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# ON THE GENERATORS OF CLIFFORD SEMIGROUPS: POLYNOMIAL RESOLVENTS AND THEIR INTEGRAL TRANSFORMS

RICCARDO GHILONI AND VINCENZO RECUPERO

ABSTRACT. This paper deals with generators  $A$  of strongly continuous right linear semigroups in Banach two-sided spaces whose set of scalars is an arbitrary Clifford algebra  $Cl(0, n)$ . We study the invertibility of operators of the form  $P(A)$ , where  $P(x) \in \mathbb{R}[x]$  is any real polynomial, and we give an integral representation for  $P(A)^{-1}$  by means of a Laplace-type transform of the semigroup  $T(t)$  generated by  $A$ . In particular, we deduce a new integral representation for the spherical quadratic resolvent of  $A$  (also called pseudoresolvent of  $A$ ). As an immediate consequence, we also obtain a new proof of the well-known integral representation for the spherical resolvent of  $A$ .

*dedicated to Professor Klaus Gürlebeck*

## 1. INTRODUCTION AND MAIN RESULTS

Quaternionic functional analysis has probably its original motivation in the seminal paper [4], where it is pointed out that quantum mechanics may be formulated, not only on complex Hilbert spaces, but also on Hilbert spaces whose set of scalars is  $\mathbb{H}$ , the noncommutative real algebra of quaternions.

Many papers have been devoted to the development of quantum mechanics in the quaternionic framework (see, e.g., [20, 18, 33, 1]), whose natural setting is a Hilbert two-sided  $\mathbb{H}$ -module  $X$ , with the space of bounded linear operators acting on it replaced by the set  $\mathcal{L}^r(X)$  of bounded *right* linear operators. However, a full development of quaternionic quantum mechanics was prevented by the lack of suitable quaternionic spectral notions, indeed, as observed in [9, 23], the classical definitions of spectrum and resolvent operator do not allow to define a noncommutative functional calculus.

A first rigorous formulation of a quaternionic spectral theory has been provided only in [8] where one can find the first definition of the notions of *spherical resolvent set*  $\rho_S(A)$ , *quadratic resolvent operator*  $Q_q(A)$ , *spherical resolvent operator*  $C_q(A)$  and *spherical spectrum*  $\sigma_S(A)$  of a right linear operator  $A$  on a quaternionic Banach space  $X$ . They are

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given by

$$\begin{aligned}\rho_s(\mathbf{A}) &:= \{q \in \mathbb{H} : \exists(\mathbf{A}^2 - 2\operatorname{Re}(q)\mathbf{A} + |q|^2)^{-1} \in \mathcal{L}^r(X)\}, \\ \mathbf{Q}_q(\mathbf{A}) &:= (\mathbf{A}^2 - 2\operatorname{Re}(q)\mathbf{A} + |q|^2)^{-1}, \quad q \in \rho_s(\mathbf{A}), \\ \mathbf{C}_q(\mathbf{A}) &:= \mathbf{Q}_q(\mathbf{A})\bar{q} - \mathbf{A}\mathbf{Q}_q(\mathbf{A}), \quad q \in \rho_s(\mathbf{A}).\end{aligned}$$

In [8] the name *pseudoresolvent* is used for  $\mathbf{Q}_q(\mathbf{A})$  but we choose the term *quadratic resolvent* since in the classical complex case the name “pseudoresolvent” has a different meaning. The above definitions permit to develop a noncommutative functional calculus for right linear operators on a Banach two-sided module over  $\mathbb{H}$  (and over a Clifford algebra as well, cf. [14, 11, 12, 9, 10, 15, 23]) and to deduce in [2, 24] the spectral representation theorems for normal operators in the quaternionic Hilbert setting (cf. [2, 24]). A wider bibliography can be found in the recent accounts on the theory [5, 6].

The mentioned noncommutative functional calculus is intimately connected to the theory of slice regular functions, introduced in [22], which extends to quaternions the classical concept of holomorphic function. They form a class of functions admitting a local power series expansion at every point of their domain of definition (cf. [21]), including polynomials with quaternionic coefficients on one side, and they admit a Cauchy-type integral representation formula with a suitable quaternionic version of the kernel proved for the first time in [7] (see also [26]).

The next natural stage in this analysis is the development of a noncommutative theory of right linear operator semigroups which was developed in [13, 27, 28]. In the classical complex theory a fundamental tool is provided by the integral representation of the resolvent operator of a generator  $\mathbf{A}$  by means of the Laplace transform of the semigroup  $\mathbf{T}(t)$  generated by  $\mathbf{A}$ . An analogous integral representation in the quaternionic case for the spherical resolvent operator  $\mathbf{C}_q(\mathbf{Q})$  is shown in [13] and a proof is provided in [28] using techniques from slice regular function theory.

The purpose of the present paper is to study the invertibility of operators of the form  $P(\mathbf{A})$  where  $P(x)$  is an arbitrary polynomial with real coefficients of degree at least 2, including  $P(x) = \Delta_q(x) = x^2 - 2\operatorname{Re}(q)x + |q|^2$ . Under natural conditions, we prove that  $P(\mathbf{A})^{-1}$  exists and belongs to  $\mathcal{L}^r(X)$ . Furthermore, we provide an integral representation for  $P(\mathbf{A})^{-1}$  by means of a Laplace-type transform of the semigroup  $\mathbf{T}(t)$  generated by  $\mathbf{A}$ . We extend this integral representation to operators of the form  $\sum_{j=0}^{d-1} \mathbf{A}^j P(\mathbf{A})^{-1} p_j$ , where  $d$  is the degree of  $P$  and  $p_0, \dots, p_{d-1}$  are arbitrarily chosen quaternions. In the case  $P(x) = \Delta_q(x)$ , we obtain a new integral representation for  $\mathbf{Q}_q(\mathbf{A})$  and, setting  $p_0 := \bar{q}$  and  $p_1 := -1$ , we discover again the well-known integral representation for  $\mathbf{C}_q(\mathbf{A})$  via the quaternionic Laplace transform. This gives also a new proof of the integral representation for  $\mathbf{C}_q(\mathbf{A})$ , which avoids the use of slice regular function techniques. Our results are valid not only on quaternions but also on a class of real associative  $*$ -algebras including, as the main examples, all Clifford algebras  $\mathcal{C}\ell(0, n)$ .

Let  $n \in \mathbb{N}$ , let  $\mathbb{R}_n$  be the Clifford algebra  $\mathcal{C}\ell(0, n)$  equipped with the Clifford conjugation and the Clifford operator norm  $|\cdot|$ . Consider a Banach two-sided  $\mathbb{R}_n$ -module  $X$  with norm  $\|\cdot\|$  and the set  $\mathcal{L}^r(X)$  of all bounded right linear operators on  $X$  (all the precise definitions will be recalled in the next section).

Let  $m \in \mathbb{N}$  and let  $P(x) = \sum_{k=0}^{m+2} x^k a_k \in \mathbb{R}[x]$  be a polynomial with real coefficients in the indeterminate  $x$ . Suppose  $P$  has degree  $m+2$ , that is,  $a_{m+2} \neq 0$ . Given a right linear

operator  $A : D(A) \rightarrow X$ , we define the right linear operator  $P(A) : D(A^{m+2}) \rightarrow X$  simply by replacing  $x$  with  $A$ , that is,  $P(A) := \sum_{k=0}^{m+2} A^k a_k$ . Denote by  $C^\infty([0, \infty[; \mathbb{R})$  the set of all infinitely many times differentiable functions  $g : [0, \infty[ \rightarrow \mathbb{R}$ . Consider the following ODE with constant coefficients in the variable  $g \in C^\infty([0, \infty[; \mathbb{R})$ :

$$\begin{cases} P\left(-\frac{d}{dt}\right)(g) = 0 & \text{on } [0, \infty[, \\ g(0) = g'(0) = \dots = g^{(m)}(0) = 0, \\ g^{(m+1)}(0) = (-1)^m (a_{m+2})^{-1}, \end{cases} \quad (1.1)$$

where  $g^{(k)}$  is the  $k^{\text{th}}$ -derivative of  $g$  and  $P\left(-\frac{d}{dt}\right)(g) := \sum_{k=0}^{m+2} g^{(k)}(-1)^k a_k$ . Denote by  $g_P \in C^\infty([0, \infty[; \mathbb{R})$  the unique solution of (1.1). Recall that, if  $\lambda_1, \dots, \lambda_h$  are the complex roots of the polynomial  $P(x)$  with multiplicity  $m_1, \dots, m_h$ , then there exist complex polynomials  $Q_1(x), \dots, Q_h(x) \in \mathbb{C}[x]$  such that the degree of each  $Q_j(x)$  is  $< m_j$  and

$$g_P(t) = \sum_{j=1}^h Q_j(t) e^{-\lambda_j t}. \quad (1.2)$$

Define  $r_P \in \mathbb{R}$  by

$$r_P := \min\{\Re(\lambda_1), \dots, \Re(\lambda_h)\}, \quad (1.3)$$

where  $\Re(\lambda_j)$  is the real part of the complex number  $\lambda_j$ .

Recall also that if  $\mathbb{T} : [0, \infty[ \rightarrow \mathcal{L}^r(X)$  is a strongly continuous right linear semigroup then there exists  $\omega \in \mathbb{R}$  such that  $\sup_{t \in [0, \infty[} \|\mathbb{T}(t)\| e^{-\omega t} < \infty$ , see [27, Thm 4.5(b)].

Our main result reads as follows.

**Theorem 1.1.** *Let  $\mathbb{T} : [0, \infty[ \rightarrow \mathcal{L}^r(X)$  be a strongly continuous right linear semigroup, let  $A : D(A) \rightarrow X$  be its generator and let  $\omega \in \mathbb{R}$  be a real constant such that  $M := \sup_{t \in [0, \infty[} \|\mathbb{T}(t)\| e^{-\omega t} < \infty$ . Then, if  $r_P > \omega$ , the operator  $P(A)$  is bijective,  $P(A)^{-1} \in \mathcal{L}^r(X)$  and it holds:*

$$P(A)^{-1}x = \int_0^\infty \mathbb{T}(t) g_P(t) x \, dt \quad \forall x \in X \quad (1.4)$$

and

$$\|P(A)^{-1}\| \leq M \sum_{j=1}^h \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(r_P - \omega)^k}, \quad (1.5)$$

where  $c_{-k}^{(j)}$  is the residue at  $\lambda_j$  of the meromorphic function  $z \mapsto a_{m+2}(z - \lambda_j)^{k-1}/P(-z)$ .

