1$. Let $\Phi:\partial B\to\partial B_f\subset\mathbb{R}^N$ be the map defined by $\Phi(x)=(1+\ f(x))x$, by direct computations one can prove that

$$J\Phi(x)=(1+f(x))^{N-1}(1+(1+f(x))^{-2}|\nabla f(x)|^{1/2}.$$

For every $\psi\in C^1(\partial B)$, let

$$X:\mathbb{R}^N\to\mathbb{R}^N,\quad X(x):=\frac{x}{|x|}\psi\left(\frac{x}{|x|}\right).$$

Employing the area formula we get

$$\begin{aligned}
\partial P^s(B_f)[\psi]&=\int_{\partial B_f}H^s_{B_f}\nu_{B_f}\cdot X\,\mathrm{d}\mathcal{H}^{N-1}\\
&=\int_{\partial B}H^s_{B_f}(p)\nu_{B_f}(p)\cdot x\psi(x)J\Phi(x)\,\mathrm{d}\mathcal{H}^{N-1}_x\\
&=\int_{\partial B}H^s_{B_f}(p)\,\psi(x)\,(1+f(x))^{N-1}\,\mathrm{d}\mathcal{H}^{N-1}_x,
\end{aligned}$$

where we have set $p = (1 + f(x))x$ (for more details see [27, Section 1] and [14, Section 3]). Now, by a simple Taylor expansion we obtain

$$\partial P^s(B_f)[\psi] = \int_{\partial B} H^s_{B_f}(p)\, \psi(x)\, (1 + (N-1)f(x) + O(f^2))\, \mathrm{d}\mathcal{H}^{N-1}_x. \tag{2.8}$$

We recall that

$$H^s_B(x) = (N-s)\frac{P^s(B)}{P(B)} \quad \text{for all } x \in \partial B.$$

If $\psi = 1$, by combining formulas (2.8) and (2.3), we infer

$$\int_{\partial B} (H^s_{B_f}(p) - H^s_B)(1 + (N-1)f(x) + O(f^2))\, \mathrm{d}\mathcal{H}^{N-1}_x = \int_{\partial B} O(f)\, \mathrm{d}\mathcal{H}^{N-1} + O([f]^2_{\frac{1+s}{2}}) \tag{2.9}$$

and if $\psi = f$, by combining equations (2.8) and (2.4), we get

$$\begin{aligned} &\int_{\partial B}\int_{\partial B} \frac{(f(x)-f(y))^2}{|x-y|^{N+s}}\, \mathrm{d}\mathcal{H}^{N-1}_x\, \mathrm{d}\mathcal{H}^{N-1}_y - s(N-s)\frac{P^s(B)}{P(B)}\int_{\partial B} f^2\, \mathrm{d}\mathcal{H}^{N-1} \\ &= \int_{\partial B} \Big(H^s_{B_f}(p) - H^s_B\Big)\, (1 + (N-1)f(x) + O(f^2)) f(x)\, \mathrm{d}\mathcal{H}^{N-1}_x \\ &\quad + O([f]^2_{\frac{1+s}{2}})\|f\|_{C^1}. \end{aligned} \tag{2.10}$$

Using the same arguments of the proof of [14, Theorem 1.3] (see also [27, Theorem 1.3]) we can conclude. Anyway for the interested reader we present a sketch of the proof.

By (2.9), for δ sufficiently small, using Hölder's inequality we obtain

$$\begin{aligned} \left|\bar{H}^s_{B_f} - H^s_B\right| &\le \left| -\!\!\!\!\!\!\int_{\partial B} (H^s_{B_f} - H^s_B)((N-1)f + O(f^2))\, \mathrm{d}\mathcal{H}^{N-1}\right| \\ &\quad + \int_{\partial B} O(|f|)\, \mathrm{d}\mathcal{H}^{N-1} + O([f]^2_{\frac{1+s}{2}}) \\ &\le \left| -\!\!\!\!\!\!\int_{\partial B} (H^s_{B_f} - \bar{H}^s_{B_f})((N-1)f + O(f^2))\, \mathrm{d}\mathcal{H}^{N-1}\right| \\ &\quad + \left| -\!\!\!\!\!\!\int_{\partial B} (\bar{H}^s_{B_f} - H^s_B)((N-1)f + O(f^2))\, \mathrm{d}\mathcal{H}^{N-1}\right| \\ &\quad + \int_{\partial B} O(|f|)\, \mathrm{d}\mathcal{H}^{N-1} + O([f]^2_{\frac{1+s}{2}}) \\ &\le \delta\, \frac{N-1+C\delta}{P(B)^{1/2}}\|H^s_{B_f} - \bar{H}^s_{B_f}\|_{L^2} + \delta\,(N-1+C\delta)\,|\bar{H}^s_{B_f} - H^s_B| \\ &\quad + \int_{\partial B} O(|f|)\, \mathrm{d}\mathcal{H}^{N-1} + O([f]^2_{\frac{1+s}{2}}), \end{aligned}$$

with $C = C(N)$. For δ small enough, recalling that $\|H_{B_f} - \bar{H}_{B_f}\|_{L^2} \le 1$, the previous inequality implies

$$\frac{1}{2}|\bar{H}^s_{B_f} - H^s_B| \le C\delta\|H^s_{B_f} - \bar{H}^s_{B_f}\|_{L^2} + \int_{\partial B} O(|f|)\, \mathrm{d}\mathcal{H}^{N-1} + O([f]^2_{\frac{1+s}{2}}) \le C\delta. \tag{2.11}$$

By (2.10), using again Hölder's inequality and by the previous remark, we get

$$\begin{aligned}
&\int_{\partial B}\int_{\partial B}\frac{(f(x)-f(y))^2}{|x-y|^{N+s}}\,\mathrm{d}\mathcal{H}^{N-1}_x\,\mathrm{d}\mathcal{H}^{N-1}_y - s(N-s)\frac{P^s(B)}{P(B)}\int_{\partial B} f^2\,\mathrm{d}\mathcal{H}^{N-1}\\
&=\int_{\partial B}\Big(H^s_{B_f}(p)-H^s_B\Big)(1+(N-1)f+O(f^2))f\,\mathrm{d}\mathcal{H}^{N-1}\\
&\quad+O([f]^2_{\frac{1+s}{2}})\|f\|_{C^1}\\
&=\int_{\partial B}(H^s_{B_f}(p)-\bar H^s_{B_f})(1+(N-1)f+O(f^2))f\,\mathrm{d}\mathcal{H}^{N-1}\\
&\quad+\int_{\partial B}(\bar H^s_{B_f}-H^s_B)(1+(N-1)f+O(f^2))f\,\mathrm{d}\mathcal{H}^{N-1}\\
&\quad+O([f]^2_{\frac{1+s}{2}})\|f\|_{C^1}\\
&\le C\|H^s_{B_f}-\bar H^s_{B_f}\|_{L^2}\|f\|_{L^2}+|\bar H^s_{B_f}-H^s_B|\int_{\partial B}(1+(N-1)f+O(f^2))f\,\mathrm{d}\mathcal{H}^{N-1}\\
&\quad+O([f]^2_{\frac{1+s}{2}})\|f\|_{C^1}. \qquad (2.12)
\end{aligned}$$

Since $|B_f|=\omega_N$, we have

$$\Big|\int_{\partial B} f\,\mathrm{d}\mathcal{H}^{N-1}\Big| = \int_{\partial B} O(f^2)\,\mathrm{d}\mathcal{H}^{N-1}. \tag{2.13}$$

By (2.13) and (2.11), we obtain

$$|\bar H^s_{B_f}-H^s_B|\int_{\partial B}(f+O(f^2))\,\mathrm{d}\mathcal{H}^{N-1}\le\delta\int_{\partial B}O(f^2).$$

Finally, by the above inequality, (2.13) again and by combining (2.12) with (2.11) we deduce that, for any $\eta>0$, it holds

$$\begin{aligned}
&\int_{\partial B}\int_{\partial B}\frac{(f(x)-f(y))^2}{|x-y|^{N+s}}\,\mathrm{d}\mathcal{H}^{N-1}_x\,\mathrm{d}\mathcal{H}^{N-1}_y - s(N-s)\frac{P^s(B)}{P(B)}\int_{\partial B} f^2\,\mathrm{d}\mathcal{H}^{N-1}\\
&\le C\|H^s_{B_f}-\bar H^s_{B_f}\|_{L^2}\|f\|_{L^2}+C\delta(\|f\|^2_{L^2}+[f]^2_{\frac{1+s}{2}}) \qquad (2.14)\\
&\le \frac{1}{\eta}C^2\|H^s_{B_f}-\bar H^s_{B_f}\|^2_{L^2}+\eta\|f\|^2_{L^2}+C\delta(\|f\|^2_{L^2}+[f]^2_{\frac{1+s}{2}}). \qquad (2.15)
\end{aligned}$$

