On the generators of Clifford semigroups: Polynomial resolvents and their integral transforms

*Original*
On the generators of Clifford semigroups: Polynomial resolvents and their integral transforms / Ghiloni, R., Recupero, V.. - In: JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS. - ISSN 0022-247X. - ELETTRONICO. - 521:1(2023), pp. 1-19. [10.1016/j.jmaa.2022.126905]

*Availability:*
This version is available at: 11583/2973840 since: 2023-12-05T09:54:02Z

*Publisher:*
Elsevier

*Published*
DOI:10.1016/j.jmaa.2022.126905



(Article begins on next page)

# ON THE GENERATORS OF CLIFFORD SEMIGROUPS: POLYNOMIAL RESOLVENTS AND THEIR INTEGRAL TRANSFORMS

RICCARDO GHILONI AND VINCENZO RECUPERO

Abstract. This paper deals with generators $\mathsf{A}$ of strongly continuous right linear semigroups in Banach two-sided spaces whose set of scalars is an arbitrary Clifford algebra $C\ell(0,n)$. We study the invertibility of operators of the form $P(\mathsf{A})$, where $P(\mathrm{x}) \in \mathbb{R}[\mathrm{x}]$ is any real polynomial, and we give an integral representation for $P(\mathsf{A})^{-1}$ by means of a Laplace-type transform of the semigroup $\mathsf{T}(t)$ generated by $\mathsf{A}$. In particular, we deduce a new integral representation for the spherical quadratic resolvent of $\mathsf{A}$ (also called pseudoresolvent of $\mathsf{A}$). As an immediate consequence, we also obtain a new proof of the well-known integral representation for the spherical resolvent of $\mathsf{A}$.

*dedicated to Professor Klaus Gürlebeck*

## 1. Introduction and main results

Quaternionic functional analysis has probably its original motivation in the seminal paper [4], where it is pointed out that quantum mechanics may be formulated, not only on complex Hilbert spaces, but also on Hilbert spaces whose set of scalars is $\mathbb{H}$, the noncommutative real algebra of quaternions.

Many papers have been devoted to the development of quantum mechanics in the quaternionic framework (see, e.g., [20, 18, 33, 1]), whose natural setting is a Hilbert two-sided $\mathbb{H}$-module $X$, with the space of bounded linear operators acting on it replaced by the set $\mathscr{L}^r(X)$ of bounded *right* linear operators. However, a full development of quaternionic quantum mechanics was prevented by the lack of suitable quaternionic spectral notions, indeed, as observed in [9, 23], the classical definitions of spectrum and resolvent operator do not allow to define a noncommutative functional calculus.

A first rigorous formulation of a quaternionic spectral theory has been provided only in [8] where one can find the first definition of the notions of *spherical resolvent set* $\rho_{\mathcal{S}}(\mathsf{A})$, *quadratic resolvent operator* $\mathsf{Q}_q(\mathsf{A})$, *spherical resolvent operator* $\mathsf{C}_q(\mathsf{A})$ and *spherical spectrum* $\sigma_{\mathcal{S}}(\mathsf{A})$ of a right linear operator $\mathsf{A}$ on a quaternionic Banach space $X$. They are

2010 *Mathematics Subject Classification.* 30G35, 47D03, 47A60, 47A10.

*Key words and phrases.* Semigroups in the noncommutative setting, Slice regular semigroups; Spectrum, resolvent; Laplace transform, Functional calculus, Quaternions, Clifford algebras; Functions of hypercomplex variables.

The first author is a member of GNSAGA-INdAM. The second author is a member of GNAMPA-INdAM.

given by

$$\begin{aligned}
&\rho_{\mathcal{S}}(\mathsf{A}) := \{q \in \mathbb{H} \ : \ \exists (\mathsf{A}^2 - 2\operatorname{Re}(q)\,\mathsf{A} + |q|^2)^{-1} \in \mathscr{L}^{r}(X)\},\\
&\mathsf{Q}_q(\mathsf{A}) := (\mathsf{A}^2 - 2\operatorname{Re}(q)\,\mathsf{A} + |q|^2)^{-1}, \qquad q \in \rho_{\mathcal{S}}(\mathsf{A}),\\
&\mathsf{C}_q(\mathsf{A}) := \mathsf{Q}_q(\mathsf{A})\overline{q} - \mathsf{A}\mathsf{Q}_q(\mathsf{A}), \qquad q \in \rho_{\mathcal{S}}(\mathsf{A}).
\end{aligned}$$

In [8] the name *pseudoresolvent* is used for $\mathsf{Q}_q(\mathsf{A})$ but we choose the term *quadratic resolvent* since in the classical complex case the name "pseudoresolvent" has a different meaning. The above definitions permit to develop a noncommutative functional calculus for right linear operators on a Banach two-sided module over $\mathbb{H}$ (and over a Clifford algebra as well, cf. [14, 11, 12, 9, 10, 15, 23]) and to deduce in [2, 24] the spectral representation theorems for normal operators in the quaternionic Hilbert setting (cf. [2, 24]). A wider bibliography can be found in the recent accounts on the theory [5, 6].

The mentioned noncommutative functional calculus is intimately connected to the theory of slice regular functions, introduced in [22], which extends to quaternions the classical concept of holomorphic function. They form a class of functions admitting a local power series expansion at every point of their domain of definition (cf. [21]), including polynomials with quaternionic coefficients on one side, and they admit a Cauchy-type integral representation formula with a suitable quaternionic version of the kernel proved for the first time in [7] (see also [26]).

The next natural stage in this analysis is the development of a noncommutative theory of right linear operator semigroups which was developed in [13, 27, 28]. In the classical complex theory a fundamental tool is provided by the integral representation of the resolvent operator of a generator $\mathsf{A}$ by means of the Laplace transform of the semigroup $\mathsf{T}(t)$ generated by $\mathsf{A}$. An analogous integral representation in the quaternionic case for the spherical resolvent operator $\mathsf{C}_q(\mathsf{Q})$ is shown in [13] and a proof is provided in [28] using techniques from slice regular function theory.

The purpose of the present paper is to study the invertibility of operators of the form $P(\mathsf{A})$ where $P(\mathrm{x})$ is an arbitrary polynomial with real coefficients of degree at least 2, including $P(\mathrm{x}) = \Delta_q(\mathrm{x}) = \mathrm{x}^2 - 2\operatorname{Re}(q)\mathrm{x} + |q|^2$. Under natural conditions, we prove that $P(\mathsf{A})^{-1}$ exists and belongs to $\mathscr{L}^{r}(X)$. Furthermore, we provide an integral representation for $P(\mathsf{A})^{-1}$ by means of a Laplace-type transform of the semigroup $\mathsf{T}(t)$ generated by $\mathsf{A}$. We extend this integral representation to operators of the form $\sum_{j=0}^{d-1}\mathsf{A}^j P(\mathsf{A})^{-1}p_j$, where $d$ is the degree of $P$ and $p_0, \ldots, p_{d-1}$ are arbitrarily chosen quaternions. In the case $P(\mathrm{x}) = \Delta_q(\mathrm{x})$, we obtain a new integral representation for $\mathsf{Q}_q(\mathsf{A})$ and, setting $p_0 := \overline{q}$ and $p_1 := -1$, we discover again the well-known integral representation for $\mathsf{C}_q(\mathsf{A})$ via the quaternionic Laplace transform. This gives also a new proof of the integral representation for $\mathsf{C}_q(\mathsf{A})$, which avoids the use of slice regular function techniques. Our results are valid not only on quaternions but also on a class of real associative $^*$-algebras including, as the main examples, all Clifford algebras $C\ell(0,n)$.

Let $n \in \mathbb{N}$, let $\mathbb{R}_n$ be the Clifford algebra $C\ell(0,n)$ equipped with the Clifford conjugation and the Clifford operator norm $|\cdot|$. Consider a Banach two-sided $\mathbb{R}_n$-module $X$ with norm $\|\cdot\|$ and the set $\mathscr{L}^{r}(X)$ of all bounded right linear operators on $X$ (all the precise definitions will be recalled in the next section).

Let $m \in \mathbb{N}$ and let $P(\mathrm{x}) = \sum_{k=0}^{m+2}\mathrm{x}^k a_k \in \mathbb{R}[\mathrm{x}]$ be a polynomial with real coefficients in the indeterminate x. Suppose $P$ has degree $m+2$, that is, $a_{m+2} \neq 0$. Given a right linear operator $\mathsf{A} : D(\mathsf{A}) \longrightarrow X$, we define the right linear operator $P(\mathsf{A}) : D(\mathsf{A}^{m+2}) \longrightarrow X$ simply by replacing x with $\mathsf{A}$, that is, $P(\mathsf{A}) := \sum_{k=0}^{m+2} \mathsf{A}^k a_k$. Denote by $C^\infty([0,\infty[\,;\mathbb{R})$ the set of all infinitely many times differentiable functions $g : [0,\infty[ \longrightarrow \mathbb{R}$. Consider the following ODE with constant coefficients in the variable $g \in C^\infty([0,\infty[\,;\mathbb{R})$:

$$\begin{cases} P\left(-\frac{d}{dt}\right)(g) = 0 \text{ on } [0,\infty[\,, \\ g(0) = g'(0) = \ldots = g^{(m)}(0) = 0\,, \\ g^{(m+1)}(0) = (-1)^m (a_{m+2})^{-1}\,, \end{cases} \tag{1.1}$$

where $g^{(k)}$ is the $k^{\text{th}}$-derivative of $g$ and $P\left(-\frac{d}{dt}\right)(g) := \sum_{k=0}^{m+2} g^{(k)}(-1)^k a_k$. Denote by $g_P \in C^\infty([0,\infty[\,;\mathbb{R})$ the unique solution of (1.1). Recall that, if $\lambda_1, \ldots, \lambda_h$ are the complex roots of the polynomial $P(\mathrm{x})$ with multiplicity $m_1, \ldots, m_h$, then there exist complex polynomials $Q_1(\mathrm{x}), \ldots, Q_h(\mathrm{x}) \in \mathbb{C}[\mathrm{x}]$ such that the degree of each $Q_j(\mathrm{x})$ is $< m_j$ and

$$g_P(t) = \sum_{j=1}^{h} Q_j(t) e^{-\lambda_j t}. \tag{1.2}$$

Define $r_P \in \mathbb{R}$ by

$$r_P := \min\{\Re(\lambda_1), \ldots, \Re(\lambda_h)\}, \tag{1.3}$$

where $\Re(\lambda_j)$ is the real part of the complex number $\lambda_j$.

Recall also that if $\mathsf{T} : [0,\infty[ \longrightarrow \mathscr{L}^r(X)$ is a strongly continuous right linear semigroup then there exists $\omega \in \mathbb{R}$ such that $\sup_{t\in[0,\infty[} \|\mathsf{T}(t)\| e^{-\omega t} < \infty$, see [27, Thm 4.5(b)].

