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# An extension theorem from connected sets and homogenization of non-local functionals

Andrea Braides\*, Valeria Chiadò Piat and Lorenza D’Elia†

**Abstract.** We study the asymptotic behaviour of convolution-type functionals defined on general periodic domains by proving an extension theorem.

**Keywords:** homogenization, perforated domains, non-local functionals, extension operators

**AMS Classifications.** 49J45, 49J55, 74Q05, 35B27, 35B40, 45E10

## 1 Introduction

In this paper we consider energies of convolution-type whose prototypes are functionals of the form

$$\frac{1}{\varepsilon^{d+p}} \int_{\Omega \times \Omega} a\left(\frac{y-x}{\varepsilon}\right) |u(y) - u(x)|^p dx dy, \quad (1)$$

where  $a$  is a non-negative convolution kernel,  $p \in (1, +\infty)$ ,  $\varepsilon$  is a scaling parameter and  $\Omega$  is a Lipschitz domain in  $\mathbb{R}^d$ . The kernel  $a : \mathbb{R}^d \rightarrow [0, +\infty[$ , describing the strength of the interaction at a given distance, satisfies

$$\int_{\mathbb{R}^d} a(\xi)(1 + |\xi|^p) d\xi < +\infty, \quad (2)$$

and

$$a(\xi) \geq c > 0, \quad \text{if } |\xi| \leq r_0, \quad (3)$$

for some  $r_0 > 0$  and  $c > 0$ .

Functionals of this form have been used as an approximation of the  $L^p$ -norm of the gradient as  $\varepsilon \rightarrow 0$  and as such give an alternative way of defining Sobolev spaces (see *e.g.* [2, 10]). In the case  $p = 2$  perturbations of such energies (1) arise from models in population dynamics where the macroscopic properties are reduced to studying the

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evolution of the first-correlation function describing the population density  $u$  in the system [11], and recently they have also been used in problems in Data Science [12]. Furthermore discrete versions of such energies have been extensively studied in a general setting (see *e.g.* [3, 5] and related works).

A rather complete analysis of perturbations of functionals (1), more precisely, of functionals that are dominated from below and above by functionals of type (1), is presented in [4]. In this paper we consider another type of perturbation of (1) in the framework of the so-called *perforated domains*, that cannot be reduced to the analysis in [4] since it is ‘degenerate’ on the complement of a periodic connected set.

In our analysis we consider a typical situation arising in the study of inhomogeneous media with a periodic microstructure, when one sets the model in a domain obtained by removing inclusions representing sites with which the system does not interact. Usually, such a periodically perforated domain is obtained by intersecting  $\Omega$  with a periodic open subset  $E_\delta = \delta E$  of  $\mathbb{R}^d$ , where  $E$  is a periodic set with Lipschitz boundary and  $\delta$  is the (small) period of the microstructure. In the setting of energies (1) the relevant scale of the period  $\delta$  is of order  $\varepsilon$ . Indeed, in the other cases we have a multi-scale problem that can be decomposed into two separate limit analyses that fall within known results corresponding to letting first  $\delta \rightarrow 0$  and then  $\varepsilon \rightarrow 0$ , or the converse (see [8]). Hence, we will consider energies whose prototypes are of the form

$$F_\varepsilon(u) = \frac{1}{\varepsilon^{d+p}} \int_{(\Omega \cap \varepsilon E) \times (\Omega \cap \varepsilon E)} a\left(\frac{y-x}{\varepsilon}\right) |u(x) - u(y)|^p dy dx, \quad (4)$$

where  $\Omega$  is a fixed domain in  $\mathbb{R}^d$ .

In order to study the asymptotic analysis of such energies, it is necessary to prove that sequences with equi-bounded energy (and equi-bounded  $L^p$ -norm) are precompact. For the analog energy on Sobolev spaces

$$F_\varepsilon^{\text{Sob}}(u) = \int_{\Omega \cap \varepsilon E} |\nabla u|^p dy dx.$$

this has been done in [1] through the construction of suitable extension operators  $T_\varepsilon : L^p(\Omega \cap \varepsilon E) \rightarrow L^p(\Omega)$  which, for each  $\Omega'$  compactly contained in  $\Omega$ , provide an embedding of  $W^{1,p}(\Omega')$  in  $W^{1,p}(\Omega \cap \varepsilon E)$  uniformly for  $\varepsilon$  small enough (below a threshold explicitly depending on the distance between  $\Omega'$  and  $\partial\Omega$ ). The compact embedding of  $W^{1,p}(\Omega')$  in  $L^p(\Omega')$  then provides the desired compactness property. In our case, since the energies are non-local, a more complex statement is necessary. After noting that by condition (3) it is sufficient to prove compactness when  $a$  is the characteristic function of a ball centered in 0 and given radius  $r_0$ , we prove the existence of extension operators  $T_\varepsilon : L^p(\Omega \cap \varepsilon E) \rightarrow L^p(\Omega)$  with the property that  $R$  and  $C$  exists such that for each  $\Omega'$  compactly contained in  $\Omega$ ,

$$\begin{aligned} & \int_{\Omega' \times \Omega'} \chi_{B_R}\left(\frac{y-x}{\varepsilon}\right) |T_\varepsilon u(x) - T_\varepsilon u(y)|^p dy dx \\ & \leq C \int_{(\Omega \cap \varepsilon E) \times (\Omega \cap \varepsilon E)} \chi_{B_{r_0}}\left(\frac{y-x}{\varepsilon}\right) |u(x) - u(y)|^p dy dx, \end{aligned} \quad (5)$$

for  $\varepsilon$  small enough, with  $C$  and  $R$  independent of  $\varepsilon$  (here  $B_\rho$  denotes the ball of centre 0 and radius  $\rho$  and  $\chi_A$  is the characteristic function of the set  $A$ ). The precise statement of this result is given in Theorem 2.2. It provides a uniform bound for energies of the type (1) on  $\Omega'$  in terms of energies (4), which in turn allows to apply the compactness results in [4] (see Section 2.2). Moreover, the asymptotic analysis of functionals (1) ensure that limits of functions with equibounded energies are in  $W^{1,p}(\Omega')$  with a uniform bound and hence they belong to  $W^{1,p}(\Omega)$ .

The case  $p = 2$  in (4) and with compact perforations; *i.e.*, with  $E$  of the form  $E = \mathbb{R}^d \setminus (K_0 + \mathbb{Z}^d)$ , where  $K_0$  is a compact subset of  $\mathbb{R}^d$  with Lipschitz boundary such that  $(K_0 + i) \cap (K_0 + j) = \emptyset$  if  $i, j \in \mathbb{Z}^d$  and  $i \neq j$ , has been studied in [8], together with some variants that allow to consider random perforations [9]. The main feature of our paper is the proof of the extension theorem under the only assumption that the periodic set  $E$  is connected and with Lipschitz boundary, and holds for any  $p > 1$ . The construction of  $T_\varepsilon$  is inspired by the arguments of [1], consisting in proving a local extension result on cubes and then using a periodic partition of the unity. The non-locality of the energies adds further technical difficulties to the possible non-connectedness or non-regularity of the restriction of  $E$  to cubes, already present in the case of Sobolev functions, and forces the introduction of the radius of interaction  $R$  in inequality (5).

As an application, we study the asymptotic behaviour of energies of the form

$$H_\varepsilon(u) = \frac{1}{\varepsilon^d} \int_{(\Omega \cap_\varepsilon E)^2} h\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{u(y) - u(x)}{\varepsilon}\right) dx dy,$$

with  $u \in L^p(\Omega; \mathbb{R}^m)$ , upon some structure hypotheses on  $h$  as those considered in [4], that allow  $H_\varepsilon$  to be compared with  $F_\varepsilon$ . In Section 3 we obtain a homogenization theorem for  $H_\varepsilon$  as  $\varepsilon \rightarrow 0$  proving that the  $\Gamma$ -limit of  $H_\varepsilon$  is defined on  $W^{1,p}(\Omega; \mathbb{R}^m)$  and has a standard local form

$$\int_{\Omega} h_{\text{hom}}(Du) dx,$$

with  $h_{\text{hom}}$  characterized by non-local homogenization formulas and of  $p$ -growth by (2) and (3). The proof is obtained by a perturbation argument that allows to use homogenization theorems proved in [4] for the corresponding energies defined on ‘solid’ domains, applied to functionals of the form  $H_\varepsilon + \delta F_\varepsilon$ . The Extension Theorem provides uniform estimates that allow to invert the passage to the limit as  $\varepsilon \rightarrow 0$  and  $\delta \rightarrow 0$ . We note that a discrete analog of this result can be found in [6], where the discrete setting allows easier extension results from the discrete version of a perforated domain.

Before stating and proving the main result we gather some of the notation used in the following.

## Notation

- $Q = (0, 1)^d$  denotes the unit cube in  $\mathbb{R}^d$ .

- $\chi_A$  denotes the characteristic function of the set  $A$ .
- $\lfloor t \rfloor$  denotes the integer part of  $t \in \mathbb{R}$ .
- $\mathbf{M}^{m \times d}$  is the space of  $m \times d$  real matrices.
- if  $\Xi \in \mathbf{M}^{m \times d}$  and  $x \in \mathbb{R}^d$  then  $\Xi x \in \mathbb{R}^m$  is defined by the usual row-by-column product.
- For any open set  $\Omega \subset \mathbb{R}^d$  and for any  $\lambda > 0$ ,  $\lambda\Omega$  denotes the  $\lambda$ -homothetic set

$$\lambda\Omega := \{\lambda x : x \in \Omega\},$$

and  $\Omega(\lambda)$  is the retracted set

$$\Omega(\lambda) := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \lambda\}. \quad (6)$$

- For  $R > 0$ ,  $D_R$  denotes the set of points in  $\mathbb{R}^d \times \mathbb{R}^d$  whose distance is less than  $R$ ; *i.e.*,

$$D_R := \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |x - y| \leq R\}.$$

- Given an open set with finite Lebesgue measure  $|A| < \infty$ , the mean value of  $u$  over  $A$  is given by

$$u_A = \frac{1}{|A|} \int_A u(x) dx. \quad (7)$$

- We say that a set  $E \subset \mathbb{R}^d$  is periodic (more precisely,  $Q$ -periodic) if  $E + e_i = E$  for every  $i = 1, 2, \dots, d$  where  $(e_i)_{i=1}^d$  is the canonical basis of  $\mathbb{R}^d$ .

## 2 The extension theorem

In this section, we prove the existence of an extension operator for non-local functionals defined on general connected domains. The main result of the paper is Theorem 2.2, from which we deduce a compactness result in Section 2.2. Before stating it, we recall the definition of a set with Lipschitz boundary.

**Definition 2.1.** *An open set  $E \subset \mathbb{R}^n$  has Lipschitz boundary at  $x \in \partial E$  if  $\partial E$  is locally the graph of a Lipschitz function, in the sense that there exist a coordinate system  $(y_1, \dots, y_d)$ , a Lipschitz function  $\Phi$  of  $d - 1$  variables, and an open rectangle  $U_x$  in the  $y$ -coordinates, centred at  $x$ , such that  $E \cap U_x = \{y : y_n < \Phi(y_1, \dots, y_{d-1})\}$  and that  $\partial E$  splits  $U_x$  into two connected sets,  $E \cap U_x$  and  $U_x \setminus \overline{E}$ . If this property holds for every  $x \in \partial E$  with the same Lipschitz constant, we say that  $E$  has Lipschitz boundary.*

**Theorem 2.2.** *Let  $E$  be a periodic open subset of  $\mathbb{R}^d$  with Lipschitz boundary and let  $\Omega$  be a bounded open subset of  $\mathbb{R}^d$ . Then, there exist  $R = R(E) > 0$  and  $k_0 > 0$  such that for all  $\varepsilon > 0$  there exists a linear and continuous extension operator  $T_\varepsilon : L^p(\Omega \cap \varepsilon E) \rightarrow L^p(\Omega)$  such that for all  $r > 0$  and for all  $u \in L^p(\Omega \cap \varepsilon E)$ ,*

$$T_\varepsilon(u) = u \quad \text{a.e. in } \Omega \cap \varepsilon E, \quad (8)$$

$$\int_{\Omega(\varepsilon k_0)} |T_\varepsilon(u)|^p dx \leq c_1 \int_{\Omega \cap \varepsilon E} |u|^p dx, \quad (9)$$

$$\int_{(\Omega(\varepsilon k_0))^2 \cap D_{\varepsilon R}} |T_\varepsilon(u)(x) - T_\varepsilon(u)(y)|^p dx dy \leq c_2(r) \int_{(\Omega \cap \varepsilon E)^2 \cap D_{\varepsilon r}} |u(x) - u(y)|^p dx dy, \quad (10)$$

where we use notation (6). The positive constants  $c_1$  and  $c_2$  depend on  $E$  and  $d$  and, in addition,  $c_2$  depends also on  $r$ , but both are independent of  $\varepsilon$ .

