On the Coupling of the Curved Virtual Element Method with the One-Equation Boundary Element Method for 2D Exterior Helmholtz Problems

ON THE COUPLING OF THE CURVED VIRTUAL ELEMENT METHOD WITH THE ONE-EQUATION BOUNDARY ELEMENT METHOD FOR 2D EXTERIOR HELMHOLTZ PROBLEMS*

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Abstract. We consider the Helmholtz equation with a non-constant coefficient, defined in unbounded domains external to 2D bounded ones, endowed with a Dirichlet condition on the boundary and the Sommerfeld radiation condition at infinity. To solve it, we reduce the infinite region, in which the solution is defined, to a bounded computational one, delimited by a curved smooth artificial boundary and we impose on this latter a non reflecting condition of boundary integral type. Then, we apply the curved virtual element method in the finite computational domain, combined with the one-equation boundary element method on the artificial boundary. We present the theoretical analysis of the proposed approach and we provide an optimal convergence error estimate in the energy norm. The numerical tests confirm the theoretical results and show the effectiveness of the new proposed approach.

Key words. exterior Helmholtz problems, curved virtual element method, boundary element method, non reflecting boundary condition.

AMS subject classifications. 65N38, 65N99, 65N12, 65N15

1. Introduction.

Frequency-domain wave propagation problems defined in unbounded regions, external to bounded obstacles, turn out to be a difficult physical and numerical task due to the issue of determining the solution in an infinite domain. One of the typical techniques to solve such problems is the Boundary Integral Equation (BIE) method, which reduces by one the dimension of the problem, requiring only the discretization of the obstacle boundary. Once the boundary distribution is retrieved by means of a Boundary Element Method (BEM) [48], the solution of the original problem at each point of the exterior domain is obtained by computing a boundary integral. However, this procedure may result not efficient, especially when the solution has to be evaluated in a region surrounding the obstacle. Moreover, in order to apply a BEM, one needs to know the Green representation of the solution in the corresponding unbounded region. For problems with non-constant coefficients, like those here considered, this representation could be unknown.

During the last decades much effort has been concentrated on developing alternative approaches. Among these we mention those based on the coupling of the BEM with domain methods such as Finite Difference Method (FDM), Finite Element Method (FEM), Discontinuous Galerkin (DG) and the recent Virtual Element Method (VEM). These are obtained by reducing the unbounded domain to a bounded computational one, delimited by an artificial boundary, on which a suitable Boundary Integral-Non Reflecting Boundary Condition (BI-NRBC) is imposed. The most popular approaches for such a coupling, associated with the use of the FEM in the interior

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domain, are often referred to as the Johnson & Nédélec Coupling (JNC) [40] or the Costabel & Han Coupling (CHC) [25, 37]. The JNC is based on a single BIE, involving the single and the double layer integral operators associated with the fundamental solution. It is known as the *one equation coupling* and it gives rise to a non-symmetric final linear system. On the contrary, the CHC is based on a couple of BIEs, one of which involves the second order normal derivative of the fundamental solution (hence a hypersingular integral operator), and it yields to a symmetric scheme. Despite the fact that an integration by parts strategy can be applied to weaken the hypersingularity, the CHC approach turns out to be quite onerous from the computational point of view, especially in the case of frequency-domain wave problems for which the accuracy of the BEM is strictly connected to the frequency parameter and to the density of discretization points per wavelength. Even if the CHC has been applied in several contexts, among which we mention [36], where the theoretical analysis has been derived for the solution of the Helmholtz problem by means of a VEM, the JNC is simpler to implement and represents an appealing approach to solve engineering problems (see, for example, [2] and [31]).

In this paper we propose a new approach based on the JNC between the Galerkin BEM and the Curved Virtual Element Method (CVEM) in the interior of the computational domain. This choice is based on the fact that the VEM allows for broadening the classical family of the FEM for the discretization of partial differential equations for what concerns both the decomposition of domains with complex geometry and the definition of local high order discrete spaces. In the standard VEM formulation the discrete spaces, built on meshes made of polygonal or polyhedral elements, are similar to the usual finite element spaces with the addition of suitable non-polynomial functions. One of the main advantages of the VEM with respect to the FEM consists in defining discrete spaces and degrees of freedom in such a way that the elementary stiffness and mass matrices can be computed using only the degrees of freedom, without the need of explicitly knowing the non-polynomial functions (whence the "virtual" word comes), with a consequent easiness of implementation even for high approximation orders. Moreover, the nature of the VEM allows in principle for decoupling the approximation orders and the mesh grids of the domain and boundary methods without the need of using special auxiliary variables (like mortar ones) for the coupling. Indeed, by exploiting the peculiar construction of the VEM, it is possible to add intermediate nodes on the edges of the elements that belong to the artificial boundary, without significantly modifying the structure of the interior mesh.

Originally developed as a variational reformulation of the nodal Mimetic Finite Difference (MFD) method [10, 18, 43], the VEM has been applied to a wide variety of interior problems (among the most recent papers we refer the reader to [3, 12, 22]). On the contrary, only few papers deal with VEM applied to exterior problems, among which [35, 36, 31]. In this latter the JNC between the collocation BEM and the VEM has been numerically investigated for the approximation of the solution of Dirichlet boundary value problems defined by the Helmholtz equation in 2D polygonal domains.

The very satisfactory results we have obtained in [31] have stimulated us to further investigate on the application of the VEM to the solution of exterior problems. For this reason, we propose here a novel approach in the CVEM-Galerkin context that we have studied both from the theoretical and the numerical point of view. In particular, the choice of the CVEM, instead of the standard (polygonal) VEM, relies on the fact that the use of curvilinear elements allows us to avoid the sub-optimal rate of convergence for high approximation orders, when curvilinear obstacles are considered. Various CVEM approaches have been investigated, among which we mention those proposed in [11] and [9], where the exact representation of the curvilinear edges is taken into account. Although the latter deals with local polynomial preserving VEM spaces, we choose the former, being it well-suited for our problem that is characterized by computational domain with fixed curved boundaries. For applications of the CVEM proposed in [11], see also [27], [28] and [4]. Furthermore, to the best of our knowledge, besides the curved versions of VEM, in literature there are few methods which are able to make use of polytopal meshes with curvilinear edges, e.g. [15], [19], [20], and this is a current field of research.

