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The stiff Neumann problem: asymptotic specialty and "kissing"domains

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Abstract

We study the stiff spectral Neumann problem for the Laplace operator in a smooth bounded domain $\Omega \subset \mathbb{R}^d$ which is divided into two subdomains: an annulus Ω_1 and a core Ω_0. The density and the stiffness constants are of order ε^{-2m} and ε^{-1} in Ω_0, while they are of order 1 in Ω_1. Here $m \in \mathbb{R}$ is fixed and $\varepsilon > 0$ is small. We provide asymptotics for the eigenvalues and the corresponding eigenfunctions as $\varepsilon \to 0$ for any m. In dimension 2 the case when Ω_0 touches the exterior boudary $\partial\Omega$ and Ω_1 gets two cusps at a point $\mathcal{O}$ is included into consideration. The possibility to apply the same asymptotic procedure as in the "smooth" case is based on the structure of eigenfunctions in the vicinity of the irregular part. The full asymptotic series as $x \to \mathcal{O}$ for solutions of the mixed boundary value problem for the Laplace operator in the cuspidal domain is given.

1 Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^d$ and let Ω_1 and Ω_0 be two bounded domains in $\mathbb{R}^d$ with smooth boundaries Γ_1 and Γ_0 respectively such that $\partial\Omega = \Gamma_1$, $\overline{\Omega}_0 \subset \Omega$ and $\Omega = \Omega_0 \cup \Omega_1 \cup \Gamma_0$. We refer to Ω_1 as the annulus and Ω_0 as the core. A typical geometrical situation is drawn in Fig. 1, where the annulus is shaded. We consider the spectral Neumann problem in $\Omega_1 \cup \Omega_0$ with natural transmission conditions for a second order differential operator with piecewise constant coefficients

$$-\Delta_x u_1^\varepsilon(x) = \lambda^\varepsilon u_1^\varepsilon(x), \qquad x \in \Omega_1, \tag{1}$$

$$-\varepsilon^{-1}\Delta_x u_0^\varepsilon(x) = \lambda^\varepsilon \varepsilon^{-2m} u_0^\varepsilon(x), \qquad x \in \Omega_0, \tag{2}$$

$$\partial_{\nu_1} u_1^\varepsilon(x) = 0, \qquad x \in \Gamma_1, \tag{3}$$

$$u_0^\varepsilon(x) = u_1^\varepsilon(x), \qquad \varepsilon^{-1}\partial_{\nu_0} u_0^\varepsilon(x) = \partial_{\nu_0} u_1^\varepsilon(x), \qquad x \in \Gamma_0, \tag{4}$$

where ∂_{ν_1} and ∂_{ν_0} denote the derivatives along outward and inward normal vectors ν_1 and ν_0 to Γ_1 and Γ_0 respectively, λ^ε is the spectral parameter and $-2m \in \mathbb{R}$ a fixed exponent. From a physical point of view, the factor ε^{-2m} reflects the dead-weight of the material, i.e. increasing m makes the material heavier. In the first part of the paper, we discuss the asymptotic behavior as $\varepsilon \to 0$ of the eigenpairs $(\lambda^\varepsilon, u^\varepsilon)$ of problem (1)-(4). We identify the (real) eigenfunctions u^ε with the pairs of functions $\{u_0^\varepsilon, u_1^\varepsilon\}$, where u_i^ε stands for the restriction of u^ε to Ω_i, $i = 0, 1$.

Depending on the orders of the relative density, namely ε^{-2m}, we predict a different asymptotic behaviour of the eigenpairs $(\lambda^\varepsilon, u^\varepsilon)$, as $\varepsilon \to 0$, putting a special emphasis on the case

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$0 < m < 1/2$. We also characterize the eigenpairs of all the limit spectral problems which have a discrete spectrum. Both for $m \in (0, 1/2)$ and $m \leq 0$, the limit problem is described by an eigenvalue problem for the Laplace operator in the annulus Ω_1 with mixed boundary conditions, while in Ω_0 the zero-order term of asymptotics of eigenfunction u_0^ε is a solution of the Laplace equation with Neumann condition. For $m > 1/2$ an eigenvalue problem for the Laplace operator posed in the core Ω_0 characterizes the limit problem for $\varepsilon = 0$, while the leading term in the annulus Ω_1 is a harmonic function. The case $m = 1/2$ is investigated in the book [32] in more general setting. Here we give an independent proof for the reader's convenience. In this case the stiffness and the density constants are of the same order, i.e. ε^{-1}, in the equation (2); two limit problems appear: the spectral Neumann problem for the Laplace operator in Ω_0 and the spectral problem with the mixed Dirichlet-Neumann boundary conditions in Ω_1. We derive estimates of convergerce rates in different situations. The spectral problems (1)-(4) are of interest in many area of physiscs. For instance, they are considered in the study of reinforcement and elasticity problems (cf. [1, 3, 4, 29]). In [20], estimates of convergence rates of the spectrum of stiff elasticity problems are obtained. We also mention the papers [12, 14], where the authors deal with the asymptotics of spectral stiff problem in domains surrounded by a thin band depending on ε. For a study of asymptotics for vibrating systems containing a stiff region independent of the small parameter ε, we refer to Sections V.7-V.10 in [32] and the papers [13, 21, 31]. The problem considered in this paper arises also in the study of different properties of porous media. They are particularly treated in the homogenization theory (cf. [2, 8, 9, 10, 30]). In the context of second order differential operator with double periodic coefficients, we also mention [5, 6, 15, 16, 34], where the authors investigate how to give rise to spectral gaps in the essential spectrum.
In the last part of the paper, we handle the same stiff problem (1)-(4) but with a geometry of the domain Ω, which differs from that drawn in Fig 1. A irregular point appears on the boundary $\partial\Omega$, consisting of the point $\mathcal{O}$ of tangency of the two "kissing" disks Ω_0 and Ω_1 in $\mathbb{R}^2$ (see Fig.2). The main feature is that the ansätze obtained when the boundary $\partial\Omega$ is smooth are still valid. It is worth to mention that in a certain sense the problem (1)-(4) can be reduced to a regular perturbation in an operator setting depending on the exponent m. In this way the full asymptotic series for eigenpairs of the problem can be readily derived in the "smooth" case after constructing the main asymptotic and first correction terms. However, in the case of "kissing" domains the perturbation analysis becomes much more involved because of possible singularities of solutions at the irregular point $\mathcal{O}$. We succeed to prove that these singularities do not interest our asymptotic procedure in the Neumann stiff problem and explain why it does not work directly for the Dirichlet stiff problem, namely when the condiction (3) is replaced by $u_1^\varepsilon(x) = 0$, $x \in \Gamma_1$. Further investigation of Dirichlet stiff problem are left as open questions to be considered. We provide the asymptotic expansion as $x \to \mathcal{O}$ of the eigenfunction of the Laplace operator along with Neumann boundary condition on the exterior boundary Γ_1 and a constant trace on the interior boundary Γ_0. The ansatz is made of particular functions depending on the geometry of the domain and the boundary conditions. Moreover, we show that all eigenfunctions decay exponentially as $x \to \mathcal{O}$ when we set a homogeneous Dirichlet condition on interior boundary Γ_0.
We mention the paper [28] in which the authors investigate the asymptotic behaviour of the eigenfunctions of Laplace operator along with Neumann boundary conditions in a bounded domain with a cuspidal point (cf [24, 26, 27]). The paper [11] discusses the regularity in the space of infinitely smooth functions in the case of cuspidal edges and the paper [22] investigates the regularity of solution of bi-harmonic operator in domains with cusps. We refer to the monographs [18, 25] for a detailed study of elliptic boundary problems in domains with other type of singularities.
The paper is organized as follows. In Section 2 we introduce the weak formulation of the problem (1)-(4). We deduce the formal asymptotic expansions for the eigenelements in the most interesting case $m \in (0, 1/2)$, whose leading terms are determined by the constant c_0 (see (14)) in Ω_0 and via Neumann spectral problem for Laplacian in Ω_1. We also discuss

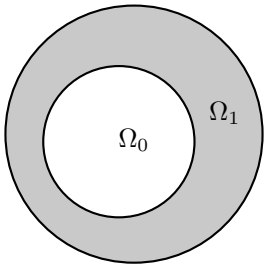

Figure 1: Annulus domain Ω_1 and core domain Ω_0

briefly the infinite asymptotic series. Section 3 contains the main result which is formulated in Theorem 3.1 and the justification of the asymptotics for $m \in (0, 1/2)$. In Sections $4-7$ we present the asymptotic expansions of eigenelements for the remaining values of m. We introduce the problems which determine the leading and the first-order correction terms and we justify the expansions. In Section 8 we derive and justify the asymptotic expansion of the eigenfunctions of the Laplace operator in $\Omega_1 \setminus \mathcal{O}$ along with the homogenuous Neumann condition on Γ_1 and the non-homogeneous Dirichlet boundary condition on Γ_0. Moreover we discuss some open questions.

2 Formal asymptotics in the case $0 < m < 1/2$

2.1 Setting of the problem

The variational formulation of problem (1)-(4) reads: find $\lambda^\varepsilon \in \mathbb{R}$ and $\{u_0^\varepsilon, u_1^\varepsilon\} \in H^1(\Omega)\setminus\{0\}$ satisfying

$$(\nabla_x u_1^\varepsilon, \nabla_x \varphi_1)_{\Omega_1} + \varepsilon^{-1}(\nabla_x u_0^\varepsilon, \nabla_x \varphi_0)_{\Omega_0} = \lambda^\varepsilon((u_1^\varepsilon, \varphi_1)_{\Omega_1} + \varepsilon^{-2m}(u_0^\varepsilon, \varphi_0)_{\Omega_0}) \tag{5}$$

$\forall \varphi \in H^1(\Omega)$. Here, $(\cdot,\cdot)_{\Omega_i}$ denotes the natural inner product of Lesbegue space $L^2(\Omega_i)$, $i = 0, 1$ and $m \in \mathbb{R}$. For each $\varepsilon > 0$ the bilinear form on the left-hand side of (5) is positive, symmetric and closed in $H^1(\Omega)$. Due to comptacness of the embeddings $H^1(\Omega_i) \hookrightarrow L^2(\Omega_i), i = 0, 1$, the problem (1)-(4) is associated with a self-adjoint operator whose spectrum consists of the monotone increasing unbounded sequence of eigenvalues (cf., for example, [7, Theorems 10.1.5 and 10.2.2])

$$0 = \lambda_1^\varepsilon < \lambda_2^\varepsilon \le \cdots \le \lambda_n^\varepsilon \le \cdots \to \infty \tag{6}$$

repeated according to their multiplicity. The corresponding eigenfunctions $\{u_0^\varepsilon, u_1^\varepsilon\}$ are subject to the orthonormalization conditions

$$(u_{1,i}^\varepsilon, u_{1,j}^\varepsilon)_{\Omega_1} + \varepsilon^{-2m}(u_{0,i}^\varepsilon, u_{0,j}^\varepsilon)_{\Omega_0} = \delta_{i,j}, \qquad i, j \in \mathbb{N}, \tag{7}$$

where $\delta_{i,j}$ is the Kronecker symbol. The orthonormalization condition (7) suggests to perform the replacements

$$v_1^\varepsilon(x) = u_1^\varepsilon(x), \quad x \in \Omega_1, \qquad\qquad v_0^\varepsilon(x) = \varepsilon^{-m} u_0^\varepsilon(x), \quad x \in \Omega_0. \tag{8}$$

Hence, $\{v_0^\varepsilon, v_1^\varepsilon\}$ satify the orthonormalization condition in $L^2(\Omega)$ which does not depend anymore on ε. Equations (1)-(2) remain unchanged, while the transmission conditions (4) turn into

$$\varepsilon^m v_0^\varepsilon(x) = v_1^\varepsilon(x), \qquad\qquad \varepsilon^{m-1}\partial_{\nu_0} v_0^\varepsilon(x) = \partial_{\nu_1} v_1^\varepsilon(x), \quad x \in \Gamma_0.$$

We look for the asymptotic expansion of eigenfunctions $\{v_0^\varepsilon, v_1^\varepsilon\}$ in the form

$$v_0^\varepsilon(x) = \varepsilon^m v_0^0(x) + \varepsilon^{1-m} v_0'(x) + \cdots, \qquad x \in \Omega_0, \tag{9}$$

$$v_1^\varepsilon(x) = v_1^0(x) + \varepsilon^{2m} v_1'(x) + \cdots, \qquad x \in \Omega_1. \tag{10}$$

We assume that the eigenvalue λ^ε admits the asymptotic ansatz

$$\lambda^\varepsilon = \lambda^0 + \varepsilon^{2m}\lambda' + \cdots. \tag{11}$$

By inserting expansions (9), (10), (11) in the spectral problem (1)-(4), we collect coefficients of the alike powers of ε and gather boundary value problems for v_0^0, v_0' and v_1^0, v_1'.

2.2 Problem for v_0^0 and v_0'

The leading term in (9) is a solution of the problem

$$-\Delta_x v_0^0(x) = 0, \quad x \in \Omega_0, \qquad \partial_{\nu_0} v_0^0(x) = 0, \quad x \in \Gamma_0, \tag{12}$$

and hence $v_0^0 = c_0$. At this stage, c_0 is an arbitrary constant in $\mathbb{R}$. The first-order correction term in (9) satisfies the boundary value problem

$$-\Delta_x v_0'(x) = \lambda^0 v_0^0(x), \quad x \in \Omega_0, \qquad \partial_{\nu_0} v_0'(x) = \partial_{\nu_0} v_1^0(x), \quad x \in \Gamma_0. \tag{13}$$

From the compatibility condition for inhomogeneous Neumann problem, we determine the constant c_0:

$$c_0 = \frac{1}{\lambda^0 |\Omega_0|} \int_{\Gamma_0} \partial_{\nu_0} v_1^0 ds_x, \tag{14}$$

where $|\cdot|$ stands for Lebesgue measure of a set and $\lambda^0 \neq 0$ is an eigenvalue of the problem (15)-(16).

