Clearing Payments in Dynamic Financial Networks

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Giuseppe Calafiore, Giulia Fracastoro, and Anton V. Proskurnikov

***Abstract*— This paper proposes a novel dynamical model for determining clearing payments in financial networks. We extend the classical Eisenberg-Noe model of financial contagion to multiple time periods, allowing financial operations to continue after possible initial *pseudo defaults*, thus permitting nodes to recover and eventually fulfil their liabilities. Optimal clearing payments in our model are computed by solving a suitable linear program, both in the full matrix payments case and in the pro-rata constrained case. We prove that the proposed model obeys the *priority of debt claims* requirement, that is, each node at every step either pays its liabilities in full, or it pays out all its balance. In the pro-rata case, the optimal dynamic clearing payments are unique, and can be determined via a time-decoupled sequential optimization approach.**

I. Introduction

The current global financial system is a highly interconnected network of institutions that are linked together via a structure of mutual debts or liabilities. Such interconnected structure makes the system potentially prone to "cascading defaults," whereby a shock at a node (e.g., an expected incoming payment that gets cancelled or delayed for some reason) may provoke a default at that node, which then cannot pay its liabilities to neighbouring nodes, which in turn default, and so on in an avalanche fashion. Since the consequences of these cascading events can be catastrophic, modeling and analyzing such behavior is of crucial importance. The seminal work [1] introduced a simple model for studying the financial contagion. In particular, they focused on defining a clearing procedure between financial entities. Clearing consists in defining a procedure for settling claims in the case of defaults, on the basis of a set of rules and prevailing regulations. In [1], the authors showed that there exist a clearing vector which defines the mutual interbank payments, under certain assumptions. Among such assumptions, an important one is the fact that the debts of all nodes of the system are paid simultaneously.

The basic model presented in [1] has become a cornerstone in the analysis of financial contagion and it has been later extended in various directions. In particular, many works added some non-trivial features in order to make the model more realistic [2]–[8]. However, the vast majority of these works considers this problem only in a static, or single-period, setting. This assumption is quite unrealistic, since it supposes that all liabilities are claimed and due at the same time. In addition, static models are only able to capture the immediate consequences of a financial shock. For these reasons, recently some works proposed time-dynamic extensions of the Eisenberg-Noe model. In [9] a continuous-time model of clearing in financial networks is presented. This work has later been extended by considering liquid assets [10], heterogeneous network structures over time and early defaults [11]. Other works [12] propose to combine the interbank Eisenberg-Noe model and the dynamic mean field approach. Instead, [13] uses a continuous-time model for price-mediated contagion. A different line of research extended the Eisenberg-Noe model considering a discrete-time setting. In [14], [15] a multi-period clearing framework is introduced. Using a similar approach, [16] considers the case where interbank liabilities can have multiple maturities, considering both long-term and short-term liabilities. Also [17] considers a multi-period setting, formulating liability clearing as an optimal control problem with convex objectives and constraints.

In this work, we focus on a discrete-time setting and introduce a multi-period model whereby financial operations are allowed for a given number of time periods after the initial theoretical default (named here *pseudo default*). This allows to reduce the effects of a financial shock, since some nodes may possibly recover and eventually fulfill their debts. We first consider the general case where payment matrices are unconstrained. This scenario has been introduced in the static case in [18], where its advantages over the proportional rule in terms of the overall system loss have been highlighted. The optimal sequence of payment matrices satisfies the absolute priority of debt claims rule, and hence the proposed method produces proper clearing matrices at each stage.

We then consider the situation in which a proportionality rule is enforced, whereby nodes must pay the claimant institutions proportionally to their nominal claims (pro-rata rule). Under the pro-rata rule, the optimal payments are again proper clearing payments, they are unique and, further, the multi-stage optimization problem can be decoupled in time into an equivalent series of LP problems.

Due to space limitations, the proofs of main results are **omitted** and are available in the extended document [19].

The remainder of the paper is organized as follows. Section 2 introduces some preliminary notions and the notation that will be used in the next sections. In Section 3 we introduce the Eisenberg-Noe financial network model. Then, in Section 4 we illustrate the proposed dynamic model, considering both the unrestricted case and the case with the pro-rata rule imposed. A schematic example is proposed in Section 5 in order to illustrate the proposed model. Conclusions are drawn in Section 6. For ease of reading, we collected the proofs of all technical results in an appendix.

The authors are with the Department of Electronics and Telecommunications, Politecnico di Torino, Turin, Italy. E-mails:{giuseppe.calafiore, giulia.fracastoro, anton.proskurnikov}@polito.it.