For the discretization of the BI-NRBC, we consider a classical BEM associated with Lagrangian nodal basis functions, this latter being a well-established tool. The main challenge in the theoretical analysis is the lack of ellipticity of the associated bilinear form. However, using the Fredholm theory for integral operators, it is possible to prove the well-posedness of the problem in case of computational domains with smooth artificial boundaries, provided that the square of the wave number is not equal to an eigenvalue of the associated Dirichlet-Laplace problem in the interior domain. We present the theoretical analysis of the method in a quite general framework and we provide an optimal error estimate in the energy norm. Since the analysis is based on that of the pioneering paper by Johnson and Nédélec, the smoothness properties of the artificial boundary represent a key requirement. We remark that, for the classical Galerkin approach, the breakthrough in the theoretical analysis that validates the stability of the JNC also in case of non-smooth boundaries, was proved by Sayas in [49]. However, since we deal with a generalized Galerkin method, the same analysis can not be straightforwardly applied and needs further investigations.

The paper is organized as follows: in the next section we present the model problem for the Helmholtz equation and its reformulation in a bounded region, by the introduction of the artificial boundary and the associated one equation BI-NRBC. In Section 3 we introduce the variational formulation of the problem restricted to the finite computational domain, recalling the corresponding main theoretical issues, among which existence and uniqueness of the solution. In Section 4 we apply the Galerkin method providing an error estimate in the energy norm, for a quite generic class of approximation spaces. In Section 5 we describe the choice of the CVEM-BEM approximation spaces and we prove the validity of the error analysis in this specific context. In Section 6 we present some numerical results highlighting the effectiveness of the proposed approach and the validation of the theoretical results. We show that the optimal convergence order of the scheme is guaranteed also when polygonal computational domains are considered. Moreover, we present a numerical algorithm which allows for detecting whether the square of the wavenumber is a critical value and, in the case, for choosing properly the artificial boundary in such a way that the new problem can be successfully solved. Finally, some conclusions are drawn in Section 7.

2. The model problem. Let $\Omega_0 \subset \mathbf{R}^2$ be an open bounded domain with Lipschitz boundary Γ_0 having positive Hausdorff measure, and $\Omega_e := \mathbf{R}^2 \setminus \overline{\Omega}_0$ the exterior unbounded domain. We consider the frequency-domain wave propagation problem:

$$
\begin{cases}
\Delta u_e(\mathbf{x}) + \kappa^2 \theta(\mathbf{x}) u_e(\mathbf{x}) = -f(\mathbf{x}) & \mathbf{x} \in \Omega_e, \qquad (2.1a)\\
u_e(\mathbf{x}) = g(\mathbf{x}) & \mathbf{x} \in \Gamma_0, \qquad (2.1b)\\
\lim\limits_{\|\mathbf{x}\| \to \infty} \|\mathbf{x}\|^{1/2} \left(\nabla u_e(\mathbf{x}) \cdot \dfrac{\mathbf{x}}{\|\mathbf{x}\|} - \imath \kappa u_e(\mathbf{x}) \right) = 0. & \qquad (2.1c)
\end{cases}
$$

Equation (2.1a) is known as the Helmholtz equation, with source term $f \in L^2(\Omega_e)$, Equation (2.1b) represents a boundary condition of Dirichlet type with datum g, and Equation (2.1c) is the Sommerfeld radiation condition, that ensures the appropriate behaviour of the complex-valued unknown function u_e at infinity. We assume κ real, positive and constant and θ, which takes into account the non-homogeneity of the medium, satisfying $\theta \in C^{m_1+1}(\Omega_e)$, with $m_1 \geq 1$, and $\operatorname{Re}\theta(\mathbf{x}) > 0$ and $\operatorname{Im}\theta(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \Omega_e$. We also require that both $(1-\theta)$ and f have compact support in Ω_e. Moreover we assume $g \in H^{1/2}(\Gamma_0)$, to guarantee existence and uniqueness of the solution u_e of Problem (2.1) in the Sobolev space $H^1_{\text{loc}}(\Omega_e)$ (see [24]).

Aiming at determining the solution u_e of Problem (2.1) in a bounded subregion of Ω_e surrounding Ω_0, we introduce an artificial boundary Γ which allows for decomposing Ω_e into a finite computational domain Ω, bounded internally by Γ_0 and externally by Γ, and an infinite residual one, denoted by Ω_∞. We choose Γ of class C^∞ and such that $\operatorname{supp}(f)$ and $\operatorname{supp}(1-\theta)$ are bounded subsets of Ω. We assume that Γ_0 consists of a finite number of curves of class C^{m_2+1}, with $m_2 \geq 0$.

In the following we will use a numerical method whose order of accuracy will be denoted by $k \in \mathbf{N}$. It is worth to point out that a necessary condition to obtain the optimal convergence rate of the proposed approach is $\min(m_1, m_2) > k$. From now on we suppose m_1 and m_2 large enough such that the above condition is guaranteed.

Denoting by u and u_∞ the restrictions of the solution u_e to Ω and Ω_∞ respectively, and by $\mathbf{n}$ and $\mathbf{n}_\infty$ the unit normal vectors on Γ pointing outside Ω and Ω_∞, we impose the following compatibility and equilibrium conditions on Γ (recall that $\mathbf{n}_\infty = -\mathbf{n}$):

$$u(\mathbf{x}) = u_\infty(\mathbf{x}), \qquad \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = -\frac{\partial u_\infty}{\partial \mathbf{n}_\infty}(\mathbf{x}), \qquad \mathbf{x} \in \Gamma. \tag{2.2}$$