The conclusion then follows combining (2.15) with Lemma 2.3 and taking δ and η sufficiently small. □

Remark 2.4. By slightly changing the last step in the previous proof we can prove the quantitative Alexandrov result in the classical case, see Theorem 0.2. First, we remark that (2.14) can be read as

$$\begin{aligned}
&(1-s)\Big(\int_{\partial B}\int_{\partial B}\frac{(f(x)-f(y))^2}{|x-y|^{N+s}}\,\mathrm{d}\mathcal{H}^{N-1}_x\,\mathrm{d}\mathcal{H}^{N-1}_y - s(N-s)\frac{P^s(B)}{P(B)}\int_{\partial B} f^2\,\mathrm{d}\mathcal{H}^{N-1}\Big)\\
&\le C\Big\|(1-s)\Big(H^s_{B_f}-\bar H^s_{B_f}\Big)\Big\|_{L^2}\|f\|_{L^2}+C\delta(1-s)(\|f\|^2_{L^2}+[f]^2_{\frac{1+s}{2}}),
\end{aligned}$$

from which we obtain

$$\begin{aligned}
\frac{1-s}{4}\Big(\lambda^s_1\|f\|^2_{L^2}+[f]^2_{\frac{1+s}{2}}\Big)\le&\frac{C^2}{\eta}\Big\|(1-s)\Big(H^s_{B_f}-\bar H^s_{B_f}\Big)\Big\|^2_{L^2}+\eta\|f\|^2_{L^2}\\
&+C\delta(1-s)(\|f\|^2_{L^2}+[f]^2_{\frac{1+s}{2}}). \qquad (2.16)
\end{aligned}$$

By recalling the definition of λ^s_1 (see (2.7)), and by Theorem 1.4 we obtain

$$\lim_{s\to1}(1-s)\lambda^s_1=(N-1)\omega_{N-1}.$$

Finally, using Theorems 1.5, 1.6, we can take the limit as $s \to 1^-$ in the inequality 2.16 and get

$$\frac{1}{4}\left((N-1)\omega_{N-1}\|f\|_{L^2}^2 + C\|\nabla f\|_{L^2}^2\right) \le \frac{C^2}{\eta}\left\|\omega_{N-1}\left(H_{B_f} - \bar{H}_{B_f}\right)\right\|_{L^2}^2 + \eta\|f\|_{L^2}^2 + C\delta\|\nabla f\|_{L^2}^2,$$

where $C = C(N)$ and we also used that, by uniform convergence, $(1-s)\bar{H}^s_{B_f} \to \omega_{N-1}\bar{H}_{B_f}$. We then conclude by taking η and δ sufficiently small. Finally, the hypothesis $f \in C^2(\partial B)$ can be weakened to $f \in C^1(\partial B) \cap H^2(\partial B)$ by approximation.

3. The asymptotic of the discrete volume-preserving mean curvature flow

In this section we start by introducing the incremental minimum problem which defines the discrete-in-time approximation of the volume preserving fractional mean curvature flow.

Let $E \neq \emptyset$ be a bounded, measurable subset of $\mathbb{R}^N$. In the following we will always assume that E coincides with its Lebesgue representative. Fixed $h > 0$, $m > 0$, we consider the minimum problem

$$\min\left\{P^s(F) + \frac{1}{h}\int_F \mathrm{sd}_E(x)\,\mathrm{d}x + \frac{1}{h^{\frac{s}{s+1}}}||F| - m| : F \subset \mathbb{R}^N\right\}, \tag{3.1}$$

where $\mathrm{sd}_E(x) := \mathrm{dist}_E(x) - \mathrm{dist}_{E^c}(x)$ is the signed distance from the set E. Observe that the minimum problem (3.1) is equivalent to the problem

$$\min\left\{P^s(F) + \frac{1}{h}\int_{F\triangle E} \mathrm{dist}_{\partial E}(x)\,\mathrm{d}x + \frac{1}{h^{\frac{s}{s+1}}}||F| - m| : F \subset \mathbb{R}^N\right\}.$$

We set $\mathcal{F}_h(\cdot,\, E) : \mathrm{M}(\mathbb{R}^N) \to (-\infty,\, +\infty]$ the functional

$$\mathcal{F}_h(F,\, E) = P^s(F) + \frac{1}{h}\int_F \mathrm{sd}_E(x)\,\mathrm{d}x + \frac{1}{h^{\frac{s}{s+1}}}||F| - m|.$$

Let $E,\, F \in \mathrm{M}(\mathbb{R}^N)$, we set

$$\mathcal{D}(E,\, F) := \int_{E\Delta F} \mathrm{dist}_{\partial E}(x)\,\mathrm{d}x.$$

The following proposition recalls some properties of minimizers of problem (3.1).

Proposition 3.1. *Let $M > 0, h > 0, s \in (0,1)$ and $m > 0$. Let $E \subset \mathbb{R}^N$ be a bounded, measurable set such that $P^s(E) \le M$ and $|E| \le M$. Then, there exists a minimizer F of (3.1). Moreover, it is bounded and satisfies the following properties:*

i) *There exists $\Lambda = \Lambda(h, N, s) > 0$ such that F is a Λ-minimizer of the fractional perimeter, namely*
$$P^s(F) \le P^s(F') + \Lambda|F\triangle F'|$$
for all measurable set $F' \subset \mathbb{R}^N$ such that $\mathrm{diam}(F\triangle F') \le 1$.

ii) *The boundary ∂F is of class $C^{2,\alpha}$ for any $\alpha \in (0, s)$ outside of a closed set Σ of Hausdorff dimension at most $N-3$. Moreover, there exists $s_0 \in (0,1)$ such that, if $s \in (s_0, 1)$, then ∂F is of class $C^{1,\alpha}$ for any $\alpha \in (0,1)$ outside a closed set Σ of Hausdorff dimension at most $N-8$.*

iii) *There exist $c_0 = c_0(N, s) > 0$ and a radius $r_0 = r_0(h, N, s) > 0$ such that for every $x \in \partial F \setminus \Sigma$ and $r \in (0, r_0]$ we have*
$$|B_r(x) \cap F| \ge c_0 r^N \quad \text{and} \quad |B_r(x) \setminus F| \ge c_0 r^N.$$

iv) *The following Euler-Lagrange equation holds: for all $X \in C^1_c(\mathbb{R}^N, \mathbb{R}^N)$ we have*

$$\int_{\partial F} \frac{\mathrm{sd}_E}{h} X\cdot\nu_F\,\mathrm{d}\mathcal{H}^{N-1} + \int_{\partial F} H^s_F X\cdot\nu_F\,\mathrm{d}\mathcal{H}^{N-1} = \lambda\int_{\partial F} X\cdot\nu_F\,\mathrm{d}\mathcal{H}^{N-1}, \tag{3.2}$$

where $\lambda = \fint(H^s_F + \frac{1}{h}\mathrm{sd}_E)$ and, if $|F| \neq m$, it also holds $\lambda = \mathrm{sgn}(m - |F|)h^{-\frac{s}{1+s}}$.

v) *There exist* $k_0 = k_0(h, N, s, M, m) \in \mathbb{N}$ *and* $d_0 = d_0(h, N, s, M, m) > 0$ *such that* F *is made up of at most* k_0 *connected components having diameter larger than* d_0.

Proof. For the existence of minimizers of (3.1) see for example [8, Theorem 1.1]. The $\Lambda-$minimality property is easily deduced, for instance we can choose $\Lambda = 2(h^{-1} + h^{-\frac{s}{1+s}})$. Concerning property *ii*), it follows from [8, Theorem 1.1] and [6, Theorem 5]. The density esimates can be found in [5, Theorem 4.1]. Item iv) can be proved as in the local case (see [28, Lemma 3.7]). The bound on the number of connected components and on the diameter of the components follows from a covering argument as in [27, Proposition 2.3]. □

By induction we can now define the discrete-in-time, volume preserving fractional mean curvature flow.

Definition 3.2. Fixed $h > 0$ and $m > 0$, let $E_0 \subset \mathbb{R}^N$ be a measurable set such that $|E_0| = m$. Let $E_1^{(h)}$ be a solution of the problem (3.1) with E_0 instead of E. Assume that $E_k^{(h)}$ is defined for $1 \leq k \leq n-1$, let $E_n^{(h)}$ be a solution of (3.1) with E replaced by $E_{n-1}^{(h)}$. The sequence $\{E_n^{(h)}\}_{n\in\mathbb{N}}$ will be called a *discrete flow*.

