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On the convergence of a novel time-slicing approximation scheme for Feynman path integrals

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In this note we study the properties of a sequence of approximate propagators for the Schrödinger equation, in the spirit of Feynman's path integrals. Precisely, we consider Hamiltonian operators arising as the Weyl quantization of a quadratic form in phase space, plus a bounded potential perturbation in the form of a pseudodifferential operator with a rough symbol. The corresponding Schrödinger propagator belongs to the class of generalized metaplectic operators, a fact that naturally motivates the introduction of a manageable time-slicing approximation scheme consisting of operators of the same type. By leveraging on this design and techniques of wave packet analysis we are able to prove several convergence results with precise rates in terms of the mesh size of the time slicing subdivision, even stronger then those which can be achieved under the same assumptions using the standard Trotter approximation scheme. In particular, we prove convergence in the norm operator topology in L^2, as well as pointwise convergence of the corresponding integral kernels for non-exceptional times.

1 Introduction

The rigorous analysis of Feynman path integrals has been, and still is, a challenging source of intriguing problems for mathematicians from manifold areas. The knowledge accumulated over the last seventy years encompasses several aspects, ranging from foundational issues to applied ones; the interested reader can consult the monographs [2, 39] for in-depth studies on the topic.

Inspired by the custom in physics, the *sequential approach* to path integrals is an operator-theoretic framework aimed at providing explicit representation formulae for the Schrödinger evolution operator in terms of sequences of approximate propagators on L^2 – see for instance [27, 28, 34, 37] in this connection.

In the Euclidean d-dimensional setting we consider the Cauchy problem for the Schrödinger equation

$$\begin{cases} i\hbar\partial_t\psi = (H_0+V)\psi \\ \psi(s,x) = \psi_s(x) \end{cases} \tag{1}$$

with initial datum $\psi_s \in \mathcal{S}(\mathbb{R}^d)$ at time $t=s$. As an illustration of the sequential approach we recall a classical result due to Nelson [41], which relies on the Trotter formula. Under suitable assumptions on the Hamiltonian operator $H = H_0 + V$ (see e.g., [14, 46]) we have that the sequence of approximate propagators

$$T_n(t,s) := \left(e^{-\frac{i}{\hbar}\frac{t-s}{n}H_0} e^{-\frac{i}{\hbar}\frac{t-s}{n}V} \right)^n$$

converges to the Schrödinger evolution operator $U(t,s) = e^{-\frac{i}{\hbar}(t-s)(H_0+V)}$ generated by H in the strong topology of operators on $L^2(\mathbb{R}^d)$:

$$U(t,s)f = \lim_{n\to\infty} T_n(t,s)f, \quad f \in L^2(\mathbb{R}^d).$$

A careful design of the sequence of approximate propagators is of paramount importance in order to obtain stronger convergence results. For instance, the time-slicing approximations introduced by Fujiwara in [25, 26] rely on techniques of oscillatory integral operators and semiclassical analysis. The use of sophisticated mathematical techniques is repaid by deep convergence results in the norm operator topology and also at the subtler level of kernels, even in the semiclassical regime $\hbar \to 0$.

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1.1 The role of Gabor analysis

Only in recent times the techniques of Gabor wave packet analysis have been successfully brought into play in the study of mathematical path integrals, leading to relevant advances and new directions to be explored [45].

By way of illustration let us mention here the papers [42] and [44] – an expository overview of these and other related results can be found in [54]. The power of a phase-space approach is fully displayed in the first contribution, where it allows one to "break the barriers" of the standard L^2 setting and derive convergence results for Fujiwara's time-slicing approximations in L^p-Sobolev spaces with $p \neq 2$.

The second paper focuses on the pointwise convergence of integral kernels of the Feynman-Trotter approximate propagators $T_n(t,s)$ introduced above. Generally speaking, this issue has been heuristically conjectured by Feynman himself [20, 21] and remained a widely open problem for a long time, at least until the pioneering results by Fujiwara already mentioned above – which actually involve sufficiently regular potentials and different, more refined parametrices. In the paper [44] the problem has been solved in the case where H_0 is a quadratic Hamiltonian and V is a bounded potential perturbations with low regularity, mainly using tools and function spaces of Gabor analysis [10, 30].

This is a convenient stage where to briefly discuss the basic notions of time-frequency analysis, both for clarity and future reference. Recall that a phase-space representation of a signal u can be obtained as a decomposition with respect to *Gabor wave packets* of the form

$$\pi(z)g(y) = e^{2\pi i\xi\cdot y}g(y-x), \quad z=(x,\xi)\in\mathbb{R}^{2d},$$

where g is a fixed (non-trivial) function on $\mathbb{R}^d$ that is well localized in the time-frequency space $\mathbb{R}^d\times\widehat{\mathbb{R}^d}\simeq\mathbb{R}^{2d}$. To be more precise, the function $V_gf(z) \coloneqq \langle u,\pi(z)g\rangle$ of z resulting from duality pairing between $f\in\mathcal{S}'(\mathbb{R}^d)$ and $\pi(z)g$ with $g\in\mathcal{S}(\mathbb{R}^d)$ is called the *Gabor transform* of f.

A concrete, alternative point of view on the procedure just described comes from signal analysis. Consider for instance the case where $f\in L^2(\mathbb{R}^d)$. Then we have explicitly

$$V_gf(x,\xi) = \int_{\mathbb{R}^d} e^{-2\pi i\xi\cdot y}f(y)\overline{g(y-x)}\,dy = \mathcal{F}(f\cdot\overline{g(\cdot-x)})(\xi),$$

and thus the Gabor transform corresponds to the Fourier transform of the variable slice of the signal f obtained by localization near x with a sliding window function g. This remark explains why the Gabor transform is also widely known as the *short-time Fourier transform.*

Modulation spaces can be naturally introduced at this stage as spaces of distributions characterized by prescribed summability conditions for the corresponding phase-space representations [3]. In the simplest setting, for $1\le p\le\infty$ and a fixed $g\in\mathcal{S}(\mathbb{R}^d)\setminus\{0\}$ we set

$$M^p(\mathbb{R}^d) \coloneqq \{f\in\mathcal{S}'(\mathbb{R}^d) : \|u\|_{M^p}<\infty\}, \quad \|f\|_{M^p} \coloneqq \|V_gf\|_{L^p(\mathbb{R}^{2d})}.$$

Modulation spaces constitute a family of Banach spaces, increasing in p, whose norm is stable under change of window – precisely, different choices of g result in equivalent norms. We emphasize that many typical function spaces of harmonic analysis turn out to coincide with (generalized) modulation spaces. The most important example is $M^2(\mathbb{R}^d)\simeq L^2(\mathbb{R}^d)$. Together with the extremal spaces $M^1(\mathbb{R}^d)$ (the original Feichtinger algebra [16, 38]) and $M^\infty(\mathbb{R}^d)$ (the space of *mild distributions* [19]) they provide a convenient framework for basic Fourier analysis in the form of a Banach-Gelfand triple [22], also known in quantum physics and spectral theory as the formalism of rigged Hilbert spaces.

1.2 The Schrödinger propagator and the *FIO* class

Modulation spaces are widely used in the aforementioned papers [42, 44] both to rigorously frame the problem of path integrals from a phase space perspective and to tune the regularity of potential perturbations, following an established series of results for the Schrödinger equation - see the monograph [10] for a state-of-the-art account.

We focus here on the problem (1) with $\hbar = 1/2\pi$ for consistency with the previous section, under similar yet more general assumptions than those in [44]. Precisely, we assume that H_0 is the Weyl quantization Q^{w} of a homogeneous quadratic polynomial $Q:\mathbb{R}^{2d}\to\mathbb{R}$ and the potential perturbation V is a pseudodifferential operator with Weyl symbol $\sigma(t,\cdot)=\sigma_t(\cdot)$, in the *Sjöstrand class* $M^{\infty,1}(\mathbb{R}^{2d})$ [51] – that is a modulation space characterized by a mixed Lebesgue-type norm in phase-space:

$$\|\sigma_t\|_{M^{\infty,1}} \coloneqq \int_{\mathbb{R}^{2d}}\sup_{z\in\mathbb{R}^{2d}}|V_\Phi\sigma_t(z,\zeta)|\,d\zeta<\infty, \quad \Phi\in\mathcal{S}(\mathbb{R}^{2d})\setminus\{0\}.$$

It is additionally assumed that the correspondence $\mathbb{R} \ni t \mapsto \sigma(t,\cdot) \in M^{\infty,1}(\mathbb{R}^{2d})$ is continuous in a mild sense - see Definition 2.4 below for further details. We highlight the article [35] for the pioneering use of Weyl operators in path integral problems.

For concreteness, let us briefly discuss here the case where $V = V_t\times$ is a standard multiplication operator by $V_t \in M^{\infty,1}(\mathbb{R}^d)$ (see Section 2.2). To give a flavour of typical potentials in this family we remark that the latter includes bounded functions that locally coincide with the Fourier transform of an integrable function, hence also continuous. This is in fact the least regularity level of such potentials, since members of $M^{\infty,1}(\mathbb{R}^d)$ are not differentiable in general – for instance, any piecewise linear function belongs to the Sjöstrand class, see Remark 2.2 for details and further examples. Nevertheless, a relevant subclass of $M^{\infty,1}(\mathbb{R}^d)$ is the space $C_{\mathrm{b}}^{\infty}(\mathbb{R}^d)$ of smooth bounded functions with bounded derivatives of any order. We also have $\mathcal{FM}(\mathbb{R}^d) \subset M^{\infty,1}(\mathbb{R}^d)$, where $\mathcal{FM}(\mathbb{R}^d)$ is the space of Fourier transforms of complex finite measures (cf. [44, Proposition 3.4]). Potentials of this type already appeared several times in the literature on mathematical path integrals, especially in connection with the Parseval duality approach introduced by Itô [36] and developed by Albeverio et al. [1, 2].

