ON THE GENERATORS OF CLIFFORD SEMIGROUPS: POLYNOMIAL RESOLVENTS AND THEIR INTEGRAL TRANSFORMS

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Abstract. This paper deals with generators A of strongly continuous right linear semigroups in Banach two-sided spaces whose set of scalars is an arbitrary Clifford algebra $C\ell(0,n)$. We study the invertibility of operators of the form $P(\mathsf{A})$, where $P(\mathrm{x}) \in \mathbb{R}[\mathrm{x}]$ is any real polynomial, and we give an integral representation for $P(\mathsf{A})^{-1}$ by means of a Laplace-type transform of the semigroup $\mathsf{T}(t)$ generated by A. In particular, we deduce a new integral representation for the spherical quadratic resolvent of A (also called pseudoresolvent of A). As an immediate consequence, we also obtain a new proof of the well-known integral representation for the spherical resolvent of A.

dedicated to Professor Klaus Gürlebeck

1. Introduction and main results

Quaternionic functional analysis has probably its original motivation in the seminal paper [4], where it is pointed out that quantum mechanics may be formulated, not only on complex Hilbert spaces, but also on Hilbert spaces whose set of scalars is $\mathbb{H}$, the noncommutative real algebra of quaternions.

Many papers have been devoted to the development of quantum mechanics in the quaternionic framework (see, e.g., [20, 18, 33, 1]), whose natural setting is a Hilbert two-sided $\mathbb{H}$-module X, with the space of bounded linear operators acting on it replaced by the set $\mathscr{L}^r(X)$ of bounded *right* linear operators. However, a full development of quaternionic quantum mechanics was prevented by the lack of suitable quaternionic spectral notions, indeed, as observed in [9, 23], the classical definitions of spectrum and resolvent operator do not allow to define a noncommutative functional calculus.

A first rigorous formulation of a quaternionic spectral theory has been provided only in [8] where one can find the first definition of the notions of *spherical resolvent set* $\rho_{\mathcal{S}}(\mathsf{A})$, *quadratic resolvent operator* $\mathsf{Q}_q(\mathsf{A})$, *spherical resolvent operator* $\mathsf{C}_q(\mathsf{A})$ and *spherical spectrum* $\sigma_{\mathcal{S}}(\mathsf{A})$ of a right linear operator A on a quaternionic Banach space X. They are

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2010 *Mathematics Subject Classification.* 30G35, 47D03, 47A60, 47A10.

Key words and phrases. Semigroups in the noncommutative setting, Slice regular semigroups; Spectrum, resolvent; Laplace transform, Functional calculus, Quaternions, Clifford algebras; Functions of hypercomplex variables.

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given by

$$\rho_{\mathcal{S}}(\mathsf{A}) := \{q \in \mathbb{H} \ : \ \exists (\mathsf{A}^2 - 2\,\mathrm{Re}(q)\,\mathsf{A} + |q|^2)^{-1} \in \mathscr{L}^r(X)\},$$
$$\mathsf{Q}_q(\mathsf{A}) := (\mathsf{A}^2 - 2\,\mathrm{Re}(q)\,\mathsf{A} + |q|^2)^{-1}, \qquad q \in \rho_{\mathcal{S}}(\mathsf{A}),$$
$$\mathsf{C}_q(\mathsf{A}) := \mathsf{Q}_q(\mathsf{A})\overline{q} - \mathsf{A}\mathsf{Q}_q(\mathsf{A}), \qquad q \in \rho_{\mathcal{S}}(\mathsf{A}).$$

In [8] the name *pseudoresolvent* is used for $\mathsf{Q}_q(\mathsf{A})$ but we choose the term *quadratic resolvent* since in the classical complex case the name "pseudoresolvent" has a different meaning. The above definitions permit to develop a noncommutative functional calculus for right linear operators on a Banach two-sided module over $\mathbb{H}$ (and over a Clifford algebra as well, cf. [14, 11, 12, 9, 10, 15, 23]) and to deduce in [2, 24] the spectral representation theorems for normal operators in the quaternionic Hilbert setting (cf. [2, 24]). A wider bibliography can be found in the recent accounts on the theory [5, 6].

The mentioned noncommutative functional calculus is intimately connected to the theory of slice regular functions, introduced in [22], which extends to quaternions the classical concept of holomorphic function. They form a class of functions admitting a local power series expansion at every point of their domain of definition (cf. [21]), including polynomials with quaternionic coefficients on one side, and they admit a Cauchy-type integral representation formula with a suitable quaternionic version of the kernel proved for the first time in [7] (see also [26]).

The next natural stage in this analysis is the development of a noncommutative theory of right linear operator semigroups which was developed in [13, 27, 28]. In the classical complex theory a fundamental tool is provided by the integral representation of the resolvent operator of a generator A by means of the Laplace transform of the semigroup $\mathsf{T}(t)$ generated by A. An analogous integral representation in the quaternionic case for the spherical resolvent operator $\mathsf{C}_q(\mathsf{Q})$ is shown in [13] and a proof is provided in [28] using techniques from slice regular function theory.

The purpose of the present paper is to study the invertibility of operators of the form $P(\mathsf{A})$ where $P(\mathrm{x})$ is an arbitrary polynomial with real coefficients of degree at least 2, including $P(\mathrm{x}) = \Delta_q(\mathrm{x}) = \mathrm{x}^2 - 2\,\mathrm{Re}(q)\mathrm{x} + |q|^2$. Under natural conditions, we prove that $P(\mathsf{A})^{-1}$ exists and belongs to $\mathscr{L}^r(X)$. Furthermore, we provide an integral representation for $P(\mathsf{A})^{-1}$ by means of a Laplace-type transform of the semigroup $\mathsf{T}(t)$ generated by A. We extend this integral representation to operators of the form $\sum_{j=0}^{d-1} \mathsf{A}^j P(\mathsf{A})^{-1} p_j$, where d is the degree of P and $p_0, \ldots, p_{d-1}$ are arbitrarily chosen quaternions. In the case $P(\mathrm{x}) = \Delta_q(\mathrm{x})$, we obtain a new integral representation for $\mathsf{Q}_q(\mathsf{A})$ and, setting $p_0 := \overline{q}$ and $p_1 := -1$, we discover again the well-known integral representation for $\mathsf{C}_q(\mathsf{A})$ via the quaternionic Laplace transform. This gives also a new proof of the integral representation for $\mathsf{C}_q(\mathsf{A})$, which avoids the use of slice regular function techniques. Our results are valid not only on quaternions but also on a class of real associative *-algebras including, as the main examples, all Clifford algebras $C\ell(0, n)$.

Let $n \in \mathbb{N}$, let $\mathbb{R}_n$ be the Clifford algebra $C\ell(0, n)$ equipped with the Clifford conjugation and the Clifford operator norm $|\cdot|$. Consider a Banach two-sided $\mathbb{R}_n$-module X with norm $\|\cdot\|$ and the set $\mathscr{L}^r(X)$ of all bounded right linear operators on X (all the precise definitions will be recalled in the next section).

Let $m \in \mathbb{N}$ and let $P(\mathrm{x}) = \sum_{k=0}^{m+2} \mathrm{x}^k a_k \in \mathbb{R}[\mathrm{x}]$ be a polynomial with real coefficients in the indeterminate x. Suppose P has degree $m + 2$, that is, $a_{m+2} \neq 0$. Given a right linear operator $\mathsf{A} : D(\mathsf{A}) \longrightarrow X$, we define the right linear operator $P(\mathsf{A}) : D(\mathsf{A}^{m+2}) \longrightarrow X$ simply by replacing x with A, that is, $P(\mathsf{A}) := \sum_{k=0}^{m+2} \mathsf{A}^k a_k$. Denote by $C^\infty([0,\infty[\,;\mathbb{R})$ the set of all infinitely many times differentiable functions $g : [0,\infty[\longrightarrow \mathbb{R}$. Consider the following ODE with constant coefficients in the variable $g \in C^\infty([0,\infty[\,;\mathbb{R})$:

$$\begin{cases} P\left(-\frac{d}{dt}\right)(g) = 0 \ \text{ on } [0,\infty[\,, \\ g(0) = g'(0) = \ldots = g^{(m)}(0) = 0\,, \\ g^{(m+1)}(0) = (-1)^m (a_{m+2})^{-1}\,, \end{cases} \tag{1.1}$$

where $g^{(k)}$ is the k^{th}-derivative of g and $P\left(-\frac{d}{dt}\right)(g) := \sum_{k=0}^{m+2} g^{(k)}(-1)^k a_k$. Denote by $g_P \in C^\infty([0,\infty[\,;\mathbb{R})$ the unique solution of (1.1). Recall that, if $\lambda_1, \ldots, \lambda_h$ are the complex roots of the polynomial $P(\mathrm{x})$ with multiplicity $m_1, \ldots, m_h$, then there exist complex polynomials $Q_1(\mathrm{x}), \ldots, Q_h(\mathrm{x}) \in \mathbb{C}[\mathrm{x}]$ such that the degree of each $Q_j(\mathrm{x})$ is $< m_j$ and

$$g_P(t) = \sum_{j=1}^{h} Q_j(t) e^{-\lambda_j t}. \tag{1.2}$$

Define $r_P \in \mathbb{R}$ by

$$r_P := \min\{\Re(\lambda_1), \ldots, \Re(\lambda_h)\}, \tag{1.3}$$

where $\Re(\lambda_j)$ is the real part of the complex number λ_j.

Recall also that if $\mathsf{T} : [0,\infty[\longrightarrow \mathscr{L}^r(X)$ is a strongly continuous right linear semigroup then there exists $\omega \in \mathbb{R}$ such that $\sup_{t\in[0,\infty[} \|\mathsf{T}(t)\| e^{-\omega t} < \infty$, see [27, Thm 4.5(b)].