In the above relations and in the sequel we omit, for simplicity, the use of the trace operator to indicate the restriction of H^1 functions to the boundary Γ from the exterior or interior. It is known that the solution u_∞ in Ω_∞ in which, according to the choice of Γ, $\theta(\mathbf{x}) \equiv 1$, can be represented by the Kirchhoff formula (see [24]):

$$u_\infty(\mathbf{x}) = \int_\Gamma G_\kappa(\mathbf{x},\mathbf{y}) \frac{\partial u_\infty}{\partial \mathbf{n}_\infty}(\mathbf{y})\, \mathrm{d}\Gamma_{\mathbf{y}} - \int_\Gamma \frac{\partial G_\kappa}{\partial \mathbf{n}_{\infty,\mathbf{y}}}(\mathbf{x},\mathbf{y}) u_\infty(\mathbf{y})\, \mathrm{d}\Gamma_{\mathbf{y}} \quad \mathbf{x} \in \Omega_\infty \setminus \Gamma, \tag{2.3}$$

where G_κ is the fundamental solution of the 2D Helmholtz problem and $\mathbf{n}_{\infty,\mathbf{y}}$ denotes the normal unit vector with initial point in $\mathbf{y} \in \Gamma$. The expression of G_κ and of its normal derivative in (2.3) are given by

$$G_\kappa(\mathbf{x},\mathbf{y}) := \frac{\imath}{4} H_0^{(1)}(\kappa r) \quad \text{and} \quad \frac{\partial G_\kappa}{\partial \mathbf{n}_{\infty,\mathbf{y}}}(\mathbf{x},\mathbf{y}) = \frac{\imath\kappa}{4} \frac{\mathbf{r}\cdot\mathbf{n}_{\infty,\mathbf{y}}}{r} H_1^{(1)}(\kappa r),$$

where $r = |\mathbf{r}| = |\mathbf{x}-\mathbf{y}|$ and $H_m^{(1)}$ denotes the m-th order Hankel function of the first kind. We introduce the single-layer and double-layer integral operators $\mathrm{V}_\kappa \colon H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and $\mathrm{K}_\kappa \colon H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$

$$\mathrm{V}_\kappa\psi(\mathbf{x}) := \int_\Gamma G_\kappa(\mathbf{x},\mathbf{y})\psi(\mathbf{y})\, \mathrm{d}\Gamma_{\mathbf{y}}, \qquad \mathrm{K}_\kappa\varphi(\mathbf{x}) := -\int_\Gamma \frac{\partial G_\kappa}{\partial \mathbf{n}_{\infty,\mathbf{y}}}(\mathbf{x},\mathbf{y})\varphi(\mathbf{y})\, \mathrm{d}\Gamma_{\mathbf{y}} \quad \mathbf{x} \in \Gamma,$$

which are continuous for all $\kappa > 0$ (see [39]). Then, the trace of (2.3) on Γ reads

$$\frac{1}{2} u_\infty(\mathbf{x}) - \mathrm{V}_\kappa \frac{\partial u_\infty}{\partial \mathbf{n}_\infty}(\mathbf{x}) - \mathrm{K}_\kappa u_\infty(\mathbf{x}) = 0, \qquad \mathbf{x} \in \Gamma. \tag{2.4}$$

Equation (2.4) is imposed on Γ as an exact (non local) BI-NRBC to solve Problem (2.1) in the finite computational domain. Thus, taking into account the compatibility and equilibrium conditions (2.2), and denoting $\lambda := \frac{\partial u}{\partial \mathbf{n}}$, the new problem defined in the domain of interest Ω takes the form:

$$\begin{cases} \Delta u(\mathbf{x}) + \kappa^2\theta(\mathbf{x})u(\mathbf{x}) = -f(\mathbf{x}) & \mathbf{x} \in \Omega, \quad (2.5a) \\ u(\mathbf{x}) = g(\mathbf{x}) & \mathbf{x} \in \Gamma_0, \quad (2.5b) \\ \frac{1}{2}u(\mathbf{x}) + \mathrm{V}_\kappa\lambda(\mathbf{x}) - \mathrm{K}_\kappa u(\mathbf{x}) = 0 & \mathbf{x} \in \Gamma. \quad (2.5c) \end{cases}$$

We point out that λ, which is defined on the boundary Γ in general by means of a trace operator (see [46]), is an additional unknown function.

For the theoretical analysis we will present, we further need to introduce the fundamental solution G_0 of the Laplace equation and its normal derivative:

$$G_0(\mathbf{x},\mathbf{y}) := -\frac{1}{2\pi}\log r \quad \text{and} \quad \frac{\partial G_0}{\partial \mathbf{n}_{\infty,\mathbf{y}}}(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\frac{\mathbf{r}\cdot\mathbf{n}_{\infty,\mathbf{y}}}{r^2}.$$

Denoting by V_0 and K_0 the associated single and double layer operators, the following regularity property of the operators $\mathrm{V}_\kappa - \mathrm{V}_0$ and $\mathrm{K}_\kappa - \mathrm{K}_0$ holds.

LEMMA 2.1. *For $s \in \mathbf{R}$, the operators $\mathrm{V}_\kappa - \mathrm{V}_0 : H^s(\Gamma) \to H^{s+2}(\Gamma)$ and $\mathrm{K}_\kappa - \mathrm{K}_0 : H^s(\Gamma) \to H^{s+1}(\Gamma)$ are continuous.*

Proof. We preliminary recall that the Hankel functions $H_m^{(1)}$, with $m = 0, 1$, have the following asymptotic behaviour when $r \to 0$ (see formulae (2.14), (2.15) in [47]):

$$H_0^{(1)}(r) = \frac{\imath 2}{\pi}\log r + 1 + \frac{\imath 2}{\pi}(\gamma - \log 2) + O(r^2), \quad H_1^{(1)}(r) = -\frac{\imath 2}{\pi r} + O(1)$$

where $\gamma \simeq 0.577216$ is the Euler constant. Then it easily follows that, when $r \to 0$

$$\begin{aligned} &G_\kappa(\mathbf{x},\mathbf{y}) - G_0(\mathbf{x},\mathbf{y}) = \tfrac{\imath}{4} - \tfrac{1}{2\pi}\left(\gamma - \log\left(\tfrac{\kappa}{2}\right)\right) + O(r^2), \\ &\frac{\partial G_\kappa}{\partial \mathbf{n}_{\infty,\mathbf{y}}}(\mathbf{x},\mathbf{y}) - \frac{\partial G_0}{\partial \mathbf{n}_{\infty,\mathbf{y}}}(\mathbf{x},\mathbf{y}) = O(1). \end{aligned}$$

