KÄHLER-RICCI SOLITONS INDUCED BY INFINITE DIMENSIONAL COMPLEX SPACE FORMS

ANDREA LOI, FILIPPO SALIS, AND FABIO ZUDDAS

Abstract. We exhibit families of non trivial (i.e. not Kähler-Einstein) radial Kähler-Ricci solitons (KRS), both complete and not complete, which can be Kähler immersed into infinite dimensional complex space forms. This result shows that the triviality of a KRS induced by a finite dimensional complex space form proved in [12] does not hold when the ambient space is allowed to be infinite dimensional. Moreover, we show that the radial potential of a radial KRS induced by a non-elliptic complex space form is necessarily defined at the origin.

Contents

1. Introduction

The study of those complex manifolds M equipped with a Kähler-Einstein (KE) metric g induced by a complex space form, namely such that (M, g) can be Kähler

2000 *Mathematics Subject Classification.* 53C55, 32Q15, 32T15.
Key words and phrases. Kähler metric, Kähler-Ricci solitons; Einstein metrics; Calabi's diastasis function; complex space forms.
The first and the third authors were supported by STAGE - Funded by Fondazione di Sardegna and by KASBA- Funded by Regione Autonoma della Sardegna. The second author was supported by PRIN 2017 "Real and Complex Manifolds: topology, geometry and holomorphic dynamics" and MIUR grant "Dipartimenti di Eccellenza 2018-2022". All the three authors were supported by INdAM GNSAGA - Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni.

immersed[1] into a finite or infinite dimensional complex space form (S, g), is a classical problem in complex differential geometry. When the ambient complex space form is assumed to be finite dimensional and of non positive holomorphic sectional curvature M. Umehara [24] shows that (M, g) is forced to be totally geodesic and hence is itself an open subset of a complex space form. On the other hand a classification of those KE manifolds Kähler immersed into the finite dimensional complex projective space is still missing. The general conjecture is that such a KE manifold is forced to be an open subset of a compact homogeneous Kähler manifold, i.e. it is acted upon transitively by its group of holomorphic isometries. Roughly speaking when the ambient space is finite dimensional one has (locally) a finite number of holomorphic functions describing the Kähler immersion which seems to force the potential of a KE metric (which satisfies a Monge-Ampere equation) to have symmetries, i.e. to be the potential of a homogeneous metric. Many authors have proved the validity of this conjecture under additional assumptions (see, e.g. [4], [20], [23], [8], [9], [18]). When the ambient space is infinite dimensional the situation changes drastically: there exist continuous families of complete not homogeneous KE metrics projectively induced by an infinite dimensional complex projective space[2] (see [15] and [7]).

Therefore it is natural to impose some extra conditions on the KE metric g in order to recover the loss of symmetries due to the infinite dimensional assumption of the ambient space. One natural condition is to require that the metric is radial, i.e. g admits a global Kähler potential $f(r)$ which depends only on the sum $r = |z|^2 = |z_1|^2 + \cdots + |z_n|^2$ of the local coordinates' moduli. Notice that since the manifold M is assumed to be connected the potential $f(r)$ is defined on an open interval $(r_{\inf}, r_{\sup})$, $0 < r_{\inf} < r_{\sup}$. The prototypes of radial KE metrics are of course the finite dimensional complex space forms and any homogeneous Kähler manifold with a radial potential is indeed a complex space form. The main result in this regard found by the authors of the present paper can be summarized as follows (see Definition 1 in the next section and Definition 2 in Section 5 for the notions of well-behaved or c-stable projectively induced metrics).

Theorem A. (see [14, Theorem 1.3 and Theorem 1.4]) *Let g be a radial KE metric on a complex manifold M and assume that (M, g) can be Kähler immersed into an infinite dimensional complex space form (S^∞, g_c^∞) with constant holomorphic sectional curvature c.*

(1) If $c \leq 0$ then (M, g) is a complex space form.

[1]Throughout the paper the Kähler manifold M is not necessarly compact (or complete) and the Kähler immersion is not required to be injective or an embedding.
[2]We still do not know if similar phenomena can also happen in the infinite dimensional non elliptic case.

(2) If $c > 0$ and the metric g is either well-behaved or c-stable projectively induced then (M, g) is a complex space form.

We believe that the assumptions that g is well-behaved or c-stable projectively induced in Theorem A are superfluous (cfr. [14, Conjecture 2]). This is true if the Einstein constant of g vanishes: indeed in [17] we prove that a projectively induced Ricci flat metric is forced to be flat. It is also worth mentioning that the KE condition in Theorem A can be weakened to constant scalar curvature (cscK) case [14, Theorem 1.3] but not to the case of Calabi's extremal metrics, see [14, Example 1]. Both cscK and extremal metrics are generalization of KE metrics. Another natural extension is that of *Kahler-Ricci soliton* (KRS); therefore it is natural to study radial KRS induced by infinite dimensional complex space forms. This is what we do in the present paper. Recall that a KRS on a complex manifold M is a pair (g, X) consisting of a Kähler metric g and a holomorphic vector field X, called the *solitonic vector field*, such that

$$\rho = \lambda\omega + L_X\omega \tag{1}$$

for some $\lambda \in \mathbb{R}$, called the *solitonic constant*. Here ω and ρ are respectively the Kähler form and the Ricci form of the metric g and $L_X\omega$ denotes the Lie derivative of ω with respect to X. KRS are special solutions of the Kähler-Ricci flow and they generalize Kähler–Einstein (KE) metrics. Indeed any KE metric g on a complex manifold M gives rise to a trivial KRS by choosing $X = 0$ or X Killing with respect to g. Obviously if the automorphism group of M is discrete then a KRS (g, X) is nothing but a KE metric g. The reader is referred to [10, 3, 25, 19, 21, 22] for more information on KRS.

It turns out that a radial KRS with given solitonic constant λ on a n-dimensional complex manifold M is uniquely determined by $(\mu, \nu, k) \in \mathbb{R}^3$ if $n = 1$ and $(\mu, \nu) \in \mathbb{R}^2$ if $n \geq 2$ (cfr. Proposition 2.2 in the next section). Further the KRS is not trivial if μ and ν are not zero.

The following theorem is the first main result of the paper.

Theorem 1.1. *Let (g, X) be a non trivial KRS with solitonic constant λ on an n-dimensional complex manifold M. Assume that*[3]

$$\nu = \frac{n!(\mu - \lambda)}{\mu^{n+1}}. \tag{2}$$

Then the following facts hold true.

[3]It turns out that the condition (2) is equivalent to the fact that the Kähler potential $f(r)$ of the metric g is defined at the origin, namely at $r_{\inf} = 0$ (see Proposition 2.2 below).

(a) *if μ and ν are strictly positive and $k = 0$ then (M, g) can be Kähler immersed into any infinite dimensional complex space forms of non-negative holomorphic sectional curvature.*

(b) *if $\lambda \leq 0$, $\mu = n + 1$ and $k = 0$ then (M, g) can be Kähler immersed into any infinite dimensional complex space form.*

Theorem 1.1 shows that the same conclusions of Theorem A (namely the constancy of the holomorphic sectional curvature) cannot be achieved if one weakens Einstein's condition with that of KRS, even if one requires that the radial potential $f(r)$ of the metric g is defined at the origin (which is stronger than well-behaveness) and that the metric g is induced by any complex space form (which is much stronger than c-stability, see Remark 4 in Section 5 below). The theorem also shows that the main result in [12] due to the first author and R. Mossa, asserting that a KRS induced by a *finite* dimensional (even indefinite) complex space form is trivial, does not extend to the infinite dimensional setting.

One can show that the KRS in Theorem 1.1 are not complete, namely their Kähler metric are not complete. Thus it is natural to see if there exist complete KRS induced by some infinite dimensional complex space form. The following theorem, our second main result, shows that this is indeed the case.