Our main result reads as follows.

Theorem 1.1. *Let $\mathsf{T} : [0,\infty[\longrightarrow \mathscr{L}^r(X)$ be a strongly continuous right linear semigroup, let $\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ be its generator and let $\omega \in \mathbb{R}$ be a real constant such that $M := \sup_{t\in[0,\infty[} \|\mathsf{T}(t)\| e^{-\omega t} < \infty$. Then, if $r_P > \omega$, the operator $P(\mathsf{A})$ is bijective, $P(\mathsf{A})^{-1} \in \mathscr{L}^r(X)$ and it holds:*

$$P(\mathsf{A})^{-1} x = \int_0^\infty \mathsf{T}(t) g_P(t) x \,\mathrm{d}t \qquad \forall x \in X \tag{1.4}$$

and

$$\|P(\mathsf{A})^{-1}\| \le M \sum_{j=1}^{h} \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(r_P - \omega)^k}\,, \tag{1.5}$$

where $c_{-k}^{(j)}$ is the residue at λ_j of the meromorphic function $z \mapsto a_{m+2}(z-\lambda_j)^{k-1}/P(-z)$.

Furthermore, given $(p_0, p_1, \ldots, p_{m+1}) \in (\mathbb{R}_n)^{m+2}$, we have

$$\sum_{j=0}^{m+1} \mathsf{A}^k P(\mathsf{A})^{-1} p_j x = \int_0^\infty \mathsf{T}(t) \left(\sum_{j=0}^{m+1} g_P^{(j)} (-1)^j p_j x \right) \mathrm{d}t \qquad \forall x \in X. \tag{1.6}$$

Let $Q_{\mathbb{R}_n}$ be the quadratic cone of $\mathbb{R}_n$ (see (2.5) for the definition) and let $q \in Q_{\mathbb{R}_n}$. Define the function $g_q : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$g_q(t) := t e^{-\operatorname{Re}(q) t} \operatorname{sinc}(t|\operatorname{Im}(q)|), \qquad t \in \mathbb{R}, \tag{1.7}$$

where $\operatorname{Re}(q)$ and $\operatorname{Im}(q)$ are the real and imaginary parts of q, respectively. We recall that $\operatorname{sinc} : \mathbb{R} \longrightarrow \mathbb{R}$ is the *unnormalized sinc function*, that is, the real-valued continuous function ξ on $\mathbb{R}$ defined by $\xi(0) = 1$ and $\xi(r) = \sin(r)/r$ for all $r \neq 0$.

Thanks to the preceding result, we are able to prove the following:

Theorem 1.2. *Let* $\mathsf{T} : [0, \infty[\longrightarrow \mathscr{L}^r(X)$ *be a strongly continuous right linear semigroup, let* $\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ *be its generator, and let* $\omega \in \mathbb{R}$ *be a real constant such that* $M := \sup_{t\in[0,\infty[} \|\mathsf{T}(t)\| e^{-\omega t} < \infty$. *Consider any* $q \in Q_{\mathbb{R}_n}$ *and set* $a := \mathrm{Re}(q)$ *and* $b := |\,\mathrm{Im}(q)|$. *If* $\mathrm{Re}(q) > \omega$, *then we have that* $q \in \rho_S(\mathsf{A})$ *and it holds:*

$$\mathsf{Q}_q(\mathsf{A})x = \int_0^\infty \mathsf{T}(t) g_q(t) x \,\mathrm{d}t = \int_0^\infty \mathsf{T}(t) t e^{-ta} \operatorname{sinc}(tb) x \,\mathrm{d}t, \tag{1.8}$$

$$\mathsf{A}\mathsf{Q}_q(\mathsf{A})x = -\int_0^\infty \mathsf{T}(t) g_q'(t) x \,\mathrm{d}t = -\int_0^\infty \mathsf{T}(t) e^{-ta}(\cos(tb) - at\operatorname{sinc}(tb)) x \,\mathrm{d}t, \tag{1.9}$$

$$\mathsf{C}_q(\mathsf{A})x = \int_0^\infty \mathsf{T}(t) e^{-tq} x \,\mathrm{d}t \tag{1.10}$$

for every $x \in X$. *Moreover, we have:*

$$\|\mathsf{Q}_q(\mathsf{A})\| \le \frac{M}{(\mathrm{Re}(q) - \omega)^2}\,, \tag{1.11}$$

$$\|\mathsf{C}_q(\mathsf{A})\| \le \frac{M}{\mathrm{Re}(q) - \omega}\,. \tag{1.12}$$

Here is the plan of the paper. In the following section we recall all the needed precise definitions. In Section 3 we present the preceding theorems in the more general case of certain real $*$-algebras, including all the $\mathbb{R}_n$'s. Section 4 is devoted to the proofs of these theorems. Finally, in Section 5 we apply the main theorem in order to derive an integral representation of the integer powers of the quadratic resolvent and the estimate of their norms; this extends to $\mathsf{Q}_q(\mathsf{A})$ our Theorem 6.6 in [28] concerning the Laplace-type transform for the integer slice powers of $\mathsf{C}_q(\mathsf{A})$.

2. Preliminaries

Throughout all the paper we will assume that $\mathbb{A}$ is a nontrivial finite dimensional $\mathbb{R}$-vector space endowed with a bilinear product $\mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{A} : (p, q) \longmapsto pq$ with unit $1_{\mathbb{A}}$, and with a mapping $\mathbb{A} \longrightarrow \mathbb{A} : q \longmapsto q^c$ called **-involution*, which is an $\mathbb{R}$-linear mapping such that $(q^c)^c = q$, $(pq)^c = q^c p^c$, and $r^c = r$ for all $p, q \in \mathbb{A}, r \in \mathbb{R} \subseteq \mathbb{A}$, where we are identifying $\mathbb{R}$ with the subalgebra of $\mathbb{A}$ generated by $1_{\mathbb{A}}$ by means of the algebra isomorphism $\mathbb{R} \longrightarrow \mathbb{R}1_{\mathbb{A}} : r \longmapsto r1_{\mathbb{A}}$. Therefore we can write $1 = 1_{\mathbb{A}}$ and we summarize the previous assumptions by saying that

$$\begin{gathered}\mathbb{A} \text{ is a finite dimensional associative nontrivial real } {}^*\text{-algebra} \\ \text{with } {}^*\text{-involution } q \longmapsto q^c \text{ and unit } 1.\end{gathered} \tag{2.1}$$

Under the previous assumptions we define the *imaginary sphere* $\mathbb{S}_{\mathbb{A}}$ in $\mathbb{A}$ by

$$\mathbb{S}_{\mathbb{A}} := \left\{ q \in \mathbb{A} \ : \ q^c = -q, \ q^2 = -1 \right\}. \tag{2.2}$$

In the remainder of the paper we will assume that

$$\mathbb{S}_{\mathbb{A}} \neq \varnothing. \tag{2.3}$$

Condition (2.3) in particular implies that $\mathbb{A}$ cannot be equal to $\mathbb{R}$. We set

$$\mathbb{C}_{\mathbf{j}} := \{r + s\mathbf{j} \in \mathbb{A} \ : \ r, s \in \mathbb{R}\}, \qquad \mathbf{j} \in \mathbb{S}_{\mathbb{A}}, \tag{2.4}$$

$$Q_{\mathbb{A}} := \bigcup_{\mathbf{j} \in \mathbb{S}_{\mathbb{A}}} \mathbb{C}_{\mathbf{j}}, \tag{2.5}$$

the set $Q_{\mathbb{A}}$ being called *quadratic cone of* $\mathbb{A}$. The *real part* $\mathrm{Re}(q)$ and the *imaginary part* $\mathrm{Im}(q)$ of an element $q \in \mathbb{A}$ are defined by

$$\mathrm{Re}(q) := (q + q^c)/2, \quad \mathrm{Im}(q) := (q - q^c)/2, \qquad q \in \mathbb{A}. \tag{2.6}$$

Notice that in general $\mathrm{Re}(q)$ and $\mathrm{Im}(q)$ are not real numbers, at variance with the customary complex notations $\Re(z) := (z + \overline{z})/2 \in \mathbb{R}$ and $\Im(z) := (z - \overline{z})/2i \in \mathbb{R}$ for $z \in \mathbb{C}$.