Following [39] (see Section 7.1), we can therefore deduce that $G_\kappa - G_0$ and $\dfrac{\partial G_\kappa}{\partial \mathbf{n}_{\infty,\mathbf{y}}} - \dfrac{\partial G_0}{\partial \mathbf{n}_{\infty,\mathbf{y}}}$ have pseudo-homogeneous expansions of degree 2 and 0, respectively. From this, and proceeding as in [48] (see Remark 3.13), the assertion follows. ∎

Remark 2.2. Similarly, since G_0 has a pseudo-homogeneous expansion of degree 0, we deduce that the operator $\mathrm{V}_0 : H^s(\Gamma) \to H^{s+1}(\Gamma)$ is continuous for all $s \in \mathbf{R}$.

Remark 2.3. It is worth to point out that the continuity constants of the BEM operators V_κ, K_κ, as well as of $(\mathrm{V}_\kappa - \mathrm{V}_0)$, $(\mathrm{K}_\kappa - \mathrm{K}_0)$ in Lemma 2.1, depend on the wavenumber κ. We are aware of some explicit control of such constants with respect to the wavenumber (see for example [33, 38]). However, since our theoretical analysis is based on the Fredholm theory, we are not able to keep track of such dependence in the final convergence estimates. Such an analysis could be provided in the same spirit of [49], but this would require a deeper investigation, which is beyond the aim of the present paper.

3. The weak formulation of the model problem. Without loss of generality, we reduce the non homogeneous boundary condition on Γ_0 in (2.1) to a homogeneous one by splitting u_e as the sum of a suitable fixed function in $H^1_{g,\Gamma_0}(\Omega) := \{u \in H^1(\Omega) : u = g \text{ on } \Gamma_0\}$ satisfying the Sommerfeld radiation condition and of an unknown function belonging to the space $H^1_{0,\Gamma_0}(\Omega)$. Therefore, from now on, we consider Problem (2.5) with $g = 0$.

To derive the weak form of Problem (2.5), we introduce the bilinear forms $a : H^1(\Omega) \times H^1(\Omega) \to \mathbf{C}$ and $m : L^2(\Omega) \times L^2(\Omega) \to \mathbf{C}$ given by

$$a(u,v) := \int_\Omega \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x})\, \mathrm{d}\mathbf{x} \qquad \text{and} \qquad m(u,v) := \int_\Omega \theta(\mathbf{x}) u(\mathbf{x}) v(\mathbf{x})\, \mathrm{d}\mathbf{x}, \tag{3.1}$$

and the duality pairing $\langle \cdot, \cdot \rangle_\Gamma$ on $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$. The variational formulation of Problem (2.5) consists in finding $u \in H^1_{0,\Gamma_0}(\Omega)$ and $\lambda \in H^{-1/2}(\Gamma)$ such that

$$\begin{cases} a(u,v) - \kappa^2 m(u,v) - \langle \lambda, v\rangle_\Gamma = (f,v)_{L^2(\Omega)} & \forall\, v \in H^1_{0,\Gamma_0}(\Omega), \quad (3.2\text{a}) \\ \langle \mu, \left(\frac{1}{2}I - \mathrm{K}_\kappa\right) u\rangle_\Gamma + \langle \mu, \mathrm{V}_\kappa \lambda\rangle_\Gamma = 0 & \forall\, \mu \in H^{-1/2}(\Gamma), \quad (3.2\text{b}) \end{cases}$$

where I is the identity operator. To reformulate (3.2) in operator form, following [40], we consider the Hilbert space $V := H^1_{0,\Gamma_0}(\Omega) \times H^{-1/2}(\Gamma)$, equipped with the norm

$$\|\hat{u}\|^2_V := \|u\|^2_{H^1(\Omega)} + \|\lambda\|^2_{H^{-1/2}(\Gamma)}, \quad \text{for } \hat{u} = (u, \lambda).$$

We introduce the bilinear form $\mathcal{A}_\kappa : V \times V \to \mathbf{C}$ defined by

$$\mathcal{A}_\kappa(\hat{u},\hat{v}) := a(u,v) - \kappa^2 m(u,v) - \langle \lambda, v\rangle_\Gamma + \langle \mu, u\rangle_\Gamma + 2\langle \mu, \mathrm{V}_\kappa \lambda\rangle_\Gamma - 2\langle \mu, \mathrm{K}_\kappa u\rangle_\Gamma,$$

for $\hat{u} = (u, \lambda)$ and $\hat{v} = (v, \mu)$, and the linear continuous operator $\mathcal{L}_f : V \to \mathbf{C}$

$$\mathcal{L}_f(\hat{v}) := (f, v)_{L^2(\Omega)}, \quad \text{for } \hat{v} = (v, \mu).$$

Thus, Problem (3.2) can be rewritten as follows: find $\hat{u} \in V$ such that

$$\mathcal{A}_\kappa(\hat{u},\hat{v}) = \mathcal{L}_f(\hat{v}) \qquad \forall\, \hat{v} \in V. \tag{3.3}$$

The well-posedness of the above problem has been proved in [44] (see Theorem 3.2), provided that κ^2 is not an eigenvalue of the Dirichlet-Laplace problem in $\Omega \cup \overline{\Omega}_0$.

Remark 3.1. It is worth to point out that in the proof of the mentioned Theorem 3.2 in [44], the assumptions on the regularity of the artificial boundary Γ and on the value of the wave number κ are necessary to prove theoretically the well-posedness of the problem. While concerning the former we have numerically observed that the choice of a polygonal Γ does not affect the effectiveness of the scheme (see Example 6.1), if κ^2 is taken equal to one of the mentioned eigenvalues, the success of the method is compromised. To avoid this drawback, a possible remedy could be the approach proposed in [45] and [32], where an impedance boundary condition has been considered. As already remarked for the CHC approach, this procedure entails the use of the full Calderón system, which is more expensive from a computational point of view than the JNC one we consider here. Furthermore, we remark that Helmholtz problems with frequencies that should be in principle excluded in our approach, can be solved by suitably modifying the choice of the artificial boundary, the condition on κ^2 being related only to this latter and not to the nature of the physical obstacle Γ_0. For such a remedy we refer the reader to Remark 6.1, where a practical algorithm is proposed and described. This consists in detecting whether κ^2 is a critical wavenumber and, in the case, in modifying properly and efficiently the choice of the artificial boundary.