We recall the density estimate holding for one-sided minimizers of the fractional perimeter, which can be found in [5, Theorem 4.1] .

Proposition 3.3. *There exists a constant* $C = C(N, s) > 0$ *with the following property: given* $E \subset \mathbb{R}^N$, R, $\mu > 0$ *and* $x_0 \in \partial E$ *such that*

$$P^s(E) \leq P^s(E \setminus B_r(x_0)) + \mu|E \cap B_r(x_0)| \quad \forall 0 < r < R,$$

then

$$Cr^N \leq |E \cap B_r(x_0)| \quad \forall 0 < r < \min\{R, \mu^{-1/s}\}.$$

We employ the density estimates above to bound the distance function between two consecutive sets of the discrete flow. The proof follows the line of [28, Proposition 3.2] where it is proved in the of local case, see also [26].

Proposition 3.4. *There exists a constant* $\gamma = \gamma(N, s) > 0$ *with the following property. Let* $F \subset \mathbb{R}^N$ *be a bounded set of finite fractional perimeter and let* E *be a minimizer of* $\mathcal{F}_h(\cdot, F)$, *then*

$$\sup_{E\triangle F} \mathrm{dist}_{\partial F} \leq \gamma h^{1/1+s}.$$

Proof. Let $\gamma = \max\{3, 2^{s+1/s} P^s(B)^{1/s} C^{-1/s}\}$, where $C = C(N, s)$ is the constant given by the Proposition 3.3. Let $c > \gamma$ and $x_0 \in E\triangle F$. Suppose by contradiction that $\mathrm{dist}_{\partial F}(x_0) > ch^{1/1+s}$. Since the other case is analogous, we assume $x_0 \in E \setminus F$. We then have

$$\mathrm{sd}_F(x_0) > ch^{1/1+s} \tag{3.3}$$

and thus any ball $B_r(x_0)$ of radius $r \leq ch^{1/1+s}/2$ is contained in F^c. By the minimality of E, we have $\mathcal{F}_h(E, F) \leq \mathcal{F}_h(E \setminus B_r(x_0), F)$, therefore

$$P^s(E) \leq P^s(E \setminus B_r(x_0)) - \frac{1}{h}\int_{E\cap B_r(x_0)} \mathrm{sd}_F \,\mathrm{d}x + \frac{1}{h^{s/1+s}}|E \cap B_r(x_0)|.$$

We use (3.3) and $r \leq ch^{1/1+s}/2$ to infer that

$$-\frac{1}{h}\int_{E\cap B_r(x_0)} \mathrm{sd}_F \,\mathrm{d}x < -\frac{c}{2h^{s/1+s}}|E \cap B_r(x_0)|.$$

Then we have

$$P^s(E) \leq P^s(E \setminus B_r(x_0)) - \frac{1}{h^{s/1+s}}\left(\frac{c}{2} - 1\right)|E \cap B_r(x_0)|. \tag{3.4}$$

By assumption $c > 3$ and we can apply Proposition 3.3 with $\mu = 0$ and obtain

$$Cr^N \leq |E \cap B_r(x_0)| \qquad \forall 0 < r < \frac{c}{2} h^{1/1+s}. \tag{3.5}$$

On the other hand, from (3.4) we deduce, for every $0 < r < ch^{1/1+s}/2$, that

$$\frac{1}{h^{s/1+s}} \left(\frac{c}{2} - 1\right) |E \cap B_r(x_0)| \leq P^s(E \setminus B_r(x_0)) - P^s(E) \leq P^s(B_r^c) = P^s(B) r^{N-s} \tag{3.6}$$

(where the last inequality follows from the subadditivity of the perimeter on E and B_r^c). Combining (3.5) and (3.6), we get that

$$Cr^N \leq |E \cap B_r(x_0)| \leq P^s(B) \left(\frac{c}{2} - 1\right)^{-1} h^{s/1+s} r^{N-s} \leq 2P^s(B) h^{s/1+s} r^{N-s}$$

for all $0 < r < ch^{1/1+s}/2$, which gives the desired contradiction to the choice of c as soon as $r \to ch^{1/1+s}/2$. □

As a corollary of the previous result we obtain the following density estimates, their proof is an adaptation of the one of [28, Corollary 3.3].

Corollary 3.5. *Let $E \subset \mathbb{R}^N$ be a bounded set of finite fractional perimeter and let F be a minimizer of $\mathcal{F}_h(\cdot, E)$. Then for every $r \in (0, \gamma h^{1/1+s})$ and for every $x_0 \in \partial^* F$, it holds*

$$\min\{|B_r(x_0) \setminus F|, |F \cap B_r(x_0)|\} \geq cr^N \tag{3.7}$$

$$cr^{N-s} \leq P^s(F, B_r(x_0)) \leq Cr^{N-s}, \tag{3.8}$$

where γ is the constant given by Proposition 3.4 and the constants c, C only depend on N and s.

Proof. Since F is a minimizer of $\mathcal{F}_h(\cdot, E)$, for any $x_0 \in \partial F$, it holds that $\mathcal{F}_h(F, E) \leq \mathcal{F}_h(F \cup B_r(x_0), E)$, which implies

$$\begin{aligned} P^s(F) &\leq P^s(F \cup B_r(x_0)) + \frac{1}{h} \int_{B_r(x_0) \setminus F} \mathrm{sd}_E \, \mathrm{d}x + \frac{1}{h^{s/1+s}} |B_r(x_0) \setminus F| \\ &\leq P^s(F \cup B_r(x_0)) + \frac{C}{h^{s/1+s}} |B_r(x_0) \setminus F|, \end{aligned}$$

where we bounded $\mathrm{sd}_E \leq \gamma h^{1/1+s}$ by Proposition 3.4. Analogously, one can show that

$$\begin{aligned} P^s(F) &\leq P^s(F \setminus B_r(x_0)) + \frac{C}{h^{s/1+s}} |F \cap B_r(x_0)| \\ &= \mathcal{L}_s(F \setminus B_r(x_0), F^c \setminus B_r(x_0)) + \mathcal{L}_s(F \setminus B_r(x_0), B_r(x_0)) + \frac{C}{h^{s/1+s}} |F \cap B_r(x_0)| \end{aligned} \tag{3.9}$$

Therefore, by Proposition 3.3, we deduce

$$\min\{|F \cap B_r(x_0)|, |B_r(x_0) \setminus F|\} \geq cr^N \qquad \forall 0 < r < \gamma h^{1/1+s}.$$

The first inequality in (3.8) is now an immediate consequence of the relative isoperimetric inequality. To prove the second inequality, by (3.9) we get

$$\begin{aligned} P^s(F, B_r(x_0)) &= \mathcal{L}_s(F \cap B_r(x_0), F^c) + \mathcal{L}_s(F \setminus B_r(x_0), F^c \cap B_r(x_0)) \\ &= P^s(F) - \mathcal{L}_s(F \setminus B_r(x_0), F^c \setminus B_r(x_0)) \\ &\leq \mathcal{L}_s(F \setminus B_r(x_0), B_r(x_0)) + \frac{C}{h^{s/1+s}} |B_r(x_0) \setminus F| \\ &\leq P^s(B_r(x_0)) + \frac{C\gamma^s}{r^s} \omega_N r^N \leq C(N, s) r^{N-s}, \end{aligned}$$

where we used that $r \leq \gamma h^{1/1+s}$. □

Remark 3.6. From the monotonicity of the energy $P^s(\cdot) + h^{-\frac{s}{1+s}}||\cdot| - m|$ along the discrete flow starting from E_0 with $|E_0| = m,\ P^s(E_0) \leq M$, one can observe that $|E_n^{(h)}| \in (m/2, 3m/2)$ for all $n \in \mathbb{N}$ and for $h = h(m, M)$ small,.

We now characterize the stationary sets E for the discrete flow. We say that E is a *stationary set* for the discrete flow if it is a fixed set for the functional (3.1), that is,

$$E = E_n^{(h)} \quad \forall n \in \mathbb{N}.$$

In the following, we will always assume that either:

- $N = 2$;
- $N \leq 7$ and $s \in (s_0, 1)$, where s_0 is the constant of Proposition 3.1, item *ii)*.

This hypothesis is essential for the proof of the following result.