We finally observe that $qq^c \in \mathbb{R}$ for every $q \in Q_{\mathbb{A}}$ and we assume that

$$\begin{aligned} &\mathbb{A} \text{ is endowed with a complete norm } |\cdot| \text{ such that} \\ &|q_1 q_2| \leq |q_1||q_2| \text{ for every } q_1, q_2 \in \mathbb{A} \text{ and } |q|^2 = qq^c \text{ for every } q \in Q_{\mathbb{A}}. \end{aligned} \tag{2.7}$$

The equivalence of the above definitions with other presentations (e.g. [25] is provided in [28]). We recall here that

$$\mathbb{C}_{\mathbf{j}} \cap \mathbb{C}_{\mathbf{k}} = \mathbb{R} \qquad \forall \mathbf{j}, \mathbf{k} \in \mathbb{S}_{\mathbb{A}}, \ \mathbf{j} \neq \pm \mathbf{k}. \tag{2.8}$$

Example 2.1 (Clifford algebras)**.** For $n \in \mathbb{N} \setminus \{0\}$ let $\mathcal{P}(n)$ be the power set of $\{1, \ldots, n\}$. If we identify $\mathbb{R}$ with the vector subspace $\mathbb{R} \times \{0\}$ of $\mathbb{R}^{2^n} = \mathbb{R} \times \mathbb{R}^{2^n - 1}$ and we set $e_{\varnothing} := 1$, then we denote by $\{e_K\}_{K \in \mathcal{P}(n)}$ the canonical basis of $\mathbb{R}^{2^n}$. For convenience, we set $e_k := e_{\{k\}}$ if $k \in \{1, \ldots, n\}$ and we define a real bilinear and associative product on $\mathbb{R}^{2^n}$ by imposing that 1 is the neutral element and that

$$\begin{aligned} &e_k^2 = -1 \text{ and } e_k e_h = -e_h e_k \text{ if } k, h \in \{1, \ldots, n\} \text{ with } k \neq h, \\ &e_K = e_{k_1} \cdots e_{k_s} \text{ if } K = \{k_1, \ldots, k_s\} \in \mathcal{P}(n) \setminus \{\varnothing\} \text{ with } k_1 < \ldots < k_s. \end{aligned}$$

The *Clifford conjugation* of $\mathbb{R}^{2^n}$ is the *-involution $q \longmapsto q^c := \overline{q}$ defined by

$$\overline{q} := \textstyle\sum_{K \in \mathcal{P}(n)} (-1)^{|K|(|K|+1)/2} a_K e_k \quad \text{if } q = \sum_{K \in \mathcal{P}(n)} a_K e_k \in \mathbb{R}_n, \ a_K \in \mathbb{R},$$

where $|K|$ indicates the cardinality of the set K. Endowing $\mathbb{R}^{2^n}$ with the above defined product and with the Clifford conjugation, we obtain a real *-algebra $\mathbb{A}$ satisfying (2.1), called *Clifford algebra $C\ell(0,n)$ of signature* $(0,n)$, which is denoted also by $\mathbb{R}_n$. Observe that $\mathbb{R}_1$ and $\mathbb{R}_2$ are isomorphic to $\mathbb{C}$ and $\mathbb{H}$, respectively. Moreover $\mathbb{R}_n$ is not commutative if $n \geq 2$. If $n \geq 3$ then $\mathbb{R}_n$ has zero divisors, indeed $(1 - e_{\{1,2,3\}})(1 + e_{\{1,2,3\}}) = 0$. One verifies that a point $q = \sum_{K \in \mathcal{P}(n)} a_K e_K$ of $\mathbb{R}_n$ with $a_K \in \mathbb{R}$ belongs to the quadratic cone $Q_{\mathbb{R}_n}$ of $\mathbb{R}_n$ if and only if it satisfies the following conditions

$$a_K = 0 \quad \text{and} \quad \langle q, qe_K \rangle_{2^n} = 0 \quad \text{for every } K \in \mathcal{P}(n) \setminus \{\varnothing\} \text{ with } e_K^2 = 1,$$

where $\langle \cdot, \cdot \rangle_{2^n}$ denotes the standard scalar product on $\mathbb{R}^{2^n}$. On $\mathbb{R}_n$ it is defined the following submultiplicative norm, called *Clifford operator norm*: $|q|_{C\ell} := \sup\{|qa|_{2^n} \in \mathbb{R} \ : \ |a|_{2^n} = 1\}$, where $|\cdot|_{2^n}$ indicates the Euclidean norm of $\mathbb{R}^{2^n}$. It turns out that:

(a) $Q_{\mathbb{R}_n} = \mathbb{R}_n$ if and only if $n \in \{1, 2\}$. In particular, $\mathbb{R}_1$ and $\mathbb{R}_2$ are division algebras.
(b) $|q|_{C\ell} = |x| = \sqrt{x\overline{x}}$ for every $x \in Q_{\mathbb{R}_n}$ and hence $|\cdot|_{C\ell} = |\cdot|$ if $n \in \{1, 2\}$.

Notice that if $n \geq 3$ then the Euclidean norm $|\cdot|_{2^n}$ of $\mathbb{R}_n$ is not submultiplicative (e.g. $|(1+e_{\{1,2,3\}})^2| = \sqrt{8} > 2 = |1+e_{\{1,2,3\}}|^2$). Endowing $\mathbb{R}_n$ with Clifford conjugation and Clifford operator norm, we obtain a real *-algebra $\mathbb{A}$ satisfying (2.3) and (2.7). For further details we refer the reader to [29, 31].

Example 2.2 (Complex numbers and quaternions)**.** If $n \in \mathbb{N} \setminus \{0\}$ and $\mathbb{R}_n$ denotes the Clifford algebra of signature $(0,n)$ recalled in the previous Example 2.1, then we have:

(i) $\mathbb{R}_1 = \mathbb{C}$ with $e_1 = i$, where $z \longmapsto z^c = \bar{z}$ is the standard conjugation and $|\cdot|$ is the Euclidean norm;

(ii) $\mathbb{R}_2$ is the algebra of quaternions $\mathbb{H}$ with $i := e_1$, $j := e_2$, $k := e_3$, where $q = a + bi + cj + dk \longmapsto q^c = \bar{q} = a - bi - cj - dk$ and $|\cdot|$ is the euclidean norm.

Definition 2.3. *If $\mathbb{A}$ satisfies* (2.1) *then a* two-sided $\mathbb{A}$-module *is a commutative group $(X,+)$ endowed with a left scalar multiplication $\mathbb{A} \times X \longrightarrow X : (q,x) \longmapsto qx$ and a right scalar multiplication $X \times \mathbb{A} \longrightarrow X : (x,q) \longmapsto xq$ such that*

$$
\begin{array}{llll}
q(x+y) = qx+qy, & (x+y)q = xq+yq & \forall x,y \in X, & \forall q \in \mathbb{A}, \\
(p+q)x = px+qx, & x(p+q) = xp+xq & \forall x \in X, & \forall p,q \in \mathbb{A}, \\
1x = x = x1 & & \forall x \in X, & \\
p(qx) = (pq)x, & (xp)q = x(pq) & \forall x \in X, & \forall p,q \in \mathbb{A}, \\
p(xq) = (px)q & & \forall x \in X, & \forall p,q \in \mathbb{A}, \\
rx = xr & & \forall x \in X, & \forall r \in \mathbb{R}.
\end{array}
$$

If Y is a commutative subgroup of X then Y is called a left $\mathbb{A}$-submodule *if $qx \in Y$ whenever $x \in Y$ and $q \in \mathbb{A}$. Instead Y is called a* right $\mathbb{A}$-submodule *of X if $xq \in Y$ whenever $x \in Y$ and $q \in \mathbb{A}$. Finally Y is called a* two-sided $\mathbb{A}$-submodule *of X if it is both a left and a right $\mathbb{A}$-submodule of X.*

Definition 2.4. *Assume* (2.1) *and* (2.7) *hold. A two-sided $\mathbb{A}$-module X is called a* normed two-sided $\mathbb{A}$-module *if it is endowed with a* $\mathbb{A}$-norm on X*, that is, a function $\|\cdot\| : X \longrightarrow [0,\infty[$ such that*

$$
\begin{array}{lll}
\|x\| = 0 \iff x = 0, & & \\
\|x+y\| \leq \|x\| + \|y\| & \forall x,y \in X, & \\
\|qx\| \leq |q|\,\|x\|, \quad \|xq\| \leq |q|\,\|x\| & \forall x \in X, & \forall q \in \mathbb{A}.
\end{array}
\tag{2.9}
$$

We say that X is a Banach two-sided $\mathbb{A}$-module *if the metric $d : X \times X \longrightarrow [0,\infty[: (x,y) \longmapsto \|x-y\|$ is complete.*

Let us recall the following result (cf. [28, Lemma 3.3]).

Lemma 2.5. *Assume* (2.1) *and* (2.7) *hold, and let X be a normed two-sided $\mathbb{A}$-module. Then*

$$\|qx\| = \|xq\| = |q|\|x\| \qquad \forall x \in X, \quad \forall q \in Q_{\mathbb{A}}. \tag{2.10}$$

Definition 2.6. *Assume* (2.1) *holds and that X is a two-sided $\mathbb{A}$-module. Let $D(\mathsf{A})$ be a right $\mathbb{A}$-submodule of X. We say that $\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ is* right linear *if it is additive and*

$$\mathsf{A}(xq) = \mathsf{A}(x)q \qquad \forall x \in D(\mathsf{A}), \quad \forall q \in \mathbb{A}.$$

As usual, the notation $\mathsf{A}x$ *is often used in place of* $\mathsf{A}(x)$. *We use the symbol* $\mathit{End}^{\mathrm{r}}(X)$ *to denote the set of right linear operators* A *with* $D(\mathsf{A}) = X$. *The identity operator is right linear and is denoted by* Id_X *or simply by* Id. *Moreover, if* X *is a normed two-sided* $\mathbb{A}$*-module, then we say that* $\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ *is* closed *if its graph is closed in* $X \times X$. *As in the classical theory, we set* $D(\mathsf{A}^n) := \{x \in D(\mathsf{A}^{n-1}) \ : \ \mathsf{A}^{n-1}x \in D(\mathsf{A})\}$ *for every* $n \in \mathbb{N} \smallsetminus \{0\}$, *where* $\mathsf{A}^0 := \mathsf{Id}$.

Let us also recall the following definition (see, e.g., [3, Chapter 1, p. 55-57]).

Definition 2.7. *Let* $D(\mathsf{A})$ *be a right* $\mathbb{A}$*-submodule of* X *and let* $q \in \mathbb{A}$. *If* $\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ *is a right linear operator, then we define the mapping* $q\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ *by setting*

$$(q\mathsf{A})(x) := q\mathsf{A}(x), \qquad x \in D(\mathsf{A}). \tag{2.11}$$

If $D(\mathsf{A})$ *is also a left* $\mathbb{A}$*-submodule of* X, *then we can define* $\mathsf{A}q : D(\mathsf{A}) \longrightarrow X$ *by setting*

$$(\mathsf{A}q)(x) := \mathsf{A}(qx), \qquad x \in D(\mathsf{A}). \tag{2.12}$$

The sum of operators is defined in the usual way.