II. Preliminaries and notation

Given a finite set $\mathcal{V}$, the symbol $|\mathcal{V}|$ stands for its cardinality. The set of families $(a_\xi)_{\xi\in\Xi}$, $a_\xi \in \mathbb{R}$, is denoted by $\mathbb{R}^\Xi$. For two such families $(a_\xi), (b_\xi)$, we write $a \le b$ (b dominates a, or a is dominated by b) if $a_\xi \le b_\xi$, $\forall \xi \in \Xi$. We write $a \lneq b$ if $a \le b$ and $a \neq b$. The operations $\min, \max$ are also defined elementwise, e.g., $\min(a, b) \doteq (\min(a_\xi, b_\xi))_{\xi\in\Xi}$. These notation symbols apply to both vectors (usually, $\Xi = \{1, \dots, n\}$) and matrices (usually, $\Xi = \{1, \dots, n\} \times \{1, \dots, n\}$).

Every nonnegative square matrix $A = (a_{ij})_{i,j\in\mathcal{V}}$ corresponds to a weighted digraph $\mathcal{G}[A] = (\mathcal{V}, \mathcal{E}[A], A)$ whose nodes are indexed by $\mathcal{V}$ and whose set of arcs is defined as $\mathcal{E}[A] = \{(i, j) \in \mathcal{V} \times \mathcal{V} : a_{ij} > 0\}$. The value a_{ij} can be interpreted as the weight of arc $i \to j$. A sequence of arcs $i_0 \to i_1 \to \dots \to i_{s-1} \to i_s$ constitute a *walk* between nodes i_0 and i_s in graph $\mathcal{G}[A]$. The set of nodes $J \subseteq \mathcal{V}$ is *reachable* from node i if $i \in J$ or a walk from i to some element $j \in J$ exists; J is called *globally reachable* in the graph if it is reachable from every node $i \notin J$.

III. The static model of a financial network

We start by considering the "static" case introduced in the seminal work of Eisenberg and Noe [1]. In this setting, n nodes, representing financial entities (banks), are connected via a complex structure of mutual liabilities. The payment due from node i to node j is denoted by $\bar{p}_{ij} \ge 0$, and such liabilities are supposed to be due at the end of a fixed time period. These interbank liabilities form the *liability matrix* $\bar{P} \in \mathbb{R}^{n\times n}$, such that $[\bar{P}]_{ij} = \bar{p}_{ij}\, \forall i \neq j$, and $[\bar{P}]_{ii} = 0\, \forall i$.

Following the notation introduced in [20, Section 5], we let $c \in \mathbb{R}^n_+$ be the vector whose ith component $c_i \ge 0$ represents the total payments due to node i from non-financial entities (i.e., from any other entity, different from the n banks). Payments from banks to the external sector are instead modeled by introducing a fictitious node that represents the external sector and owes no liability to the other nodes (the corresponding row of $\bar{P}$ is zero).

The nominal cash in-flow and out-flow at a node i are, respectively,

$$\bar{\phi}_i^{\text{in}} \doteq c_i + \sum\nolimits_{k\neq i} \bar{p}_{ki}, \quad \bar{p}_i \doteq \bar{\phi}_i^{\text{out}} \doteq \sum\nolimits_{k\neq i} \bar{p}_{ik}.$$

In regular operations, the in-flow at each bank is no smaller than its out-flow (i.e., $\bar{\phi}_i^{\text{in}} \ge \bar{\phi}_i^{\text{out}}$), each bank remains solvable and is able to pay its liabilities in full. A critical situation occurs instead when (due to, e.g., a drop in the external liquidity in-flow c_i) some bank i has not enough incoming liquidity to fully pay its liabilities. In this situation, the actual payments to other banks have to be remodulated to lesser values than their nominal values $\bar{p}_{ij}$. The *clearing payments* are a set of mutual payments which settle the mutual claims in case of defaults, by enforcing a set of rules [1], [8], which are: *(i)* payments cannot exceed the corresponding liabilities, *(ii) limited liability*, i.e., the balance at each node cannot be negative, *(iii) absolute priority*: each node pays its liabilities in full or pays out all its balance.

We let $p_{ij} \in [0, \bar{p}_{ij}]$, $i \neq j = 1, \dots, n$, denote the actual inter-bank payments executed at the end of the period, which we shall collect in matrix $P \in \mathbb{R}^{n,n}$. At each node i we write a flow balance equation, involving the actual cash in-flow and out-flow, defined respectively as

$$\phi_i^{\text{in}} \doteq c_i + \sum\nolimits_{k\neq i} p_{ki}, \tag{1}$$

$$\phi_i^{\text{out}} \doteq p_i \doteq \sum\nolimits_{k\neq i} p_{ik}. \tag{2}$$

The cash balance represents the net worth w_i of the ith bank, which is defined as

$$w_i \doteq \phi_i^{\text{in}} - \phi_i^{\text{out}} = c_i + \sum\nolimits_{k\neq i} p_{ki} - \sum\nolimits_{k\neq i} p_{ik}. \tag{3}$$

The limited liability rule (ii) requires that $w_i \ge 0$, $\forall i$.