2.3 Problem for v_1^0 and v_1'

The leading terms in (10) and (11) verify the spectral mixed boundary value problem

$$-\Delta_x v_1^0(x) = \lambda^0 v_1^0(x), \qquad x \in \Omega_1, \tag{15}$$

$$\partial_{\nu_1} v_1^0(x) = 0, \quad x \in \Gamma_1, \qquad v_1^0(x) = 0, \quad x \in \Gamma_0. \tag{16}$$

The variational setting implies the integral identity

$$(\nabla_x v_1^0, \nabla_x \varphi)_{\Omega_1} = \lambda^0 (v_1^0, \varphi)_{\Omega_1}, \qquad \varphi \in H_0^1(\Omega_1, \Gamma_0),$$

where $H_0^1(\Omega_1, \Gamma_0) := \{u \in H^1(\Omega_1) : u_{|\Gamma_0} = 0\}$. The spectrum of problem (15)-(16) is discrete and turns into a monotone unbounded sequences of eigenvalues

$$0 < \lambda_1^0 < \lambda_2^0 \leq \cdots \leq \lambda_n^0 \leq \cdots \to +\infty, \tag{17}$$

and the corresponding eigenfunctions $v_{1,1}^0, v_{1,2}^0, \ldots$ are subject to the orthonormalization conditions

$$(v_{1,i}^0, v_{1,j}^0)_{\Omega_1} = \delta_{i,j}, \qquad i, j \in \mathbb{N}. \tag{18}$$

The correction term in (10) is determined by the boundary value problem

$$-\Delta_x v_1'(x) - \lambda^0 v_1'(x) = \lambda' v_1^0(x), \qquad x \in \Omega_1, \tag{19}$$

$$\partial_{\nu_1} v_1'(x) = 0, \quad x \in \Gamma_1, \qquad v_1'(x) = v_0^0(x), \quad x \in \Gamma_0. \tag{20}$$

Since $v_0^0 = c_0$ is fixed and defined by (14), the boundary condition (20) becomes $v_1'(x) = c_0$, $x \in \Gamma_0$. The correction term λ' is determined through the compatibility condition in the

problem (19)-(20). First, we assume that the eigenvalue $\lambda_n^0 \neq 0$ of problem (15)-(16) is simple. Then the problem (19)-(20) has a unique solution if and only if

$$\lambda_n' \int_{\Omega_1} |v_{1,n}^0(x)|^2 dx = c_0 \int_{\Gamma_0} \partial_{\nu_0} v_{0,n}'(x) ds_x = -c_0 \int_{\Omega_0} \Delta_x v_{0,n}'(x) dx = c_0^2 \lambda_n^0 |\Omega_0|.$$

Thus, the perturbation term in the ansatz (11) takes the form

$$\lambda_n' = c_0^2 \lambda_n^0 |\Omega_0| = \frac{1}{\lambda_n^0 |\Omega_0|} \left(\int_{\Gamma_0} \partial_{\nu_0} v_1^0 ds_x \right)^2. \tag{21}$$

2.3.1 Multiple eingenvalues

In the case $\lambda_n^0 \neq 0$ is a multiple eigenvalue with multiplicity $\tau > 1$, i.e.

$$\lambda_{n-1}^0 < \lambda_n^0 = \lambda_{n+1}^0 = \cdots = \lambda_{n+\tau-1}^0 < \lambda_{n+\tau}^0, \tag{22}$$

the expansions (9)-(10) are still valid. However we predict that the leading terms of $v_{1,n}^\varepsilon, v_{1,n+1}^\varepsilon, \ldots, v_{1,n+\tau-1}^\varepsilon$ are linear combinations of the eigenfunctions $v_{1,n}^0, v_{1,n+1}^0, \ldots, v_{1,n+\tau-1}^0$ of the problem (15)-(16) associated to eigenvalue λ_n^0, i.e.

$$V_{1,j}^0(x) = a_n^j v_{1,n}^0(x) + \cdots + a_{n+\tau-1}^j v_{1,n+\tau-1}^0(x), \qquad j = n, \ldots, n+\tau-1. \tag{23}$$

Furthermore, we require that the columns

$$a^j = (a_n^j, \ldots, a_{n+\tau-1}^j)^\top \in \mathbb{R}^\tau, \qquad j = n, \ldots, n+\tau-1,$$

satisfy the orthonormalization conditions

$$(a^j, a^i) := \sum_{k=n}^{n+\tau-1} a_k^j a_k^i = \delta_{j,i}, \qquad j, i = n, \ldots, n+\tau-1. \tag{24}$$

As a consequence, the linear combinations (23) with $j = n, \ldots, n+\tau-1$ are a new orthonormal basis in the eigenspace of the eigenvalue λ_n^0.
Bearing in mind the linear combinations (23), the compatibility conditions in the problem (13) yield the new constant leading terms $v_{0,n}^0, \ldots, v_{0,n+\tau-1}^0$ of the ansatz (9)

$$v_{0,j}^0 = \frac{1}{\lambda_n^0 |\Omega_0|} \sum_{k=n}^{n+\tau-1} a_k^j \int_{\Gamma_0} \partial_{\nu_0} v_{1,k}^0 ds_x, \qquad j = 1, \ldots, n+\tau-1. \tag{25}$$

The correction term $V_{1,j}'$ is determined from the problem

$$-\Delta_x V_{1,j}'(x) - \lambda_n^0 V_{1,j}'(x) = \lambda_j' V_{1,j}^0(x), \qquad x \in \Omega_1, \tag{26}$$

$$\partial_{\nu_1} V_{1,j}'(x) = 0, \quad x \in \Gamma_1, \qquad V_{1,j}'(x) = v_{0,j}^0, \quad x \in \Gamma_0. \tag{27}$$

The Fredholm alternative leading to the necessary and sufficient condition for $V_{1,j}', j = n, \ldots, n+\tau-1$ to exist, is given by

$$\lambda_j' (V_{1,j}^0, v_{1,p}^0)_{\Omega_1} = \int_{\Gamma_0} V_{1,j}' \partial_{\nu_0} v_{1,p}^0(x) ds_x, \qquad p = n, \ldots, n+\tau-1.$$

Owing to (25) and the orthonormalization condition (18), the above formulas become

$$\lambda_j' a_p^j = \sum_{k=n}^{n+\tau-1} a_k^j \frac{1}{\lambda_n^0 |\Omega_0|} \int_{\Gamma_0} \partial_{\nu_0} v_{1,k}^0(x) ds_x \int_{\Gamma_0} \partial_{\nu_0} v_{1,p}^0(x) ds_x, \qquad p = n \ldots, n+\tau-1. \tag{28}$$

We represent the relations (28) as an algebraic spectral system

$$\mathcal{M}a^j = \lambda'_j a^j, \qquad j = n, \dots, n+\tau-1,$$

with the matrix $\mathcal{M}$ of size $\tau \times \tau$ defined by

$$\mathcal{M}_{pk} = \frac{1}{\lambda_n^0 |\Omega_0|} \int_{\Gamma_0} \partial_{\nu_0} v^0_{1,p}(x) ds_x \int_{\Gamma_0} \partial_{\nu_0} v^0_{1,k}(x) ds_x, \qquad p,k = n, \dots, n+\tau-1.$$

It is clear that $\mathcal{M}$ is a symmetric matrix, i.e. $\mathcal{M}_{pk} = \mathcal{M}_{kp}$. Therefore, it has τ real eigenvalues, $\lambda'_n, \lambda'_{n+1}, \dots, \lambda'_{n+\tau-1}$, with eigenvectors $a^n, a^{n+1}, \dots, a^{n+\tau-1}$ satisfying the orthonormalization conditions (24). Since the determinant of the matrix $\mathcal{M}$ and all its minors of order k, $1 \le k \le \tau-1$, are equal to 0, the characteristic polynomial of $\mathcal{M}$ is simply

$$(\lambda')^\tau - tr(\mathcal{M})(\lambda')^{\tau-1} = 0, \tag{29}$$

with $tr(\mathcal{M})$ being the trace of the matrix $\mathcal{M}$. It follows that the roots of (29) λ'_j , $j = n, \dots, n+\tau-1$, are given by

$$\lambda'_n = \dots = \lambda'_{n+\tau-2} = 0, \qquad \lambda'_{n+\tau-1} = tr(\mathcal{M}) = \frac{1}{\lambda_n^0 |\Omega_0|} \sum_{k=n}^{n+\tau-1} \left(\int_{\Gamma_0} \partial_{\nu_0} v^0_{1,k}(x) ds_x \right)^2. \tag{30}$$

2.4 Final remarks

The asymptotic procedure described above can be continued to construct infinite asymptotic series for eigenvalues and eigenfunctions of the problem (1)-(4). If the eigenvalues λ_n^0 is simple, the analysis just repeats the explained steps and provides the formal series

$$\sum_{j,k=0}^{\infty} \varepsilon^{jm+k(1-2m)} \lambda_n^{(j,k)}, \tag{31}$$

and the difference between the true eigenvalue λ^ε and the partial sum of the series (31) can be estimated in a way, quite similar to Section 3.
The same can be readily done in the case $\tau = 2$ when the correction term λ'_{n+1} in (30) does not vanish so that both the eigenvalues λ_n^ε and $\lambda_{n+1}^\varepsilon$ become simple and therefore can be examined independently. However, if λ_n^0 has multiplicity $\tau > 2$ or $\tau = 2$ with $\lambda'_{n+1} = 0$ (cf. (30)), the coefficients of the linear combination (23) are not completely determined. In order to compute them, the coefficients $a_n^j, \dots, a_{n+\tau-1}^j$ are assumed to be a linear combination of the eigencolumns associated to the eigenvalue 0 of the matrix $\mathcal{M}$, obtaining the coefficients and the next term of the expansion of λ^ε. Nevertheless, there is no argument ensuring that the new matrix has distint eigenvalues and hence the coefficients of linear combination of $a_n^j, \dots, a_{n+\tau-1}^j$ can not be uniquely defined, so that an iteration of the previous procedure is needed again.

3 Main result

We present the main result of this paper, which is valid for any value $m \in \mathbb{R}$.

Theorem 3.1. *For $m \in \mathbb{R}$ and for any $N \in \mathbb{N}$ there exist $\varepsilon_{N,m} > 0$ and $C_{N,m} > 0$ such that the estimate*

$$|\lambda_n^\varepsilon - \varepsilon^\alpha \lambda_n^0 - \varepsilon^\beta \lambda'_n| \le C_{N,m} \varepsilon^\gamma, \qquad n = 1, \dots, N, \tag{32}$$

holds for some α, β and γ, depending only on m, and $\varepsilon \in (0, \varepsilon_{N,m})$.

Remark 3.2. *In the estimate* (32), λ_n^ε *is the n-th eigenvalue of the problem* (1)-(4), λ_n^0 *and* λ_n' *are the corresponding leading and first-order correction terms, appearing in the different ansätze for* λ_n^ε*, which we will define in the forthcoming sections.*

In the next subsection we provide the proof of the Theorem 3.1 with $m \in (0, 1/2)$, where $\alpha = 0$, $\beta = 2m$, $\gamma = \min\{3m, 1\}$, and λ_n' is given by formula (21) for a simple eigenvalues and formulas (30) for multiple ones. The proof is split in two steps. The first one consists in proving partially that the eigenpairs $(\lambda^\varepsilon, \{u_0^\varepsilon, u_1^\varepsilon\})$ converge to $(\lambda^0, \{0, u_1^0\})$, where (λ^0, u_1^0) is an eigenpair of the limit problem (15)-(16). In the second step, we will use the so-called Lemma about near eigenvalues and eigenfunctions (cf.[33]) in order to conclude with the proof of the Theorem 3.1.

3.1 Justification of asymptotics in the case $m \in (0, 1/2)$

3.1.1 Step 1: Convergence theorem

In this subsection, we show that for fixed $n \in \mathbb{N}$ the eigenvalue λ_n^ε converges to λ_n^0, as $\varepsilon \to 0$, and the corresponding eigenfunctions converge strongly in $L^2(\Omega_1)$.

Proposition 3.3. *The eigenvalues* λ_n^ε *of the problem* (1)-(4) *and the eigenvalues* λ_n^0 *of the problem* (15)-(16) *are related by passing to the limit*

$$\lambda_n^\varepsilon \to \lambda_n^0, \qquad as \quad \varepsilon \to 0, \qquad n \in \mathbb{N}.$$

We begin to show the following lemma.

Lemma 3.4. *Assume that for any* $n \in \mathbb{N}$ *there exist* $\varepsilon_n > 0$ *and* $C_n > 0$ *such that*

$$0 < \lambda_n^\varepsilon \le C_n \qquad for \ \ \varepsilon \in (0, \varepsilon_n). \tag{33}$$

Then, we have that $\lambda_n^\varepsilon \to \lambda_{\bar n}^0$*, for some* $\bar n \in \mathbb{N}$*, as* $\varepsilon \to 0$*.*

Proof. By virtue of estimate (33), whose proof will be given in Remark 3.6, we extract an infinitesimal positive sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ such that

$$\lambda_n^{\varepsilon_k} \to \lambda_{\bar n}^0, \qquad \varepsilon_k \to 0. \tag{34}$$

In order to simplify the notation we write λ_n^ε in place of $\lambda_n^{\varepsilon_k}$. The normalization condition (7), the estimate (33) and the weak formulation (5) of the spectral problem (1)-(4) yield

$$\|\nabla_x u_{1,n}^\varepsilon\|_{\Omega_1}^2 + \varepsilon^{-1}\|\nabla_x u_{0,n}^\varepsilon\|_{\Omega_0}^2 = \lambda_n^\varepsilon \le C_n.$$

As a consequence,

$$\|\nabla_x u_{1,n}^\varepsilon\|_{\Omega_1}^2 \le C_n, \qquad \|u_{1,n}^\varepsilon\|_{\Omega_1}^2 \le 1.$$

The norms $\|u_{1,n}^\varepsilon\|_{H^1(\Omega_1)}$ are uniformly bounded in $\varepsilon \in (0, \varepsilon_n)$ for a fixed n. Then, up to subsequence, $u_{1,n}^\varepsilon$ converges weakly in $H_0^1(\Omega_1, \Gamma_0)$ and strongly in $L^2(\Omega_1)$ to some function g_1^0, which can be identified as an eigenfunction $u_{1,\bar n}^0$ associated to $\lambda_{\bar n}^0$. In fact if we take an arbitrary function $\varphi_1 \in H_0^1(\Omega_1, \Gamma_0)$, and $\varphi_0 = 0$ in Ω_0 as a test functions in the integral identity (5), it admits the limit passage as $\varepsilon \to 0$, yielding the integral identity

$$(\nabla_x g_1^0, \nabla_x \varphi_1)_{\Omega_1} = \lambda_{\bar n}^0 (g_1^0, \varphi_1)_{\Omega_1}. \tag{35}$$

The equality (35) gives rise to the problem

$$-\Delta_x g_1^0(x) = \lambda_{\bar n}^0 g_1^0(x), \qquad x \in \Omega_1,$$
$$\partial_{\nu_1} g_1^0(x) = 0, \quad x \in \Gamma_1, \qquad\qquad g_1^0(x) = 0, \quad x \in \Gamma_0,$$

which implies that $g_1^0 = u_{1,\bar{n}}^0$. In other terms, $\lambda_{\bar{n}}^0$ is an eigenvalue of the limit problem (15)-(16) with corresponding eigenfunction $u_{1,\bar{n}}^0$. Concerning the function $u_{0,n}^\varepsilon$, we find that

$$\varepsilon^{-1}\|\nabla_x u_{0,n}^\varepsilon\|_{\Omega_0}^2 \le C, \qquad \varepsilon^{-2m}\|u_{0,n}^\varepsilon\|_{\Omega_0}^2 \le 1,$$

so that $u_{0,n}^\varepsilon$ converges to 0 strongly in $H_0^1(\Omega_0)$ and hence in $L^2(\Omega_0)$ (if necessary, we can again pass to a subsequence).
The eigenfunction $u_{1,\bar{n}}^0$ is also normalized in L^2-norm. Indeed, bearing in mind the replacement (8), we deduce that

$$\|v_{0,n}^\varepsilon\|_{\Omega_0}^2 \le 1, \qquad \|\nabla_x v_{0,n}^\varepsilon\|_{\Omega_0}^2 \le \varepsilon^{1-2m}C,$$

from which it follows that $v_{0,n}^\varepsilon$ converges strongly in $L^2(\Omega_0)$ to some constant $\tilde{c}$. In order to prove that $\tilde{c} = 0$, we take $\varphi_1 = \varphi_0 = \varepsilon^m\tilde{c}$ as test functions in (5), obtaining

$$0 = \lambda_n^\varepsilon \left(\varepsilon^m \tilde{c} \int_{\Omega_1} u_{1,n}^\varepsilon dx + \tilde{c} \int_{\Omega_0} v_{0,n}^\varepsilon dx \right).$$

Passing to the limit as $\varepsilon \to 0$, we find that $\tilde{c} = 0$. As a consequence, $\varepsilon^{-2m}\|u_{0,n}^\varepsilon\|_{\Omega_0}^2 \to 0$, as $\varepsilon \to 0$ and the normalization condition (7) leads to $\|u_{1,\bar{n}}^0\|_{L^2(\Omega_1)} = 1$. □

The goal of the next subsection is to check that $n = \bar{n}$, concluding hence the proofs of Proposition 3.3 and of Theorem 3.1.