It is easy to see that the operators defined in (2.11) and (2.12) are right linear.

Definition 2.8. *Assume* X *is normed with* $\mathbb{A}$*-norm* $\|\cdot\|$. *For every* $\mathsf{B} \in \mathit{End}^{\mathrm{r}}(X)$, *we set*

$$\|\mathsf{B}\| := \sup_{x \neq 0} \frac{\|\mathsf{B}x\|}{\|x\|} \tag{2.13}$$

and we define the set $\mathscr{L}^{\mathrm{r}}(X) := \{\mathsf{B} \in \mathit{End}^{\mathrm{r}}(X) \ : \ \|\mathsf{B}\| < \infty\}$.

3. Main results in their general form

In order to state our main result in its general form we recall the noncommutative spectral notions given for the first time in [8] for quaternions and in [14] for arbitrary Clifford algebras $\mathbb{R}_n$. Here we consider the general case introduced in [27, Definition 2.26]. We will assume that

$\mathbb{A}$ *satisfies* (2.1), (2.3) *and* (2.7), *and* X *is a Banach two-sided* $\mathbb{A}$*-module.*

Definition 3.1. *Let* $D(\mathsf{A})$ *be a right* $\mathbb{A}$*-submodule of* X *and let* $\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ *be a closed right linear operator.*

(i) *Given* $q \in Q_{\mathbb{A}}$, *the right linear operator* $\Delta_q(\mathsf{A}) : D(\mathsf{A}^2) \longrightarrow X$ *is defined by*

$$\Delta_q(\mathsf{A}) := \mathsf{A}^2 - 2\,\mathrm{Re}(q)\,\mathsf{A} + |q|^2\,\mathsf{Id}, \qquad q \in Q_{\mathbb{A}}.$$

(ii) *The* spherical resolvent set $\rho_{\mathcal{S}}(\mathsf{A})$ of A *is defined by*

$$\rho_{\mathcal{S}}(\mathsf{A}) := \{q \in Q_{\mathbb{A}} \ : \ \Delta_q(\mathsf{A}) \text{ is bijective, } \Delta_q(\mathsf{A})^{-1} \in \mathscr{L}^{\mathrm{r}}(X)\}$$

and the spherical spectrum $\sigma_{\mathcal{S}}(\mathsf{A})$ of A *by* $\sigma_{\mathcal{S}}(\mathsf{A}) := Q_{\mathbb{A}} \smallsetminus \rho_{\mathcal{S}}(\mathsf{A})$.

(iii) *Given* $q \in \rho_{\mathcal{S}}(\mathsf{A})$, *the* quadratic resolvent *(or* spherical pseudoresolvent*) of* A *at* q *is the operator* $\mathsf{Q}_q(\mathsf{A}) : X \longrightarrow X$ *defined by*

$$\mathsf{Q}_q(\mathsf{A}) := \Delta_q(\mathsf{A})^{-1}, \qquad q \in \rho_{\mathcal{S}}(\mathsf{A}).$$

(iv) *Given* $q \in \rho_{\mathcal{S}}(\mathsf{A})$, *the* spherical resolvent of A at q *is the operator* $\mathsf{C}_q(\mathsf{A}) : X \longrightarrow X$ *defined by*

$$\mathsf{C}_q(\mathsf{A}) := \mathsf{Q}_q(\mathsf{A})q^c - \mathsf{A}\mathsf{Q}_q(\mathsf{A}), \qquad q \in \rho_{\mathcal{S}}(\mathsf{A}).$$

Let us observe that

$$\mathsf{Q}_q(\mathsf{A}) \in \mathscr{L}^{\mathrm{r}}(X), \qquad \mathsf{C}_q(\mathsf{A}) \in \mathscr{L}^{\mathrm{r}}(X) \qquad \forall q \in \rho_{\mathcal{S}}(\mathsf{A}). \tag{3.1}$$

Indeed, by definition, $\mathsf{Q}_q(\mathsf{A})$ is bounded and if we endow $(X, +)$ with the (left) real scalar multiplication $\mathbb{R} \times X \longrightarrow X : (r, x) \longmapsto rx = xr$, then thanks to (2.10) X can be considered as a real Banach space and A is a closed $\mathbb{R}$-linear operator on it, thus the closed graph theorem implies that $\mathsf{A}\mathsf{Q}_q(\mathsf{A})$ is continuous and consequently $\mathsf{C}_q(\mathsf{A})$ is also continuous. Since all these operators are also $\mathbb{A}$-right linear we infer (3.1).

The introduction of the name "quadratic resolvent" for $\mathsf{Q}_q(\mathsf{A})$, which we slightly prefer, is due to the fact that in the classical complex literature the term "pseudoresolvent" is already used for a different class of operators (cf., e.g., [36, Section 1.9]). We also mention that a definition that has some similarities with the spherical spectrum was given in [34] in the context of real *-algebras.

We now recall the natural definition of right linear operator semigroup (cf. [13] for the quaternionic case and [27] for the general case).

Definition 3.2. *A mapping* $\mathsf{S} : [0, \infty[\longrightarrow \mathscr{L}^{\mathrm{r}}(X)$ *is called* strongly continuous *if the* $t \longmapsto \mathsf{S}(t)x$ *is continuous from* $[0, \infty[$ *into* X *for every* $x \in X$.

Definition 3.3. *A mapping* $\mathsf{T} : [0, \infty[\longrightarrow \mathscr{L}^{\mathrm{r}}(X)$ *is called* right linear strongly continuous (operator) semigroup *if* T *is strongly continuous and if*

$$\begin{aligned} &\mathsf{T}(t+s) = \mathsf{T}(t)\mathsf{T}(s) \qquad \forall t, s > 0, \\ &\mathsf{T}(0) = \mathsf{Id}. \end{aligned}$$

The generator of T *is the right linear operator* $\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ *defined by*

$$\begin{aligned} &D(\mathsf{A}) := \left\{ x \in X \ : \ \exists \lim_{h \to 0} \frac{1}{h}(\mathsf{T}(h)x - x) \in X \right\}, \\ &\mathsf{A}x := \lim_{h \to 0} \frac{1}{h}(\mathsf{T}(h)x - x), \qquad x \in D(\mathsf{A}). \end{aligned}$$

For the classical theory of semigroups in the complex framework we refer, e.g., to [32, 16, 36, 30, 35, 37, 19].

The next result includes Theorem 1.1.

Theorem 3.4. *Let* $m \in \mathbb{N}$, *let* $P(\mathrm{x}) = \sum_{k=0}^{m+2} \mathrm{x}^k a_k \in \mathbb{R}[\mathrm{x}]$ *such that* $a_{m+2} \neq 0$ *and let* $g_P \in C^\infty([0, \infty[\,; \mathbb{R})$ *be the unique solution of* (1.1). *Let* $\mathsf{T} : [0, \infty[\longrightarrow \mathscr{L}^r(X)$ *be a strongly continuous right linear semigroup, let* $\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ *be its generator and let* $\omega \in \mathbb{R}$ *be a real constant such that* $M := \sup_{t \in [0,\infty[} \|\mathsf{T}(t)\| e^{-\omega t} < \infty$. *Then, if* $r_P > \omega$, *the operator* $P(\mathsf{A}) = \sum_{k=0}^{m+2} \mathsf{A}^k a_k$ *is bijective,* $P(\mathsf{A})^{-1} \in \mathscr{L}^r(X)$ *and it holds:*

(a) $P(\mathsf{A})^{-1} = \mathsf{L}(g_P)$, *that is,*

$$P(\mathsf{A})^{-1}x = \int_0^\infty \mathsf{T}(t) g_P(t) x \,\mathrm{d}t \qquad \forall x \in X. \tag{3.2}$$

(b) *Given* $(p_0, p_1, \ldots, p_{m+1}) \in \mathbb{A}^{m+2}$, *we have*

$$\sum_{j=0}^{m+1} \mathsf{A}^k P(\mathsf{A})^{-1} p_j x = \mathsf{L}\left(\sum_{j=0}^{m+1} g_P^{(j)} (-1)^j p_j \right) x \qquad \forall x \in X. \tag{3.3}$$

Moreover,

$$\|P(\mathsf{A})^{-1}\| \leq M \sum_{j=1}^{h} \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(r_P - \omega)^k} \tag{3.4}$$

where $c_{-k}^{(j)}$ *is the residue at* λ_j *of the complex rational function* $a_{m+2}(z-\lambda_j)^{k-1}/P(-z)$.

Furthermore we have

Theorem 3.5. *The statement of Theorem 1.2 holds true replacing* $\mathbb{R}_n$ *with* $\mathbb{A}$.

4. Proofs

Let us start with a lemma on strongly continuous mapping. The symbol $C([0,\infty[\,;\mathbb{A})$ denotes the space of continuous functions from $[0,\infty[$ to $\mathbb{A}$, both endowed with the topology induced by the Euclidean distance.

Lemma 4.1. *If* $\mathsf{T} : [0,\infty[\longrightarrow \mathscr{L}^r(X)$ *is a strongly continuous and* $g \in C([0,\infty[\,;\mathbb{A})$, *then the following statements hold true.*

(a) *The function* $t \longmapsto \|\mathsf{T}(t)\| |g(t)|$ *is Lebesgue measurable on* $[0,\infty[$.
(b) *For every* $x \in X$ *the function* $t \longmapsto \mathsf{T}(t)g(t)x$ *is continuous from* $[0,\infty[$ *into* X.