In vector notation, the vectors of actual and nominal in/out-flows and the vector of net worths are

$$\phi^{\text{in}} = c + P^\top \mathbf{1}, \quad \bar{\phi}^{\text{in}} = c + \bar{P}^\top \mathbf{1} \tag{4}$$

$$\phi^{\text{out}} = p = P\mathbf{1}, \quad \bar{\phi}^{\text{out}} = \bar{p} = \bar{P}\mathbf{1} \tag{5}$$

$$w = \phi^{\text{in}} - \phi^{\text{out}} = (c + P^\top \mathbf{1}) - P\mathbf{1}, \tag{6}$$

where $\mathbf{1}$ denotes a vector of ones of suitable dimension.

The above mentioned conditions *(i), (ii)* on the payments are written in compact vector form as $0 \le P \le \bar{P}$ and $P\mathbf{1} \le c + P^\top \mathbf{1}$, that is the payment matrix P is restricted to belong to the following convex polytope

$$\begin{aligned}\mathcal{P}(c, \bar{P}) \doteq \big\{ &P \in \mathbb{R}^{n\times n} : 0 \le P \le \bar{P}, \\ &P\mathbf{1} \le c + P^\top \mathbf{1},\, P_{ii} = 0,\ i = 1, \dots, n \big\}.\end{aligned} \tag{7}$$

A payment matrix $P \in \mathcal{P}(c, \bar{P})$ is a *clearing matrix*, or matrix of clearing payments, if it complies with the *absolute priority* of debt claims rule *(iii)*, that is,

$$P\mathbf{1} = \min(\bar{P}\mathbf{1}, c + P^\top \mathbf{1}). \tag{8}$$

It can be shown [18] that a clearing matrix can be found by solving an optimization problem of the form

$$\begin{aligned}\min_P \quad & f(P) \\ \text{subject to:} \quad & P \in \mathcal{P}(c, \bar{p})\end{aligned} \tag{9}$$

where f is a decreasing function of the matrix argument P on $[0, \bar{P}]$, i.e., a function such that $\bar{P} \ge P^{(2)} > P^{(1)} \ge 0$, $P^{(2)} \neq P^{(1)}$, implies $f(P^{(2)}) < f(P^{(1)})$. It can be shown that for any choice of f the solution to (9) is automatically a clearing matrix, that is, (8) holds. Possible choices for f in (9) are for instance $f(P) = \|\bar{\phi}^{\text{in}} - \phi^{\text{in}}\|_1$ and $f(P) = \|\bar{\phi}^{\text{in}} - \phi^{\text{in}}\|_2^2$, where $\phi^{\text{in}}(P) = c + P^\top \mathbf{1}$. The optimal solution of (9), however, may be non unique.

A. The pro-rata rule

In practice, payments under default are subject to additional prevailing regulations. A common one is the so called *proportionality* (or, pro-rata) rule, according to which payments are made in proportion to the original outstanding claims. Denoting by

$$a_{ij} \doteq \begin{cases} \dfrac{\bar{p}_{ij}}{\bar{p}_i} & \text{if } \bar{p}_i > 0 \\ 1 & \text{if } \bar{p}_i = 0 \text{ and } i = j \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

the relative proportion of payment due nominally by node i to node j, we form the *relative liability* matrix $A = [a_{ij}]$. By definition, A is row-stochastic, that is $A\mathbf{1} = \mathbf{1}$. The pro-rata rule imposes the relations

$$p_{ij} = a_{ij} p_i, \quad \forall i, j, \tag{11}$$

where p_i is the out-flow defined in (2). In matrix notation, the pro-rata rule corresponds to a linear equality constraint on the entries of P, that is $P = \mathrm{diag}(P\mathbf{1})A$. Under pro-rata rule, the problem of clearing payments can be rewritten in terms of the total out-payments vector $p = P\mathbf{1}$, which is said to be feasible if it belongs to

$$\mathcal{P}_{\mathrm{pr}}(c, \bar{p}) \doteq \{p \in \mathbb{R}^n : 0 \leq p \leq \bar{p},\ p \leq c + A^\top p\}, \tag{12}$$