Furthermore, given  $(p_0, p_1, \dots, p_{m+1}) \in (\mathbb{R}_n)^{m+2}$ , we have

$$\sum_{j=0}^{m+1} A^j P(A)^{-1} p_j x = \int_0^\infty \mathbb{T}(t) \left( \sum_{j=0}^{m+1} g_P^{(j)}(-1)^j p_j x \right) dt \quad \forall x \in X. \quad (1.6)$$

Let  $Q_{\mathbb{R}_n}$  be the quadratic cone of  $\mathbb{R}_n$  (see (2.5) for the definition) and let  $q \in Q_{\mathbb{R}_n}$ . Define the function  $g_q : \mathbb{R} \rightarrow \mathbb{R}$  by

$$g_q(t) := t e^{-\Re(q)t} \operatorname{sinc}(t | \operatorname{Im}(q)|), \quad t \in \mathbb{R}, \quad (1.7)$$

where  $\Re(q)$  and  $\operatorname{Im}(q)$  are the real and imaginary parts of  $q$ , respectively. We recall that  $\operatorname{sinc} : \mathbb{R} \rightarrow \mathbb{R}$  is the *unnormalized sinc function*, that is, the real-valued continuous function  $\xi$  on  $\mathbb{R}$  defined by  $\xi(0) = 1$  and  $\xi(r) = \sin(r)/r$  for all  $r \neq 0$ .

Thanks to the preceding result, we are able to prove the following:

**Theorem 1.2.** *Let  $\mathbb{T} : [0, \infty[ \rightarrow \mathcal{L}^r(X)$  be a strongly continuous right linear semigroup, let  $A : D(A) \rightarrow X$  be its generator, and let  $\omega \in \mathbb{R}$  be a real constant such that  $M := \sup_{t \in [0, \infty[} \|\mathbb{T}(t)\| e^{-\omega t} < \infty$ . Consider any  $q \in Q_{\mathbb{R}_n}$  and set  $a := \operatorname{Re}(q)$  and  $b := |\operatorname{Im}(q)|$ . If  $\operatorname{Re}(q) > \omega$ , then we have that  $q \in \rho_s(A)$  and it holds:*

$$Q_q(A)x = \int_0^\infty \mathbb{T}(t)g_q(t)x \, dt = \int_0^\infty \mathbb{T}(t)t e^{-ta} \operatorname{sinc}(tb)x \, dt, \quad (1.8)$$

$$AQ_q(A)x = - \int_0^\infty \mathbb{T}(t)g'_q(t)x \, dt = - \int_0^\infty \mathbb{T}(t)e^{-ta}(\cos(tb) - at \operatorname{sinc}(tb))x \, dt, \quad (1.9)$$

$$C_q(A)x = \int_0^\infty \mathbb{T}(t)e^{-tq}x \, dt \quad (1.10)$$

for every  $x \in X$ . Moreover, we have:

$$\|Q_q(A)\| \leq \frac{M}{(\operatorname{Re}(q) - \omega)^2}, \quad (1.11)$$

$$\|C_q(A)\| \leq \frac{M}{\operatorname{Re}(q) - \omega}. \quad (1.12)$$

Here is the plan of the paper. In the following section we recall all the needed precise definitions. In Section 3 we present the preceding theorems in the more general case of certain real  $*$ -algebras, including all the  $\mathbb{R}_n$ 's. Section 4 is devoted to the proofs of these theorems. Finally, in Section 5 we apply the main theorem in order to derive an integral representation of the integer powers of the quadratic resolvent and the estimate of their norms; this extends to  $Q_q(A)$  our Theorem 6.6 in [28] concerning the Laplace-type transform for the integer slice powers of  $C_q(A)$ .

## 2. PRELIMINARIES

Throughout all the paper we will assume that  $\mathbb{A}$  is a nontrivial finite dimensional  $\mathbb{R}$ -vector space endowed with a bilinear product  $\mathbb{A} \times \mathbb{A} \rightarrow \mathbb{A} : (p, q) \mapsto pq$  with unit  $1_{\mathbb{A}}$ , and with a mapping  $\mathbb{A} \rightarrow \mathbb{A} : q \mapsto q^c$  called  $*$ -involution, which is an  $\mathbb{R}$ -linear mapping such that  $(q^c)^c = q$ ,  $(pq)^c = q^c p^c$ , and  $r^c = r$  for all  $p, q \in \mathbb{A}, r \in \mathbb{R} \subseteq \mathbb{A}$ , where we are identifying  $\mathbb{R}$  with the subalgebra of  $\mathbb{A}$  generated by  $1_{\mathbb{A}}$  by means of the algebra isomorphism  $\mathbb{R} \rightarrow \mathbb{R}1_{\mathbb{A}} : r \mapsto r1_{\mathbb{A}}$ . Therefore we can write  $1 = 1_{\mathbb{A}}$  and we summarize the previous assumptions by saying that

$$\begin{aligned} \mathbb{A} \text{ is a finite dimensional associative nontrivial real } * \text{-algebra} \\ \text{with } * \text{-involution } q \mapsto q^c \text{ and unit } 1. \end{aligned} \quad (2.1)$$

Under the previous assumptions we define the *imaginary sphere*  $\mathbb{S}_{\mathbb{A}}$  in  $\mathbb{A}$  by

$$\mathbb{S}_{\mathbb{A}} := \{q \in \mathbb{A} : q^c = -q, q^2 = -1\}. \quad (2.2)$$

In the remainder of the paper we will assume that

$$\mathbb{S}_{\mathbb{A}} \neq \emptyset. \quad (2.3)$$

Condition (2.3) in particular implies that  $\mathbb{A}$  cannot be equal to  $\mathbb{R}$ . We set

$$\mathbb{C}_{\mathbf{j}} := \{r + s\mathbf{j} \in \mathbb{A} : r, s \in \mathbb{R}\}, \quad \mathbf{j} \in \mathbb{S}_{\mathbb{A}}, \quad (2.4)$$

$$Q_{\mathbb{A}} := \bigcup_{\mathbf{j} \in \mathbb{S}_{\mathbb{A}}} \mathbb{C}_{\mathbf{j}}, \quad (2.5)$$

the set  $Q_{\mathbb{A}}$  being called *quadratic cone of  $\mathbb{A}$* . The *real part*  $\operatorname{Re}(q)$  and the *imaginary part*  $\operatorname{Im}(q)$  of an element  $q \in \mathbb{A}$  are defined by

$$\operatorname{Re}(q) := (q + q^c)/2, \quad \operatorname{Im}(q) := (q - q^c)/2, \quad q \in \mathbb{A}. \quad (2.6)$$

Notice that in general  $\operatorname{Re}(q)$  and  $\operatorname{Im}(q)$  are not real numbers, at variance with the customary complex notations  $\Re(z) := (z + \bar{z})/2 \in \mathbb{R}$  and  $\Im(z) := (z - \bar{z})/2i \in \mathbb{R}$  for  $z \in \mathbb{C}$ .

We finally observe that  $qq^c \in \mathbb{R}$  for every  $q \in Q_{\mathbb{A}}$  and we assume that

$$\begin{aligned} &\mathbb{A} \text{ is endowed with a complete norm } |\cdot| \text{ such that} \\ &|q_1 q_2| \leq |q_1| |q_2| \text{ for every } q_1, q_2 \in \mathbb{A} \text{ and } |q|^2 = qq^c \text{ for every } q \in Q_{\mathbb{A}}. \end{aligned} \quad (2.7)$$

The equivalence of the above definitions with other presentations (e.g. [25] is provided in [28]). We recall here that

$$\mathbb{C}_{\mathbf{j}} \cap \mathbb{C}_{\mathbf{k}} = \mathbb{R} \quad \forall \mathbf{j}, \mathbf{k} \in \mathbb{S}_{\mathbb{A}}, \mathbf{j} \neq \pm \mathbf{k}. \quad (2.8)$$

**Example 2.1** (Clifford algebras). For  $n \in \mathbb{N} \setminus \{0\}$  let  $\mathcal{P}(n)$  be the power set of  $\{1, \dots, n\}$ . If we identify  $\mathbb{R}$  with the vector subspace  $\mathbb{R} \times \{0\}$  of  $\mathbb{R}^{2^n} = \mathbb{R} \times \mathbb{R}^{2^n-1}$  and we set  $e_{\emptyset} := 1$ , then we denote by  $\{e_K\}_{K \in \mathcal{P}(n)}$  the canonical basis of  $\mathbb{R}^{2^n}$ . For convenience, we set  $e_k := e_{\{k\}}$  if  $k \in \{1, \dots, n\}$  and we define a real bilinear and associative product on  $\mathbb{R}^{2^n}$  by imposing that 1 is the neutral element and that

$$\begin{aligned} e_k^2 &= -1 \text{ and } e_k e_h = -e_h e_k \text{ if } k, h \in \{1, \dots, n\} \text{ with } k \neq h, \\ e_K &= e_{k_1} \cdots e_{k_s} \text{ if } K = \{k_1, \dots, k_s\} \in \mathcal{P}(n) \setminus \{\emptyset\} \text{ with } k_1 < \dots < k_s. \end{aligned}$$