3.1.2 Step 2: Lemma about near eigenvalues and eigenfunctions

Let $\mathcal{H}_\varepsilon$ denote the Hilbert space $H^1(\Omega)$ endowed with the inner product

$$\langle U, V\rangle_\varepsilon = (\nabla_x U_1, \nabla_x V_1)_{\Omega_1} + \varepsilon^{-1}(\nabla_x U_0, \nabla_x V_0)_{\Omega_0} + (U_1, V_1)_{\Omega_1} + \varepsilon^{-2m}(U_0, V_0)_{\Omega_0}. \quad (36)$$

We introduce the operator $\mathcal{K}_\varepsilon$ in $\mathcal{H}_\varepsilon$ by the formula

$$\langle \mathcal{K}_\varepsilon U, V\rangle_\varepsilon = (U_1, V_1)_{\Omega_1} + \varepsilon^{-2m}(U_0, V_0)_{\Omega_0} \qquad \forall\, U, V \in \mathcal{H}_\varepsilon, \quad (37)$$

and the new spectral parameter

$$k_n^\varepsilon = (1 + \lambda_n^\varepsilon)^{-1}. \quad (38)$$

It is easy to verify that $\mathcal{K}_\varepsilon$ is a continuous, self-adjoint, positive and compact operator. Thus, the spectrum of operator $\mathcal{K}_\varepsilon$ consists of the essential spectrum $\sigma_{\rm ess}(\mathcal{K}_\varepsilon) = \{0\}$ and an infinitesimal positive sequence of real eigenvalues

$$k_1^\varepsilon \ge k_2^\varepsilon \ge \cdots \ge k_n^\varepsilon \ge \cdots \to 0.$$

Taking into account formulas (36)-(38), the integral identity (5) is equivalent to the abstract equation

$$\mathcal{K}_\varepsilon U^\varepsilon = k_n^\varepsilon U^\varepsilon.$$

The following statement is known as "lemma about near eigenvalues and eigenvectors"(cf. [33]) and follows from the spectral decomposition of the resolvent, cf. [7, Chapter 6].

Lemma 3.5. *Assume $\mathfrak{U}^\varepsilon \in \mathcal{H}_\varepsilon$ and $\mathfrak{k}^\varepsilon \in \mathbb{R}_+$ such that*

$$\|\mathfrak{U}^\varepsilon\|_{\mathcal{H}_\varepsilon} = 1, \qquad \|\mathcal{K}_\varepsilon\mathfrak{U}^\varepsilon - \mathfrak{k}^\varepsilon\mathfrak{U}^\varepsilon\|_{\mathcal{H}_\varepsilon} =: \delta^\varepsilon \in (0, \mathfrak{k}^\varepsilon).$$

Then in the segment $[\mathfrak{k}^\varepsilon - \delta^\varepsilon, \mathfrak{k}^\varepsilon + \delta^\varepsilon]$ there is at least one eigenvalue of the operator $\mathcal{K}_\varepsilon$. Moreover, for any $\delta'_\varepsilon \in (\delta^\varepsilon, \mathfrak{k}^\varepsilon)$ there exist coefficients $\mathfrak{a}_{J^\varepsilon}^\varepsilon, \cdots, \mathfrak{a}_{J^\varepsilon+K^\varepsilon-1}^\varepsilon$ such that

$$\left\| \mathfrak{U}^\varepsilon - \sum_{j=J^\varepsilon}^{J^\varepsilon+K^\varepsilon-1} \mathfrak{a}_j^\varepsilon \mathfrak{u}_j^\varepsilon \right\|_{\mathcal{H}_\varepsilon} \le 2\frac{\delta^\varepsilon}{\delta'_\varepsilon}, \qquad \sum_{j=J^\varepsilon}^{J^\varepsilon+K^\varepsilon-1} |\mathfrak{a}_j^\varepsilon|^2 = 1,$$

where $\mathfrak{u}^\varepsilon_{J^\varepsilon},\cdots,\mathfrak{u}^\varepsilon_{J^\varepsilon+K^\varepsilon-1}$ *are eigenvectors associated to all eigenvalues* $k^\varepsilon_{J^\varepsilon},\cdots,k^\varepsilon_{J^\varepsilon+K^\varepsilon-1}$ *of the operator* $\mathcal{K}_\varepsilon$ *situated in* $[\mathfrak{k}^\varepsilon-\delta^\varepsilon,\mathfrak{k}^\varepsilon+\delta^\varepsilon]$. *The eigenvectors are subject to the orthonormalization conditions*

$$\langle \mathfrak{u}^\varepsilon_i,\mathfrak{u}^\varepsilon_j\rangle_\varepsilon=\delta_{i,j}. \tag{39}$$

In the case of a simple eigenvalue $\lambda^0_{\bar n}$ of the problem (15)-(16), the approximate eigenvalue $\mathfrak{k}^\varepsilon_{\bar n}$ is

$$(1+\lambda^0_{\bar n}+\varepsilon^{2m}\lambda'_{\bar n})^{-1}, \tag{40}$$

where $\lambda'_{\bar n}$ is the asymptotic correction (21) and the approximate eigenfunction $\mathfrak{U}^\varepsilon_{\bar n}=(\mathfrak{U}^\varepsilon_{0,\bar n},\mathfrak{U}^\varepsilon_{1,\bar n})$ is defined by

$$(\varepsilon^{2m}c_{0,\bar n}+\varepsilon u'_{0,\bar n}+\varepsilon^{2-2m}\mathscr{U}^\varepsilon_{0,\bar n},\ \ u^0_{1,\bar n}+\varepsilon^{2m}u'_{1,\bar n}+\varepsilon\mathscr{U}_{1,\bar n}+\varepsilon^{2-2m}\mathscr{U}'_{1,\bar n}), \tag{41}$$

where $c_{0,\bar n}$ is given by (14), $u'_{0,\bar n}$ is the solution to the problem (13), $u^0_{1,\bar n}$ solves the limit problem (15)-(16) and $u'_{1,\bar n}$ is characterized by the problem (19)-(20). The arbitrary (but fixed) functions $\mathscr{U}_{1,\bar n},\mathscr{U}'_{1,\bar n}$ in $H^1(\Omega_1)$ are such that

$$\mathscr{U}_{1,\bar n}(x)=u'_{0,\bar n}(x),\qquad \mathscr{U}'_{1,\bar n}(x)=\mathscr{U}^\varepsilon_{0,\bar n}(x),\qquad x\in\Gamma_0,$$

and $\mathscr{U}^\varepsilon_{0,\bar n}$ is the solution to the Neumann problem for the Helmholtz operator

$$-\Delta_x\mathscr{U}^\varepsilon_{0,\bar n}(x)-\varepsilon^{1-2m}\lambda^0_{\bar n}\mathscr{U}^\varepsilon_{0,\bar n}(x)=\lambda^0_{\bar n}u'_{0,\bar n}(x),\qquad x\in\Omega_0, \tag{42}$$

$$\partial_{\nu_0}\mathscr{U}^\varepsilon_{0,\bar n}(x)=0,\qquad x\in\Gamma_0. \tag{43}$$

Denoting by $L^2_\perp(\Omega_0)$ the subspace $\{u\in L^2(\Omega_0):\int_{\Omega_0}u(x)dx=0\}$ and setting $H^2_\perp(\Omega_0)=H^2(\Omega_0)\cap L^2_\perp(\Omega_0)$, the Neumann Laplacian $\Delta_x:H^2_\perp(\Omega_0)\to L^2_\perp(\Omega_0)$ is an isomorphism. Consequently, for small $\varepsilon>0$ the mapping $-\Delta_x-\varepsilon^{1-2m}\lambda^0_{\bar n}\mathrm{Id}$ is also an isomorphism, i.e. $\mathscr{U}^\varepsilon_{0,\bar n}$ is the unique solution to the problem (42)-(43) (cf., e.g., [17, Theorem 3.6.1]). Furthermore, the estimate

$$\|\mathscr{U}^\varepsilon_{0,\bar n}\|_{H^2_\perp(\Omega_0)}\le c\lambda^0_n\|u'_{0,\bar n}\|_{L^2_\perp(\Omega_0)}$$

holds, where the constant c is independent of the parameter ε.
If $\lambda^0_{\bar n}$ is a multiple eigenvalue (cf. (22)) and $\lambda'_{\bar n}$ in (40) is given by (30), then the functions $u^0_{1,j},c_{0,j},u'_{1,j}$ in (41) are replaced with $V^0_{1,j},v^0_{0,j}$ defined by the formulas (23), (25) and the solution $V'_{1,j}$ to the problem (26)-(27) for any $j=\bar n,\cdots,\bar n+\tau-1$.
The almost eigenfunction $\mathfrak{U}^\varepsilon_{\bar n}$ belongs to Hilbert space $\mathcal{H}_\varepsilon$ but in generally it does not satisfy the normalization condition. Then, we apply Lemma 3.5 with $\|\mathfrak{U}^\varepsilon_{\bar n}\|^{-1}_{\mathcal{H}_\varepsilon}\mathfrak{U}^\varepsilon_{\bar n}\in\mathcal{H}_\varepsilon$. Note that for sufficiently small ε the estimate

$$\|\mathfrak{U}^\varepsilon_{\bar n}\|_{\mathcal{H}_\varepsilon}\ge\frac12, \tag{44}$$

follows from formula (45). Indeed, the inner product

$$\begin{aligned}\langle\mathfrak{U}^\varepsilon_i,\mathfrak{U}^\varepsilon_j\rangle_\varepsilon&=(\nabla_x\mathfrak{U}^\varepsilon_{1,i},\ \ \nabla_x\mathfrak{U}^\varepsilon_{1,j})_{\Omega_1}+\varepsilon^{-1}(\nabla_x\mathfrak{U}^\varepsilon_{0,i},\ \ \nabla_x\mathfrak{U}^\varepsilon_{0,j})_{\Omega_0}+(\mathfrak{U}^\varepsilon_{1,i},\ \ \mathfrak{U}^\varepsilon_{1,j})_{\Omega_1}\\&\quad+\varepsilon^{-2m}(\mathfrak{U}^\varepsilon_{0,i},\ \ \mathfrak{U}^\varepsilon_{0,j})_{\Omega_0}\\&=(\nabla_xu^0_{1,i},\nabla_xu^0_{1,j})_{\Omega_1}+(u^0_{1,i},u^0_{1,j})_{\Omega_1}+O(\varepsilon^{2m})\\&=(1+\lambda^0_p)(u^0_{1,p},u^0_{1,q})_{\Omega_1}+O(\varepsilon^{2m})\\&=(1+\lambda^0_i)\delta_{ij}+O(\varepsilon^{2m}),\qquad\qquad i,j=1,2,\dots\end{aligned} \tag{45}$$

where the last equality is due to orthonormalization conditions (18). Note that $O(\varepsilon^{2m})$ contains the terms listed below multiplied by some power of ε and they can be easily estimated:

$$
\begin{aligned}
\varepsilon^{2m} &: (\nabla_x u^0_{1,i}, \nabla_x u^0_{1,j})_{\Omega_1} + (\nabla_x u'_{1,i}, \nabla_x u^0_{1,j})_{\Omega_1} + (u^0_{1,i}, u'_{1,j})_{\Omega_1} + (u'_{1,i}, u^0_{1,j})_{\Omega_1};\\
\varepsilon^{4m} &: (\nabla_x u'_{1,i}, \nabla_x u'_{1,j})_{\Omega_1} + (u'_{1,i}, u'_{1,j})_{\Omega_1}; \qquad \varepsilon^{4-6m} : (\mathscr{U}^\varepsilon_{0,i}, \mathscr{U}^\varepsilon_{0,j})_{\Omega_0};\\
\varepsilon &: (\nabla_x u^0_{1,i}, \nabla_x \mathscr{U}_{1,j})_{\Omega_1} + (\nabla_x \mathscr{U}'_{1,i}, \nabla_x u^0_{1,j})_{\Omega_1} + (\nabla_x u'_{0,i}, \nabla_x u'_{0,j})_{\Omega_0}\\
&\quad + (u^0_{1,i}, \mathscr{U}_{1,j})_{\Omega_1} + (\mathscr{U}_{1,i}, u^0_{1,j})_{\Omega_1};\\
\varepsilon^{2m+1} &: (\nabla_x u'_{1,i}, \nabla_x \mathscr{U}_{1,j})_{\Omega_1} + (\nabla_x \mathscr{U}_{1,i}, \nabla_x \mathscr{U}_{1,j})_{\Omega_1} + (u'_{1,i}, \mathscr{U}_{1,j})_{\Omega_1} + (\mathscr{U}'_{1,i}, u'_{1,j})_{\Omega_1};\\
\varepsilon^{2-2m} &: (\nabla_x u^0_{1,i}, \nabla_x \mathscr{U}'_{1,j})_{\Omega_1} + (\nabla_x \mathscr{U}'_{1,i}, \nabla_x u^0_{1,j})_{\Omega_1} + (\nabla_x u'_{0,i}, \nabla_x \mathscr{U}^\varepsilon_{0,j})_{\Omega_0}\\
&\quad + (\nabla_x \mathscr{U}^\varepsilon_{0,i}, \nabla_x u'_{0,j})_{\Omega_0} + (u^0_{1,i}, \mathscr{U}'_{1,j})_{\Omega_1} + (\mathscr{U}'_{1,i}, u^0_{1,j})_{\Omega_1} + (u'_{0,i}, u'_{0,j})_{\Omega_0};\\
\varepsilon^{2} &: (\nabla_x u'_{1,i}, \nabla_x \mathscr{U}'_{1,j})_{\Omega_1} + (\nabla_x \mathscr{U}_{1,i}, \nabla_x \mathscr{U}_{1,j})_{\Omega_1} + (\nabla_x \mathscr{U}'_{1,i}, \nabla_x u'_{1,j})_{\Omega_1}\\
&\quad + (u'_{1,i}, \mathscr{U}'_{1,j})_{\Omega_1} + (\mathscr{U}_{1,i}, \mathscr{U}_{1,j})_{\Omega_1} + (\mathscr{U}'_{1,i}, \mathscr{U}'_{1,j})_{\Omega_1};\\
\varepsilon^{3-4m} &: (\nabla_x \mathscr{U}^\varepsilon_{0,i}, \nabla_x \mathscr{U}^\varepsilon_{0,j})_{\Omega_0} + (u'_{0,i}, \mathscr{U}^\varepsilon_{0,j})_{\Omega_0} + (\mathscr{U}^\varepsilon_{0,i}, u'_{0,j})_{\Omega_0};\\
\varepsilon^{4-4m} &: (\nabla_x \mathscr{U}_{1,i}, \nabla_x \mathscr{U}'_{1,j})_{\Omega_1} + (\nabla_x \mathscr{U}'_{1,i}, \nabla_x \mathscr{U}_{1,j})_{\Omega_1} + (\mathscr{U}_{1,i}, \mathscr{U}'_{1,j})_{\Omega_1} + (\mathscr{U}'_{1,i}, \mathscr{U}_{1,j})_{\Omega_1};\\
\varepsilon^{4-4m} &: (\nabla_x \mathscr{U}'_{1,i}, \nabla_x \mathscr{U}'_{1,j})_{\Omega_1} + (\mathscr{U}'_{1,i}, \mathscr{U}'_{1,j})_{\Omega_1}.
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
\delta^\varepsilon_{\bar n} &= \|\mathfrak{U}^\varepsilon_{\bar n}\|^{-1}_{\mathcal{H}_\varepsilon} \|\mathcal{K}_\varepsilon \mathfrak{U}^\varepsilon_{\bar n} - \mathfrak{k}^\varepsilon_{\bar n} \mathfrak{U}^\varepsilon_{\bar n}\|_{\mathcal{H}_\varepsilon}\\
&= \|\mathfrak{U}^\varepsilon_{\bar n}\|^{-1}_{\mathcal{H}_\varepsilon} \sup_{\substack{W_\varepsilon \in \mathcal{H}_\varepsilon\\ \|W_\varepsilon\|_{\mathcal{H}_\varepsilon}=1}} |\langle \mathcal{K}_\varepsilon \mathfrak{U}^\varepsilon_{\bar n} - \mathfrak{k}^\varepsilon_{\bar n} \mathfrak{U}^\varepsilon_{\bar n}, W^\varepsilon\rangle_\varepsilon|\\
&= \|\mathfrak{U}^\varepsilon_{\bar n}\|^{-1}_{\mathcal{H}_\varepsilon} (\mathfrak{k}^\varepsilon_{\bar n})^{-1} \sup_{\substack{W_\varepsilon \in \mathcal{H}_\varepsilon\\ \|W_\varepsilon\|_{\mathcal{H}_\varepsilon}=1}} |(\mathfrak{k}^\varepsilon_{\bar n})^{-1} \langle \mathcal{K}_\varepsilon \mathfrak{U}^\varepsilon_{\bar n}, W^\varepsilon\rangle_\varepsilon - \langle \mathfrak{U}^\varepsilon_{\bar n}, W^\varepsilon\rangle_\varepsilon|\\
&\le c \sup_{\substack{W_\varepsilon \in \mathcal{H}_\varepsilon\\ \|W_\varepsilon\|_{\mathcal{H}_\varepsilon}=1}} |(\mathfrak{k}^\varepsilon_{\bar n})^{-1} \langle \mathcal{K}_\varepsilon \mathfrak{U}^\varepsilon_{\bar n}, W^\varepsilon\rangle_\varepsilon - \langle \mathfrak{U}^\varepsilon_{\bar n}, W^\varepsilon\rangle_\varepsilon|,
\end{aligned}
$$