Our main result reads as follows.

**Theorem 1.1.** *Let $\mathsf{T} : [0,\infty[ \longrightarrow \mathscr{L}^r(X)$ be a strongly continuous right linear semigroup, let $\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ be its generator and let $\omega \in \mathbb{R}$ be a real constant such that $M := \sup_{t\in[0,\infty[} \|\mathsf{T}(t)\| e^{-\omega t} < \infty$. Then, if $r_P > \omega$, the operator $P(\mathsf{A})$ is bijective, $P(\mathsf{A})^{-1} \in \mathscr{L}^r(X)$ and it holds:*

$$P(\mathsf{A})^{-1} x = \int_0^\infty \mathsf{T}(t) g_P(t) x \, \mathrm{d}t \qquad \forall x \in X \tag{1.4}$$

*and*

$$\|P(\mathsf{A})^{-1}\| \leq M \sum_{j=1}^{h} \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(r_P - \omega)^k}\,, \tag{1.5}$$

*where $c_{-k}^{(j)}$ is the residue at $\lambda_j$ of the meromorphic function $z \mapsto a_{m+2}(z-\lambda_j)^{k-1}/P(-z)$.*

*Furthermore, given $(p_0, p_1, \ldots, p_{m+1}) \in (\mathbb{R}_n)^{m+2}$, we have*

$$\sum_{j=0}^{m+1} \mathsf{A}^k P(\mathsf{A})^{-1} p_j x = \int_0^\infty \mathsf{T}(t) \left( \sum_{j=0}^{m+1} g_P^{(j)} (-1)^j p_j x \right) \mathrm{d}t \qquad \forall x \in X. \tag{1.6}$$

Let $Q_{\mathbb{R}_n}$ be the quadratic cone of $\mathbb{R}_n$ (see (2.5) for the definition) and let $q \in Q_{\mathbb{R}_n}$. Define the function $g_q : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$g_q(t) := t e^{-\operatorname{Re}(q)t} \operatorname{sinc}(t|\operatorname{Im}(q)|), \qquad t \in \mathbb{R}, \tag{1.7}$$

where $\operatorname{Re}(q)$ and $\operatorname{Im}(q)$ are the real and imaginary parts of $q$, respectively. We recall that $\operatorname{sinc} : \mathbb{R} \longrightarrow \mathbb{R}$ is the *unnormalized sinc function*, that is, the real-valued continuous function $\xi$ on $\mathbb{R}$ defined by $\xi(0) = 1$ and $\xi(r) = \sin(r)/r$ for all $r \neq 0$.

Thanks to the preceding result, we are able to prove the following:

**Theorem 1.2.** *Let* $\mathsf{T} : [0,\infty[ \longrightarrow \mathscr{L}^r(X)$ *be a strongly continuous right linear semigroup, let* $\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ *be its generator, and let* $\omega \in \mathbb{R}$ *be a real constant such that* $M := \sup_{t\in[0,\infty[} \|\mathsf{T}(t)\| e^{-\omega t} < \infty$. *Consider any* $q \in Q_{\mathbb{R}_n}$ *and set* $a := \mathrm{Re}(q)$ *and* $b := |\,\mathrm{Im}(q)|$. *If* $\mathrm{Re}(q) > \omega$, *then we have that* $q \in \rho_S(\mathsf{A})$ *and it holds:*

$$\mathsf{Q}_q(\mathsf{A})x = \int_0^\infty \mathsf{T}(t) g_q(t) x \,\mathrm{d}t = \int_0^\infty \mathsf{T}(t) t e^{-ta} \operatorname{sinc}(tb) x \,\mathrm{d}t, \tag{1.8}$$

$$\mathsf{A}\mathsf{Q}_q(\mathsf{A})x = -\int_0^\infty \mathsf{T}(t) g_q'(t) x \,\mathrm{d}t = -\int_0^\infty \mathsf{T}(t) e^{-ta}(\cos(tb) - at\operatorname{sinc}(tb)) x \,\mathrm{d}t, \tag{1.9}$$

$$\mathsf{C}_q(\mathsf{A})x = \int_0^\infty \mathsf{T}(t) e^{-tq} x \,\mathrm{d}t \tag{1.10}$$

*for every* $x \in X$. *Moreover, we have:*

$$\|\mathsf{Q}_q(\mathsf{A})\| \le \frac{M}{(\mathrm{Re}(q) - \omega)^2}\,, \tag{1.11}$$

$$\|\mathsf{C}_q(\mathsf{A})\| \le \frac{M}{\mathrm{Re}(q) - \omega}\,. \tag{1.12}$$

Here is the plan of the paper. In the following section we recall all the needed precise definitions. In Section 3 we present the preceding theorems in the more general case of certain real $*$-algebras, including all the $\mathbb{R}_n$'s. Section 4 is devoted to the proofs of these theorems. Finally, in Section 5 we apply the main theorem in order to derive an integral representation of the integer powers of the quadratic resolvent and the estimate of their norms; this extends to $\mathsf{Q}_q(\mathsf{A})$ our Theorem 6.6 in [28] concerning the Laplace-type transform for the integer slice powers of $\mathsf{C}_q(\mathsf{A})$.

## 2. Preliminaries

Throughout all the paper we will assume that $\mathbb{A}$ is a nontrivial finite dimensional $\mathbb{R}$-vector space endowed with a bilinear product $\mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{A} : (p,q) \longmapsto pq$ with unit $1_{\mathbb{A}}$, and with a mapping $\mathbb{A} \longrightarrow \mathbb{A} : q \longmapsto q^c$ called ${}^*$*-involution*, which is an $\mathbb{R}$-linear mapping such that $(q^c)^c = q$, $(pq)^c = q^c p^c$, and $r^c = r$ for all $p, q \in \mathbb{A}, r \in \mathbb{R} \subseteq \mathbb{A}$, where we are identifying $\mathbb{R}$ with the subalgebra of $\mathbb{A}$ generated by $1_{\mathbb{A}}$ by means of the algebra isomorphism $\mathbb{R} \longrightarrow \mathbb{R}1_{\mathbb{A}} : r \longmapsto r1_{\mathbb{A}}$. Therefore we can write $1 = 1_{\mathbb{A}}$ and we summarize the previous assumptions by saying that

$$\begin{aligned}&\mathbb{A} \text{ is a finite dimensional associative nontrivial real } {}^*\text{-algebra}\\ &\text{with } {}^*\text{-involution } q \longmapsto q^c \text{ and unit } 1.\end{aligned} \tag{2.1}$$

Under the previous assumptions we define the *imaginary sphere* $\mathbb{S}_{\mathbb{A}}$ in $\mathbb{A}$ by

$$\mathbb{S}_{\mathbb{A}} := \left\{ q \in \mathbb{A} \ : \ q^c = -q,\ q^2 = -1 \right\}. \tag{2.2}$$

In the remainder of the paper we will assume that

$$\mathbb{S}_{\mathbb{A}} \neq \varnothing. \tag{2.3}$$

Condition (2.3) in particular implies that $\mathbb{A}$ cannot be equal to $\mathbb{R}$. We set

$$\mathbb{C}_{\mathbf{j}} := \{r + s\mathbf{j} \in \mathbb{A} \ : \ r, s \in \mathbb{R}\}, \qquad \mathbf{j} \in \mathbb{S}_{\mathbb{A}}, \tag{2.4}$$

$$Q_{\mathbb{A}} := \bigcup_{\mathbf{j} \in \mathbb{S}_{\mathbb{A}}} \mathbb{C}_{\mathbf{j}}, \tag{2.5}$$

the set $Q_{\mathbb{A}}$ being called *quadratic cone of* $\mathbb{A}$. The *real part* $\mathrm{Re}(q)$ and the *imaginary part* $\mathrm{Im}(q)$ of an element $q \in \mathbb{A}$ are defined by

$$\mathrm{Re}(q) := (q + q^c)/2, \quad \mathrm{Im}(q) := (q - q^c)/2, \qquad q \in \mathbb{A}. \tag{2.6}$$

Notice that in general $\mathrm{Re}(q)$ and $\mathrm{Im}(q)$ are not real numbers, at variance with the customary complex notations $\Re(z) := (z + \overline{z})/2 \in \mathbb{R}$ and $\Im(z) := (z - \overline{z})/2i \in \mathbb{R}$ for $z \in \mathbb{C}$.

We finally observe that $qq^c \in \mathbb{R}$ for every $q \in Q_{\mathbb{A}}$ and we assume that

$$\begin{aligned} &\mathbb{A} \text{ is endowed with a complete norm } |\cdot| \text{ such that} \\ &|q_1 q_2| \leq |q_1||q_2| \text{ for every } q_1, q_2 \in \mathbb{A} \text{ and } |q|^2 = qq^c \text{ for every } q \in Q_{\mathbb{A}}. \end{aligned} \tag{2.7}$$

The equivalence of the above definitions with other presentations (e.g. [25] is provided in [28]). We recall here that

$$\mathbb{C}_{\mathbf{j}} \cap \mathbb{C}_{\mathbf{k}} = \mathbb{R} \qquad \forall \mathbf{j}, \mathbf{k} \in \mathbb{S}_{\mathbb{A}}, \ \mathbf{j} \neq \pm\mathbf{k}. \tag{2.8}$$

**Example 2.1** (Clifford algebras)**.** For $n \in \mathbb{N} \setminus \{0\}$ let $\mathcal{P}(n)$ be the power set of $\{1, \ldots, n\}$. If we identify $\mathbb{R}$ with the vector subspace $\mathbb{R} \times \{0\}$ of $\mathbb{R}^{2^n} = \mathbb{R} \times \mathbb{R}^{2^n - 1}$ and we set $e_\varnothing := 1$, then we denote by $\{e_K\}_{K \in \mathcal{P}(n)}$ the canonical basis of $\mathbb{R}^{2^n}$. For convenience, we set $e_k := e_{\{k\}}$ if $k \in \{1, \ldots, n\}$ and we define a real bilinear and associative product on $\mathbb{R}^{2^n}$ by imposing that 1 is the neutral element and that

$$\begin{aligned} &e_k^2 = -1 \text{ and } e_k e_h = -e_h e_k \text{ if } k, h \in \{1, \ldots, n\} \text{ with } k \neq h, \\ &e_K = e_{k_1} \cdots e_{k_s} \text{ if } K = \{k_1, \ldots, k_s\} \in \mathcal{P}(n) \setminus \{\varnothing\} \text{ with } k_1 < \ldots < k_s. \end{aligned}$$