For what follows, it is useful to rewrite $\mathcal{A}_\kappa$ by means of the bilinear forms $\mathcal{B}_\kappa, \mathcal{K}_\kappa :$ $V \times V \to \mathbf{C}$, defined as:

$$\mathcal{A}_\kappa(\hat{u},\hat{v}) := \mathcal{B}_\kappa(\hat{u},\hat{v}) + \mathcal{K}_\kappa(\hat{u},\hat{v}), \tag{3.4a}$$

$$\mathcal{B}_\kappa(\hat{u},\hat{v}) := a(u,v) - \kappa^2 m(u,v) - \langle \lambda, v\rangle_\Gamma + \langle \mu, u\rangle_\Gamma + 2\langle \mu, \mathrm{V}_\kappa \lambda\rangle_\Gamma, \tag{3.4b}$$

$$\mathcal{K}_\kappa(\hat{u},\hat{v}) := -2\langle \mu, \mathrm{K}_\kappa u\rangle_\Gamma \tag{3.4c}$$

for $\hat{u} = (u,\lambda), \hat{v} = (v,\mu) \in V$. From the continuity of V_κ and K_κ, by using the trace theorem and the Cauchy-Schwarz inequality, it follows that the corresponding linear mappings $\mathcal{A}_\kappa, \mathcal{B}_\kappa, \mathcal{K}_\kappa : V \to V^{'}$, defined by

$$\left(\mathcal{A}_\kappa \hat{u}\right)(\hat{v}) := \mathcal{A}_\kappa(\hat{u},\hat{v}), \qquad \left(\mathcal{B}_\kappa \hat{u}\right)(\hat{v}) := \mathcal{B}_\kappa(\hat{u},\hat{v}), \qquad \left(\mathcal{K}_\kappa \hat{u}\right)(\hat{v}) := \mathcal{K}_\kappa(\hat{u},\hat{v}),$$

are continuous from V to its dual $V^{'}$. Finally, we introduce the adjoint operators $\mathcal{A}_\kappa^*, \mathcal{B}_\kappa^* : V \to V^{'}$ defined by:

$$\left(\mathcal{A}_\kappa^* \hat{v}\right)(\hat{u}) := \left(\mathcal{A}_\kappa \hat{u}\right)(\hat{v}) = \mathcal{A}_\kappa(\hat{u},\hat{v}), \quad \left(\mathcal{B}_\kappa^* \hat{v}\right)(\hat{u}) := \left(\mathcal{B}_\kappa \hat{u}\right)(\hat{v}) = \mathcal{B}_\kappa(\hat{u},\hat{v}).$$

In the following remarks we recall classical results on the afore introduced maps.

Remark 3.2. Theorem 3.2 in [44] and the closed graph theorem ensure that, if κ^2 is not an eigenvalue of the Dirichlet-Laplace problem in $\Omega \cup \overline{\Omega}_0$, the inverse linear mappings $\mathcal{A}_\kappa^{-1}, \mathcal{A}_\kappa^{*-1} : V^{'} \to V$ are continuous.

Remark 3.3. Denoting by $H_0^{-1/2}(\Gamma) := \left\{\lambda \in H^{-1/2}(\Gamma) \ : \ \langle \lambda, 1\rangle_\Gamma = 0\right\}$, we set $\widetilde{V} :=$ $H^1_{0,\Gamma_0}(\Omega) \times H_0^{-1/2}(\Gamma)$. It has been proved in [40] (see Lemmas 1, 2 and 3) that the mappings $\mathcal{A}_0, \mathcal{A}_0^*, \mathcal{B}_0, \mathcal{B}_0^* : \widetilde{V} \to \widetilde{V}^{'}$ are isomorphisms. Moreover, for $s \geq 0$, the mappings $\mathcal{A}_0^{-1}, \mathcal{A}_0^{*-1}, \mathcal{B}_0^{-1}, \mathcal{B}_0^{*-1} : H^{s-1}(\Omega) \times H^{s-1/2}(\Gamma) \times H^{s+1/2}(\Gamma) \to H^{s+1}(\Omega) \times H^{s-1/2}(\Gamma)$ are continuous. Finally, we recall that $\mathcal{B}_0$ is coercive in the $\widetilde{V}$-norm.

4. The Galerkin method. In what follows, the notation $Q_1 \lesssim Q_2$ (resp. $Q_1 \gtrsim Q_2$) means that Q_1 is bounded from above (resp. from below) by $c\,Q_2$, where c is a positive constant that, as already remarked, may depend on κ but, unless explicitly stated, does not depend on any other relevant parameter involved in the definition of Q_1 and Q_2. To describe the Galerkin approach applied to (3.3), we introduce a sequence of unstructured meshes $\{\mathcal{T}_h\}_{h>0}$, that represent coverages of the domain Ω with a finite number of elements E, having diameter h_E. The mesh width $h > 0$, related to the spacing of the grid, is $h := \max_{E\in\mathcal{T}_h} h_E$. Moreover, we denote by $\mathcal{T}_h^\Gamma$ the decomposition of the artificial boundary Γ, inherited from $\mathcal{T}_h$, which consists of curvilinear parts. We assume there exists a constant $\varrho > 0$ such that, for each h and for each element $E \in \mathcal{T}_h$:

(A.1) E is star-shaped with respect to a ball of radius greater than ϱh_E;
(A.2) the length of any (eventually curved) edge of E is greater than ϱh_E.