Theorem 1.2. *For any $n \geq 2$ there exist complete and radial KRS on $\mathbb{C}^n$ induced by any infinite dimensional complex space forms of non-negative holomorphic sectional curvature*[4].

We believe that the requirement (2) is a necessary condition for the Kähler metric g to be induced by a complex space form, as expressed by the following:

Conjecture 1: *The potential of a radial KRS induced by an infinite dimensional complex space form is defined at the origin.*

In this regard we are able to prove the following theorem which represents our third and last result.

Theorem 1.3. *Let (g, X) be a radial KRS on a complex manifold M of complex dimension $n \geq 2$. If the metric g is c-stable projectively induced, for some $c > 0$, then its Kähler potential $f(r)$ is defined at the origin and hence (2) holds true.*

Since a Kähler metric induced by a non-elliptic complex space form is c-stable for any $c > 0$ (see Remark 4 in Section 5 below), Theorem 1.3 gives the following corollary, which provides us with a partial answer to the conjecture and also shows

[4]We do not know if there exist complete KRS induced by the infinite dimensional complex hyperbolic space.

that in order to prove its validity one can restrict to the case when the ambient complex space form is the complex projective space.

Corollary 1. *If a radial KRS on a complex manifold of complex dimension $n \geq 2$ is induced by either an infinite dimensional flat space or by an infinite dimensional complex hyperbolic space then its Kähler potential is defined at the origin.*

The paper is organized as follows. In the next section we describe the basic facts on radial Kähler metrics and we provide the classification of radial KRS (Proposition 2.2). In Section 3, after recalling some necessary and sufficient conditions for a radial Kähler metric to be induced by a complex space form (Lemma 3.1) we prove Theorem 1.1. In Section 4 we show that the Kähler manifolds appearing in the proof of Theorem 1.1 are not complete and we prove Theorem 1.2. Finally, Section 5 is dedicated to the proof of Theorem 1.3. The paper ends with an appendix with two lemmata needed in the proof of Proposition 2.2.

2. Radial KRS

Let g be a radial Kähler metric on a connected complex manifold M, equipped with complex coordinates $z_1, \dots, z_n$ and let ω and ρ be respectively the Kähler form and the Ricci form associated to g. Then there exists a smooth radial function

$$f : (r_{\inf}, r_{\sup}) \to \mathbb{R},\ 0 \leq r_{\inf} < r_{\sup} \leq \infty,$$

where $(r_{\inf}, r_{\sup})$ is the maximal domain where $f(r)$ is defined such that

$$\omega = \frac{i}{2}\partial\bar{\partial} f(r),\ r = |z|^2 = |z_1|^2 + \cdots + |z_n|^2, \tag{3}$$

i.e. $f(r)$ is a radial potential for the metric g.

One can easily see that the matrix of the metric g and of the Ricci form ρ read as

$$\omega_{i\bar{j}} = f'(r)\delta_{ij} + f''(r)\bar{z}_i z_j. \tag{4}$$

$$\rho_{i\bar{j}} = L'(r)\delta_{ij} + L''(r)\bar{z}_i z_j, \tag{5}$$

where $L(r) = -\log(\det g)(r)$.

Set

$$y(r) := r f'(r). \tag{6}$$

Definition 1. A radial Kähler metric g is *well-behaved* if $y(r) \to 0$ for $r \to r_{\inf}^+$.

Clearly if a radial metric g is defined at $r_{\inf} = 0$ then it is well-behaved. In particular any metric of constant holomorphic sectional curvature is well-behaved and even real analytic on $[0, r_{\sup})$. Notice that it is not hard to see that a radial KE metric defined at the origin is indeed a complex space form. Also set

$$\psi(r) := r y'(r). \tag{7}$$

Then

$$\psi(r) = \frac{dy}{dt}, \; r = e^t. \tag{8}$$

The fact that g is a metric is equivalent to $y(r) > 0$ and $\psi(r) > 0$, $\forall r \in (r_{\inf}, r_{\sup})$. Then

$$\lim_{r \to r_{\inf}^+} y(r) = y_{\inf} \tag{9}$$

is a non negative real number. Similarly set

$$\lim_{r \to r_{\sup}^-} y(r) = y_{\sup} \in (0, +\infty]. \tag{10}$$

Therefore we can invert the map

$$(r_{\inf}, r_{\sup}) \to (y_{\inf}, y_{\sup}), \; r \mapsto y(r) = rf'(r)$$

on $(r_{\inf}, r_{\sup})$ and think r as a function of y, i.e. $r = r(y)$.

Hence we can set

$$\psi(y) := \psi(r(y)). \tag{11}$$

The following lemma will be used in the proof of Theorem 1.3 below.

Lemma 2.1. *Assume that the function $\psi(y)$ is continuous at $y_{\inf}$. If $\lim_{y \to y_{\inf}^+} \psi(y) \neq 0$ then $y_{\inf} = 0$.*

Proof. Assume by contradiction that $y_{\inf} \neq 0$. Note first that $t_{\inf} := \lim_{r \to r_{\inf}} \log r = -\infty$: otherwise (if $t_{\inf} \in \mathbb{R}$) the function $y(t)$ could be prolonged to an open interval containing $t_{\inf}$ being the solution of the Cauchy problem

$$\begin{cases} y'(t) = \psi(y(t)) \\ y(t_{\inf}) = y_{\inf} > 0. \end{cases} \tag{12}$$

Thus, by the continuity of $\psi(y)$ at $y_{\inf} \neq 0$,

$$\lim_{y \to y_{\inf}^+} \psi(y) = \lim_{t \to -\infty} \psi(y(t)) = \lim_{t \to -\infty} y'(t) = 0,$$

where the last equality follows by (9) when $t_{\inf} = -\infty$, the desired contradiction. $\square$

Finally, from (4), we easily get

$$(\det g_{i\bar{j}})(r) = \frac{(y(r))^{n-1}\psi(y)}{r^n}. \tag{13}$$

The following proposition, which represents a key tool in the proof of our main results, provide us with the explicit expressions of radial KRS in terms of the the functions y and $\psi(y)$ defined by (6) and (7).

Proposition 2.2. *Let g be a radial KRS with solitonic constant λ. Then the following facts hold true.*

If $n=1$ then there exist $\mu, k \in \mathbb{R}$ such that

$$\dot{\psi}(y) = \mu\psi(y) + k + 1 - \lambda y \tag{14}$$

and if $\mu = 0$ then the soliton is trivial (i.e. a complex space form). If $\mu \neq 0$ then

$$\psi(y) = \nu e^{\mu y} + \frac{\lambda}{\mu} y + \left(\frac{\lambda}{\mu^2} - \frac{k+1}{\mu}\right) \tag{15}$$

and the soliton is trivial iff it is flat iff $\nu = 0$.

If $n \geq 2$ then there exists $\mu \in \mathbb{R}$ such that

$$\dot{\psi}(y) = \left(\mu - \frac{n-1}{y}\right)\psi(y) + n - \lambda y \tag{16}$$

and if $\mu = 0$ the soliton is trivial (i.e. KE). If $\mu \neq 0$ then

$$\psi(y) = \frac{\nu e^{\mu y}}{y^{n-1}} + \frac{\lambda}{\mu} y + \frac{\lambda - \mu}{\mu^{1+n}} \sum_{j=0}^{n-1} \frac{n!}{j!} \mu^j y^{j+1-n}. \tag{17}$$

and the soliton is trivial iff it is flat iff $\nu = 0$ and $\mu = \lambda$.

Moreover, the KRS is defined at the origin, i.e. at $r_{\inf} = 0$, iff $\nu = \frac{n!(\mu-\lambda)}{\mu^{n+1}}$ (namely (2) in Theorem 1.1 is satisfied).

In order to prove the proposition we need the following two technical lemmata whose proofs are relegated to Appendix A below.