Consider now the Hamiltonian $H_0 = Q^{\mathrm{w}}$ as above. This setting encompasses some fundamental settings like the free particle and the harmonic oscillator, possibly in the presence of a uniform magnetic field. A classical result of harmonic analysis states that the evolution operator $U_0(t,s) = e^{-2\pi i(t-s)H_0}$ is a *metaplectic operator* associated with the phase space flow $\mathbb{R} \ni \tau \mapsto S_\tau \in \mathrm{Sp}(d,\mathbb{R})$ of the corresponding classical system, that is $U_0(t,s) = c\mu(S_{t-s})$ for some $c = c(t-s) \in \mathbb{C}$ with $|c| = 1$ - see [11, 24] and Section 2.4 for an expanded account.

The structure of the full Schrödinger propagator $U(t)$ in the presence of a potential V as above can be investigated by means of standard arguments from the theory of operator semigroups that have their roots the *perturbation method* of quantum mechanics. The main steps of this analysis are outlined in Section 2.5 below, whereas here we recall the final result, that is

$$\psi(t,x) = U(t,s)\psi_s(x), \quad U(t,s) = U_0(t,s)a(t,s)^{\mathrm{w}}, \tag{2}$$

where $a(t,s)$ is a symbol in the Sjöstrand class (cf. Lemma 2.6 below) defined by a Dyson-Phillips expansion:

$$\begin{aligned} a(t,s) &= \mathcal{T}\exp\left(-2\pi i \int_s^t b(\tau,s)d\tau\right) \\ &\coloneqq 1 + \sum_{n\geq 1} (-2\pi i)^n \int_s^t \int_s^{t_1} \ldots \int_s^{t_{n-1}} b(t_1,s)\#\cdots\#b(t_n,s)\, dt_n \ldots dt_1, \end{aligned} \tag{3}$$

where $b(\tau,s) \coloneqq \sigma_\tau \circ S_{\tau-s}$ and $\#$ denotes Weyl's twisted product of symbols (see Section 2.2 for further details). The symbol $\mathcal{T}\exp$ entails the key notion of causal time ordering [13, 23], which is implemented here by integration over a bounded simplex in the exponential-like series above.

It is intuitively clear that $U(t,s)$ is intimately connected to the homogenous propagator $U_0(t,s)$ described before. The underlying relationship has been completely elucidated in the papers [5, 6], leading to the introduction of the class $FIO(S_{t-s})$ of *generalized metaplectic operators* associated with the flow S_{t-s} - more details are collected in Section 2.5 for convenience. Roughly speaking, an operator $T \in FIO(S)$, $S \in \mathrm{Sp}(d,\mathbb{R})$, can be thought of as a perturbation of $\mu(S)$ that still retain some of its good features, such as continuity on modulation spaces ($T \in \mathcal{L}(M^p(\mathbb{R}^d))$), stability under composition, explicit representation as a Fourier integral operator, approximate localization in phase space near the graph of S by Gabor wave packets.

These properties played a crucial role in the proof of the pointwise convergence of kernels of the Trotter parametrices in [44]. Indeed, it turns out that these approximate propagators are generalized metapletic operators as well. Consider for simplicity the case where $V = \sigma^{\mathrm{w}}$ with $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ and set $\rho_\nu = e^{-2\pi i\nu\sigma}$, $\nu = (t-s)/n$. In light of the fact that the Sjöstrand class is an algebra of Weyl symbols [32] one can prove that $\rho_\nu \in M^{\infty,1}(\mathbb{R}^{2d})$ with

$$e^{-2\pi i\nu V} = \rho_\nu^{\mathrm{w}} = I + 2\pi i\nu\rho_0(t,s)^{\mathrm{w}},$$

where $\rho_0(t,s)$ belong to a bounded subset of $M^{\infty,1}(\mathbb{R}^{2d})$, uniformly with respect to n, we have

$$T_n(t,s) = U_0(t,s)\rho_n(t,s)^{\mathrm{w}} \in FIO(S_{t-s}),$$

$$\rho_n(t,s) = \prod_{k=0}^{n-1} \left(1 + 2\pi i\nu\left(\rho_0 \circ S_{k\nu}\right)\right) \in M^{\infty,1}(\mathbb{R}^{2d}),$$

where the Weyl product (with ordering from left to right along increasing values of k) is understood.

While the previous remarks should shed some light on the suitability and effectiveness of techniques of Gabor analysis in the setting considered in [44], there are some aspects that seem worthy of further attention. Above all, while it was proved that the integral kernels of $T_n(t,s)$ converge to that of the Schrödinger propagator $U(t,s)$, uniformly on compact subsets of $\mathbb{R}^{2d}$, the result comes with no information on the rate at which convergence occurs. Moreover, even if the strong convergence result implied by the Trotter formula has a role in allowing the transfer to the level of kernels, it seems not clear how to cover other operator topologies. The lack of information in connection with rates of convergence or different topologies is not completely surprising, as it is a well-known limitation of the Trotter formula – in spite of its widespread use in the path integral literature, it just provides a qualitative strong convergence result that can be hardly refined or extended whenever concerned with the unitary setting, cf. for instance [56, Appendix D].

In summary, while approximating the Schrödinger propagator $U(t,s)$ with a sequence of operators belonging to the same class sounds wise and actually leads to interesting results, the dissimilarity between the symbols $a(t,s)$ and $\rho_n(t,s)$ makes one wonder if the parametrices arising from the Trotter are too inflexible due to a limited approximation power by design.

1.3 Main results for novel FIO-type approximate propagators

Motivated by the previous discussion, we are lead to design a novel time slicing approximation aimed at better grasping the structure of the target propagator $U(t,s)$. In particular, it is natural to consider a short-time FIO-type approximate propagator $E(t,s)$ such as

$$E(t,s) \coloneqq U_0(t,s)e(t,s)^{\mathrm{w}},$$

where the symbol $e(t,s)$ is defined by mimicking the structure of $a(t,s)$, that is

$$e(t,s) \coloneqq \exp\left(-2\pi i \int_s^t b(\tau,s)d\tau\right). \tag{4}$$

We prove below that $e(t,s) \in M^{\infty,1}(\mathbb{R}^{2d})$. More importantly, there is a crucial gain in the short-time approximation power: in Lemma 3.1 below we prove that $\|a(t,s)-e(t,s)\|_{M^{\infty,1}} = O(|t-s|^2)$, whereas a similar argument shows that $\|a(t,s)-\rho_{n\nu}\|_{M^{\infty,1}} = O(|t-s|)$. While this fact might be judged as a minor improvement, it will be clear below that this is the cornerstone of a multilevel enhancement of convergence results for such novel approximation scheme.

We consider here a more general time slicing pattern than the uniform one associated with the Trotter formula. Without loss of generality assume that $s<t$ and for a given integer $L \ge 1$ consider a subdivision $\Omega_L = \{t_0,\dots,t_L\}$ of the interval $[s,t]$ such that $s=t_0<t_1<\ldots<t_L=t$. Note that the subdivision of $t-s$ underlying the Trotter formula is uniform, namely $\omega(\Omega_L)=(t-s)/L$. We accordingly define the time-slicing approximate propagator as

$$E(\Omega_L;t,s) \coloneqq E(t_L,t_{L-1})\cdots E(t_1,t_0). \tag{5}$$

The stability of generalized metaplectic operators under composition (see Theorem 2.7 below) implies that $E(\Omega_L;t,s)$ is again an operator in the same class and there exists a symbol $e(\Omega_L;t,s) \in M^{\infty,1}(\mathbb{R}^{2d})$ such that $E(\Omega_L;t,s) = U_0(t,s)e(\Omega_L;t,s)^{\mathrm{w}}$ – see Lemma 3.3 below. To be precise, iterated application of the symplectic covariance property (18) yields

$$e(\Omega_L;t,s) = \widetilde{e}(t_L,t_{L-1})\#\cdots\#\widetilde{e}(t_1,t_0), \tag{6}$$

where the modified symbols $\widetilde{e}(t_{j+1},t_j)$, $j=0,\dots,L-1$, are defined by

$$\widetilde{e}(t_{j+1},t_j) \coloneqq e(t_{j+1},t_j)\circ S_{t_j-t_0}, \quad j=0,\dots,L-1. \tag{7}$$

We stress that the parametrices $E(\Omega_L;t,s)$ are different from those considered by Fujiwara as well as from those associated with the Trotter formula. Still, the short-time propagator $E(t,s)$ is quite easy to handle thanks to the natural algebraic properties of the FIO family and that integral formulae for the Weyl product [55] can be used to derive explicit representations for the symbol $e(\Omega_L;t,s)$.

We present here our main convergence results for this family of approximations.

Theorem 1.1. Fix $T>0$ and let $s,t\in\mathbb{R}$ be such that $0<t-s\le T$. Consider a subdivision $\Omega_L=\{t_0,\dots,t_L\}$ of the interval $[s,t]$ such that $s=t_0<t_1<\ldots<t_L=t$ and set $\omega(\Omega_L)\coloneqq\sup\{t_{j+1}-t_j : 0\le j\le L-1\}$. Consider $a(t,s)$ and $e(\Omega_L;t,s)$ as defined in (3) and (6) respectively. There exists $C=C(T)>0$ such that

$$\|e(\Omega_L;t,s)-a(t,s)\|_{M^{\infty,1}} \le C\omega(\Omega_L)(t-s). \tag{8}$$

As a result, $e(\Omega_L;t,s) \to a(t,s)$ uniformly in $\mathbb{R}^{2d}$ if $\omega(\Omega_L) \to 0$ and there exists $C' = C'(T) > 0$ such that

$$\|E(\Omega_L;t,s) - U(t,s)\|_{M^p \to M^p} \le C'\omega(\Omega_L)(t-s), \tag{9}$$

for every $1 \le p \le \infty$ – in particular, for $M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$. □

While convergence at the level of symbols in (8) is ultimately a consequence of the short-time approximation power of $e(t,s)$ up to second order terms (Lemma 3.1), the composition comes with an unavoidable loss resulting in a first order rate with respect to the mesh size $\omega(\Omega_L)$. Higher order rates, namely $\omega(\Omega_L)^N$ for any integer $N \ge 2$, can be similarly obtained if time ordering is partially retained in the construction of the parametrix symbol in (4), as discussed in Remark 3.2 and Section 3.4 below. In any case, one is now able to obtain a convergence result (with the same rate) in the uniform operator topology that is generally out of reach when the Trotter formula is taken into account.