We introduce the splitting of the bilinear forms a and m defined in (3.1) into a sum of local ones $a^{\mathrm{E}}, m^{\mathrm{E}} : H^1(E) \times H^1(E) \to \mathbf{C}$, associated with E:

$$a(u,v) = \sum_{E\in\mathcal{T}_h} a^{\mathrm{E}}(u,v) := \sum_{E\in\mathcal{T}_h} \int_E \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x})\,\mathrm{d}\mathbf{x},$$
$$m(u,v) = \sum_{E\in\mathcal{T}_h} m^{\mathrm{E}}(u,v) := \sum_{E\in\mathcal{T}_h} \int_E \theta(\mathbf{x}) u(\mathbf{x}) v(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

Then, for any $k \in \mathbf{N}$, denoting by $P_k(E)$ the space of polynomials of degree k defined on E, we introduce the local polynomial H^1-projection $\Pi_k^{\nabla,E} : H^1(E) \to P_k(E)$, defined such that for $v \in H^1(E)$:

$$\begin{cases} \displaystyle\int_E \nabla \Pi_k^{\nabla,E} v \cdot \nabla q \,\mathrm{d}E = \int_E \nabla v \cdot \nabla q\,\mathrm{d}E \qquad \forall\, q \in P_k(E), \\ \displaystyle\int_{\partial E} \Pi_k^{\nabla,E} v\,\mathrm{d}s = \int_{\partial E} v\,\mathrm{d}s \end{cases} \tag{4.1}$$

and the local polynomial L^2-projection operator $\Pi_k^{0,E} : L^2(E) \to P_k(E)$ for $v \in L^2(E)$

$$\int_E \Pi_k^{0,E} v\, q\,\mathrm{d}E = \int_E v\, q\,\mathrm{d}E \qquad \forall\, q \in P_k(E). \tag{4.2}$$

The local projectors $\Pi_k^{\nabla,E}$ and $\Pi_k^{0,E}$ can be extended to the global ones $\Pi_k^{\nabla} : H^1(\Omega) \to P_k(\mathcal{T}_h)$ and $\Pi_k^0 : L^2(\Omega) \to P_k(\mathcal{T}_h)$ as follows:

$$\left(\Pi_k^{\nabla} v\right)_{|_E} := \Pi_k^{\nabla,E} v_{|_E} \quad \forall\, v \in H^1(\Omega), \quad \left(\Pi_k^{0} v\right)_{|_E} := \Pi_k^{0,E} v_{|_E} \quad \forall\, v \in L^2(\Omega),$$

$P_k(\mathcal{T}_h)$ being the space of piecewise polynomials with respect to the decomposition $\mathcal{T}_h$ of Ω. Since we shall deal with functions in the space $H^1(\mathcal{T}_h) := \prod_{E\in\mathcal{T}_h} H^1(E)$, we introduce the broken H^1-norm

$$\|v\|^2_{H^1(\mathcal{T}_h)} := \sum_{E\in\mathcal{T}_h} \|v\|^2_{H^1(E)}.$$

The following lemma, whose proof immediately descends from Lemma 4.3.8 in [17] and equation (2.45) in [16], provides a polynomial approximation property of the above defined projectors.

Lemma 4.1. *Assuming* (A.1), *for all $v \in H^{s+1}(\Omega)$ it holds:*

$$\left\|v - \Pi_k^0 v\right\|_{L^2(\Omega)} \lesssim h^{s+1} \|v\|_{H^{s+1}(\Omega)}, \quad \text{with } 0 \le s \le k \tag{4.3}$$

$$\left\|v - \Pi_k^0 v\right\|_{H^1(\mathcal{T}_h)} \lesssim h^{s} \|v\|_{H^{s+1}(\Omega)}, \quad \text{with } 1 \le s \le k. \tag{4.4}$$

4.1. The discrete variational formulation. For generality of presentation, we introduce here a class of Galerkin type discretizations of Problem (3.3), which includes the particular CVEM we adopt in the interior domain. For what concerns the BEM, besides the standard approach we consider here, based on the Lagrangian approximating basis functions, other approaches for which suitable polynomial consistencies of the integral operators hold, could be applied as well. Let $Q_h^k \subset H^1_{0,\Gamma_0}(\Omega)$ and $X_h^k \subset H^{-1/2}(\Gamma)$ denote two finite dimensional spaces associated with the meshes

$\mathcal{T}_h$ and $\mathcal{T}_h^\Gamma$, respectively. Introducing the discrete space $V_h^k := Q_h^k \times X_h^k$, the Galerkin method applied to Problem (3.2) reads: find $\hat{u}_h \in V_h^k$ such that

$$\mathcal{A}_{\kappa,h}(\hat{u}_h, \hat{v}_h) := \mathcal{B}_{\kappa,h}(\hat{u}_h, \hat{v}_h) + \mathcal{K}_\kappa(\hat{u}_h, \hat{v}_h) = \mathcal{L}_{f,h}(\hat{v}_h) \quad \forall\, \hat{v}_h \in V_h^k, \tag{4.5}$$

where $\mathcal{A}_{\kappa,h}, \mathcal{B}_{\kappa,h} : V_h^k \times V_h^k \to \mathbf{C}$ and $\mathcal{L}_{f,h} : V_h^k \to \mathbf{C}$ are suitable approximations of $\mathcal{A}_\kappa$, $\mathcal{B}_\kappa$ and $\mathcal{L}_f$, respectively.

To prove existence and uniqueness of the solution $\hat{u}_h \in V_h^k$, we introduce some assumptions on the discrete spaces Q_h^k, X_h^k and $\widetilde{X}_h^k := X_h^k \cap H_0^{-1/2}(\Gamma)$, and on $\mathcal{B}_{\kappa,h}$.