The proof, which will be given in the next subsection, is quite technical and it is split into several lemmas.

## 2.1 Technical lemmas and proof of the main result

In order to give an idea of the construction of the extension operator, we assume that  $E \cap 2Q$  is connected and has Lipschitz boundary. Under these assumptions, there exists a linear and continuous operator  $\Phi : L^p(E \cap 2Q) \rightarrow L^p(2Q)$  satisfying, in particular, an estimate analogous to (10) (see Lemma 2.5). Then, we consider the family  $\Phi^\alpha$  of the extension operator obtained by traslating  $\Phi$  by an integer vector  $\alpha \in \mathbb{Z}^d$ . Finally, thanks to a periodic partition of unity, the construction of a global extension operator is achieved glueing together  $\Phi^\alpha$  (see Lemma 2.7). Now, the assumptions that  $E \cap 2Q$  is connected and has Lipschitz boundary in general are not satisfied (unless the complement of  $E$  is a disjoint union of compact sets, which is the case studied in [8]), so that the first step consists to overcome the lack of connectedness of  $E \cap 2Q$  and the regularity of its boundary. To this end, we state a slightly modified version of [1, Lemma 2.3], which is a key tool for the construction of the extension operator. The proof remains analogous to that of [1, Lemma 2.3] and is not repeated here.

**Lemma 2.3.** *Let  $E$  be a connected open subset of  $\mathbb{R}^d$  with Lipschitz boundary. Then, there exists  $k \in \mathbb{N}$ ,  $k \geq 4$ , such that  $3Q \cap E$  is contained in a single connected component  $C$  of  $kQ \cap E$ . Moreover,  $C$  has Lipschitz boundary at each point of  $\partial C \cap 3\bar{Q}$ .*

We denote henceforth by  $\tilde{C}$  the positive constant given by  $\tilde{C} := 2\sqrt{d}k$ , where  $k$  is defined as Lemma 2.3.

The next lemma is an easy consequence of the Hölder inequality.

**Lemma 2.4.** *Let  $A$  be an open subset of  $\mathbb{R}^d$ . Assume that  $A$  has finite and positive Lebesgue measure  $|A| < \infty$ . Then, for every  $u \in L^p(A)$ , with  $1 < p < \infty$ ,*

$$\int_A |u_A - u(x)|^p dx \leq \frac{1}{|A|} \int_{A \times A} |u(x) - u(y)|^p dx dy. \quad (11)$$

*Proof.* Denote by  $p'$  the conjugate exponent of  $p$ . Thanks to Hölder's inequality, we deduce

$$\begin{aligned} \int_A |u_A - u(x)|^p dx &= \frac{1}{|A|^p} \int_A \left| \int_A (u(y) - u(x)) dy \right|^p dx \\ &\leq \frac{|A|^{p/p'}}{|A|^p} \int_A \int_A |u(y) - u(x)|^p dy dx \\ &= \frac{1}{|A|} \int_{A \times A} |u(y) - u(x)|^p dx dy, \end{aligned}$$

which concludes the proof.  $\square$

The next lemma shows the existence of an extension operator  $\Phi$  on general sets of  $\mathbb{R}^d$ . It is an adaptation of [1, Lemma 2.6].

**Lemma 2.5.** *Let  $B, \omega, \omega'$  be bounded open subsets of  $\mathbb{R}^d$ . Assume that  $\partial B$  is Lipschitz-continuous at each point of  $\partial B \cap \bar{\omega}$  and  $\omega' \subset \subset \omega$ . Then, there exist a positive real number  $R > 0$  and a linear and continuous extension operator  $\Phi : L^p(B) \rightarrow L^p(\omega')$  such that, for all  $u \in L^p(B)$ ,*

$$\Phi(u) = u \quad \text{a.e. in } B \cap \omega', \quad (12)$$

$$\int_{\omega'} |\Phi(u)|^p dx \leq c_1 \int_{B \cap \omega} |u|^p dx, \quad (13)$$

$$\int_{(\omega' \times \omega') \cap D_R} |\Phi(u)(x) - \Phi(u)(y)|^p dx dy \leq c_2 \int_{(B \cap \omega)^2} |u(x) - u(y)|^p dx dy, \quad (14)$$

where  $c_1$  and  $c_2$  are positive constant depending only on  $B, \omega', \omega$  and  $p$ .

*Proof.* Since  $\partial B$  has Lipschitz boundary at each point of  $\partial B \cap \bar{\omega}$ , there exist a neighbourhood  $U$  of  $\partial B \cap \bar{\omega}$  and a bi-lipschitz map  $\mathcal{R} : U \cap B \rightarrow U \setminus B$  such that, for any  $x_1, x_2 \in U \cap B$ ,

$$\frac{1}{2} |\mathcal{R}(x_1) - \mathcal{R}(x_2)| \leq |x_1 - x_2| \leq 2 |\mathcal{R}(x_1) - \mathcal{R}(x_2)|.$$

For fixed  $t > 0$  chosen below, we consider the set

$$A_t := \{x \in \omega \setminus B : \text{dist}(x, \partial B) < t\}. \quad (15)$$

We may fix  $t > 0$  small enough such that

$$A_t \cap \omega' \subset U \setminus B \quad \text{and} \quad \mathcal{R}^{-1}(A_t \cap \omega') \subset B \cap \omega. \quad (16)$$

Let  $\varphi$  be a  $C^\infty$  function such that  $0 \leq \varphi \leq 1$ ,  $\varphi \equiv 1$  in  $\bar{B}$  and  $\varphi \equiv 0$  in  $\{x \in \mathbb{R}^d \setminus B : \text{dist}(x, \partial B) \geq t\}$ . We define the operator  $\Phi : L^p(B) \rightarrow L^p(\omega')$  as follows

$$\Phi(u)(x) := \begin{cases} u(x), & x \in B \cap \omega', \\ \varphi(x)u(\mathcal{R}^{-1}(x)) + (1 - \varphi(x))u_{B \cap \omega}, & x \in A_t \cap \omega', \\ u_{B \cap \omega}, & x \in \omega' \setminus A_t, \end{cases} \quad (17)$$

where  $u_{B \cap \omega}$  denotes the mean value of the function  $u$  over  $B \cap \omega$  (see (7)). It follows that  $\Phi(u) \in L^p(\omega')$  and  $\Phi(u) = u$  a.e. in  $B \cap \omega'$ ; *i.e.*, condition (12) is satisfied.

We now show condition (13). To this end, note that  $\omega'$  can be written as

$$\omega' = (B \cap \omega') \cup (A_t \cap \omega') \cup (\omega' \setminus A_t).$$

This, combined with the Jensen inequality and the definition (17) of  $\Phi$ , yields

$$\begin{aligned} \int_{\omega'} |\Phi(u)(x)|^p dx &= \int_{B \cap \omega'} |\Phi(u)(x)|^p dx + \int_{A_t \cap \omega'} |\Phi(u)(x)|^p dx + \int_{\omega' \setminus A_t} |\Phi(u)(x)|^p dx \\ &= \int_{B \cap \omega'} |u(x)|^p dx + \int_{A_t \cap \omega'} |\varphi(x)u(\mathcal{R}^{-1}(x)) + (1 - \varphi(x))u_{B \cap \omega}|^p d\xi \\ &\quad + |\omega' \setminus A_t| |u_{B \cap \omega}|^p \\ &\leq \int_{B \cap \omega'} |u(x)|^p dx + 2^{p-1} \int_{A_t \cap \omega'} |u(\mathcal{R}^{-1}(x))|^p dx \\ &\quad + |u_{B \cap \omega}|^p (2^{p-1} |\omega' \cap A_t| + |\omega' \setminus A_t|). \end{aligned} \tag{18}$$

Since  $\mathcal{R}$  is a bi-Lipschitz map, the Jacobian  $\left| \frac{\partial \mathcal{R}}{\partial x}(x) \right|$  is a bounded function; *i.e.*, there exists a positive constant  $c_{\mathcal{R}}$  such that

$$\left| \frac{\partial \mathcal{R}}{\partial x}(x) \right| \leq c_{\mathcal{R}}, \tag{19}$$

so that, thanks to the change of variables  $x' = \mathcal{R}^{-1}(x)$  and properties (16), we have

$$\int_{A_t \cap \omega'} |u(\mathcal{R}^{-1}(x))|^p dx \leq c_{\mathcal{R}} \int_{B \cap \omega} |u(x')|^p dx'.$$

This, along with (18), implies that

$$\begin{aligned} \int_{\omega'} |\Phi(u)(x)|^p dx &\leq (c_{\mathcal{R}} 2^{p-1} + 1) \int_{B \cap \omega'} |u(x)|^p dx + |u_{B \cap \omega}|^p (2^{p-1} |\omega' \cap A_t| + |\omega' \setminus A_t|) \\ &\leq c_1 \int_{B \cap \omega} |u(x)|^p dx, \end{aligned}$$

where  $c_1$  denotes a positive constant depending only on  $p, \omega', B$  and  $\mathcal{R}$ . Hence, condition (13) is proven.

To conclude the proof, it remains to check condition (14). Fix  $R < t$ . For  $(x, y) \in (\omega' \times \omega') \cap D_R$ , it is enough to estimate the integral in the left-hand side of (14) by



examining separately the sets

$$\begin{aligned}
S_1 &= ((B \cap \omega') \times (B \cap \omega')) \cap D_R, \\
S_2 &= ((B \cap \omega') \times (A_t \cap \omega')) \cap D_R, \\
S'_2 &= ((A_t \cap \omega') \times (B \cap \omega')) \cap D_R, \\
S_3 &= ((A_t \cap \omega') \times (A_t \cap \omega')) \cap D_R, \\
S_4 &= ((A_t \cap \omega') \times (\omega' \setminus A_t)) \cap D_R, \\
S'_4 &= ((\omega' \setminus A_t) \times (A_t \cap \omega')) \cap D_R, \\
S_5 &= ((\omega' \setminus A_t) \times (\omega' \setminus A_t)) \cap D_R.
\end{aligned}$$

Note that the other cases do not occur since the distance between the points is greater than  $R$ . Indeed, take, for example,  $(x, y) \in (B \cap \omega') \times (A_t \setminus \omega')$ . Due to definition of  $A_t$  and since  $R < t$ , the distance  $|x - y|$  is greater than  $R$ .

Now, we evaluate the left-hand side of (14) on the set  $S_i$  defined above. In view of the definition (17) of  $\Phi$ , we have

$$\begin{aligned}
\int_{S_1} |\Phi(u)(x) - \Phi(u)(y)|^p dx dy &= \int_{S_1} |u(x) - u(y)|^p dx dy \\
&\leq \int_{(B \cap \omega)^2} |u(x) - u(x)|^p dx dy.
\end{aligned}$$

Here, we used the fact that  $S_1 \subset (B \cap \omega')^2 \subset (B \cap \omega)^2$ .