A natural problem at this stage is the study of convergence at the level of kernels for $E(\Omega_L;t,s)$. As already mentioned, generalized metaplectic operators do have a typical explicit form as integral operators - see Section 2.5 for further details. Let $\mathfrak{E}$ be the set of *exceptional times* for $FIO(S_\tau)$, namely if $S_\tau = \begin{bmatrix} A_\tau & B_\tau \\ C_\tau & D_\tau \end{bmatrix}$ is the block decomposition of the classical flow, then $\mathfrak{E} = \{\tau \in \mathbb{R} : \det B_\tau = 0\}$. It can be proved that if $U(t,s) \in FIO(S_{t-s})$ and $t-s \in \mathbb{R} \setminus \mathfrak{E}$ then there exist a phase factor $c = c(t-s) \in \mathbb{C}$, $|c| = 1$, and an amplitude function $a'(t,s) = a'(t,s;\cdot) \in M^{\infty,1}(\mathbb{R}^{2d})$ such that

$$U(t,s)f(x) = \int_{\mathbb{R}^d} u(t,s,x,y)f(y)\,dy, \tag{10}$$

$$u(t,s,x,y) \coloneqq c|\det B_{t-s}|^{-1/2} e^{2\pi i \Phi_{t-s}(x,y)} a'(t,s,x,y),$$

where $\Phi_{t-s}(x,y)$ is a quadratic polynomial whose coefficients depend only on the entries of S_{t-s}. Under the same assumptions one similarly proves that there exists $e'(\Omega_L;t,s) \in M^{\infty,1}(\mathbb{R}^{2d})$ such that

$$E(\Omega_L;t,s)f(x) = \int_{\mathbb{R}^d} k(\Omega_L;t,s,x,y)f(y)\,dy, \tag{11}$$

$$k(\Omega_L;t,s,x,y) \coloneqq c|\det B_{t-s}|^{-1/2} e^{2\pi i \Phi_{t-s}(x,y)} e'(\Omega_L;t,s,x,y).$$

The estimates in Theorem 1.1 can thus be used to infer refined convergence results at the level of integral kernels.

Theorem 1.2. For $0 < t-s \le T$ with $t-s \notin \mathfrak{E}$, let $u(t,s) = u(t,s,\cdot)$ and $k(\Omega_L;t,s) = k(\Omega_L;t,s,\cdot)$ be the integral kernels of $U(t,s)$ and $E(\Omega_L;t,s)$ in (10) and (11) respectively. Under the same assumptions of Theorem 1.1, for any real-valued function Ψ on $\mathbb{R}^{2d}$ with compact support there exist constants $C_1 = C_1(t-s,\Psi) > 0$ and $C_2 = C(T,\Psi) > 0$ such that

$$\|[k(\Omega_L;t,s) - u(t,s)]\Psi\|_{\mathcal{F}L^1} \le C_1\omega(\Omega_L)(t-s), \tag{12}$$

$$\|[e'(\Omega_L;t,s) - a'(t,s)]\Psi\|_{\mathcal{F}L^1} \le C_2\omega(\Omega_L)(t-s). \tag{13}$$

In particular, the latter bounds imply uniform convergence on compact subsets of $\mathbb{R}^{2d}$ as $\omega(\Omega_L) \to 0$. □

This result should be compared with those proved in [44] in the Feynman-Trotter scheme. They are similar in nature and to some extent also for what concerns the proof strategy – which relies here on a generalized version of the one pioneered in the aforementioned works by Fujiwara. A key difference is that here we are able to control the rate of convergence in (12), thanks to quantitative estimates on the convergence of amplitudes such as (8). This is a further illustration of how the advantages of a carefully designed time-slicing approximation reflect into better convergence results, eventually overcoming inherent limitations such as those associated with the unitary Trotter formula. We also stress that while it is expected that the bound in (12) for the full kernels cannot hold uniformly with respect to $t-s$ in view of the possibly degenerate behaviour of the oscillatory integral phase Φ_{t-s} and "normalization" $|\det B_{t-s}|^{-1/2}$, the finer bound for the amplitudes in (13) is in fact locally uniform with respect to $t-s$ in $(0,T) \setminus \mathfrak{E}$.

1.4 Further comments

We emphasize that setting $\hbar = 1/2\pi$ is not restrictive as soon as $\hbar$ is interpreted as a small fixed parameter. The more challenging problem of the semiclassical limit, namely convergence also for $\hbar \to 0$, would require the occurrence of *positive* powers of $\hbar$ in bounds such as (8). While this issue would certainly fall within the scope of our analysis, it seems clear that approximate propagators must be carefully revised in order to embrace such a different perspective. Consider again Fujiwara's short-time parametrices [27], which have the form of oscillatory integral operators where the phase depends on the action functional $S(t,s,x,y)$ of the system along the classical path γ such that $\gamma(t) = x$ and $\gamma(s) = y$ – that is unique if $|t-s|$ is small enough. This subtler design reflects into a time slicing approximation scheme along *piecewise classical paths* in spacetime, rather than broken lines as in Nelson's approach. This deeper connection with the classical dynamics is certainly accountable for the notable performance in the semiclassical regime. Generalized metaplectic operators such as $E(t,s) \in FIO(S_{t-s})$ share some formal similarities with Fujiwara's propagators, but there are some fundamental differences. In particular, the integral representation of operators in $FIO(S_{t-s})$ for $t-s \notin \mathfrak{E}$ gives rise to an (oscillatory) integral operator with phase $\Phi_{t-s}(x,y)$. Such generating function depends only on the metaplectic Hamiltonian H_0 and actually coincides with the variation of action along a corresponding classical path with endpoints y and x at times s and t respectively, the latter being unique if $|t-s|$ is sufficiently small – see [12, Chapter 4] for further details. The effect of the potential perturbation is then encoded by a suitable amplitude function in the integral formula for $E(t,s)$, hence it does not affect the piecewise classical path that is modelled on the dynamics underlying H_0. As a matter of fact, while the technical difficulties in Fujiwara's analysis are indeed related to the control of phases and amplitudes, the starring role in our FIO-based approach is played by the symbols involved in the deformation of a basic metaplectic operator – as evidenced by estimates like (8) and (27). In a sense, the FIO-type approximation scheme outlined here lies somewhere between the one relying on the Trotter formula and that based on oscillatory integrals introduced by Fujiwara.

In addition to the previous remarks, let us highlight that the effort of a refinement of the parametrices $E(t,s)$ having in mind a semiclassical setting is not expected to be successful, since the underlying framework of perturbation theory inherently involves expansions and estimates with *negative* powers of $\hbar$. In view of the lack of semiclassical approximation power for our parametrices, we also preferred to abstain from a careful revision of the basic notions of Gabor analysis in order to keep track of $\hbar$ in a consistent way as in [4].

In spite of these remarks, an intriguing related problem is that of paralleling the Gabor wave packet approach in the context of *Hamiltonian path integrals*, where paths in phase space are taken into account. Metaplectic operators already proved to be powerful models in this setting, see for instance [47]. Since the problem and the techniques are different from those discussed here, we prefer to postpone related investigations to a separate manuscript.

2 Preliminary results

2.1 Notation

We set $x \cdot y$ for the inner product on $\mathbb{R}^d$. The bracket $\langle f,g\rangle$ denotes the extension to $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ of the inner product $\langle f,g\rangle = \int_{\mathbb{R}^d} f(t)\overline{g(t)}dt$ on $L^2(\mathbb{R}^d)$.

We choose the following normalization for the Fourier transform:

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} f(x)dx, \quad \xi \in \mathbb{R}^d.$$

2.2 Weyl pseudodifferential operators

The usual definition of the Weyl operator $\sigma^{\mathrm{w}} = \mathrm{op}_{\mathrm{w}}(\sigma)$ with symbol $\sigma : \mathbb{R}^{2d} \to \mathbb{C}$ is

$$\sigma^{\mathrm{w}} f(x) := \int_{\mathbb{R}^{2d}} e^{2\pi i (x-y)\cdot\xi} \sigma\left(\frac{x+y}{2}, \xi\right) f(y)\, dy d\xi. \tag{14}$$

The way to rigorously interpret this (formal) integral operator relies on the function spaces to which the symbol σ and the function f belong. For instance, classical symbol classes such as Hörmander's $S^m_{\rho,\delta}(\mathbb{R}^{2d})$ [33] are usually defined by means of decay/smoothness conditions.

One can also approach the issue from the point of view of time-frequency analysis [31]: a straightforward computation (yet formal in general) shows that

$$\langle \sigma^{\mathrm{w}} f, g\rangle = \langle \sigma, W(g,f)\rangle, \tag{15}$$

where we have introduced the (cross-) *Wigner transform* of f, g:

$$W(f,g)(x,\xi) := \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot y} f\left(x+\frac{y}{2}\right) \overline{g\left(x-\frac{y}{2}\right)} dy.$$

This is in fact a standard representation in the phase-space formulation of quantum mechanics [57]. It turns out that there is an intimate connection between the Gabor transform and the Wigner distribution. For instance, the L^p norm of the (cross-)Wigner transform can be equivalently used to measure the modulation space regularity of a signal [11].