The *Clifford conjugation* of $\mathbb{R}^{2^n}$ is the $^*$-involution $q \longmapsto q^c := \overline{q}$ defined by

$$\overline{q} := \textstyle\sum_{K \in \mathcal{P}(n)} (-1)^{|K|(|K|+1)/2} a_K e_k \quad \text{if } q = \sum_{K \in \mathcal{P}(n)} a_K e_k \in \mathbb{R}_n, \ a_K \in \mathbb{R},$$

where $|K|$ indicates the cardinality of the set $K$. Endowing $\mathbb{R}^{2^n}$ with the above defined product and with the Clifford conjugation, we obtain a real $^*$-algebra $\mathbb{A}$ satisfying (2.1), called *Clifford algebra* $C\ell(0, n)$ *of signature* $(0, n)$, which is denoted also by $\mathbb{R}_n$. Observe that $\mathbb{R}_1$ and $\mathbb{R}_2$ are isomorphic to $\mathbb{C}$ and $\mathbb{H}$, respectively. Moreover $\mathbb{R}_n$ is not commutative if $n \geq 2$. If $n \geq 3$ then $\mathbb{R}_n$ has zero divisors, indeed $(1 - e_{\{1,2,3\}})(1 + e_{\{1,2,3\}}) = 0$. One verifies that a point $q = \sum_{K \in \mathcal{P}(n)} a_K e_K$ of $\mathbb{R}_n$ with $a_K \in \mathbb{R}$ belongs to the quadratic cone $Q_{\mathbb{R}_n}$ of $\mathbb{R}_n$ if and only if it satisfies the following conditions

$$a_K = 0 \quad \text{and} \quad \langle q, q e_K \rangle_{2^n} = 0 \quad \text{for every } K \in \mathcal{P}(n) \setminus \{\varnothing\} \text{ with } e_K^2 = 1,$$

where $\langle \cdot, \cdot \rangle_{2^n}$ denotes the standard scalar product on $\mathbb{R}^{2^n}$. On $\mathbb{R}_n$ it is defined the following submultiplicative norm, called *Clifford operator norm*: $|q|_{C\ell} := \sup\{|qa|_{2^n} \in \mathbb{R} \ : \ |a|_{2^n} = 1\}$, where $|\cdot|_{2^n}$ indicates the Euclidean norm of $\mathbb{R}^{2^n}$. It turns out that:

(a) $Q_{\mathbb{R}_n} = \mathbb{R}_n$ if and only if $n \in \{1, 2\}$. In particular, $\mathbb{R}_1$ and $\mathbb{R}_2$ are division algebras.
(b) $|q|_{C\ell} = |x| = \sqrt{x\overline{x}}$ for every $x \in Q_{\mathbb{R}_n}$ and hence $|\cdot|_{C\ell} = |\cdot|$ if $n \in \{1, 2\}$.

Notice that if $n \geq 3$ then the Euclidean norm $|\cdot|_{2^n}$ of $\mathbb{R}_n$ is not submultiplicative (e.g. $|(1+e_{\{1,2,3\}})^2| = \sqrt{8} > 2 = |1+e_{\{1,2,3\}}|^2$). Endowing $\mathbb{R}_n$ with Clifford conjugation and Clifford operator norm, we obtain a real $^*$-algebra $\mathbb{A}$ satisfying (2.3) and (2.7). For further details we refer the reader to [29, 31].

**Example 2.2** (Complex numbers and quaternions)**.** If $n \in \mathbb{N} \setminus \{0\}$ and $\mathbb{R}_n$ denotes the Clifford algebra of signature $(0,n)$ recalled in the previous Example 2.1, then we have:

(i) $\mathbb{R}_1 = \mathbb{C}$ with $e_1 = i$, where $z \longmapsto z^c = \bar{z}$ is the standard conjugation and $|\cdot|$ is the Euclidean norm;

(ii) $\mathbb{R}_2$ is the algebra of quaternions $\mathbb{H}$ with $i := e_1$, $j := e_2$, $k := e_3$, where $q = a + bi + cj + dk \longmapsto q^c = \bar{q} = a - bi - cj - dk$ and $|\cdot|$ is the euclidean norm.

**Definition 2.3.** *If* $\mathbb{A}$ *satisfies* (2.1) *then a* two-sided $\mathbb{A}$-module *is a commutative group* $(X,+)$ *endowed with a left scalar multiplication* $\mathbb{A} \times X \longrightarrow X : (q,x) \longmapsto qx$ *and a right scalar multiplication* $X \times \mathbb{A} \longrightarrow X : (x,q) \longmapsto xq$ *such that*

$$\begin{array}{llll}
q(x+y) = qx+qy, & (x+y)q = xq+yq & \forall x,y \in X, & \forall q \in \mathbb{A}, \\
(p+q)x = px+qx, & x(p+q) = xp+xq & \forall x \in X, & \forall p,q \in \mathbb{A}, \\
1x = x = x1 & & \forall x \in X, & \\
p(qx) = (pq)x, & (xp)q = x(pq) & \forall x \in X, & \forall p,q \in \mathbb{A}, \\
p(xq) = (px)q & & \forall x \in X, & \forall p,q \in \mathbb{A}, \\
rx = xr & & \forall x \in X, & \forall r \in \mathbb{R}.
\end{array}$$

*If* $Y$ *is a commutative subgroup of* $X$ *then* $Y$ *is called a* left $\mathbb{A}$-submodule *if* $qx \in Y$ *whenever* $x \in Y$ *and* $q \in \mathbb{A}$*. Instead* $Y$ *is called a* right $\mathbb{A}$-submodule *of* $X$ *if* $xq \in Y$ *whenever* $x \in Y$ *and* $q \in \mathbb{A}$*. Finally* $Y$ *is called a* two-sided $\mathbb{A}$-submodule *of* $X$ *if it is both a left and a right* $\mathbb{A}$*-submodule of* $X$*.*

**Definition 2.4.** *Assume* (2.1) *and* (2.7) *hold. A two-sided* $\mathbb{A}$*-module* $X$ *is called a* normed two-sided $\mathbb{A}$-module *if it is endowed with a* $\mathbb{A}$-norm on $X$*, that is, a function* $\|\cdot\| : X \longrightarrow [0,\infty[$ *such that*

$$\begin{aligned}
&\|x\| = 0 \iff x = 0, \\
&\|x+y\| \leq \|x\| + \|y\| && \forall x,y \in X, \\
&\|qx\| \leq |q|\,\|x\|, \quad \|xq\| \leq |q|\,\|x\| && \forall x \in X, \quad \forall q \in \mathbb{A}.
\end{aligned} \tag{2.9}$$

*We say that* $X$ *is a* Banach two-sided $\mathbb{A}$-module *if the metric* $d : X \times X \longrightarrow [0,\infty[\ : (x,y) \longmapsto \|x-y\|$ *is complete.*

Let us recall the following result (cf. [28, Lemma 3.3]).

**Lemma 2.5.** *Assume* (2.1) *and* (2.7) *hold, and let* $X$ *be a normed two-sided* $\mathbb{A}$*-module. Then*

$$\|qx\| = \|xq\| = |q|\|x\| \qquad \forall x \in X, \quad \forall q \in Q_{\mathbb{A}}. \tag{2.10}$$

**Definition 2.6.** *Assume* (2.1) *holds and that* $X$ *is a two-sided* $\mathbb{A}$*-module. Let* $D(\mathsf{A})$ *be a right* $\mathbb{A}$*-submodule of* $X$*. We say that* $\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ *is* right linear *if it is additive and*

$$\mathsf{A}(xq) = \mathsf{A}(x)q \qquad \forall x \in D(\mathsf{A}), \quad \forall q \in \mathbb{A}.$$

*As usual, the notation* $\mathsf{A}x$ *is often used in place of* $\mathsf{A}(x)$. *We use the symbol* $\mathit{End}^{\mathrm{r}}(X)$ *to denote the set of right linear operators* $\mathsf{A}$ *with* $D(\mathsf{A}) = X$. *The identity operator is right linear and is denoted by* $\mathsf{Id}_X$ *or simply by* $\mathsf{Id}$. *Moreover, if* $X$ *is a normed two-sided* $\mathbb{A}$*-module, then we say that* $\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ *is* closed *if its graph is closed in* $X \times X$. *As in the classical theory, we set* $D(\mathsf{A}^n) := \{x \in D(\mathsf{A}^{n-1}) \ : \ \mathsf{A}^{n-1}x \in D(\mathsf{A})\}$ *for every* $n \in \mathbb{N} \setminus \{0\}$, *where* $\mathsf{A}^0 := \mathsf{Id}$.

Let us also recall the following definition (see, e.g., [3, Chapter 1, p. 55-57]).

**Definition 2.7.** *Let* $D(\mathsf{A})$ *be a right* $\mathbb{A}$*-submodule of* $X$ *and let* $q \in \mathbb{A}$. *If* $\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ *is a right linear operator, then we define the mapping* $q\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ *by setting*

$$(q\mathsf{A})(x) := q\mathsf{A}(x), \qquad x \in D(\mathsf{A}). \tag{2.11}$$

*If* $D(\mathsf{A})$ *is also a left* $\mathbb{A}$*-submodule of* $X$, *then we can define* $\mathsf{A}q : D(\mathsf{A}) \longrightarrow X$ *by setting*

$$(\mathsf{A}q)(x) := \mathsf{A}(qx), \qquad x \in D(\mathsf{A}). \tag{2.12}$$

*The sum of operators is defined in the usual way.*

It is easy to see that the operators defined in (2.11) and (2.12) are right linear.

**Definition 2.8.** *Assume* $X$ *is normed with* $\mathbb{A}$*-norm* $\|\cdot\|$. *For every* $\mathsf{B} \in \mathit{End}^{\mathrm{r}}(X)$, *we set*

$$\|\mathsf{B}\| := \sup_{x \neq 0} \frac{\|\mathsf{B}x\|}{\|x\|} \tag{2.13}$$

*and we define the set* $\mathscr{L}^{\mathrm{r}}(X) := \{\mathsf{B} \in \mathit{End}^{\mathrm{r}}(X) \ : \ \|\mathsf{B}\| < \infty\}$.