Proposition 3.7. *Every stationary set E for the discrete flow is a critical set of the $s-$perimeter, that is, a single ball.*

Proof. It is an immediate consequence of the Euler-Lagrange equation (3.2). Since E is a stationary point for the discrete flow, it satisfies

$$\int_{\partial E} H_E^s \, \mathrm{d}\mathcal{H}^{N-1} = \lambda \int_{\partial E} X \cdot \nu_E \, \mathrm{d}\mathcal{H}^{N-1}$$

for all $X \in C_c^1(\mathbb{R}^N, \mathbb{R}^N)$, i.e. E is a critical point for the $s-$perimeter. By [4, Theorem 1.1] and [13, Theorem 1.1], we conclude that E is a single ball having constant fractional mean curvature $H_E^s = \lambda$. □

Before proving the convergence of the flow up to translations, we recall [27, Lemma 3.5] that will be used in the proof of the next proposition. The proof in the fractional setting is analogous and will be omitted.

Lemma 3.8. *Let $\{E_n^{(h)}\}_{n\in\mathbb{N}}$ be a discrete flow starting from E_0 and let $E_{k_n}^{(h)}$ be a subsequence such that $E_{k_n}^{(h)} + \tau_n \to F$ in L^1 for some set F and a suitable sequence $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$. Then $\mathrm{dist}_{\partial E_{k_n-1}^{(h)}}(\cdot + \tau_n) \to \mathrm{dist}_{\partial F}$ uniformly.*

The following result proves the convergence of the discrete flow to a union of disjointed balls, all having the same radius. The proof follows closely the one of [27, Proposition 3.6]. Moreover, we prove that the flow eventually has fixed volume. A this point, we can not rule out that the flow is converging to different balls (each at infinite distance from the others) and that the translations introduced are different along different subsequences. We will provide a sharper result in the final theorem.

Proposition 3.9. *Let m, $M > 0$ and E_0 be an initial bounded set with $P^s(E_0) \leq M$, $|E_0| = m$. Then there exists $h^* = h^*(s, M, m) > 0$ such that, for any $h < h^*$ and for any discrete flow $E_n^{(h)}$ starting from E_0, the following hold:*

i) *for n sufficiently large $|E_n^{(h)}| = m$;*
ii) *there exists*
$$P_\infty^s = \lim_{n\to\infty} P^s(E_n^{(h)});$$
iii) *$E_n^{(h)}$ is made of $K = (P_\infty^s/\omega_N^s)^{\frac{N}{s}}(\omega_N/m)^{\frac{N}{s}-1}$ distinct connected components $E_{n,i}^{(h)}$, and $E_{n,i}^{(h)} - \mathrm{bar}(E_{n,i}^{(h)})$ converges in C^k, for every $k \in \mathbb{N}$, to the ball centered at the origin and having mass m/K.*

Proof. Let $\{E^{(h)}_{k_n}\}_{n\in\mathbb{N}}$ be any given subsequence of $\{E^{(h)}_n\}_{n\in\mathbb{N}}$. By Proposition 3.1, each set $E^{(h)}_{k_n}$ is made up of $l_n \le k_0$ connected components having diameter uniformly bounded by d_0. Therefore, there exist l_n balls $\{B_{d_0}(\xi^i_n)\}$, each containing a different component of $E^{(h)}_{k_n}$ and such that $E^{(h)}_{k_n} \subset \cup_{i=1}^{l_n} B_{d_0}(\xi^i_n)$. Up to subsequences, we can assume that $l_n = \tilde{l}$, and for all $1 \le i < j \le \tilde{l}$ the following limits exist

$$\limsup_{n\to\infty} |\xi^i_n - \xi^j_n| =: d^{i,j} \in [0, +\infty].$$

Now we define the following equivalence classes: we say that $i \equiv j$ if and only if $d^{i,j} < +\infty$. Denote by $l \le \tilde{l}$ the number of such equivalence classes, let $j(i)$ be a representative for each class $i \in \{1, \ldots, l\}$, and set $\sigma^i_n := \xi^{j(i)}$ for $i = 1, \ldots, l$. We have constructed a subsequence $E^{(h)}_{k_n}$ satisfying $E^{(h)}_{k_n} \subset \cup_{i=1}^{l} B_R(\sigma^i_n)$, where $R = d_0 + \max\{d^{i,j} : d^{i,j} < +\infty\} + 1$, and for all $i \not\equiv j$ it holds $|\sigma^i_n - \sigma^j_n| \to +\infty$ as $n \to +\infty$.

Now, fix $1 \le i \le l$, and set

$$F^i_n := E^{(h)}_{k_n} - \sigma^i_n, \quad \tilde{F}^i_n := (E^{(h)}_{k_n} - \sigma^i_n) \cap B_R, \quad m^i_n := |\tilde{F}^i_n|.$$

Up to a subsequence, we have $m^i_n \to m^i > 0$. Moreover, by Lemma 3.8 and by the compactness of sets of equi-bounded fractional perimeters, there exist $\tilde{F}^i \Subset B_R$ such that, up to a subsequence,

$$\tilde{F}^i_n \to \tilde{F}^i \text{ in } L^1, \quad \mathrm{sd}_{E^{(h)}_{k_n-1}}(\cdot + \sigma^i_n) \to \mathrm{sd}_{\tilde{F}^i}(\cdot) \text{ locally uniformly.} \tag{3.10}$$

Let $\tilde{G}^i$ be any bounded set with $|\tilde{G}^i| = m^i_n$ and let $\tilde{G}^i_n := \left(\frac{m^i_n}{m^i}\right)^{\frac{1}{N}} \tilde{G}^i$. We set now $G^i_n := (F^i_n \setminus \tilde{F}^i_n) \cup \tilde{G}^i_n$ so that, for n sufficiently large, $|F^i_n| = |G^i_n|$. By the minimality of $E^{(h)}_{k_n}$ we have

$$P^s(F^i_n) + \frac{1}{h}\int_{F^i_n} \mathrm{sd}_{E^{(h)}_{k_n-1}}(x + \sigma^i_n)\,\mathrm{d}x \le P^s(G^i_n) + \frac{1}{h}\int_{G^i_n} \mathrm{sd}_{E^{(h)}_{k_n-1}}(x + \sigma^i_n)\,\mathrm{d}x.$$

For n sufficiently large, we obtain

$$\begin{aligned} P^s(\tilde{F}^i_n) &+ \int_{\tilde{F}^i_n}\int_{F^i_n\setminus\tilde{F}^i_n} \frac{1}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{h}\int_{\tilde{F}^i_n} \mathrm{sd}_{E^{(h)}_{k_n-1}}(x+\sigma^i_n)\,\mathrm{d}x \\ &\le P^s(\tilde{G}^i_n) + \int_{\tilde{G}^i_n}\int_{F^i_n\setminus\tilde{F}^i_n} \frac{1}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{h}\int_{\tilde{G}^i_n} \mathrm{sd}_{E^{(h)}_{k_n-1}}(x+\sigma^i_n)\,\mathrm{d}x. \end{aligned}$$

Passing to the limit as $n \to \infty$, using (3.10) and the uniform boundedness of $\tilde{F}^i_n$ and $\tilde{G}^i_n$, we deduce that

$$P^s(\tilde{F}^i) + \frac{1}{h}\int_{\tilde{F}^i} \mathrm{sd}_{\tilde{F}^i}(x)\,\mathrm{d}x \le P^s(G^i) + \frac{1}{h}\int_{G^i} \mathrm{sd}_{\tilde{F}^i}(x)\,\mathrm{d}x.$$

This minimality property extends by density to all competitors G^i with finite perimeter and volume m^i, so that we deduce that $\tilde{F}^i$ is a fixed point for the discrete scheme with prescribed volume m^i, and, whence by Proposition 3.7, it is a ball. Moreover, since $\tilde{F}^i$ are uniform $\Lambda-$minimizer by Proposition 3.1, we also deduce that $\tilde{F}^i_n$ converge to $\tilde{F}^i$ in $C^{1,\alpha}$ for every $\alpha \in (0,1)$. In particular, for n large enough, $\tilde{F}^i_n$ has only one connected component.