The *Clifford conjugation* of  $\mathbb{R}^{2^n}$  is the \*-involution  $q \mapsto q^c := \bar{q}$  defined by

$$\bar{q} := \sum_{K \in \mathcal{P}(n)} (-1)^{|K|(|K|+1)/2} a_K e_K \quad \text{if } q = \sum_{K \in \mathcal{P}(n)} a_K e_K \in \mathbb{R}_n, \quad a_K \in \mathbb{R},$$

where  $|K|$  indicates the cardinality of the set  $K$ . Endowing  $\mathbb{R}^{2^n}$  with the above defined product and with the Clifford conjugation, we obtain a real \*-algebra  $\mathbb{A}$  satisfying (2.1), called *Clifford algebra  $\mathcal{Cl}(0, n)$  of signature  $(0, n)$* , which is denoted also by  $\mathbb{R}_n$ . Observe that  $\mathbb{R}_1$  and  $\mathbb{R}_2$  are isomorphic to  $\mathbb{C}$  and  $\mathbb{H}$ , respectively. Moreover  $\mathbb{R}_n$  is not commutative if  $n \geq 2$ . If  $n \geq 3$  then  $\mathbb{R}_n$  has zero divisors, indeed  $(1 - e_{\{1,2,3\}})(1 + e_{\{1,2,3\}}) = 0$ . One verifies that a point  $q = \sum_{K \in \mathcal{P}(n)} a_K e_K$  of  $\mathbb{R}_n$  with  $a_K \in \mathbb{R}$  belongs to the quadratic cone  $Q_{\mathbb{R}_n}$  of  $\mathbb{R}_n$  if and only if it satisfies the following conditions

$$a_K = 0 \quad \text{and} \quad \langle q, q e_K \rangle_{2^n} = 0 \quad \text{for every } K \in \mathcal{P}(n) \setminus \{\emptyset\} \text{ with } e_K^2 = 1,$$

where  $\langle \cdot, \cdot \rangle_{2^n}$  denotes the standard scalar product on  $\mathbb{R}^{2^n}$ . On  $\mathbb{R}_n$  it is defined the following submultiplicative norm, called *Clifford operator norm*:  $|q|_{\mathcal{Cl}} := \sup\{|qa|_{2^n} \in \mathbb{R} : |a|_{2^n} = 1\}$ , where  $|\cdot|_{2^n}$  indicates the Euclidean norm of  $\mathbb{R}^{2^n}$ . It turns out that:

- (a)  $Q_{\mathbb{R}_n} = \mathbb{R}_n$  if and only if  $n \in \{1, 2\}$ . In particular,  $\mathbb{R}_1$  and  $\mathbb{R}_2$  are division algebras.
- (b)  $|q|_{\mathcal{Cl}} = |x| = \sqrt{x\bar{x}}$  for every  $x \in Q_{\mathbb{R}_n}$  and hence  $|\cdot|_{\mathcal{Cl}} = |\cdot|$  if  $n \in \{1, 2\}$ .

Notice that if  $n \geq 3$  then the Euclidean norm  $|\cdot|_{2^n}$  of  $\mathbb{R}_n$  is not submultiplicative (e.g.  $|(1 + e_{\{1,2,3\}})^2| = \sqrt{8} > 2 = |1 + e_{\{1,2,3\}}|^2$ ). Endowing  $\mathbb{R}_n$  with Clifford conjugation and Clifford operator norm, we obtain a real  $*$ -algebra  $\mathbb{A}$  satisfying (2.3) and (2.7). For further details we refer the reader to [29, 31].

**Example 2.2** (Complex numbers and quaternions). If  $n \in \mathbb{N} \setminus \{0\}$  and  $\mathbb{R}_n$  denotes the Clifford algebra of signature  $(0, n)$  recalled in the previous Example 2.1, then we have:

- (i)  $\mathbb{R}_1 = \mathbb{C}$  with  $e_1 = i$ , where  $z \mapsto z^c = \bar{z}$  is the standard conjugation and  $|\cdot|$  is the Euclidean norm;
- (ii)  $\mathbb{R}_2$  is the algebra of quaternions  $\mathbb{H}$  with  $i := e_1$ ,  $j := e_2$ ,  $k := e_3$ , where  $q = a + bi + cj + dk \mapsto q^c = \bar{q} = a - bi - cj - dk$  and  $|\cdot|$  is the euclidean norm.

**Definition 2.3.** If  $\mathbb{A}$  satisfies (2.1) then a two-sided  $\mathbb{A}$ -module is a commutative group  $(X, +)$  endowed with a left scalar multiplication  $\mathbb{A} \times X \rightarrow X : (q, x) \mapsto qx$  and a right scalar multiplication  $X \times \mathbb{A} \rightarrow X : (x, q) \mapsto xq$  such that

$$\begin{aligned}
q(x + y) &= qx + qy, & (x + y)q &= xq + yq & \forall x, y \in X, & \forall q \in \mathbb{A}, \\
(p + q)x &= px + qx, & x(p + q) &= xp + xq & \forall x \in X, & \forall p, q \in \mathbb{A}, \\
1x &= x = x1 & & & \forall x \in X, \\
p(qx) &= (pq)x, & (xp)q &= x(pq) & \forall x \in X, & \forall p, q \in \mathbb{A}, \\
p(xq) &= (px)q & & & \forall x \in X, & \forall p, q \in \mathbb{A}, \\
rx &= xr & & & \forall x \in X, & \forall r \in \mathbb{R}.
\end{aligned}$$

If  $Y$  is a commutative subgroup of  $X$  then  $Y$  is called a left  $\mathbb{A}$ -submodule if  $qx \in Y$  whenever  $x \in Y$  and  $q \in \mathbb{A}$ . Instead  $Y$  is called a right  $\mathbb{A}$ -submodule of  $X$  if  $xq \in Y$  whenever  $x \in Y$  and  $q \in \mathbb{A}$ . Finally  $Y$  is called a two-sided  $\mathbb{A}$ -submodule of  $X$  if it is both a left and a right  $\mathbb{A}$ -submodule of  $X$ .

**Definition 2.4.** Assume (2.1) and (2.7) hold. A two-sided  $\mathbb{A}$ -module  $X$  is called a normed two-sided  $\mathbb{A}$ -module if it is endowed with a  $\mathbb{A}$ -norm on  $X$ , that is, a function  $\|\cdot\| : X \rightarrow [0, \infty[$  such that

$$\begin{aligned}
\|x\| &= 0 \iff x = 0, \\
\|x + y\| &\leq \|x\| + \|y\| & \forall x, y \in X, \\
\|qx\| &\leq |q| \|x\|, \quad \|xq\| \leq |q| \|x\| & \forall x \in X, \quad \forall q \in \mathbb{A}.
\end{aligned} \tag{2.9}$$

We say that  $X$  is a Banach two-sided  $\mathbb{A}$ -module if the metric  $d : X \times X \rightarrow [0, \infty[ : (x, y) \mapsto \|x - y\|$  is complete.

Let us recall the following result (cf. [28, Lemma 3.3]).

**Lemma 2.5.** Assume (2.1) and (2.7) hold, and let  $X$  be a normed two-sided  $\mathbb{A}$ -module. Then

$$\|qx\| = \|xq\| = |q| \|x\| \quad \forall x \in X, \quad \forall q \in Q_{\mathbb{A}}. \tag{2.10}$$

**Definition 2.6.** Assume (2.1) holds and that  $X$  is a two-sided  $\mathbb{A}$ -module. Let  $D(\mathbb{A})$  be a right  $\mathbb{A}$ -submodule of  $X$ . We say that  $\mathbf{A} : D(\mathbb{A}) \rightarrow X$  is right linear if it is additive and

$$\mathbf{A}(xq) = \mathbf{A}(x)q \quad \forall x \in D(\mathbb{A}), \quad \forall q \in \mathbb{A}.$$

As usual, the notation  $Ax$  is often used in place of  $A(x)$ . We use the symbol  $\text{End}^r(X)$  to denote the set of right linear operators  $A$  with  $D(A) = X$ . The identity operator is right linear and is denoted by  $\text{Id}_X$  or simply by  $\text{Id}$ . Moreover, if  $X$  is a normed two-sided  $\mathbb{A}$ -module, then we say that  $A : D(A) \rightarrow X$  is closed if its graph is closed in  $X \times X$ . As in the classical theory, we set  $D(A^n) := \{x \in D(A^{n-1}) : A^{n-1}x \in D(A)\}$  for every  $n \in \mathbb{N} \setminus \{0\}$ , where  $A^0 := \text{Id}$ .

Let us also recall the following definition (see, e.g., [3, Chapter 1, p. 55-57]).

**Definition 2.7.** Let  $D(A)$  be a right  $\mathbb{A}$ -submodule of  $X$  and let  $q \in \mathbb{A}$ . If  $A : D(A) \rightarrow X$  is a right linear operator, then we define the mapping  $qA : D(A) \rightarrow X$  by setting

$$(qA)(x) := qA(x), \quad x \in D(A). \quad (2.11)$$

If  $D(A)$  is also a left  $\mathbb{A}$ -submodule of  $X$ , then we can define  $Aq : D(A) \rightarrow X$  by setting

$$(Aq)(x) := A(qx), \quad x \in D(A). \quad (2.12)$$

The sum of operators is defined in the usual way.

It is easy to see that the operators defined in (2.11) and (2.12) are right linear.

**Definition 2.8.** Assume  $X$  is normed with  $\mathbb{A}$ -norm  $\|\cdot\|$ . For every  $B \in \text{End}^r(X)$ , we set

$$\|B\| := \sup_{x \neq 0} \frac{\|Bx\|}{\|x\|} \quad (2.13)$$

and we define the set  $\mathcal{L}^r(X) := \{B \in \text{End}^r(X) : \|B\| < \infty\}$ .

### 3. MAIN RESULTS IN THEIR GENERAL FORM

In order to state our main result in its general form we recall the noncommutative spectral notions given for the first time in [8] for quaternions and in [14] for arbitrary Clifford algebras  $\mathbb{R}_n$ . Here we consider the general case introduced in [27, Definition 2.26]. We will assume that

$\mathbb{A}$  satisfies (2.1), (2.3) and (2.7), and  $X$  is a Banach two-sided  $\mathbb{A}$ -module.

**Definition 3.1.** Let  $D(A)$  be a right  $\mathbb{A}$ -submodule of  $X$  and let  $A : D(A) \rightarrow X$  be a closed right linear operator.