## 3. Main results in their general form

In order to state our main result in its general form we recall the noncommutative spectral notions given for the first time in [8] for quaternions and in [14] for arbitrary Clifford algebras $\mathbb{R}_n$. Here we consider the general case introduced in [27, Definition 2.26]. We will assume that

$\mathbb{A}$ *satisfies* (2.1), (2.3) *and* (2.7), *and* $X$ *is a Banach two-sided* $\mathbb{A}$*-module.*

**Definition 3.1.** *Let* $D(\mathsf{A})$ *be a right* $\mathbb{A}$*-submodule of* $X$ *and let* $\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ *be a closed right linear operator.*

(i) *Given* $q \in Q_{\mathbb{A}}$, *the right linear operator* $\Delta_q(\mathsf{A}) : D(\mathsf{A}^2) \longrightarrow X$ *is defined by*

$$\Delta_q(\mathsf{A}) := \mathsf{A}^2 - 2\,\mathrm{Re}(q)\,\mathsf{A} + |q|^2\,\mathsf{Id}, \qquad q \in Q_{\mathbb{A}}.$$

(ii) *The* spherical resolvent set $\rho_{\mathcal{S}}(\mathsf{A})$ of $\mathsf{A}$ *is defined by*

$$\rho_{\mathcal{S}}(\mathsf{A}) := \{q \in Q_{\mathbb{A}} \ : \ \Delta_q(\mathsf{A}) \textit{ is bijective, } \Delta_q(\mathsf{A})^{-1} \in \mathscr{L}^{\mathrm{r}}(X)\}$$

*and the* spherical spectrum $\sigma_{\mathcal{S}}(\mathsf{A})$ of $\mathsf{A}$ *by* $\sigma_{\mathcal{S}}(\mathsf{A}) := Q_{\mathbb{A}} \setminus \rho_{\mathcal{S}}(\mathsf{A})$.

(iii) *Given* $q \in \rho_{\mathcal{S}}(\mathsf{A})$, *the* quadratic resolvent *(or* spherical pseudoresolvent*) of* $\mathsf{A}$ *at* $q$ *is the operator* $\mathsf{Q}_q(\mathsf{A}) : X \longrightarrow X$ *defined by*

$$\mathsf{Q}_q(\mathsf{A}) := \Delta_q(\mathsf{A})^{-1}, \qquad q \in \rho_{\mathcal{S}}(\mathsf{A}).$$

(iv) *Given* $q \in \rho_{\mathcal{S}}(\mathsf{A})$, *the* spherical resolvent of $\mathsf{A}$ at $q$ *is the operator* $\mathsf{C}_q(\mathsf{A}) : X \longrightarrow X$ *defined by*

$$\mathsf{C}_q(\mathsf{A}) := \mathsf{Q}_q(\mathsf{A})q^c - \mathsf{A}\mathsf{Q}_q(\mathsf{A}), \qquad q \in \rho_{\mathcal{S}}(\mathsf{A}).$$

Let us observe that

$$\mathsf{Q}_q(\mathsf{A}) \in \mathscr{L}^r(X), \qquad \mathsf{C}_q(\mathsf{A}) \in \mathscr{L}^r(X) \qquad \forall q \in \rho_{\mathcal{S}}(\mathsf{A}). \tag{3.1}$$

Indeed, by definition, $\mathsf{Q}_q(\mathsf{A})$ is bounded and if we endow $(X, +)$ with the (left) real scalar multiplication $\mathbb{R} \times X \longrightarrow X : (r, x) \longmapsto rx = xr$, then thanks to (2.10) $X$ can be considered as a real Banach space and $\mathsf{A}$ is a closed $\mathbb{R}$-linear operator on it, thus the closed graph theorem implies that $\mathsf{A}\mathsf{Q}_q(\mathsf{A})$ is continuous and consequently $\mathsf{C}_q(\mathsf{A})$ is also continuous. Since all these operators are also $\mathbb{A}$-right linear we infer (3.1).

The introduction of the name "quadratic resolvent" for $\mathsf{Q}_q(\mathsf{A})$, which we slightly prefer, is due to the fact that in the classical complex literature the term "pseudoresolvent" is already used for a different class of operators (cf., e.g., [36, Section 1.9]). We also mention that a definition that has some similarities with the spherical spectrum was given in [34] in the context of real $^*$-algebras.

We now recall the natural definition of right linear operator semigroup (cf. [13] for the quaternionic case and [27] for the general case).

**Definition 3.2.** *A mapping* $\mathsf{S} : [0, \infty[ \longrightarrow \mathscr{L}^r(X)$ *is called* strongly continuous *if the* $t \longmapsto \mathsf{S}(t)x$ *is continuous from* $[0, \infty[$ *into* $X$ *for every* $x \in X$.

**Definition 3.3.** *A mapping* $\mathsf{T} : [0, \infty[ \longrightarrow \mathscr{L}^r(X)$ *is called* right linear strongly continuous (operator) semigroup *if* $\mathsf{T}$ *is strongly continuous and if*

$$\begin{aligned} &\mathsf{T}(t + s) = \mathsf{T}(t)\mathsf{T}(s) \qquad \forall t, s > 0, \\ &\mathsf{T}(0) = \mathsf{Id}. \end{aligned}$$

*The* generator of $\mathsf{T}$ *is the right linear operator* $\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ *defined by*

$$\begin{aligned} D(\mathsf{A}) &:= \left\{ x \in X \ : \ \exists \lim_{h \to 0} \frac{1}{h}(\mathsf{T}(h)x - x) \in X \right\}, \\ \mathsf{A}x &:= \lim_{h \to 0} \frac{1}{h}(\mathsf{T}(h)x - x), \qquad x \in D(\mathsf{A}). \end{aligned}$$

For the classical theory of semigroups in the complex framework we refer, e.g., to [32, 16, 36, 30, 35, 37, 19].

The next result includes Theorem 1.1.

**Theorem 3.4.** *Let* $m \in \mathbb{N}$, *let* $P(\mathrm{x}) = \sum_{k=0}^{m+2} \mathrm{x}^k a_k \in \mathbb{R}[\mathrm{x}]$ *such that* $a_{m+2} \neq 0$ *and let* $g_P \in C^\infty([0, \infty[\,; \mathbb{R})$ *be the unique solution of* (1.1). *Let* $\mathsf{T} : [0, \infty[ \longrightarrow \mathscr{L}^r(X)$ *be a strongly continuous right linear semigroup, let* $\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ *be its generator and let* $\omega \in \mathbb{R}$ *be a real constant such that* $M := \sup_{t \in [0,\infty[} \|\mathsf{T}(t)\| e^{-\omega t} < \infty$. *Then, if* $r_P > \omega$, *the operator* $P(\mathsf{A}) = \sum_{k=0}^{m+2} \mathsf{A}^k a_k$ *is bijective,* $P(\mathsf{A})^{-1} \in \mathscr{L}^r(X)$ *and it holds:*

(a) $P(\mathsf{A})^{-1} = \mathsf{L}(g_P)$, *that is,*

$$P(\mathsf{A})^{-1}x = \int_0^\infty \mathsf{T}(t) g_P(t) x \,\mathrm{d}t \qquad \forall x \in X. \tag{3.2}$$

(b) *Given* $(p_0, p_1, \ldots, p_{m+1}) \in \mathbb{A}^{m+2}$, *we have*

$$\sum_{j=0}^{m+1} \mathsf{A}^k P(\mathsf{A})^{-1} p_j x = \mathsf{L}\left( \sum_{j=0}^{m+1} g_P^{(j)} (-1)^j p_j \right) x \qquad \forall x \in X. \tag{3.3}$$

*Moreover,*

$$\|P(\mathsf{A})^{-1}\| \leq M \sum_{j=1}^{h} \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(r_P - \omega)^k} \tag{3.4}$$

*where $c_{-k}^{(j)}$ is the residue at $\lambda_j$ of the complex rational function $a_{m+2}(z-\lambda_j)^{k-1}/P(-z)$.*

Furthermore we have

**Theorem 3.5.** *The statement of Theorem 1.2 holds true replacing $\mathbb{R}_n$ with $\mathbb{A}$.*

## 4. Proofs

Let us start with a lemma on strongly continuous mapping. The symbol $C([0,\infty[\,;\mathbb{A})$ denotes the space of continuous functions from $[0,\infty[$ to $\mathbb{A}$, both endowed with the topology induced by the Euclidean distance.

**Lemma 4.1.** *If $\mathsf{T} : [0,\infty[ \longrightarrow \mathscr{L}^r(X)$ is a strongly continuous and $g \in C([0,\infty[\,;\mathbb{A})$, then the following statements hold true.*

(a) *The function $t \longmapsto \|\mathsf{T}(t)\||g(t)|$ is Lebesgue measurable on $[0,\infty[$.*
(b) *For every $x \in X$ the function $t \longmapsto \mathsf{T}(t)g(t)x$ is continuous from $[0,\infty[$ into $X$.*

*Proof.* Since $\mathsf{T} : [0,\infty[ \longrightarrow \mathscr{L}^r(X)$ is strongly continuous, by the Banach-Steinhaus theorem it follows that $\|\mathsf{T}(t)\| \leq \liminf_{\tau\to t} \|\mathsf{T}(\tau)\|$ for every $t \geq 0$, i.e. $t \longmapsto \|\mathsf{T}(t)\|$ is lower semicontinuous, and hence it is Lebesgue measurable. Thus (a) is proved. In order to prove (b) fix an arbitrary $t_0 \geq 0$. Since $\mathsf{T}$ is strongly continuous, by the uniform boundedness principle there exists $C > 0$ such that for every $t \geq 0$ with $|t - t_0| < 1$ we have $\|T(t)\| \leq C$ and

$$\begin{aligned} \|\mathsf{T}(t)g(t)x - \mathsf{T}(t_0)g(t_0)x\| &\leq \|\mathsf{T}(t)g(t)x - \mathsf{T}(t)g(t_0)x\| + \|\mathsf{T}(t)g(t_0)x - \mathsf{T}(t_0)g(t_0)x\| \\ &\leq C|g(t) - g(t_0)|\|x\| + \|\mathsf{T}(t)g(t_0)x - \mathsf{T}(t_0)g(t_0)x\|. \end{aligned}$$

Thus the continuity of $t \longmapsto \mathsf{T}(t)g(t)x$ at $t_0$ follows from the continuity of $g$ and from the strong continuity of $\mathsf{T}$. □

If $\mathsf{T} : [0,\infty[ \longrightarrow \mathscr{L}^r(X)$ is a strongly continuous right linear semigroup, $t \geq 0$, $g \in C([0,\infty[\,;\mathbb{A})$, and $x \in X$, then Lemma 4.1 and estimate $\|\mathsf{T}(t)g(t)x\| \leq \|\mathsf{T}(t)\||g(t)|\|x\|$ allow to give the following definition.