where in the last inequality we used (44) and $(\mathfrak{k}^\varepsilon_{\bar n})^{-1} \ge 1$. Now, we focus only on the absolute value. Using formulas (36) and (37), we find

$$
\begin{aligned}
|(\mathfrak{k}^\varepsilon_{\bar n})^{-1} \langle \mathcal{K}_\varepsilon \mathfrak{U}^\varepsilon_{\bar n}, W^\varepsilon\rangle_\varepsilon - \langle \mathfrak{U}^\varepsilon_{\bar n}, W^\varepsilon\rangle_\varepsilon| = |&J_0 + \varepsilon^{2m} J_1 + \varepsilon^{1-2m} J_2 + \varepsilon^{4m} \lambda'_{\bar n}(u'_{1,\bar n}, W^\varepsilon_1)_{\Omega_1}\\
&+ \varepsilon^{2-4m} \lambda^0_{\bar n}(\mathscr{U}^\varepsilon_{0,\bar n}, W^\varepsilon_0)_{\Omega_0} + \varepsilon J_3 + \varepsilon^{1+2m} \lambda'_{\bar n}(\mathscr{U}_{1,\bar n}, W^\varepsilon_1)_{\Omega_1}\\
&\varepsilon^{2-2m} J_4 + \varepsilon^2 \lambda'_{\bar n}(\mathscr{U}'_{1,\bar n}, W^\varepsilon_1)_{\Omega_1}|. \qquad (46)
\end{aligned}
$$

Here

$$
\begin{aligned}
J_0 &= \lambda^0_{\bar n}(u^0_{1,\bar n}, W^\varepsilon_1)_{\Omega_1} + \lambda^0_{\bar n}(c_{0,\bar n}, W^\varepsilon_0)_{\Omega_0} - (\nabla_x u^0_{1,\bar n}, \nabla_x W^\varepsilon_1)_{\Omega_1} - (\nabla_x u'_{0,\bar n}, \nabla_x W^\varepsilon_0)_{\Omega_0};\\
J_1 &= \lambda'_{\bar n}(c_{0,\bar n}, W^\varepsilon_0)_{\Omega_0} + \lambda^0_{\bar n}(u'_{1,\bar n}, W^\varepsilon_1)_{\Omega_1} + \lambda'_{\bar n}(u^0_{1,\bar n}, W^\varepsilon_1)_{\Omega_1} - (\nabla_x u'_{1,\bar n}, \nabla_x W^\varepsilon_1)_{\Omega_1};\\
J_2 &= \lambda^0_{\bar n}(u'_{0,\bar n}, W^\varepsilon_0)_{\Omega_0} - (\nabla_x \mathscr{U}^\varepsilon_{0,\bar n}, \nabla_x W^\varepsilon_0)_{\Omega_0};\\
J_3 &= \lambda^0_{\bar n}(\mathscr{U}_{1,\bar n}, W^\varepsilon_1)_{\Omega_1} + \lambda'_{\bar n}(u'_{0,\bar n}, W^\varepsilon_0)_{\Omega_0} - (\nabla_x \mathscr{U}_{1,\bar n}, \nabla_x W^\varepsilon_1)_{\Omega_1};\\
J_4 &= \lambda^0_{\bar n}(\mathscr{U}'_{1,\bar n}, W^\varepsilon_1)_{\Omega_1} + \lambda'_{\bar n}(\mathscr{U}^\varepsilon_{0,\bar n}, W^\varepsilon_0)_{\Omega_0} - (\nabla_x \mathscr{U}'_{1,\bar n}, \nabla_x W^\varepsilon_1)_{\Omega_1}.
\end{aligned}
$$

Integrating by parts the problems (15)-(16), (13) and (19)-(20), the expression under the modulus sign on the right-hand side of (46) becomes

$$
\begin{aligned}
|(\mathfrak{k}^\varepsilon_{\bar n})^{-1} \langle \mathcal{K}_\varepsilon \mathfrak{U}^\varepsilon_{\bar n}, W^\varepsilon\rangle_\varepsilon - \langle \mathfrak{U}^\varepsilon_{\bar n}, W^\varepsilon\rangle_\varepsilon| = |&\varepsilon^{2m} J'_1 + \varepsilon^{1-2m} J_2 + \varepsilon^{4m} \lambda'_{\bar n}(u'_{1,\bar n}, W^\varepsilon_1)_{\Omega_1}\\
&+ \varepsilon^{2-4m} \lambda^0_{\bar n}(\mathscr{U}^\varepsilon_{0,\bar n}, W^\varepsilon_0)_{\Omega_0} + \varepsilon J_3 + \varepsilon^{1+2m} \lambda'_{\bar n}(\mathscr{U}_{1,\bar n}, W^\varepsilon_1)_{\Omega_1}\\
&+ \varepsilon^{2-2m} J_4 + \varepsilon^2 \lambda'_{\bar n}(\mathscr{U}'_{1,\bar n}, W^\varepsilon_1)_{\Omega_1}|,
\end{aligned}
$$

with $J_1' = \lambda_{\bar n}'(c_{0,\bar n}, W_0^\varepsilon)_{\Omega_0} - (\partial_{\nu_0} u_{1,\bar n}', W_1^\varepsilon)_{\Gamma_0}$. Note that $\varepsilon^{1-2m} J_2 + \varepsilon^{2-4m}\lambda_{\bar n}^0(\mathscr{U}_{0,\bar n}^\varepsilon, W_0^\varepsilon)_{\Omega_0} = 0$ due to the fact that $\mathscr{U}_{0,\bar n}^\varepsilon$ can be written as Neumann series (cf. [17, Theorem 3.6.1]). Moreover, the definition of the inner product (36) in the Hilbert space $\mathcal{H}_\varepsilon$ yields the following estimates of the classical norm in $L^2(\Omega_i)$, $i = 0, 1$

$$\|W_1^\varepsilon\|_{\Omega_1} \le \|W\|_{\mathcal{H}_\varepsilon}, \quad \|\nabla_x W_1^\varepsilon\|_{\Omega_1} \le \|W^\varepsilon\|_{\mathcal{H}_\varepsilon}, \quad \|W^\varepsilon\|_{\Omega_0} \le \varepsilon^m \|W^\varepsilon\|_{\mathcal{H}_\varepsilon}.$$

Finally,

$$\delta_{\bar n}^\varepsilon \le C_1\varepsilon^{3m} + C_2\varepsilon + C_3\varepsilon^{4m} + C_4\varepsilon^{1+m} + C_5\varepsilon^{1+2m} + C_6\varepsilon^{2-2m} + C_7\varepsilon^{2-m} + C_8\varepsilon^2 \le C\varepsilon^\gamma,$$

where $\gamma = \min\{3m, 1\}$. Then, the first part of Lemma 3.5 implies that there exists at least one eigenvalue k_n^ε of $\mathcal{K}_\varepsilon$ such that

$$|\mathfrak{k}_{\bar n}^\varepsilon - k_n^\varepsilon| \le C\varepsilon^\gamma. \tag{47}$$

Bearing in mind Lemma 3.4, the inequality (47) can be written as

$$|\lambda_n^\varepsilon - \lambda_{\bar n}^0 - \varepsilon^{2m}\lambda_{\bar n}'| \le C|1 + \lambda_n^\varepsilon||1 + \lambda_{\bar n}^0 + \varepsilon^{2m}\lambda_{\bar n}'|\varepsilon^\gamma \le C_n\varepsilon^\gamma \qquad \forall \varepsilon \in (0, \varepsilon_N). \tag{48}$$

In order to show (32) and to conclude the proof of Theorem 3.1, we must check that the indices n and $\bar n$ in the inequality (48) coincide. To this end, we will apply the second part of Lemma 3.5.
Assume that $\lambda_{\bar n}^0$ is an eigenvalue of multiplicity $\tau \ge 2$ of the problem (15)-(16). We can associate τ copies of almost eigenfunctions $\mathfrak{U}_p^\varepsilon$, given by (41), and introduce

$$\delta_{\bar n\varepsilon}' = T\max\{\delta_{\bar n}^\varepsilon, \cdots, \delta_{\bar n+\tau-1}^\varepsilon\},$$

where T is large and fixed (independent of ε). The second part of Lemma 3.5 gives the normalized columns $\mathfrak{a}_{J^\varepsilon}^{p\varepsilon}, \cdots, \mathfrak{a}_{J^\varepsilon+K^\varepsilon-1}^{p\varepsilon}$ verifying the inequality

$$\left\| \mathfrak{U}_p^\varepsilon - \sum_{j=J^\varepsilon}^{J^\varepsilon+K^\varepsilon-1} \mathfrak{a}_j^{p\varepsilon}\mathfrak{u}_j^\varepsilon \right\|_{\mathcal{H}_\varepsilon} \le \frac{2}{T}, \qquad p = \bar n, \cdots, \bar n + \tau - 1. \tag{49}$$

We aim to show that in the closure of a $\delta_{\bar n\varepsilon}'$-neighbourhood of the point $\mathfrak{k}_{\bar n}^\varepsilon$ there are at least τ eigenvalues of the operator $\mathcal{K}_\varepsilon$, i.e. in the inequality (49) the number K^ε is such that $K^\varepsilon \ge \tau$. The equality (45) implies the estimate

$$|\langle \mathfrak{U}_p^\varepsilon, \mathfrak{U}_q^\varepsilon\rangle_\varepsilon - (1 + \lambda_p^0)\delta_{p,q}| \le C\varepsilon^{2m}, \qquad p, q = \bar n, \ldots, \bar n + \tau - 1. \tag{50}$$

Set

$$S_p^\varepsilon = \sum_{j=N^\varepsilon}^{N^\varepsilon+K^\varepsilon-1} \mathfrak{a}_j^{p\varepsilon}\mathfrak{u}_j^\varepsilon, \qquad p = \bar n, \ldots, \bar n + \tau - 1.$$

In view of the estimate (50) and orthonormalization condition (39) of $\mathfrak{u}^\varepsilon$, we find

$$\begin{aligned}
\left| \sum_{j=N^\varepsilon}^{N^\varepsilon+K^\varepsilon-1} \mathfrak{a}_j^{p\varepsilon}\overline{\mathfrak{a}_j^{q\varepsilon}} - (1 + \lambda_{\bar n}^0)\delta_{p,q} \right| &= \left| \langle \sum_{j=N^\varepsilon}^{N^\varepsilon+K^\varepsilon-1} a_j^{p\varepsilon} u_j^\varepsilon, \sum_{j=N^\varepsilon}^{N^\varepsilon+K^\varepsilon-1} a_j^{q\varepsilon} u_j^\varepsilon\rangle_\varepsilon - (1 + \lambda_{\bar n}^0)\delta_{p,q} \right| \\
&= |\langle S_p^\varepsilon, S_q^\varepsilon\rangle_\varepsilon - \langle S_p^\varepsilon, \mathfrak{U}_q^\varepsilon\rangle_\varepsilon + \langle S_p^\varepsilon, \mathfrak{U}_q^\varepsilon\rangle_\varepsilon - \langle \mathfrak{U}_p^\varepsilon, \mathfrak{U}_q^\varepsilon\rangle_\varepsilon \\
&\quad + \langle \mathfrak{U}_p^\varepsilon, \mathfrak{U}_q^\varepsilon\rangle_\varepsilon - (1 + \lambda_{\bar n}^0)\delta_{p,q}| \\
&\le |\langle S_p^\varepsilon, S_q^\varepsilon - \mathfrak{U}_q^\varepsilon\rangle_\varepsilon + \langle S_p^\varepsilon - \mathfrak{U}_p^\varepsilon, \mathfrak{U}_q^\varepsilon\rangle_\varepsilon| + |\langle \mathfrak{U}_p^\varepsilon, \mathfrak{U}_q^\varepsilon\rangle_\varepsilon - (1 + \lambda_{\bar n}^0)\delta_{p,q}| \\
&\le \|S_p^\varepsilon\|_\varepsilon \|S_q^\varepsilon - \mathfrak{U}_q^\varepsilon\|_\varepsilon + \|S_p^\varepsilon - \mathfrak{U}_p^\varepsilon\|_\varepsilon \|\mathfrak{U}_q^\varepsilon\|_\varepsilon + O(\varepsilon^{2m}) \\
&\le C(\varepsilon^{2m} + \frac{4}{T}).
\end{aligned}$$