We are thus lead to interpret (14) in the weak sense and consider (15) as a definition of σ^{w} for a generalized symbol $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$, for any $f, g \in \mathcal{S}(\mathbb{R}^d)$. Note that this choice paves the way to using modulation spaces both as symbol classes as well as target spaces for Weyl operators; the interested reader can find more details on this perspective in [30, 10].

By way of illustration let us mention that the multiplication by a function $V(x)$ is a special example of Weyl operator with symbol

$$\sigma_V(x,\xi) = (V \otimes 1)(x,\xi) = V(x), \quad (x,\xi) \in \mathbb{R}^{2d}.$$

One may similarly prove that a Fourier multiplier with symbol $m(\xi)$ is a Weyl operator with symbol $(1 \otimes m)(x,\xi) = m(\xi)$. The symbolic calculus relies on the composition of Weyl transforms, which provides a bilinear form on symbols known as the *Weyl* (or *twisted*) *product*:

$$\sigma^{\mathrm{w}} \circ \rho^{\mathrm{w}} = (\sigma \# \rho)^{\mathrm{w}}, \quad \sigma \# \rho = \mathcal{F}^{-1}(\hat{\sigma} \natural \hat{\rho}),$$

where the *twisted convolution* [30] of $\hat{\sigma}$ and $\hat{\rho}$ is (formally) defined by

$$(\hat{\sigma} \natural \hat{\rho})(x,\xi) := \int_{\mathbb{R}^{2d}} e^{\pi i (x,\xi)\cdot(\eta,-y)} \hat{\sigma}(y,\eta) \hat{\rho}(x-y,\xi-\eta)\, dy d\eta.$$

We remark that in phase space quantum mechanics and deformation quantization it is customary to refer to $\sigma \# \rho$ as the *Moyal star product* of σ and ρ after [40].

2.3 The Sjöstrand class

We already introduced the Sjöstrand class in Section 1. As the name suggests, it was first presented by Sjöstrand in [51] as an exotic class of non-smooth symbols still giving bounded pseudodifferential operators in $L^2(\mathbb{R}^d)$. It was later rediscovered in Gabor analysis by Gröchenig [29, 32], who obtained novel proofs of known results but also a number of new important characterizations (e.g., almost diagonalization in phase space of the corresponding Weyl operators and the Wiener property).

We recall that, as a fully fledged modulation space, $M^{\infty,1}(\mathbb{R}^d)$ can be designed the collection of $f \in \mathcal{S}'(\mathbb{R}^d)$ such that, for some (hence any) non-trivial $g \in \mathcal{S}(\mathbb{R}^d)$,

$$\|f\|_{M^{\infty,1}} := \int_{\mathbb{R}^d} \sup_{x \in \mathbb{R}^d} |V_g f(x,\xi)|\, d\xi < \infty.$$

Using the properties of the Gabor transform [30, Lemma 3.1.1], it is not difficult to show that if $f \in M^{\infty,1}(\mathbb{R}^d)$ then $\|\hat{f}\|_{W^{\infty,1}} < \infty$, where we set

$$\|h\|_{W^{\infty,1}} := \int_{\mathbb{R}^d} \sup_{\xi \in \mathbb{R}^d} |V_g h(x,\xi)| dx.$$

Equivalently, reversing the order of time and frequency variables in the norm of $M^{\infty,1}(\mathbb{R}^d)$ provides a natural norm for the space $W^{\infty,1}(\mathbb{R}^d) := \mathcal{F}M^{\infty,1}(\mathbb{R}^d)$. In general, for $1 \le p \le \infty$ we set $W^p(\mathbb{R}^d) := \mathcal{F}M^p(\mathbb{R}^d)$, the latter being special examples of *Wiener amalgam spaces* [15, 17]. To be precise, we have that $W^p(\mathbb{R}^d) = W^{p,p}(\mathbb{R}^d)$, where $W^{p,q}(\mathbb{R}^d) = W(\mathcal{F}L^p, L^q)(\mathbb{R}^d)$ is the amalgam space with local and global components $\mathcal{F}L^p(\mathbb{R}^d)$ and $L^q(\mathbb{R}^d)$ respectively. Further details on the topic can be found for instance in [10, Section 2.4].

We list here some properties that will be used below. The reader can consult the aforementioned papers for a more comprehensive account.

Proposition 2.1. (i) $M^{\infty,1}(\mathbb{R}^d) \subset (\mathcal{F}L^1(\mathbb{R}^d))_{\mathrm{loc}} \cap L^\infty(\mathbb{R}^d) \subset C_{\mathrm{b}}(\mathbb{R}^d)$, where $C_{\mathrm{b}}(\mathbb{R}^d)$ is the space of bounded and continuous functions $\mathbb{R}^d \to \mathbb{C}$. More precisely, $M^{\infty,1}(\mathbb{R}^d) \subset W^{1,\infty}(\mathbb{R}^d)$.

(ii) $(M^{\infty,1}(\mathbb{R}^d))_{\mathrm{loc}} = (\mathcal{F}L^1(\mathbb{R}^d))_{\mathrm{loc}}$.

(iii) $\mathcal{F}\mathcal{M}(\mathbb{R}^d) \subset M^{\infty,1}(\mathbb{R}^d)$, where $\mathcal{M}(\mathbb{R}^d)$ is the space of complex finite measures on $\mathbb{R}^d$.

(iv) $M^{\infty,1}(\mathbb{R}^d)$ is a Banach algebra under pointwise multiplication:

$$f, g \in M^{\infty,1}(\mathbb{R}^d) \Rightarrow f \cdot g \in M^{\infty,1}(\mathbb{R}^d).$$

(v) $M^{\infty,1}(\mathbb{R}^d)$ is a Banach algebra under the Weyl product of symbols:

$$\rho, \sigma \in M^{\infty,1}(\mathbb{R}^d) \Rightarrow \rho\#\sigma \in M^{\infty,1}(\mathbb{R}^d).$$

(vi) If $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ then σ^{w} is a bounded operator on $L^2(\mathbb{R}^d)$ and, in general, on $M^p(\mathbb{R}^d)$ for $1 \le p \le \infty$. To be precise, there exists $C > 0$ independent of σ and p such that

$$\|\sigma^{\mathrm{w}}\|_{M^p \to M^p} \le C\|\sigma\|_{M^{\infty,1}}. \tag{16}$$

□

Proof. (i) It is a direct consequence of the definition. The refined inclusion $M^{\infty,1}(\mathbb{R}^d) \subset W^{1,\infty}(\mathbb{R}^d)$ is readily obtained by means of a straightforward application of Minkowski's integral inequality.

(ii) See [3, Proposition 2.9] for a proof.

(iii) A direct proof can be found in [44, Proposition 3.4]. Arguing in terms of Wiener amalgam spaces, since $\mathcal{M}(\mathbb{R}^d) = W(\mathcal{M}, L^1)(\mathbb{R}^d)$ by [18, Theorem 1] we can resort to [15, Remark 1.2] to equivalently conclude that $\mathcal{M}(\mathbb{R}^d) = W(\mathcal{M}, L^1)(\mathbb{R}^d) \subset W(\mathcal{F}L^\infty, L^1) = \mathcal{F}M^{\infty,1}(\mathbb{R}^d)$.

(iv) This is a special case of a more general characterization, see [48, Theorem 3.5 and Corollary 2.10].

(v) Proofs can be found in the original paper [51] by Sjöstrand as well as in the already mentioned paper [32] by Gröchenig.

(vi) The same comments of the previous item apply here. For a streamlined textbook proof see also [30, Theorem 14.5.2]. ■

Remark 2.2. As a further illustration of the Sjöstrand class regularity, it may be useful to observe that any function $f = g * h$ obtained by smoothing a mild distribution $h \in M^\infty(\mathbb{R}^d)$ with a filter $g \in M^1(\mathbb{R}^d)$ belongs to $M^{\infty,1}(\mathbb{R}^d)$. The proof is easily carried on the spectral side by means of the properties of Wiener amalgam spaces (cf. for instance [10, Theorem 2.4.9]), namely

$$\hat{f} = \hat{g} \cdot \hat{h} \in W^1(\mathbb{R}^d) \cdot W^\infty(\mathbb{R}^d) \subset W^{\infty,1}(\mathbb{R}^d).$$

This remark gives us the opportunity to build several interesting examples of functions in the Sjöstrand class in a straightforward way. Consider for instance functions that are obtained via periodization of a given $g \in M^1(\mathbb{R}^d)$ along a regular lattice $\Lambda = \alpha\mathbb{Z}^d$ with $\alpha > 0$, that is $f(x) = g * Ш_\Lambda(x) := \sum_{k\in\mathbb{Z}^d} g(x - \alpha k)$, where we introduced the Dirac comb $Ш_\Lambda := \sum_{k\in\mathbb{Z}^d} \delta_{\alpha k} \in M^\infty(\mathbb{R}^d)$. More generally, we have that $f = g * h \in M^{\infty,1}(\mathbb{R}^d)$ for any $g \in M^1(\mathbb{R}^d)$ and $h = \sum_{k\in\mathbb{Z}^d} v_k \delta_{\alpha k}$ with $(v_k)_{k\in\mathbb{Z}^d} \in \ell^\infty(\mathbb{Z})$, since $h \in W(\mathcal{M}, \ell^\infty)(\mathbb{R}^d) \subset M^\infty(\mathbb{R}^d)$ as well.