where $\bar{p} \doteq \bar{P}\mathbf{1}$. Among the feasible payment vectors $p \in \mathcal{P}_{\mathrm{pr}}(c, \bar{p})$, a vector of clearing payments, or simply *clearing vector* is a vector $p \in \mathcal{P}_{\mathrm{pr}}(c, \bar{p})$ such that

$$p = \min(\bar{p}, c + A^\top p). \tag{13}$$

A clearing vector p^* can be found [20] by solving an optimization problem of the form

$$\begin{aligned} &\min_p f(p) \\ &\text{subject to:} \quad p \in \mathcal{P}_{\mathrm{pr}}(c, \bar{p}) \end{aligned} \tag{14}$$

where $f : [0, \bar{p}] \to \mathbb{R}$ is any decreasing function, that is, a function such that $p^{(1)}, p^{(2)} \in [0, \bar{p}]$ and $p^{(1)} \leq p^{(2)}$ imply $f(p^{(1)}) \geq f(p^{(2)})$, and the latter inequality is strict unless $p^{(1)} = p^{(2)}$. Possible choices for f are for instance $f(p) = \|\bar{\phi}^{\mathrm{in}} - \phi^{\mathrm{in}}(p)\|_2^2$, and $f(p) = \sum_{i=1}^n (\bar{\phi}_i^{\mathrm{in}} - \phi_i^{\mathrm{in}}(p))$, where $\phi^{\mathrm{in}}(p) = c + A^\top p$. The following proposition holds.

IV. Dynamic financial networks

A key observation is that the default and clearing model discussed in the previous section, which coincides with the mainstream one studied in the literature [20] is an *instantaneous* one. By instantaneous we mean that the described process assumes that at one point in time (say, at the end of a day), all liabilities are claimed and due simultaneously, and that the entire network of banks becomes aware of the claims and possible defaults and instantaneously agrees on the clearing payments. On the one hand such an instantaneous model may be quite unrealistic, and on the other hand the implied default mechanism is such that all financial operations of defaulted nodes are instantaneously frozen, which possibly induces propagation of the default to other neighboring nodes, in an avalanche fashion, see, e.g. [21].

One motivation for the dynamic model we propose in this paper is that one may expect that if financial operations are allowed for a given number of time periods after the initial theoretical defaults, some nodes may actually *recover* and eventually manage to fulfill their obligations. The overall system-level advantage of such strategy is that the catastrophic effects of avalanche defaults are possibly mitigated, as shown by examples in Section V.

In our dynamic multi-period model described below, if a theoretical default condition (we shall call this a *pseudo-default*) happens at some time $t < T$, where T is the final time, we do not freeze operations. Instead, we carry over the residual liabilities for the next period and let the nodes continue their mutual payments operations, and so on until the final time T. The key elements of this model are:

- $t = 0, 1, \ldots, T$, denote discrete time instants delimiting periods of fixed length (e.g, one day, one month, etc.);
- $T \geq 0$ denotes the final horizon;
- $c(t) \in \mathbb{R}^n \geq 0$ represents the cash in-flow at the nodes at the beginning of period t;
- matrix $\bar{P}(t) = (\bar{p}_{ij}(t)) \in \mathbb{R}^{n,n}$ describes the liabilities (i.e., the mutual payment obligations) among the nodes at period t, i.e., $\bar{p}_{ij}(t)$ is the nominal amount due from i to j at the end of period t. $\bar{P} \doteq \bar{P}(0)$ denotes the initial liabilities at $t = 0$;
- matrix $P(t) = (p_{ij}(t)) \in \mathbb{R}^{n,n}$ contains the actual payments from i to j performed at period t;
- the vectors of actual and nominal in-flows and out-flows $\phi^{\mathrm{in}}(t), \phi^{\mathrm{out}}(t), \bar{\phi}^{\mathrm{in}}(t), \bar{\phi}^{\mathrm{out}}(t)$ at period $t = 0, \ldots, T-1$, are defined similarly to (4) and (5);
- the net worth $w_i(t)$ of node i at the beginning of period t evolves in accordance with

$$w_i(t+1) = w_i(t) + \phi_i^{\mathrm{in}}(t) - \phi_i^{\mathrm{out}}(t) \tag{15}$$

  or, in the equivalent vector form

$$w(t+1) = w(t) + c(t) + P(t)^\top \mathbf{1} - P(t)\mathbf{1}. \tag{16}$$

Similar to the single-period case discussed in Section III, the limited liability condition requires that $w(t) \geq 0$ at all t. It may therefore happen that a payment $p_{ij}(t)$ has to be lower than the corresponding liability $\bar{p}_{ij}(t)$ in order to guarantee $w_i(t) \geq 0$. When this happens at some $t < T$, instead of declaring default and freezing the financial system, we allow operations to continue up to the final time T, updating the due payments according to the equation