We conclude that for sufficiently large T, the columns $\mathfrak{a}^{p\varepsilon}$ turn out to be almost orthonormalized that is possible only if $K^\varepsilon \geq \tau$. Consequently, for τ-multiple eigenvalue $\lambda^0_{\bar n}$ there are at least τ distinct eigenvalues $k^\varepsilon_j, \dots, k^\varepsilon_{j+\tau-1}$ of the operator $\mathcal{H}_\varepsilon$ such that

$$|\mathfrak{k}^\varepsilon_{\bar n} - k^\varepsilon_j| \leq T c_n \varepsilon^\gamma, \qquad j = n, \dots, n+\tau-1. \tag{51}$$

Remark 3.6. *The formula* (51) *leads to check the inequality* (33)*. Indeed, for each eigenvalue λ^0_n of the sequence* (17)*, one can associate the eigenvalue $\lambda^\varepsilon_{M(n)}$ such that $\lambda^\varepsilon_{M(n)} \leq \lambda^0_n + C_n\varepsilon^\gamma$. Moreover $M(n_1) < M(n_2)$ if $n_1 < n_2$. Consequently $n < M(n)$ and*

$$\lambda^\varepsilon_n \leq \lambda^\varepsilon_{M(n)} \leq \lambda^0_n + C_n\varepsilon^{2m} \leq \lambda^0_n + C_n,$$

which implies (33)*.*

In order to conclude the proof of Theorem 3.1, we will check that the eigenvalues $\lambda^\varepsilon_n, \dots, \lambda^\varepsilon_{n+\tau-1}$ of the sequence (6) verify the estimate (32). In other words, the equality $\bar n = n$ holds true in (48).
Let $\bar n$ be some index such that $\lambda^0_{\bar n}$ is τ-multiple eigenvalue of the problem (15)-(16). If we assume $M(\bar n+\tau-1) > \bar n+\tau-1$, then there exists an eigenvalue $\lambda^\varepsilon_{J^\varepsilon}$ with $J^\varepsilon \leq M(\bar n+\tau-1)$, such that

$$\lambda^\varepsilon_{J^\varepsilon} \leq \lambda^0_{\bar n+\tau-1} + \varepsilon^{2m}\lambda'_{\bar n+\tau-1} + C\varepsilon^\gamma < \lambda^0_{\bar n+\tau}.$$

From Lemma 3.4, the eigenpair $(\lambda^\varepsilon_{J^\varepsilon}, U^\varepsilon_{J^\varepsilon})$ converges to eigenelement $(\lambda^*_{J^0}, U^*)$ of the limit problem (15)-(16), where U^* is orthogonal to $U^0_1, \cdots, U^0_{\bar n+\tau-1}$ in $L^2(\Omega_1)$. This last claim is invalid, because of the min-max principle (see, e.g. [7])

$$\lambda^*_{J^0} = \max_{\substack{E\subset H^1_0(\Omega_1)\\ dim E = J^\varepsilon - 1}} \min_{\substack{v\in E^\perp\\ v\neq 0}} \frac{\|\nabla_x v\|_{L^2(\Omega_1)}}{\|v\|_{L^2(\Omega_1)}}$$

and the inequality $\lambda^*_{J^0} < \lambda^0_{\bar n+\tau}$. Thus, $\bar n = n$ and Theorem 3.1 is proved.

4 Asymptotic expansion for $m < 0$

We briefly describe the behaviour of the eigenpairs of the problem (1)-(4) for $m < 0$. The proof of Theorem 3.1 uses the same argument as in Section 3. We consider an asymptotic expansion for an eigenvalue λ^ε and the corresponding eigenfunction $\{u^\varepsilon_0, u^\varepsilon_1\}$ of the form

$$\lambda^\varepsilon = \lambda^0 + \varepsilon\lambda' + \cdots, \tag{52}$$

$$u^\varepsilon_0(x) = u^0_0(x) + \varepsilon u'_0(x) + \cdots, \qquad x \in \Omega_0, \tag{53}$$

$$u^\varepsilon_1(x) = u^0_1(x) + \varepsilon u'_1(x) + \cdots, \qquad x \in \Omega_1. \tag{54}$$

The formulas (52)-(53) mean that the eigenpair $(\lambda^\varepsilon, \{u^\varepsilon_0, u^\varepsilon_1\})$ is expected to depend on the parameter ε continuously. By replacing the expansions (52)-(53) in the problem (1)-(4) and collecting the coefficients of the same powers of ε, the leading term in (53) is a solution to the homogenuous Neumann problem (12), i.e. a constant c_0. The correction term u'_0 defined up to an additive constant satisfies

$$-\Delta_x u'_0(x) = 0, \quad x \in \Omega_0, \qquad \partial_{\nu_0} u'_0(x) = \partial_{\nu_0} u^0_1(x), \quad x \in \Gamma_0.$$

The compatibility condition reads

$$\int_{\Gamma_0} \partial_{\nu_0} u^0_1(x) ds_x = 0. \tag{55}$$

The leading terms λ_n^0 and u_1^0 in the ansätze (52),(54) are obtained from the spectral problem

$$-\Delta_x u_1^0(x) = \lambda^0 u_1^0(x), \qquad x \in \Omega_1, \tag{56}$$

$$\partial_{\nu_1} u_1^0(x) = 0, \quad x \in \Gamma_1, \qquad u_1^0(x) = u_0^0(x), \quad x \in \Gamma_0, \tag{57}$$

along with the integral condition (55). To write down the variational formulation of the problem (56)-(57), we set $H^1_\bullet(\Omega_1, \Gamma_0)$ as the subspace of functions in $H^1(\Omega_1)$ with a constant trace on the boundary Γ_0. For $\varphi \in H^1_\bullet(\Omega_1, \Gamma_0)$, the Green formula provides

$$\begin{aligned} -\int_{\Omega_1} \Delta_x u_1^0(x)\varphi(x)dx &= \int_{\Omega_1} \nabla_x u_1^0(x)\nabla_x\varphi(x)dx - \int_{\Gamma_0} \partial_{\nu_0} u_1^0(x)\varphi(x)ds_x \\ &= \lambda^0 \int_{\Omega_1} u_1^0(x)\varphi(x)dx. \end{aligned}$$

Since φ is a constant on the boundary Γ_0, it follows that

$$\int_{\Gamma_0} \partial_{\nu_0} u_1^0(x)\varphi(x)ds_x = \text{const} \int_{\Gamma_0} \partial_{\nu_0} u_1^0(x)ds_x = 0.$$

As a consequence, the variational formulation reads: find the eigenvalue $\lambda^0 \in \mathbb{R}$ and the corresponding eigenfunction $u_1^0 \in H^1_\bullet(\Omega_1, \Gamma_0) \setminus \{0\}$ such that

$$(\nabla_x u_1^0, \nabla_x \varphi)_{\Omega_1} = \lambda^0 (u_1^0, \varphi)_{\Omega_1} \qquad \forall \varphi \in H^1_\bullet(\Omega_1, \Gamma_0). \tag{58}$$

Note that the integral equality (58) implies the condition (55). The problem (58) admits the eigenvalues sequence (17). The corresponding eigenfunctions u_1^0 are orthonormalized in $L^2(\Omega_1)$.
The correction term u_1' verifies the problem

$$-\Delta_x u_1'(x) - \lambda^0 u_1'(x) = \lambda' u_1^0(x), \qquad x \in \Omega_1, \tag{59}$$

$$\partial_{\nu_1} u_1'(x) = 0, \quad x \in \Gamma_1, \qquad u_1'(x) = u_0'(x), \quad x \in \Gamma_0. \tag{60}$$

Before computing λ', we investigate term of higher order in the asymptotic (53). We assume that $-2m > 1$. The term u_0'' of order ε^2 solves the problem

$$-\Delta_x u_0''(x) = 0, \quad x \in \Omega_0, \qquad \partial_{\nu_0} u_0''(x) = \partial_{\nu_0} u_1'(x), \quad x \in \Gamma_0.$$

The compatibility condition reads as

$$\int_{\Gamma_0} \partial_{\nu_0} u_1'(x)ds_x = 0. \tag{61}$$

When $-2m = 1$ the compatibility condition becomes inhomegeneous, i.e.

$$\int_{\Gamma_0} \partial_{\nu_0} u_1'(x)ds_x = \lambda^0 c_0 |\Omega_0|, \tag{62}$$

since u_0'' solves the problem

$$-\Delta_x u_0''(x) = \lambda_0 u_0^0(x), \quad x \in \Omega_0, \qquad \partial_{\nu_0} u_0''(x) = \partial_{\nu_0} u_1'(x), \quad x \in \Gamma_0. \tag{63}$$

In the last case, i.e. $-2m < 1$, the term u_0'' has order ε^{-2m+1} and it solves the problem (63), yielding the compatibility condition (62).
In case of simple eigenvalue λ_n^0 and $-2m > 1$, the Fredholm alternative leads to the following expression for the correction term λ_n':

$$\lambda_n' = -(\partial_{\nu_0} u_1^0, u_1')_{\Gamma_0} = -(\partial_{\nu_0} u_0', u_0')_{\Gamma_0} = -(\nabla_x u_0', \nabla_x u_0')_{\Omega_0} = -\|\nabla_x u_0'\|^2_{L^2(\Omega_0)}. \tag{64}$$

If $-2m \leq 1$, the term λ' becomes

$$\lambda' = -(\partial_{\nu_0} u_1', u_1^0)_{\Gamma_0} - (\partial_{\nu_0} u_1^0, u_1')_{\Gamma_0} = -\lambda^0 c_0^2 |\Omega_0| - \|\nabla_x u_0'\|^2_{L^2(\Omega_0)},$$

due to the compatibility condition (62).
Suppose that λ_n^0 is a τ-multiple eigenvalue. As in Section 2, the leading terms of the asymptotics of the eigenfunctions $u_{1,n}^\varepsilon, \dots, u_{1,n+\tau-1}^\varepsilon$ are predicted in the form of linear combinations

$$U_{1,j}^0(x) = a_n^j u_{1,n}^0(x) + \cdots + a_{n+\tau-1}^j u_{1,n+\tau-1}^0(x), \qquad j = n, \dots, n+\tau-1,$$

of the eigenfunctions $u_{1,n}^0, \dots, u_{1,n+\tau-1}^0$ of the limit problem (56)-(57). The coefficients $a_n^j, \dots, a_{n+\tau-1}^j$ satisfy the orthonormalization condition (24). The first order corrector $U_{1,j}'$ in (54) verifies the problem

$$-\Delta_x U_{1,j}'(x) - \lambda_n^0 U_{1,j}'(x) = \lambda_j' U_{1,j}^0(x), \qquad x \in \Omega_1,$$

$$\partial_{\nu_1} U_{1,j}'(x) = 0, \quad x \in \Gamma_1, \qquad U_{1,j}'(x) = \sum_{k=n}^{n+\tau-1} a_k^j u_{0,k}'(x), \quad x \in \Gamma_0.$$

In the case $-2m > 1$, from the Fredholm alternative and the formula (61), we get the τ compatibility conditions

$$\lambda_j'(U_{1,j}^0, u_{1,q}^0)_{\Omega_1} = (\partial_{\nu_0} u_{1,q}^0, U_{1,j}')_{\Gamma_0} = \sum_{k=n}^{n+\tau-1} a_k^j (\partial_{\nu_0} u_{0,q}', u_{0,j}')_{\Gamma_0} = \sum_{k=n}^{n+\tau-1} a_k^j (\nabla_x u_{0,j}', \nabla_x u_{0,q}')_{\Omega_0}.$$

It follows that

$$\lambda_j' a_q^j = \sum_{k=n}^{n+\tau-1} a_k^j (\nabla_x u_{0,j}', \nabla_x u_{0,q}')_{\Omega_0},$$

which can be written in the form of the linear system of τ algebraic equations

$$G a^j = \lambda_j' a^j, \qquad j = n, \dots, n+\tau-1. \tag{65}$$

Here, G is the Gram matrix whose entries are

$$G_{i,j} = (\nabla_x u_{0,i}', \nabla_x u_{0,j}')_{\Omega_0}, \qquad i, j = n, \dots, n+\tau-1.$$

Since G is a symmetric $\tau \times \tau$ matrix, its eigenvalues $\lambda_n', \dots, \lambda_{n+\tau-1}'$ are real and positive. Indeed the derivatives $\partial_{\nu_0} u_{1,n}^0, \dots, \partial_{\nu_0} u_{1,n+\tau-1}^0$ are linearly independent in $L^2(\Gamma_0)$. Otherwise, a linear combination

$$U(x) = \sum_{k=n}^{n+\tau-1} a_i u_{1,i}^0(x), \qquad x \in \Omega_1,$$

satisfies the equation $-\Delta_x U(x) = \lambda^0 U(x)$, $x \in \Omega_1$, and simultaneously two boundary conditions $U(x) = \text{const}$ and $\partial_{\nu_0} U(x) = 0$, $x \in \Gamma_0$. This is a contraddiction due to the theorem on strong unique continuation (e.g. cf. [35]). Hence, $\nabla_x u_{0,n}', \dots, \nabla_x u_{0,n+\tau-1}'$ are linearly indepedent in $L^2(\Omega_0)^d$ and the matrix G is positive-definite matrix. We emphasize that $\nabla_x u_{0,i}'$, $i = n, \cdots, n+\tau-1$, are defined uniquely, although $u_{0,i}'$ are defined up to a constant.
If $-2m \leq 1$, the Fredholm alternative and the expression (62) yield to

$$\lambda_j' = -(\partial_{\nu_0} U_{1,j}', u_{1,q}^0)_{\Gamma_0} - (\partial_{\nu_0} u_{1,q}^0, U_{1,j}')_{\Gamma_0} = \lambda_n^0 (u_{0,q}^0)^2 |\Omega_0| - \sum_{k=n}^{n+\tau-1} a_k^j (\nabla_x u_{0,j}', \nabla_x u_{0,q}')_{\Omega_0}. \tag{66}$$

As far as the justification procedure is concerned, the estimate (32) of the Theorem 3.1 for $m < 0$ is valid with $\alpha = 0$, $\beta = 1$, $\gamma = \min\{1-m, 2\}$, λ_n^0 being an eigenvalue of the problem (56)-(57) and λ_n' the correction term in (52), given by formula (64) for a simple eigenvalue and by formulas (65)-(66) for a multiple ones.