Lemma 2.3. *Let $G(z) = \Phi(z) + \bar{\Phi}(z)$, where $\Phi(z)$, $z \in \mathbb{C}^n$, is a holomorphic function and $G(x_1, \ldots, x_n) = G(|z_1|^2, \ldots, |z_n|^2)$ is a rotation invariant function, $x_j = |z_j|^2$. Then*

$$G(x_1, \ldots, x_n) = \sum_{j=1}^{n} c_j \log x_j + d, \tag{18}$$

for some $c_j, d \in \mathbb{R}$. In particular, if $G(x_1 + \cdots + x_n) = G(|z_1|^2 + \cdots + |z_n|^2)$ is radial and $n \geq 2$, then the c_j's must vanish and $G = d$ is constant.

Lemma 2.4. *Let the equality*

$$\sum_{k=1}^{n} (\bar{z}_k Y_k(z) + z_k \bar{Y}_k(z)) = \phi(r) \tag{19}$$

hold, where Y_k, $k = 1, \ldots, n$, is a holomorphic function and $\phi(r)$ is a radial function. Then $\phi(r) = \alpha r$ for some $\alpha \in \mathbb{R}$.

Proof of Proposition 2.2. Let $X = \sum_k X_k \frac{\partial}{\partial z_k} + \bar{X}_k \frac{\partial}{\partial \bar{z}_k}$ be a real holomorphic vector field ($X_k = X_k(z)$ are holomorphic functions). If we take local complex coordinates $z_1, \ldots, z_n$ and use the fact that

$$L_X\omega(Y, Z) = X(\omega(Y, Z)) - \omega([X, Y], Z) - \omega(Y, [X, Z]).$$

we get

$$(L_X\omega)_{i\bar{j}} = X(\omega_{i\bar{j}}) + \frac{\partial X_k}{\partial z_i}\omega_{k\bar{j}} + \frac{\partial \bar{X}_k}{\partial \bar{z}_j}\omega_{i\bar{k}} \tag{20}$$

By substituting (4) in (20) we obtain

$$(L_X\omega)_{i\bar{j}} = (\sum_k X_k\bar{z}_k + \bar{X}_k z_k)(f''(r)\delta_{ij} + f'''(r)\bar{z}_i z_j) + f''(r)(X_j\bar{z}_i + \bar{X}_i z_j) +$$

$$+\frac{\partial X_k}{\partial z_i}(f'(r)\delta_{kj} + f''(r)\bar{z}_k z_j) + \frac{\partial \bar{X}_k}{\partial \bar{z}_j}(f'(r)\delta_{ik} + f''(r)\bar{z}_i z_k).$$

Now, it is easy to see from a straight calculation that this last expression is equal to

$$\frac{\partial^2}{\partial z_i \partial \bar{z}_j}\sum_k f'(r)(\bar{z}_k X_k + z_k \bar{X}_k)$$

and then the soliton equation (1) writes

$$\frac{\partial^2}{\partial z_i \partial \bar{z}_j}\sum_k f'(r)(\bar{z}_k X_k + z_k \bar{X}_k) = \rho_{i\bar{j}} - \lambda\omega_{i\bar{j}},$$

By taking into account (4) and (5) one gets

$$\frac{\partial^2}{\partial z_i \partial \bar{z}_j}\sum_k f'(r)(\bar{z}_k X_k + z_k \bar{X}_k) = a(r)\delta_{ij} + a'(r)\bar{z}_i z_j,$$

where we set $a(r) := L'(r) - \lambda f'(r)$.

Let $\Psi := \sum_k f'(\bar{z}_k X_k + z_k \bar{X}_k)$, we have

$$\frac{\partial^2 \Psi}{\partial z_i \partial \bar{z}_j} = a(r)\delta_{ij} + a'(r)\bar{z}_i z_j$$

for every i and j. Thus

$$\frac{\partial^2 \Psi}{\partial z_i \partial \bar{z}_j} = \frac{\partial^2 A}{\partial z_i \partial \bar{z}_j}$$

where $A(r) = \int a(r) = L(r) - \lambda f(r) + \gamma$ ($\gamma \in \mathbb{R}$) and then one concludes that

$$\Psi = A(r) + F + \bar{F} \tag{21}$$

for some holomorphic function F. Thus we can write

$$-\log\det g - \lambda f + F + \bar{F} = \sum_k f'(\bar{z}_k X^k + z_k \bar{X}^k).$$

By averaging with respect to the action of the unitary group $U(n)$, we get

$$-(\log\det g)(r) - \lambda f(r) + \int_{U(n)} \left(F(Az) + \bar{F}(Az)\right) dA = \sum_k f'(r)\left(\bar{z}_k Y^k(z) + z_k \bar{Y}^k(z)\right), \tag{22}$$

where $Y^k(z) = \int_{U(n)} \bar{A}^k_h X^h(Az) dA$.

Equation (22) can be rewritten as $\bar{z}_k Y^k + z_k \bar{Y}^k = \phi(r)$ where $\phi(r)$ is radial and the Y^k's are holomorphic. Then Lemma 2.4 above applies and we conclude that $\bar{z}_k Y^k + z_k \bar{Y}^k = \alpha r$, so that (22) reads as

$$-(\log\det g)(r) - \lambda f(r) + \int_{U(n)} \left(F(Az) + \bar{F}(Az)\right) dA = \alpha r f'(r) \tag{23}$$

If $n = 1$, then by Lemma 2.3 the real part of $\int_{U(n)} F(Az)dA$ is equal to $h + k\log r$, with h, k constants. In this case, (23) gives

$$-\log\left[f'(r) + rf''(r)\right] - \lambda f(r) + h + k\log r = \alpha r f'(r).$$

By derivating both sides of this equation with respect to r and by multiplying by r

$$-\frac{f''(r) + (f''(r)r)'}{f'(r) + f''(r)r} r - \lambda r f'(r) + k = \alpha (f'(r)r)' r,$$

by $y(r) = rf'(r)$ and $\psi(r) = r(rf'(r))'$ we get

$$\dot{\psi}(y) = -\alpha\psi(y) + k + 1 - \lambda y.$$

Then (14) follows by setting $\mu = -\alpha$. Moreover, if $\mu = 0$ equation (14) integrates and gives

$$\psi(y) = -\lambda\frac{y^2}{2} + (k+1)y + c,\ c \in \mathbb{R},$$

which by [14, Lemma 2.1] implies g has constant scalar curvature and hence is KE.

If $\mu \neq 0$, one easily integrates (14) and gets (15). By (15) and by taking into account [14, Lemma 2.1] we deduce that g is cscK iff it is flat iff $\nu = 0$.

Let us now assume $n \geq 2$. By applying again Lemma 2.3 to $\Phi = \int_{U(n)} F(Az)dA$, we get that the real part of $\int_{U(n)} F(Az)dA$ is constant and hence (22) reads as

$$-\log\det g - \lambda f(r) + k = \alpha r f'(r).$$

If we derivate this equation (with respect to r) and multiply both sides by r we get

$$r[-\log\det(g)]' - \lambda r f'(r) = \alpha r(rf'(r))'$$

i.e.

$$r[-\log\det(g)]' - \lambda y = \alpha\psi(y).$$

Now, by using (13) one obtains

$$-r[\log(y^{n-1}(r)\psi(y))]' + n - \lambda y = \alpha\psi(y).$$

Since $\frac{d}{dr} = \frac{dy}{dr}\frac{d}{dy} = (rf'(r))'\frac{d}{dy} = \frac{\psi(y)}{r}\frac{d}{dy}$, we can rewrite the previous expression as

$$\dot{\psi}(y) = -\left(\alpha + \frac{n-1}{y}\right)\psi(y) + n - \lambda y$$

which gives (16) by setting $\mu = -\alpha$,

By integrating (16) one gets

$$\psi(y) = \frac{e^{\mu y}}{y^{n-1}} \left[\nu + \int (n - \lambda y) e^{-\mu y} y^{n-1} dy \right], \tag{24}$$

for some constant $\nu \in \mathbb{R}$.