$$\bar{p}_{ij}(t+1) = \alpha \left(\bar{p}_{ij}(t) - p_{ij}(t) \right), \tag{17}$$

where $\alpha \geq 1$ is the interest rate applied on past due payments. The previous relation can be written as

$$\bar{P}(t+1) = \alpha \left(\bar{P}(t) - P(t) \right), \quad t \in \mathcal{T}, \tag{18}$$

where $\mathcal{T} \doteq \{0, \ldots, T-1\}$. The meaning of equation (18) is that if a due payment at t is not paid in full, then the residual debt is added to the nominal liability for the next period, possibly increased by an interest factor $\alpha \geq 1$. This mechanism allows for a node which is technically in default at a time t to continue operations and (possibly) repay its dues in subsequent periods. Notice that time-varying $\bar{P}(t)$ depends on the *actual* payment matrices $P(0), \ldots, P(t-1)$. The final nominal matrix $\bar{P}(T)$ contains the residual debts at the end of the final period. The recursions (16) and (18) are initialized with $w(0) = 0$, $\bar{P}(0) = \bar{P}$, where $\bar{P}$ is the initial liability matrix.

Vectors of external payments $c(t)$ are considered as given inputs, while actual payments matrices $P(t)$ are to be determined, being subject to the constraints

$$P(t) \geq 0, \quad P(t) \leq \bar{P}(t), \quad t \in \mathcal{T} \tag{19}$$

$$P(t)\mathbf{1} \leq w(t) + c(t) + P(t)^\top \mathbf{1}, \quad t \in \mathcal{T}, \tag{20}$$

where (19) represents the requirement that actual payments never exceed the nominal liabilities, and (20) represents the requirement that $w(t+1)$, as given in (16), remains nonnegative at all t. Conditions (19), (20) can be made explicit by eliminating the variables $w(t)$ and $\bar{P}(t)$, which by using (16)–(18) can be expressed as

$$\bar{P}(t) = \alpha^t \bar{P}(0) - \sum\nolimits_{k=0}^{t-1} \alpha^{t-k} P(k), \tag{21}$$

$$w(t) = C(t-1) + \sum\nolimits_{k=0}^{t-1} \left(P^\top(k) - P(k)\right)\mathbf{1}, \tag{22}$$

$$C(t) \doteq \sum\nolimits_{k=0}^{t} c(k), \quad t = 0, \ldots, T. \tag{23}$$

Conditions (19), (20) can thus be rewritten as

$$P(t) \geq 0, \tag{24}$$

$$\sum\nolimits_{k=0}^{t} \alpha^{t-k} P(k) \leq \alpha^t \bar{P} \tag{25}$$

$$C(t) + \sum\nolimits_{k=0}^{t} \left(P(k)^\top - P(k)\right)\mathbf{1} \geq 0 \tag{26}$$

$$\forall t \in \mathcal{T}.$$

For brevity, we denote

$$[P] \doteq (P(0), \ldots, P(T-1)), \; [c] \doteq (c(0), \ldots, c(T-1)).$$

Definition 1: We call a sequence of payment matrices $[P]$ *admissible* if conditions (24)–(26) hold. Let

$$\mathcal{P}([c], \bar{P}) \doteq \{[P] : \text{(24)–(26) hold}\}$$

stand for the polyhedral set of all admissible matrix sequences $[P]$ that correspond to the given sequence of vectors $[c]$ and initial liability matrix $\bar{P}$.

The system-level cost that we consider is the cumulative sum of deviations of the actual in-flows at nodes from the nominal ones, that is

$$L([P]) \doteq \sum_{t=0}^{T-1} \sum_{i=1}^{n} (\bar{\phi}_i^{\text{in}}(t) - \phi_i^{\text{in}}(t)).$$

From the definition (4) of in-flow vectors and from (21) we obtain that

$$\begin{aligned} L([P]) &= \sum_{t=0}^{T-1} \mathbf{1}^\top (\bar{\phi}^{\text{in}}(t) - \phi^{\text{in}}(t)) = \\ &= \sum_{t=0}^{T-1} \mathbf{1}^\top (\alpha^t \bar{P} - \sum_{k=0}^{t} \alpha^{t-k} P(k))\mathbf{1} \\ &= a_0 \mathbf{1}^\top \bar{P}\mathbf{1} - \sum_{t=0}^{T-1} a_t \mathbf{1}^\top P(t)\mathbf{1}, \end{aligned} \tag{27}$$

where the constants $a_0 > a_1 > \ldots > a_{T-1}$ are defined as

$$a_t \doteq \sum_{j=0}^{T-t-1} \alpha^j = \begin{cases} \frac{\alpha^{T-t}-1}{\alpha-1}, & \text{if } \alpha > 1 \\ T-t, & \text{if } \alpha = 1. \end{cases} \tag{28}$$