5 Asymptotic expansion for $m = 0$

If $m = 0$, the ansätze (52)-(53) are still correct. As a consequence, the problem (12) is satisfied by the leading term u_0^0. The main difference comes from the problem satisfied by the correction term u_0'

$$-\Delta_x u_0'(x) = \lambda^0 u_0^0(x), \quad x \in \Omega_0, \qquad \partial_{\nu_0} u_0'(x) = \partial_{\nu_0} u_1^0(x), \quad x \in \Gamma_0,$$

whose compatibility condition reads as

$$c_0 = \frac{1}{\lambda^0 |\Omega_0|} \int_{\Gamma_0} \partial_{\nu_0} u_1^0 ds_x. \tag{67}$$

The leading terms λ^0 and u_1^0 in (52) and(54) solve the problem (56)-(57) along with the integral condition (67). Therefore, the variational formulation reads as

$$(\nabla_x u_1^0, \nabla_x \varphi)_{\Omega_1} = \lambda^0 (u_1^0, \varphi)_{\Omega_0} + \lambda^0 |\Omega_0| \overline{u}_1^0 \overline{\varphi}, \qquad \forall \varphi \in H^1_\bullet(\Omega_1, \Gamma_0). \tag{68}$$

Here $\overline{u}$ denotes the constant trace of function $u \in H^1(\Omega_1)$ on the boundary Γ_0. The problem (68) admits discrete spectrum given by (17). The correction term u_1' satisfies the problem (59)-(60). The claim of Theorem 3.1 is still true and the estimate (32) becomes

$$|\lambda^\varepsilon - \lambda_n^0 - \varepsilon \lambda_n'| \le C_N \varepsilon^{3/2}.$$

6 Asymptotic expansion for $m = 1/2$

This case is discussed in more abstract setting in the textbook [32, Chapter VII], but for the convenience of the reader a simple and independent proof is presented for the problem under consideration. The Helmholtz equation (2) gets rid of the small parameter ε

$$-\Delta_x u_0^\varepsilon(x) = \lambda^\varepsilon u_0^\varepsilon(x), \qquad x \in \Omega_0.$$

We perfom the replacement (8), i.e. $v_0^\epsilon(x) = \varepsilon^{-1/2} u_0^\varepsilon(x)$ and $v_1^\varepsilon(x) = u_1^\varepsilon(x)$. The asymptotics of eigenpairs $(\lambda^\varepsilon, \{u_0^\varepsilon, u_1^\varepsilon\})$ take the form

$$\lambda^\varepsilon = \lambda^0 + \varepsilon^{1/2}\lambda' + \cdots, \qquad v_1^\epsilon(x) = v_1^0(x) + \varepsilon^{1/2} v_1'(x) + \cdots, \qquad x \in \Omega_1,$$
$$v_0^\epsilon(x) = v_0^0(x) + \varepsilon^{1/2} v_0'(x) + \cdots, \qquad x \in \Omega_0.$$

The essential difference with respect to the other cases is the presence of two spectral limit problems. In fact, the leading term v_0^0 is determined from the problem

$$-\Delta_x v_0^0(x) = \lambda^0 v_0^0(x), \quad x \in \Omega_0, \qquad \partial_{\nu_0} v_0^0(x) = 0 \qquad x \in \Gamma_0. \tag{69}$$

The leading term v_1^0 solves the problem

$$-\Delta_x v_1^0(x) = \lambda^0 v_1^0(x), \qquad x \in \Omega_1, \tag{70}$$
$$\partial_{\nu_1} v_1^0(x) = 0, \quad x \in \Gamma_1, \qquad v_1^0(x) = 0, \quad x \in \Gamma_0. \tag{71}$$

The problem for the correction terms v_0' is

$$-\Delta_x v_0'(x) - \lambda^0 v_0'(x) = \lambda' v_0^0(x), \quad x \in \Omega_0, \qquad \partial_{\nu_0} v_0'(x) = \partial_{\nu_0} v_1^0(x), \quad x \in \Gamma_0.$$

The correction term v_1' is determined by the problem (19)-(20). Owing to the two limit problems, the procedure made for the convergence theorem must be slightly modified. We explain it briefly.

According to the convergence (34) of eigenvalues λ_n^ε, the weak formulation (5) and the normalization condition (7) with $m = 1/2$, we deduce

$$\|\nabla_x v_{1,n}^\varepsilon\|^2_{L^2(\Omega_1)} + \varepsilon^{-1}\|\nabla_x v_{0,n}^\varepsilon\|^2_{L^2(\Omega_0)} \leq C_n.$$

As in Section 3.1.1, $v_{0,n}^\varepsilon$ converges to zero strongly in $H^1(\Omega_0)$ and hence in $L^2(\Omega_0)$, while $v_{1,n}^\varepsilon$ converges to some $v_{1,\bar{n}}^0$ weakly in $H^1(\Omega_1)$ and strongly in $L^2(\Omega_1)$. If $v_{1,\bar{n}}^0 \neq 0$, the continuity of trace operator ensures that $v_{0,n}^\varepsilon$ converges to 0 in $L^2(\Gamma_0)$. Then the boundary condition (4) yields the strong convergence of $v_{1,n}^\varepsilon$ to 0 in $L^2(\Gamma_0)$, i.e. $v_{1,\bar{n}}^0 \in H_0^1(\Omega_1, \Gamma_0)$. Using the same arguments as in Section 3.1.1, we deduce that the leading terms $\lambda_{\bar{n}}^0$ and $v_{1,\bar{n}}^0$, with $v_{1,\bar{n}}^0 \neq 0$, are characterized as the eigenelements of the spectral problem (70)-(71).
Assume, now, that $v_{1,\bar{n}}^0 = 0$. The previous arguments fail and we introduce the new normalization condition

$$\|v_{1,n}^\varepsilon\|^2_{L^2(\Omega_1)} + \varepsilon^{-1}\|v_{0,n}^\varepsilon\|^2_{L^2(\Omega_0)} = \varepsilon^{-1}. \tag{72}$$

The weak formulation (5) implies the bound

$$\|\nabla_x v_{1,n}^\varepsilon\|^2_{L^2(\Omega_1)} + \varepsilon^{-1}\|\nabla_x v_{0,n}^\varepsilon\|^2_{L^2(\Omega_0)} \leq C_n\varepsilon^{-1}. \tag{73}$$

Multiplying the inequalities (72) and (73) by ε, the quantities $\|v_{0,n}^\varepsilon\|_{H^1(\Omega_0)}$ and $\|\nabla_x v_{0,n}^\varepsilon\|_{H^1(\Omega_0)}$ are bounded and then $v_{0,n}^\varepsilon$ converges weakly in $H^1(\Omega_0)$ and strongly to $L^2(\Omega_0)$ to some function $v_{0,\bar{n}}^0$. Moreover, the trace of $v_{0,n}^\varepsilon$ converges to the trace of $v_{0,\bar{n}}^0$ in $L^2(\Gamma_0)$. Finally, the limit passage as $\varepsilon \to 0$ in the weak formulation (5) leads to characterize $v_{0,\bar{n}}^0$ as the eigenfunction with associated eigenvalue $\lambda_{\bar{n}}^0$ of the problem (69) and the eigenfunctions $v_{0,\bar{n}}^0$ are normalized in $L^2(\Omega_0)$. Indeed bearing in mind that $\lambda_{\bar{n}}^0$ does not belong to the spectrum of the problem (19)-(20) and the convergence (34), for small $\varepsilon > 0$ λ_n^ε is not an eigenvalue of the problem

$$\begin{aligned} -\Delta_x v_{1,n}^\varepsilon(x) &= \lambda_n^\varepsilon v_{1,n}^\varepsilon(x), \quad && x \in \Omega_1, \\ \partial_{\nu_1} v_{1,n}^\varepsilon(x) = 0, \quad x \in \Gamma_1, \qquad & v_{1,n}^\varepsilon(x) = 0, && x \in \Gamma_0. \end{aligned}$$

As a consequence, we have

$$\|v_{1,n}^\varepsilon\|_{H^1(\Omega_1)} \leq c\|v_{0,n}^\varepsilon\|_{H^{1/2}(\Gamma_0)} \leq c\|v_{0,n}^\varepsilon\|_{H^1(\Omega_0)} \leq c. \tag{74}$$

The inequalities (72) and (74) lead to check the normalization condition of the eigenfunction $v_{0,\bar{n}}^0$. The Theorem 3.1 is still valid with the estimate

$$|\lambda^\varepsilon - \lambda_n^0 - \varepsilon^{1/2}\lambda_n'| \leq C_N\varepsilon.$$

7 Asymptotic expansion for $m > 1/2$

We postulate the asymptotic expansions for the eigenvalue λ^ε

$$\lambda^\varepsilon = \varepsilon^{2m-1}\lambda^0 + \varepsilon^{2m}\lambda' + \cdots. \tag{75}$$

For the corresponding eigenfunctions $\{u_0^\varepsilon, u_1^\varepsilon\}$ we consider an asymptotic expansion of the form

$$u_0^\varepsilon = u_0^0 + \varepsilon u_0' + \cdots, \qquad x \in \Omega_0, \tag{76}$$

$$u_1^\varepsilon = u_1^0 + \varepsilon^{\min\{1,2m-1\}}u_1' + \cdots, \qquad x \in \Omega_1. \tag{77}$$

Using the same procedure as in the other cases, we find that the leading terms λ^0, u_0^0 in (75), (76) are characterized as the solution to the spectral problem

$$-\Delta_x u_0^0(x) = \lambda^0 u_0^0(x), \quad x \in \Omega_0, \qquad \partial_{\nu_0} u_0^0(x) = 0, \quad x \in \Gamma_0. \tag{78}$$

The problem (78) in the Sobolev space $H^1(\Omega_0)$ has a discrete spectrum

$$0 = \lambda_1^0 < \lambda_2^0 \leq \cdots \leq \lambda_n^0 \leq \cdots \to \infty$$

and the corresponding eigenfunctions $u_{0,n}^0$ are subject to the orthonormalization condition in $L^2(\Omega_0)$. The leading term u_1^0 in (77) is defined as a unique solution of the problem

$$-\Delta_x u_1^0(x) = 0, \qquad x \in \Omega_1,$$
$$\partial_{\nu_1} u_1^0(x) = 0, \quad x \in \Gamma_1, \qquad u_1^0(x) = u_0^0(x) \quad x \in \Gamma_0.$$

If $\min\{1, 2m-1\} = 2m-1$, the problem for the correction term u_1' in (77) is

$$-\Delta_x u_1'(x) = \lambda^0 u_1^0(x), \qquad x \in \Omega_1,$$
$$\partial_{\nu_1} u_1'(x) = 0, \quad x \in \Gamma_1, \qquad u_1'(x) = 0, \quad x \in \Gamma_0.$$

If $\min\{1, 2m-1\} = 1$, the correction term u_1' is characterized as the solution to the problem

$$-\Delta_x u_1'(x) = 0, \qquad x \in \Omega_1,$$
$$\partial_{\nu_1} u_1'(x) = 0, \quad x \in \Gamma_1, \qquad u_1'(x) = u_0'(x), \quad x \in \Gamma_0.$$

We point out that when $m = 1$, the problem satisfied by u_1' turns into

$$-\Delta_x u_1'(x) = \lambda^0 u_1^0, \qquad x \in \Omega_1,$$
$$\partial_{\nu_1} u_1'(x) = 0, \quad x \in \Gamma_1, \qquad u_1'(x) = u_0', \quad x \in \Gamma_0.$$

The correction term u_0' in the asymptotic expansion (76) is determined from the problem

$$-\Delta_x u_0'(x) - \lambda^0 u_0'(x) = \lambda' u_0^0(x), \quad x \in \Omega_0, \qquad \partial_\nu u_0'(x) = \partial_\nu u_1^0(x), \quad x \in \Gamma_0. \tag{79}$$

The compatibility condition in the problem (79) provides the correction term λ'. Indeed, if the eigenvalue λ_n^0 is simple then we get

$$\lambda' = -\|\nabla_x u_1^0\|^2_{L^2(\Omega_1)}. \tag{80}$$

Assume now that the eigenvalue λ_n^0 has multiplicity $\tau > 1$, i.e. $\lambda_{n-1}^0 < \lambda_n^0 = \cdots = \lambda_{n+\tau-1}^0 < \lambda_{n+\tau}^0$. The leading term in the expansions (76) are predicted in the form of linear combinations of the eigenfunctions $u_{0,n}^0, \ldots, u_{0,n+\tau-1}^0$

$$U_{0,j}^0(x) = a_n^j u_{0,n}^0(x) + \cdots + a_{n+\tau-1}^j u_{0,n+\tau-1}^0(x), \qquad j = n, \ldots, n+\tau-1.$$

Therefore, $U_{0,j}'$ is the solution to the problem

$$-\Delta_x U_{0,j}'(x) - \lambda_n^0 U_{0,j}'(x) = \lambda_j' U_{0,j}^0(x), \quad x \in \Omega_0, \qquad \partial_\nu U_{0,j}'(x) = \sum_{k=n}^{n+\tau-1} a_k^j \partial_{\nu_0} u_{1,k}^0(x), \quad x \in \Gamma_0.$$

According to the Fredhom alternative, the τ compatibility conditions are

$$\lambda_j'(U_{0,j}^0, u_{0,p}^0)_{\Omega_0} = (\partial_{\nu_0} U_{0,j}', u_{0,p}^0)_{\Gamma_0} = \sum_{k=n}^{n+\tau-1} a_n^j (\partial_{\nu_0} u_{1,k}^0, u_{0,p}^0)_{\Gamma_0}$$
$$= \sum_{k=n}^{n+\tau-1} a_n^j (\nabla_x u_{1,k}^0, \nabla_x u_{0,p}^0)_{\Omega_1}, \quad j = n, \ldots, n+\tau-1 \tag{81}$$

The previous relation can be written as the formula (65) with a different Gram matrix. In other words, the τ correction terms are the eigenvalues of the Gram matrix G whose entries are given by

$$G_{i.j} = (\nabla_x u_{1,i}^0, \nabla_x u_{1,j}^0)_{\Omega_1}, \qquad i, j = n, \ldots, n+\tau-1,$$

with a_n^j being the corresponding eigenvectors. The estimate (32) of Theorem 3.1 holds with $\alpha = 2m-1$, $\beta = m$, $\gamma = \min\{4m-1, 1+m\}$ for $m \in (1/2, 1)$ and $\gamma = 2m+1$ for $m \geq 1$, λ_n^0 being the eigenvalue of the problem (78) and λ_n' being the correction term, given by formula (80) if λ_n^0 is a simple eigenvalue and (81) if λ_n^0 is a multiple one.

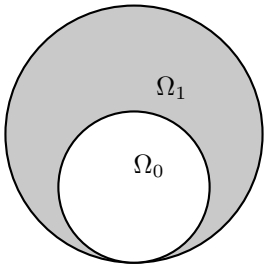

Figure 2: Kissing domains

8 Kissing domains in $\mathbb{R}^2$

A distinguishing feature of the stiff Neumann problem is that all asymptotic forms derived and justified in previuous sections, are preserved when $\mathrm{dist}(\Gamma_0, \Gamma_1) \to 0$, i.e in the limit the core Ω_0 touches the exterior boundary Γ_1, cf. Fig. 2, so that Ω_0 and Ω form the interior kiss of two domains. This peculiar conclusion is certainly based on the exterior Neumann condition (3) and the prevaling stiffness of th annulus Ω_1. Changing particular details in problem's statement may quit the above-mentioned limit passage: in Section 8.4 we will discuss a serious issue in the case when the Neumann condition (3) is replaced with the homogeneous Dirichlet one. In this way, in many cases the Dirichlet problem of the type (1)-(4) remains a fully open question.
The performed asymptotic analysis demonstrates that in all situations under consideration the limit problem in the cuspidal annulus reads

$$-\Delta_x u(x) = \lambda u(x), \qquad x \in \Omega_1 \tag{82}$$

$$\partial_{\nu_1} u(x) = 0, \quad x \in \Gamma_1 \setminus \mathcal{O} \qquad\qquad u(x) = g(x), \quad x \in \Gamma_0 \setminus \mathcal{O}, \tag{83}$$

where $\lambda \geq 0$ and $g = 0$ or $g = \mathrm{const}$ on the boundary Γ_0. Denoting by $G \in H^1(\mathbb{R}^d \setminus \Omega_0)$ an extension of g onto the exterior of Ω_0, the variational formulation of the problem (82)-(83) reads, cf. [19]: find $u \in H^1(\Omega_1)$ such that $u - G \in H^1(\Omega_1; \Gamma_0)$ and the following integral identity is valid

$$(\nabla_x u, \nabla_x v)_{\Omega_1} = \lambda (u, v)_{\Omega_1} \qquad \forall v \in \mathbf{H}, \tag{84}$$

with $\mathbf{H} = H^1_0(\Omega_1; \Gamma_0)$ if $g = 0$ on Γ_0 or $\mathbf{H} = H^1_\bullet(\Omega_1, \Gamma_0)$ if $g = \mathrm{const}$ on Γ_0 Due to the Dirichlet condition on Γ_0, the space $H^1_0(\Omega_1; \Gamma_0)$ is compactly embedded into $L^2(\Omega_1)$ [1]. Hence, all necessary properties of the problem (82)-(83) are kept in the cuspidal domain Ω_1 and these allow us to repeat with easily predictable modifications our calculations and argumentation in previous sections, to conclude the analog of the Theorem 3.1 in the case of the tounching boundaries Γ_0 and Γ_1.
In the sequel we will describe the asymptotic behaviour as $x \to \mathcal{O}$ of solutions to the mixed boundary value problem (82)-(83) and a similar Dirichlet problem in Ω_1. However, to reduce cumbersome and long computations, we deal with the 2D case only while a needed modification for multi-dimensional cases can be found in the papers [24, 26, 27, 28] and others.