In this connection, a concrete family of relevant potentials in dimension $d = 1$ with $M^{\infty,1}$ regularity is given by continuous piecewise linear functions f with values $(v_k)_{k\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ at nodes $\{\alpha k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$, for some $\alpha > 0$. Indeed, any such function can be represented as $f = g_\alpha * h$, where $g_\alpha(x) = \max\{1 - |x|/\alpha, 0\}$ is a suitable triangular function (with $\mathcal{F}L^1$ regularity and compact support, hence $g_\alpha \in M^1(\mathbb{R})$) and $h = \sum_{k\in\mathbb{Z}} v_k \delta_{\alpha k} \in M^\infty(\mathbb{R})$ as above.

□

Remark 2.3. Let $(A, \star)$ be a unital Banach algebra. From the very definition we have that there exists $C > 0$ such that

$$\|a_1 \star a_2\|_A \le C\|a_1\|_A\|a_2\|_A, \quad a_1, a_2 \in A.$$

Recall that it is always possible to introduce an equivalent norm on A for which the product estimate above holds with $C = 1$ and the unit has norm equal to 1 (cf. e.g. [50, Theorem 10.2]). From now on we will tacitly consider such equivalent norm whenever concerned with a Banach algebra. □

It was already noted in [51] that $\mathcal{S}(\mathbb{R}^d)$ is not dense in $M^{\infty,1}(\mathbb{R}^d)$ with the norm topology. This issue can be fixed if one introduces the following notion [7, 51].

Definition 2.4. Fix $I \subseteq \mathbb{R}$. The map $I \ni \nu \mapsto \sigma_\nu \in M^{\infty,1}(\mathbb{R}^d)$ is said to be continuous in the sense of the narrow convergence if:

(i) it is weakly continuous in $\mathcal{S}'(\mathbb{R}^d)$ (i.e., the map $\nu \mapsto \langle \sigma_\nu, g\rangle$ is continuous for every $g \in \mathcal{S}(\mathbb{R}^d)$), and

(ii) there exists a function $h \in L^1(\mathbb{R}^d)$ such that for some (hence any) window $g \in \setminus\{0\}$ one has $\sup_{x\in\mathbb{R}^d} |V_g\sigma_\nu(x,\xi)| \le h(\xi)$ for any $\nu \in I$ and a.e. $\xi \in \mathbb{R}^d$.

The latter condition implies that σ_ν belongs to a bounded subset of $M^{\infty,1}(\mathbb{R}^d)$, uniformly with respect to ν – precisely, $\sup_{\nu\in I} \|\sigma_\nu\|_{M^{\infty,1}} \le \|h\|_{L^1} < \infty$.

□

Finally, we recall from [6, Lemma 2.2] and [44, Lemma 3.1] a result on uniform estimates for linear changes of variable in the Sjöstrand class that will be used later.

Lemma 2.5. Let $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ and $t \mapsto M_t \in \mathrm{Sp}(d,\mathbb{R})$ be a continuous mapping defined on the compact interval $[-T,T]$, $T>0$. For any $t \in [-T,T]$, we have $\sigma \circ M_t \in M^{\infty,1}(\mathbb{R}^{2d})$; precisely, there exists $C(T)>0$ such that

$$\|\sigma \circ M_t\|_{M^{\infty,1}} \le C(T)\|\sigma\|_{M^{\infty,1}}. \tag{17}$$

□

2.4 Metaplectic operators and quadratic Hamiltonian operators

There are several equivalent ways to define metaplectic operators. The reader is referred to the monographs [11, 24, 49, 53] for comprehensive treatments of the topic. Here we confine ourselves to recall that the metaplectic group coincides with the two-fold covering of the symplectic group $\mathrm{Sp}(d,\mathbb{R})$. In particular, there exists a faithful and strongly continuous unitary representation of $\mathrm{Mp}(d,\mathbb{R})$ on $L^2(\mathbb{R}^d)$, called the Shale-Weil *metaplectic representation*, which allows us to recast the issue in terms of a correspondence between any symplectic matrix $S \in \mathrm{Sp}(d,\mathbb{R})$ and a pair of unitary *metaplectic operators* differing only by the sign, both denoted by $\mu(S)$ with a slight abuse of notation.

Among the several properties satisfied by metaplectic operators, it is important to our purposes to highlight that the Weyl quantization satisfies a characterizing intertwining relationship called *symplectic covariance* (cf. [11, Theorem 215]): for every symbol $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ we have

$$(\sigma \circ S)^{\mathrm{w}} = \mu(S)^{-1}\sigma^{\mathrm{w}}\mu(S). \tag{18}$$

While boundedness results on modulation spaces for metaplectic operators can be readily derived from phase-space estimates obtained by localization via Gabor wave packets [9, Theorem 2.3], we need below a more precise bound where the dependence on S of the underlying constants is completely clarified. A result in this spirit has been recently obtained in [8, Corollary 3.4] and reads as follows: for every $S \in \mathrm{Sp}(d,\mathbb{R})$ and $1 \le p \le \infty$ there exists an absolute constant $C>0$ such that, for every $f \in M^p(\mathbb{R}^d)$,

$$\|\mu(S)f\|_{M^p} \le C(\sigma_1(S)\cdots\sigma_d(S))^{|1/2-1/p|}\|f\|_{M^p}, \tag{19}$$

where $\sigma_1(S) \ge \ldots \ge \sigma_d(S) \ge 1$ are the d largest singular values of S.

A concrete characterization of metaplectic operators can be given in terms of Schrödinger propagators with quadratic Hamiltonian [47]. Consider the Schrödinger equation $i\partial_t\psi = 2\pi H_0\psi$ where $H_0 = Q^{\mathrm{w}}$ is the Weyl quantization a real-valued, time-independent, quadratic homogeneous polynomial Q on $\mathbb{R}^{2d}$. Precisely, if

$$Q(x,\xi) = \frac{1}{2}A\xi\cdot\xi + Bx\cdot\xi + \frac{1}{2}Cx\cdot x,$$

for some matrices $A,B,C \in \mathbb{R}^{d\times d}$ with $A = A^\top$ and $C = C^\top$, then

$$H_0 = Q^{\mathrm{w}} = -\frac{1}{8\pi^2}\sum_{j,k=1}^d A_{j,k}\partial_j\partial_k - \frac{i}{2\pi}\sum_{j,k=1}^d B_{j,k}x_j\partial_k + \frac{1}{2}\sum_{j,k=1}^d C_{j,k}x_jx_k - \frac{i}{4\pi}\mathrm{Tr}(B). \tag{20}$$

It is well known that the associated propagator is a metaplectic operator, precisely

$$U_0(t,s) = e^{-2\pi i(t-s)Q^{\mathrm{w}}} = c\mu(S_{t-s}),$$

for some $c = c(t-s) \in \mathbb{C}$ with $|c| = 1$, where the (continuous) mapping

$$\mathbb{R} \ni \tau \mapsto S_\tau = \begin{bmatrix} A_\tau & B_\tau \\ C_\tau & D_\tau \end{bmatrix} \in \mathrm{Sp}(d, \mathbb{R}) \tag{21}$$

is the phase-space flow determined by the Hamilton equations for the corresponding classical system with Hamiltonian $Q(x, \xi)$.

Another classical result concerns the explicit characterization of the propagator $U_0(t,s)$ as an integral operator, cf. [24, Theorems 4.51 and 4.53] and [11]. Recall that the matrix S_τ is said to be *free* if the upper-right block B_τ in (21) is invertible. In this case, we have

$$U_0(t,s)f(x) = c|\det B_{t-s}|^{-1/2} \int_{\mathbb{R}^d} e^{2\pi i \Phi_{t-s}(x,y)} f(y) dy, \qquad f \in \mathcal{S}(\mathbb{R}^d), \tag{22}$$

again for some $c = c(t-s) \in \mathbb{C}$ with $|c| = 1$, where Φ_{t-s} is a quadratic form on $\mathbb{R}^{2d}$ known as *generating function* of S_{t-s}:

$$\Phi_{t-s}(x,y) = \frac{1}{2} D_{t-s} B_{t-s}^{-1} x \cdot x - B_{t-s}^{-1} x \cdot y + \frac{1}{2} B_{t-s}^{-1} A_{t-s} y \cdot y.$$

We accordingly define the set of *exceptional times* for S_τ to be

$$\mathfrak{E} = \{\tau \in \mathbb{R} \,:\, \det B_\tau = 0\},$$

that is the collection of values of τ such that S_τ is *not* a free symplectic matrix. The exceptional nature of these values does not just concern the lack of integral representations as in (10), since the subset of free symplectic matrices has codimension 1 in $\mathrm{Sp}(d, \mathbb{R})$ [12, Proposition 171]. Some of the properties of $\mathfrak{E}$ can be immediately inferred from the fact that it actually coincides with the zero set of an analytic function: apart from the case $\mathfrak{E} = \mathbb{R}$ (which trivially occurs when $H_0 = 0$), $\mathfrak{E}$ is a discrete (hence at most countable) subset of $\mathbb{R}$ which always includes $\tau = 0$ and possibly only this value (this is the case of the free particle).

2.5 Generalized metaplectic operators

Let us consider the perturbed problem in (1) with the assumptions stated in Section 1, recalled below.

Assumption 1. We consider the Hamiltonian operator $H = H_0 + V$, where $H_0 = Q^{\mathrm{w}}$ is the Weyl quantization of a real quadratic form on $\mathbb{R}^{2d}$ as in (20) and $V = \sigma_t^{\mathrm{w}}$ for a one-parameter family of symbols $\sigma_t \in M^{\infty,1}(\mathbb{R}^{2d})$ such that the correspondence $\mathbb{R} \ni t \mapsto \sigma_t \in M^{\infty,1}(\mathbb{R}^{2d})$ is continuous in the sense of the narrow convergence. □

Standard arguments of perturbation theory can be used to give a rigorous proof of the following facts (cf. [6, Theorem 4.1] for the details): the problem under consideration is globally backward and forward well-posed in $L^2(\mathbb{R}^d)$ and the corresponding full Schrödinger propagator $U(t,s)$ is a one-parameter strongly continuous group of automorphisms of $L^2(\mathbb{R}^d)$ – in fact, of any modulation space $M^p(\mathbb{R}^d)$, $1 \le p \le \infty$, hence the phase-space concentration is preserved under the evolution.