We have shown that, for n large enough, $E^{(h)}_{k_n}$ is made up by a fixed number K of connected components $E^{(h),i}_{k_n}$, $i = 1, \ldots, K$ and $E^{(h),i}_{k_n} - \mathrm{bar}(E^{(h),i}_{k_n}) \to B_{R_i}$ where $|B_{R_i}| = m_i$. Now, we show that all the radii R_i are equal to R. To this aim, we consider the Euler-Lagrange equation (3.2)

$$\frac{1}{h}\mathrm{sd}_{E^{(h)}_{k_n-1}} + H^s_{E^{(h)}_{k_n}} = \lambda_n \quad \text{on } \partial E^{(h)}_{k_n}.$$

By Proposition 3.4, we deduce that

$$|\lambda_n| \le h^{-1}\|\mathrm{sd}_{E^{(h)}_{k_n-1}}\|_{L^\infty(\partial E^{(h)}_{k_n})} + \|H^s_{E^{(h)}_{k_n}}\|_{L^\infty(\partial E^{(h)}_{k_n})} \le c + \|H^s_{E^{(h)}_{k_n}}\|_{L^\infty(\partial E^{(h)}_{k_n})}.$$

To bound the right hand side, we use the $\Lambda-$minimality of $E^{(h)}_{k_n}$ to obtain

$$\|H^s_{E^{(h)}_{k_n}}\|_{L^\infty(\partial E^{(h)}_{k_n})} \le \Lambda.$$

Therefore, by passing to a further subsequence, we can assume $\lambda_n \to \lambda \in \mathbb{R}$. Arguing as before, we can localize the Euler-Lagrange equation to each single F^i_n and obtain

$$\frac{1}{h}\mathrm{sd}_{E^{(h)}_{k_n-1}}(x+\sigma^i_n) + H^s_{F^i_n}(x) = \lambda_n \quad x \in \partial F^i_n.$$

We can pass to the limit as $n \to \infty$ thanks to Lemma 3.8 and the continuity property of the fractional mean curvature (see e.g. [13, Lemma 2.1]). Thus, taking into account that $\tilde{F}^i$ is a fixed set for (3.1), we deduce that

$$H^s_{\tilde{F}^i} = \lambda \quad \text{on } \partial\tilde{F}^i.$$

In particular, this shows that $R_i = c\lambda^{-s}$, for a suitable constant c depending only on s and N. In order to prove that, eventually, $|E^{(h)}_n| = m$, we proceed as follows. Set $|B_{R_i}| = c_1\lambda^{-sN}$ and $P^s(B_{R_i}) = c_2\lambda^{-s(N-s)}$, for suitable constants c_1, c_2 depending on N, s. From Remark 3.6, we take $h = h(s, M)$ small enough such that

$$|E^{(h)}_{k_n}| \in \left[\frac{m}{2}, \frac{3m}{2}\right], \quad P^s(E^{(h)}_{k_n}) \le P^s(E_0) \le M$$

and, for n large enough, this implies

$$\sum_{i=1}^K m^i_n \in \left[\frac{m}{2}, \frac{3m}{2}\right], \quad \sum_{i=1}^K P^s(\tilde{F}^i_n) \le M.$$

Passing to the limit as $n \to \infty$ we obtain

$$Kc_1\lambda^{-sN} \in \left[\frac{m}{2}, \frac{3m}{2}\right], \quad Kc_2\lambda^{-s(N-s)} \le M,$$

which implies

$$\lambda^{s^2} \le \frac{2c_1\, M}{m\, c_2}. \tag{3.11}$$

If we suppose that $|E^{(h)}_{k_n}| \ne m$ for infinitely many indexes, then $\lambda = \mathrm{sgn}(m - |E^{(h)}_{k_n}|)h^{-\frac{s}{1+s}}$ which is a contradiction to (3.11) if h is sufficiently small. We have thus proved item *i)*. Since, for n large enough, $|E^{(h)}_n| = m$, the sequence $\{P^s(E^{(h)}_n)\}_{n\in\mathbb{N}}$ is eventually non-increasing, from which item *ii)* follows. Knowing the exact values of the volume and $s-$perimeter of any limit point, we are able to compute K and obtain the convergence in L^1 of the whole sequence. Moreover, arguing as in [8] we conclude the convergence in C^k for every $k \in \mathbb{N}$ via a bootstrap method. □

In order to prove the main theorem, we need to recall some results of [27].

Lemma 3.10. *Let $\eta > 0$. There exists $\delta > 0$ with the following property: if $f_1, f_2 \in C^1(\partial B)$ with $\|f_i\|_{C^1(\partial B)} \le \delta$ and $|B_{f_i}| = |B|$ for $i = 1, 2$ we have*

$$C_1(1-\eta)\|f_1 - f_2\|^2_{L^2(B)} \le \mathcal{D}(B_{f_1}, B_{f_2}) \le C_1(1+\eta)\|f_1 - f_2\|^2_{L^2(B)} \tag{3.12}$$

$$\frac{1-\eta}{2}\int_{\partial B_{f-1}} \mathrm{sd}^2_{B_{f_2}}\,\mathrm{d}\mathcal{H}^{N-1} \le \mathcal{D}(B_{f_1}, B_{f_2}) \le \frac{1+\eta}{2}\int_{\partial B_{f-1}} \mathrm{sd}^2_{B_{f_2}}\,\mathrm{d}\mathcal{H}^{N-1} \tag{3.13}$$

$$|\mathrm{bar}(B_{f_1}) - \mathrm{bar}(B_{f_2})|^2 \le C_2\|f_1 - f_2\|^2_{L^2(B)} \le \frac{C_2}{C_1(1-\eta)}\mathcal{D}(B_{f_1}, B_{f_2})$$

for suitable constants C_1, $C_2 > 0$.

The following lemma proves the crucial dissipation-dissipation inequality (3.14). This result will play a central role in the proof of Theorem 0.1. Its proof is based on the Alexandrov-type estimate contained in Theorem 0.3, and it is the same of [14, Lemma 5.4].

Lemma 3.11. *Let $h > 0$. There exist constants $C(h, m, s)$, $\delta > 0$ with the following property: given two normal deformations $B^{(m)}_{f_1}$, $B^{(m)}_{f_2}$ of $B^{(m)}$ with $f_i \in C^2(\partial B^{(m)})$, $\|f_i\|_{C^1(\partial B^{(m)})} \le \delta$, and such that $|B^{(m)}_{f_2}| = m$, $\mathrm{bar}(B^{(m)}_{f_2}) = 0$ and*

$$H^s_{B^{(m)}_{f_2}} + \frac{\mathrm{sd}_{B^{(m)}_{f_1}}}{h} = \lambda \quad \text{on} \quad \partial B^{(m)}_{f_2} \tag{3.14}$$

for some $\lambda \in \mathbb{R}$, we have

$$\mathcal{D}(B^{(m)}, B^{(m)}_{f_2}) \le C\mathcal{D}(B^{(m)}_{f_2}, B^{(m)}_{f_1}).$$

Proof. By Theorem 0.3, for δ sufficiently small, we get by using (3.14)

$$\begin{aligned}\|f_2\|^2_{L^2(\partial B^{(m)})} &\le C\|H^s_{B^{(m)}_{f_2}} - \overline{H}^s_{B^{(m)}_{f_2}}\|^2_{L^2(\partial B^{(m)})} \le C\|H^s_{B^{(m)}_{f_2}} - \lambda\|^2_{L^2(\partial B^{(m)})} \\ &\le C\|H^s_{B^{(m)}_{f_2}} - \lambda\|^2_{L^2(\partial B^{(m)}_{f_2})} = \frac{C}{h^2}\int_{\partial B^{(m)}_{f_2}} \mathrm{sd}^2_{B^{(m)}_{f_1}} \, \mathrm{d}\mathcal{H}^{N-1},\end{aligned}$$

where the third inequality follows by bounding the Jacobian of the change of variables by 1 (up to taking δ sufficiently small). By combining the previous inequalities with (3.12) and (3.13), we obtain the thesis. □

We now prove Theorem 0.1. We will follow closely the proofs of [27, Theorem 3.3] and [14, Theorem 1.1]. The main difference is that we use the fractional perimeter framework previously studied instead of the classical one. We present a sketch of the proof.

Proof. We start by sketching the proof of the exponential decay of the dissipations following Step 1 in [27, Theorem 3.3].