8.1 Asymptotics of solutions at the cusp in Neumann case

We consider the spectral problem (82)-(83), where $g = c_0$ on $\Gamma_0 \setminus \mathcal{O}$ and c_0 is an arbitrary constant. Set R_0, R_1 the radii of the disks Ω_0, Ω_1 respectively such that $R_0 < R_1$. The

[1] This fact is true for $H^1(\Omega_1)$ as well, see, e.g. [23]

boundaries Γ_0 and Γ_1 are described by

$$H_i(x_1) = \frac{|x_1|^2}{2R_i} + O(|x_1|^4), \qquad i = 0, 1. \tag{85}$$

The thickness is defined as $H(x_1) = H_0(x_1) - H_1(x_1)$. We write down the representation

$$u(x) = c_0 + \cdots, \qquad x \to \mathcal{O}, \tag{86}$$

where the dots denote the lower-order terms. The distinguished asymptotic term on the right-hand side of (86) satisfies the boundary conditions (83) but generates the residual

$$\lambda c_0 + \cdots$$

in the differential equation (82). Then, we introduce a new term $\mathcal{U}_1(x_1, \eta)$ in (86), involving the stretched coordinate

$$\eta = \frac{x_2 - H_1(x_1)}{H(x_1)} \in (0, 1).$$

Then the asymptotic (86) turns into

$$u(x) = c_0 + \mathcal{U}_1(x_1, \eta) + \cdots, \qquad x \to \mathcal{O}. \tag{87}$$

In order to rewrite (82) in the new variables (x_1, η), we evalute

$$\frac{\partial}{\partial x_2} = \frac{\partial \eta}{\partial x_2}\frac{\partial}{\partial \eta} = \frac{1}{H(x_1)}\frac{\partial}{\partial \eta}, \qquad \frac{\partial^2}{\partial x_2^2} = \frac{1}{H(x_1)^2}\frac{\partial^2}{\partial \eta^2}, \tag{88}$$

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_1} - H(x_1)^{-1}(H_1'(x_1) + \eta H'(x_1))\frac{\partial}{\partial \eta},$$

$$\frac{\partial^2}{\partial x_2^2} = \left(\frac{\partial}{\partial x_1} - H(x_1)^{-1}(H_1'(x_1) + \eta H'(x_1))\frac{\partial}{\partial \eta}\right)^2$$

$$+ \left(\frac{2H'(x_1)H_1'(x_1) + 2(H'(x_1))^2\eta - H_1''(x_1)H(x_1) - H''(x_1)H(x_1)\eta}{H(x_1)^2}\right)\frac{\partial}{\partial \eta}, \tag{89}$$

where $H'(x_1) = \frac{dH(x_1)}{dx_1}$. Owing to (85), (88), (89), the Laplace operator $\Delta_{(x_1,x_2)}$ in the new variables (x_1, η) is written as

$$\Delta_{(x_1,\eta)} = \frac{1}{H^p(x_1)^2}\frac{\partial^2}{\partial \eta^2} + \sum_{j=1}^{\infty} L_j(x_1, \eta, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial \eta}), \tag{90}$$

where we replaced the thickness function $H(x_1)$ with its principal part

$$H^p(x_1) = \frac{1}{2}\left(\frac{1}{R_0} - \frac{1}{R_1}\right)|x_1|^2.$$

The normal derivative on the lower boundary Γ_1 can be written as

$$\partial_{\nu_1} = \frac{1}{(1 + |H_1'(x_1)|^2)^{1/2}}\left(\frac{\partial}{\partial x_2} - H_1'(x_1)\frac{\partial}{\partial x_1}\right)$$

$$= \frac{1}{(1 + |H_1'(x_1)|^2)^{1/2}}\left(\frac{1}{H(x_1)}\frac{\partial}{\partial \eta} - H_1'(x_1)\frac{\partial}{\partial x_1} + H_1'(x_1)H(x_1)^{-1}(H_1'(x_1) + \eta H'(x_1))\frac{\partial}{\partial \eta}\right). \tag{91}$$

In view of (90) and (91), we insert the expansion ansatz (87) into the problem (82)-(83), obtaining the problem

$$-\frac{1}{H^p(x_1)^2}\frac{\partial^2}{\partial \eta^2}\mathcal{U}_1(x_1, \eta) = \lambda c_0, \tag{92}$$

$$\frac{\partial}{\partial \eta}\mathcal{U}_1(x_1, \eta)\Big|_{\eta=0} = 0, \qquad \mathcal{U}_1(x_1, \eta)\Big|_{\eta=1} = 0.$$

By a direct computation, the solution $\mathcal{U}_1$ is given by

$$\mathcal{U}_1(x_1,x_2) = -\frac{\lambda c_0}{2}[x_2^2 - 2H_1^p(x_1)(x_2 + H_0^p(x_1)) - H_0^p(x_1)^2], \tag{93}$$

where $H_i^p(x_1) = |x_1|^2/(2R_i)$ denotes the principal part of $H_i(x_1)$, $i = 0, 1$. Note that the first-order correction term $\mathcal{U}_1(x_1, x_2)$ is of order $|x_1|^4$. Iterating this procedure, we are able to construct the formal infinite series of the eigenfunction u of the problem (82)-(83)

$$u(x) = c_0 + \sum_{j=1}^{\infty} \mathcal{U}_j(x), \tag{94}$$

containing the already chosen main term (93). Keeping in mind the decomposition (90), (91) and replacing the eigenfunction u with its formal series (94) into the equation (82), we find that the term $\mathcal{U}_2$ is solution of the problem

$$-\frac{1}{H^p(x_1)^2}\frac{\partial^2}{\partial\eta^2}\mathcal{U}_2(x_1,\eta) = L_1(x_1,\eta,\frac{\partial}{\partial x_1},\frac{\partial}{\partial\eta},\lambda)\mathcal{U}_1(x_1,\eta),$$
$$\partial_\eta\mathcal{U}_2(x_1,\eta)\big|_{\eta=0} = N_1(\frac{\partial}{\partial x_1},\frac{\partial}{\partial\eta})\mathcal{U}_1(x_1,\eta)\big|_{\eta=0}, \qquad \mathcal{U}_2(x_1,\eta)\big|_{\eta=1} = 0,$$

where

$$L_1(x_1,\eta,\frac{\partial}{\partial x_1},\frac{\partial}{\partial\eta},\lambda) = \left(\frac{\partial}{\partial x_1} - H^p(x_1)^{-1}((H_1^p)'(x_1) + \eta(H^p)'(x_1))\frac{\partial}{\partial\eta}\right)^2,$$
$$+\left(\frac{2(H^p(x_1))'(H_1^p(x_1))' + 2(H'^p(x_1))^2\eta}{H^p(x_1)^2}\right.$$
$$\left. - \frac{(H_1^p)''(x_1)H^p(x_1) + (H^p(x_1))''H^p(x_1)\eta}{H^p(x_1)^2}\right)\frac{\partial}{\partial\eta} + \frac{|x_1|^6}{8R^4}\lambda c_0$$
$$N_1(\frac{\partial}{\partial x_1},\frac{\partial}{\partial\eta}) = (H_1^p)'(x_1)\left(\frac{\partial}{\partial x_1} + \frac{(H_1^p)'(x_1)}{(H^p)'(x_1)}\frac{\partial}{\partial\eta}\right).$$

The other terms of the series (94) are determined by the problems

$$-\frac{1}{H^p(x_1)}\partial_\eta\mathcal{U}_j(x_1,\eta) = L_{j-1}(x_1,\eta,\frac{\partial}{\partial x_1},\frac{\partial}{\partial\eta},\lambda)\mathcal{U}_{j-1}(x_1,\eta) + \lambda\mathcal{U}_{j-2}(x_1,\eta), \qquad j = 3,4,\cdots,$$
$$\partial_\eta\mathcal{U}_j(x_1,\eta)\big|_{\eta=0} = N_{j-1}(0,\frac{\partial}{\partial x_1},\frac{\partial}{\partial\eta})\mathcal{U}_{j-1}(x_1,\eta)\big|_{\eta=0}, \qquad \mathcal{U}_j(x_1,\eta)\big|_{\eta=1} = 0.$$

We point out that the terms $\mathcal{U}_j$, $j = 2, 3, \cdots$, of the series (94) are of order $|x_1|^{2j+2}$.

8.2 Justification of Asymptotics

Let χ be a smooth cut-off function such that $0 \le \chi(x_1) \le 1$ and

$$\chi(x_1) = 0, \ \text{ if } |x_1| \ge R_0, \qquad \chi(x_1) = 1 \ \text{ if } |x_1| \le \frac{R_0}{2}.$$

We set

$$u(x) = c_0 + \chi(x_1)\mathcal{U}_1(x) + \tilde{u}(x), \tag{95}$$

with $\tilde{u}(x)$ being the remainder.

Theorem 8.1. *The solution u of the spectral problem* (82)-(83) *admits the asymptotic form* (95). *More specifically, there exists an exponent $N > 0$ such that the norm*

$$\|\rho(x)^{-N}\tilde{u}\|_{H_0^1(\Omega_1,\Gamma_0)} < \infty$$

and the functions $\rho(x)^{-N}(u(x) - c_0)$ and $\rho(x)^{-N}\mathcal{U}_1(x_1,\eta)$ do not belong to the Sobolev space H^1 in a neighbourhood of the cusp $\mathcal{O}$.

Proof. The remainder $\tilde{u}$ verifies the following equation

$$-\Delta_x \tilde{u}(x) - \lambda \tilde{u}(x) = \lambda c_0 + \lambda \mathcal{U}_1(x)\chi(x_1) + \Delta_x(\chi(x_1)\mathcal{U}_1(x)), \qquad x \in \Omega_1 \setminus \mathcal{O}, \tag{96}$$

along with homogeneous boundary conditions

$$\partial_{\nu_1} \tilde{u}(x) = 0, \quad x \in \Gamma_1 \setminus \mathcal{O}, \qquad \tilde{u}(x) = 0, \quad x \in \Gamma_0 \setminus \mathcal{O}.$$

Multiplying (96) by an arbitrary test function $v \in H_0^1(\Omega_1, \Gamma_0)$ and integrating in Ω_1, we find

$$\begin{aligned}(-\Delta_x \tilde{u}, v)_{\Omega_1} - \lambda(\tilde{u}, v)_{\Omega_1} &= (\Delta_x(\mathcal{U}_1\chi), v)_{\Omega_1} + \lambda(c_0, v)_{\Omega_1} + \lambda(\mathcal{U}_1\chi, v)_{\Omega_1} \\ &= (\chi\Delta_x\mathcal{U}_1, v)_{\Omega_1} + ([\Delta_x, \chi]\mathcal{U}_1, v)_{\Omega_1} + \lambda(c_0, v)_{\Omega_1} + \lambda(\mathcal{U}_1\chi, v)_{\Omega_1}.\end{aligned} \tag{97}$$

The commutator $[\Delta_x, \chi]$ is

$$[\Delta_x, \chi]\mathcal{U} = 2\nabla_x \mathcal{U} \cdot \nabla_x \chi + \mathcal{U}\Delta_x\chi.$$

Let ρ denote a smooth positive function on Ω_1 which coincides with the distance to the origin of the Cartesian coordinate system in a neighbourhood of the cuspidal point $\mathcal{O}$ and introduce the weight function

$$T_\delta(x) = \begin{cases} \delta^{-N}, & \text{if} \quad \rho(x) \le \delta, \\ \rho(x)^{-N}, & \text{if} \quad \delta < \rho(x) \le R_0/2, \\ (R_0/2)^{-N}, & \text{if} \quad \rho(x) > R_0/2, \end{cases}$$

where the parameter $\delta > 0$ is small and will be sent to 0. Later on, we will impose some constraints on the exponent N. We point out that the derivative of T_δ vanishes for $\rho(x) \le \delta$, $\rho(x) > R_0/2$ and satisfies the inequality

$$|\nabla_x T_\delta(x)| \le C T_\delta(x) \rho(x)^{-1}, \qquad \delta < \rho(x) \le R_0. \tag{98}$$

Since $\tilde{u} \in H_0^1(\Omega_1, \Gamma_0)$, we choose as a test function $V = T_\delta \tilde{v} \in H_0^1(\Omega_1, \Gamma_0)$, with $\tilde{v} = T_\delta \tilde{u}$. After algebraic transformations, the left-hand side of (97) can be written as

$$\begin{aligned}-(\Delta_x \tilde{u}, V)_{\Omega_1} - \lambda(\tilde{u}, V)_{\Omega_1} &= (\nabla_x \tilde{u}, \nabla_x V)_{\Omega_1} - \lambda(\tilde{v}, \tilde{v})_{\Omega_1} \\ &= (\nabla_x \tilde{u}, \tilde{v}\nabla_x T_\delta)_{\Omega_1} + (\nabla_x \tilde{u}, T_\delta \nabla_x \tilde{v})_{\Omega_1} - \lambda(\tilde{v}, \tilde{v})_{\Omega_1} \\ &= (T_\delta \nabla_x \tilde{u}, \tilde{v} T_\delta^{-1} \nabla_x T_\delta)_{\Omega_1} + (T_\delta \nabla_x \tilde{u}, \nabla_x \tilde{v})_{\Omega_1} - \lambda(\tilde{v}, \tilde{v})_{\Omega_1} \\ &= (\nabla_x \tilde{v}, \tilde{v} T_\delta^{-1} \nabla_x T_\delta)_{\Omega_1} - (\tilde{u}\nabla_x T_\delta, \tilde{v} T_\delta^{-1} \nabla_x T_\delta)_{\Omega_1} + (\nabla_x \tilde{v}, \nabla_x \tilde{v})_{\Omega_1} \\ &\quad - (\tilde{u}\nabla_x T_\delta, \nabla_x \tilde{v})_{\Omega_1} - \lambda(\tilde{v}, \tilde{v})_{\Omega_1} \\ &= (\nabla_x \tilde{v}, \nabla_x \tilde{v})_{\Omega_1} - (\tilde{u}\nabla_x T_\delta, \tilde{v} T_\delta^{-1} \nabla_x T_\delta)_{\Omega_1} - \lambda(\tilde{v}, \tilde{v})_{\Omega_1} \\ &= \|\nabla_x \tilde{v}\|^2_{L^2(\Omega_1)} - \|\tilde{v} T_\delta^{-1} \nabla_x T_\delta\|^2_{L^2(\Omega_1)} - \lambda \|\tilde{v}\|^2_{L^2(\Omega_1)}.\end{aligned} \tag{99}$$