By taking $\mu = 0$ in (24) we get

$$\psi(y) = y + \frac{\nu + c}{y^{n-1}} - \frac{\lambda y^2}{n+1},$$

which, together with [14, Lemma 2.1], implies that the metric g is KE (with Einstein constant 2λ).

If $\mu \neq 0$ then (17) follows, after a long but straight computation, by (24) and by

$$\int e^{-\mu y} y^k dy = \frac{e^{-\mu y}}{\alpha} y^k - \sum_{j=0}^{k-1} \frac{k!}{(k-j-1)! \mu^{j+2}} e^{-\mu y} y^{k-j-1}.$$

Finally, by combining (16) with [14, Lemma 2.1] one gets that g is never KE unless it is flat and this happens exactly when $\nu = 0$ and $\mu = \lambda$.

In order to prove the last assertion of the Proposition, first notice that (17) rewrites as

$$\psi(y) = \frac{\nu e^{\mu y} + \frac{\lambda}{\mu} y^n + \frac{\lambda - \mu}{\mu^{1+n}} \sum_{j=0}^{n-1} \frac{n!}{j!} \mu^j y^j}{y^{n-1}}. \tag{25}$$

If the metric g is defined at the origin, then both $y(r) = r f'(r)$ and $\psi(r) = r(r f'(r))'$ are defined and vanish at $r_{\inf} = 0$ and (25) yields $\nu = n! \frac{\mu - \lambda}{\mu^{1+n}}$.

Conversely, if $\nu = n! \frac{\mu - \lambda}{\mu^{1+n}}$ then (25) implies that $\psi(y) = y + O(y^2)$ and (by the Hartman-Grobman linearisation theorem, see also [6], Section 4.2) the differential equation $\frac{dy}{dt} = \psi(y)$ can be conjugated in a neighbourhood of $y = 0$ to the linear equation $\frac{dz}{dt} = z(t)$ by a diffeomorphism $y = \Phi(z)$ satisfying $\Phi(0) = 0$. Thus $\Phi(z) = z\tilde{\Phi}(z)$ for some smooth $\tilde{\Phi}$ and $y(t) = ce^t \tilde{\Phi}(ce^t)$. By $e^t = r$ and $y(r) = r f'(r)$ we finally get $f'(r) = c\tilde{\Phi}(cr)$ which implies that $f(r)$ is smooth in $r = 0$. $\square$

Remark 1. A different proof of equation (17) is obtained by Feldman-Ilmanen-Knopf [6, Section 3.2] when the vector field X is assumed to be gradient (see also [3] for the case of radial steady KRS, i.e. $\lambda = 0$). In fact, it is not hard to see that the vector field Y appearing in the proof of Proposition 2.2 is a gradient vector field. However, in this paper we include the proof of Proposition 2.2 for reader's convenience and to make the paper as self contained as possible.

3. The proof of Theorem 1.1

A finite or infinite dimensional complex space form (S^N, g_c^N) is a manifold of constant holomorphic sectional curvature c and complex dimension $N \leq \infty$. By the word "induced" we mean that the Kähler manifold (M, g) can be Kähler immersed into (S^N, g_c^N), i.e. there exists a holomorphic map $\varphi : M \to S^N$ such that $\varphi^* g_c^N = g$ (see [2] or the book [16] for an updated material on the subject).

If one assumes that (S^N, g_c^N) is complete and simply-connected one has the corresponding three cases, depending on the sign of c:

- for $c = 0$, $S^N = \mathbb{C}^N$ ($S^\infty = \ell^2(\mathbb{C})$) and g_0^N is the flat metric with associated Kähler form

$$\omega_0 = \frac{i}{2}\partial\bar{\partial}|z|^2,\ |z|^2 = \sum_{j=1}^{N} |z_j|^2,\ N \leq \infty;$$

- for $c < 0$, $S^N = \mathbb{C}H^N$ is the N-dimensional complex hyperbolic space, namely the unit ball of $\mathbb{C}^N$ with the metric g_c^N with associated Kähler form

$$\omega_c = \frac{i}{2c}\partial\bar{\partial}\log(1 - |z|^2);$$

- for $c > 0$, $S^N = \mathbb{C}P^N$ is the N-dimensional complex projective space and g_c^N is the metric with associated Kähler form ω_c, given in homogeneous coordinates by:

$$\omega_c = \frac{i}{2c}\partial\bar{\partial}\log(|Z_0|^2 + \cdots + |Z_N|^2).$$

Notice that when $c = 1$ (resp. $c = -1$) the metric g_c^N is the standard Fubini-Study metric g_{FS} (respectively hyperbolic metric g_{hyp}) of holomorphic sectional curvature 4 (resp. -4). Throughout the paper we will say that a metric g on a complex (connected) manifold is *projectively induced* if (M, g) admits a Kähler immersion into $(\mathbb{C}\mathrm{P}^\infty, g_{FS})$.

Let $\epsilon \in \{-1, 0, 1\}$ and define recursively the following function in y

$$Q_1^\epsilon(y) := y; \qquad Q_{k+1}^\epsilon(y) = (\epsilon y - k)Q_k^\epsilon(y) + \dot{Q}_k^\epsilon(y)\psi(y), \tag{26}$$

with $\psi(y)$ given by (11).

Lemma 3.1. *Let M be an n-dimensional ($n \geq 1$) complex manifold equipped with a Kähler metric g with radial Kähler potential $f(r)$ which is real analytic in $(r_{\inf}, r_{\sup})$. If (M, g) can be Kähler immersed into (S^N, g_ϵ^N) then the $Q_k^\epsilon(y)$ are nonnegative for $y \in (y_{\inf}, y_{\sup})$. Moreover, if $r_{\inf} = 0$ and $f(r)$ is defined in $[0, r_{\sup})$ the converse holds true.*

Proof. See [13, Lemma 3.2] for a proof. □

Proof of Theorem 1.1. By the last part of Proposition 2.2 the assumption (2) implies that the potential $f(r)$ is defined at the origin and thus the metric g is real-analytic on $[0, r_{\sup})$ (see, e.g. [11, Corollary 1.3] for a proof). We first show that $y(r)$ and all its derivatives w.r.t. r are non-negative on $[0, r_{\sup})$.

We only treat the case $n \geq 2$ (the case $n = 1$ is obtained similarly and it is omitted). Under the assumption (2) equation (17) reads as

$$\psi(y) = y + \sum_{k=1}^{\infty} \frac{\nu \mu^{n+k}}{(n+k)!} y^{k+1} \tag{27}$$

Moreover, by $\nu, \mu > 0$ we have $\psi(y) \geq 0$.

We claim that for every k one has

$$y^{(k)}(r) = \frac{F_k(y(r))}{r^k} \tag{28}$$

with

$$F_k(y) = O(y^k), \quad F_k(y) \geq 0. \tag{29}$$

Formulae (28) and (29) hold true for $k = 1$ with $F_1(y) = \psi(y) \geq 0$ since $y'(r) = \frac{dy}{dt}\frac{dt}{dr} = \frac{y'(t)}{r} = \frac{\psi(y)}{r}$, and $\psi(y) = O(y) \geq 0$ by (27). Assuming now that (28) and (29) are true for some k, we have

$$y^{(k+1)}(r) = \frac{d}{dr}\left(\frac{F_k(y)}{r^k}\right) = \frac{\frac{dF_k}{dy} y'(r) r^k - F_k(y) k r^{k-1}}{r^{2k}} =$$

(recalling that $y'(r) = \frac{\psi(y)}{r}$)

$$= \frac{\frac{dF_k}{dy}\psi(y) - kF_k(y)}{r^{k+1}}. \tag{30}$$

This shows that (28) holds true also for $k+1$, with

$$F_{k+1}(y) = \frac{dF_k}{dy}\psi(y) - kF_k(y),$$

By inserting (27) and $F_k(y) = \sum_{j=k}^{\infty} a_j^k y^j$ in this recursion relation it is now easy to see that $F_{k+1}(y) = O(y^{k+1})$ and $F_{k+1}(y) \geq 0$, which concludes the proof by induction that (28) and (29) are true for every $k \geq 1$ and then that the derivatives $y^{(k)}(r)$ are non-negative on $[0, r_{\sup})$ for $k \geq 1$.