**Definition 4.2.** *Let $\mathsf{T} : [0,\infty[ \longrightarrow \mathscr{L}^r(X)$ be a strongly continuous right linear semigroup. We denote by $L_{\mathsf{T}}([0,\infty[\,;\mathbb{A})$ the real vector space of all continuous functions $g : [0,\infty[ \longrightarrow \mathbb{A}$ such that the function $t \longmapsto \|\mathsf{T}(t)\||g(t)|$ belongs to $L^1([0,\infty[\,;\mathbb{R})$. For every $g \in L_{\mathsf{T}}([0,\infty[\,;\mathbb{A})$ we define the operator $\mathsf{L}(g) : X \longrightarrow X$ by setting*

$$\mathsf{L}(g)x := \int_0^\infty \mathsf{T}(t)g(t)x \,\mathrm{d}t, \qquad x \in X. \tag{4.1}$$

Notice that the assumptions implies that the integral in (4.1) is a convergent Lebesgue integral for functions with values in the Banach space $(X,+)$ endowed with the real scalar multiplication $\mathbb{R} \times X \longrightarrow X : (r,x) \longmapsto rx = xr$ (thanks to (2.10) $\|\cdot\|$ is a norm on this real vector space). The symbol $L^1(J;X)$ denotes the space of Lebesgue integrable functions from an interval $J \subseteq \mathbb{R}$ into this real Banach space.

In the remainder of the paper, for $g \in C([0,\infty[\,;\mathbb{A})$, the symbols $g'$ and $g''$ will denote the first and second derivative of $g$, respectively.

**Lemma 4.3.** *If* $\mathsf{T} : [0, \infty[ \longrightarrow \mathscr{L}^r(X)$ *is a strongly continuous right linear semigroup, then the following statements hold true.*

(a) *If* $g \in L_{\mathsf{T}}([0, \infty[\,; \mathbb{A})$ *then* $t \longmapsto \mathsf{T}(t)g(t-h)x$ *belongs to* $L^1([h, \infty[\,; X)$ *for every* $h > 0$ *and for every* $x \in X$*.*

(b) *If* $g, g' \in L_{\mathsf{T}}([0, \infty[\,; \mathbb{A})$ *then*

$$\lim_{h \to 0} \left\| \int_h^\infty \mathsf{T}(t) \frac{g(t-h) - g(t)}{h} x \, \mathrm{d}t + \mathsf{L}(g')x \right\| = 0 \qquad \forall x \in X.$$

*Proof.* The claim (a) follows trivially by the estimate $\|\mathsf{T}(t)g(t-h)x\| \le \|\mathsf{T}(h)\| \|\mathsf{T}(t-h)\| |g(t-h)| \|x\|$ holding for every $x \in X$, $h > 0$, and $t > h$. In order to prove (b) fix $x \in X$ and an arbitrary $\varepsilon > 0$, and let $T > 0$ be such that $\int_T^\infty \|\mathsf{T}(t)\| |g(t)| \, \mathrm{d}t < \varepsilon$. Since $\mathsf{T}$ is a strongly continuous semigroup, there exists $M \ge 1$ such that $\|\mathsf{T}(s)\| \le M$ whenever $0 \le s \le 1$, therefore for every $h \in \,]0, 1[$ and every $t > h$ we have

$$\begin{aligned} \left\| \mathsf{T}(t) \frac{g(t-h) - g(t)}{h} x \right\| &= \left\| \int_0^1 \mathsf{T}(\xi h) \mathsf{T}(t - \xi h) g'(t - \xi h) x \, \mathrm{d}\xi \right\| \\ &\le M \|x\| \int_0^1 \|\mathsf{T}(t - \xi h)\| |g'(t - \xi h)| \, \mathrm{d}\xi \end{aligned}$$

hence it follows that

$$\begin{aligned} \left\| \int_T^\infty \mathsf{T}(t) \frac{g(t-h) - g(t)}{h} x \, \mathrm{d}t \right\| &\le M\|x\| \int_T^\infty \int_0^1 \|\mathsf{T}(t - \xi h)\| |g'(t - \xi h)| \, \mathrm{d}\xi \, \mathrm{d}t \\ &= M\|x\| \int_0^1 \int_T^\infty \|\mathsf{T}(t - \xi h)\| |g'(t - \xi h)| \, \mathrm{d}t \, \mathrm{d}\xi \\ &= M\|x\| \int_0^1 \int_{T - \xi h}^\infty \|\mathsf{T}(\tau)\| |g'(\tau)| \, \mathrm{d}\tau \, \mathrm{d}\xi \\ &\le M\|x\| \int_0^1 \int_T^\infty \|\mathsf{T}(\tau)\| |g'(\tau)| \, \mathrm{d}\tau \, \mathrm{d}\xi \\ &\le M\|x\| \int_T^\infty \|\mathsf{T}(\tau)\| |g'(\tau)| \, \mathrm{d}\tau \le M\|x\|\varepsilon. \end{aligned} \tag{4.2}$$

Moreover it is easily found a $\delta \in \,]0, 1[$ such that for every $h \in \,]0, \delta[$ we have

$$\begin{aligned} &\left\| \int_h^T \mathsf{T}(t) \frac{g(t-h) - g(t)}{h} x \, \mathrm{d}t + \int_0^T \mathsf{T}(t) g'(t) x \, \mathrm{d}t \right\| \\ &\le \left\| \int_h^T \mathsf{T}(t) \left( \frac{g(t-h) - g(t)}{h} + g'(t) \right) x \, \mathrm{d}t \right\| + \left\| \int_0^h \mathsf{T}(t) g'(t) x \, \mathrm{d}t \right\| \le \varepsilon. \end{aligned} \tag{4.3}$$

Hence assertion (b) follows from (4.2)–(4.3) and from the following estimate

$$\begin{aligned} &\left\| \int_h^\infty \mathsf{T}(t) \frac{g(t-h) - g(t)}{h} x \, \mathrm{d}t + \mathsf{L}(g') x \, \mathrm{d}t \right\| \\ &\le \left\| \int_h^T \mathsf{T}(t) \frac{g(t-h) - g(t)}{h} x \, \mathrm{d}t + \int_0^T \mathsf{T}(t) g'(t) x \, \mathrm{d}t \right\| \\ &\quad + \left\| \int_T^\infty \mathsf{T}(t) \frac{g(t-h) - g(t)}{h} x \, \mathrm{d}t \right\| + \left\| \int_T^\infty \mathsf{T}(t) g'(t) x \, \mathrm{d}t \right\|. \end{aligned}$$

□

The next lemma plays a key role in the proof of Theorem 3.5.

**Lemma 4.4.** *Let* $\mathsf{T} : [0,\infty[ \longrightarrow \mathscr{L}^r(X)$ *be a strongly continuous right linear semigroup and for every* $g \in L_{\mathsf{T}}([0,\infty[\,;\mathbb{A})$ *let* $\mathsf{L}(g) : X \longrightarrow X$ *be defined by* (4.1)*. Then* $\mathsf{L}(g) \in \mathscr{L}^r(X)$ *for every* $g \in L_{\mathsf{T}}([0,\infty[\,;\mathbb{A})$ *and the resulting mapping* $\mathsf{L} : L_{\mathsf{T}}([0,\infty[\,;\mathbb{A}) \longrightarrow \mathscr{L}^r(X)$ *is* $\mathbb{R}$*-linear. Moreover the following assertions hold.*

(a) *If* $g, g' \in L_{\mathsf{T}}([0,\infty[\,;\mathbb{A})$*, then*
$$\mathsf{L}(g)(X) \subseteq D(\mathsf{A}), \tag{4.4}$$
$$\mathsf{A}\mathsf{L}(g)x = -g(0)x - \mathsf{L}(g')x \qquad \forall x \in X. \tag{4.5}$$

(b) *If* $m \in \mathbb{N}$, $g, g', \ldots, g^{(m+2)} \in L_{\mathsf{T}}([0,\infty[\,;\mathbb{A})$*, and* $g(0) = g'(0) = \cdots = g^{(m)}(0) = 0$*, then*
$$\mathsf{L}(g)(X) \subseteq D(\mathsf{A}^{m+2}), \tag{4.6}$$
$$\mathsf{A}^{m+2}\mathsf{L}(g)x = (-1)^{m+2}\left(g^{(m+1)}(0)x + \mathsf{L}(g^{(m+2)})x\right) \qquad \forall x \in X, \tag{4.7}$$
$$\mathsf{A}^k\mathsf{L}(g)x = (-1)^k\mathsf{L}(g^{(k)}) \qquad \forall x \in X,\ \forall k \in \{0,\ldots,m+1\}. \tag{4.8}$$

(c) *If* $g, g', g'' \in L_{\mathsf{T}}([0,\infty[\,;\mathbb{A})$*,* $g(0) = 0$*, and* $g'(0) = 1$*, then*
$$\Delta_q(\mathsf{A})\mathsf{L}(g)x = x + \mathsf{L}(g'' + 2\operatorname{Re}(q)g' + |q|^2 g)x \qquad \forall x \in X,\ \forall q \in Q_{\mathbb{A}}. \tag{4.9}$$

(d) *If* $g, g' \in L_{\mathsf{T}}([0,\infty[\,;\mathbb{A})$ *and* $g$ *is real-valued, then*
$$\mathsf{A}\mathsf{L}(g)x = \mathsf{L}(g)\mathsf{A}x \qquad \forall x \in D(\mathsf{A}). \tag{4.10}$$

(e) *If* $m \in \mathbb{N}$, $g, g', \ldots, g^{(m+2)} \in L_{\mathsf{T}}([0,\infty[\,;\mathbb{A})$*,* $g(0) = g'(0) = \cdots = g^{(m)}(0) = 0$*, and* $g$ *is real valued, then*
$$\mathsf{A}^k\mathsf{L}(g)x = \mathsf{L}(g)\mathsf{A}^k x \qquad \forall x \in D(\mathsf{A}^k),\ \forall k \in \{1,\ldots,m+2\}. \tag{4.11}$$

*Proof.* For every $g \in L_{\mathsf{T}}([0,\infty[\,;\mathbb{A})$ the right linearity of $\mathsf{L}(g)$ follows from the right linearity of $\mathsf{T}(t)$ and from the definition of the $X$-valued Lebesgue integral. For every $x \in X$ we have
$$\|\mathsf{L}(g)x\| \le \|x\| \int_0^\infty \|\mathsf{T}(t)\||g(t)|\,\mathrm{d}t,$$
hence $\mathsf{L}(g)$ is also continuous and $\|\mathsf{L}(g)\| \le \int_0^\infty \|\mathsf{T}(t)\||g(t)|\,\mathrm{d}t$. The real linearity of $\mathsf{L}$ is straightforward. Now in the following list of items we prove the assertions from (a) to (e).
(a) For every $h > 0$ and for every $x \in X$ we have
$$\begin{aligned}\frac{\mathsf{T}(h) - \mathsf{Id}}{h}\mathsf{L}(g)x &= \frac{1}{h}\int_0^\infty \mathsf{T}(t+h)g(t)x\,\mathrm{d}t - \frac{1}{h}\int_0^\infty \mathsf{T}(t)g(t)x\,\mathrm{d}t\\ &= \frac{1}{h}\int_h^\infty \mathsf{T}(t)g(t-h)x\,\mathrm{d}t - \frac{1}{h}\int_0^\infty \mathsf{T}(t)g(t)x\,\mathrm{d}t\\ &= -\frac{1}{h}\int_0^h \mathsf{T}(t)g(t)x\,\mathrm{d}t + \int_h^\infty \mathsf{T}(t)\left(\frac{g(t-h)-g(t)}{h}\right)x\,\mathrm{d}t.\end{aligned}$$
Hence, taking the limit as $h \to 0$, thanks to Lemma 4.1 we find that
$$\mathsf{A}\mathsf{L}(g)x = -\mathsf{T}(0)g(0)x - \int_0^\infty \mathsf{T}(t)g'(t)x\,\mathrm{d}t = -g(0)x - \mathsf{L}(g')x.$$