Proof. Since $\mathsf{T} : [0,\infty[\longrightarrow \mathscr{L}^r(X)$ is strongly continuous, by the Banach-Steinhaus theorem it follows that $\|\mathsf{T}(t)\| \leq \liminf_{\tau\to t} \|\mathsf{T}(\tau)\|$ for every $t \geq 0$, i.e. $t \longmapsto \|\mathsf{T}(t)\|$ is lower semicontinuous, and hence it is Lebesgue measurable. Thus (a) is proved. In order to prove (b) fix an arbitrary $t_0 \geq 0$. Since T is strongly continuous, by the uniform boundedness principle there exists $C > 0$ such that for every $t \geq 0$ with $|t - t_0| < 1$ we have $\|T(t)\| \leq C$ and

$$\begin{aligned}
\|\mathsf{T}(t)g(t)x - \mathsf{T}(t_0)g(t_0)x\| &\leq \|\mathsf{T}(t)g(t)x - \mathsf{T}(t)g(t_0)x\| + \|\mathsf{T}(t)g(t_0)x - \mathsf{T}(t_0)g(t_0)x\| \\
&\leq C|g(t) - g(t_0)|\|x\| + \|\mathsf{T}(t)g(t_0)x - \mathsf{T}(t_0)g(t_0)x\|.
\end{aligned}$$

Thus the continuity of $t \longmapsto \mathsf{T}(t)g(t)x$ at t_0 follows from the continuity of g and from the strong continuity of T. □

If $\mathsf{T} : [0,\infty[\longrightarrow \mathscr{L}^r(X)$ is a strongly continuous right linear semigroup, $t \geq 0$, $g \in C([0,\infty[\,;\mathbb{A})$, and $x \in X$, then Lemma 4.1 and estimate $\|\mathsf{T}(t)g(t)x\| \leq \|\mathsf{T}(t)\||g(t)|\|x\|$ allow to give the following definition.

Definition 4.2. *Let* $\mathsf{T} : [0,\infty[\longrightarrow \mathscr{L}^r(X)$ *be a strongly continuous right linear semigroup. We denote by* $L_{\mathsf{T}}([0,\infty[\,;\mathbb{A})$ *the real vector space of all continuous functions* $g : [0,\infty[\longrightarrow \mathbb{A}$ *such that the function* $t \longmapsto \|\mathsf{T}(t)\||g(t)|$ *belongs to* $L^1([0,\infty[\,;\mathbb{R})$. *For every* $g \in L_{\mathsf{T}}([0,\infty[\,;\mathbb{A})$ *we define the operator* $\mathsf{L}(g) : X \longrightarrow X$ *by setting*

$$\mathsf{L}(g)x := \int_0^\infty \mathsf{T}(t)g(t)x \,\mathrm{d}t, \qquad x \in X. \tag{4.1}$$

Notice that the assumptions implies that the integral in (4.1) is a convergent Lebesgue integral for functions with values in the Banach space $(X,+)$ endowed with the real scalar multiplication $\mathbb{R} \times X \longrightarrow X : (r,x) \longmapsto rx = xr$ (thanks to (2.10) $\|\cdot\|$ is a norm on this real vector space). The symbol $L^1(J;X)$ denotes the space of Lebesgue integrable functions from an interval $J \subseteq \mathbb{R}$ into this real Banach space.

In the remainder of the paper, for $g \in C([0,\infty[\,;\mathbb{A})$, the symbols g' and g'' will denote the first and second derivative of g, respectively.

Lemma 4.3. *If $\mathsf{T} : [0, \infty[\longrightarrow \mathscr{L}^r(X)$ is a strongly continuous right linear semigroup, then the following statements hold true.*

(a) *If $g \in L_{\mathsf{T}}([0, \infty[\,; \mathbb{A})$ then $t \longmapsto \mathsf{T}(t)g(t-h)x$ belongs to $L^1([h, \infty[\,; X)$ for every $h > 0$ and for every $x \in X$.*

(b) *If $g, g' \in L_{\mathsf{T}}([0, \infty[\,; \mathbb{A})$ then*

$$\lim_{h \to 0} \left\| \int_h^\infty \mathsf{T}(t) \frac{g(t-h) - g(t)}{h} x \, \mathrm{d}t + \mathsf{L}(g')x \right\| = 0 \qquad \forall x \in X.$$

Proof. The claim (a) follows trivially by the estimate $\|\mathsf{T}(t)g(t-h)x\| \le \|\mathsf{T}(h)\| \|\mathsf{T}(t-h)\| |g(t-h)| \|x\|$ holding for every $x \in X$, $h > 0$, and $t > h$. In order to prove (b) fix $x \in X$ and an arbitrary $\varepsilon > 0$, and let $T > 0$ be such that $\int_T^\infty \|\mathsf{T}(t)\| |g(t)| \, \mathrm{d}t < \varepsilon$. Since T is a strongly continuous semigroup, there exists $M \ge 1$ such that $\|\mathsf{T}(s)\| \le M$ whenever $0 \le s \le 1$, therefore for every $h \in\,]0, 1[$ and every $t > h$ we have

$$\begin{aligned}
\left\| \mathsf{T}(t) \frac{g(t-h) - g(t)}{h} x \right\| &= \left\| \int_0^1 \mathsf{T}(\xi h) \mathsf{T}(t - \xi h) g'(t - \xi h) x \, \mathrm{d}\xi \right\| \\
&\le M \|x\| \int_0^1 \|\mathsf{T}(t - \xi h)\| |g'(t - \xi h)| \, \mathrm{d}\xi
\end{aligned}$$

hence it follows that

$$\begin{aligned}
\left\| \int_T^\infty \mathsf{T}(t) \frac{g(t-h) - g(t)}{h} x \, \mathrm{d}t \right\| &\le M \|x\| \int_T^\infty \int_0^1 \|\mathsf{T}(t - \xi h)\| |g'(t - \xi h)| \, \mathrm{d}\xi \, \mathrm{d}t \\
&= M \|x\| \int_0^1 \int_T^\infty \|\mathsf{T}(t - \xi h)\| |g'(t - \xi h)| \, \mathrm{d}t \, \mathrm{d}\xi \\
&= M \|x\| \int_0^1 \int_{T - \xi h}^\infty \|\mathsf{T}(\tau)\| |g'(\tau)| \, \mathrm{d}\tau \, \mathrm{d}\xi \\
&\le M \|x\| \int_0^1 \int_T^\infty \|\mathsf{T}(\tau)\| |g'(\tau)| \, \mathrm{d}\tau \, \mathrm{d}\xi \\
&\le M \|x\| \int_T^\infty \|\mathsf{T}(\tau)\| |g'(\tau)| \, \mathrm{d}\tau \le M \|x\| \varepsilon. \qquad (4.2)
\end{aligned}$$

Moreover it is easily found a $\delta \in\,]0, 1[$ such that for every $h \in\,]0, \delta[$ we have

$$\begin{aligned}
&\left\| \int_h^T \mathsf{T}(t) \frac{g(t-h) - g(t)}{h} x \, \mathrm{d}t + \int_0^T \mathsf{T}(t) g'(t) x \, \mathrm{d}t \right\| \\
&\le \left\| \int_h^T \mathsf{T}(t) \left(\frac{g(t-h) - g(t)}{h} + g'(t) \right) x \, \mathrm{d}t \right\| + \left\| \int_0^h \mathsf{T}(t) g'(t) x \, \mathrm{d}t \right\| \le \varepsilon. \qquad (4.3)
\end{aligned}$$

Hence assertion (b) follows from (4.2)–(4.3) and from the following estimate

$$\begin{aligned}
&\left\| \int_h^\infty \mathsf{T}(t) \frac{g(t-h) - g(t)}{h} x \, \mathrm{d}t + \mathsf{L}(g') x \, \mathrm{d}t \right\| \\
&\le \left\| \int_h^T \mathsf{T}(t) \frac{g(t-h) - g(t)}{h} x \, \mathrm{d}t + \int_0^T \mathsf{T}(t) g'(t) x \, \mathrm{d}t \right\| \\
&\quad + \left\| \int_T^\infty \mathsf{T}(t) \frac{g(t-h) - g(t)}{h} x \, \mathrm{d}t \right\| + \left\| \int_T^\infty \mathsf{T}(t) g'(t) x \, \mathrm{d}t \right\|.
\end{aligned}$$

□

The next lemma plays a key role in the proof of Theorem 3.5.