From Proposition 3.9 we know that any limit point of the discrete flow is given by the union of K disjoint balls, all having volume m/K. We then use two competitors to obtain a discrete Gronwall-type inequality. Firstly, testing the minimality of $E^{(h)}_n$ with $E^{(h)}_{n-1}$ and summing from n to infinity, we obtain

$$\sum_{k \ge n-1} \mathcal{D}(E^{(h)}_k, E^{(h)}_{n-1}) \le P(E^{(h)}_n) - P^s_\infty = P(E^{(h)}_n) - KP^s(B^{(m/K)}).$$

On the other hand, recalling Proposition 3.9, the sets $(E^{(h)}_n)^i - \mathrm{bar}((E^{(h)}_n)^i) =: (E^{(h)}_n)^i - \xi^i_n$ are eventually $C^{1,\alpha}-$deformations of $B^{(m/K)}$, having volume $|(E^{(h)}_n)^i| = m^i_n$. We consider the admissible competitor for $E^{(h)}_n$ given by

$$\mathcal{B}_n = \bigcup_{i=1}^K \left(B^{(m^i_{n-1})} + \xi^i_{n-1}\right).$$

Testing the minimality of $E^{(h)}_n$ against $\mathcal{B}_n$, one can obtain, by employing Lemma 3.11, that

$$P^s(E^{(h)}_n) - P^s(\mathcal{B}_n) \le C\mathcal{D}(E^{(h)}_{n-1}, E^{(h)}_{n-2}).$$

Recalling that, if a measurable set F has L disjointed connected components F^i, $i = 1, \dots, L$, then

$$P^s(F) = \sum_{i=1}^L P^s(F^i) - 2\sum_{i<j} \int_{F^i}\int_{F^j} \frac{1}{|x-y|^{N+s}} \, \mathrm{d}x \, \mathrm{d}y,$$

by concavity, we estimate

$$P^s(\mathcal{B}_n) \le \sum_{i=1}^{K} P^s(B^{(m^i_{n-1})}) \le K P^s(B^{(m/K)}).$$

Thus, combining the previous two estimates, we obtain the discrete Gronwall-type estimate

$$\sum_{k\ge n-1} \mathcal{D}(E_k^{(h)}, E_{n-1}^{(h)}) \le C\mathcal{D}(E_{n-1}^{(h)}, E_{n-2}^{(h)}).$$

Finally, employing [27, Lemma 3.10] we conclude the exponential convergence of the dissipations

$$\mathcal{D}(E_n^{(h)}, E_{n-1}^{(h)}) \le \left(1 - \frac{1}{C+1}\right)^{\frac{n}{2}} (P^s(E_0) - K P^s(B^{(m/K)})).$$

From now on, one can follow directly the proof of [27, Theorem 3.3] employing Lemma 3.10 to conclude that the discrete flow $E_n^{(h)}$ is eventually contained in a compact set and converges in C^k to a union of K disjoint balls. Now, from Proposition 3.7 we deduce that the limit point is indeed a single ball, having volume equal to m, thus reaching the conclusion of the proof. □

As a corollary of the previous result, we prove the following characterization of the long-time behaviour of the discrete volume-preserving mean curvature flow associated with the classical perimeter. The definition of the discrete flow is the analogous of Definition 3.2 in the local case, where we consider the following minimization problem

$$\min\left\{P(F) + \frac{1}{h}\int_F \mathrm{sd}_E(x)\,\mathrm{d}x + \frac{1}{\sqrt{h}}||F| - m| : F \subset \mathbb{R}^N\right\}, \tag{3.15}$$

which is essentially (3.1) but with P replacing P^s. This is the same functional considered in [28]. The previous definition differs from the one considered in [27], as in that case they required the competitors to have fixed volume. One can see (compare [27, Proposition 2.2]) that the volume-constrained problem is equivalent to a penalized one *with a large enough penalization constant.* Namely, their incremental problem can be written as

$$\min\left\{P(F) + \frac{1}{h}\int_F \mathrm{sd}_E(x)\,\mathrm{d}x + \Lambda||F| - m| : F \subset \mathbb{R}^N\right\},$$

where Λ some large enough constant. It is likely that for h small enough the two problems above are equivalent but at the moment it has not been proved. A careful study of the previous results in the nonlocal case highlights that the classical perimeter behaves essentially in the same manner. In particular, Proposition 3.9 still holds for $s = 1$, replacing "P^1" with the classical perimeter. Thus, we conclude the following.

Proposition 3.12. *Let m, $M > 0$ and let E_0 be an initial set with $P(E_0) \le M$, $|E_0| = m$. Then, for $h = h(M, m)$ small enough the following holds: for any discrete flow $E_n^{(h)}$ starting from E_0, there exist $x_i \in \mathbb{R}^N$, $i = 1, \dots, K$, where $K = N^{-N}\omega_N m^{1-N} P_\infty^N$, such that*

$$E_n^{(h)} \to \bigcup_{i=1}^{K} \left(B^{(m/K)} + x_i\right)$$

in C^k for all $k \in \mathbb{N}$. Moreover, the convergence is exponentially fast.

The proof is essentially the same, the major difference being that the limit point of the flow now are union of disjoint balls (see [27, Proposition 3.1] for details) and not a single ball.

Appendix A. Existence of flat flows

This appendix is devoted to prove the existence of a flat flow, as previously defined in the introduction. We follow the lines of [28, Section 3]. In this appendix we drop the assumption on the dimension and work in $\mathbb{R}^N$, $N \geq 2$. We start by remarking that the minimality of the sets $E_n^{(h)}$ implies

$$\begin{aligned}
&P^s(E_t^{(h)}) + \frac{1}{h}\int_{E_t^{(h)}\triangle E_{t-h}^{(h)}} \mathrm{dist}_{\partial E_{t-h}^{(h)}}(x)\,\mathrm{d}x + \frac{1}{h^{s/1+s}}||E_t^{(h)}| - m| \\
&\leq P^s(E_{t-h}^{(h)}) + \frac{1}{h^{s/1+s}}||E_{t-h}^{(h)}| - m| \qquad \forall t \in [h, +\infty)
\end{aligned} \tag{A.1}$$

and, by iterating (A.1) and using $|E_0| = m$, we get

$$P^s(E_t^{(h)}) \leq P^s(E_0), \tag{A.2}$$

$$\frac{1}{h^{s/1+s}}||E_t^{(h)}| - m| \leq P^s(E_0), \tag{A.3}$$

$$\int_h^T \int_{E_t^{(h)}\triangle E_{t-h}^{(h)}} \frac{\mathrm{dist}_{\partial E_{t-h}^{(h)}}(x)}{h}\,\mathrm{d}x \leq P^s(E_0),$$

for all $t \geq 0$ and for every $T > h$. We are then able to bound the L^1-distance between two consecutive sets of the discrete flow by classical arguments (see [26]).

Proposition A.1. *Let $t > 0$. Then*

$$|E_t^{(h)}\triangle E_{t-h}^{(h)}| \leq C\left(l^s P^s(E_{t-h}^{(h)}) + \frac{1}{l}\int_{E_t^{(h)}\triangle E_{t-h}^{(h)}} \mathrm{dist}_{\partial E_{t-h}^{(h)}}(x)\,\mathrm{d}x\right) \qquad \forall l \leq \gamma h^{1/1+s}.$$

Proof. In order to estimate $|E_t^{(h)}\triangle E_{t-h}^{(h)}|$, we split it into two parts:

$$|E_t^{(h)}\triangle E_{t-h}^{(h)}| \leq |\{x \in E_t^{(h)}\triangle E_{t-h}^{(h)} : \mathrm{dist}_{\partial F}(x) \leq l\}| + |\{x \in E_t^{(h)}\triangle E_{t-h}^{(h)} : \mathrm{dist}_{\partial E_{t-h}^{(h)}}(x) \geq l\}|.$$

The second term is estimated by

$$|\{x \in E_t^{(h)}\triangle E_{t-h}^{(h)} : \mathrm{dist}_{\partial E_{t-h}^{(h)}}(x) \geq l\}| \leq \frac{1}{l}\int_{E_t^{(h)}\triangle E_{t-h}^{(h)}} \mathrm{dist}_{\partial E_{t-h}^{(h)}}(x)\,\mathrm{d}x.$$

To estimate the first term, we use a covering argument to find a collection of disjoint balls $\{B_l(x_i)\}_{i\in I}$ with $x_i \in \partial E_{t-h}^{(h)}$ and $I \subset \mathbb{N}$ a finite set such that $\partial E_{t-h}^{(h)} \subset \cup_{i\in I} B_{2l}(x_i)$. Observe that, by (3.7), (3.8) and the relative isoperimetric inequality, for every $i \in I$, we get

$$\begin{aligned}
|B_{3l}(x_i)| &\leq C\min\{|E_{t-h}^{(h)} \cap B_l(x_i)|, |B_l(x_i) \setminus E_{t-h}^{(h)}|\} \\
&\leq C\mathcal{L}_s(E_{t-h}^{(h)} \cap B_l(x_i), B_l(x_i) \setminus E_{t-h}^{(h)})^{N/N-s} \\
&\leq Cl^s\,\mathcal{L}_s(E_{t-h}^{(h)} \cap B_l(x_i), B_l(x_i) \setminus E_{t-h}^{(h)}) \\
&\leq Cl^s\,\mathcal{L}_s(E_{t-h}^{(h)} \cap B_l(x_i), \mathbb{R}^N \setminus E_{t-h}^{(h)}).
\end{aligned}$$

Since the set $\{x \in E_t^{(h)} \triangle E_{t-h}^{(h)} : \mathrm{dist}_{\partial E_{t-h}^{(h)}}(x) \le l\}$ is covered by $\{B_{3l}(x_i)\}_{i\in I}$, by summing over i and by the choice of the balls $\{B_l(x_i)\}_{i\in I}$, we obtain

$$\begin{aligned}|\{x \in E_t^{(h)} \triangle E_{t-h}^{(h)} : \mathrm{dist}_{\partial E_{t-h}^{(h)}}(x) \le l\}| &\le \sum_{i\in I} |B_{3l}(x_i)| \\ &\le Cl^s \sum_{i\in I} \mathcal{L}_s(E_{t-h}^{(h)} \cap B_l(x_i), \mathbb{R}^N \setminus E_{t-h}^{(h)}) \\ &\le Cl^s P^s(E_{t-h}^{(h)}).\end{aligned}$$

□

The above results yields, in turn, uniform Holder continuity in time for the discrete flow.