The optimal payment matrices are thus obtained as a solution to the following optimization problem

$$\max_{[P]} \sum_{t=0}^{T-1} a_t \mathbf{1}^\top P(t)\mathbf{1} \quad \text{s.t.:} \quad [P] \in \mathcal{P}([c], \bar{P}), \tag{29}$$

which is equivalent to minimization of the overall "system loss" $L([P])$ over the set of all admissible payment matrices.

Observe that, from a numerical point of view, finding an optimal sequence of payment matrices amounts to solving the linear programming (LP) problem (29). Notice also that in the case $T = 1$ the set $\mathcal{P}([c])$ reduces to the polytope of matrices (7), and the optimization problem (29) is a special case of (9), where $f(P) = -\mathbf{1}^\top P(0)\mathbf{1}$.

We note that a dynamic model similar to the one presented in this section has been introduced in [17], where the problem of finding a suitable sequence of payment matrices is formulated as a convex optimal problem. However, [17] assumes that entities cannot pay other entities more than the cash they have on hand. This means that in such a setting cash cannot make multiple steps through the network at once.

We next establish a fundamental property of the payment matrices resulting from (29).

A. The absolute priority rule

Recall that in the static (single period) case the optimal payment matrix automatically satisfies the absolute priority rule (8). A natural question arises whether a counterpart of this rule can be proved for the dynamical model in question: is it true that a bank failing to meet the nominal obligation has to nevertheless pay the maximal possible amount? Mathematically, this means that for all $t = 0, \ldots, T-1$ the following implication holds:

$$\phi_i^{\text{out}}(t) < \bar{\phi}_i^{\text{out}}(t) \Longrightarrow \phi_i^{\text{out}}(t) = \phi_i^{\text{in}}(t) + w(t). \tag{30}$$

The affirmative answer is given by the following theorem.

Theorem 1: Suppose that $[P] = (P(t))_{t=0}^{T-1}$ is an optimal solution of (29), and let $(\bar{P}(t))_{t=0}^{T}$ be the corresponding sequence of nominal liability matrices, defined in accordance to (18). For a given bank i, let $t_* = t_*(i)$ be the first instant when i pays its debt to the other banks

$$p_{ij}(t_*) = \bar{p}_{ij}(t_*) \quad \forall j \neq i$$

(if such an instant fails to exist, we formally define $t_* = T$). Then, either $t_* = 0$ (the debt is paid immediately) or

$$\phi_i^{\text{out}}(t) = \phi_i^{\text{in}}(t) + w_i(t) \;\; \forall t = 0, \ldots, (t_* - 1). \tag{31}$$

In particular, the implication (30) holds for any optimal sequence of payments matrices $[P]$. Furthermore, for each $t \geq 1$ the graph $\mathcal{G}[P(t)]$ contains no directed cycles.

A proof of Theorem 1 can be found in [19].

Remark 1: Implication (30) implies that each bank pays its nominal liability at the earliest period t when such a payment is possible: $w_i(t) + \phi_i^{\text{in}}(t) \geq \bar{\phi}_i^{\text{out}}(t)$. The requirement of minimal system loss thus pushes the banks towards paying the claims as early as possible. ◊

B. Dynamic networks with pro-rated payments

The pro-rata rule discussed in Subsection III-A can be introduced also in the dynamic network setting. Here, we let the pro-rata matrix be fixed according to the *initial* liabilities, that is the A matrix is given by (10) with $\bar{P} = \bar{P}(0)$. Then, the pro-rata rule is nothing but a linear equality constraint on the payment matrices, that is

$$P(t) = \mathrm{diag}(P(t)\mathbf{1})A, \quad t = 0, \dots, T-1. \qquad (32)$$

In view of the definition of A, one has $\bar{P}(0) = \mathrm{diag}(\bar{P}(0)\mathbf{1})A$. Using induction on t and equation (21), it can be easily shown that (32) entails the equations

$$\bar{P}(t) = \mathrm{diag}(\bar{P}(t)\mathbf{1})A, \quad t = 0, \dots, T.$$

Hence, payment matrices $P(t)$ and $\bar{P}(t)$ are uniquely determined by the actual and nominal payment vectors

$$p(t) \doteq P(t)\mathbf{1} = \phi^{\mathrm{out}}(t), \quad \bar{p}(t) \doteq \bar{P}(t)\mathbf{1} = \bar{\phi}^{\mathrm{out}}(t). \quad (33)$$