By $rf'(r) = y(r) = \sum_{k=1}^{\infty} \frac{y^{(k)}(0)}{k!} r^k$, one immediately deduces that the derivatives $f^{(k)}(r)$ are non-negative for $k \geq 1$.

Now we claim that the functions $Q_k^{\epsilon}(y)$ defined in (26) satisfy $Q_k^0(y) = r^k f^{(k)}(r)$ for every $k \geq 1$: this will prove that these functions are non-negative and by Lemma 3.1, (M, g) can be Kähler immersed into $\ell^2(\mathbb{C})$ and hence also in $\mathbb{C}P^{\infty}$ by a result of Calabi [2] (this proves (a) of Theorem 1.1).

In order to prove the claim, notice that (26) reads

$$Q_1^0(y) := y; \qquad Q_{k+1}^0(y) = -kQ_k^0(y) + \dot{Q}_k^0(y)\psi(y) \tag{31}$$

Since $y = rf'(r)$, we have $Q_1^0(y) = rf^{'}(r)$; assuming now that $Q_k^0(y) = r^k f^{(k)}(r)$ is true for some k, we have by (31)

$$Q^0_{k+1}(y) = -kr^k f^{(k)}(r) + \frac{d}{dr}\left(r^k f^{(k)}(r)\right)\frac{dr}{dy}\psi(y) =$$

(by using $\frac{dr}{dy} = \frac{r}{\psi}$)

$$= -kr^k f^{(k)}(r) + kr^k f^{(k)}(r) + r^{k+1} f^{(k+1)}(r) = r^{k+1} f^{(k+1)}(r)$$

which proves the claim and ends the proof of part (a) of Theorem 1.1.

Assume now that the parameters of the radial KRS (g, X) satisfy

$$\nu = \frac{n!(n+1-\lambda)}{(n+1)^{n+1}},\ \lambda \leq 0,\ \mu = n+1,\ k = 0. \tag{32}$$

To prove (b) of Theorem 1.1 we need to show that the Kähler manifold (M, g) can be Kähler immersed into $(\mathbb{C}\mathrm{H}^\infty, g_{hyp})$. Indeed this would imply that it can be Kähler immersed into the infinite dimensional flat space by a result of Bochner [1] and into any infinite dimensional complex projective space by [5, Lemma 8].

Notice that by (a) the radial potential $f(r)$ of the metric g of the family of KRS given by (32) is real-analytic on $[0, r_{\sup})$. Hence, by Lemma 3.1, we must prove that $Q^{-1}_k(y)$ is non-negative $\forall k \in \mathbb{Z}^+$ on $[0, y_{\sup})$. The proof is by induction on k. We treat only the case $n \geq 2$ (the case $n = 1$ is treated similarly). First, let us notice that (27) under the assumptions (32) writes

$$\psi(y) = y + \frac{n!(n+1-\lambda)}{n+1}\sum_{k=1}^{\infty}\frac{(n+1)^k}{(n+k)!}y^{k+1} = y + \left(1 - \frac{\lambda}{n+1}\right)y^2 + O(y^3).$$

We now assume by induction that the coefficients in the expansion of $Q^{-1}_k(y)$ are all nonnegative and that it vanishes at $y = 0$ with order greater or equal to k. This property is clearly verified for $k = 1$, since $Q^{-1}_1(y) = y$ by construction. Then, if $Q^{-1}_k(y) = \sum_{j\geq k} a_j y^j$, by (26) with $\epsilon = -1$, we get

$$\begin{aligned}
Q^{-1}_{k+1}(y) &= -(y+k)Q^{-1}_k(y) + \dot{Q}^{-1}_k(y)\psi(y) = \\
&-\sum_{j\geq k} a_j y^{j+1} - \sum_{j\geq k} k a_j y^j + \sum_{j\geq k} j a_j y^{j-1}\left(y + \left(1 - \frac{\lambda}{n+1}\right)y^2 + O(y^3)\right) = \\
&\sum_{j\geq k+1}(j-k)a_j y^j + \sum_{j\geq k}\left(j - \frac{\lambda j}{n+1} - 1\right)a_j y^{j+1} + O(y^{k+2}).
\end{aligned}$$

and we notice that all the coefficients of $O(y^3)$ in the second line are positive, so the coefficients in $O(y^{k+2})$ in the last line are all nonnegative. □

Remark 2. Combining Lemma 3.1 with the fact that (M, g) cannot be Kähler immersed into $\mathbb{C}H^N$, with $N < \infty$ (see [12] for a proof), we deduce that the number of positive $Q^{-1}_k(y)$ in the proof of the previous theorem is forced to be infinite.

4. Complete KRS solitons and the proof of Theorem 1.2

Let us now investigate the completeness of our radial metrics.

The matrix of a radial metric with radial potential $f(r)$ is given by (see (4))

$$g_{i\bar{j}} = f'(r)\delta_{ij} + f''(r)\bar{z}_i z_j$$

Take now a curve $\gamma(s) = (z(s), 0, \ldots, 0)$, $z(s) \in \mathbb{R}$, $z'(s) > 0$, where $s \in (s_1, s_2)$. By definition, its length is given by

$$l(\gamma) = \int_{s_1}^{s_2} \sqrt{g_{1\bar{1}} z'^2(s)} ds = \int_{s_1}^{s_2} \sqrt{f'(z^2(s)) + f''(z^2(s)) z^2(s)} z'(s) ds$$

i.e. by the change of variable $z = z(s)$

$$l(\gamma) = \int_{z_1}^{z_2} \sqrt{f'(z^2) + f''(z^2) z^2} dz$$

where we have set $z_1 = z(s_1)$, $z_2 = z(s_2)$.

Now, set $r = z^2$ so that $dz = \frac{1}{2\sqrt{r}} dr$ and

$$l(\gamma) = \frac{1}{2} \int_{r_1}^{r_2} \sqrt{\frac{f'(r) + f''(r) r}{r}} dr = \frac{1}{2} \int_{r_1}^{r_2} \sqrt{\frac{(rf')'}{r}} dr$$

Now, in order to rewrite this integral in terms of the functions $y(r) = rf'(r)$ and $\psi(y(r)) = r(rf'(r))'$, we make the change of variable $y = y(r)$. Notice that $\frac{dy}{dr} = (rf'(r))' = \frac{\psi f((r))}{r}$, so we get

$$l(\gamma) = \frac{1}{2} \int_{y_1}^{y_2} \sqrt{\frac{\psi}{r^2}} \frac{r}{\psi} dy = \frac{1}{2} \int_{y_1}^{y_2} \sqrt{\frac{1}{\psi}} dy$$

where $y_1 = y(r_1)$ and $y_2 = y(r_2)$.

Therefore we deduce that a radial metric corresponding to the function $\psi(y)$ defined on $[0, y_{sup})$ is complete if and only if

$$\lim_{y_2 \to y_{\sup}} \int_{y_1}^{y_2} \sqrt{\frac{1}{\psi}} dy = +\infty. \tag{33}$$

Example 1. Let (g, X) be the KRS of the complex manifolds M of dimension $n \geq 2$ given by Theorem 1.1. As we have seen at the beginning of the proof of Theorem 1.1, assumption (2) implies that ψ is given by (27). Then, for every $t_0 \in \mathbb{R}$ and $y_0 > 0$, the function

$$\Psi(y) := \int_{y_0}^{y} \frac{dy}{\psi(y)} \tag{34}$$

(which gives an implicit solution $\Psi(y) = t - t_0$ of the Cauchy problem $\frac{dy}{dt} = \psi(y)$, $y(t_0) = y_0$) is defined for every $y > 0$ since, by (27), under the assumption $\nu >$

$0, \mu > 0$ the denominator $\psi(y)$ of the integrand in (34) vanishes only for $y = 0$ and then the integral (34) is finite for every $y > 0$.