Lemma 4.4. *Let $\mathsf{T} : [0,\infty[\longrightarrow \mathscr{L}^r(X)$ be a strongly continuous right linear semigroup and for every $g \in L_{\mathsf{T}}([0,\infty[\,;\mathbb{A})$ let $\mathsf{L}(g) : X \longrightarrow X$ be defined by (4.1). Then $\mathsf{L}(g) \in \mathscr{L}^r(X)$ for every $g \in L_{\mathsf{T}}([0,\infty[\,;\mathbb{A})$ and the resulting mapping $\mathsf{L} : L_{\mathsf{T}}([0,\infty[\,;\mathbb{A}) \longrightarrow \mathscr{L}^r(X)$ is $\mathbb{R}$-linear. Moreover the following assertions hold.*

(a) *If $g, g' \in L_{\mathsf{T}}([0,\infty[\,;\mathbb{A})$, then*

$$\mathsf{L}(g)(X) \subseteq D(\mathsf{A}), \tag{4.4}$$

$$\mathsf{A}\mathsf{L}(g)x = -g(0)x - \mathsf{L}(g')x \qquad \forall x \in X. \tag{4.5}$$

(b) *If $m \in \mathbb{N}$, $g, g', \dots, g^{(m+2)} \in L_{\mathsf{T}}([0,\infty[\,;\mathbb{A})$, and $g(0) = g'(0) = \cdots = g^{(m)}(0) = 0$, then*

$$\mathsf{L}(g)(X) \subseteq D(\mathsf{A}^{m+2}), \tag{4.6}$$

$$\mathsf{A}^{m+2}\mathsf{L}(g)x = (-1)^{m+2}\left(g^{(m+1)}(0)x + \mathsf{L}(g^{(m+2)})x\right) \qquad \forall x \in X, \tag{4.7}$$

$$\mathsf{A}^k\mathsf{L}(g)x = (-1)^k\mathsf{L}(g^{(k)}) \qquad \forall x \in X,\ \forall k \in \{0,\dots,m+1\}. \tag{4.8}$$

(c) *If $g, g', g'' \in L_{\mathsf{T}}([0,\infty[\,;\mathbb{A})$, $g(0) = 0$, and $g'(0) = 1$, then*

$$\Delta_q(\mathsf{A})\mathsf{L}(g)x = x + \mathsf{L}(g'' + 2\,\mathrm{Re}(q)g' + |q|^2 g)x \qquad \forall x \in X,\ \forall q \in Q_{\mathbb{A}}. \tag{4.9}$$

(d) *If $g, g' \in L_{\mathsf{T}}([0,\infty[\,;\mathbb{A})$ and g is real-valued, then*

$$\mathsf{A}\mathsf{L}(g)x = \mathsf{L}(g)\mathsf{A}x \qquad \forall x \in D(\mathsf{A}). \tag{4.10}$$

(e) *If $m \in \mathbb{N}$, $g, g', \dots, g^{(m+2)} \in L_{\mathsf{T}}([0,\infty[\,;\mathbb{A})$, $g(0) = g'(0) = \cdots = g^{(m)}(0) = 0$, and g is real valued, then*

$$\mathsf{A}^k\mathsf{L}(g)x = \mathsf{L}(g)\mathsf{A}^k x \qquad \forall x \in D(\mathsf{A}^k),\ \forall k \in \{1,\dots,m+2\}. \tag{4.11}$$

Proof. For every $g \in L_{\mathsf{T}}([0,\infty[\,;\mathbb{A})$ the right linearity of $\mathsf{L}(g)$ follows from the right linearity of $\mathsf{T}(t)$ and from the definition of the X-valued Lebesgue integral. For every $x \in X$ we have

$$\|\mathsf{L}(g)x\| \le \|x\| \int_0^\infty \|\mathsf{T}(t)\||g(t)|\,\mathrm{d}t,$$

hence $\mathsf{L}(g)$ is also continuous and $\|\mathsf{L}(g)\| \le \int_0^\infty \|\mathsf{T}(t)\||g(t)|\,\mathrm{d}t$. The real linearity of L is straightforward. Now in the following list of items we prove the assertions from (a) to (e).
(a) For every $h > 0$ and for every $x \in X$ we have

$$\begin{aligned}
\frac{\mathsf{T}(h) - \mathsf{Id}}{h}\,\mathsf{L}(g)x &= \frac{1}{h}\int_0^\infty \mathsf{T}(t+h)g(t)x\,\mathrm{d}t - \frac{1}{h}\int_0^\infty \mathsf{T}(t)g(t)x\,\mathrm{d}t \\
&= \frac{1}{h}\int_h^\infty \mathsf{T}(t)g(t-h)x\,\mathrm{d}t - \frac{1}{h}\int_0^\infty \mathsf{T}(t)g(t)x\,\mathrm{d}t \\
&= -\frac{1}{h}\int_0^h \mathsf{T}(t)g(t)x\,\mathrm{d}t + \int_h^\infty \mathsf{T}(t)\left(\frac{g(t-h) - g(t)}{h}\right)x\,\mathrm{d}t.
\end{aligned}$$

Hence, taking the limit as $h \to 0$, thanks to Lemma 4.1 we find that

$$\mathsf{A}\mathsf{L}(g)x = -\mathsf{T}(0)g(0)x - \int_0^\infty \mathsf{T}(t)g'(t)x\,\mathrm{d}t = -g(0)x - \mathsf{L}(g')x.$$

(b) We proceed by induction on $m \in \{-1\} \cup \mathbb{N}$. The case $m = -1$ follows from (a). Let us assume that the result is true for $m - 1$, and we prove it for m. Therefore if g satisfies the assumptions, in particular we have $\mathsf{L}(g)x \in D(\mathsf{A}^{m+1})$ and $\mathsf{A}^{m+1}\mathsf{L}(g)x = (-1)^{m+1}(g^{(m)}(0)x + \mathsf{L}(g^{(m+1)})x$ for every $x \in X$. But $g^{(m)}(0) = 0$ hence $\mathsf{A}^{m+1}\mathsf{L}(g)x = (-1)^{m+1}\mathsf{L}(g^{(m+1)})x$ and $\mathsf{L}(g^{(m+1)})x \in D(\mathsf{A})$ by virtue of an application of (4.4) with g replaced by $g^{(m+1)}$. Thus $\mathsf{A}^{m+1}\mathsf{L}(g)x \in D(\mathsf{A})$ and (4.6) follows. Using again the validity of the statement for $m - 1$ and the identity $g^{(m)}(0) = 0$ we have

$$\begin{aligned}\mathsf{A}^{m+2}\mathsf{L}(g)x = \mathsf{A}\mathsf{A}^{m+1}\mathsf{L}(g)x &= (-1)^{m+1}\mathsf{A}\mathsf{L}(g^{(m+1)})x \\ &= (-1)^{m}(g^{(m+1)}(0)x + \mathsf{L}(g^{(m+2)}))x,\end{aligned}$$

where in the last equality we have used (4.5) with g replaced by $g^{(m+1)}$. Therefore (4.7) is proved. Formula (4.8) is trivial for $k = 0$ and follows from (a) for $k = 1$, while for $2 \le k \le m + 1$ follows from (4.7) which we have already proved.

(c) Now fix $x \in X$ and $q \in Q_{\mathbb{A}}$. From (b) we obtain $\mathsf{A}^2\mathsf{L}(g)x = x + \mathsf{L}(g'')x$, hence, exploiting again (4.5) and the $\mathbb{R}$-linearity of L, we obtain

$$\begin{aligned}\Delta_q(\mathsf{A})\mathsf{L}(g)x &= \mathsf{A}^2\mathsf{L}(g)x - 2\,\mathrm{Re}(q)\mathsf{A}\mathsf{L}(g)x + |q|^2\mathsf{L}(g)x \\ &= x + \mathsf{L}(g'')x - 2\,\mathrm{Re}(q)(-\mathsf{L}(g')x) + |q|^2\mathsf{L}(g)x \\ &= x + \mathsf{L}(g'' + 2\,\mathrm{Re}(q)g' + |q|^2 g)x.\end{aligned}$$

(d) If $x \in D(\mathsf{A})$, then for every $h > 0$ and for every $t > 0$ we have $\mathsf{T}(t)g(t)\mathsf{T}(h)x = \mathsf{T}(t)\mathsf{T}(h)g(t)x = \mathsf{T}(t+h)g(t)x = \mathsf{T}(h)\mathsf{T}(t)g(t)x$, because g is real-valued and T is a semigroup. Therefore

$$\begin{aligned}\mathsf{L}(g)\frac{\mathsf{T}(h) - \mathsf{Id}}{h}x &= \frac{1}{h}\int_0^\infty \mathsf{T}(t)g(t)\mathsf{T}(h)x\,\mathrm{d}t - \frac{1}{h}\int_0^\infty \mathsf{T}(t)g(t)x\,\mathrm{d}t \\ &= \frac{1}{h}\int_0^\infty \mathsf{T}(h)\mathsf{T}(t)g(t)x\,\mathrm{d}t - \frac{1}{h}\int_0^\infty \mathsf{T}(t)g(t)x\,\mathrm{d}t \\ &= \frac{\mathsf{T}(h) - \mathsf{Id}}{h}\mathsf{L}(g)x,\end{aligned}$$

and the assertion follows taking the limit as $h \to 0$ and invoking (4.4).

(e) By induction on $m \in \{-1\} \cup \mathbb{N}$. The case $m = -1$ is true by virtue of (d). Let us assume that the result is true for $m - 1$, and we prove it for m. Therefore if g satisfies the assumptions, in particular we have that $\mathsf{A}^k\mathsf{L}(g) = \mathsf{L}(g)\mathsf{A}^k$ on $D(\mathsf{A}^k)$ for all $k \in \{1, \ldots, m+1\}$. Hence if $x \in D(\mathsf{A}^{m+2}) \subset D(\mathsf{A}^{m+1})$ then $\mathsf{A}^{m+1}x \in D(\mathsf{A})$ and we have that

$$\mathsf{L}(g)\mathsf{A}^{m+2}x = \mathsf{L}(g)\mathsf{A}\mathsf{A}^{m+1}x = \mathsf{A}\mathsf{L}(g)\mathsf{A}^{m+1}x = \mathsf{A}\mathsf{A}^{m+1}\mathsf{L}(g)x = \mathsf{A}^{m+2}\mathsf{L}(g)x,$$

where in the second equality we have used again (d). □

Proof of Theorem 3.4. Let us first recall that the existence of constant $\omega \in \mathbb{R}$ such that $M := \sup_{t \ge 0} \|\mathsf{T}(t)\|e^{-\omega t} < \infty$ is well know (cf. [27, Thm 4.5(b)]). From (1.2) it follows that $g_P \in C^{m+2}([0, \infty[\,; \mathbb{A})$ and the Leibniz formula for the higher derivatives of a product yields the existence of a polynomial $p(t, \lambda_1, \ldots, \lambda_h)$ such that