Due to definition (17) of  $\Phi$ , an application of Jensen's inequality yields

$$\begin{aligned}
\int_{S_2} |\Phi(u)(x) - \Phi(u)(y)|^p dx dy &= \int_{S_2} |u(x) - u(\mathcal{R}^{-1}(y)) + (1 - \varphi(y))(u(\mathcal{R}^{-1}(y)) - u_{B \cap \omega})|^p dx dy \\
&\leq 2^{p-1} \int_{S_2} |u(x) - u(\mathcal{R}^{-1}(y))|^p dx dy \\
&\quad + 2^{p-1} \int_{S_2} |1 - \varphi(y)|^p |u(\mathcal{R}^{-1}(y)) - u_{B \cap \omega}|^p dx dy \quad (20)
\end{aligned}$$

Using the change of variables  $y' = \mathcal{R}^{-1}(y)$  and properties (16) and (19), the first integral in the left-hand side of (20) can be estimated as

$$\begin{aligned}
\int_{S_2} |u(x) - u(\mathcal{R}^{-1}(y))|^p dx dy &\leq \int_{B \cap \omega'} \left( \int_{A_t \cap \omega'} |u(x) - u(\mathcal{R}^{-1}(y))|^p dy \right) dx \\
&\leq c_{\mathcal{R}} \int_{(B \cap \omega')^2} |u(x) - u(y')|^p dx dy' \\
&\leq c_{\mathcal{R}} \int_{(B \cap \omega)^2} |u(x) - u(y)|^p dx dy. \quad (21)
\end{aligned}$$

By applying Lemma 2.4 and taking into account condition (19), the second integral in the right-hand side of (20) can be estimated as

$$\begin{aligned}
\int_{S_2} |1 - \varphi(y)|^p |u(\mathcal{R}^{-1}(y)) - u_{B \cap \omega}|^p dx dy &\leq |B \cap \omega'| \int_{A_t \cap \omega'} |u(\mathcal{R}^{-1}(y)) - u_{B \cap \omega}|^p dy \\
&\leq c_{\mathcal{R}} |B \cap \omega'| \int_{B \cap \omega} |u(y') - u_{B \cap \omega}|^p dy' \\
&\leq c_{\mathcal{R}} \frac{|B \cap \omega'|}{|B \cap \omega|} \int_{(B \cap \omega)^2} |u(x) - u(y)|^p dx dy.
\end{aligned}$$

Combined with (20) and (21), this implies

$$\int_{S_2} |\Phi u(x) - \Phi u(y)|^p dx dy \leq c \int_{(B \cap \omega)^2} |u(x) - u(y)|^p dx dy,$$

where  $c$  is a positive constant depending on  $p, B, \omega, \omega'$  and  $\mathcal{R}$ . Similarly, we have that

$$\int_{S_2'} |\Phi u(x) - \Phi u(y)|^p dx dy \leq c \int_{(B \cap \omega)^2} |u(x) - u(y)|^p dx dy.$$

Now, consider  $(x, y) \in S_3$ . From the definition (17) of  $\Phi$ , we have

$$\Phi u(x) - \Phi u(y) = F_1(x, y) + F_2(x, y), \tag{22}$$

where  $F_1(x, y)$  and  $F_2(x, y)$  are given by

$$\begin{aligned}
F_1(x, x) &:= (u(\mathcal{R}^{-1}(x)) - u_{B \cap \omega})(\varphi(x) - \varphi(y)), \\
F_2(x, y) &:= \varphi(y) (u(\mathcal{R}^{-1}(x)) - u(\mathcal{R}^{-1}(y))).
\end{aligned}$$

Thanks to Lemma 2.4 and due to properties (16) and the estimate  $|\varphi(x) - \varphi(y)| \leq 2$ , we deduce that

$$\begin{aligned}
\int_{S_3} |F_1(x, y)|^p dx dy &\leq 2^p \int_{(A_t \cap \omega')^2} |u(\mathcal{R}^{-1}(x)) - u_{B \cap \omega}|^p dx dy \\
&= 2^p |A_t \cap \omega'| \int_{(A_t \cap \omega')} |u(\mathcal{R}^{-1}(x)) - u_{B \cap \omega}|^p dx \\
&\leq 2^p |A_t \cap \omega'| c_{\mathcal{R}} \int_{B \cap \omega} |u(x') - u_{B \cap \omega}|^p dx' \\
&\leq 2^p c_{\mathcal{R}} \frac{|A_t \cap \omega'|}{|B \cap \omega|} \int_{(B \cap \omega)^2} |u(x') - u(y)|^p dx' dy. \tag{23}
\end{aligned}$$

On the other hand, using the changes of variables  $x' = \mathcal{R}^{-1}(x)$  and  $y' = \mathcal{R}^{-1}(y)$ , we get

$$\begin{aligned}
\int_{S_3} |F_2(x, y)|^p dx dy &\leq \int_{(A_t \cap \omega')^2} |u(\mathcal{R}^{-1}(x)) - u(\mathcal{R}^{-1}(y))|^p dx dy \\
&\leq c_{\mathcal{R}}^2 \int_{(B \cap \omega)^2} |u(x') - u(y')|^p dx' dy'. \tag{24}
\end{aligned}$$

In view of (22), an application of Jensen's inequality combined with (23) and (24) leads to

$$\begin{aligned} \int_{S_3} |\Phi(u)(x) - \Phi(u)(y)|^p dx dy &\leq 2^{p-1} \left( \int_{S_3} |F_1(x, y)|^p dx dy + \int_{S_3} |F_2(x, y)|^p dx dy \right) \\ &\leq c \int_{(B \cap \omega)^2} |u(x) - u(y)|^p dx dy, \end{aligned} \quad (25)$$

where  $c$  denotes a positive constant depending only on  $p, B, \omega, \omega'$  and  $\mathcal{R}$ .

Take now  $(x, y) \in S_4$ . Applying Lemma 2.4 and using the change of variables  $x' = \mathcal{R}^{-1}(x)$ , from the definition (17) of  $\Phi$ , we deduce that

$$\begin{aligned} \int_{S_4} |\Phi(u)(x) - \Phi(u)(y)|^p dx dy &= |\omega' \setminus A_t| \int_{A_t \cap \omega'} |u(\mathcal{R}^{-1}(x)) - u_{B \cap \omega}|^p dx \\ &\leq c_{\mathcal{R}} |\omega' \setminus A_t| \int_{B \cap \omega} |u(x') - u_{B \cap \omega}|^p dx' \\ &\leq c_{\mathcal{R}} \frac{|\omega' \setminus A_t|}{|B \cap \omega|} \int_{(B \cap \omega)^2} |u(x) - u(y)|^p dx dy \end{aligned}$$

Similarly, we also get

$$\int_{S'_4} |\Phi(u)(x) - \Phi(u)(y)|^p dx dy \leq c_{\mathcal{R}} \frac{|\omega' \cap A_t|}{|B \cap \omega|} \int_{(B \cap \omega)^2} |u(x) - u(y)|^p dx dy.$$

Now, take  $(x, y) \in S_5$ . Hence, we have that  $\Phi(x) - \Phi(y) = 0$  for a.e.  $x, y \in \omega' \setminus A_t$ . Finally, gathering all the previous estimates, we conclude that

$$\begin{aligned} \int_{(\omega' \times \omega') \cap D_R} |\Phi(u)(x) - \Phi(u)(y)|^p dx dy &= \sum_{i=1}^5 \int_{S_i} |\Phi(u)(x) - \Phi(u)(y)|^p dx dy \\ &\quad + \int_{S'_2 \cup S'_4} |\Phi(u)(x) - \Phi(u)(y)|^p dx dy \\ &\leq c_2 \int_{(B \cap \omega)^2} |u(x) - u(y)|^p dx dy, \end{aligned}$$

where  $c_2$  is a constant depending on  $p, \omega', \omega$  and  $B$ . This shows (14) and concludes the proof.  $\square$

The reflection argument that we used to construct the operator  $\Phi$  cannot be used to prove the existence of a map  $\Phi : L^p(B) \rightarrow L^p(\omega)$  since estimate (14) may not hold with  $\omega' = \omega$ , as showed in the following example.

**Example 2.6.** Let  $B$  be the ball in  $\mathbb{R}^2$  centered at 0 and of radius 1 and let  $\omega$  be the set of  $\mathbb{R}^2$  defined by

$$\omega := \{(x, y) \in \mathbb{R}^2 : x \in (-1, 2), -x + 1 \leq y \leq -x + 2\}.$$

We define  $u \in L^p(B)$  as

$$u(x) := \begin{cases} 1, & x \in B \setminus \omega, \\ 0, & x \in B \cap \omega. \end{cases}$$

If  $\Phi(u)$  is the extension of  $u$  out of  $B$  by reflection, then we have

$$\int_{\omega^2 \cap D_R} |\Phi u(x) - \Phi u(y)|^p dx dy > 0,$$

since  $u$  is not identically constant in the neighbourhood of the points  $(1, 0)$  and  $(0, 1)$ , while

$$\int_{(B \cap \omega)^2 \cap D_R} |u(x) - u(y)|^p dx dy = 0,$$

so that the condition (14) is not satisfied.

**Lemma 2.7.** *Let  $E$  be a periodic, connected open subset of  $\mathbb{R}^d$  with Lipschitz boundary. Let  $\Omega, \Omega'$  be open subsets of  $\mathbb{R}^d$  such that  $\Omega' \subset\subset \Omega$  and  $\text{dist}(\Omega', \partial\Omega) > \tilde{C}$ . Then there exist  $R = R(E) > 0$  and a linear and continuous operator*

$$L : L^p(\Omega \cap E) \rightarrow L^p(\Omega')$$

such that for all  $r > 0$  and for all  $u \in L^p(\Omega \cap E)$ ,

$$Lu = u, \quad \text{a.e. in } \Omega' \cap E, \quad (26)$$

$$\int_{\Omega'} |Lu|^p dx \leq c_1 \int_{\Omega \cap E} |u|^p dx, \quad (27)$$

$$\int_{(\Omega' \times \Omega') \cap D_R} |Lu(x) - Lu(y)|^p dx dy \leq c_2(r) \int_{(\Omega \cap E)^2 \cap D_r} |u(x) - u(y)|^p dx dy, \quad (28)$$

where  $c_1$  and  $c_2$  are positive constants depending on  $E$  and  $d$  and, in addition,  $c_2$  depends also on  $r$ . The constant  $R$  depends only on the set  $E$ .

*Proof.* In view of Lemma 2.3, there exists  $k \in \mathbb{N}$ ,  $k \geq 4$ , such that  $3Q \cap E$  is contained in a single connected component  $C$  of  $kQ \cap E$ . Since  $C$  has Lipschitz boundary at each point of  $C \cap 3\bar{Q}$ , we can apply Lemma 2.5 with  $B = C$ ,  $\omega' = 2Q$  and  $\omega = 3Q$ . Hence, there exist  $R > 0$  and a linear and continuous operator  $\Phi : L^p(C) \rightarrow L^p(2Q)$  defined by (17) such that, for any  $u \in L^p(C)$ ,

$$\Phi(u) = u \quad \text{a.e. in } C \cap 2Q, \quad (29)$$

$$\int_{2Q} |\Phi(u)|^p dx \leq c_1 \int_{C \cap 3Q} |u|^p dx, \quad (30)$$

$$\int_{(2Q \times 2Q) \cap D_R} |\Phi(u)(x) - \Phi(u)(y)|^p dx dy \leq c_2 \int_{(C \cap 3Q) \times (C \cap 3Q)} |u(x) - u(y)|^p dx dy, \quad (31)$$

where the positive constants  $c_1$  and  $c_2$  depend on  $C$  and  $2Q$ .

Let  $(Q_2^\alpha)_{\alpha \in \mathbb{Z}^d}$  be the open cover of  $\mathbb{R}^d$  obtained by translating the cube  $2Q$  by the vector  $\alpha \in \mathbb{Z}^d$ . For every set  $\Omega \subset \mathbb{R}^d$ , for every  $\alpha \in \mathbb{Z}^d$  and for every real number  $h > 0$ , we use the notation

$$\Omega_h^\alpha := \alpha + h\Omega. \quad (32)$$

For  $h = 1$  we simply write  $\Omega^\alpha = \Omega_1^\alpha$ , while, for  $\alpha = 0$ ,  $\Omega_h = \Omega_h^0$ . For every set  $A \subseteq \mathbb{R}^d$ , we define the set

$$I(A) := \{\alpha \in \mathbb{Z}^d : Q_2^\alpha \cap A \neq \emptyset\}.$$

Since  $\text{dist}(\Omega', \partial\Omega) > \tilde{C} = 2\sqrt{dk}$ , for every  $\alpha \in I(\Omega')$ , we have that  $Q_{2k}^\alpha \subset \Omega$ .