(i) Given  $q \in Q_{\mathbb{A}}$ , the right linear operator  $\Delta_q(A) : D(A^2) \rightarrow X$  is defined by

$$\Delta_q(A) := A^2 - 2\text{Re}(q)A + |q|^2 \text{Id}, \quad q \in Q_{\mathbb{A}}.$$

(ii) The spherical resolvent set  $\rho_S(A)$  of  $A$  is defined by

$$\rho_S(A) := \{q \in Q_{\mathbb{A}} : \Delta_q(A) \text{ is bijective, } \Delta_q(A)^{-1} \in \mathcal{L}^r(X)\}$$

and the spherical spectrum  $\sigma_S(A)$  of  $A$  by  $\sigma_S(A) := Q_{\mathbb{A}} \setminus \rho_S(A)$ .

(iii) Given  $q \in \rho_S(A)$ , the quadratic resolvent (or spherical pseudoresolvent) of  $A$  at  $q$  is the operator  $Q_q(A) : X \rightarrow X$  defined by

$$Q_q(A) := \Delta_q(A)^{-1}, \quad q \in \rho_S(A).$$

(iv) Given  $q \in \rho_S(A)$ , the spherical resolvent of  $A$  at  $q$  is the operator  $C_q(A) : X \rightarrow X$  defined by

$$C_q(A) := Q_q(A)q^c - AQ_q(A), \quad q \in \rho_S(A).$$

Let us observe that

$$\mathbf{Q}_q(\mathbf{A}) \in \mathcal{L}^r(X), \quad \mathbf{C}_q(\mathbf{A}) \in \mathcal{L}^r(X) \quad \forall q \in \rho_s(\mathbf{A}). \quad (3.1)$$

Indeed, by definition,  $\mathbf{Q}_q(\mathbf{A})$  is bounded and if we endow  $(X, +)$  with the (left) real scalar multiplication  $\mathbb{R} \times X \rightarrow X : (r, x) \mapsto rx = xr$ , then thanks to (2.10)  $X$  can be considered as a real Banach space and  $\mathbf{A}$  is a closed  $\mathbb{R}$ -linear operator on it, thus the closed graph theorem implies that  $\mathbf{A}\mathbf{Q}_q(\mathbf{A})$  is continuous and consequently  $\mathbf{C}_q(\mathbf{A})$  is also continuous. Since all these operators are also  $\mathbb{A}$ -right linear we infer (3.1).

The introduction of the name ‘‘quadratic resolvent’’ for  $\mathbf{Q}_q(\mathbf{A})$ , which we slightly prefer, is due to the fact that in the classical complex literature the term ‘‘pseudoresolvent’’ is already used for a different class of operators (cf., e.g., [36, Section 1.9]). We also mention that a definition that has some similarities with the spherical spectrum was given in [34] in the context of real  $*$ -algebras.

We now recall the natural definition of right linear operator semigroup (cf. [13] for the quaternionic case and [27] for the general case).

**Definition 3.2.** *A mapping  $\mathbf{S} : [0, \infty[ \rightarrow \mathcal{L}^r(X)$  is called strongly continuous if the  $t \mapsto \mathbf{S}(t)x$  is continuous from  $[0, \infty[$  into  $X$  for every  $x \in X$ .*

**Definition 3.3.** *A mapping  $\mathbf{T} : [0, \infty[ \rightarrow \mathcal{L}^r(X)$  is called right linear strongly continuous (operator) semigroup if  $\mathbf{T}$  is strongly continuous and if*

$$\begin{aligned} \mathbf{T}(t+s) &= \mathbf{T}(t)\mathbf{T}(s) \quad \forall t, s > 0, \\ \mathbf{T}(0) &= \text{Id}. \end{aligned}$$

*The generator of  $\mathbf{T}$  is the right linear operator  $\mathbf{A} : D(\mathbf{A}) \rightarrow X$  defined by*

$$\begin{aligned} D(\mathbf{A}) &:= \left\{ x \in X : \exists \lim_{h \rightarrow 0} \frac{1}{h} (\mathbf{T}(h)x - x) \in X \right\}, \\ \mathbf{A}x &:= \lim_{h \rightarrow 0} \frac{1}{h} (\mathbf{T}(h)x - x), \quad x \in D(\mathbf{A}). \end{aligned}$$

For the classical theory of semigroups in the complex framework we refer, e.g., to [32, 16, 36, 30, 35, 37, 19].

The next result includes Theorem 1.1.

**Theorem 3.4.** *Let  $m \in \mathbb{N}$ , let  $P(x) = \sum_{k=0}^{m+2} x^k a_k \in \mathbb{R}[x]$  such that  $a_{m+2} \neq 0$  and let  $g_P \in C^\infty([0, \infty[; \mathbb{R})$  be the unique solution of (1.1). Let  $\mathbf{T} : [0, \infty[ \rightarrow \mathcal{L}^r(X)$  be a strongly continuous right linear semigroup, let  $\mathbf{A} : D(\mathbf{A}) \rightarrow X$  be its generator and let  $\omega \in \mathbb{R}$  be a real constant such that  $M := \sup_{t \in [0, \infty[} \|\mathbf{T}(t)\| e^{-\omega t} < \infty$ . Then, if  $r_P > \omega$ , the operator  $P(\mathbf{A}) = \sum_{k=0}^{m+2} \mathbf{A}^k a_k$  is bijective,  $P(\mathbf{A})^{-1} \in \mathcal{L}^r(X)$  and it holds:*

(a)  $P(\mathbf{A})^{-1} = \mathbf{L}(g_P)$ , that is,

$$P(\mathbf{A})^{-1}x = \int_0^\infty \mathbf{T}(t)g_P(t)x \, dt \quad \forall x \in X. \quad (3.2)$$

(b) Given  $(p_0, p_1, \dots, p_{m+1}) \in \mathbb{A}^{m+2}$ , we have

$$\sum_{j=0}^{m+1} \mathbf{A}^j P(\mathbf{A})^{-1} p_j x = \mathbf{L} \left( \sum_{j=0}^{m+1} g_P^{(j)}(-1)^j p_j \right) x \quad \forall x \in X. \quad (3.3)$$

Moreover,

$$\|P(\mathbf{A})^{-1}\| \leq M \sum_{j=1}^h \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(r_P - \omega)^k} \quad (3.4)$$

where  $c_{-k}^{(j)}$  is the residue at  $\lambda_j$  of the complex rational function  $a_{m+2}(z - \lambda_j)^{k-1}/P(-z)$ .

Furthermore we have

**Theorem 3.5.** *The statement of Theorem 1.2 holds true replacing  $\mathbb{R}_n$  with  $\mathbb{A}$ .*

#### 4. PROOFS

Let us start with a lemma on strongly continuous mapping. The symbol  $C([0, \infty[; \mathbb{A})$  denotes the space of continuous functions from  $[0, \infty[$  to  $\mathbb{A}$ , both endowed with the topology induced by the Euclidean distance.

**Lemma 4.1.** *If  $\mathbb{T} : [0, \infty[ \rightarrow \mathcal{L}^r(X)$  is a strongly continuous and  $g \in C([0, \infty[; \mathbb{A})$ , then the following statements hold true.*

- (a) *The function  $t \mapsto \|\mathbb{T}(t)\| |g(t)|$  is Lebesgue measurable on  $[0, \infty[$ .*
- (b) *For every  $x \in X$  the function  $t \mapsto \mathbb{T}(t)g(t)x$  is continuous from  $[0, \infty[$  into  $X$ .*

*Proof.* Since  $\mathbb{T} : [0, \infty[ \rightarrow \mathcal{L}^r(X)$  is strongly continuous, by the Banach-Steinhaus theorem it follows that  $\|\mathbb{T}(t)\| \leq \liminf_{\tau \rightarrow t} \|\mathbb{T}(\tau)\|$  for every  $t \geq 0$ , i.e.  $t \mapsto \|\mathbb{T}(t)\|$  is lower semicontinuous, and hence it is Lebesgue measurable. Thus (a) is proved. In order to prove (b) fix an arbitrary  $t_0 \geq 0$ . Since  $\mathbb{T}$  is strongly continuous, by the uniform boundedness principle there exists  $C > 0$  such that for every  $t \geq 0$  with  $|t - t_0| < 1$  we have  $\|\mathbb{T}(t)\| \leq C$  and

$$\begin{aligned} \|\mathbb{T}(t)g(t)x - \mathbb{T}(t_0)g(t_0)x\| &\leq \|\mathbb{T}(t)g(t)x - \mathbb{T}(t)g(t_0)x\| + \|\mathbb{T}(t)g(t_0)x - \mathbb{T}(t_0)g(t_0)x\| \\ &\leq C|g(t) - g(t_0)|\|x\| + \|\mathbb{T}(t)g(t_0)x - \mathbb{T}(t_0)g(t_0)x\|. \end{aligned}$$

Thus the continuity of  $t \mapsto \mathbb{T}(t)g(t)x$  at  $t_0$  follows from the continuity of  $g$  and from the strong continuity of  $\mathbb{T}$ .  $\square$

If  $\mathbb{T} : [0, \infty[ \rightarrow \mathcal{L}^r(X)$  is a strongly continuous right linear semigroup,  $t \geq 0$ ,  $g \in C([0, \infty[; \mathbb{A})$ , and  $x \in X$ , then Lemma 4.1 and estimate  $\|\mathbb{T}(t)g(t)x\| \leq \|\mathbb{T}(t)\| |g(t)| \|x\|$  allow to give the following definition.

**Definition 4.2.** *Let  $\mathbb{T} : [0, \infty[ \rightarrow \mathcal{L}^r(X)$  be a strongly continuous right linear semigroup. We denote by  $L_{\mathbb{T}}([0, \infty[; \mathbb{A})$  the real vector space of all continuous functions  $g : [0, \infty[ \rightarrow \mathbb{A}$  such that the function  $t \mapsto \|\mathbb{T}(t)\| |g(t)|$  belongs to  $L^1([0, \infty[; \mathbb{R})$ . For every  $g \in L_{\mathbb{T}}([0, \infty[; \mathbb{A})$  we define the operator  $\mathbf{L}(g) : X \rightarrow X$  by setting*

$$\mathbf{L}(g)x := \int_0^\infty \mathbb{T}(t)g(t)x dt, \quad x \in X. \quad (4.1)$$

Notice that the assumptions implies that the integral in (4.1) is a convergent Lebesgue integral for functions with values in the Banach space  $(X, +)$  endowed with the real scalar multiplication  $\mathbb{R} \times X \rightarrow X : (r, x) \mapsto rx = xr$  (thanks to (2.10)  $\|\cdot\|$  is a norm on this real vector space). The symbol  $L^1(J; X)$  denotes the space of Lebesgue integrable functions from an interval  $J \subseteq \mathbb{R}$  into this real Banach space.