Moreover, by (17) we have

$$\lim_{y \to +\infty} \int_{y_0}^{+\infty} \sqrt{\frac{y^{n-1} dy}{\nu \left[e^{\mu y} - \sum_{j=0}^{n} \frac{\mu^j}{j!} y^j \right] + y^n}} < +\infty$$

because the integrand goes to zero as $\sqrt{\frac{y^{n-1}}{e^{\mu y}}}$, and the integral $\int_{y_0}^{+\infty} \sqrt{\frac{y^{n-1}}{e^{\mu y}}} dy$ converges. Thus, by (33), the metric is not complete.

Example 2. Let (g, X) be the KRS of the complex manifold M of dimension $n \geq 2$ associated to $(\nu, \mu) \in \mathbb{R}^2$ satisfying

$$\nu = n! \frac{\mu - \lambda}{\mu^{1+n}}, \quad \lambda = \mu - n - 1 < 0, \; \mu < 0. \tag{35}$$

We want to show that $M = \mathbb{C}^n$ and that the metric g is complete.

First, by assumption (2) the metric is defined at the origin and ψ is given by (27).

Moreover, the function ψ vanishes only for $y = 0$: indeed, assume by contradiction that under the assumptions $\nu = n! \frac{\mu - \lambda}{\mu^{1+n}}, \quad \mu < 0, \quad \lambda < 0$ there exists another positive zero $y = a$ for ψ. Then, by (16) we get $\psi'(a) = n - \lambda a > 0$, which is not possible (if ψ starts positive from zero, then it must be decreasing in a neighbourhood of the positive zero a). It follows that ψ is defined and positive for $y \in (0, +\infty)$. Now, the implicit definition of the solution $y(t)$, i.e. $\int_{y_0}^{y} \frac{dy}{\psi(y)} = t - t_0$, rewrites

$$\int_{y_0}^{y} \frac{y^{n-1}}{\nu \left[e^{\mu y} - \sum_{j=0}^{n} \frac{\mu^j}{j!} y^j \right] + y^n} dy = t - t_0$$

and then, since $\mu < 0$, one immediately sees that the integrand goes to zero as $1/y$ for $y \to +\infty$, so it diverges and $t_{\sup} = +\infty$. In terms of $r = e^t$ we get $(r_{\inf}, r_{\sup}) = (0, +\infty)$ and the metric is defined on all $\mathbb{C}^n$.

As for completeness, by (33) we need to check that the integral

$$\int_{y_0}^{+\infty} \sqrt{\frac{y^{n-1}}{\nu \left[e^{\mu y} - \sum_{j=0}^{n} \frac{\mu^j}{j!} y^j \right] + y^n}} dy \tag{36}$$

diverges. But this is clear since, as already observed above, for $y \to +\infty$ the function $\frac{1}{\psi}$ goes to zero as $1/y$, so that the integrand in (36) goes to zero as $1/\sqrt{y}$, and then the conclusion follows by $\int_{y_0}^{+\infty} \frac{1}{\sqrt{y}} dy = [2\sqrt{y}]_{y_0}^{+\infty} = +\infty$.

Remark 3. Let us notice that this soliton is an example of the complete expanding soliton on all of $\mathbb{C}^n$ found by Cao in [3] characterized by the values of the parameters (in our notation) $\lambda = -1$, $\mu < 0, \nu = n! \frac{\mu - \lambda}{\mu^{1+n}}$.

Proof of Theorem 1.2. Take the complete non trivial KRS $(\mathbb{C}^n, g_\mu)$ in Example 2 and set

$$\psi(y,\mu) := \psi(y) = y + \sum_{j=2}^{\infty} \frac{(n+1)!}{(n+j-1)!}\mu^{j-2}y^j. \tag{37}$$

In order to prove the theorem we will show that for a suitable choice of μ the Kähler manifold $(\mathbb{C}^n, g_\mu)$ can be Kähler immersed into $\ell^2(\mathbb{C})$ (and hence into $\mathbb{C}P^\infty$).

By using Weierstrass M-test one sees that $\frac{\partial^h \psi}{\partial y^h}$ are continuous with respect to μ, for all $h \in \mathbb{N}$ in the interval $[-1,1]$. Then, by definition of Q_k^0, namely

$$Q_{k+1}^0(y,\mu) = \dot{Q}_k^0\psi(y,\mu) - kQ_k^0(y,\mu), \tag{38}$$

every derivative w.r.t. y of $Q_k^0(y,\mu)$ is continuous w.r.t. μ.

Since $\psi(y,0) = y + y^2$ and

$$Q_k^0(y,0) = (k-1)!y^k, \tag{39}$$

we deduce that for every $k \in \mathbb{N}$, there exists a real negative constant $-1 \leq \epsilon_k < 0$ such that

$$\frac{\partial^k Q_k^0}{\partial y^k}\Big|_{(0,\mu)} > 0, \quad \forall \mu \in (\epsilon_k, 0).$$

Moreover, we claim that

$$I := \bigcap_{k\in\mathbb{N}} (\epsilon_k, 0) \neq \emptyset.$$

Indeed, if by contradiction $\lim_{k\to\infty} \epsilon_k = 0$ then one can easily find a sequence $\mu_k \in [-1,0)$ such that $\lim_{k\to\infty} \frac{\partial^k Q_k^0}{\partial y^k}\Big|_{(0,\mu_k)} = 0$. On the other hand, by (39) we deduce that $\lim_{k\to\infty} \frac{\partial^k Q_k^0}{\partial y^k}\Big|_{(0,0)} = +\infty$, yielding the desired contradiction and proving the claim.

Finally notice that by (37) and (38) we have that $Q_k^0(y) = O(y^k)$ independently of μ and then by Lemma 3.1 we deduce that g_μ with $\mu \in I$ is induced by $\ell^2(\mathbb{C})$. □

5. Projectively induced KRS and the proof of Theorem 1.3

We start by describing some necessary conditions for a radial KRS to be projectively induced.

Proposition 5.1. *Let M be an n-dimensional complex manifold with $n \geq 2$ and let g be the Kähler metric of the radial KRS corresponding to the function $\psi(y)$, $y \in (y_{\inf}, y_{\sup})$ with solitonic constant λ. Let us assume that g is projectively induced. Set $y_{\inf} := h$. Then the following facts hold true:*

(i) $\psi(h) = 0$;
(ii) $h \in \mathbb{N}$;
(iii) $\dot{\psi}(h) \in \mathbb{Z}$;

(iv) *if* $h \neq 0$ *then* $\lambda \in \mathbb{Q}$.

Proof. Throughout the proof we will denote the $Q_k^{+1}(y)$ appearing in (26) simply by $Q_k(y)$. Assume by contradiction that $\psi(h) \neq 0$. Then by Lemma 2.1 one has $h = 0$. Thus, by (16) when $n \geq 2$ and the fact that $\psi(y) > 0$ we see that

$$\lim_{y \to h^+} \dot{\psi}(y) = \lim_{y \to 0^+} \dot{\psi}(y) = -\infty \tag{40}$$

By deriving (26) with $\epsilon = 1$ and $k = 1$, we get

$$\dot{Q}_2(y) = 2y - 1 + \dot{\psi}(y)$$

which combined with (40) yields $\lim_{y \to 0^+} \dot{Q}_2(y) = -\infty$.