For any  $\alpha \in I(\Omega')$ , we define the extension operator  $\Phi^\alpha : L^p(C^\alpha) \rightarrow L^p(Q_2^\alpha)$  by translating the operator  $\Phi$  by the integer vector  $\alpha$ . In other words, for any  $u \in L^p(C^\alpha)$ ,

$$\Phi^\alpha(u) := (\Phi(u \circ \pi^{-\alpha})) \circ \pi^{-\alpha}, \quad (33)$$

where, for every  $\alpha \in \mathbb{Z}^d$  and for every real number  $h > 0$ , we use the notation

$$\pi_h^\alpha(x) := \alpha + hx \quad \text{for } x \in \mathbb{R}^d. \quad (34)$$

If  $h = 1$ , we write  $\pi^\alpha = \pi_1^\alpha$  and if  $\alpha = 0$ , we set  $\pi_h = \pi_h^0$ . For simplicity, for  $u \in L^p(\Omega \cap E)$  we denote by  $u^\alpha$  the function

$$u^\alpha := \Phi^\alpha(u|_{C^\alpha}) \in L^p(Q_2^\alpha). \quad (35)$$

From (17) and (33), the explicit expression of  $u^\alpha$  is given by

$$u^\alpha(x) := \begin{cases} u|_{C^\alpha}(x), & x \in (2Q \cap C)^\alpha, \\ \varphi(x - \alpha)u(\mathcal{R}^{-1}(x - \alpha) + \alpha) + (1 - \varphi(x - \alpha))u_{(3Q \cap C)^\alpha}, & x \in (2Q \cap A_t)^\alpha, \\ u_{(3Q \cap C)^\alpha}, & x \in (2Q \setminus (C \cup A_t))^\alpha, \end{cases}$$

where  $A_t$  is given by (15) with  $B = C^\alpha$ ,  $\omega = 3Q^\alpha$ , and  $u_{(3Q \cap C)^\alpha}$  is the mean value of  $u|_{C^\alpha}$  over  $(3Q \cap C)^\alpha$ ; *i.e.*,

$$u_{(3Q \cap C)^\alpha} := \int_{(3Q \cap C)^\alpha} u|_{C^\alpha}(x) dx.$$

We now define the global extension operator  $L : L^p(\Omega \cap E) \rightarrow L^p(\Omega')$ . To this end, let  $(\psi^\alpha)_{\alpha \in \mathbb{Z}^d}$  be a partition of unity associated to  $(Q_2^\alpha)_{\alpha \in \mathbb{Z}^d}$  such that  $\psi^\beta = \psi^\alpha \circ \pi^{\alpha - \beta}$ , for any  $\alpha, \beta \in \mathbb{Z}^d$ . Then, the map  $L : L^p(\Omega \cap E) \rightarrow L^p(\Omega')$  is defined by

$$Lu := \sum_{\alpha \in I(\Omega')} u^\alpha \psi^\alpha,$$

where  $u^\alpha$  is given by (35). Note that  $L$  is a linear and continuous operator from  $L^p(\Omega \cap E)$  to  $L^p(\Omega')$  and that condition (26) is satisfied. Indeed, in view of (35) and due to (29), we have

$$Lu(x) = \sum_{\alpha \in I(\Omega')} u^\alpha(x) \psi^\alpha(x) = \sum_{\alpha \in I(\Omega')} u(x) \psi^\alpha(x) = u(x)$$

for a.e.  $x \in \Omega' \cap E$ .

Now, we show condition (27). To this end, fix  $\beta \in I(\Omega')$  and note that, for any  $\alpha \in I(Q_2^\beta)$ , we have  $Q_k^\alpha \subset Q_{2k}^\beta$ . Combined with estimate (30) and Jensen's inequality, this implies that, for any  $u \in L^p(\Omega \cap E)$ ,

$$\begin{aligned} \int_{Q_2^\beta} |Lu|^p dx &\leq N^{p-1} \sum_{\alpha \in I(Q_2^\beta)} \int_{Q_2^\beta \cap Q_2^\alpha} |u^\alpha|^p dx \leq c_1 N^{p-1} \sum_{\alpha \in I(Q_2^\beta)} \int_{(C \cap 3Q)^\alpha} |u|^p dx \\ &\leq c_1 N^{p-1} \sum_{\alpha \in I(Q_2^\beta)} \int_{Q_k^\alpha \cap E} |u|^p dx \leq c_1 N^p \int_{Q_{2k}^\beta \cap E} |u|^p dx, \end{aligned}$$

where  $N$  denotes, henceforth, the cardinality of the set  $I(Q_2^\beta)$ . Taking the sum over  $\beta \in I(\Omega')$  in the previous inequality, we deduce that

$$\begin{aligned} \int_{\Omega'} |Lu|^p dx &\leq \sum_{\beta \in I(\Omega')} \int_{Q_2^\beta} |Lu|^p dx \\ &\leq c_1 N^p \sum_{\beta \in I(\Omega')} \int_{Q_{2k}^\beta \cap E} |u|^p dx \leq N^p (2k)^d c_1 \int_{\Omega \cap E} |u|^p dx. \end{aligned}$$

The factor  $(2k)^d$  is due to the fact that each point  $x \in \mathbb{R}^d$  is contained in at most  $(2k)^d$  cubes of the form  $(Q_{2k}^\beta)_{\beta \in \mathbb{Z}^d}$ .

To conclude the proof, it remains to show condition (28). To this end, we state the following estimate whose proof is given in Lemma 2.8 below: for all  $r > 0$  there exists a positive constant  $c = c(r)$  such that

$$\int_{((C \cap Q_3)^\alpha)^2} |u(x) - u(y)|^p dx dy \leq c(r) \int_{(Q_k^\alpha \cap E)^2 \cap D_r} |u(x) - u(y)|^p dx dy. \quad (36)$$

Fix  $\beta \in \mathbb{Z}^d$ . Since

$$Lu(x) - Lu(y) = \sum_{\alpha \in I(Q_2^\beta)} (u^\alpha(x) - u^\alpha(y)) \psi^\alpha(x) - \sum_{\alpha \in I(Q_2^\beta)} u^\alpha(y) (\psi^\alpha(y) - \psi^\alpha(x))$$

for a.e.  $x, y \in Q_2^\beta$ , an application of Jensen's inequality leads to

$$\begin{aligned} &\int_{(Q_2^\beta)^2 \cap D_R} |Lu(x) - Lu(y)|^p dx dy \\ &\leq 2^{p-1} \int_{(Q_2^\beta)^2 \cap D_R} \left| \sum_{\alpha \in I(Q_2^\beta)} (u^\alpha(x) - u^\alpha(y)) \psi^\alpha(x) \right|^p dx dy \\ &\quad + 2^{p-1} \int_{(Q_2^\beta)^2 \cap D_R} \left| \sum_{\alpha \in I(Q_2^\beta)} u^\alpha(y) (\psi^\alpha(y) - \psi^\alpha(x)) \right|^p dx dy. \end{aligned} \quad (37)$$

Due to Jensen's inequality and in view of (31) and (36), the first integral is estimated as follows

$$\begin{aligned}
& \int_{(Q_2^\beta)^2 \cap D_R} \left| \sum_{\alpha \in I(Q_2^\beta)} (u^\alpha(x) - u^\alpha(y)) \psi^\alpha(x) \right|^p dx dy \\
& \leq N^{p-1} \sum_{\alpha \in I(Q_2^\beta)} \int_{(Q_2^\beta \cap Q_2^\alpha)^2 \cap D_R} |u^\alpha(x) - u^\alpha(y)|^p dx dy \\
& \leq N^{p-1} \sum_{\alpha \in I(Q_2^\beta)} \int_{(Q_2^\alpha \times Q_2^\alpha) \cap D_R} |u^\alpha(x) - u^\alpha(y)|^p dx dy \\
& \leq c_2 N^{p-1} \sum_{\alpha \in I(Q_2^\beta)} \int_{((Q_3 \cap C)^\alpha)^2} |u(x) - u(y)|^p dx dy \\
& \leq c_2 c(r) N^{p-1} \sum_{\alpha \in I(Q_2^\beta)} \int_{(Q_k^\alpha \cap E)^2 \cap D_r} |u(x) - u(y)|^p dx dy \\
& \leq c_2 c(r) N^p \int_{(Q_{2k}^\beta \cap E)^2 \cap D_r} |u(x) - u(y)|^p dx dy. \tag{38}
\end{aligned}$$

We evaluate the second integral. Since  $\text{supp}(\psi^\alpha) \subset Q_2^\alpha$  for any  $\alpha \in \mathbb{Z}^d$ , we have that, for any  $x, y \in Q_2^\beta$ ,

$$\sum_{\alpha \in I(Q_2^\beta)} (\psi^\alpha(x) - \psi^\alpha(y)) = 0,$$

which implies that

$$\begin{aligned}
\sum_{\alpha \in I(Q_2^\beta)} u^\alpha(y) (\psi^\alpha(y) - \psi^\alpha(x)) &= \sum_{\alpha \in I(Q_2^\beta)} u^\alpha(y) (\psi^\alpha(y) - \psi^\alpha(x)) - u^\beta(x) \sum_{\alpha \in I(Q_2^\beta)} (\psi^\alpha(y) - \psi^\alpha(x)) \\
&= \sum_{\alpha \in I(Q_2^\beta)} (u^\alpha(y) - u^\beta(x)) (\psi^\alpha(y) - \psi^\alpha(x)),
\end{aligned}$$

for a.e.  $x, y \in Q_2^\beta$ . Thanks to the Jensen inequality, we obtain that

$$\begin{aligned}
& \int_{(Q_2^\beta)^2 \cap D_R} \left| \sum_{\alpha \in I(Q_2^\beta)} u^\alpha(x) (\psi^\alpha(y) - \psi^\alpha(x)) \right|^p dx dy \\
& \leq N^{p-1} \sum_{\alpha \in I(Q_2^\beta)} \int_{(Q_2^\beta \cap Q_2^\alpha)^2 \cap D_R} |u^\alpha(y) - u^\beta(x)|^p |\psi^\alpha(y) - \psi^\alpha(x)|^p dx dy \\
& \leq c N^{p-1} \sum_{\alpha \in I(Q_2^\beta)} \int_{(Q_2^\beta \cap Q_2^\alpha)^2 \cap D_R} |u^\alpha(y) - u^\beta(x)|^p dx dy. \tag{39}
\end{aligned}$$

In order to estimate the integral on the right-hand side of (39), we perform computations analogous to that of Lemma 2.5. The difference is that  $u^\alpha$  and  $u^\beta$  are extensions of  $u$  which belong to two different translated cubes  $Q_2^\alpha$  and  $Q_2^\beta$ . Hence, we separately evaluate the integral on the right-hand side of (39) on the following sets, which take into account the fact that  $u^\alpha$  and  $u^\beta$  are the extension of  $u \in L^p(\Omega' \cap E)$  on different translated cubes,

$$\begin{aligned}
S_1^{\alpha,\beta} &= (Q_2^\alpha \cap Q_2^\beta \cap C)^2 \cap D_R; \\
S_2^{\alpha,\beta} &= (((2Q \cap C)^\alpha \cap Q_2^\beta) \times (Q_2^\alpha \cap (2Q \cap A_t)^\beta)) \cap D_R; \\
S_3^{\alpha,\beta} &= (((2Q \cap A_t)^\alpha \cap Q_2^\beta) \times (Q_2^\alpha \cap (2Q \cap C)^\beta)) \cap D_R; \\
S_4^{\alpha,\beta} &= (((2Q \cap A_t)^\alpha \cap Q_2^\beta) \times (Q_2^\alpha \cap (2Q \cap A_t)^\beta)) \cap D_R; \\
S_5^{\alpha,\beta} &= (((2Q \cap A_t)^\alpha \cap Q_2^\beta) \times (Q_2^\alpha \cap (2Q \setminus (C \cup A_t))^\beta)) \cap D_R; \\
S_6^{\alpha,\beta} &= (((2Q \setminus (C \cup A_t))^\alpha \cap Q_2^\beta) \times (Q_2^\alpha \cap (2Q \cap A_t)^\beta)) \cap D_R; \\
S_7^{\alpha,\beta} &= (((2Q \setminus (C \cup A_t))^\alpha \cap Q_2^\beta) \times (Q_2^\alpha \cap (2Q \setminus (C \cup A_t))^\beta)) \cap D_R.
\end{aligned}$$

Note that, as in Lemma 2.5, the other combinations do not occur since  $R$  is chosen such that  $R < t$ .

Consider the case  $(x, y) \in S_1^{\alpha,\beta}$ . Since  $u^\alpha = u^\beta$  a.e. in  $Q_2^\alpha \cap Q_2^\beta \cap C$  and due to estimate (36), we have

$$\begin{aligned}
\int_{S_1^{\alpha,\beta}} |u^\alpha(x) - u^\beta(y)|^p dx dy &= \int_{S_1^{\alpha,\beta}} |u(x) - u(y)|^p dx dy \\
&\leq \int_{(2Q \cap C)^\beta \times (2Q \cap C)^\beta} |u(x) - u(y)|^p dx dy \\
&\leq \int_{((Q_3 \cap C)^\beta)^2} |u(x) - u(y)|^p dx dy \\
&\leq c(r) \int_{(Q_k^\beta \cap E)^2 \cap D_r} |u(x) - u(y)|^p dx dy \\
&\leq c(r) \int_{(Q_{2k}^\beta \cap E)^2 \cap D_r} |u(x) - u(y)|^p dx dy.
\end{aligned}$$

Here, we have used the fact that  $S_1^{\alpha,\beta} \subset (2Q \cap C)^\beta \times (2Q \cap C)^\beta$ .