From formulas (97) and (99), we find that

$$\begin{aligned}\|\nabla_x \tilde{v}\|^2_{L^2(\Omega_1)} = (\chi\Delta_x\mathcal{U}_1, V)_{\Omega_1} + ([\Delta_x, \chi]\mathcal{U}_1, V)_{\Omega_1} + \lambda(c_0, V)_{\Omega_1} + \lambda(\mathcal{U}_1\chi, V)_{\Omega_1} \\ + \|\tilde{v} T_\delta^{-1} \nabla_x T_\delta\|^2_{L^2(\Omega_1)} + \lambda \|\tilde{v}\|^2_{L^2(\Omega_1)}.\end{aligned} \tag{100}$$

We estimate each terms in the previous equality. Bearing in mind that the correction term $\mathcal{U}_1$ is the solution of (92), we obtain

$$(\chi\Delta_x\mathcal{U}_1, V)_{\Omega_1} = (\chi\frac{\partial^2}{\partial x_1^2}\mathcal{U}_1, V)_{\Omega_1} + (\chi\frac{\partial^2}{\partial x_2^2}\mathcal{U}_1, V)_{\Omega_1} = (\chi\frac{\partial^2}{\partial x_1^2}\mathcal{U}_1, V)_{\Omega_1} - (\chi\lambda c_0, V)_{\Omega_1}.$$

Therefore,

$$|\lambda(c_0,V)_{\Omega_1}-(\chi\lambda c_0,V)_{\Omega_1}|\leq|\lambda(c_0,V)_{\Omega_1\cap\{x:\rho(x)\geq R_0/2\}}|\leq\lambda c_0(R_0/2)^{-2N}\|\tilde{u}\|^2_{L^2(\Omega_1)}<\infty.$$

Moreover, from Poincarè's inequality

$$\|H(R_0)^{-1}\tilde{v}(x)\|_{L^2(\Omega_1)}\leq C\|\nabla_x\tilde{v}(x)\|_{L^2(\Omega_1)} \tag{101}$$

we find

$$\begin{aligned}\left|(\chi\frac{\partial^2}{\partial x^2}\mathcal{U}_1,V)_{\Omega_1}\right|&=|(\chi T_\delta\frac{\partial^2}{\partial x^2}\mathcal{U}_1,\tilde{v})_{\Omega_1}|=|(\chi T_\delta H(x_1)\frac{\partial^2}{\partial x_1^2}\mathcal{U}_1,H(x_1)^{-1}\tilde{v})_{\Omega_1}|\\&\leq\|\chi T_\delta H(x_1)\frac{\partial^2}{\partial x_1^2}\mathcal{U}_1\|_{L^2(\Omega_1)}\|H(x_1)^{-1}\tilde{v}\|_{L^2(\Omega_1)}\\&\leq C\|\chi T_\delta H(x_1)\frac{\partial^2}{\partial x_1^2}\mathcal{U}_1\|_{L^2(\Omega_1)}\|\nabla_x\tilde{v}\|_{L^2(\Omega_1)}.\end{aligned}$$

Taking into account that $H(x_1)=O(|x_1|^2)$ and $\frac{\partial^2}{\partial x_1^2}\mathcal{U}_1=O(|x_1|^2)$, the norm

$$\begin{aligned}\|\chi T_\delta H(x_1)\frac{\partial^2}{\partial x_1^2}\mathcal{U}_1\|_{L^2(\Omega_1)}\leq{}&\|\rho^{-N}H(x_1)\frac{\partial^2}{\partial x_1^2}\mathcal{U}_1\|_{L^2(\Omega_1\cap\{x:\rho(x)\leq R_0/2\})}\\&+(R_0/2)^{-2N}\|H(x_1)\frac{\partial^2}{\partial x_1^2}\mathcal{U}_1\|_{L^2(\Omega_1\cap\{x:\rho(x)>R_0/2\})}\end{aligned}$$

is finite for $N<\frac{11}{5}$. The term $([\Delta_x,\chi]\mathcal{U}_1,V)_{\Omega_1}$ involves the derivatives of the cut-off function χ. Then it does not vanish only if $R_0/2<\rho(x)<R_0$ and

$$\begin{aligned}|([\Delta_x,\chi]\mathcal{U}_1,V)_{\Omega_1}|&=|([\Delta_x,\chi]\mathcal{U}_1,V)_{\{x\in\Omega_1:R_0/2<\rho(x)<R_0\}}|\\&=|(T_\delta[\Delta_x,\chi]\mathcal{U}_1,\tilde{v})_{\{x\in\Omega_1:R_0/2<\rho(x)<R_0\}}|\\&\leq\|T_\delta[\Delta_x,\chi]H(x_1)\mathcal{U}_1\|_{L^2(\{x\in\Omega_1:R_0/2<\rho(x)<R_0\})}\\&\quad\times\|H^{-1}(x_1)\tilde{v}\|_{L^2(\{x\in\Omega_1:R_0/2<\rho(x)<R_0\})}\\&\leq C\|T_\delta[\Delta_x,\chi]H(x_1)\mathcal{U}_1\|_{L^2(\{x\in\Omega_1:R_0/2<\rho(x)<R_0\})}\\&\quad\times\|\nabla_x\tilde{v}\|_{L^2(\{x\in\Omega_1:R_0/2<\rho(x)<R_0\})},\end{aligned}$$

which is finite for any value of N since $T_\delta(x)=(R_0/2)^{-N}$ if $\rho(x)>R_0/2$. According to the inequality (98), we have

$$\|\tilde{v}T_\delta^{-1}\nabla_xT_\delta\|^2_{L^2(\Omega_1)}\leq C\|\rho^{-1}\tilde{v}\|^2_{L^2(\Omega_1)}.$$

Choosing R_0 such that $\lambda\leq CH(R_0)^{-2}$, from (101) we deduce

$$\lambda\|\tilde{v}\|^2_{L^2(\Omega_1)}\leq CH(R_0)^{-2}\|\tilde{v}\|^2_{L^2(\Omega_1)}\leq C\|\nabla_x\tilde{v}\|^2_{L^2(\Omega_1)}$$

Finally, we have

$$\begin{aligned}|(\mathcal{U}_1\chi,V)_{\Omega_1}|&=|(T_\delta\mathcal{U}_1\chi,\tilde{v})_{\Omega_1}|=|(H(x_1)T_\delta\mathcal{U}_1\chi,H(x_1)^{-1}\tilde{v})_{\Omega_1}|\\&\leq\|H(x_1)T_\delta\mathcal{U}_1\chi\|_{L^2(\Omega_1)}\|H(x_1)^{-1}\tilde{v}\|_{L^2(\Omega_1)}\leq C\|H(x_1)T_\delta\mathcal{U}_1\chi\|_{L^2(\Omega_1)}\|\nabla_x\tilde{v}\|_{L^2(\Omega_1)}.\end{aligned}$$

The norm

$$\begin{aligned}\|\chi H(x_1)T_\delta\mathcal{U}_1\|_{L^2(\Omega_1)}\leq{}&\|\rho^{-N}H(x_1)\mathcal{U}_1\|_{L^2(\Omega_1\cap\{x:\rho(x)\leq R_0/2\})}\\&+(R_0/2)^{-2N}\|H(x_1)\mathcal{U}_1\|_{L^2(\Omega_1\cap\{x:\rho(x)>R_0/2\})}\end{aligned}$$

is finite if and only if $N<15/2$. Setting $N<11/2$, the relation (100) implies that

$$\|\rho^{-1}\tilde{v}\|^2_{L^2(\Omega_1)}+\|\nabla_x\tilde{v}\|^2_{L^2(\Omega_1)}\leq C<\infty.$$

Since T_δ is monotone increasing as $\delta\to0$, the limit of the last, bounded expression exists.

□

Since terms in the formal series (94) are polynomials in x, we deduce the smoothness of the solution u to the problem (82)-(83). In virtue of estimate of Theorem 8.1, the estimate of the reimander in a weighted H^2 follows from standard local estimates near smooth parts of the boundary (cf. [28]).

8.3 The Dirichlet case

If we replace the boundary condition (83) on Γ_0 with a homogeneous Dirichlet condition, i.e. $c_0 = 0$, then all eigenfunctions u of the problem

$$-\Delta_x u(x) = \lambda u(x), \quad x \in \Omega_1, \tag{102}$$

$$\partial_{\nu_1} u(x) = 0, \qquad x \in \Gamma_1 \setminus \mathcal{O}, \qquad u(x) = 0, \qquad x \in \Gamma_0 \setminus \mathcal{O}, \tag{103}$$

decay exponentially as $x \to \mathcal{O}$.

Proposition 8.2. *The eigenfunction $u \in H^1_0(\Omega_1, \Gamma_0)$ of the problem* (102)-(103) *decays exponentially as $x \to \mathcal{O}$.*

Proof. Let T_δ be the weight function defined by

$$T_\delta(x) = \begin{cases} e^{\frac{\beta}{\delta}}, & |x_1| \le \delta, \\ e^{\frac{\beta}{|x_1|}}, & \delta < |x_1| \le R, \\ e^{\frac{\beta}{R}}, & |x_1| > R. \end{cases}$$

Here, the parameter δ is small, positive and it will be sent to 0 and $\beta > 0$. Note that T_δ is a continuous function such that

$$|\nabla_x T_\delta(x)| \le \beta |x_1|^{-2} T_\delta(x), \qquad e^{\frac{\beta}{R}} \le T_\delta(x) \le e^{\frac{\beta}{\delta}}.$$

We insert into the integral identity (84) the test function $v = T_\delta U \in H^1_0(\Omega_1, \Gamma_0)$, with $U = T_\delta u$, obtaining

$$\begin{aligned} \lambda(u,v)_{\Omega_1} &= (\nabla_x u, \nabla_x v)_{\Omega_1} = (\nabla_x u, U\nabla_x T_\delta)_{\Omega_1} + (\nabla_x u, T_\delta \nabla_x U)_{\Omega_1} \\ &= (T_\delta \nabla_x u, T_\delta^{-1} U \nabla_x T_\delta)_{\Omega_1} + (T_\delta \nabla_x u, \nabla_x U)_{\Omega_1} \\ &= (\nabla_x U, T_\delta^{-1} U \nabla_x T_\delta)_{\Omega_1} - (u\nabla_x T_\delta, T_\delta^{-1} U \nabla_x T_\delta)_{\Omega_1} \\ &\quad + (\nabla_x U, \nabla_x U)_{\Omega_1} - (u \nabla_x T_\delta, \nabla_x U)_{\Omega_1} \\ &= (\nabla_x U, \nabla_x U)_{\Omega_1} - (T_\delta^{-1} U \nabla_x T_\delta, T_\delta^{-1} U \nabla_x T_\delta)_{\Omega_1}. \end{aligned}$$

Hence,

$$\|\nabla_x U\|^2_{L^2(\Omega_1)} = \lambda \|U\|^2_{L^2(\Omega_1)} + \|T_\delta^{-1} U \nabla_x T_\delta\|^2_{L^2(\Omega_1)}.$$

Taking into account the Poincarè's inequality (101), we find

$$(c - \beta^2) \||x_1|^{-2} U\|^2_{L^2(\Omega_1)} \le \lambda \|U\|^2_{L^2(\Omega_1)} \le \lambda e^{\frac{2\beta}{\delta}} \|u\|^2_{L^2(\Omega_1)} < \infty.$$

In particular, choosing β such that $0 \le \beta^2 < c$, we get

$$e^{-\frac{2\beta}{\delta}} \int_{\Omega_1} |x_1|^{-4} |U(x)|^2 dx \le c_\lambda < \infty.$$

It implies that both of the integrals

$$e^{-\frac{2\beta}{\delta}} \int_{\Omega_1 \cap \{x : |x_1| \le e^{-\frac{\beta}{2\delta}}\}} |x_1|^{-4} |U(x)|^2 dx, \qquad e^{-\frac{2\beta}{\delta}} \int_{\Omega_1 \cap \{x : |x_1| > e^{-\frac{\beta}{2\delta}}\}} |x_1|^{-4} |U(x)|^2 dx,$$

are bounded for all $\delta > 0$. The first one gives

$$\int_{\Omega_1 \cap \{x:|x_1| \le e^{-\frac{\beta}{2\delta}}\}} |U(x)|^2 dx \le e^{-\frac{2\beta}{\delta}} \int_{\Omega_1 \cap \{x:|x_1| \le e^{-\frac{\beta}{2\delta}}\}} |x_1|^{-4} |U(x)|^2 dx < \infty$$

for all $\delta > 0$. Since T_δ is monotone increase as $\delta \to 0$, we conclude that the eigenfunction u has an exponential decay in L^2-norm in a neighbourhood of the cusp $\mathcal{O}$. □

The eigenfunctions u are thus smooth at any distance of $\mathcal{O}$ and vanish at cusp point $\mathcal{O}$ with all their derivatives due to the exponential decay. We conclude that also in this case the asymptotic anzätze for $(\lambda^\varepsilon, u^\varepsilon)$ and the procedure given in the Sections $2-3$ are corect.

8.4 Open Questions

Due to the shape of the boundary Γ_1, the solution of the problem (82) − (83) behaves in substantially different way from the solution of the problem

$$-\Delta_x u(x) = \lambda u(x), \quad x \in \Omega_1, \tag{104}$$

$$u(x) = 0, \quad x \in \Gamma_1 \setminus \mathcal{O}, \qquad u(x) = c_0, \quad x \in \Gamma_0 \setminus \mathcal{O}. \tag{105}$$

Here we have simply replaced the Neumann boundary condition (3) of the problem (82)-(83) with a homogeneous Dirichlet condition. Indeed an approximation of the solution u of the problem (104)-(105) is to be found in such a way that the boundary conditions are satisfied exactly while discrepancies in the equation (104) is reduced as much as possible. As a consequence, a solution u with the asymptotic

$$u(x) = c_0 \frac{x_2 - H_1(x_1)}{H(x_1)} + \cdots, \qquad x \to \mathcal{O},$$

cannot belong to the Sobolev space $H^1(\Omega_1)$. Indeed the integral

$$\int_0^{1/3} \int_{H_1(x_1)}^{H_0(x_1)} \left| \frac{\partial}{\partial x_2} \left(c_0 \frac{x_2 - H_1(x_1)}{H(x_1)} \right) \right|^2 dx_2 dx_1 = 2c_0^2 \int_0^{1/3} \frac{1}{H(x_1)^2} dx_1$$

is divergent because the integrand has nonadmissible singularity $O(|x_1|^{-4})$. The derivation of the ansatz for the eigenfunction u of the problem (104)-(105) is still an open problem.

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