$$\|\mathsf{T}(t)\||g_P^k(t)| \le Me^{\omega t}|p(t, \lambda_1, \ldots, \lambda_h)|e^{-tr_P} = M|p(t, \lambda_1, \ldots, \lambda_h)|e^{(\omega - r_P)t}$$

for all $t \geq 0$ and for all $k \in \{0, 1, \ldots, m+2\}$. Thus $g_P, g_P', \ldots, g_P^{(m+2)} \in L_{\mathsf{T}}([0,\infty[\,;\mathbb{A})$ and we can apply part (b) of Lemma 4.4 taking into account of initial conditions for g_P. We infer that

$$\begin{aligned} P(\mathsf{A})\mathsf{L}(g_P)x &= \sum_{k=0}^{m+2} \mathsf{A}^k a_k \mathsf{L}(g_P)x \\ &= \sum_{k=0}^{m+1} (-1)^k \mathsf{L}(g_P^{(k)}) a_k x + (-1)^{m+2}\left(g_P^{(m+1)}(0) + \mathsf{L}(g_P^{(m+2)})\right) a_{m+2} x \\ &= \mathsf{L}\left(\sum_{k=0}^{m+2} g_P^{(k)} (-1)^k a_k\right) x + (-1)^{m+1} x = x g_P^{(m+1)}(0)(-1)^m a_{m+2} = x. \end{aligned}$$

On the other hand since g_P is real-valued we can also apply parts (d) and (e) of Lemma 4.4 and infer that $\mathsf{L}(g_P)P(\mathsf{A}) = P(\mathsf{A})\mathsf{L}(g_P)$ thus $\mathsf{L}(g_P) = P(\mathsf{A})^{-1}$ and (3.2) is proved. Now take $(p_0, p_1, \ldots, p_{m+1}) \in \mathbb{A}^{m+2}$. Using Lemma 4.4(b) we deduce:

$$\sum_{j=0}^{m+1} \mathsf{A}^k P(\mathsf{A})^{-1} p_j x = \sum_{j=0}^{m+1} \mathsf{A}^k \mathsf{L}(g_P) p_j x = \sum_{j=0}^{m+1} (-1)^k \mathsf{L}(g_P^{(k)}) p_j x = \mathsf{L}\left(\sum_{j=0}^{m+1} (-1)^k g_P^{(k)} p_j\right) x$$

and (3.3) is proved. It is well known that

$$Q_j(t) = \sum_{k=1}^{m_j} \frac{c_{-k}^{(j)}}{(k-1)!} t^{k-1},$$

where $c_{-k}^{(j)}$ is the coefficient of $(z-\lambda_j)^{-k}$ within the partial frations decomposition of $a_{m+2}/P(-z)$, i.e. the residue at λ_j of the function $a_{m+2}(z-\lambda_j)^{k-1}/P(-z)$. Therefore

$$|g_P(t)| \leq \sum_{j=1}^{h} \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(k-1)!} t^{k-1} e^{-r_P t} \qquad \forall t \geq 0$$

and

$$\begin{aligned} \|P(\mathsf{A})^{-1}\| &\leq \int_0^\infty \|\mathsf{T}(t)\| \sum_{j=1}^{h} \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(k-1)!} t^{k-1} e^{-r_P t}\,\mathrm{d}t \\ &\leq M \sum_{j=1}^{h} \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(k-1)!} \int_0^\infty t^{k-1} e^{-(r_P-\omega)t}\,\mathrm{d}t \\ &= M \sum_{j=1}^{h} \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(r_P-\omega)^k} \frac{1}{(k-1)!} \int_0^\infty t^{k-1} e^{-t}\,\mathrm{d}t \\ &= M \sum_{j=1}^{h} \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(r_P-\omega)^k} \end{aligned}$$

and the theorem is completely proved. □

Now we address the case when $P(\mathrm{x}) = \mathrm{x}^2 - 2\,\mathrm{Re}(q) + |q|^2$ for some $q \in Q_{\mathbb{A}}$ which is related to part (c) of Lemma 4.4. We first present a simple lemma whose proof is a trivial calculus exercise.

Lemma 4.5. *For every fixed $q \in Q_{\mathbb{A}}$, the unique solution of the Cauchy problem*

$$g'' + 2\operatorname{Re}(q)g' + |q|^2 g = 0, \quad g(0) = 0, \quad g'(0) = 1. \tag{4.12}$$

is the function $g_q : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$g_q(t) := te^{-\operatorname{Re}(q)t}\operatorname{sinc}(t|\operatorname{Im}(q)|), \qquad t \in \mathbb{R}, \tag{4.13}$$

where we recall that sinc : $\mathbb{R} \longrightarrow \mathbb{R}$ *is the* unnormalized sinc function, *i.e. the only continuous real function ξ on $\mathbb{R}$ such that $\xi(r) = (\sin r)/r$ for all $r \neq 0$. Moreover we have that $g_q, g_q', g_q'' \in L_{\mathsf{T}}([0,\infty[\,;\mathbb{A})$.*

Now we are in position to find the integral representations of the quadratic resolvent and of the spherical resolvent operators as a simple consequence of our main Theorem 3.4.

Proof of Theorem 3.5. In order to prove (1.8) it is enough to apply part (a) of Theorem 3.4 with $P(\mathrm{x}) = \mathrm{x}^2 - 2\operatorname{Re}(q) + |q|^2$ taking into account of Lemma 4.5. Formula (1.9) is a straighforward application of part (b) of Theorem 3.4 with $p_0 = 0$, $p_1 = 1$ and $p_2 = 0$. Instead (1.10) is obtained taking in part (b) of Theorem 3.4 $p_0 = q^c$, $p_1 = -1$, and $p_2 = 0$. Finally

$$\|\mathsf{Q}_q(\mathsf{A})\| \le \int_0^\infty \|\mathsf{T}(t)\||g_p(t)|\,\mathrm{d}t \le \int_0^\infty Mte^{(\omega-\operatorname{Re}(q))t}\,\mathrm{d}t = \frac{M}{(\operatorname{Re}(q)-\omega)^2},$$

i.e. (1.11) holds. Finally estimate (1.12) can be obtained in a similar way. □

Remark 4.6. By exploiting (4.13) of our Lemma 4.5 we can write the integral representation (1.8) in a more explicit way for every $q \in Q_{\mathbb{A}}$ such that $\operatorname{Re}(q) > \omega$:

$$\mathsf{Q}_q(\mathsf{A})x = -\int_0^\infty \mathsf{T}(t)\frac{e^{-\operatorname{Re}(q)t}\sin(|\operatorname{Im}(q)|t)}{|\operatorname{Im}(q)|}x\,\mathrm{d}t \qquad \forall x \in X,\ q \notin \mathbb{R}, \tag{4.14}$$

$$\mathsf{Q}_q(\mathsf{A})x = -\int_0^\infty \mathsf{T}(t)te^{-qt}x\,\mathrm{d}t \qquad \forall x \in X,\ q \in \mathbb{R}. \tag{4.15}$$

The following lemma will connect the integral representation of $\mathsf{Q}_q(\mathsf{A})$ to the so called *spherical derivative* of $q \longmapsto e^{-tq}$ (cf. [25]).

Lemma 4.7. *For every $t \in \mathbb{R}$ let $\exp^t : Q_{\mathbb{A}} \longrightarrow \mathbb{A}$ be the function defined by*

$$\exp^t(q) := \sum_{n=0}^\infty \frac{t^n}{n!}q^n = \sum_{n=0}^\infty \frac{(tq)^n}{n!}, \qquad q \in Q_{\mathbb{A}}. \tag{4.16}$$

If $(\exp^t)_s' : Q_{\mathbb{A}} \smallsetminus \mathbb{R} \longrightarrow \mathbb{A}$ denotes the function defined by

$$(\exp^t)_s'(q) := (q-q^c)^{-1}(\exp^t(q) - \exp^t(q^c)), \qquad q \in Q_{\mathbb{A}} \smallsetminus \mathbb{R}, \tag{4.17}$$

which is also called spherical derivative *of $\exp^t$, then $(\exp^t)_s'$ extends to a unique continuous function on $Q_{\mathbb{A}}$, which we still denote by $(\exp^t)_s' : Q_{\mathbb{A}} \longrightarrow \mathbb{A}$, and we have*

$$(\exp^t)_s'(q) = e^{t\operatorname{Re}(q)}\sum_{n=0}^\infty \frac{t^{2n+1}\operatorname{Im}(q)^{2n}}{(2n+1)!} \in \mathbb{R} \qquad \forall q \in Q_{\mathbb{A}} \smallsetminus \mathbb{R} \tag{4.18}$$

and $(\exp^t)_s'(q) = te^{t\operatorname{Re}(q)}$ for every $q \in \mathbb{R}$. In particular $(\exp^t)_s'$ is a real-valued. By abuse of notation, we write $\exp_s'(t,q)$ to indicate the element $(\exp^t)_s'(q)$ of $\mathbb{A}$ for every $t \in \mathbb{R}$ and for every $q \in Q_{\mathbb{A}}$, respectively.