Now, take  $(x, y) \in S_2^{\alpha,\beta}$ . Hence,

$$\begin{aligned}
u^\alpha(x) - u^\beta(y) &= u(x) - \varphi(y - \beta)u(\mathcal{R}^{-1}(y - \beta) + \beta) - (1 - \varphi(y - \beta))u_{(3Q \cap C)^\beta} \\
&= [u(x) - u_{(3Q \cap C)^\alpha}] + [u_{(3Q \cap C)^\alpha} - u_{(3Q \cap C)^\beta}] \\
&\quad \varphi(y - \beta)[u(\mathcal{R}^{-1}(y - \beta) + \beta) - u_{(3Q \cap C)^\beta}],
\end{aligned}$$



which implies that

$$\begin{aligned}
\int_{S_2^{\alpha,\beta}} |u^\alpha(x) - u^\beta(y)|^p dx dy &\leq 3^{p-1} |2Q \cap A_t| \int_{(2Q \cap C)^\alpha} |u(x) - u_{(3Q \cap C)^\alpha}|^p dx \\
&\quad + 3^{p-1} |2Q \cap C| |2Q \cap A_t| |u_{(3Q \cap C)^\alpha} - u_{(3Q \cap C)^\beta}|^p \\
&\quad + 3^{p-1} |2Q \cap C| \int_{(2Q \cap A_t)^\beta} |\varphi(y - \beta)|^p |u(\mathcal{R}^{-1}(y - \beta) + \beta) - u_{(3Q \cap C)^\beta}|^p dy.
\end{aligned} \tag{40}$$

Taking Lemma 11 and estimate (36) into account, we immediately deduce that

$$\begin{aligned}
\int_{(2Q \cap C)^\alpha} |u(x) - u_{(3Q \cap C)^\alpha}|^p dx &\leq \int_{(3Q \cap C)^\alpha} |u(x) - u_{(3Q \cap C)^\alpha}|^p dx \\
&\leq \frac{1}{|3Q \cap C|} \int_{((Q_3 \cap C)^\alpha)^2} |u(x) - u(y)|^p dx dy \\
&\leq \frac{c(r)}{|3Q \cap C|} \int_{(Q_k^\alpha \cap E)^2 \cap D_r} |u(x) - u(y)|^p dx dy \\
&\leq \frac{c(r)}{|3Q \cap C|} \int_{(Q_{2^k}^\beta \cap E)^2 \cap D_r} |u(x) - u(y)|^p dx dy.
\end{aligned} \tag{41}$$

By (19), we already know that  $\mathcal{R}$  has bounded Jacobian and  $R^{-1}(2Q \cap A_t) \subset (3Q \cap C)$ . Then, in view of (36) and Lemma 11, it follows, after the changes of variables  $y' = y - \beta$  and then  $y'' = \mathcal{R}^{-1}(y') + \beta$ , that

$$\begin{aligned}
&\int_{(2Q \cap A_t)^\beta} |\varphi(y - \beta)|^p |u(\mathcal{R}^{-1}(y - \beta) + \beta) - u_{(3Q \cap C)^\beta}|^p dy \\
&= \int_{2Q \cap A_t} |\varphi(y')|^p |u(\mathcal{R}^{-1}(y') + \beta) - u_{(3Q \cap C)^\beta}|^p dy' \\
&\leq c_{\mathcal{R}} \int_{(3Q \cap C)^\beta} |u(y'') - u_{(3Q \cap C)^\beta}|^p dy'' \\
&\leq \frac{c_{\mathcal{R}}}{|3Q \cap C|} \int_{((Q_3 \cap C)^\beta)^2} |u(x) - u(y)|^p dx dy \\
&\leq \frac{c_{\mathcal{R}}}{|3Q \cap C|} c(r) \int_{(Q_k^\beta \cap E)^2 \cap D_r} |u(x) - u(y)|^p dx dy \\
&\leq \frac{c_1}{|3Q \cap C|} c(r) \int_{(Q_{2^k}^\beta \cap E)^2 \cap D_r} |u(x) - u(y)|^p dx dy.
\end{aligned} \tag{42}$$

In order to estimate the term  $|u_{(3Q \cap C)^\alpha} - u_{(3Q \cap C)^\beta}|^p$ , note that

$$\begin{aligned} |u_{(3Q \cap C)^\alpha} - u_{(3Q \cap C)^\beta}|^p &= \frac{1}{|3Q \cap C|^p} \left| \int_{(3Q \cap C)^\alpha \times (3Q \cap C)^\beta} u_{|_{C^\alpha}}(x) - u_{|_{C^\beta}}(y) dx dy \right|^p \\ &\leq \frac{1}{|3Q \cap C|^p} \int_{(3Q \cap C)^\alpha \times (3Q \cap C)^\beta} |u_{|_{C^\alpha}}(x) - u_{|_{C^\beta}}(y)|^p dx dy. \end{aligned} \quad (43)$$

Since  $u_{|_{C^\alpha}} = u_{|_{C^\beta}}$  a.e. on  $Q_3^\alpha \cap Q_3^\beta \cap C$ , the last integral can be estimated as follows

$$\begin{aligned} &\int_{(3Q \cap C)^\alpha \times (3Q \cap C)^\beta} |u_{|_{C^\alpha}}(x) - u_{|_{C^\beta}}(y)|^p dx dy \\ &= \frac{1}{|Q_3^\alpha \cap Q_3^\beta \cap C|} \int_{Q_3^\alpha \cap Q_3^\beta \cap C} \int_{(3Q \cap C)^\alpha \times (3Q \cap C)^\beta} |u_{|_{C^\alpha}}(x) - u(z) + u(z) - u_{|_{C^\beta}}(y)|^p dx dy dz \\ &\leq \frac{2^{p-1} |3Q \cap C|}{|Q_3^\alpha \cap Q_3^\beta \cap C|} \int_{(Q_3^\alpha \cap Q_3^\beta \cap C) \times (3Q \cap C)^\alpha} |u_{|_{C^\alpha}}(x) - u(z)|^p dx dz \\ &\quad + \frac{2^{p-1} |3Q \cap C|}{|Q_3^\alpha \cap Q_3^\beta \cap C|} \int_{(Q_3^\alpha \cap Q_3^\beta \cap C) \times (3Q \cap C)^\beta} |u_{|_{C^\beta}}(y) - u(z)|^p dy dz. \end{aligned}$$

Since  $Q_3^\alpha \cap Q_3^\beta \cap C$  is contained in  $(3Q \cap C)^\alpha$ , an application of estimate (36) leads to

$$\begin{aligned} \int_{(Q_3^\alpha \cap Q_3^\beta \cap C) \times (3Q \cap C)^\alpha} |u_{|_{C^\alpha}}(x) - u(z)|^p dx dz &\leq \int_{((Q_3 \cap C)^\alpha)^2} |u(x) - u(z)|^p dx dz \\ &\leq c(r) \int_{(Q_k^\alpha \cap E)^2 \cap D_r} |u(x) - u(z)|^p dx dz \\ &\leq c(r) \int_{(Q_{2k}^\beta \cap E)^2 \cap D_r} |u(x) - u(z)|^p dx dz. \end{aligned}$$

Similarly, we also deduce that

$$\int_{(Q_3^\alpha \cap Q_3^\beta \cap C) \times (3Q \cap C)^\beta} |u_{|_{C^\beta}}(y) - u(z)|^p dy dz \leq c(r) \int_{(Q_{2k}^\beta \cap E)^2 \cap D_r} |u(y) - u(z)|^p dy dz.$$

Finally, from (43) we get

$$|u_{(3Q \cap C)^\alpha} - u_{(3Q \cap C)^\beta}|^p \leq \frac{2^p c(r)}{|3Q \cap C|^{p-1} |Q_3^\alpha \cap Q_3^\beta \cap C|} \int_{(Q_{2k}^\beta \cap E)^2 \cap D_r} |u(x) - u(y)|^p dx dy. \quad (44)$$

Gathering estimates (41), (42) and (44), from (40) we conclude that

$$\int_{S_2^{\alpha, \beta}} |u^\alpha(x) - u^\beta(y)|^p dx dy \leq c_1(r) \int_{(Q_{2k}^\beta \cap E)^2 \cap D_r} |u(x) - u(y)|^p dx dy,$$

where  $c_1(r)$  is a positive constant depending on  $p, E$  and  $r$ . The same arguments also show that

$$\int_{S_3^{\alpha, \beta}} |u^\alpha(x) - u^\beta(y)|^p dx dy \leq c_1(r) \int_{(Q_{2^k}^\beta \cap E)^2 \cap D_r} |u(x) - u(y)|^p dx dy.$$

Now consider  $(x, y) \in S_4^{\alpha, \beta}$ . We have that

$$\begin{aligned} u^\alpha(x) - u^\beta(y) &= \varphi(x - \alpha)[u(\mathcal{R}^{-1}(x - \alpha) + \alpha) - u_{(3Q \cap C)^\alpha}] + (u_{(3Q \cap C)^\alpha} - u_{(3Q \cap C)^\beta}) \\ &\quad \varphi(y - \beta)[u(\mathcal{R}^{-1}(y - \beta) + \beta) - u_{(3Q \cap C)^\beta}]. \end{aligned}$$

In view of inequalities (42) and (44), we obtain that

$$\begin{aligned} \int_{S_4^{\alpha, \beta}} |u^\alpha(x) - u^\beta(y)| dx dx &\leq 3^{p-1} |2Q \cap A_t| \int_{(2Q \cap A_t)^\alpha} |\varphi(x - \alpha)|^p |u(\mathcal{R}^{-1}(x - \alpha) + \alpha) - u_{(3Q \cap C)^\alpha}|^p dx \\ &\quad 3^{p-1} |2Q \cap A_t|^2 |u_{(3Q \cap C)^\alpha} - u_{(3Q \cap C)^\beta}|^p \\ &\quad 3^{p-1} |2Q \cap A_t| \int_{(2Q \cap A_t)^\beta} |\varphi(y - \beta)|^p |u(\mathcal{R}^{-1}(y - \beta) + \beta) - u_{(3Q \cap C)^\beta}|^p dy \\ &\leq c_1(r) \int_{(Q_{2^k}^\beta \cap E)^2 \cap D_r} |u(x) - u(y)|^p dx dy, \end{aligned}$$

where  $c_1$  is a positive constant depending on  $p, E$  and  $r$ .