By (26) with $\epsilon = 1$ and $k = 2$

$$Q_3(y) = (y - 2)Q_2^1(y) + \psi(y)\dot{Q}_2(y).$$

We then immediately deduce that $\lim_{y \to 0^+} Q_3(y) = -\infty$, in contrast with Lemma 3.1, since the metric is projectively induced, and (i) is proved.

Notice now that by (i) and (26) (with $\epsilon = 1$) one can prove by induction on $k \in \mathbb{Z}^+$ that

$$Q_{k+1}(h) = (h - k)(h - k + 1) \cdots (h - 1)h. \tag{41}$$

Assume by contradiction that $h \notin \mathbb{Z}$. Then by (41) there exists k such that $Q_{k+1}(h) < 0$ again in contrast with the projectively induced assumption on g. Thus (ii) is proved.

By (41) we also deduce that

$$Q_j(h) = 0, \forall j \geq h + 1. \tag{42}$$

We claim that

$$\dot{Q}_{h+j}(h) = (\dot{\psi}(h) - 1)(\dot{\psi}(h) - 2) \cdots (\dot{\psi}(h) - j + 1)\dot{Q}_{h+1}(h), \tag{43}$$

for all $j \geq 2$.

Indeed, by deriving (26) we have

$$\dot{Q}_{h+2}(y) = Q_{h+1}(y) + (y - h - 1)\dot{Q}_{h+1}(y) + \psi(y)\ddot{Q}_{h+1}(y) + \dot{\psi}(y)\dot{Q}_{h+1}(y) \tag{44}$$

and the assertion follows for $j = 2$ by letting $y = h$ and using $Q_{h+1}(h) = 0$ (by (42)) and $\psi(h) = 0$ (by (i)). Assuming that (43) is true for some j, by deriving (26) w.r.t. y we get

$$\dot{Q}_{h+j+1}(y) = Q_{h+j}(y) + (y - h - j)\dot{Q}_{h+j}(y) + \psi(y)\ddot{Q}_{h+j}(y) + \dot{\psi}(y)\dot{Q}_{h+j}(y) \tag{45}$$

By taking $y = h$ we see that

$$\dot{Q}_{h+j+1}(h) = (\dot{\psi}(h) - j)\dot{Q}_{h+j}(h),$$

which, together with the inductive assumption, proves our claim.

By (26) and its derivative (with $k = j$) with respect to y and taking into account (i), i.e. $\psi(h) = 0$, on easily gets

$$Q_j(h) = Q_{j-1}(h)(h - j + 1)$$
$$\dot{Q}_j(h) = \dot{Q}_{j-1}(h)(h - j + 1) + Q_{j-1}(h) + \dot{\psi}(h)\dot{Q}_{j-1}(h).$$

Combining these two equalities and using $Q_1(y) = y$ we find

$$(Q_j + \dot{\psi}\dot{Q}_j)(h) = (h + \dot{\psi}(h))(h - 1 + \dot{\psi}(h)) \cdots (h - j + 1 + \dot{\psi}(h)).$$

Taking $j = h + 1$ and using $Q_{h+1}(h) = 0$ we get

$$\dot{Q}_{h+1}(h) = (h + \dot{\psi}(h))(h - 1 + \dot{\psi}(h)) \cdots (1 + \dot{\psi}(h)).$$

Now if by contradiction (iii) is false, i.e. $\dot{\psi}(h) \notin \mathbb{Z}$, we get $\dot{Q}_{h+1}(h) \neq 0$. Thus by (43) we deduce $\dot{Q}_{h+j}(h) < 0$ for some j, which combined with (42) implies that $Q_{h+j}(y) < 0$ on a right neighborhood of h, in contrast with fact that g is projectively induced.

By combining (i) and (16) (here we are using the assumption that $n \geq 2$) one deduces that $\dot{\psi}(h) = n - \lambda h$ and hence (iv) readily follows by (ii) and (iii). $\square$

Before proving Theorem 1.3 we recall the definition of c-stable projectively induced metric.

Definition 2. *Let $c > 0$. A Kähler metric g is said to be* c-stable projectively induced *if there exists $\epsilon > 0$ such that αg is induced by $(\mathbb{C}P^\infty, g_c^\infty)$ for all $\alpha \in (1-\epsilon, 1+\epsilon)$. A Kähler metric g is said to be unstable if it is not c-stable projectively induced for any $c > 0$. When $c = 1$ we simply say that g is* stable-projectively induced.

Remark 4. Notice that a Kähler metric which can be Kähler immersed into a non-elliptic (finite or infinite dimensional) complex space form is authomatically c-stable projectivelly induced (the reader is referred to [13] for details).

Proof of Theorem 1.3. Without loss of generality we can assume that g is induced by $(\mathbb{C}\mathrm{P}^\infty, g_{FS})$ and hence g is stable projectively induced. Therefore, by multiplying the metric by a suitable positive constant β, we can assume that βg is still projectively induced and with solitonic constant $\frac{\lambda}{\beta} \in \mathbb{R} \setminus \mathbb{Q}$. Hence by (i) and (iv) of Proposition 5.1 $y_{\inf} = 0$ and $\psi(y_{\inf}) = 0$. Thus, as seen in the last part of the proof of Proposition 2.2, $f(r)$ is defined at $r_{\inf} = 0$.

$\square$

Appendix A. The proofs of Lemma 2.3 and Lemma 2.4

Proof of Lemma 2.3. If we derivate the equality $G(z) = \Phi(z) + \bar{\Phi}(z)$ with respect to z_j and $\bar{z}_k$ we obtain

$$0 = \frac{\partial^2 G}{\partial x_j \partial x_k} z_k \bar{z}_j + \frac{\partial G}{\partial x_j} \delta_{jk} \tag{46}$$

Now, if $n \geq 2$ we can take $k \neq j$ in this equation and deduce $\frac{\partial^2 G}{\partial x_j \partial x_k} = 0$, so that $\frac{\partial G}{\partial x_j} = \Gamma_j(x_j)$, for some smooth function $\Gamma_j(x_j)$, $j = 1, \dots n$. This combined with (46) for $k = j$ yields

$$0 = \Gamma_j'(x_j) x_j + \Gamma_j(x_j), \forall j = 1, \dots n, \tag{47}$$

i.e.,

$$(\Gamma_j(x_j) x_j)' = 0, \ \forall j = 1, \dots n, \tag{48}$$

and then

$$\Gamma_j(x_j) = \frac{\partial G}{\partial x_j} = \frac{c_j}{x_j}, \ \forall j = 1, \dots n,$$

from which (18) follows immediately.

If $n = 1$, equation (46) writes $G''(x)x + G'(x) = 0$, which can be treated as equation (47) to deduce that $G'(x) = \frac{c}{x}$ (for some constant c) and obtain the same conclusion.

The last assertion in the statement follows immediately by noticing that (18) with G rotation invariant can be satisfied only if the c_j's vanish or $n = 1$. $\square$

Proof of Lemma 2.4. By deriving (19) with respect to z_l and $\bar{z}_l$ we get

$$\frac{\partial Y_l}{\partial z_l} + \frac{\partial \bar{Y}_l}{\partial \bar{z}_l} = \phi''(r)|z_l|^2 + \phi'(r), \ l = 1, \dots n. \tag{49}$$

Since the right-hand side is a rotation invariant function and $\frac{\partial Y_l}{\partial z_l}$ is holomorphic, we can apply Lemma 2.3 to deduce that

$$\frac{\partial Y_l}{\partial z_l} = \sum_{j=1}^{n} c_{lj} \log z_j + d_l, \ l = 1, \dots, n$$

for some $c_{lj}, d_l \in \mathbb{C}$. Then (49) writes

$$\sum_{j=1}^{n} (c_{lj} \log z_j + \bar{c}_{lj} \log \bar{z}_j) + 2\tilde{d}_l = \phi''(r)|z_l|^2 + \phi'(r), \ l = 1, \dots, n,$$

where $2\tilde{d}_l = d_l + \bar{d}_l$.