(b) We proceed by induction on $m \in \{-1\} \cup \mathbb{N}$. The case $m = -1$ follows from (a). Let us assume that the result is true for $m - 1$, and we prove it for $m$. Therefore if $g$ satisfies the assumptions, in particular we have $\mathsf{L}(g)x \in D(\mathsf{A}^{m+1})$ and $\mathsf{A}^{m+1}\mathsf{L}(g)x = (-1)^{m+1}(g^{(m)}(0)x + \mathsf{L}(g^{(m+1)})x$ for every $x \in X$. But $g^{(m)}(0) = 0$ hence $\mathsf{A}^{m+1}\mathsf{L}(g)x = (-1)^{m+1}\mathsf{L}(g^{(m+1)})x$ and $\mathsf{L}(g^{(m+1)})x \in D(\mathsf{A})$ by virtue of an application of (4.4) with $g$ replaced by $g^{(m+1)}$. Thus $\mathsf{A}^{m+1}\mathsf{L}(g)x \in D(\mathsf{A})$ and (4.6) follows. Using again the validity of the statement for $m - 1$ and the identity $g^{(m)}(0) = 0$ we have

$$\begin{aligned}\mathsf{A}^{m+2}\mathsf{L}(g)x = \mathsf{A}\mathsf{A}^{m+1}\mathsf{L}(g)x &= (-1)^{m+1}\mathsf{A}\mathsf{L}(g^{(m+1)})x \\ &= (-1)^{m}(g^{(m+1)}(0)x + \mathsf{L}(g^{(m+2)}))x,\end{aligned}$$

where in the last equality we have used (4.5) with $g$ replaced by $g^{(m+1)}$. Therefore (4.7) is proved. Formula (4.8) is trivial for $k = 0$ and follows from (a) for $k = 1$, while for $2 \le k \le m + 1$ follows from (4.7) which we have already proved.

(c) Now fix $x \in X$ and $q \in Q_{\mathbb{A}}$. From (b) we obtain $\mathsf{A}^2\mathsf{L}(g)x = x + \mathsf{L}(g'')x$, hence, exploiting again (4.5) and the $\mathbb{R}$-linearity of $\mathsf{L}$, we obtain

$$\begin{aligned}\Delta_q(\mathsf{A})\mathsf{L}(g)x &= \mathsf{A}^2\mathsf{L}(g)x - 2\operatorname{Re}(q)\mathsf{A}\mathsf{L}(g)x + |q|^2\mathsf{L}(g)x \\ &= x + \mathsf{L}(g'')x - 2\operatorname{Re}(q)(-\mathsf{L}(g')x) + |q|^2\mathsf{L}(g)x \\ &= x + \mathsf{L}(g'' + 2\operatorname{Re}(q)g' + |q|^2 g)x.\end{aligned}$$

(d) If $x \in D(\mathsf{A})$, then for every $h > 0$ and for every $t > 0$ we have $\mathsf{T}(t)g(t)\mathsf{T}(h)x = \mathsf{T}(t)\mathsf{T}(h)g(t)x = \mathsf{T}(t+h)g(t)x = \mathsf{T}(h)\mathsf{T}(t)g(t)x$, because $g$ is real-valued and $\mathsf{T}$ is a semigroup. Therefore

$$\begin{aligned}\mathsf{L}(g)\frac{\mathsf{T}(h) - \mathsf{Id}}{h}x &= \frac{1}{h}\int_0^\infty \mathsf{T}(t)g(t)\mathsf{T}(h)x\,\mathrm{d}t - \frac{1}{h}\int_0^\infty \mathsf{T}(t)g(t)x\,\mathrm{d}t \\ &= \frac{1}{h}\int_0^\infty \mathsf{T}(h)\mathsf{T}(t)g(t)x\,\mathrm{d}t - \frac{1}{h}\int_0^\infty \mathsf{T}(t)g(t)x\,\mathrm{d}t \\ &= \frac{\mathsf{T}(h) - \mathsf{Id}}{h}\mathsf{L}(g)x,\end{aligned}$$

and the assertion follows taking the limit as $h \to 0$ and invoking (4.4).

(e) By induction on $m \in \{-1\} \cup \mathbb{N}$. The case $m = -1$ is true by virtue of (d). Let us assume that the result is true for $m - 1$, and we prove it for $m$. Therefore if $g$ satisfies the assumptions, in particular we have that $\mathsf{A}^k\mathsf{L}(g) = \mathsf{L}(g)\mathsf{A}^k$ on $D(\mathsf{A}^k)$ for all $k \in \{1, \dots, m+1\}$. Hence if $x \in D(\mathsf{A}^{m+2}) \subset D(\mathsf{A}^{m+1})$ then $\mathsf{A}^{m+1}x \in D(\mathsf{A})$ and we have that

$$\mathsf{L}(g)\mathsf{A}^{m+2}x = \mathsf{L}(g)\mathsf{A}\mathsf{A}^{m+1}x = \mathsf{A}\mathsf{L}(g)\mathsf{A}^{m+1}x = \mathsf{A}\mathsf{A}^{m+1}\mathsf{L}(g)x = \mathsf{A}^{m+2}\mathsf{L}(g)x,$$

where in the second equality we have used again (d). $\square$

*Proof of Theorem 3.4.* Let us first recall that the existence of constant $\omega \in \mathbb{R}$ such that $M := \sup_{t \ge 0} \|\mathsf{T}(t)\|e^{-\omega t} < \infty$ is well know (cf. [27, Thm 4.5(b)]). From (1.2) it follows that $g_P \in C^{m+2}([0, \infty[\,; \mathbb{A})$ and the Leibniz formula for the higher derivatives of a product yields the existence of a polynomial $p(t, \lambda_1, \dots, \lambda_h)$ such that

$$\|\mathsf{T}(t)\||g_P^k(t)| \le Me^{\omega t}|p(t, \lambda_1, \dots, \lambda_h)|e^{-tr_P} = M|p(t, \lambda_1, \dots, \lambda_h)|e^{(\omega - r_P)t}$$

for all $t \geq 0$ and for all $k \in \{0, 1, \ldots, m+2\}$. Thus $g_P, g'_P, \ldots, g_P^{(m+2)} \in L_{\mathsf{T}}([0, \infty[\,; \mathbb{A})$ and we can apply part (b) of Lemma 4.4 taking into account of initial conditions for $g_P$. We infer that

$$\begin{aligned} P(\mathsf{A})\mathsf{L}(g_P)x &= \sum_{k=0}^{m+2} \mathsf{A}^k a_k \mathsf{L}(g_P)x \\ &= \sum_{k=0}^{m+1} (-1)^k \mathsf{L}(g_P^{(k)}) a_k x + (-1)^{m+2} \left( g_P^{(m+1)}(0) + \mathsf{L}(g_P^{(m+2)}) \right) a_{m+2} x \\ &= \mathsf{L}\left( \sum_{k=0}^{m+2} g_P^{(k)} (-1)^k a_k \right) x + (-1)^{m+1} x = x g_P^{(m+1)}(0)(-1)^m a_{m+2} = x. \end{aligned}$$

On the other hand since $g_P$ is real-valued we can also apply parts (d) and (e) of Lemma 4.4 and infer that $\mathsf{L}(g_P)P(\mathsf{A}) = P(\mathsf{A})\mathsf{L}(g_P)$ thus $\mathsf{L}(g_P) = P(\mathsf{A})^{-1}$ and (3.2) is proved. Now take $(p_0, p_1, \ldots, p_{m+1}) \in \mathbb{A}^{m+2}$. Using Lemma 4.4(b) we deduce:

$$\sum_{j=0}^{m+1} \mathsf{A}^k P(\mathsf{A})^{-1} p_j x = \sum_{j=0}^{m+1} \mathsf{A}^k \mathsf{L}(g_P) p_j x = \sum_{j=0}^{m+1} (-1)^k \mathsf{L}(g_P^{(k)}) p_j x = \mathsf{L}\left( \sum_{j=0}^{m+1} (-1)^k g_P^{(k)} p_j \right) x$$

and (3.3) is proved. It is well known that

$$Q_j(t) = \sum_{k=1}^{m_j} \frac{c_{-k}^{(j)}}{(k-1)!} t^{k-1} \,,$$

where $c_{-k}^{(j)}$ is the coefficient of $(z - \lambda_j)^{-k}$ within the partial frations decomposition of $a_{m+2}/P(-z)$, i.e. the residue at $\lambda_j$ of the function $a_{m+2}(z-\lambda_j)^{k-1}/P(-z)$. Therefore

$$|g_P(t)| \leq \sum_{j=1}^{h} \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(k-1)!} t^{k-1} e^{-r_P t} \qquad \forall t \geq 0$$

and

$$\begin{aligned} \|P(\mathsf{A})^{-1}\| &\leq \int_0^\infty \|\mathsf{T}(t)\| \sum_{j=1}^{h} \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(k-1)!} t^{k-1} e^{-r_P t} \, \mathrm{d}t \\ &\leq M \sum_{j=1}^{h} \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(k-1)!} \int_0^\infty t^{k-1} e^{-(r_P - \omega)t} \, \mathrm{d}t \\ &= M \sum_{j=1}^{h} \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(r_P - \omega)^k} \frac{1}{(k-1)!} \int_0^\infty t^{k-1} e^{-t} \, \mathrm{d}t \\ &= M \sum_{j=1}^{h} \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(r_P - \omega)^k} \end{aligned}$$

and the theorem is completely proved. □

Now we address the case when $P(\mathrm{x}) = \mathrm{x}^2 - 2\,\mathrm{Re}(q) + |q|^2$ for some $q \in Q_{\mathbb{A}}$ which is related to part (c) of Lemma 4.4. We first present a simple lemma whose proof is a trivial calculus exercise.