Now, consider  $(x, y) \in S_5^{\alpha, \beta}$ . Hence,

$$u^\alpha(x) - u^\beta(y) = \varphi(x - \alpha)[u(\mathcal{R}^{-1}(x - \alpha) + \alpha) - u_{(3Q \cap C)^\alpha}] + (u_{(3Q \cap C)^\alpha} - u_{(3Q \cap C)^\beta}),$$

which, thanks to (42) and (44), implies that

$$\int_{S_5^{\alpha, \beta}} |u^\alpha(x) - u^\beta(y)| dx dx \leq c(r) \int_{(Q_{2^k}^\beta \cap E)^2 \cap D_r} |u(x) - u(y)|^p dx dy.$$

Similarly, if  $(x, y) \in S_6^{\alpha, \beta}$ , we have

$$\int_{S_6^{\alpha, \beta}} |u^\alpha(x) - u^\beta(y)| dx dx \leq c(r) \int_{(Q_{2^k}^\beta \cap E)^2 \cap D_r} |u(x) - u(y)|^p dx dy.$$

If  $(x, y) \in S_7^{\alpha, \beta}$ , then (44) shows the desired inequality on  $S_7^{\alpha, \beta}$ . Finally, gathering all the previous estimate on  $S_i^{\alpha, \beta}$ ,  $i = 1, \dots, 7$ , from (39) it follows that

$$\int_{(Q_2^\beta)^2 \cap D_R} \left| \sum_{\alpha \in I(Q_2^\beta)} u^\alpha(y)(\psi^\alpha(x) - \psi^\alpha(y)) \right|^p dx dy \leq c_2(r) \int_{(Q_{2^k}^\beta \cap E)^2 \cap D_r} |u(x) - u(y)|^p dx dy,$$

where  $c_2$  denotes a positive constant depending on  $E$ ,  $p$  and  $r$ . In view of (37), the previous estimate combined with (38) leads us to

$$\int_{(Q_2^\beta \times Q_2^\beta) \cap D_R} |Lu(x) - Lu(y)|^p dx dy \leq c_2(r) \int_{(Q_{2k}^\beta \cap E)^2 \cap D_r} |u(x) - u(y)|^p dx dy,$$

with  $c_2(r)$  being a positive constant depending on  $p$ ,  $E$  and  $r$ . Finally, summing up over  $\beta \in I(\Omega')$  in the last inequality, we conclude the

$$\begin{aligned} \int_{(\Omega' \times \Omega') \cap D_R} |Lu(x) - Lu(y)|^p dx dy &\leq \sum_{\beta \in I(\Omega')} \int_{(Q_2^\beta \times Q_2^\beta) \cap D_R} |Lu(x) - Lu(y)|^p dx dy \\ &\leq c_2(r) \sum_{\beta \in I(\Omega')} \int_{(Q_{2k}^\beta \cap E)^2 \cap D_r} |u(x) - u(y)|^p dx dy \\ &\leq (2k)^{2d} c_2(r) \int_{(\Omega \cap E)^2 \cap D_r} |u(x) - u(y)|^p dx dy, \end{aligned}$$

where  $c_2(r)$  denotes the positive constant depending on  $p$ ,  $E$  and  $r$  and the factor  $(2k)^{2d}$  is due to the fact that each point  $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$  is contained in at most  $(2k)^{2d}$  cubes of the form  $(Q_{2k}^\beta \times Q_{2k}^\beta)_{\beta \in \mathbb{Z}^d}$ . This concludes the proof.  $\square$

The next result proves estimate (36).

**Lemma 2.8.** *Let  $C$  be the connected component of  $kQ \cap E$ ,  $k \geq 4$ , such that  $3Q \cap E \subset C$  and  $C$  has Lipschitz boundary at each point of  $\partial C \cap 3\bar{Q}$ . For any  $r > 0$  there exists a constant  $c(r) > 0$  such that the following inequality holds*

$$\int_{(3Q \cap C)^2} |u(x) - u(y)|^p dx dy \leq c(r) \int_{(kQ \cap E)^2 \cap D_r} |u(x) - u(y)|^p dx dy. \quad (45)$$

*Proof.* We adapt the proof of [8, Lemma 3.3].

Note that for any function  $u$  the integral on the right-hand side of (45) is an increasing function of  $r$ . Hence, it is sufficient to prove (45) for  $r > 0$  small enough. For fixed  $r > 0$ , there exists  $r_1 \in (0, \frac{1}{3}r)$  and  $\nu \in (0, 1]$  which depends on the Lipschitz constant of  $\partial C \cap 3\bar{Q}$  such that for any two points  $\eta', \eta'' \in 3Q \cap C$  there exists a discrete path from  $\eta'$  to  $\eta''$ ; *i.e.*, a set of points

$$\eta_0 = \eta', \eta_1, \dots, \eta_N, \eta_{N+1} = \eta''$$

such that

- i)  $|\eta_{j+1} - \eta_j| \leq r_1$ , for  $j = 0, 1, \dots, N$ ;
- ii) for any  $j = 1, \dots, N$  the ball  $B_{\nu r_1}(\eta_j) = \{\eta \in \mathbb{R}^d : |\eta - \eta_j| \leq \nu r_1\}$  is contained in  $kQ \cap C$ ;
- iii) there exists  $\bar{N} = \bar{N}(r_1)$  such that  $N \leq \bar{N}$  for all  $\eta', \eta'' \in 3Q \cap C$ .

Let  $\xi_j \in B_{\nu r_1}(\eta_j)$ , for  $j = 1, \dots, N$ . Hence, thanks to the Jensen inequality and the condition *ii*) above, we deduce, for  $\eta', \eta'' \in 3Q \cap C$ ,

$$\begin{aligned}
& \int_{(3Q \cap C) \cap B_{\nu r_1}(\eta') \times (3Q \cap C) \cap B_{\nu r_1}(\eta'')} |u(\xi_0) - u(\xi_{N+1})|^p d\xi_0 d\xi_{N+1} \\
&= c_d(\nu r_1)^{-dN} \int_{B_{\nu r_1}(\eta_1)} \cdots \int_{B_{\nu r_1}(\eta_N)} \int_{(3Q \cap C) \cap B_{\nu r_1}(\eta') \times (3Q \cap C) \cap B_{\nu r_1}(\eta'')} |u(\xi_0) - u(\xi_1) + u(\xi_1) - \cdots \\
&\quad - u(\xi_N) + u(\xi_N) - u(\xi_{N+1})|^p d\xi_0 d\xi_{N+1} d\xi_N \cdots d\xi_1 \\
&\leq (N+1)^{p-1} c_d(\nu r_1)^{-dN} \int_{(kQ \cap E) \cap B_{\nu r_1}(\eta_0)} \cdots \int_{(kQ \cap E) \cap B_{\nu r_1}(\eta_{N+1})} \sum_{j=1}^{N+1} |u(\xi_j) - u(\xi_{j-1})|^p d\xi_0 d\xi_{N+1} \cdots d\xi_1 \\
&= c(N+1)^{p-1} \sum_{j=1}^{N+1} \int_{(kQ \cap E) \cap B_{\nu r_1}(\eta_j) \times (kQ \cap E) \cap B_{\nu r_1}(\eta_{j-1})} |u(\xi_j) - u(\xi_{j-1})|^p d\xi_j d\xi_{j-1}. \tag{46}
\end{aligned}$$

In view of assumption (i), for  $\xi_{j-1} \in (kQ \cap E) \cap B_{\nu r_1}(\eta_{j-1})$  and  $\xi_j \in (kQ \cap E) \cap B_{\nu r_1}(\eta_j)$ , we have

$$|\xi_j - \xi_{j-1}| \leq |\xi_j - \eta_j| + |\eta_j - \eta_{j-1}| + |\eta_{j-1} - \xi_{j-1}| \leq 2\nu r_1 + r_1 \leq r,$$

which implies that  $(kQ \cap E) \cap B_{\nu r_1}(\eta_j) \times (kQ \cap E) \cap B_{\nu r_1}(\eta_{j-1})$  is contained in  $(kQ \cap E)^2 \cap D_r$ . In view of (46) and due to item (iii), we get

$$\begin{aligned}
& c(N+1)^{p-1} \sum_{j=1}^{N+1} \int_{(kQ \cap E) \cap B_{\nu r_1}(\eta_j) \times (kQ \cap E) \cap B_{\nu r_1}(\eta_{j-1})} |u(\xi_j) - u(\xi_{j-1})|^p d\xi_j d\xi_{j-1} \\
&\leq c(N+1)^{p-1} \sum_{j=1}^{N+1} \int_{(kQ \cap E)^2 \cap D_r} |u(\xi) - u(\eta)|^p d\xi d\eta \\
&\leq c(N+1)^p \int_{(kQ \cap E)^2 \cap D_r} |u(\xi) - u(\eta)|^p d\xi d\eta \\
&\leq c(\bar{N}+1)^p \int_{(kQ \cap E)^2 \cap D_r} |u(\xi) - u(\eta)|^p d\xi d\eta.
\end{aligned}$$

This implies that

$$\begin{aligned}
& \int_{(3Q \cap C) \cap B_{\nu r_1}(\eta') \times (3Q \cap C) \cap B_{\nu r_1}(\eta'')} |u(\xi_0) - u(\xi_{N+1})|^p d\xi_0 d\xi_{N+1} \\
&\leq c(\bar{N}+1)^p \int_{(kQ \cap E)^2 \cap D_r} |u(\xi) - u(\eta)|^p d\xi d\eta.
\end{aligned}$$

Covering  $3Q \cap C$  with a finite number of balls of radius  $\nu r_1$  and summing up the last inequality over all pairs of these balls gives the desired estimate (28).  $\square$

Now, we may prove Theorem 2.2.

*Proof of Theorem 2.2.* The proof follows the lines of that of Theorem 2.1 in [1]. Fix  $\varepsilon > 0$  and set  $k_0 = 2\tilde{C}$ . First, let us show that there exist  $R = R(E) > 0$ , independent of  $\varepsilon$ , and a linear and continuous extension operator  $L_\varepsilon : L^p(\Omega \cap \varepsilon E) \rightarrow L^p(\Omega(\varepsilon k_0/2))$  such that, for all  $r > 0$  and for any  $u \in L^p(\Omega \cap \varepsilon E)$ ,

$$L_\varepsilon(u) = u \quad \text{a.e. in } \Omega(\varepsilon k_0/2) \cap \varepsilon E, \quad (47)$$

$$\int_{\Omega(\varepsilon k_0/2)} |L_\varepsilon(u)|^p dx \leq c_1 \int_{\Omega \cap \varepsilon E} |u|^p dx, \quad (48)$$

$$\int_{(\Omega(\varepsilon k_0/2))^2 \cap D_{\varepsilon R}} |L_\varepsilon(u)(x) - L_\varepsilon(u)(y)|^p dx dy \leq c_2(r) \int_{(\Omega \cap \varepsilon E)^2 \cap D_{\varepsilon r}} |u(x) - u(y)|^p dx dy. \quad (49)$$

To this end, note that for every  $u \in L^p(\Omega \cap \varepsilon E)$ , we have  $u \circ \pi_\varepsilon \in L^p(\varepsilon^{-1}\Omega \cap E)$ , where we use the notation (34) for the map  $\pi_\varepsilon$ . Moreover,  $\text{dist}(\varepsilon^{-1}\Omega(\varepsilon k_0/2), \partial(\varepsilon^{-1}\Omega)) > k_0 = 2\tilde{C}$ . Hence, we can apply Lemma 2.7, so that there exist  $R = R(E) > 0$ , independent of  $\varepsilon$ , and a linear and continuous operator  $L : L^p(\varepsilon^{-1}\Omega \cap E) \rightarrow L^p(\varepsilon^{-1}\Omega(\varepsilon k_0/2))$  such that, for all  $r > 0$  and for all  $u \in L^p(\varepsilon^{-1}\Omega \cap E)$ ,

$$L(u) = u, \quad \text{a.e. in } \varepsilon^{-1}\Omega(\varepsilon k_0/2) \cap E,$$

$$\int_{\varepsilon^{-1}\Omega(\varepsilon k_0/2)} |L(u)|^p dx \leq c_1 \int_{\varepsilon^{-1}\Omega \cap E} |u|^p dx,$$

$$\int_{(\varepsilon^{-1}\Omega(\varepsilon k_0/2))^2 \cap D_R} |L(u)(x) - L(u)(y)|^p dx dy \leq c_2(r) \int_{(\varepsilon^{-1}\Omega \cap E)^2 \cap D_r} |u(x) - u(y)|^p dx dy,$$

where the constants  $c_1$  and  $c_2$  are given by Lemma (2.7) and they are, in particular, independent of  $\varepsilon$ . Hence, we set  $L_\varepsilon u = (L(u \circ \pi_\varepsilon)) \circ \pi_{1/\varepsilon}$ . Note that  $L_\varepsilon u \in L^p(\Omega(\varepsilon k_0/2))$  and (47), (48), (49) are satisfied.

Now, we define the extension operator  $T_\varepsilon : L^p(\Omega \cap \varepsilon E) \rightarrow L^p(\Omega)$  by  $T_\varepsilon(u) := L_\varepsilon(u)$  a.e. in  $\Omega(\varepsilon k_0)$  and extended by zero out of  $\Omega(\varepsilon k_0)$ . Hence, we have that  $T_\varepsilon(u) \in L^p(\Omega)$  and (8), (9) and (10) follow directly from (47), (48) and (49) and this concludes the proof.  $\square$

## 2.2 Compactness

In this section we prove a compactness result which in particular implies the equi-coerciveness of families of non-local functionals as those in the homogenization result in the next section. The proof is based on the extension Theorem 2.2 and on the following compactness result proved in [9] for the case  $p = 2$  and in [4] for general  $p > 1$ .