Proof. For every $q \in Q_{\mathbb{A}} \smallsetminus \mathbb{R}$ there exists $\mathbf{j} \in \mathbb{S}_{\mathbb{A}}$ and $a, b \in \mathbb{R}$ such that $b > 0$ and $q = a + b\mathbf{j}$. Hence $q^c = a^c - \mathbf{j}^c b^c = a - b\mathbf{j}$, $\mathrm{Re}(q) = a$, $\mathrm{Im}(q) = b\mathbf{j}$, and $|\operatorname{Im}(q)| = \sqrt{(b\mathbf{j})(b\mathbf{j})^c} = b$. Since $\mathbb{C}_{\mathbf{j}}$ and $\mathbb{C}$ are isomorphic real algebras, we find that $\exp^t(q) = e^{tq} = e^{ta}(\cos(tb) + \sin(tb)\mathbf{j})$ and $\exp^t(q^c) = e^{tq^c} = e^{ta}(\cos(tb) - \sin(tb)\mathbf{j}) = (e^{tq})^c$, therefore

$$(\exp^t)'_s(q) = (q - q^c)^{-1}(e^{tq} - e^{tq^c}) = e^{t\,\mathrm{Re}(q)} \sin(t|\operatorname{Im}(q)|)|\operatorname{Im}(q)|^{-1} \qquad \forall q \in Q_{\mathbb{A}} \smallsetminus \mathbb{R}, \quad (4.19)$$

which proves the first equality in (4.18). The right-hand side of (4.19) immediately extends by continuity to te^{ta} for $q \in \mathbb{R}$, thus $(\exp^t)'_s$ is a real-valued function. As $|\operatorname{Im}(q)|^2 = -\operatorname{Im}(q)^2$, by (4.19) we have

$$(\exp^t)'_s(q) = e^{t\,\mathrm{Re}(q)} \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1} |\operatorname{Im}(q)|^{2n}}{(2n+1)!} = e^{t\,\mathrm{Re}(q)} \sum_{n=0}^{\infty} \frac{t^{2n+1} \operatorname{Im}(q)^{2n}}{(2n+1)!} \qquad \forall q \in Q_{\mathbb{A}} \smallsetminus \mathbb{R},$$

which implies the second equality of (4.18). □

Corollary 4.8. *Under the assumption of Theorem 3.5, for every $q \in Q_{\mathbb{A}}$ such that $\mathrm{Re}(q) > \omega$ we have*

$$\mathsf{Q}_q(\mathsf{A})x = -\int_0^\infty \mathsf{T}(t) \exp'_s(-t, q)\, x \,\mathrm{d}t \qquad \forall x \in X. \quad (4.20)$$

5. Integral representation of the powers of $\mathsf{Q}_q(\mathsf{A})$

In this section we look for an integral representation of the integer powers of the quadratic resolvent operator $\mathsf{Q}_q(\mathsf{A})$. In order to find this representation we need the following simple lemma.

Lemma 5.1. *If $f, g \in L_{\mathsf{T}}([0, \infty[\,; \mathbb{A})$ and f is real-valued, then*

$$\mathsf{L}(f)\mathsf{L}(g)x = \mathsf{L}(f \star g)x = \mathsf{L}(g \star f)x \qquad \forall x \in X,$$

where we recall that $f \star g : [0, \infty[\longrightarrow \mathbb{A}$, the convolution of f and g, is defined by $(f \star g)(t) := \int_0^t f(t-s)g(s)\,\mathrm{d}s$.

Proof. Using the fact that f is real-valued, we see at once that $f \star g = g \star f$. In addition, bearing in mind the semigroup law for T, we find

$$\begin{aligned}
\mathsf{L}(f)\mathsf{L}(g)x &= \int_0^\infty \mathsf{T}(t) f(t) \int_0^\infty \mathsf{T}(s) g(s) x \,\mathrm{d}s\,\mathrm{d}t \\
&= \int_0^\infty \int_0^\infty \mathsf{T}(t)\mathsf{T}(s) f(t) g(s) x \,\mathrm{d}s\,\mathrm{d}t \\
&= \int_0^\infty \int_0^\infty \mathsf{T}(t+s) f(t) g(s) x \,\mathrm{d}s\,\mathrm{d}t\,.
\end{aligned}$$

Thus a change of variable and an application of Fubini theorem yields

$$\begin{aligned}
\mathsf{L}(f)\mathsf{L}(g)x &= \int_0^\infty \int_t^\infty \mathsf{T}(s)f(t)g(s-t)x \,\mathrm{d}s\,\mathrm{d}t \\
&= \int_0^\infty \int_0^\infty \chi_{[0,s]}(t)\mathsf{T}(s)f(t)g(s-t)x \,\mathrm{d}s\,\mathrm{d}t \\
&= \int_0^\infty \int_0^\infty \chi_{[0,s]}(t)\mathsf{T}(s)f(t)g(s-t)x \,\mathrm{d}t\,\mathrm{d}s \\
&= \int_0^\infty \int_0^t \mathsf{T}(s)f(t)g(s-t)x \,\mathrm{d}t\,\mathrm{d}s \\
&= \int_0^\infty \mathsf{T}(s)\int_0^t f(t)g(s-t)x \,\mathrm{d}t\,\mathrm{d}s \\
&= \mathsf{L}(f\star g) = \mathsf{L}(g\star f).
\end{aligned}$$

The proof is complete. □

Given $n \in \mathbb{N}\smallsetminus\{0\}$ and $f \in L_{\mathsf{T}}([0,\infty[\,;\mathbb{A})$, we define $f^{\star n} \in L_{\mathsf{T}}([0,\infty[\,;\mathbb{A})$ by

$$f^{\star n} := \underbrace{f\star f\star\cdots\star f}_{n \text{ times}}.$$

Corollary 5.2. *Let $\mathsf{T} : [0,\infty[\, \longrightarrow \mathscr{L}^r(X)$ be a strongly continuous right linear semigroup, let $\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ be its generator, and let $\omega \in \mathbb{R}$ be a real constant such that $M := \sup_{t\in[0,\infty[} \|\mathsf{T}(t)\|e^{-\omega t} < \infty$. Given any $q \in Q_{\mathbb{A}}$ with $\mathrm{Re}(q) > \omega$, we have that $q \in \rho_S(\mathsf{A})$ and*

$$\mathsf{Q}_q(\mathsf{A})^n x = (-1)^n \int_0^\infty \mathsf{T}(t)\exp_s'(-t,q)^{\star n} x \,\mathrm{d}t \tag{5.1}$$

where $\exp_s'(-t,q)^{\star n} \in \mathbb{A}$ indicates the value of $((\exp^{-t})_s')^{\star n}$ at q.

Moreover for every $q \in Q_{\mathbb{A}}$ with $\mathrm{Re}(q) > \omega$ we have

$$\|\mathsf{Q}_q(\mathsf{A})^n\| \le \frac{M}{(\mathrm{Re}(q)-\omega)^{2n}} \qquad \forall n \in \mathbb{N}\smallsetminus\{0\}. \tag{5.2}$$

Proof. Formula (5.1) follows immediately from n applications of Theorem 3.5 and Lemma 5.1. In order to prove (5.2) let us observe that, given $q \in Q_{\mathbb{A}}$, if $a = \mathrm{Re}(q)$ and $b = |\,\mathrm{Im}(q)|$, and $g(t) = -\exp_s'(-t,q)$ for $t \ge 0$, then

$$\begin{aligned}
|(g\star g)(t)| &\le \int_0^t |g(t-s)||g(s)| \,\mathrm{d}s \\
&\le \int_0^t (t-s)e^{-(t-s)a} s e^{-sa} \,\mathrm{d}s \\
&= e^{-ta}\int_0^t s(t-s)\,\mathrm{d}s = \frac{1}{2\cdot 3}t^3 e^{-ta}.
\end{aligned}$$

Le us assume by induction that $|g^{\star(n-1)}| \leq ((2n-3)!)^{-1}t^{2n-3}e^{-at}$. Therefore for every n

$$\begin{aligned}
|g^{\star n}(t)| &\leq \int_0^t |g(t-s)||g^{\star(n-1)}(s)| \,\mathrm{d}s \\
&\leq \frac{1}{(2n-3)!} \int_0^t (t-s)e^{-(t-s)a} s^{2n-3} e^{-as} \,\mathrm{d}s \\
&= \frac{e^{-ta}}{(2n-3)!} \int_0^t (t-s)s^{2n-3} \,\mathrm{d}s \\
&= \frac{e^{-ta}}{(2n-3)!} \frac{t^{2n-1}}{(2n-2)(2n-1)} = \frac{t^{2n-1}e^{-ta}}{(2n-1)!}.
\end{aligned}$$

Thus $|g^{\star n}(t)| \leq \frac{1}{(2n-1)!}t^{2n-1}e^{-ta}$ for every $t \geq 0$ and every $n \in \mathbb{N} \setminus \{0\}$ and, recalling that $\int_0^\infty t^{2n-1}e^{-t}\,\mathrm{d}t = (2n-1)!$, we have

$$\begin{aligned}
\|\mathsf{Q}_q(\mathsf{A})^n\| &\leq \int_0^\infty \|\mathsf{T}(t)\| |g^{\star n}(t)| \,\mathrm{d}t \\
&\leq \int_0^\infty Me^{\omega t} \frac{1}{(2n-1)!} t^{2n-1} e^{-ta} \,\mathrm{d}t \\
&= \frac{M}{(2n-1)!} \int_0^\infty t^{2n-1} e^{-(a-\omega)t} \,\mathrm{d}t \\
&= \frac{M}{(2n-1)!} \int_0^\infty \frac{t^{2n-1}}{(a-\omega)^{2n}} e^{-t} \,\mathrm{d}t \\
&= \frac{M}{(a-\omega)^{2n}},
\end{aligned}$$

and we are done. □

Acknowledgments. The idea of writing this work about a relationship between Semigroups and Clifford Algebras came to our mind when the first author participated to the *International conference on Hypercomplex Analysis in Mathematics and in the Applied Sciences, Celebrating the scientific life of Klaus Gürlebeck.*

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