**Lemma 4.5.** *For every fixed $q \in Q_{\mathbb{A}}$, the unique solution of the Cauchy problem*
$$g'' + 2\operatorname{Re}(q)g' + |q|^2 g = 0, \quad g(0) = 0, \quad g'(0) = 1. \tag{4.12}$$
*is the function $g_q : \mathbb{R} \longrightarrow \mathbb{R}$ defined by*
$$g_q(t) := te^{-\operatorname{Re}(q)t}\operatorname{sinc}(t|\operatorname{Im}(q)|), \qquad t \in \mathbb{R}, \tag{4.13}$$
*where we recall that* $\operatorname{sinc} : \mathbb{R} \longrightarrow \mathbb{R}$ *is the* unnormalized sinc function, *i.e. the only continuous real function $\xi$ on $\mathbb{R}$ such that $\xi(r) = (\sin r)/r$ for all $r \neq 0$. Moreover we have that $g_q, g_q', g_q'' \in L_{\mathsf{T}}([0,\infty[\,;\mathbb{A})$.*

Now we are in position to find the integral representations of the quadratic resolvent and of the spherical resolvent operators as a simple consequence of our main Theorem 3.4.

*Proof of Theorem 3.5.* In order to prove (1.8) it is enough to apply part (a) of Theorem 3.4 with $P(\mathrm{x}) = \mathrm{x}^2 - 2\operatorname{Re}(q) + |q|^2$ taking into account of Lemma 4.5. Formula (1.9) is a straighforward application of part (b) of Theorem 3.4 with $p_0 = 0$, $p_1 = 1$ and $p_2 = 0$. Instead (1.10) is obtained taking in part (b) of Theorem 3.4 $p_0 = q^c$, $p_1 = -1$, and $p_2 = 0$. Finally
$$\|\mathsf{Q}_q(\mathsf{A})\| \le \int_0^\infty \|\mathsf{T}(t)\||g_p(t)|\,\mathrm{d}t \le \int_0^\infty Mte^{(\omega-\operatorname{Re}(q))t}\,\mathrm{d}t = \frac{M}{(\operatorname{Re}(q)-\omega)^2},$$
i.e. (1.11) holds. Finally estimate (1.12) can be obtained in a similar way. □

**Remark 4.6.** By exploiting (4.13) of our Lemma 4.5 we can write the integral representation (1.8) in a more explicit way for every $q \in Q_{\mathbb{A}}$ such that $\operatorname{Re}(q) > \omega$:
$$\mathsf{Q}_q(\mathsf{A})x = -\int_0^\infty \mathsf{T}(t)\frac{e^{-\operatorname{Re}(q)t}\sin(|\operatorname{Im}(q)|t)}{|\operatorname{Im}(q)|}x\,\mathrm{d}t \qquad \forall x \in X,\ q \notin \mathbb{R}, \tag{4.14}$$
$$\mathsf{Q}_q(\mathsf{A})x = -\int_0^\infty \mathsf{T}(t)te^{-qt}x\,\mathrm{d}t \qquad \forall x \in X,\ q \in \mathbb{R}. \tag{4.15}$$

The following lemma will connect the integral representation of $\mathsf{Q}_q(\mathsf{A})$ to the so called *spherical derivative* of $q \longmapsto e^{-tq}$ (cf. [25]).

**Lemma 4.7.** *For every $t \in \mathbb{R}$ let $\exp^t : Q_{\mathbb{A}} \longrightarrow \mathbb{A}$ be the function defined by*
$$\exp^t(q) := \sum_{n=0}^\infty \frac{t^n}{n!}q^n = \sum_{n=0}^\infty \frac{(tq)^n}{n!}, \qquad q \in Q_{\mathbb{A}}. \tag{4.16}$$
*If $(\exp^t)_s' : Q_{\mathbb{A}} \setminus \mathbb{R} \longrightarrow \mathbb{A}$ denotes the function defined by*
$$(\exp^t)_s'(q) := (q-q^c)^{-1}(\exp^t(q) - \exp^t(q^c)), \qquad q \in Q_{\mathbb{A}} \setminus \mathbb{R}, \tag{4.17}$$
*which is also called* spherical derivative *of $\exp^t$, then $(\exp^t)_s'$ extends to a unique continuous function on $Q_{\mathbb{A}}$, which we still denote by $(\exp^t)_s' : Q_{\mathbb{A}} \longrightarrow \mathbb{A}$, and we have*
$$(\exp^t)_s'(q) = e^{t\operatorname{Re}(q)}\sum_{n=0}^\infty \frac{t^{2n+1}\operatorname{Im}(q)^{2n}}{(2n+1)!} \in \mathbb{R} \qquad \forall q \in Q_{\mathbb{A}} \setminus \mathbb{R} \tag{4.18}$$
*and $(\exp^t)_s'(q) = te^{t\operatorname{Re}(q)}$ for every $q \in \mathbb{R}$. In particular $(\exp^t)_s'$ is a real-valued. By abuse of notation, we write $\exp_s'(t,q)$ to indicate the element $(\exp^t)_s'(q)$ of $\mathbb{A}$ for every $t \in \mathbb{R}$ and for every $q \in Q_{\mathbb{A}}$, respectively.*

*Proof.* For every $q \in Q_{\mathbb{A}} \smallsetminus \mathbb{R}$ there exists $\mathbf{j} \in \mathbb{S}_{\mathbb{A}}$ and $a, b \in \mathbb{R}$ such that $b > 0$ and $q = a + b\mathbf{j}$. Hence $q^c = a^c - \mathbf{j}^c b^c = a - b\mathbf{j}$, $\mathrm{Re}(q) = a$, $\mathrm{Im}(q) = b\mathbf{j}$, and $|\mathrm{Im}(q)| = \sqrt{(b\mathbf{j})(b\mathbf{j})^c} = b$. Since $\mathbb{C}_{\mathbf{j}}$ and $\mathbb{C}$ are isomorphic real algebras, we find that $\exp^t(q) = e^{tq} = e^{ta}(\cos(tb) + \sin(tb)\mathbf{j})$ and $\exp^t(q^c) = e^{tq^c} = e^{ta}(\cos(tb) - \sin(tb)\mathbf{j}) = (e^{tq})^c$, therefore

$$(\exp^t)'_s(q) = (q - q^c)^{-1}(e^{tq} - e^{tq^c}) = e^{t\,\mathrm{Re}(q)} \sin(t|\mathrm{Im}(q)|)|\mathrm{Im}(q)|^{-1} \qquad \forall q \in Q_{\mathbb{A}} \smallsetminus \mathbb{R}, \quad (4.19)$$

which proves the first equality in (4.18). The right-hand side of (4.19) immediately extends by continuity to $te^{ta}$ for $q \in \mathbb{R}$, thus $(\exp^t)'_s$ is a real-valued function. As $|\mathrm{Im}(q)|^2 = -\mathrm{Im}(q)^2$, by (4.19) we have

$$(\exp^t)'_s(q) = e^{t\,\mathrm{Re}(q)} \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1} |\mathrm{Im}(q)|^{2n}}{(2n+1)!} = e^{t\,\mathrm{Re}(q)} \sum_{n=0}^{\infty} \frac{t^{2n+1}\,\mathrm{Im}(q)^{2n}}{(2n+1)!} \qquad \forall q \in Q_{\mathbb{A}} \smallsetminus \mathbb{R},$$

which implies the second equality of (4.18). □

**Corollary 4.8.** *Under the assumption of Theorem 3.5, for every $q \in Q_{\mathbb{A}}$ such that $\mathrm{Re}(q) > \omega$ we have*

$$\mathsf{Q}_q(\mathsf{A})x = -\int_0^{\infty} \mathsf{T}(t) \exp'_s(-t, q)\, x \,\mathrm{d}t \qquad \forall x \in X. \quad (4.20)$$

## 5. Integral representation of the powers of $\mathsf{Q}_q(\mathsf{A})$

In this section we look for an integral representation of the integer powers of the quadratic resolvent operator $\mathsf{Q}_q(\mathsf{A})$. In order to find this representation we need the following simple lemma.

**Lemma 5.1.** *If $f, g \in L_{\mathsf{T}}([0, \infty[\,; \mathbb{A})$ and $f$ is real-valued, then*

$$\mathsf{L}(f)\mathsf{L}(g)x = \mathsf{L}(f \star g)x = \mathsf{L}(g \star f)x \qquad \forall x \in X,$$

*where we recall that $f \star g : [0, \infty[ \longrightarrow \mathbb{A}$, the convolution of $f$ and $g$, is defined by $(f \star g)(t) := \int_0^t f(t - s)g(s)\,\mathrm{d}s$.*

*Proof.* Using the fact that $f$ is real-valued, we see at once that $f \star g = g \star f$. In addition, bearing in mind the semigroup law for $\mathsf{T}$, we find

$$\begin{aligned}
\mathsf{L}(f)\mathsf{L}(g)x &= \int_0^{\infty} \mathsf{T}(t)f(t) \int_0^{\infty} \mathsf{T}(s)g(s)x \;\mathrm{d}s\,\mathrm{d}t \\
&= \int_0^{\infty}\int_0^{\infty} \mathsf{T}(t)\mathsf{T}(s)f(t)g(s)x \;\mathrm{d}s\,\mathrm{d}t \\
&= \int_0^{\infty}\int_0^{\infty} \mathsf{T}(t+s)f(t)g(s)x \;\mathrm{d}s\,\mathrm{d}t\,.
\end{aligned}$$

Thus a change of variable and an application of Fubini theorem yields

$$\begin{aligned}
\mathsf{L}(f)\mathsf{L}(g)x &= \int_0^\infty \int_t^\infty \mathsf{T}(s)f(t)g(s-t)x \,\mathrm{d}s\,\mathrm{d}t \\
&= \int_0^\infty \int_0^\infty \chi_{[0,s]}(t)\mathsf{T}(s)f(t)g(s-t)x \,\mathrm{d}s\,\mathrm{d}t \\
&= \int_0^\infty \int_0^\infty \chi_{[0,s]}(t)\mathsf{T}(s)f(t)g(s-t)x \,\mathrm{d}t\,\mathrm{d}s \\
&= \int_0^\infty \int_0^t \mathsf{T}(s)f(t)g(s-t)x \,\mathrm{d}t\,\mathrm{d}s \\
&= \int_0^\infty \mathsf{T}(s) \int_0^t f(t)g(s-t)x \,\mathrm{d}t\,\mathrm{d}s \\
&= \mathsf{L}(f\star g) = \mathsf{L}(g\star f).
\end{aligned}$$

The proof is complete. □

Given $n \in \mathbb{N} \setminus \{0\}$ and $f \in L_{\mathsf{T}}([0,\infty[\,;\mathbb{A})$, we define $f^{\star n} \in L_{\mathsf{T}}([0,\infty[\,;\mathbb{A})$ by

$$f^{\star n} := \underbrace{f \star f \star \cdots \star f}_{n \text{ times}}.$$