**Theorem 2.9.** Let  $\Omega$  be an open set with Lipschitz boundary, and assume that for a family  $\{w_\varepsilon\}_{\varepsilon>0}$ ,  $w_\varepsilon \in L^p(\Omega)$ , the estimate

$$\int_{\Omega(\varepsilon k)} \int_{D_R} \left| \frac{w_\varepsilon(x + \xi) - w_\varepsilon(x)}{\varepsilon} \right|^p d\xi dx \leq c \quad (50)$$

is satisfied with some  $k > 0$  and  $R > 0$ . Assume moreover that the family  $\{w_\varepsilon\}$  is bounded in  $L^p(\Omega)$ . Then for any sequence  $\varepsilon_j \rightarrow 0$  as  $j \rightarrow +\infty$ , and for any open subset  $\Omega' \subset\subset \Omega$  the set  $\{w_{\varepsilon_j}\}_{j \in \mathbb{N}}$  is relatively compact in  $L^p(\Omega')$  and every its limit point is in  $W^{1,p}(\Omega)$ .

**Corollary 2.10.** Let  $u_\varepsilon$  be a family of functions in  $L^p(\Omega \cap \varepsilon E)$  such that there exists  $c > 0$  and  $r > 0$  such that  $\|u_\varepsilon\|_{L^p(\Omega \cap \varepsilon E)} \leq c$  and

$$\int_{\{|\xi| \leq r\}} \int_{(\Omega \cap \varepsilon E)_\varepsilon(\xi)} \left| \frac{u_\varepsilon(x + \varepsilon \xi) - u_\varepsilon(x)}{\varepsilon} \right|^p dx d\xi \leq c, \quad (51)$$

for all  $\varepsilon > 0$ , with  $(\Omega \cap \varepsilon E)_\varepsilon(\xi) = \{x \in \Omega \cap \varepsilon E : x + \varepsilon \xi \in \Omega \cap \varepsilon E\}$ . Then, for any sequence  $\varepsilon_j \rightarrow 0$  as  $j \rightarrow +\infty$ , and for any open subset  $\Omega' \subset\subset \Omega$  the set  $\{T_{\varepsilon_j} u_{\varepsilon_j}\}_{j \in \mathbb{N}}$  is relatively compact in  $L^p(\Omega')$  and every its limit point is in  $W^{1,p}(\Omega)$ .

*Proof.* Let  $u_\varepsilon$  be such that  $\|u_\varepsilon\|_{L^p(\Omega \cap \varepsilon E)} \leq c$  and (51) hold for every  $\varepsilon > 0$ . From Theorem 2.2, the extended functions  $T_\varepsilon u_\varepsilon$  satisfy the estimates

$$\int_{\Omega(\varepsilon k_0)} |T_\varepsilon u_\varepsilon|^p dx \leq c \quad (52)$$

and

$$\begin{aligned} & \frac{1}{\varepsilon^{d+p}} \int_{(\Omega(\varepsilon k_0))^2 \cap D_{\varepsilon R}} |T_\varepsilon u_\varepsilon(y) - T_\varepsilon u_\varepsilon(x)|^p dy dx \\ & \leq c(r) \int_{|\xi| \leq r} \int_{(\Omega \cap E)_\varepsilon(\xi)} \left| \frac{u_\varepsilon(x + \varepsilon \xi) - u_\varepsilon(x)}{\varepsilon} \right|^p dx d\xi \leq c, \end{aligned}$$

for some  $R > 0$  independent of  $\varepsilon$ . The latter, after the change of variables  $y = x + \varepsilon \xi$ , is equivalent to

$$\int_{\Omega(\varepsilon k_0)} \int_{|\xi| \leq R} \left| \frac{T_\varepsilon u_\varepsilon(x + \varepsilon \xi) - T_\varepsilon u_\varepsilon(x)}{\varepsilon} \right|^p d\xi dx \leq c, \quad (53)$$

which corresponds to (50), for  $w_\varepsilon = T_\varepsilon u_\varepsilon$ . Using Theorem 2.9 for  $w_\varepsilon = T_\varepsilon u_\varepsilon$  and (52), (53), we can conclude that for any sequence  $\varepsilon_j \rightarrow 0$  as  $j \rightarrow +\infty$ , and for any open subset  $\Omega' \subset\subset \Omega$ ,  $T_{\varepsilon_j} u_{\varepsilon_j}$  is relatively compact in  $L^p(\Omega')$  and every its limit point is in  $W^{1,p}(\Omega)$ .  $\square$

**Remark 2.11.** The limit  $u$  in the previous corollary does not depend on the choice of the extension. In fact, if  $\tilde{v}_\varepsilon$  is another extension of  $u_\varepsilon$  and  $v$  is its limit, then for any  $\Omega'' \subset\subset \Omega' \subset\subset \Omega$

$$\int_{\Omega'' \cap \varepsilon E} |u - v|^p dx \leq c \int_{\Omega'} |u - \tilde{u}_\varepsilon|^p dx + c \int_{\Omega'} |\tilde{v}_\varepsilon - v|^p dx$$

Passing to the limit as  $\varepsilon \rightarrow 0$ , one gets

$$|(0, 1)^d \cap E| \int_{\Omega''} |u - v|^p dx \leq 0$$

and concludes that  $u = v$ , by the arbitrariness of  $\Omega''$ .

### 3 An application to homogenization

In this section we present an application of the Extension Theorem 2.2 to the homogenization of non-local functional. Specifically, we consider a periodic integrand  $h : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^m \rightarrow [0, +\infty)$ ; *i.e.*, a Borel function such that  $h(\cdot, \xi, z)$  is  $[0, 1]^d$ -periodic for all  $\xi \in \mathbb{R}^d$  and  $z \in \mathbb{R}^m$  and satisfies the following growth conditions: there exist positive constants  $c_0, c_1, r_0$  and non-negative function  $\psi : \mathbb{R}^d \rightarrow [0, +\infty)$  such that

$$h(x, \xi, z) \leq \psi(\xi)(|z|^p + 1) \quad (54)$$

$$h(x, \xi, z) \geq c_0(|z|^p - 1) \quad \forall |\xi| \leq r_0 \quad (55)$$

with

$$\int_{\mathbb{R}^d} \psi(\xi)(|\xi|^p + 1) d\xi \leq c_1. \quad (56)$$

Let  $\Omega \subset \mathbb{R}^d$  be an open set with Lipschitz boundary. For any  $\varepsilon > 0$ , we introduce the non-local functional  $H_\varepsilon : L^p(\Omega; \mathbb{R}^m) \rightarrow [0, +\infty]$  defined as

$$H_\varepsilon(u) = \int_{\mathbb{R}^d} \int_{(\Omega \cap \varepsilon E)_\varepsilon(\xi)} h\left(\frac{x}{\varepsilon}, \xi, \frac{u(x + \varepsilon\xi) - u(x)}{\varepsilon}\right) dx d\xi, \quad (57)$$

where for each set  $B$ ,  $\varepsilon > 0$  and  $\xi \in \mathbb{R}^d$ , we use the notation

$$B_\varepsilon(\xi) = \{x \in B : x + \varepsilon\xi \in B\} \quad (58)$$

Note that the integration in (57) is performed for  $x, \xi$  such that both  $x$  and  $x + \varepsilon\xi$  belong to the perforated domain  $\Omega \cap \varepsilon E$ . Conditions (54)–(56) guarantee that functionals  $H_\varepsilon$  are estimated from above and below by functionals of the type (4).

Thanks to Corollary 2.10, our functionals  $H_\varepsilon$  are equi-coercive with respect to the  $L^p_{\text{loc}}(\Omega)$ -convergence upon identifying functions with their extensions from the perforated domain. More precisely, from each sequence  $\{u_\varepsilon\}$  with equi-bounded energy  $H_\varepsilon(u_\varepsilon)$  we can extract a subsequence such that the corresponding extensions converge in  $L^p_{\text{loc}}$  to some limit  $u \in W^{1,p}(\Omega)$ . This is implied by Corollary 2.10 applied with  $r = r_0$  to each component of the vector-valued functions  $u_\varepsilon$ , upon noting that (55) implies (51).

We now may state the homogenization result for the functional  $H_\varepsilon$  with respect to the  $L^p_{\text{loc}}(\Omega; \mathbb{R}^m)$  convergence.



**Theorem 3.1.** *The functionals  $H_\varepsilon$  defined by (57)  $\Gamma$ -converge with respect to  $L^p_{\text{loc}}(\Omega; \mathbb{R}^m)$ -convergence to the functional*

$$H_{\text{hom}}(u) = \begin{cases} \int_{\Omega} h_{\text{hom}}(Du(x)) dx & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^m) \\ +\infty & \text{otherwise,} \end{cases} \quad (59)$$

with  $h_{\text{hom}}$  satisfying the asymptotic formula

$$h_{\text{hom}}(\Xi) = \lim_{T \rightarrow +\infty} \frac{1}{T^d} \inf \left\{ \int_{(0,T)^d \cap E} \int_{(0,T)^d \cap E} h(x, y-x, v(y) - v(x)) dx dy : \right. \\ \left. v(x) = \Xi x \text{ if } \text{dist}(x, \partial(0, T)^d) < k_0 \right\} \quad (60)$$

for all  $\Xi \in \mathbf{M}^{m \times d}$ . Furthermore, if  $h$  is convex in the third variable, the cell-problem formula

$$h_{\text{hom}}(\Xi) = \inf \left\{ \int_{(0,1)^d \cap E} \int_E h(x, y-x, v(y) - v(x)) dx dy : v(x) - \Xi x \text{ is 1-periodic} \right\} \quad (61)$$

holds.

*Proof.* In [4] this theorem is proved when  $E = \mathbb{R}^d$ . We will prove Theorem 3.1 reducing to that case by a perturbation argument. For every  $\delta \geq 0$  we set

$$h^\delta(x, \xi, z) = \chi_E(x) \chi_E(x + \xi) h(x, \xi, z) + \delta \chi_{B_{R_0}}(\xi) |z|^p,$$

where  $R_0 > 0$  is fixed but arbitrary, and

$$H_\varepsilon^\delta(u) = \int_{\mathbb{R}^d} \int_{\Omega_\varepsilon(\xi)} h^\delta \left( \frac{x}{\varepsilon}, \xi, \frac{u(x + \varepsilon\xi) - u(x)}{\varepsilon} \right) dx d\xi$$

is defined for  $u \in L^p(\Omega; \mathbb{R}^m)$ , where we use the notation in (58) for the set  $\Omega_\varepsilon(\xi)$ . Note that  $H_\varepsilon^\delta \geq H_\varepsilon$ , and for  $\delta = 0$  we have  $H_\varepsilon^0 = H_\varepsilon$ . In the following, for any open set  $A$  and  $\delta \geq 0$ , we also consider the ‘localized’ functionals

$$H_\varepsilon^\delta(v, A) = \int_{\mathbb{R}^d} \int_{A_\varepsilon(\xi)} h \left( \frac{x}{\varepsilon}, \xi, \frac{u(x + \varepsilon\xi) - u(x)}{\varepsilon} \right) dx d\xi,$$

where we use the notation in (58) for the set  $A_\varepsilon(\xi)$ . If  $\delta = 0$  we write  $H_\varepsilon(v, A)$  in the place of  $H_\varepsilon^0(v, A)$ .