The structure of $U(t,s)$ is intimately related to that of the companion homogeneous problem $U_0(t,s) = c\mu(S_{t-s})$. Let us briefly review the main steps of the derivation for the sake of clarity. First, the problem (1) can be recast in integral form in accordance with Duhamel's principle, namely

$$\psi(t,x) = U_0(t,s)\psi_s(x) - 2\pi i \int_s^t U_0(t,\tau)\sigma_\tau^{\mathrm{w}} \psi(\tau, x) d\tau.$$

After setting $\varphi(t,x) \coloneqq U_0(s,t)\psi(t,x)$ (which leads to the the so-called *interaction picture*) and using the evolution property of U_0 we have

$$\varphi(t,x) = \psi_s(x) - 2\pi i \int_s^t U_0(s,\tau)\sigma_\tau^{\mathrm{w}} U_0(\tau,s)\varphi(\tau,x) d\tau. \tag{23}$$

We resort to the symplectic covariance property (18), which plays here the role of an Egorov-like result for linear symplectic transformations regulating the flow of the symbol σ_τ [58], that is

$$U_0(s,\tau)\sigma_\tau^{\mathrm{w}} U_0(\tau,s) = (\sigma_\tau \circ S_{\tau-s})^{\mathrm{w}} =: b(\tau,s)^{\mathrm{w}}.$$

The solution of the Volterra integral equation (23) is thus given by

$$\varphi(t,x) = a(t,s)^{\mathrm{w}} \psi_s(x),$$

where the symbol $a(t,s) = \mathcal{T}\exp\left(-2\pi i \int_s^t b(\tau,s)d\tau\right)$ was defined in (3). To conclude, we have that $U(t,s) = U_0(t,s)a(t,s)^{\mathrm{w}}$. Let us emphasize that that $U(t,s)$ is not a unitary propagator in general, unless V is self-adjoint - equivalently, if σ_τ is a real-valued symbol.

We prove that the symbols $a(t,s)$ belong to a bounded subset of $M^{\infty,1}(\mathbb{R}^{2d})$.

Lemma 2.6. Fix $T > 0$ and let $s,t \in \mathbb{R}$ be such that $0 < |t-s| \le T$. Let $a(t,s)$ be defined as in (3). Then $a(t,s) \in M^{\infty,1}(\mathbb{R}^{2d})$ and there exists $C = C(T) > 1$ such that $\|a(t,s)\|_{M^{\infty,1}} \le C$. □

Proof. Let us set

$$\alpha_n(t,s) := \int_s^t \int_s^{t_1} \cdots \int_s^{t_{n-1}} b(t_1,s)\# \cdots \# b(t_n,s)\, dt_n \ldots dt_1, \quad n \in \mathbb{N},$$

so that we can write $a(t,s) = 1 + \sum_{n\ge 1} (-2\pi i)^n \alpha_n(t,s)$. It is clear that $\alpha_n(t,s) \in M^{\infty,1}(\mathbb{R}^{2d})$ since $(M^{\infty,1}, \#)$ is a Banach algebra (cf. Proposition 2.1) and $b(\tau,s) = \sigma_\tau \circ S_{\tau-s} \in M^{\infty,1}(\mathbb{R}^{2d})$ for all $\tau > s$ in view of Lemma 2.5. To be precise, we have

$$\begin{aligned}
\|\alpha_n(t,s)\|_{M^{\infty,1}} &= \left\| \int_s^t \int_s^{t_1} \cdots \int_s^{t_{n-1}} b(t_1,s)\# \cdots \# b(t_n,s)\, dt_n \ldots dt_1 \right\|_{M^{\infty,1}} \\
&\le \int_s^t \int_s^{t_1} \cdots \int_s^{t_{n-1}} \|b(t_1,s)\|_{M^{\infty,1}} \cdots \|b(t_n,s)\|_{M^{\infty,1}}\, dt_n \ldots dt_1 \\
&\le \frac{|t-s|^n}{n!} \left(\sup_{\tau\in[s,t]} \|b(\tau,s)\|_{M^{\infty,1}} \right)^n \\
&\le \frac{C_0(T)^n |t-s|^n}{n!} \left(\sup_{\tau\in[s,t]} \|\sigma_\tau\|_{M^{\infty,1}} \right)^n \\
&\le \frac{C_1(T)^n}{n!} |t-s|^n,
\end{aligned} \tag{24}$$

where we set $C_1(T) = C_0(T)\left(\sup_{\tau\in[s,t]} \|\sigma_\tau\|_{M^{\infty,1}}\right)$, in particular $C_0(T)$ comes from the estimate (17) and the supremum is finite since $t \mapsto \sigma_t$ is continuous in the sense of the narrow convergence. The claim follows from (24), since

$$\|a(t,s)\|_{M^{\infty,1}} \le 1 + \sum_{n\ge 1} (2\pi)^n \|\alpha_n(t,s)\|_{M^{\infty,1}} \le e^{C_1(T)T/2\pi} =: C. \qquad \blacksquare$$

Operators of the form of $U(t,s)$ as above, namely arising as combinations of metaplectic and pseudodifferential operators with symbols in $M^{\infty,1}(\mathbb{R}^{2d})$, have been thoroughly studied in time-frequency analysis. They constitute the family $FIO(S)$ of *generalized metaplectic operators* introduced in [5, 6, 9]. Roughly speaking, $FIO(S)$ can be thought of as the largest class of "perturbations" of $\mu(S)$ that still evolve Gabor wave packets in phase space essentially following the flow associated with S. A precise condition can be given in terms of the *Gabor matrix* of an operator $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$, defined by $\langle T\pi(z)g, \pi(w)g\rangle$, $z,w \in \mathbb{R}^{2d}$, for fixed $g \in \mathcal{S}(\mathbb{R}^d)$. In particular, for every $N \in \mathbb{N}$ there exists $C = C(N) > 0$ such that

$$|\mu(S)\pi(z)g, \pi(w)g| \le C(1 + |w - Sz|)^{-N}, \quad z,w \in \mathbb{R}^{2d},$$

whereas the class $FIO(S)$ is formed by all and only the operators T such that there exists $H \in L^1(\mathbb{R}^{2d})$ satisfying

$$|T\pi(z)g, \pi(w)g| \le H(w - Sz), \quad w,z \in \mathbb{R}^{2d}.$$

In light of the previous remarks, the following properties of generalized metaplectic operators should not be surprising – proofs can be found in the aforementioned papers as well as in [10].

Theorem 2.7. Let $S, S_1, S_2 \in \mathrm{Sp}(d,\mathbb{R})$.

(i) Let $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ be a linear continuous operator. $T \in FIO(S)$ if and only if there exist $\sigma_1, \sigma_2 \in M^{\infty,1}(\mathbb{R}^{2d})$ such that

$$T = \sigma_1^{\mathrm{w}} \mu(S) = \mu(S) \sigma_2^{\mathrm{w}}.$$

In particular, $\sigma_2 = \sigma_1 \circ S$.

(ii) An operator $T \in FIO(S)$ is bounded on $M^p(\mathbb{R}^d)$ for any $1 \le p \le \infty$.

(iii) If $T_1 \in FIO(S_1)$ and $T_2 \in FIO(S_2)$, then $T_1 T_2 \in FIO(S_1 S_2)$.

(iv) If $T \in FIO(S)$ is invertible on $L^2(\mathbb{R}^d)$ then $T^{-1} \in FIO(S^{-1})$.

□

The following result provides a concrete picture of generalized metaplectic operators arising as Schrödinger propagators and clarifies the intimate relation with the representation formula (22) of the companion metapletic operator for non-exceptional times.

Theorem 2.8. Let $U(t,s) = U_0(t,s)a(t,s)^{\mathrm{w}} \in FIO(S_{t-s})$ be the Schrödinger propagator corresponding to the Hamiltonian $H = H_0 + V$ as in Assumption 1. For every $t, s \in \mathbb{R}$ with $0 < |t-s| \le T$ and $t - s \notin \mathfrak{E}$ there exists a symbol $a'(t,s) = a'(t,s,\cdot) \in M^{\infty,1}(\mathbb{R}^{2d})$ such that

$$U(t,s)f(x) = c(t-s)|\det B_{t-s}|^{-1/2} \int_{\mathbb{R}^d} e^{2\pi i \Phi_{t-s}(x,y)} a'(t,s,x,y) f(y) dy,$$

the notation being the same of (22). Moreover, there exists $C = C(T) > 0$ such that

$$\|a'(t,s)\|_{M^{\infty,1}} \le C \|a(t,s)\|_{M^{\infty,1}}. \tag{25}$$

□

Proof. The representation of $T \in FIO(S)$ as a Fourier integral operator is proved in [6, Theorem 5.1], so that the claim follows after easy modifications as detailed for instance [44, Lemma 3.1]. It remains to prove the uniformity of the estimate (25). In the aforementioned results (see also [6, Proposition 5.2]) it is shown by direct computation that

$$a'(t,s) = \mathcal{U}_2 \mathcal{U} \mathcal{U}_1 (a(t,s) \circ S_{t-s}^{-1}),$$

where $\mathcal{U}, \mathcal{U}_1, \mathcal{U}_2$ are the mappings defined on $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ by

$$\mathcal{U}_1 \sigma(x,y) = \sigma(x, y + A_{t-s}x), \quad \mathcal{U}_2 \sigma(x,y) = \sigma(x, B_{t-s}^{\top} y), \quad \widehat{\mathcal{U}\sigma}(\xi,\eta) = e^{\pi i \xi \cdot \eta} \widehat{\sigma}(\xi,\eta),$$

where A_{t-s} and B_{t-s} come from the block decomposition of S_{t-s} in (21). For what concerns $\mathcal{U}$, the proof of [30, Corollary 14.5.5] precisely shows that $M^{\infty,1}(\mathbb{R}^{2d})$ is invariant under the action of $\mathcal{U}$. The desired conclusion thus follows by repeated application of Lemma 2.5. ■

Remark 2.9. The previous representation result extends to any $T \in FIO(S)$ - in particular it holds for $E(\Omega_L; t,s) \in FIO(S_{t-s})$ in (5), as claimed in (11). We emphasize that also in this case we have that

$$\|e'(t,s)\|_{M^{\infty,1}} \le C \|e(t,s)\|_{M^{\infty,1}}, \tag{26}$$

for a constant $C = C(T) > 0$ that depends only on T, for $0 < |t-s| \le T$ with $t - s \notin \mathfrak{E}$. □

3 Proof of the main results

3.1 Preliminary estimates

For this and the subsequent sections we refer to the problem (1) under Assumption 1. We also use the notation introduced in Section 1.