**Corollary 5.2.** *Let $\mathsf{T} : [0,\infty[ \longrightarrow \mathscr{L}^r(X)$ be a strongly continuous right linear semigroup, let $\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ be its generator, and let $\omega \in \mathbb{R}$ be a real constant such that $M := \sup_{t\in[0,\infty[} \|\mathsf{T}(t)\|e^{-\omega t} < \infty$. Given any $q \in Q_{\mathbb{A}}$ with $\mathrm{Re}(q) > \omega$, we have that $q \in \rho_S(\mathsf{A})$ and*

$$\mathsf{Q}_q(\mathsf{A})^n x = (-1)^n \int_0^\infty \mathsf{T}(t) \exp'_s(-t,q)^{\star n} x \,\mathrm{d}t \tag{5.1}$$

*where $\exp'_s(-t,q)^{\star n} \in \mathbb{A}$ indicates the value of $((\exp^{-t})'_s)^{\star n}$ at $q$.*

*Moreover for every $q \in Q_{\mathbb{A}}$ with $\mathrm{Re}(q) > \omega$ we have*

$$\|\mathsf{Q}_q(\mathsf{A})^n\| \le \frac{M}{(\mathrm{Re}(q)-\omega)^{2n}} \qquad \forall n \in \mathbb{N}\setminus\{0\}. \tag{5.2}$$

*Proof.* Formula (5.1) follows immediately from $n$ applications of Theorem 3.5 and Lemma 5.1. In order to prove (5.2) let us observe that, given $q \in Q_{\mathbb{A}}$, if $a = \mathrm{Re}(q)$ and $b = |\,\mathrm{Im}(q)|$, and $g(t) = -\exp'_s(-t,q)$ for $t \ge 0$, then

$$\begin{aligned}
|(g\star g)(t)| &\le \int_0^t |g(t-s)||g(s)| \,\mathrm{d}s \\
&\le \int_0^t (t-s)e^{-(t-s)a} s e^{-sa} \,\mathrm{d}s \\
&= e^{-ta} \int_0^t s(t-s)\,\mathrm{d}s = \frac{1}{2\cdot 3} t^3 e^{-ta}.
\end{aligned}$$

Le us assume by induction that $|g^{\star(n-1)}| \leq ((2n-3)!)^{-1}t^{2n-3}e^{-at}$. Therefore for every $n$

$$\begin{aligned}
|g^{\star n}(t)| &\leq \int_0^t |g(t-s)||g^{\star(n-1)}(s)|\,\mathrm{d}s \\
&\leq \frac{1}{(2n-3)!}\int_0^t (t-s)e^{-(t-s)a}s^{2n-3}e^{-as}\,\mathrm{d}s \\
&= \frac{e^{-ta}}{(2n-3)!}\int_0^t (t-s)s^{2n-3}\,\mathrm{d}s \\
&= \frac{e^{-ta}}{(2n-3)!}\frac{t^{2n-1}}{(2n-2)(2n-1)} = \frac{t^{2n-1}e^{-ta}}{(2n-1)!}.
\end{aligned}$$

Thus $|g^{\star n}(t)| \leq \frac{1}{(2n-1)!}t^{2n-1}e^{-ta}$ for every $t \geq 0$ and every $n \in \mathbb{N}\setminus\{0\}$ and, recalling that $\int_0^\infty t^{2n-1}e^{-t}\,\mathrm{d}t = (2n-1)!$, we have

$$\begin{aligned}
\|\mathsf{Q}_q(\mathsf{A})^n\| &\leq \int_0^\infty \|\mathsf{T}(t)\||g^{\star n}(t)|\,\mathrm{d}t \\
&\leq \int_0^\infty Me^{\omega t}\frac{1}{(2n-1)!}t^{2n-1}e^{-ta}\,\mathrm{d}t \\
&= \frac{M}{(2n-1)!}\int_0^\infty t^{2n-1}e^{-(a-\omega)t}\,\mathrm{d}t \\
&= \frac{M}{(2n-1)!}\int_0^\infty \frac{t^{2n-1}}{(a-\omega)^{2n}}e^{-t}\,\mathrm{d}t \\
&= \frac{M}{(a-\omega)^{2n}},
\end{aligned}$$

and we are done. □

**Acknowledgments.** The idea of writing this work about a relationship between Semigroups and Clifford Algebras came to our mind when the first author participated to the *International conference on Hypercomplex Analysis in Mathematics and in the Applied Sciences, Celebrating the scientific life of Klaus Gürlebeck.*

## References

[1] S. Adler, "Quaternionic Quantum Field Theory", Oxford University Press, 1995.

[2] D. Alpay, F. Colombo, D.P. Kimsey, *The spectral theorem for quaternionic unbounded normal operators based on the S-spectrum*, J. Math. Phys. **57**, 023503, 27 pp. (2016).

[3] F.W. Anderson, K.R. Fuller, "Rings and Categories of Modules", Springer Verlag, New York, 1974.

[4] G. Birkhoff, J. von Neumann, *The logic of quantum mechanics*, Ann. of Math. (2), **37** (1936), 823–843.

[5] F. Colombo, J. Gantner, D.P. Kimsey, "Spectral theory on the S-spectrum for quaternionic operators". Operator Theory: Advances and Applications. Birkhäuser/Springer, 2018.

[6] F. Colombo, J. Gantner, "Quaternionic closed operators, fractional powers and fractional diffusion processes". Operator Theory: Advances and Applications. Birkhäuser/Springer, 2019.

[7] F. Colombo, G. Gentili, I. Sabadini, *A Cauchy kernel for slice regular functions*, Ann. Glob. Anal. Geom., **37** (2010), 361–378.

[8] F. Colombo, G. Gentili, I. Sabadini, D.C. Struppa, *A functional calculus in a non commutative setting*. Electron. Res. Announc. Math. Sci., **14** (2007), 60–68.

[9] F. Colombo, G. Gentili, I. Sabadini, D.C. Struppa, *Non Commutative Functional Calculus: Bounded Operators*, Complex Anal. Oper. Theory., **4** (2010), 821–843.

[10] F. Colombo, G. Gentili, I. Sabadini, D.C. Struppa, *Non-commutative functional calculus: Unbounded operators*, J. Geom. Phys., **60** (2010), 251–259.

[11] F. Colombo, I. Sabadini, *On some properties of the quaternionic functional calculus*, J. Geom. Anal., **19** (2009), 601–627.

[12] F. Colombo, I. Sabadini, *On the formulation of the quaternionic functional calculus*, J. Geom. Phys., **60** (2009), 1490–1508.

[13] F. Colombo, I. Sabadini, *The quaternionic evolution operator*, Adv. Math., **227** (2011), 1772–1805.

[14] F. Colombo, I. Sabadini, D.C. Struppa, *A new functional calculus for noncommuting operators*, J. Functional Analysis, **254** (2008), 2255–2274.

[15] F. Colombo, I. Sabadini, D.C. Struppa, "Noncommutative Functional Calculus",

[16] E.B. Davies, "One Parameter Semigroups", Academic Press, 1980.

[17] H.-D. Ebbinghaus, H. Hermes, F. Hirzebruch, M. Koecher, K. Mainzer, J. Neukirch, A. Prestel, R. Remmert, "Numbers", Grad. Texts in Math., vol. 123, Springer-Verlag, New York, 1990.

[18] G. Emch, *Mécanique quantique quaternionienne et relativité restreinte*, Elv. Phys. Acta, **36** (1963), 739–769.

[19] K.-J. Engel, R. Nagel, "One-Parameter Semigroups for Linear Evolution Equations", Springer Verlag, New York, 2000.

[20] D. Finkelstein, J.M. Jauch, S. Schiminovich, D. Speiser, *Foundations of quaternionic quantum mechanics*, J. Mathematical Phys., **3** (1962), 207–220.

[21] G. Gentili, C. Stoppato, *Power series and analyticity over the quaternions*, Math. Ann., **352** (2012), 113-131.

[22] G. Gentili, D.C. Struppa, *A new theory of regular functions of a quaternionic variable*, Adv. Math., **216** (2007), 279–301.

[23] R. Ghiloni, V. Moretti, A. Perotti, *Continuous slice functional calculus in quaternionic Hilbert spaces*, Rev. Math. Phys. **25** (2013), no. 4, 1330006, 83 pp.

[24] R. Ghiloni, V. Moretti, A. Perotti, *Spectral representations of normal operators in quaternionic Hilbert spaces via intertwining quaternionic PVMs*, arXiv:1602.02661v1

[25] R. Ghiloni, A. Perotti, *Slice regular functions on real alternative algebras*, Adv. Math., **226** (2011), 1662–1691.

[26] R. Ghiloni, A. Perotti, V. Recupero, *Noncommutative Cachy integral formula*, Complex Anal. Oper. Theory **11** (2017), 289-306.

[27] R. Ghiloni, V. Recupero, *Semigroups over real alternative *-algebras: Generation theorems and spherical sectorial operators*, Trans. Amer. Math. Soc., **368** (2016), no. 4, 2645–2678.

[28] R. Ghiloni, V. Recupero, *Slice regular semigroups*, Trans. Amer. Math. Soc., **370** (2018), no. 7, 4993–5032.

[29] J.E. Gilbert, M.A.M Murray, "Clifford algebras and Dirac operators in harmonic analysis", Cambridge University Press, Cambridge, 1991.

[30] J.A. Goldstein, "Semigroups of Operators and Applications", Oxford University Press, 1985.

[31] K. Gürlebeck, K. Habetha, W. Sprößig, "Holomorphic functions in the plane and $n$-dimensional space", Birkhäuser Verlag, Basel, xiv+394, 2008.

[32] E. Hille, R.S. Phillips, "Functional Analysis and Semigroups", Amer. Math. Soc. Coll. Publ., vol. 31, Amer. Math. Soc., 1957.

[33] L. P. Horwitz, L.C. Biedenharn, *Quaternionic quantum mechanics: Second quantization and gauge field*, Annals of Physics, **157** (1984), 432–488.

[34] I. Kaplansky, *Normed algebras*, Duke Mat. J., **16** (1949), 399–418.

[35] A. Lunardi, "Analytic Semigroups and Optimal Regularity in Parabolic Problems", Birkhauser-Verlag, 1995.

[36] A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations", Springer-Verlag, New York, 1983.

[37] K. Taira, "Analytic Semigroups and Semilinear Initial Boundary Value Problems", London Math. Soc. Lect. Notes Ser., vol. 223, Cambridge University Press, 1995.

**Riccardo Ghiloni**, Dipartimento di Matematica, Università di Trento, Via Sommarive 14, 38123 Trento, Italy.
E-mail address: riccardo.ghiloni@unitn.it

**Vincenzo Recupero**, Dipartimento di Scienze Matematiche, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy.
E-mail address: vincenzo.recupero@polito.it