Proposition A.2. *Let $h \le 1$, then it holds*

$$|E_{t_1}^{(h)} \triangle E_{t_2}^{(h)}| \le C|t_1 - t_2|^{s/s+1} \quad \forall 0 \le t_1 \le t_2 < +\infty. \tag{A.4}$$

Proof. It is enough to consider the case $t_2 - t_1 \ge h$. Let $j \in \mathbb{N}$ and $k \in \mathbb{N} \setminus \{0\}$ be such that $t_1 \in [jh, (j+1)h)$ and $t_2 \in [(j+k)h, (j+k+1)h)$. Then, we can use Proposition A.1 with $l = \gamma h/|t_1 - t_2|^{s/s+1}$ (note that $l \le \gamma h^{1/s+1}$ by the assumption $t_2 - t_1 \ge h$) and estimate in the following way

$$\begin{aligned}|E_{t_1}^{(h)} \triangle E_{t_2}^{(h)}| &\le \sum_{i=1}^k |E_{(j+i)h}^{(h)} \triangle E_{(j+i-1)h}^{(h)}| \\ &\le C \sum_{i=1}^k \frac{h^s}{|t_1 - t_2|^{s^2/s+1}} P^s(E_{(j+i-1)h}^{(h)}) \\ &\quad + C \sum_{i=1}^k \frac{|t_1 - t_2|^{s/s+1}}{h} \int_{E_{(j+i)h}^{(h)} \triangle E_{(j+i-1)h}^{(h)}} |\mathrm{sd}_{E_{(j+i-1)h}^{(h)}}| \, dx.\end{aligned}$$

By using (A.1) we estimate the sum above by

$$\begin{aligned}|E_{t_1}^{(h)} \triangle E_{t_2}^{(h)}| &\le C \sum_{i=1}^k \frac{h^s}{|t_1 - t_2|^{s^2/s+1}} P(E_0) \\ &\quad + C \sum_{i=1}^k |t_1 - t_2|^{s/s+1} (P(E_{(j+i-1)h}^{(h)}) - P(E_{(j+i)h}^{(h)})) \\ &\quad + C \sum_{i=1}^k \frac{|t_1 - t_2|^{s/s+1}}{h^{s/s+1}} \left(||E_{(j+i-1)h}^{(h)}| - m| - ||E_{(j+i)h}^{(h)}| - m| \right) \\ &\le C \frac{kh^s}{|t_1 - t_2|^{s^2/s+1}} P(E_0) + C|t_1 - t_2|^{s/s+1} (P(E_{t_1}^{(h)}) - P(E_{t_2}^{(h)})) \\ &\quad + C \frac{|t_1 - t_2|^{s/s+1}}{h^{s/s+1}} \left(||E_{t_1}^{(h)}| - m| - ||E_{t_2}^{(h)}| - m| \right).\end{aligned}$$

Now, by (A.2) and (A.3), we get

$$|E_{t_1}^{(h)} \triangle E_{t_2}^{(h)}| \le C|t_1 - t_2|^{s/s+1} P(E_0),$$

where we used that $kh^s \le 2|t_1 - t_2| h^{s-1} \le 2|t_1 - t_2|^s$ since $h \le |t_1 - t_2|$. □

We are now able to prove the existence of fractional flat flows, defined as L^1_{loc}−limit points of the discrete flow previously introduced.

Proposition A.3. *Let E_0 be a bounded initial set of finite fractional perimeter and volume m. For any $h>0$, let $\{E_t^{(h)}\}_{t\geq 0}$ be a discrete flow. Then, there exist a family of sets $\{E_t\}_{t\geq 0}$ of finite fractional perimeter and a subsequence $h_k \to 0$ such that, as $k\to\infty$, it holds*

$$E_t^{(h_k)} \to E_t \quad \text{in } L^1_{loc}, \quad \text{for all } t\in[0,+\infty).$$

Moreover, the flow satisfies $\forall\, 0\leq s\leq t$,

$$\begin{aligned}|E_t \triangle E_s| &\leq C|t-s|^{\frac{s}{1+s}},\\ P^s(E_t) &\leq P^s(E_0).\end{aligned}$$

Proof. We follow [28, Theorem 2.2]. Considering $t\in\mathbb{Q}^+$, from (A.2) and the compactness of sets of finite fractional perimeter, we find a subsequence $h_k\to 0$ such that

$$L^1_{loc} - \lim_{k\to\infty} E_t^{(h_k)} = E_t \quad \forall t\in\mathbb{Q}^+.$$

By the triangular inequality, it is easy to see that (A.4) passes to the limit. Finally, a simple continuity argument implies that the whole sequence $E_t^{(h_k)}$ converges to the sets E_t for all $t\in[0,+\infty)$. ☐

If we assume the following hypothesis:

(1) given $T>0$ and an initial bounded set E_0 of finite fractional perimeter, there exists $R>0$ independent of h such that $E_t^{(h)}\subset B_R$ for all $h>0, t\in[0,T]$,

we are able to prove that the flat flow is indeed volume-preserving.

Corollary A.4. *Under hypothesis 1, let E_0 be a bounded initial set of finite fractional perimeter and volume m. For any $h>0$, let $\{E_t^{(h)}\}_{t\geq 0}$ be a discrete flow. Then, there exist a family of sets $\{E_t\}_{t\geq 0}$ of finite fractional perimeter and a subsequence $h_k\to 0$ such that, as $k\to\infty$, it holds*

$$E_t^{(h_k)} \to E_t \quad \text{in } L^1 \quad \text{for all } t\in[0,+\infty).$$

Moreover, the flow satisfies $\forall\, 0\leq s\leq t$,

$$\begin{aligned}|E_t| &= |E_0|\\ |E_t \triangle E_s| &\leq C|t-s|^{\frac{s}{1+s}},\\ P^s(E_t) &\leq P^s(E_0).\end{aligned}$$

Proof. The proof is analogous to the one of Proposition A.3. By uniform boundedness, the limits are in L^1 instead of L^1_{loc}. Then, passing to the limit $h\to 0$ in (A.1) we conclude that $|E_t|=m$ for all $t\geq 0$. ☐

Acknowledgements. The authors warmly thank Professor Massimiliano Morini for insightful discussions and the referee for helpful suggestions that helped to improve the paper. D. De Gennaro and Anna Kubin wish to acknowledge the hospitality of the Faculty of Mathematics of the Technical University of Munich, where part of this research was carried out. Andrea Kubin and Anna Kubin are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). D. De Gennaro has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 94532. . Andrea Kubin was supported by the DFG Collaborative Research Center TRR 109 "Discretization in Geometry and Dynamics".

References

[1] N. Abatangelo and E. Valdinoci. “A notion of nonlocal curvature”. In: *Numer. Funct. Anal. Optim.* 35.7-9 (2014), pp. 793–815. ISSN: 0163-0563. DOI: 10.1080/01630563.2014.901837. URL: https://doi.org/10.1080/01630563.2014.901837.

[2] F. Almgren, J. E. Taylor, and L. Wang. “Curvature-driven flows: a variational approach”. In: *SIAM J. Control Optim.* 31.2 (1993), pp. 387–438. ISSN: 0363-0129. DOI: 10.1137/0331020. URL: https://doi.org/10.1137/0331020.

[3] J. Bourgain, H. Brezis, and P. Mironescu. “Another look at Sobolev spaces”. In: *Optimal control and partial differential equations.* IOS, Amsterdam, 2001, pp. 439–455.

[4] X. Cabré, M. M. Fall, J. Solà-Morales, and T. Weth. “Curves and surfaces with constant nonlocal mean curvature: meeting Alexandrov and Delaunay”. In: *J. Reine Angew. Math.* 745 (2018), pp. 253–280. ISSN: 0075-4102. DOI: 10.1515/crelle-2015-0117. URL: https://doi.org/10.1515/crelle-2015-0117.