The homogenization theorem in [4] ensures that for all  $\delta > 0$  there exists the  $\Gamma$ -limit

$$H_{\text{hom}}^\delta(u) = \Gamma\text{-}\lim_{\varepsilon \rightarrow 0} H_\varepsilon^\delta(u)$$

with domain  $W^{1,p}(\Omega; \mathbb{R}^m)$ , on which it is represented as

$$H_{\text{hom}}^\delta(u) = \int_{\Omega} h_{\text{hom}}^\delta(Du) dx.$$

The energy density  $h_{\text{hom}}^\delta$  satisfies

$$h_{\text{hom}}^\delta(\Xi) = \lim_{T \rightarrow +\infty} \frac{1}{T^d} \inf \left\{ \int_{(0,T)^d} \int_{(0,T)^d} h^\delta(x, y-x, v(y) - v(x)) dx dy : \right. \\ \left. v(x) = \Xi x \text{ if } \text{dist}(x, \partial(0, T)^d) < r \right\},$$

for any fixed  $r > 0$ , and

$$c_1(|\Xi|^p - 1) \leq h_{\text{hom}}^\delta(\Xi) \leq c_2(1 + |\Xi|^p)$$

with  $c_1, c_2$  independent of  $\delta$ , for  $\delta \in [0, 1]$ . Note that the independence of  $c_1$  from  $\delta$  is an immediate consequence of the Extension Theorem. Indeed, let  $u_\varepsilon^\delta \rightarrow \Xi x$  be such that

$$h_{\text{hom}}^\delta(\Xi) = \lim_{\varepsilon \rightarrow 0} H_\varepsilon^\delta(u_\varepsilon^\delta, (0, 1)^d).$$

Applying Corollary 2.10 with  $\Omega = (0, 1)^d$ , we deduce that  $T_\varepsilon u_\varepsilon^\delta$  converge to  $\Xi x$  locally in  $(0, 1)^d$  (in particular the convergence is strong *e.g.* in  $(\frac{1}{4}, \frac{3}{4})^d$ ). Hence, using (55), the Extension Theorem, and the liminf inequality of the  $\Gamma$ -limit (see *e.g.* [7]) we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} H_\varepsilon^\delta(u_\varepsilon^\delta, (0, 1)^d) &\geq \liminf_{\varepsilon \rightarrow 0} H_\varepsilon(u_\varepsilon^\delta, (0, 1)^d) \\ &\geq c_0 \liminf_{\varepsilon \rightarrow 0} \left( \frac{1}{\varepsilon^{p+d}} \int_{((0,1)^d \cap \varepsilon E)^2 \cap D_{r_0}} |u_\varepsilon^\delta(x) - u_\varepsilon^\delta(y)|^p dx dy - 1 \right) \\ &\geq \frac{c_0}{c_2(r_0)} \liminf_{\varepsilon \rightarrow 0} \left( \frac{1}{\varepsilon^{p+d}} \int_{((\frac{1}{4}, \frac{3}{4})^d)^2 \cap D_R} |T_\varepsilon u_\varepsilon^\delta(x) - T_\varepsilon u_\varepsilon^\delta(y)|^p dx dy - 1 \right) \\ &\geq \frac{c_0}{c_2(r_0)} \min \left\{ \frac{1}{2^d} c_R, 1 \right\} (|\Xi|^p - 1), \end{aligned}$$

where in the last inequality we have used that

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{p+d}} \int_{((\frac{1}{4}, \frac{3}{4})^d)^2 \cap D_R} |v(x) - v(y)|^p dx dy = c_R \int_{(\frac{1}{4}, \frac{3}{4})^d} |\nabla v|^p dx,$$

where  $c_R = \int_{\{|\xi| \leq R\}} |\xi_1|^p d\xi$  (see [4]).

Since  $h_{\text{hom}}^\delta$  is increasing with  $\delta$ , we may define

$$h_0(\Xi) = \inf_{\delta > 0} h_{\text{hom}}^\delta(\Xi) = \lim_{\delta \rightarrow 0^+} h_{\text{hom}}^\delta(\Xi),$$

and deduce (here we use the usual notation for the upper  $\Gamma$ -limit) that

$$\int_{\Omega} h_0(Du) dx \geq \Gamma\text{-}\limsup_{\varepsilon \rightarrow 0} H_{\varepsilon}(u). \quad (62)$$

If  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$  and  $u_{\varepsilon} \rightarrow u$  with  $\sup_{\varepsilon} H_{\varepsilon}(u_{\varepsilon}) < +\infty$  then for all fixed  $\Omega'$  compactly contained in  $\Omega$ , if  $R_0 < R$ , upon identifying  $u_{\varepsilon}$  with its extension given by the Extension Theorem, we obtain that,

$$\int_{\{|\xi| \leq R_0\}} \int_{(\Omega')_{\varepsilon}(\xi)} \left| \frac{u_{\varepsilon}(x + \varepsilon\xi) - u_{\varepsilon}(x)}{\varepsilon} \right|^p dx d\xi \leq c,$$

so that

$$\liminf_{\varepsilon \rightarrow 0} H_{\varepsilon}(u_{\varepsilon}) \geq \liminf_{\varepsilon \rightarrow 0} H_{\varepsilon}(u_{\varepsilon}, \Omega') \geq \liminf_{\varepsilon \rightarrow 0} H_{\varepsilon}^{\delta}(u_{\varepsilon}, \Omega') - \delta c.$$

From this inequality we obtain (in terms of the lower  $\Gamma$ -limit)

$$\Gamma\text{-}\liminf_{\varepsilon \rightarrow 0} H_{\varepsilon}(u) \geq \int_{\Omega} h_0(Du) dx$$

by the arbitrariness of  $\delta$  and  $\Omega' \subset\subset \Omega$ . Hence, recalling (62), we have proved that

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} H_{\varepsilon}(u) = \int_{\Omega} h_0(Du) dx,$$

and in particular that the  $\Gamma$ -limit exists as  $\varepsilon \rightarrow 0$  (no subsequence is involved) and it can be represented as an integral functional with a homogeneous integrand. Note moreover that the lower-semicontinuity of the  $\Gamma$ -limit implies that  $h_0$  is quasiconvex (see [7]).

We now prove that  $h_0$  coincides with  $h_{\text{hom}}$  given by the asymptotic formula. First, note that

$$\begin{aligned} h_0(\Xi) \geq \limsup_{T \rightarrow +\infty} \frac{1}{T^d} \inf \left\{ \int_{(0,T)^d \cap E} \int_{(0,T)^d \cap E} h(x, y-x, v(y) - v(x)) dx dy : \right. \\ \left. v(x) = \Xi x \text{ if } \text{dist}(x, \partial(0,T)^d) < r \right\}. \end{aligned} \quad (63)$$

If we take  $r = k_0$ , we obtain a lower bound for  $h_0$ .

To prove the opposite inequality, for any diverging sequence  $\{T_j\}$  we can consider (almost-)minimizers  $v_j$  of the problems in (63) with  $r = k_0$  and  $T = T_j$ . By Lemma 2.7 (applied componentwise) with  $\Omega = (0, T)^d$  and  $\Omega' = (\frac{k_0}{2}, T_j - \frac{k_0}{2})^d$ , recalling that  $k_0 = 2\tilde{C}$ , we can consider  $\tilde{v}_j = L(v_j) \in L^p((\frac{k_0}{2}, T_j - \frac{k_0}{2})^d; \mathbb{R}^m)$  with  $\tilde{v}_j = v_j$  on  $\Omega = (0, T)^d \cap E$  and

$$\begin{aligned} \int_{(\frac{k_0}{2}, T_j - \frac{k_0}{2})^d \cap D_R} |\tilde{v}_j(\xi) - \tilde{v}_j(\eta)|^p d\xi d\eta \\ \leq c_2(r_0) \int_{(0, T_j)^d \cap E)^2 \cap D_{r_0}} |v_j(\xi) - v_j(\eta)|^p d\xi d\eta \leq c T_j^d (1 + |\Xi|^p) \end{aligned}$$

for some  $c > 0$  independent of  $j$ . Upon choosing a larger  $k_0 > 2$  we may suppose that  $\lfloor \frac{k_0}{2} \rfloor + 1 < k_0$  so that we may consider  $w_j \in L^p((0, T_j - n)^d; \mathbb{R}^m)$ , where  $n = 2\lfloor \frac{k_0}{2} \rfloor + 2$ , defined by

$$w_j(x) = L(v_j)\left(x + \left(\lfloor \frac{k_0}{2} \rfloor + 1\right)(1, \dots, 1)\right) - \left(\lfloor \frac{k_0}{2} \rfloor + 1\right)\Xi(1, \dots, 1).$$

Having set  $\varepsilon_j = T_j - n$  we can consider the scaled functions

$$u_j(x) = \varepsilon_j w_j\left(\frac{x}{\varepsilon_j}\right).$$

By the boundedness of the energies above and noting that there exists  $c > 0$  such that  $w_j(x) = \Xi x$  if  $x \in E$  and  $\text{dist}(x, \partial(0, T_j - n)^d) < c$ , upon extracting a subsequence, we may suppose that  $u_j \rightarrow u$  and  $u \in \Xi x + W_0^{1,p}((0, 1)^d; \mathbb{R}^m)$ . We may then use the quasiconvexity inequality for  $h_0$  to obtain

$$\begin{aligned} h_0(\Xi) &\leq \int_{(0,1)^d} h_0(Du) dx \\ &\leq \liminf_j H_{\varepsilon_j}^\delta(u_j, (0, 1)^d) \\ &\leq \liminf_j H_{\varepsilon_j}(u_j, (0, 1)^d) + c\delta \\ &\leq \liminf_j \frac{1}{(T_j - n)^d} H_1(w_j, (0, T_j - n)^d) + c\delta \\ &\leq \liminf_j \frac{1}{(T_j - n)^d} H_1(v_j, (0, T_j)^d) + c\delta \\ &= \liminf_j \frac{1}{(T_j - n)^d} \inf \left\{ \int_{(0, T_j)^d \cap E} \int_{(0, T_j)^d \cap E} h(x, y - x, v(y) - v(x)) dx dy : \right. \\ &\quad \left. v(x) = \Xi x \text{ if } \text{dist}(x, \partial(0, T_j)^d) < k_0 \right\} + c\delta \\ &= \liminf_j \frac{1}{T_j^d} \inf \left\{ \int_{(0, T_j)^d \cap E} \int_{(0, T_j)^d \cap E} h(x, y - x, v(y) - v(x)) dx dy : \right. \\ &\quad \left. v(x) = \Xi x \text{ if } \text{dist}(x, \partial(0, T_j)^d) < k_0 \right\} + c\delta. \end{aligned}$$

By the arbitrariness of  $\delta$  and of the sequence  $T_j$  we obtain the desired upper bound for  $h_0$ , which, together with (63), proves the asymptotic formula.

In the convex case, again by the homogenization results in [4], we may repeat the arguments used to get (63) to obtain the lower bound for  $h_0$

$$h_0(\Xi) \geq \inf \left\{ \int_{(0,1)^d \cap E} \int_E h(x, y - x, v(y) - v(x)) dx dy : v(x) - \Xi x \text{ is 1-periodic} \right\}. \quad (64)$$

Note that this implies that the right-hand side is bounded from above by  $c_2(1 + |\Xi|^p)$ .

Now, let  $v$  be an (almost) minimizing function for (64), and set  $v_\varepsilon(x) = \varepsilon v(\frac{x}{\varepsilon})$ . After applying Theorem 2.2 to any set  $\Omega$  compactly containing  $(0, 1)^d$  to possibly redefine  $v_\varepsilon$  outside  $\varepsilon E$ , we can suppose that  $v_\varepsilon$  converge in  $L^p((0, 1)^d; \mathbb{R}^m)$  to  $\Xi x$  and that

$$\frac{1}{\varepsilon^{p+d}} \int_{((0,1)^d \times (0,1)^d) \cap D_{\varepsilon R_0}} |v_\varepsilon(x) - v_\varepsilon(y)|^p dx dy \leq c(1 + |\Xi|^p).$$

We then estimate

$$\begin{aligned} h_{\text{hom}}^\delta(\Xi) &\leq \liminf_{\varepsilon \rightarrow 0} H_\varepsilon^\delta(v_\varepsilon) \\ &\leq \int_{(0,1)^d \cap E} \int_E h(x, y - x, v(y) - v(x)) dx dy + c\delta(1 + |\Xi|^p). \end{aligned}$$

Taking the limit as  $\delta \rightarrow 0$ , we obtain the converse inequality of (64), and conclude the proof.  $\square$

**Remark 3.2.** The function  $h_{\text{hom}}$  obtained in the asymptotic formula (60) also satisfies

$$\begin{aligned} h_{\text{hom}}(\Xi) = \lim_{T \rightarrow +\infty} \frac{1}{T^d} \inf \left\{ \int_{(0,T)^d \cap E} \int_{(0,T)^d \cap E} h(x, y - x, v(y) - v(x)) dx dy : \right. \\ \left. v(x) - \Xi x \text{ is } (0, T)^d \text{ - periodic} \right\}. \end{aligned}$$

**Remark 3.3.** An example is given by the convolution functional

$$F_\varepsilon(u) = \frac{1}{\varepsilon^{d+p}} \int_{(\Omega \cap E_\varepsilon) \times (\Omega \cap E_\varepsilon)} a\left(\frac{y-x}{\varepsilon}\right) |u(x) - u(y)|^p dy dx.$$

Since the integrand function  $h(x, \xi, z) = a(\xi)|z|^p$  is convex in  $z$ , then Theorem 3.1 and (61) ensure that the integrand of the  $\Gamma$ -limit (59) of  $F_\varepsilon$  is given by

$$\inf \left\{ \int_{(0,1)^d \cap E} \int_{E - \{x\}} a(\xi) |v(x + \xi) - v(x)|^p d\xi dx : v(x) - \Xi x \text{ is } 1\text{-periodic} \right\}.$$

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