In the remainder of the paper, for  $g \in C([0, \infty[; \mathbb{A})$ , the symbols  $g'$  and  $g''$  will denote the first and second derivative of  $g$ , respectively.

**Lemma 4.3.** *If  $\mathbb{T} : [0, \infty[ \rightarrow \mathcal{L}^r(X)$  is a strongly continuous right linear semigroup, then the following statements hold true.*

- (a) *If  $g \in L_{\mathbb{T}}([0, \infty[; \mathbb{A})$  then  $t \mapsto \mathbb{T}(t)g(t-h)x$  belongs to  $L^1([h, \infty[; X)$  for every  $h > 0$  and for every  $x \in X$ .*  
(b) *If  $g, g' \in L_{\mathbb{T}}([0, \infty[; \mathbb{A})$  then*

$$\lim_{h \rightarrow 0} \left\| \int_h^\infty \mathbb{T}(t) \frac{g(t-h) - g(t)}{h} x dt + \mathbb{L}(g')x \right\| = 0 \quad \forall x \in X.$$

*Proof.* The claim (a) follows trivially by the estimate  $\|\mathbb{T}(t)g(t-h)x\| \leq \|\mathbb{T}(h)\| \|\mathbb{T}(t-h)\| \|g(t-h)\| \|x\|$  holding for every  $x \in X$ ,  $h > 0$ , and  $t > h$ . In order to prove (b) fix  $x \in X$  and an arbitrary  $\varepsilon > 0$ , and let  $T > 0$  be such that  $\int_T^\infty \|\mathbb{T}(t)\| |g(t)| dt < \varepsilon$ . Since  $\mathbb{T}$  is a strongly continuous semigroup, there exists  $M \geq 1$  such that  $\|\mathbb{T}(s)\| \leq M$  whenever  $0 \leq s \leq 1$ , therefore for every  $h \in ]0, 1[$  and every  $t > h$  we have

$$\begin{aligned} \left\| \mathbb{T}(t) \frac{g(t-h) - g(t)}{h} x \right\| &= \left\| \int_0^1 \mathbb{T}(\xi h) \mathbb{T}(t - \xi h) g'(t - \xi h) x d\xi \right\| \\ &\leq M \|x\| \int_0^1 \|\mathbb{T}(t - \xi h)\| |g'(t - \xi h)| d\xi \end{aligned}$$

hence it follows that

$$\begin{aligned} \left\| \int_T^\infty \mathbb{T}(t) \frac{g(t-h) - g(t)}{h} x dt \right\| &\leq M \|x\| \int_T^\infty \int_0^1 \|\mathbb{T}(t - \xi h)\| |g'(t - \xi h)| d\xi dt \\ &= M \|x\| \int_0^1 \int_T^\infty \|\mathbb{T}(t - \xi h)\| |g'(t - \xi h)| dt d\xi \\ &= M \|x\| \int_0^1 \int_{T-\xi h}^\infty \|\mathbb{T}(\tau)\| |g'(\tau)| d\tau d\xi \\ &\leq M \|x\| \int_0^1 \int_T^\infty \|\mathbb{T}(\tau)\| |g'(\tau)| d\tau d\xi \\ &\leq M \|x\| \int_T^\infty \|\mathbb{T}(\tau)\| |g'(\tau)| d\tau \leq M \|x\| \varepsilon. \end{aligned} \quad (4.2)$$

Moreover it is easily found a  $\delta \in ]0, 1[$  such that for every  $h \in ]0, \delta[$  we have

$$\begin{aligned} &\left\| \int_h^T \mathbb{T}(t) \frac{g(t-h) - g(t)}{h} x dt + \int_0^T \mathbb{T}(t) g'(t) x dt \right\| \\ &\leq \left\| \int_h^T \mathbb{T}(t) \left( \frac{g(t-h) - g(t)}{h} + g'(t) \right) x dt \right\| + \left\| \int_0^h \mathbb{T}(t) g'(t) x dt \right\| \leq \varepsilon. \end{aligned} \quad (4.3)$$

Hence assertion (b) follows from (4.2)–(4.3) and from the following estimate

$$\begin{aligned} &\left\| \int_h^\infty \mathbb{T}(t) \frac{g(t-h) - g(t)}{h} x dt + \mathbb{L}(g')x \right\| \\ &\leq \left\| \int_h^T \mathbb{T}(t) \frac{g(t-h) - g(t)}{h} x dt + \int_0^T \mathbb{T}(t) g'(t) x dt \right\| \\ &\quad + \left\| \int_T^\infty \mathbb{T}(t) \frac{g(t-h) - g(t)}{h} x dt \right\| + \left\| \int_T^\infty \mathbb{T}(t) g'(t) x dt \right\|. \end{aligned}$$

□

The next lemma plays a key role in the proof of Theorem 3.5.

**Lemma 4.4.** *Let  $\mathbb{T} : [0, \infty[ \rightarrow \mathcal{L}^r(X)$  be a strongly continuous right linear semigroup and for every  $g \in L_{\mathbb{T}}([0, \infty[; \mathbb{A})$  let  $\mathbf{L}(g) : X \rightarrow X$  be defined by (4.1). Then  $\mathbf{L}(g) \in \mathcal{L}^r(X)$  for every  $g \in L_{\mathbb{T}}([0, \infty[; \mathbb{A})$  and the resulting mapping  $\mathbf{L} : L_{\mathbb{T}}([0, \infty[; \mathbb{A}) \rightarrow \mathcal{L}^r(X)$  is  $\mathbb{R}$ -linear. Moreover the following assertions hold.*

(a) *If  $g, g' \in L_{\mathbb{T}}([0, \infty[; \mathbb{A})$ , then*

$$\mathbf{L}(g)(X) \subseteq D(\mathbf{A}), \quad (4.4)$$

$$\mathbf{A}\mathbf{L}(g)x = -g(0)x - \mathbf{L}(g')x \quad \forall x \in X. \quad (4.5)$$

(b) *If  $m \in \mathbb{N}$ ,  $g, g', \dots, g^{(m+2)} \in L_{\mathbb{T}}([0, \infty[; \mathbb{A})$ , and  $g(0) = g'(0) = \dots = g^{(m)}(0) = 0$ , then*

$$\mathbf{L}(g)(X) \subseteq D(\mathbf{A}^{m+2}), \quad (4.6)$$

$$\mathbf{A}^{m+2}\mathbf{L}(g)x = (-1)^{m+2} (g^{(m+1)}(0)x + \mathbf{L}(g^{(m+2)})x) \quad \forall x \in X, \quad (4.7)$$

$$\mathbf{A}^k\mathbf{L}(g)x = (-1)^k \mathbf{L}(g^{(k)})x \quad \forall x \in X, \forall k \in \{0, \dots, m+1\}. \quad (4.8)$$

(c) *If  $g, g', g'' \in L_{\mathbb{T}}([0, \infty[; \mathbb{A})$ ,  $g(0) = 0$ , and  $g'(0) = 1$ , then*

$$\Delta_q(\mathbf{A})\mathbf{L}(g)x = x + \mathbf{L}(g'' + 2\operatorname{Re}(q)g' + |q|^2g)x \quad \forall x \in X, \forall q \in \mathbb{Q}_{\mathbb{A}}. \quad (4.9)$$

(d) *If  $g, g' \in L_{\mathbb{T}}([0, \infty[; \mathbb{A})$  and  $g$  is real-valued, then*

$$\mathbf{A}\mathbf{L}(g)x = \mathbf{L}(g)\mathbf{A}x \quad \forall x \in D(\mathbf{A}). \quad (4.10)$$

(e) *If  $m \in \mathbb{N}$ ,  $g, g', \dots, g^{(m+2)} \in L_{\mathbb{T}}([0, \infty[; \mathbb{A})$ ,  $g(0) = g'(0) = \dots = g^{(m)}(0) = 0$ , and  $g$  is real valued, then*

$$\mathbf{A}^k\mathbf{L}(g)x = \mathbf{L}(g)\mathbf{A}^kx \quad \forall x \in D(\mathbf{A}^k), \forall k \in \{1, \dots, m+2\}. \quad (4.11)$$

*Proof.* For every  $g \in L_{\mathbb{T}}([0, \infty[; \mathbb{A})$  the right linearity of  $\mathbf{L}(g)$  follows from the right linearity of  $\mathbb{T}(t)$  and from the definition of the  $X$ -valued Lebesgue integral. For every  $x \in X$  we have

$$\|\mathbf{L}(g)x\| \leq \|x\| \int_0^\infty \|\mathbb{T}(t)\| |g(t)| dt,$$

hence  $\mathbf{L}(g)$  is also continuous and  $\|\mathbf{L}(g)\| \leq \int_0^\infty \|\mathbb{T}(t)\| |g(t)| dt$ . The real linearity of  $\mathbf{L}$  is straightforward. Now in the following list of items we prove the assertions from (a) to (e).