[5] L. Caffarelli, J.-M. Roquejoffre, and O. Savin. “Nonlocal minimal surfaces”. In: *Comm. Pure Appl. Math.* 63 (2010), pp. 1111–1144.

[6] L. Caffarelli and E. Valdinoci. “Regularity properties of nonlocal minimal surfaces via limiting arguments”. In: *Adv. Math.* 248 (2013), pp. 843–871. ISSN: 0001-8708. DOI: 10.1016/j.aim.2013.08.007. URL: https://doi.org/10.1016/j.aim.2013.08.007.

[7] A. Cesaroni, L. De Luca, M. Novaga, and M. Ponsiglione. “Stability results for nonlocal geometric evolutions and limit cases for fractional mean curvature flows”. In: *Comm. Partial Differential Equations* 46.7 (2021), pp. 1344–1371. ISSN: 0360-5302. DOI: 10.1080/03605302.2021.1875485. URL: https://doi.org/10.1080/03605302.2021.1875485.

[8] A. Cesaroni and M. Novaga. “Long-time behavior of the mean curvature flow with periodic forcing”. In: *Communications in Partial Differential Equations* 38.5 (2013), pp. 780–801.

[9] A. Cesaroni and M. Novaga. “Stability of the ball under volume preserving fractional mean curvature flow”. In: *preprint* (2022).

[10] A. Chambolle, M. Morini, and M. Ponsiglione. “Nonlocal curvature flows”. In: *Arch. Ration. Mech. Anal.* 218.3 (2015), pp. 1263–1329. ISSN: 0003-9527. DOI: 10.1007/s00205-015-0880-z. URL: https://doi.org/10.1007/s00205-015-0880-z.

[11] E. Cinti, C. Sinestrari, and E. Valdinoci. “Convex sets evolving by volume-preserving fractional mean curvature flows”. In: *Anal. PDE* 13.7 (2020), pp. 2149–2171. ISSN: 2157-5045. DOI: 10.2140/apde.2020.13.2149. URL: https://doi.org/10.2140/apde.2020.13.2149.

[12] E. Cinti, C. Sinestrari, and E. Valdinoci. “Neckpinch singularities in fractional mean curvature flows”. In: *Proc. Amer. Math. Soc.* 146.6 (2018), pp. 2637–2646. ISSN: 0002-9939. DOI: 10.1090/proc/14002. URL: https://doi.org/10.1090/proc/14002.

[13] G. Ciraolo, A. Figalli, F Maggi, and M. Novaga. “Rigidity and sharp stability estimates for hypersurfaces with constant and almost-constant nonlocal mean curvature”. In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2018.741 (2018), pp. 275–294. DOI: doi:10.1515/crelle-2015-0088. URL: https://doi.org/10.1515/crelle-2015-0088.

[14] D. De Gennaro and A. Kubin. “Long Time Behaviour of the Discrete Volume Preserving Mean Curvature Flow in the Flat Torus”. In: *preprint* (2022).

[15] Matias Gonzalo Delgadino and Francesco Maggi. “Alexandrov’s theorem revisited”. In: *Anal. PDE* 12.6 (2019), pp. 1613–1642. ISSN: 2157-5045. DOI: 10.2140/apde.2019.12.1613. URL: https://doi.org/10.2140/apde.2019.12.1613.

[16] Agnese Di Castro, Matteo Novaga, Berardo Ruffini, and Enrico Valdinoci. “Nonlocal quantitative isoperimetric inequalities”. In: *Calc. Var. Partial Differential Equations* 54.3 (2015), pp. 2421–2464. ISSN: 0944-2669. DOI: 10.1007/s00526-015-0870-x. URL: https://doi.org/10.1007/s00526-015-0870-x.

[17] E. Di Nezza, G. Palatucci, and E. Valdinoci. “Hitchhikers guide to the fractional Sobolev spaces”. In: *Bulletin des Sciences Mathématiques* 136.5 (2012), pp. 521–573.

[18] J. Escher and G. Simonett. "The volume preserving mean curvature flow near spheres". In: *Proc. Amer. Math. Soc.* 126.9 (1998), pp. 2789–2796. ISSN: 0002-9939. DOI: 10.1090/S0002-9939-98-04727-3. URL: https://doi.org/10.1090/S0002-9939-98-04727-3.

[19] A. Figalli, N. Fusco, F. Maggi, V. Millot, and M. Morini. "Isoperimetry and stability properties of balls with respect to nonlocal energies". In: *Comm. Math. Phys.* 336.1 (2015), pp. 441–507. ISSN: 0010-3616. DOI: 10.1007/s00220-014-2244-1. URL: https://doi.org/10.1007/s00220-014-2244-1.

[20] B.X. Han and A. Pinamonti. "On the asymptotic beahaviour of the fractional Sobolev seminorms in metric measure spaces: Bourgain-Brezis-Mironescu's Theorem revisited". In: *preprint* (2021).

[21] G. Huisken. "The volume preserving mean curvature flow". In: *J. Reine Angew. Math.* 382 (1987), pp. 35–48. ISSN: 0075-4102. DOI: 10.1515/crll.1987.382.35. URL: https://doi.org/10.1515/crll.1987.382.35.

[22] C. Imbert. "Level set approach for fractional mean curvature flows". In: *Interfaces Free Bound.* 11.1 (2009), pp. 153–176. ISSN: 1463-9963. DOI: 10.4171/IFB/207. URL: https://doi.org/10.4171/IFB/207.

[23] V. Julin and D. A. La Manna. "Short time existence of the classical solution to the fractional mean curvature flow". In: *Ann. Inst. H. Poincaré C Anal. Non Linéaire* 37.4 (2020), pp. 983–1016. ISSN: 0294-1449. DOI: 10.1016/j.anihpc.2020.02.007. URL: https://doi.org/10.1016/j.anihpc.2020.02.007.

[24] V. Julin, M. Morini, M. Ponsiglione, and E. Spadaro. "The Asymptotics of the Area-Preserving Mean Curvature and the Mullins-Sekerka Flow in Two Dimensions". In: (2021). cvgmt preprint. URL: http://cvgmt.sns.it/paper/5399/.

[25] B. Krummel and F. Maggi. "Isoperimetry with upper mean curvature bounds and sharp stability estimates". In: *Calc. Var. Partial Differential Equations* 56.2 (2017), Paper No. 53, 43. ISSN: 0944-2669. DOI: 10.1007/s00526-017-1139-3. URL: https://doi.org/10.1007/s00526-017-1139-3.

[26] S. Luckhaus and T. Sturzenhecker. "Implicit time discretization for the mean curvature flow equation". In: *Calc. Var. Partial Differential Equations* 3.2 (1995), pp. 253–271. ISSN: 0944-2669. DOI: 10.1007/BF01205007. URL: https://doi.org/10.1007/BF01205007.

[27] M. Morini, M. Ponsiglione, and E. Spadaro. "Long time behavior of discrete volume preserving mean curvature flows". In: *J. Reine Angew. Math.* 784 (2022), pp. 27–51. ISSN: 0075-4102. DOI: 10.1515/crelle-2021-0076. URL: https://doi.org/10.1515/crelle-2021-0076.

[28] L. Mugnai, C. Seis, and E. Spadaro. "Global solutions to the volume-preserving mean-curvature flow". In: *Calc. Var. Partial Differential Equations* 55.1 (2016), Art. 18, 23. ISSN: 0944-2669. DOI: 10.1007/s00526-015-0943-x. URL: https://doi.org/10.1007/s00526-015-0943-x.

[29] J. Niinikoski. "Volume preserving mean curvature flows near strictly stable sets in flat torus". In: *J. Differential Equations* 276 (2021), pp. 149–186. ISSN: 0022-0396. DOI: 10.1016/j.jde.2020.12.010. URL: https://doi.org/10.1016/j.jde.2020.12.010.

(Daniele De Gennaro) CEREMADE department, Université Paris-Dauphine, 1 Place du Maréchal de Lattre de Tassigny, 75775, Paris, France

Email address, Daniele De Gennaro: degennaro@ceremade.dauphine.fr

(Andrea Kubin) Zentrum Mathematik-M7, Technische Universität München, Boltzmannstrasse 3, 85747, Garching, Germany

Email address, Andrea Kubin: andrea.kubin@tum.de

(Anna Kubin) Dipartimento di Scienze Matematiche, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129, Turin, Italy

Email address, Anna Kubin: anna.kubin@polito.it