We assume that for $1 \le s \le k$:

(H1.a) $\inf_{v_h \in Q_h^k} \|v - v_h\|_{H^1(\Omega)} \lesssim h^s \|v\|_{H^{s+1}(\Omega)} \qquad \forall\, v \in H^{s+1}(\Omega)$;

(H1.b) $\inf_{\mu_h \in X_h^k} \|\mu - \mu_h\|_{H^{-1/2}(\Gamma)} \lesssim h^s \|\mu\|_{H^{s-1/2}(\Gamma)} \qquad \forall\, \mu \in H^{s-1/2}(\Gamma)$;

(H1.c) $\inf_{\mu_{0h} \in \widetilde{X}_h^k} \|\mu_0 - \mu_{0h}\|_{H^{-1/2}(\Gamma)} \lesssim h^s \|\mu_0\|_{H^{s-1/2}(\Gamma)} \qquad \forall\, \mu_0 \in H^{s-1/2}(\Gamma) \cap H_0^{-1/2}(\Gamma)$.

Concerning $\mathcal{B}_{\kappa,h}$, we assume that for $\kappa \ge 0$ and $0 \le s \le k$:

(H2.a) k-consistency for $\mathcal{B}_{\kappa,h}$:
$|\mathcal{B}_{\kappa,h}((\Pi_k^0 v, \mu_h), \hat{w}_h) - \mathcal{B}_\kappa((\Pi_k^0 v, \mu_h), \hat{w}_h)| \lesssim h^{s+1} \|v\|_{H^{s+1}(\Omega)} \|\hat{w}_h\|_V$
$\forall\, (v, \mu_h) \in H^{s+1}(\Omega) \times X_h^k$ and $\forall\, \hat{w}_h \in V_h^k$;

(H2.b) continuity in V-norm: $|\mathcal{B}_{\kappa,h}(\hat{v}_h, \hat{w}_h)| \lesssim \|\hat{v}_h\|_V \|\hat{w}_h\|_V \quad \forall\, \hat{v}_h, \hat{w}_h \in V_h^k$.

Remark 4.2. It is worth noting that, in Assumption (H2.a), the evaluation of the bilinear form $\mathcal{B}_\kappa$ is well defined provided that the computation of the bilinear form $a(\cdot,\cdot)$ is split into the sum of the local contributions associated with the elements E of $\mathcal{T}_h$. For simplicity of notation, here and in what follows, we assume that such splitting is considered whenever necessary. Moreover, we assume that the approximated bilinear form $\mathcal{B}_{\kappa,h}$ is well defined on the space $H^1(\mathcal{T}_h)$.

Assumptions (H2.a) and (H2.b) allow us to prove the following consistency result.

LEMMA 4.3. *Assume* (H1.a), (H2.a) *and* (H2.b). *Let* $\hat{v} := (v, \mu) \in H^{s+1}(\Omega) \times H^{-1/2}(\Gamma)$, $1 \le s \le k$, *and* v_h^I *the interpolant of* v *in* Q_h^k. *Then, for all* $\kappa \ge 0$, *it holds*

$$\left|\mathcal{B}_\kappa((v_h^I, \mu), \hat{w}_h) - \mathcal{B}_{\kappa,h}((v_h^I, \mu), \hat{w}_h)\right| \lesssim h^s \|v\|_{H^{s+1}(\Omega)} \|\hat{w}_h\|_V \qquad \forall\, \hat{w}_h \in V_h^k.$$

Proof. We start from the following inequality

$$\begin{aligned}\left|\mathcal{B}_\kappa((v_h^I, \mu), \hat{w}_h) - \mathcal{B}_{\kappa,h}((v_h^I, \mu), \hat{w}_h)\right| &\le \left|\mathcal{B}_\kappa((v_h^I, \mu), \hat{w}_h) - \mathcal{B}_\kappa((\Pi_k^0 v, \mu), \hat{w}_h)\right| \\ &\quad + \left|\mathcal{B}_\kappa((\Pi_k^0 v, \mu), \hat{w}_h) - \mathcal{B}_{\kappa,h}((\Pi_k^0 v, \mu), \hat{w}_h)\right| \\ &\quad + \left|\mathcal{B}_{\kappa,h}((\Pi_k^0 v, \mu), \hat{w}_h) - \mathcal{B}_{\kappa,h}((v_h^I, \mu), \hat{w}_h)\right| \\ &=: I + II + III. \end{aligned} \tag{4.6}$$

Concerning I and III, from the continuity of $\mathcal{B}_\kappa$ in the V-norm and (H2.b), we obtain

$$I = \left|\mathcal{B}_\kappa((\Pi_k^0 v - v_h^I, 0), \hat{w}_h)\right| \lesssim \left\|\Pi_k^0 v - v_h^I\right\|_{H^1(\mathcal{T}_h)} \|\hat{w}_h\|_V\,, \tag{4.7}$$

$$III = \left|\mathcal{B}_{\kappa,h}((\Pi_k^0 v - v_h^I, 0)\hat{w}_h)\right| \lesssim \left\|\Pi_k^0 v - v_h^I\right\|_{H^1(\mathcal{T}_h)} \|\hat{w}_h\|_V\,. \tag{4.8}$$

Using (4.4) and Assumption (H1.a) we can write

$$\left\|\Pi_k^0 v - v_h^I\right\|_{H^1(\mathcal{T}_h)} \le \left\|\Pi_k^0 v - v\right\|_{H^1(\mathcal{T}_h)} + \left\|v - v_h^I\right\|_{H^1(\Omega)} \lesssim h^s \|v\|_{H^{s+1}(\Omega)}\,.$$

Finally, bounding II by (H2.a), from (4.7), (4.8) and (4.6) the assertion follows. ∎

To prove the main results of our theoretical analysis, we need to introduce further assumptions. In particular, denoting by $\mathcal{D}_{\kappa,h} := \mathcal{B}_{\kappa,h} - \mathcal{B}_{0,h}$, we require:

(H3.a) $\mathcal{D}_{\kappa,h}$ is continuous in the weaker W-norm, with $W := L^2(\Omega) \times H^{-1/2}(\Gamma)$:

$$|\mathcal{D}_{\kappa,h}(\hat{v}_h, \hat{w}_h)| \lesssim \|\hat{v}_h\|_W \|\hat{w}_h\|_W \qquad \forall\, \hat{v}_h, \hat{w}_h \in V_h^k;$$