Then we deduce that $c_{lj} \in \mathbb{R}$, and

$$\sum_{j=1}^{n} c_{lj} \log |z_j|^2 + 2\tilde{d}_l = \phi''(r)|z_l|^2 + \phi'(r),\ l = 1, \dots, n,$$

By setting $x_j = |z_j|^2$, we can rewrite this equation as

$$\frac{\partial}{\partial x_l}(\phi'(r)x_l) = \sum_{j=1}^{n} c_{lj} \log x_j + 2\tilde{d}_l,\ l = 1, \dots, n,$$

and, by integrating, we get

$$\phi'(r)x_l = x_l \sum_{j=1}^{n} c_{lj} \log x_j + 2\tilde{d}_l x_l + F_l(x),\ l = 1, \dots, n \tag{50}$$

where $F_l(x)$ does not depend on x_l.

If $n \geq 2$ by derivating (50) with respect to x_k, $k \neq l$, we get

$$\left(\phi''(r) - \frac{c_{lk}}{x_k}\right) x_l = \frac{\partial F_l}{\partial x_k}$$

which, since $\frac{\partial F_l}{\partial x_k}$ does not depend on x_l, implies that F_l is a constant, say f_l, and $\phi''(r) = \frac{c_{lk}}{x_k}$. But, since $\phi'' = \phi''(x_1 + \cdots + x_n)$, this last equality implies that $c_{lk} = 0$ for $k \neq l$. Then (50) becomes

$$\phi'(r)x_l = c_{ll} x_l \log x_l + 2\tilde{d}_l x_l + f_l \tag{51}$$

Since $\phi' = \phi'(x_1 + \cdots + x_n)$, this equality implies that ϕ' is a constant, and then that $\phi(r) = \alpha r$ as desired.

For $n = 1$, we have the analogous of (51) with ϕ' depending on one variable x only, that is

$$x\phi'(x) = cx \log x + 2\tilde{d}x + f,$$

By derivating and setting $2\alpha = c + 2\tilde{d}$

$$\phi'(x) + x\phi''(x) = c \log x + 2\alpha$$

and then by (49)

$$\frac{\partial Y}{\partial z} + \frac{\partial \bar{Y}}{\partial \bar{z}} = c \log |z|^2 + 2\alpha$$

which by the holomorphicity of $\frac{\partial Y}{\partial z}$ implies

$$\frac{\partial Y}{\partial z} = c \log z + \alpha$$

Then $Y = c\int \log z + \alpha z + k$. Combined with the assumption $\bar{z}Y + z\bar{Y} = \phi(r)$ this yields

$$c\bar{z}\int \log z + \bar{c}z\int \log \bar{z} + \alpha|z|^2 + (k\bar{z} + \bar{k}z) = \phi(r)$$

and this can hold true only if $c = k = 0$, so that $\phi(r) = \alpha r$, as desired. $\square$

References

[1] S. Bochner, *Curvature in Hermitian metric*, Bull. Amer. Math. Soc. 53 (1947), 179-195.

[2] E. Calabi, *Isometric Imbedding of Complex Manifolds*, Ann. of Math. Vol. 58 No. 1, 1953.

[3] H. Cao, *Existence of Gradient Kähler–Ricci Solitons*, Elliptic and Parabolic Methods in Geometry, AK Peters, 1996.

[4] S. S. Chern, *On Einstein hypersurfaces in a Kähler manifold of constant scalar curvature*, J. Diff. Geom. 1 (1967), 21-31.

[5] A. J. Di Scala, H. Hishi, A. Loi, *Kähler immersions of homogeneous Kähler manifolds into complex space forms*, Asian Journal of Math. Vol. 16 No. 3 (2012), 479-488.

[6] M. Feldman, T. Ilmanen, D. Knopf, *Rotationally symmetric shrinking and expanding gradient Kähler-Ricci solitons*, J. Diff. Geom. 65, 169-209 (2003).

[7] Hao, Yihong; Wang, An; Zhang, Liyou, *On holomorphic isometric immersions of nonhomogeneous Kähler-Einstein manifolds into the infinite dimensional complex projective space*. J. Math. Anal. Appl. 423 (2015), no. 1, 547–560.

[8] J. Hano, *Einstein complete intersections in complex projective space*, Math. Ann. 216 (1975), no. 3, 197-208.

[9] D. Hulin, *Kähler–Einstein metrics and projective embeddings*, J. Geom. Anal. 10 (2000), no. 3, 525–528.

[10] N. Koiso, *On rotationally symmetric Hamilton's equation for Kähler-Einstein metrics*, Recent topics in differential and analytic geometry, Adv. Stud. Pure Math., vol. 18, Academic Press, Boston, MA, 1990, pp. 327-337.

[11] B. L. Kotschwar, *A local version of Bando's theorem on the real-analyticity of solutions to the Ricci flow*, Bull. Lond. Math. Soc. 45, no. 1, 153–158 (2013).

[12] A. Loi, R. Mossa, *Kähler immersions of Kähler-Ricci solitons into definite or indefinite complex space forms*, arXiv:2003.00841, to appear in Proc. Amer. Math. Soc.

[13] A. Loi, F. Salis, F. Zuddas, *Two conjectures on Ricci flat metrics*, Math. Z. (2018) 290, 599–613.

[14] A. Loi, F. Salis, F. Zuddas. *Extremal Kähler metrics induced by finite or infinite dimensional complex space forms*, to appear in The Journal of Geometric Analysis

[15] A. Loi, M. Zedda, *Kähler-Einstein submanifolds of the infinite dimensional projective space*, Math. Ann. 350 (2011), 145-154.

[16] A. Loi, M. Zedda, *Kähler Immersions of Kähler Manifolds into Complex Space Forms*, Lecture Notes of the Unione Matematica Italiana **23**, Springer, (2018).

[17] A. Loi, M. Zedda, F. Zuddas, *Ricci flat Calabi's metric is not projectively induced*, Tohoku Math. J. 73 (2021), 29-37.

[18] G. Manno, F. Salis, 2-*dimensional Kähler-Einstein metrics induced by finite dimensional complex projective spaces*, arXiv:2011.04323.

[19] F. Podestà, A. Spiro, *Kähler-Ricci solitons on homogeneous toric bundles*, J. Reine Angew. Math. 642 (2010), 109–127.

[20] B. Smyth, *Differential geometry of complex hypersurfaces*, Ann. of Math. (2) 85 (1967), 246-266.

[21] G. Tian and X. Zhu, *A new holomorphic invariant and uniqueness of Kähler-Ricci solitons*, Comment. Math. Helv. 77 (2002), no. 2, 297-325.

[22] G. Tian and X. Zhu, *Uniqueness of Kähler-Ricci solitons*, Acta Math. 184 (2000), no. 2, 271-305.

[23] K. Tsukada, *Einstein-Kähler Submanifolds with codimension two in a Complex Space Form*, Math. Ann. 274 (1986), 503-516.

[24] M. Umehara, *Einstein Kaehler submanifolds of a complex linear or hyperbolic space*, Tôhoku Math. J., 39 (1987), 385-389.

[25] X. J. Wang, X. Zhu, Kähler-Ricci solitons on toric manifolds with positive first Chern class, Adv. Math. 188 (2004), no. 1, 87–103.

(Andrea Loi) Dipartimento di Matematica, Università di Cagliari, Via Ospedale 72, 09124 (Italy)

Email address: `loi@unica.it`

(Filippo Salis) Dipartimento di Scienze Matematiche "G. L. Lagrange", Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino (Italy)

Email address: `filippo.salis@polito.it`

(Fabio Zuddas) Dipartimento di Matematica, Università di Cagliari, Via Ospedale 72, 09124 (Italy)

Email address: `fabio.zuddas@unica.it`