Also, it holds that $\phi^{\mathrm{in}} = P^\top(t)\mathbf{1} = A^\top p(t)$. Conditions (19), (20) can be now rewritten as

$$p(t) \geq 0, \qquad (34)$$

$$\sum\nolimits_{k=0}^{t} \alpha^{t-k} p(k) \leq \alpha^t \bar{p} \qquad (35)$$

$$C(t) + \sum\nolimits_{k=0}^{t} \left(A^\top p(k) - p(k)\right) \geq 0 \qquad (36)$$

$$\forall t \in \mathcal{T}.$$

Definition 2: We call a sequence of payment vectors $[p] \doteq (p(0), \dots, p(T-1))$ *admissible* (under the pro-rata requirement) if conditions (34)–(36). Let

$$\mathcal{P}_{\mathrm{pr}}([c], \bar{p}) \doteq \{[p] = (p(0), \dots, p(T-1)) : \text{(34)–(36) hold}\}$$

stand for the convex polytope of all admissible sequences. Optimization problem (29) can be now rewritten as

$$\max_{[p]} \sum_{k=0}^{T-1} a_k \mathbf{1}^\top p(k) \quad \text{s.t.:} \quad [p] \in \mathcal{P}_{\mathrm{pr}}([c], \bar{p}). \qquad (37)$$

This is again an LP problem, which may be solved numerically with great efficiency. The pro-rata rule drastically reduces the number of unknown variables (each zero-diagonal payment $n \times n$ matrix reduces to n-dimensional vector). Furthermore, unlike the original problem (29), the optimization problem (37) admits a *unique* maximizer $[p^*]$. Also, the solution abides by the absolute priority rule (30). These properties are summarized in the following theorem.

Theorem 2: For each sequence $[c]$, the optimization problem (37) has a unique solution $[p^*]$, moreover the optimal vector $p^*(t)$ is the unique solution of the LP:

$$p^*(t) = \arg\max_p \ 1^\top p \qquad (38)$$

$$\text{s.t.: } 0 \leq p \leq \bar{p}^*(t), \, p \leq c(t) + w^*(t) + A^\top p, \qquad (39)$$

where $w^*(0) \doteq 0$, $\bar{p}^*(0) \doteq \bar{p}$, and, for $t = 1, \dots, T-1$,

$$\bar{p}^*(t) \doteq \alpha^t \bar{p} - \sum\nolimits_{k=0}^{t-1} \alpha^{t-k} p^*(k) \qquad (40)$$

$$w^*(t) \doteq C(t-1) + \sum\nolimits_{k=0}^{t-1} \left(A^\top p^*(k) - p^*(k)\right) \qquad (41)$$

In particular, $p^*(t) \geq 0$ obeys the absolute priority rule

$$p^*(t) = \min(\bar{p}^*(t), c(t) + w^*(t) + A^\top p^*(t)). \qquad (42)$$

The proof of Theorem 2 is omitted and can be found in the extended version of this paper [19].

Theorem 2 shows that the system-level objective in the full optimization problem (37) is minimized by finding regular clearing payments at each step t, whereby the liabilities among nodes are updated at each step by considering the residual payments due to pseudo-defaults at the previous step. Note that this statement does not hold for the problem (29), e.g., the optimal $P(0)$ (and all other elements of the optimal sequence $[P]$) depends on the whole sequence $[c]$ and not only on $c(0)$ (see [19, Example 1]).

V. Numerical illustration

We consider a network of $n = 5$ nodes (including the fictitious sink node representing the external sector) with initial liability matrix

$$\bar{P} = \begin{bmatrix} 0 & 0 & 100 & 0 & 100 \\ 80 & 0 & 100 & 0 & 100 \\ 0 & 0 & 0 & 120 & 120 \\ 0 & 150 & 0 & 0 & 120 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where the last row refers to the sink. We first discuss the static case, comparing pro-rata based results obtained by solving (14) with those obtained using an unrestricted payment matrix resulting from the solution of (9). Suppose there is a nominal scenario where external cash flows are

$$c = c_{\mathrm{nom}} \doteq [120, \ 150, \ 40, \ 150, \ 0]^\top.$$

It can be readily verified that in the nominal scenario all the nodes in the network remain solvent, and the clearing payments coincide with the nominal liabilities. Consider next a situation in which "shock" happens on the in-flow at node 2, so that this in-flow reduces from 150 to 110, that is

$$c = c_{\mathrm{shock}} \doteq [120, \ 110, \ 40, \ 150, \ 0]^\top.$$

Under the pro-rata rule, the clearing payments, resulting from the solution of (14), are shown in smaller font below the nominal liabilities in the left panel of Figure 1: all nodes in the network default in a cascade fashion due to initial default of node 2. The total defaulted amount (the sum of all the unpaid liabilities) is in this case 47.28.