First, we prove that the symbols $e(t,s)$ introduced in (4) are good short-time approximations of $a(t,s)$.

Lemma 3.1. Fix $T > 0$ and let $s, t \in \mathbb{R}$ be such that $0 < |t-s| \le T$. Consider $a(t,s)$ and $e(t,s)$ as defined in (3) and (4) respectively. We have $e(t,s) \in M^{\infty,1}(\mathbb{R}^{2d})$ and there exists $C = C(T) > 0$ such that

$$\|a(t,s) - e(t,s)\|_{M^{\infty,1}} \le C|t-s|^2. \tag{27}$$

□

Proof. First of all, note that

$$e(t,s)=\exp\left(-2\pi i\int_s^t b(\tau,s)d\tau\right)=\sum_{n\ge0}(-2\pi i)^n\frac{1}{n!}\left(\int_s^t b(\tau,s)d\tau\right)^n.$$

To align the notation with the one introduced before, we define

$$\varepsilon_n(t,s):=\frac{1}{n!}\left(\int_s^t b(\tau,s)d\tau\right)^n,\quad n\in\mathbb{N},$$

so that $e(t,s)=1+\sum_{n\ge1}(-2\pi i)^n\varepsilon_n(t,s)$. Recall from Proposition 2.1 that $M^{\infty,1}$ is a Banach algebra under pointwise multiplication, then arguing as in the proof of Lemma 2.6 we readily obtain

$$\|\varepsilon_n(t,s)\|_{M^{\infty,1}}\le\frac{C_1(T)^n}{n!}|t-s|^n,$$

where the constant $C_1(T)$ equals the one appearing in (24). In particular, we similarly infer that

$$\|e(t,s)\|_{M^{\infty,1}}\le e^{C_1(T)|t-s|/2\pi}, \tag{28}$$

hence the symbols $e(t,s)$ belong to a bounded set of $M^{\infty,1}$ depending only on T. As a result, we have

$$\begin{aligned}\|e(t,s)-a(t,s)\|_{M^{\infty,1}}&=\left\|\sum_{n\ge2}(-2\pi i)^n\left[\varepsilon_n(t,s)-\alpha_n(t,s)\right]\right\|_{M^{\infty,1}}\\&\le\sum_{n\ge2}(2\pi)^{-n}\left(\|\varepsilon_n(t,s)\|_{M^{\infty,1}}+\|\alpha_n(t,s)\|_{M^{\infty,1}}\right)\\&\le2\sum_{n\ge2}\frac{C_1(T)^n}{(2\pi)^n n!}|t-s|^n\\&\le C|t-s|^2,\end{aligned}$$

where we set $C=(C_1^2/2\pi^2)e^{C_1T/2\pi}$. ■

Remark 3.2. Inspecting the previous proof suggests that partial restoration of time ordering provides a way to enhance the short-time approximation power of parametrices. For $N\in\mathbb{N}$, $N\ge2$, consider indeed the symbols

$$e^{(N)}(t,s)=\mathcal{T}^{(N)}\exp\left(-2\pi i\int_s^t b(\tau,s)d\tau\right):=1+\sum_{n=1}^{N}(-2\pi i)^n\alpha_n(t,s)+\sum_{n\ge N+1}(-2\pi i)^n\varepsilon_n(t,s).$$

Arguing as above it is thus easy to show that $e^{(N)}(t,s)\in M^{\infty,1}(\mathbb{R}^{2d})$ and

$$\|e^{(N)}(t,s)-a(t,s)\|_{M^{\infty,1}}\le C|t-s|^{N+1},$$

for some $C=C(T)>0$. □

We now consider the composition of symbols $e(t,s)$ over a subdivision of a time interval and prove that the the estimates are uniform with respect to the number of points in the subdivision.

Lemma 3.3. Fix $T>0$ and let $s,t\in\mathbb{R}$ be such that $0<t-s\le T$. Fix a positive integer L and consider the subdivision $\Omega_L=t_0,t_1,\dots,t_L$ of the interval $[s,t]$, where $s=t_0<t_1<\ldots<t_L=t$. Let $e(\Omega_L;t,s)$ be the symbol defined in (5). Then there exists a constant $C=C(T)>1$ such that $\|e(\Omega_L;t,s)\|_{M^{\infty,1}}\le C$. □

Proof. From the very definition of $e(\Omega_L;t,s)$ we have

$$\begin{aligned}\|e(\Omega_L;t,s)\|_{M^{\infty,1}}&\le\|\widetilde{e}(t_L,t_{L-1})\|_{M^{\infty,1}}\cdots\|\widetilde{e}(t_1,t_0)\|_{M^{\infty,1}}\\&=\prod_{j=0}^{L-1}\|e(t_{j+1},t_j)\circ S_{t_j-t_0}\|_{M^{\infty,1}}.\end{aligned}$$

Combining Lemma 3.1 (in particular (28)) with Lemma 2.5 we obtain that

$$\|e(t_{j+1},t_j)\circ S_{t_j-t_0}\|_{M^{\infty,1}}\le e^{C_0(t_{j+1}-t_j)},$$

for a constant $C_0=C_0(T)>0$ that depends only on T. The claim thus follows with $C=e^{C_0(T)T}$. ■

3.2 Proof of Theorem 1.1

The strategy of the proof of Theorem 1.1 relies on a time slicing argument first pioneered in [26, Lemma 3.2]. The latter was subsequently generalized in [43, Theorem 10]. Let us focus on deriving (8) first. Consider the propagator $U(t,s)$ in (2); it is not difficult to realize that the corresponding group property and the symplectic covariance of Weyl calculus (recall that $U_0(t,s)=c(t-s)\mu(S_{t-s})$) imply the following composition law for $s<\tau<t$:

$$a(t,s)=(a(t,\tau)\circ S_{\tau-s})\#a(\tau,s). \tag{29}$$

Iteration of this procedure referring to the subdivision Ω_L in the claim brings us to introduce modified short-time symbols in a similar fashion to those in (7):

$$\widetilde{a}(t_{j+1},t_j)\coloneqq a(t_{j+1},t_j)\circ S_{t_j-t_0}, \quad j=0,\ldots,L-1.$$

As a result, we obtain the decomposition

$$a(t,s)=\widetilde{a}(t_L,t_{L-1})\#\ldots\#\widetilde{a}(t_1,t_0).$$

The arguments in the proof of Lemma 3.3 provide the existence of a constant $C_0=C_0(T)>1$ such that

$$\|\widetilde{a}(t_{j+1},t_j)\|_{M^{\infty,1}}\le C_0(T), \quad j=0,\ldots,L-1. \tag{30}$$

We also remark that the symbols $\widetilde{a}$ satisfy the composition property

$$\widetilde{a}(t_{j+1},t_j)\#\widetilde{a}(t_j,t_{j-1})=\widetilde{a}(t_{j+1},t_{j-1}) \quad j=1,\ldots,L-1, \tag{31}$$

which can be easily verified as follows, using the symplectic covariance property and the composition law (29) (the latter is enough if $j=1$):

$$\begin{aligned}
\widetilde{a}(t_{j+1},t_j)^{\mathrm{w}}\widetilde{a}(t_j,t_{j-1})^{\mathrm{w}} &= U_0(t_0,t_j)a(t_{j+1},t_j)^{\mathrm{w}}U_0(t_j,t_0)U_0(t_0,t_{j-1})a(t_j,t_{j-1})^{\mathrm{w}}U_0(t_{j-1},t_0)\\
&= U_0(t_0,t_j)a(t_{j+1},t_j)^{\mathrm{w}}U_0(t_j,t_{j-1})a(t_j,t_{j-1})^{\mathrm{w}}U_0(t_{j-1},t_0)\\
&= U_0(t_0,t_j)U_0(t_j,t_{j-1})(a(t_{j+1},t_j)\circ S_{t_j-t_{j-1}})^{\mathrm{w}}a(t_j,t_{j-1})^{\mathrm{w}}U_0(t_{j-1},t_0)\\
&= U_0(t_0,t_{j-1})(a(t_{j+1},t_j)\circ S_{t_j-t_{j-1}})^{\mathrm{w}}a(t_j,t_{j-1})^{\mathrm{w}}U_0(t_{j-1},t_0)\\
&= U_0(t_0,t_{j-1})a(t_{j+1},t_{j-1})^{\mathrm{w}}U_0(t_{j-1},t_0)\\
&= (a(t_{j+1},t_{j-1})\circ S_{t_{j-1}-t_0})^{\mathrm{w}}\\
&= \widetilde{a}(t_{j+1},t_{j-1})^{\mathrm{w}}.
\end{aligned}$$

We can thus write

$$\begin{aligned}
e(\Omega_L;t,s)-a(t,s) &= \widetilde{e}(t_L,t_{L-1})\cdots\widetilde{e}(t_1,s)-\widetilde{a}(t_L,t_{L-1})\cdots\widetilde{a}(t_1,s)\\
&= (r(t_L,t_{L-1})+\widetilde{a}(t_L,t_{L-1}))\cdots(r(t_1,s)+\widetilde{a}(t_1,s))\\
&\quad -\widetilde{a}(t_L,t_{L-1})\cdots\widetilde{a}(t_1,s),
\end{aligned}$$

where we introduced the residual terms $r(t,s)\coloneqq\widetilde{e}(t,s)-\widetilde{a}(t,s)$. By (27) and Lemma 2.5 we infer the bound

$$\|r(t,s)\|_{M^{\infty,1}}\le C_1(T)(t-s)^2, \quad 0<t-s\le T, \tag{32}$$

for some $C_1=C_1(T)>0$.