(a) For every  $h > 0$  and for every  $x \in X$  we have

$$\begin{aligned} \frac{\mathbb{T}(h) - \operatorname{Id}}{h} \mathbf{L}(g)x &= \frac{1}{h} \int_0^\infty \mathbb{T}(t+h)g(t)x dt - \frac{1}{h} \int_0^\infty \mathbb{T}(t)g(t)x dt \\ &= \frac{1}{h} \int_h^\infty \mathbb{T}(t)g(t-h)x dt - \frac{1}{h} \int_0^\infty \mathbb{T}(t)g(t)x dt \\ &= -\frac{1}{h} \int_0^h \mathbb{T}(t)g(t)x dt + \int_h^\infty \mathbb{T}(t) \left( \frac{g(t-h) - g(t)}{h} \right) x dt. \end{aligned}$$

Hence, taking the limit as  $h \rightarrow 0$ , thanks to Lemma 4.1 we find that

$$\mathbf{A}\mathbf{L}(g)x = -\mathbb{T}(0)g(0)x - \int_0^\infty \mathbb{T}(t)g'(t)x dt = -g(0)x - \mathbf{L}(g')x.$$

(b) We proceed by induction on  $m \in \{-1\} \cup \mathbb{N}$ . The case  $m = -1$  follows from (a). Let us assume that the result is true for  $m - 1$ , and we prove it for  $m$ . Therefore if  $g$  satisfies the assumptions, in particular we have  $\mathbf{L}(g)x \in D(\mathbf{A}^{m+1})$  and  $\mathbf{A}^{m+1}\mathbf{L}(g)x = (-1)^{m+1}(g^{(m)}(0)x + \mathbf{L}(g^{(m+1)})x$  for every  $x \in X$ . But  $g^{(m)}(0) = 0$  hence  $\mathbf{A}^{m+1}\mathbf{L}(g)x = (-1)^{m+1}\mathbf{L}(g^{(m+1)})x$  and  $\mathbf{L}(g^{(m+1)})x \in D(\mathbf{A})$  by virtue of an application of (4.4) with  $g$  replaced by  $g^{(m+1)}$ . Thus  $\mathbf{A}^{m+1}\mathbf{L}(g)x \in D(\mathbf{A})$  and (4.6) follows. Using again the validity of the statement for  $m - 1$  and the identity  $g^{(m)}(0) = 0$  we have

$$\begin{aligned} \mathbf{A}^{m+2}\mathbf{L}(g)x &= \mathbf{A}\mathbf{A}^{m+1}\mathbf{L}(g)x = (-1)^{m+1}\mathbf{A}\mathbf{L}(g^{(m+1)})x \\ &= (-1)^m(g^{(m+1)}(0)x + \mathbf{L}(g^{(m+2)}))x, \end{aligned}$$

where in the last equality we have used (4.5) with  $g$  replaced by  $g^{(m+1)}$ . Therefore (4.7) is proved. Formula (4.8) is trivial for  $k = 0$  and follows from (a) for  $k = 1$ , while for  $2 \leq k \leq m + 1$  follows from (4.7) which we have already proved.

(c) Now fix  $x \in X$  and  $q \in Q_{\mathbb{A}}$ . From (b) we obtain  $\mathbf{A}^2\mathbf{L}(g)x = x + \mathbf{L}(g'')x$ , hence, exploiting again (4.5) and the  $\mathbb{R}$ -linearity of  $\mathbf{L}$ , we obtain

$$\begin{aligned} \Delta_q(\mathbf{A})\mathbf{L}(g)x &= \mathbf{A}^2\mathbf{L}(g)x - 2\operatorname{Re}(q)\mathbf{A}\mathbf{L}(g)x + |q|^2\mathbf{L}(g)x \\ &= x + \mathbf{L}(g'')x - 2\operatorname{Re}(q)(-\mathbf{L}(g')x) + |q|^2\mathbf{L}(g)x \\ &= x + \mathbf{L}(g'' + 2\operatorname{Re}(q)g' + |q|^2g)x. \end{aligned}$$

(d) If  $x \in D(\mathbf{A})$ , then for every  $h > 0$  and for every  $t > 0$  we have  $\mathbf{T}(t)g(t)\mathbf{T}(h)x = \mathbf{T}(t)\mathbf{T}(h)g(t)x = \mathbf{T}(t+h)g(t)x = \mathbf{T}(h)\mathbf{T}(t)g(t)x$ , because  $g$  is real-valued and  $\mathbf{T}$  is a semigroup. Therefore

$$\begin{aligned} \mathbf{L}(g)\frac{\mathbf{T}(h) - \operatorname{Id}}{h}x &= \frac{1}{h}\int_0^\infty \mathbf{T}(t)g(t)\mathbf{T}(h)x \, dt - \frac{1}{h}\int_0^\infty \mathbf{T}(t)g(t)x \, dt \\ &= \frac{1}{h}\int_0^\infty \mathbf{T}(h)\mathbf{T}(t)g(t)x \, dt - \frac{1}{h}\int_0^\infty \mathbf{T}(t)g(t)x \, dt \\ &= \frac{\mathbf{T}(h) - \operatorname{Id}}{h}\mathbf{L}(g)x, \end{aligned}$$

and the assertion follows taking the limit as  $h \rightarrow 0$  and invoking (4.4).

(e) By induction on  $m \in \{-1\} \cup \mathbb{N}$ . The case  $m = -1$  is true by virtue of (d). Let us assume that the result is true for  $m - 1$ , and we prove it for  $m$ . Therefore if  $g$  satisfies the assumptions, in particular we have that  $\mathbf{A}^k\mathbf{L}(g) = \mathbf{L}(g)\mathbf{A}^k$  on  $D(\mathbf{A}^k)$  for all  $k \in \{1, \dots, m + 1\}$ . Hence if  $x \in D(\mathbf{A}^{m+2}) \subset D(\mathbf{A}^{m+1})$  then  $\mathbf{A}^{m+1}x \in D(\mathbf{A})$  and we have that

$$\mathbf{L}(g)\mathbf{A}^{m+2}x = \mathbf{L}(g)\mathbf{A}\mathbf{A}^{m+1}x = \mathbf{A}\mathbf{L}(g)\mathbf{A}^{m+1}x = \mathbf{A}\mathbf{A}^{m+1}\mathbf{L}(g)x = \mathbf{A}^{m+2}\mathbf{L}(g)x,$$

where in the second equality we have used again (d).  $\square$

*Proof of Theorem 3.4.* Let us first recall that the existence of constant  $\omega \in \mathbb{R}$  such that  $M := \sup_{t \geq 0} \|\mathbf{T}(t)\|e^{-\omega t} < \infty$  is well known (cf. [27, Thm 4.5(b)]). From (1.2) it follows that  $g_P \in C^{m+2}([0, \infty[; \mathbb{A})$  and the Leibniz formula for the higher derivatives of a product yields the existence of a polynomial  $p(t, \lambda_1, \dots, \lambda_h)$  such that

$$\|\mathbf{T}(t)\| |g_P^k(t)| \leq M e^{\omega t} |p(t, \lambda_1, \dots, \lambda_h)| e^{-tr_P} = M |p(t, \lambda_1, \dots, \lambda_h)| e^{(\omega - r_P)t}$$

for all  $t \geq 0$  and for all  $k \in \{0, 1, \dots, m+2\}$ . Thus  $g_P, g'_P, \dots, g_P^{(m+2)} \in L_{\mathbb{T}}([0, \infty[; \mathbb{A})$  and we can apply part (b) of Lemma 4.4 taking into account of initial conditions for  $g_P$ . We infer that

$$\begin{aligned} P(\mathbf{A})\mathbf{L}(g_P)x &= \sum_{k=0}^{m+2} \mathbf{A}^k a_k \mathbf{L}(g_P)x \\ &= \sum_{k=0}^{m+1} (-1)^k \mathbf{L}(g_P^{(k)}) a_k x + (-1)^{m+2} \left( g_P^{(m+1)}(0) + \mathbf{L}(g_P^{(m+2)}) \right) a_{m+2} x \\ &= \mathbf{L} \left( \sum_{k=0}^{m+2} g_P^{(k)} (-1)^k a_k \right) x + (-1)^{m+1} x = x g_P^{(m+1)}(0) (-1)^m a_{m+2} = x. \end{aligned}$$

On the other hand since  $g_P$  is real-valued we can also apply parts (d) and (e) of Lemma 4.4 and infer that  $\mathbf{L}(g_P)P(\mathbf{A}) = P(\mathbf{A})\mathbf{L}(g_P)$  thus  $\mathbf{L}(g_P) = P(\mathbf{A})^{-1}$  and (3.2) is proved. Now take  $(p_0, p_1, \dots, p_{m+1}) \in \mathbb{A}^{m+2}$ . Using Lemma 4.4(b) we deduce:

$$\sum_{j=0}^{m+1} \mathbf{A}^k P(\mathbf{A})^{-1} p_j x = \sum_{j=0}^{m+1} \mathbf{A}^k \mathbf{L}(g_P) p_j x = \sum_{j=0}^{m+1} (-1)^k \mathbf{L}(g_P^{(k)}) p_j x = \mathbf{L} \left( \sum_{j=0}^{m+1} (-1)^k g_P^{(k)} p_j \right) x$$

and (3.3) is proved. It is well known that

$$Q_j(t) = \sum_{k=1}^{m_j} \frac{c_{-k}^{(j)}}{(k-1)!} t^{k-1},$$

where  $c_{-k}^{(j)}$  is the coefficient of  $(z - \lambda_j)^{-k}$  within the partial fractions decomposition of  $a_{m+2}/P(-z)$ , i.e. the residue at  $\lambda_j$  of the function  $a_{m+2}(z - \lambda_j)^{k-1}/P(-z)$ . Therefore

$$|g_P(t)| \leq \sum_{j=1}^h \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(k-1)!} t^{k-1} e^{-r_P t} \quad \forall t \geq 0$$

and

$$\begin{aligned} \|P(\mathbf{A})^{-1}\| &\leq \int_0^\infty \|\mathbb{T}(t)\| \sum_{j=1}^h \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(k-1)!} t^{k-1} e^{-r_P t} dt \\ &\leq M \sum_{j=1}^h \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(k-1)!} \int_0^\infty t^{k-1} e^{-(r_P - \omega)t} dt \\ &= M \sum_{j=1}^h \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(r_P - \omega)^k} \frac{1}{(k-1)!} \int_0^\infty t^{k-1} e^{-t} dt \\ &= M \sum_{j=1}^h \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(r_P - \omega)^k} \end{aligned}$$

and the theorem is completely proved.  $\square$

Now we address the case when  $P(x) = x^2 - 2 \operatorname{Re}(q) + |q|^2$  for some  $q \in Q_{\mathbb{A}}$  which is related to part (c) of Lemma 4.4. We first present a simple lemma whose proof is a trivial calculus exercise.