Then, we dropped the pro-rata rule, and we computed the clearing payments according to (9). The results in this case are shown in the right panel of Figure 1: only node 2 defaults, while all other nodes manage to pay their full liabilities. Not only we reduced the sum of all unpaid liabilities to 20 (i.e., a 57.7% decrease with respect to the pro-rata case), but we also obtained *isolation* of the contagion, since the default was confined to node 2 and did not spread to other banks.

We next considered the dynamic case. In both the pro-rata case and the full matrix case, the idea implied by the single-period (static) approach is that in case of default the financial operations of a node are frozen, that is, defaulted nodes cannot operate even if there are cash in-flows that are foreseen in the immediate future. A classical situation arises when there is a liquidity crisis, i.e., due payments

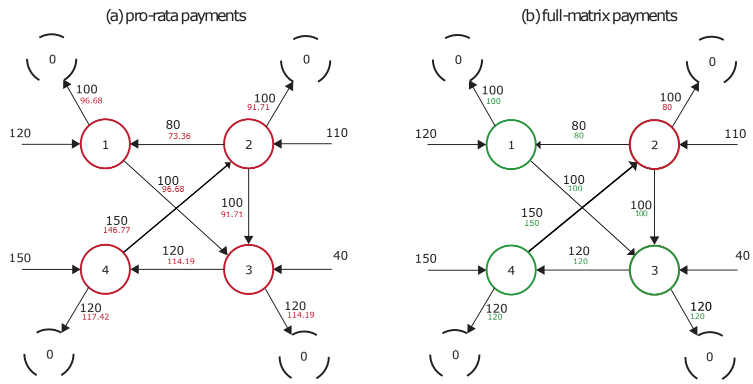

Fig. 1: Clearing payments: payments under pro-rata rule (a) vs the unrestricted clearing payments (b).

from the external sector are late and their lateness provokes defaults at some nodes, which freeze and may propagate further defaults over the network. Suppose that the in-flows in the nominal vector $c \doteq [121, 120, 41, 152, 0]^\top$ do not arrive simultaneously at time $t = 0$, due to delays, and the stream of in-flows is

$$c(0) = [60, 80, 10, 0, 0]^\top, \ c(1) = [60, 40, 10, 150, 0]^\top, \\ c(2) = [1, 0, 21, 2, 0]^\top.$$

A static approach at time $t = 0$, with unrestricted payment matrix, would result in

$$P = \begin{bmatrix} 0 & 0 & 100 & 0 & 40 \\ 80 & 0 & 100 & 0 & 20 \\ 0 & 0 & 0 & 120 & 90 \\ 0 & 120 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

with all nodes in default and a total default loss of 320. If we allow operations to continue over an horizon $T = 3$, according to the model described in Section IV, assuming an interest rate $\alpha = 1.01$ (i.e., 1% interest per period), and solving the multi-period problem (29) with full payment matrices, we obtain $P(0) = P$, and

$$P(1) = \begin{bmatrix} 0 & 0 & 0 & 0 & 60 \\ 0 & 0 & 0 & 0 & 70.3 \\ 0 & 0 & 0 & 0 & 10 \\ 0 & 30.3 & 0 & 0 & 119.7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ P(2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.61 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20.5 \\ 0 & 0 & 0 & 0 & 1.5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

After these three rounds of clearing payments, only node 2 is in default, owing a residual 10.6 to the external sector. Notice that, if we used the pro-rata rule, thus solving the multi-step problem (37), we would obtain a different set of (pro-rata) clearing payments, leading to a final situation of default at all nodes, with a total defaulted value of 22.1.

VI. Conclusions

In this paper we explored dynamic clearing mechanisms in financial networks, under both pro-rata payment rules and unrestricted matrix payments. Theorem 1 establishes some fundamental properties of the solution in the unrestricted case stating, in particular, that payments are not unnecessarily delayed (the absolute priority of debt claims), so that the solutions are indeed clearing matrices at each stage. Unrestricted optimal payments, however, are possibly non-unique and need be computed in a centralized way. Theorem 2 establishes instead key properties of the optimal pro-rated payments. The key fact is that the solution is in this case unique and, moreover, it can be computed by solving sequentially a series of LP problems (38)-(39).

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