From this point forward, the proof is substantially identical to the one given in [43]. We retrace the main steps for the sake of completeness. A careful inspection of the previous expansion reveals that it ultimately consists of the sum of ordered Weyl products of symbols, each of them having the form

$$\underbrace{\widetilde{a}\cdots\widetilde{a}}_{q_{k+1}}\,\underbrace{r\cdots r}_{p_k}\,\underbrace{\widetilde{a}\cdots\widetilde{a}}_{q_k}\cdots\underbrace{r\cdots r}_{p_1}\,\underbrace{\widetilde{a}\cdots\widetilde{a}}_{q_1}, \tag{33}$$

where $p_1,\ldots,p_k,q_1,\ldots q_k,q_{k+1}$ are non negative integers (in particular $p_j>0$) that sum to L. Note also that the symbols $\widetilde{a}$ in each block of q_j terms can be grouped using (31).

We are now in the position to bound the $M^{\infty,1}$ norm of products of the form (33) using the previously derived estimates, namely (30) and (32). Precisely, we have – recall that $C_0 \ge 1$:

$$
\begin{aligned}
&\le C_0^{k+1} \prod_{j=1}^{k} \prod_{i=1}^{p_j} C_1 (t_{J_j+i} - t_{J_j+i-1})^2 \\
&\le C_0 \prod_{j=1}^{k} \prod_{i=1}^{p_j} C_0 C_1 (t_{J_j+i} - t_{J_j+i-1})^2,
\end{aligned}
$$

where we set $J_j = p_1 + \ldots + p_{j-1} + q_1 + \ldots + q_j$ for $j \ge 2$ and $J_1 = q_1$. The sum over $p_1, \ldots, p_k, q_1, \ldots, q_{k+1}$ of terms of this type gives in turn

$$
\begin{aligned}
\|e(\Omega_L; t, s) - a(t, s)\|_{M^{\infty,1}} &\le C_0 \left\{ \prod_{j=1}^{L} (1 + C_0 C_1 (t_j - t_{j-1})^2) - 1 \right\} \\
&\le C_0 \left\{ \exp\left(\sum_{j=1}^{L} C_0 C_1 (t_j - t_{j-1})^2 \right) - 1 \right\} \\
&\le C_0 \left\{ \exp\left(C_0 C_1 \omega(\Omega_L)(t-s) \right) - 1 \right\} \\
&\le C_0^2 C_1 e^{C_0 C_1 \omega(\Omega_L)(t-s)} \omega(\Omega_L)(t-s),
\end{aligned}
$$

where in the last inequality we used $e^\tau - 1 \le \tau e^\tau$, for $\tau \ge 0$. Setting $C = C(T) = C_0^2 C_1 \exp\left(C_0 C_1 T^2\right)$ concludes the proof of (8) in the claim.

For what concerns uniform convergence of $e(\Omega_L; t, s)$ to $a(t, s)$ in $\mathbb{R}^{2d}$ as $\omega(\Omega_L) \to 0$, it is an easy consequence of (8) in view of the continuous embedding $M^{\infty,1}(\mathbb{R}^{2d}) \subset C(\mathbb{R}^{2d}) \cap L^\infty(\mathbb{R}^{2d})$, cf. Proposition 2.1.

Lastly, let us prove (9). Note that

$$
\|E(\Omega_L; t, s) - U(t, s)\|_{M^p \to M^p} \le \|U_0(t, s)\|_{M^p \to M^p} \|e(\Omega_L; t, s)^{\mathrm{w}} - a(t, s)^{\mathrm{w}}\|_{M^p \to M^p}.
$$

The term $\|U_0(t, s)\|_{M^p \to M^p}$ can be bounded by a constant C_2, possibly depending on T, since the singular values of S_{t-s} in (19) depend continuously on the entries of S_{t-s} (cf. e.g. [52, Corollary A.4.5]) and the latter are in turn continuous functions of $t - s \in \mathbb{R}$ (note that $S_0 = I$). For the remaining term, by (16) we have

$$
\|e(\Omega_L; t, s)^{\mathrm{w}} - a(t, s)^{\mathrm{w}}\|_{M^p \to M^p} \le C_3 \|e(\Omega_L; t, s) - a(t, s)\|_{M^{\infty,1}},
$$

for an absolute constant C_3. The bound in (9) thus holds with $C = C_2(T) C_3$.

3.3 **Proof of Theorem 1.2**

Let us finally provide a proof of the convergence result stated in Theorem 1.2 at the level of integral kernels. The technique is similar to that used in [44]. Fix a real-valued function $\Psi \in C_c^\infty(\mathbb{R}^{2d})$ with compact support, then choose another real-valued function $\Theta \in C_c^\infty(\mathbb{R}^{2d})$ with $\Theta = 1$ on $\mathrm{supp}\Psi$. We have

$$
\begin{aligned}
&\|\mathcal{F}\left[(k(\Omega_L; t, s) - u(t, s)) \Psi \right]\|_{L^1} \\
&\qquad = |\det B_{t-s}|^{-1/2} \left\| \mathcal{F}\left[e^{2\pi i \Phi_{t-s}} \left(e'(\Omega_L; t, s) - a'(t, s) \right) \Psi \right] \right\|_{L^1} \\
&\qquad = |\det B_{t-s}|^{-1/2} \left\| \mathcal{F}\left[\left(\Theta e^{2\pi i \Phi_{t-s}} \right) \left(e'(\Omega_L; t, s) - a'(t, s) \right) \Psi \right] \right\|_{L^1} \\
&\qquad \le |\det B_{t-s}|^{-1/2} \left\| \mathcal{F}\left[\Theta e^{2\pi i \Phi_{t-s}} \right] * \mathcal{F}\left[\left(e'(\Omega_L; t, s) - a'(t, s) \right) \Psi \right] \right\|_{L^1} \\
&\qquad \le |\det B_{t-s}|^{-1/2} \left\| \mathcal{F}\left[\Theta e^{2\pi i \Phi_{t-s}} \right] \right\|_{L^1} \left\| \mathcal{F}\left[\left(e'(\Omega_L; t, s) - a'(t, s) \right) \Psi \right] \right\|_{L^1}.
\end{aligned}
$$

Since $\Theta e^{2\pi i \Phi_{t-s}} \in C_c^\infty(\mathbb{R}^{2d})$, it is clear that we can find a constant $C_0 = C_0(t-s, \Psi)$ such that $\|\mathcal{F}\left[\Theta e^{2\pi i \Phi_{t-s}}\right]\|_{L^1} < C_0$. Finally, using (8),

$$
\begin{aligned}
\|\mathcal{F}\left[\left(e'(\Omega_L; t, s) - a'(t, s) \right) \Psi \right]\|_{L^1} &= \|V_\Psi \left(e'(\Omega_L; t, s) - a'(t, s) \right)(0, \cdot)\|_{L^1} \\
&\le C_3(\Psi) \|e'(\Omega_L; t, s) - a'(t, s)\|_{M^{\infty,1}} \\
&\le C_3(\Psi) C_4(T) \|e(\Omega_L; t, s) - a(t, s)\|_{M^{\infty,1}} \\
&\le [C_3(\Psi) C_4(T) C_5(T)] \omega(\Omega_L)(t-s),
\end{aligned}
$$

where $C_4(T)$ comes from (25) and (26), while $C_5(T)$ comes from (8). The claimed bound (13) thus follows with $C_2 = C_3(\Psi)C_4(T)C_5(T)$, while (12) is satisfied with $C_1 = C_0(t-s,\Psi)C_2(T,\Psi)$.

Uniform convergence of kernels on compact subsets is then clear: since

$$\| \left(k(\Omega_L;t,s) - u(t,s) \right) \Psi \|_{L^\infty} \le \| \mathcal{F} \left[\left(k(\Omega_L;t,s) - u(t,s) \right) \Psi \right] \|_{L^1},$$

for any compact subset $K \subset \mathbb{R}^{2d}$ it is enough to choose $\Psi \in \mathcal{S}(\mathbb{R}^{2d})$ with $\Psi \equiv 1$ on K.

3.4 Convergence results for higher order parametrices

The results proved so far easily extend to the higher order parametrices $E^{(N)}(t,s) \coloneqq U_0(t,s)e^{(N)}(t,s)$, $N \ge 2$, associated with the symbols introduced in Remark 3.2. With obvious meaning of notation, repeating the arguments in Section 3.2 with slight modifications yields a higher order counterpart of (8), namely

$$\|e^{(N)}(\Omega_L;t,s) - a(t,s)\|_{M^{\infty,1}} \le C(T)\omega(\Omega_L)^N(t-s).$$

Since the bounds for FIO symbols are the building blocks of convergence results, all the claims in Theorem 1.1 and 1.2 extend to $E^{(N)}(t,s)$ and the corresponding kernel $k^{(N)}(\Omega_L;t,s)$ with $\omega(\Omega_L)$ replaced by $\omega(\Omega_L)^N$ in the relevant estimates.

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