**Lemma 4.5.** *For every fixed  $q \in Q_{\mathbb{A}}$ , the unique solution of the Cauchy problem*

$$g'' + 2 \operatorname{Re}(q)g' + |q|^2g = 0, \quad g(0) = 0, \quad g'(0) = 1. \quad (4.12)$$

is the function  $g_q : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$g_q(t) := te^{-\operatorname{Re}(q)t} \operatorname{sinc}(t|\operatorname{Im}(q)|), \quad t \in \mathbb{R}, \quad (4.13)$$

where we recall that  $\operatorname{sinc} : \mathbb{R} \rightarrow \mathbb{R}$  is the unnormalized sinc function, i.e. the only continuous real function  $\xi$  on  $\mathbb{R}$  such that  $\xi(r) = (\sin r)/r$  for all  $r \neq 0$ . Moreover we have that  $g_q, g'_q, g''_q \in L_T([0, \infty[; \mathbb{A})$ .

Now we are in position to find the integral representations of the quadratic resolvent and of the spherical resolvent operators as a simple consequence of our main Theorem 3.4.

*Proof of Theorem 3.5.* In order to prove (1.8) it is enough to apply part (a) of Theorem 3.4 with  $P(x) = x^2 - 2 \operatorname{Re}(q)x + |q|^2$  taking into account of Lemma 4.5. Formula (1.9) is a straightforward application of part (b) of Theorem 3.4 with  $p_0 = 0, p_1 = 1$  and  $p_2 = 0$ . Instead (1.10) is obtained taking in part (b) of Theorem 3.4  $p_0 = q^c, p_1 = -1$ , and  $p_2 = 0$ . Finally

$$\|Q_q(\mathbf{A})\| \leq \int_0^\infty \|\mathbb{T}(t)\| |g_p(t)| dt \leq \int_0^\infty Mte^{(\omega - \operatorname{Re}(q))t} dt = \frac{M}{(\operatorname{Re}(q) - \omega)^2},$$

i.e. (1.11) holds. Finally estimate (1.12) can be obtained in a similar way.  $\square$

**Remark 4.6.** By exploiting (4.13) of our Lemma 4.5 we can write the integral representation (1.8) in a more explicit way for every  $q \in Q_{\mathbb{A}}$  such that  $\operatorname{Re}(q) > \omega$ :

$$Q_q(\mathbf{A})x = - \int_0^\infty \mathbb{T}(t) \frac{e^{-\operatorname{Re}(q)t} \sin(|\operatorname{Im}(q)|t)}{|\operatorname{Im}(q)|} x dt \quad \forall x \in X, \quad q \notin \mathbb{R}, \quad (4.14)$$

$$Q_q(\mathbf{A})x = - \int_0^\infty \mathbb{T}(t) te^{-qt} x dt \quad \forall x \in X, \quad q \in \mathbb{R}. \quad (4.15)$$

The following lemma will connect the integral representation of  $Q_q(\mathbf{A})$  to the so called *spherical derivative* of  $q \mapsto e^{-tq}$  (cf. [25]).

**Lemma 4.7.** *For every  $t \in \mathbb{R}$  let  $\exp^t : Q_{\mathbb{A}} \rightarrow \mathbb{A}$  be the function defined by*

$$\exp^t(q) := \sum_{n=0}^\infty \frac{t^n}{n!} q^n = \sum_{n=0}^\infty \frac{(tq)^n}{n!}, \quad q \in Q_{\mathbb{A}}. \quad (4.16)$$

If  $(\exp^t)'_s : Q_{\mathbb{A}} \setminus \mathbb{R} \rightarrow \mathbb{A}$  denotes the function defined by

$$(\exp^t)'_s(q) := (q - q^c)^{-1} (\exp^t(q) - \exp^t(q^c)), \quad q \in Q_{\mathbb{A}} \setminus \mathbb{R}, \quad (4.17)$$

which is also called *spherical derivative* of  $\exp^t$ , then  $(\exp^t)'_s$  extends to a unique continuous function on  $Q_{\mathbb{A}}$ , which we still denote by  $(\exp^t)'_s : Q_{\mathbb{A}} \rightarrow \mathbb{A}$ , and we have

$$(\exp^t)'_s(q) = e^{t \operatorname{Re}(q)} \sum_{n=0}^\infty \frac{t^{2n+1} \operatorname{Im}(q)^{2n}}{(2n+1)!} \in \mathbb{R} \quad \forall q \in Q_{\mathbb{A}} \setminus \mathbb{R} \quad (4.18)$$

and  $(\exp^t)'_s(q) = te^{t \operatorname{Re}(q)}$  for every  $q \in \mathbb{R}$ . In particular  $(\exp^t)'_s$  is a real-valued. By abuse of notation, we write  $\exp'_s(t, q)$  to indicate the element  $(\exp^t)'_s(q)$  of  $\mathbb{A}$  for every  $t \in \mathbb{R}$  and for every  $q \in Q_{\mathbb{A}}$ , respectively.

*Proof.* For every  $q \in Q_{\mathbb{A}} \setminus \mathbb{R}$  there exists  $\mathbf{j} \in \mathbb{S}_{\mathbb{A}}$  and  $a, b \in \mathbb{R}$  such that  $b > 0$  and  $q = a + b\mathbf{j}$ . Hence  $q^c = a^c - \mathbf{j}^c b^c = a - b\mathbf{j}$ ,  $\operatorname{Re}(q) = a$ ,  $\operatorname{Im}(q) = b\mathbf{j}$ , and  $|\operatorname{Im}(q)| = \sqrt{(b\mathbf{j})(b\mathbf{j})^c} = b$ . Since  $\mathbb{C}_{\mathbf{j}}$  and  $\mathbb{C}$  are isomorphic real algebras, we find that  $\exp^t(q) = e^{tq} = e^{ta}(\cos(tb) + \sin(tb)\mathbf{j})$  and  $\exp^t(q^c) = e^{tq^c} = e^{ta}(\cos(tb) - \sin(tb)\mathbf{j}) = (e^{tq})^c$ , therefore

$$(\exp^t)'_s(q) = (q - q^c)^{-1}(e^{tq} - e^{tq^c}) = e^{t\operatorname{Re}(q)} \sin(t|\operatorname{Im}(q)|) |\operatorname{Im}(q)|^{-1} \quad \forall q \in Q_{\mathbb{A}} \setminus \mathbb{R}, \quad (4.19)$$

which proves the first equality in (4.18). The right-hand side of (4.19) immediately extends by continuity to  $te^{ta}$  for  $q \in \mathbb{R}$ , thus  $(\exp^t)'_s$  is a real-valued function. As  $|\operatorname{Im}(q)|^2 = -\operatorname{Im}(q)^2$ , by (4.19) we have

$$(\exp^t)'_s(q) = e^{t\operatorname{Re}(q)} \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1} |\operatorname{Im}(q)|^{2n}}{(2n+1)!} = e^{t\operatorname{Re}(q)} \sum_{n=0}^{\infty} \frac{t^{2n+1} \operatorname{Im}(q)^{2n}}{(2n+1)!} \quad \forall q \in Q_{\mathbb{A}} \setminus \mathbb{R},$$

which implies the second equality of (4.18).  $\square$

**Corollary 4.8.** *Under the assumption of Theorem 3.5, for every  $q \in Q_{\mathbb{A}}$  such that  $\operatorname{Re}(q) > \omega$  we have*

$$Q_q(\mathbf{A})x = - \int_0^{\infty} \mathbb{T}(t) \exp'_s(-t, q) x \, dt \quad \forall x \in X. \quad (4.20)$$

## 5. INTEGRAL REPRESENTATION OF THE POWERS OF $Q_q(\mathbf{A})$

In this section we look for an integral representation of the integer powers of the quadratic resolvent operator  $Q_q(\mathbf{A})$ . In order to find this representation we need the following simple lemma.

**Lemma 5.1.** *If  $f, g \in L_{\mathbb{T}}([0, \infty[; \mathbb{A})$  and  $f$  is real-valued, then*

$$\mathbb{L}(f)\mathbb{L}(g)x = \mathbb{L}(f \star g)x = \mathbb{L}(g \star f)x \quad \forall x \in X,$$

where we recall that  $f \star g : [0, \infty[ \rightarrow \mathbb{A}$ , the convolution of  $f$  and  $g$ , is defined by  $(f \star g)(t) := \int_0^t f(t-s)g(s) \, ds$ .

*Proof.* Using the fact that  $f$  is real-valued, we see at once that  $f \star g = g \star f$ . In addition, bearing in mind the semigroup law for  $\mathbb{T}$ , we find

$$\begin{aligned} \mathbb{L}(f)\mathbb{L}(g)x &= \int_0^{\infty} \mathbb{T}(t)f(t) \int_0^{\infty} \mathbb{T}(s)g(s)x \, ds \, dt \\ &= \int_0^{\infty} \int_0^{\infty} \mathbb{T}(t)\mathbb{T}(s)f(t)g(s)x \, ds \, dt \\ &= \int_0^{\infty} \int_0^{\infty} \mathbb{T}(t+s)f(t)g(s)x \, ds \, dt. \end{aligned}$$

Thus a change of variable and an application of Fubini theorem yields

$$\begin{aligned}
L(f)L(g)x &= \int_0^\infty \int_t^\infty T(s)f(t)g(s-t)x \, ds \, dt \\
&= \int_0^\infty \int_0^\infty \chi_{[0,s]}(t)T(s)f(t)g(s-t)x \, ds \, dt \\
&= \int_0^\infty \int_0^\infty \chi_{[0,s]}(t)T(s)f(t)g(s-t)x \, dt \, ds \\
&= \int_0^\infty \int_0^t T(s)f(t)g(s-t)x \, dt \, ds \\
&= \int_0^\infty T(s) \int_0^t f(t)g(s-t)x \, dt \, ds \\
&= L(f \star g) = L(g \star f).
\end{aligned}$$

The proof is complete.  $\square$

Given  $n \in \mathbb{N} \setminus \{0\}$  and  $f \in L_T([0, \infty[; \mathbb{A})$ , we define  $f^{*n} \in L_T([0, \infty[; \mathbb{A})$  by

$$f^{*n} := \underbrace{f \star f \star \cdots \star f}_{n \text{ times}}.$$

**Corollary 5.2.** *Let  $T : [0, \infty[ \rightarrow \mathcal{L}^r(X)$  be a strongly continuous right linear semigroup, let  $A : D(A) \rightarrow X$  be its generator, and let  $\omega \in \mathbb{R}$  be a real constant such that  $M := \sup_{t \in [0, \infty[} \|T(t)\| e^{-\omega t} < \infty$ . Given any  $q \in Q_{\mathbb{A}}$  with  $\operatorname{Re}(q) > \omega$ , we have that  $q \in \rho_S(A)$  and*

$$Q_q(A)^n x = (-1)^n \int_0^\infty T(t) \exp'_s(-t, q)^{*n} x \, dt \quad (5.1)$$

where  $\exp'_s(-t, q)^{*n} \in \mathbb{A}$  indicates the value of  $((\exp^{-t})'_s)^{*n}$  at  $q$ .

Moreover for every  $q \in Q_{\mathbb{A}}$  with  $\operatorname{Re}(q) > \omega$  we have

$$\|Q_q(A)^n\| \leq \frac{M}{(\operatorname{Re}(q) - \omega)^{2n}} \quad \forall n \in \mathbb{N} \setminus \{0\}. \quad (5.2)$$

*Proof.* Formula (5.1) follows immediately from  $n$  applications of Theorem 3.5 and Lemma 5.1. In order to prove (5.2) let us observe that, given  $q \in Q_{\mathbb{A}}$ , if  $a = \operatorname{Re}(q)$  and  $b = |\operatorname{Im}(q)|$ , and  $g(t) = -\exp'_s(-t, q)$  for  $t \geq 0$ , then

$$\begin{aligned}
|(g \star g)(t)| &\leq \int_0^t |g(t-s)| |g(s)| \, ds \\
&\leq \int_0^t (t-s) e^{-(t-s)a} s e^{-sa} \, ds \\
&= e^{-ta} \int_0^t s(t-s) \, ds = \frac{1}{2 \cdot 3} t^3 e^{-ta}.
\end{aligned}$$

Let us assume by induction that  $|g^{*(n-1)}| \leq ((2n-3)!)^{-1} t^{2n-3} e^{-at}$ . Therefore for every  $n$

$$\begin{aligned} |g^{*n}(t)| &\leq \int_0^t |g(t-s)| |g^{*(n-1)}(s)| \, ds \\ &\leq \frac{1}{(2n-3)!} \int_0^t (t-s) e^{-(t-s)a} s^{2n-3} e^{-as} \, ds \\ &= \frac{e^{-ta}}{(2n-3)!} \int_0^t (t-s) s^{2n-3} \, ds \\ &= \frac{e^{-ta}}{(2n-3)!} \frac{t^{2n-1}}{(2n-2)(2n-1)} = \frac{t^{2n-1} e^{-ta}}{(2n-1)!}. \end{aligned}$$

Thus  $|g^{*n}(t)| \leq \frac{1}{(2n-1)!} t^{2n-1} e^{-ta}$  for every  $t \geq 0$  and every  $n \in \mathbb{N} \setminus \{0\}$  and, recalling that  $\int_0^\infty t^{2n-1} e^{-t} \, dt = (2n-1)!$ , we have

$$\begin{aligned} \|\mathbf{Q}_q(\mathbf{A})^n\| &\leq \int_0^\infty \|\mathbf{T}(t)\| |g^{*n}(t)| \, dt \\ &\leq \int_0^\infty M e^{\omega t} \frac{1}{(2n-1)!} t^{2n-1} e^{-ta} \, dt \\ &= \frac{M}{(2n-1)!} \int_0^\infty t^{2n-1} e^{-(a-\omega)t} \, dt \\ &= \frac{M}{(2n-1)!} \int_0^\infty \frac{t^{2n-1}}{(a-\omega)^{2n}} e^{-t} \, dt \\ &= \frac{M}{(a-\omega)^{2n}}, \end{aligned}$$

and we are done.  $\square$

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## REFERENCES

- [1] S. Adler, “Quaternionic Quantum Field Theory”, Oxford University Press, 1995.
- [2] D. Alpay, F. Colombo, D.P. Kimsey, *The spectral theorem for quaternionic unbounded normal operators based on the S-spectrum*, J. Math. Phys. **57**, 023503, 27 pp. (2016).
- [3] F.W. Anderson, K.R. Fuller, “Rings and Categories of Modules”, Springer Verlag, New York, 1974.
- [4] G. Birkhoff, J. von Neumann, *The logic of quantum mechanics*, Ann. of Math. (2), **37** (1936), 823–843.
- [5] F. Colombo, J. Gantner, D.P. Kimsey, “Spectral theory on the S-spectrum for quaternionic operators”. Operator Theory: Advances and Applications. Birkhäuser/Springer, 2018.
- [6] F. Colombo, J. Gantner, “Quaternionic closed operators, fractional powers and fractional diffusion processes”. Operator Theory: Advances and Applications. Birkhäuser/Springer, 2019.
- [7] F. Colombo, G. Gentili, I. Sabadini, *A Cauchy kernel for slice regular functions*, Ann. Glob. Anal. Geom., **37** (2010), 361–378.
- [8] F. Colombo, G. Gentili, I. Sabadini, D.C. Struppa, *A functional calculus in a non commutative setting*. Electron. Res. Announc. Math. Sci., **14** (2007), 60–68.

- [9] F. Colombo, G. Gentili, I. Sabadini, D.C. Struppa, *Non Commutative Functional Calculus: Bounded Operators*, Complex Anal. Oper. Theory., **4** (2010), 821–843.
- [10] F. Colombo, G. Gentili, I. Sabadini, D.C. Struppa, *Non-commutative functional calculus: Unbounded operators*, J. Geom. Phys., **60** (2010), 251–259.
- [11] F. Colombo, I. Sabadini, *On some properties of the quaternionic functional calculus*, J. Geom. Anal., **19** (2009), 601–627.
- [12] F. Colombo, I. Sabadini, *On the formulation of the quaternionic functional calculus*, J. Geom. Phys., **60** (2009), 1490–1508.
- [13] F. Colombo, I. Sabadini, *The quaternionic evolution operator*, Adv. Math., **227** (2011), 1772–1805.
- [14] F. Colombo, I. Sabadini, D.C. Struppa, *A new functional calculus for noncommuting operators*, J. Functional Analysis, **254** (2008), 2255–2274.
- [15] F. Colombo, I. Sabadini, D.C. Struppa, “Noncommutative Functional Calculus”,
- [16] E.B. Davies, “One Parameter Semigroups”, Academic Press, 1980.
- [17] H.-D. Ebbinghaus, H. Hermes, F. Hirzebruch, M. Koecher, K. Mainzer, J. Neukirch, A. Prestel, R. Remmert, “Numbers”, Grad. Texts in Math., vol. 123, Springer-Verlag, New York, 1990.
- [18] G. Emch, *Mécanique quantique quaternionienne et relativité restreinte*, Elv. Phys. Acta, **36** (1963), 739–769.
- [19] K.-J. Engel, R. Nagel, “One-Parameter Semigroups for Linear Evolution Equations”, Springer Verlag, New York, 2000.
- [20] D. Finkelstein, J.M. Jauch, S. Schiminovich, D. Speiser, *Foundations of quaternionic quantum mechanics*, J. Mathematical Phys., **3** (1962), 207–220.
- [21] G. Gentili, C. Stoppato, *Power series and analyticity over the quaternions*, Math. Ann., **352** (2012), 113–131.
- [22] G. Gentili, D.C. Struppa, *A new theory of regular functions of a quaternionic variable*, Adv. Math., **216** (2007), 279–301.
- [23] R. Ghiloni, V. Moretti, A. Perotti, *Continuous slice functional calculus in quaternionic Hilbert spaces*, Rev. Math. Phys. **25** (2013), no. 4, 1330006, 83 pp.
- [24] R. Ghiloni, V. Moretti, A. Perotti, *Spectral representations of normal operators in quaternionic Hilbert spaces via intertwining quaternionic PVMs*, arXiv:1602.02661v1
- [25] R. Ghiloni, A. Perotti, *Slice regular functions on real alternative algebras*, Adv. Math., **226** (2011), 1662–1691.
- [26] R. Ghiloni, A. Perotti, V. Recupero, *Noncommutative Cauchy integral formula*, Complex Anal. Oper. Theory **11** (2017), 289–306.
- [27] R. Ghiloni, V. Recupero, *Semigroups over real alternative  $*$ -algebras: Generation theorems and spherical sectorial operators*, Trans. Amer. Math. Soc., **368** (2016), no. 4, 2645–2678.
- [28] R. Ghiloni, V. Recupero, *Slice regular semigroups*, Trans. Amer. Math. Soc., **370** (2018), no. 7, 4993–5032.
- [29] J.E. Gilbert, M.A.M Murray, “Clifford algebras and Dirac operators in harmonic analysis”, Cambridge University Press, Cambridge, 1991.
- [30] J.A. Goldstein, “Semigroups of Operators and Applications”, Oxford University Press, 1985.
- [31] K. Gürlebeck, K. Habetha, W. Sprößig, “Holomorphic functions in the plane and  $n$ -dimensional space”, Birkhäuser Verlag, Basel, xiv+394, 2008.
- [32] E. Hille, R.S. Phillips, “Functional Analysis and Semigroups”, Amer. Math. Soc. Coll. Publ., vol. 31, Amer. Math. Soc., 1957.
- [33] L. P. Horwitz, L.C. Biedenharn, *Quaternionic quantum mechanics: Second quantization and gauge field*, Annals of Physics, **157** (1984), 432–488.
- [34] I. Kaplansky, *Normed algebras*, Duke Mat. J., **16** (1949), 399–418.
- [35] A. Lunardi, “Analytic Semigroups and Optimal Regularity in Parabolic Problems”, Birkhauser-Verlag, 1995.
- [36] A. Pazy, “Semigroups of Linear Operators and Applications to Partial Differential Equations”, Springer-Verlag, New York, 1983.
- [37] K. Taira, “Analytic Semigroups and Semilinear Initial Boundary Value Problems”, London Math. Soc. Lect. Notes Ser., vol. 223, Cambridge University Press, 1995.

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