(H3.b) $\mathcal{B}_{0,h}$ is $\widetilde{V}_h^k$-elliptic, with $\widetilde{V}_h^k := Q_h^k \times \widetilde{X}_h^k$:

$$\mathcal{B}_{0,h}(\hat{w}_{0h}, \hat{w}_{0h}) \gtrsim \|\hat{w}_{0h}\|_V^2 \qquad \forall\, \hat{w}_{0h} \in \widetilde{V}_h^k;$$

(H3.c) k-consistency of $\mathcal{B}_{0,h}$:

$$\mathcal{B}_{0,h}(\hat{q}, \hat{w}_h) = \mathcal{B}_0(\hat{q}, \hat{w}_h),\ \ \mathcal{B}_{0,h}(\hat{w}_h, \hat{q}) = \mathcal{B}_0(\hat{w}_h, \hat{q}),\ \forall\, \hat{q} \in P_k(\mathcal{T}_h)\times X_h^k,\ \hat{w}_h \in V_h^k.$$

Remark 4.4. We remark that Assumption (H3.a) is the discrete counterpart of the continuity property of the bilinear form $\mathcal{D}_\kappa := \mathcal{B}_\kappa - \mathcal{B}_0$. Indeed, according to the continuity of $\mathrm{V}_\kappa - \mathrm{V}_0$ and using the Cauchy-Schwarz inequality, we obtain: for $\hat{v} = (v, \mu) \in V$ and $\hat{w} = (w, \nu) \in V$

$$|\mathcal{D}_\kappa(\hat{v}, \hat{w})| = \left|\kappa^2 m(v, w) - 2\langle \nu, (\mathrm{V}_\kappa - \mathrm{V}_0)\mu\rangle_\Gamma\right| \lesssim \|\hat{v}\|_W \|\hat{w}\|_W\,. \tag{4.9}$$

Assumptions (H3.a)–(H3.c) are used to prove the following Lemmas 4.5, 4.6 and 4.7, which are then crucial to obtain the Ladyzhenskaya-Babuška-Brezzi condition for the discrete bilinear form $\mathcal{A}_{\kappa,h}$.

Lemma 4.5. *Assume* (H2.a) *and* (H3.a)*. Let* $\hat{v}_h = (v_h, \mu_h) \in V_h^k$ *with* $v_h \in H^{s+1}(\Omega)$ *for* $0 \le s \le k$*. Then, it holds*

$$|\mathcal{D}_\kappa(\hat{v}_h, \hat{w}_h) - \mathcal{D}_{\kappa,h}(\hat{v}_h, \hat{w}_h)| \lesssim h^{s+1} \|v_h\|_{H^{s+1}(\Omega)} \|\hat{w}_h\|_V\,, \quad \forall\, \hat{w}_h \in V_h^k. \tag{4.10}$$

Proof. By adding and subtracting the terms $\mathcal{D}_\kappa((\Pi_k^0 v_h, \mu_h), \hat{w}_h)$ and $\mathcal{D}_{\kappa,h}((\Pi_k^0 v_h, \mu_h), \hat{w}_h)$, using (4.9), (H2.a) and (H3.a), we get

$$\begin{aligned}
|\mathcal{D}_\kappa(\hat{v}_h, \hat{w}_h) - \mathcal{D}_{\kappa,h}(\hat{v}_h, \hat{w}_h)| &\le \left|\mathcal{D}_\kappa((v_h - \Pi_k^0 v_h, 0), \hat{w}_h)\right| \\
&\quad + \left|\mathcal{D}_\kappa((\Pi_k^0 v_h, \mu_h), \hat{w}_h) - \mathcal{D}_{\kappa,h}((\Pi_k^0 v_h, \mu_h), \hat{w}_h)\right| \\
&\quad + \left|\mathcal{D}_{\kappa,h}((v_h - \Pi_k^0 v_h, 0), \hat{w}_h)\right| \\
&\lesssim \left\|v_h - \Pi_k^0 v_h\right\|_{L^2(\Omega)} \|\hat{w}_h\|_W + h^{s+1}\|v_h\|_{H^{s+1}(\Omega)}\|\hat{w}_h\|_V.
\end{aligned}$$

Finally, from (4.3), (4.10) follows. ∎

Lemma 4.6. *Assume* (H1.a), (H1.c), (H2.a), (H2.b) *and* (H3.b)*. Let* $\hat{v}_0 = (v, \mu_0) \in \widetilde{V}$*. There exists one and only one* $\hat{v}_{0h} = (v_h, \mu_{0h}) \in \widetilde{V}_h^k$ *such that*

$$\mathcal{B}_{0,h}(\hat{w}_{0h}, \hat{v}_{0h}) = \mathcal{B}_0(\hat{w}_{0h}, \hat{v}_0) \quad \forall\, \hat{w}_{0h} \in \widetilde{V}_h^k. \tag{4.11}$$

Moreover, it holds:

$$\|\hat{v}_{0h}\|_V \lesssim \|\hat{v}_0\|_V\,, \tag{4.12a}$$

$$\|\mu_{0h} - \mu_0\|_{H^{-3/2}(\Gamma)} \lesssim h\,\|\hat{v}_0\|_V\,, \tag{4.12b}$$

$$\|v_h - v\|_{L^2(\Omega)} \lesssim h\,\|\hat{v}_0\|_V\,. \tag{4.12c}$$

Proof. Existence and uniqueness of $\hat{v}_{0h} \in \widetilde{V}_h^k$, solution of (4.11), follow from (H2.b) and (H3.b). Moreover, (4.12a) holds according to (H3.b) and the continuity of the bilinear form $\mathcal{B}_0$ (see Remark 3.3). In order to prove (4.12b), by using a duality argument, it is sufficient to show that:

$$\left|\langle \mu_{0h} - \mu_0, \eta \rangle_{H^{-3/2}(\Gamma)\times H^{3/2}(\Gamma)}\right| \lesssim h \left\|\hat{v}_{0h}\right\|_V \left\|\eta\right\|_{H^{3/2}(\Gamma)} \quad \forall \eta \in H^{3/2}(\Gamma). \tag{4.13}$$

Starting from $\eta \in H^{3/2}(\Gamma)$, we consider $\widetilde{w} := (0, 0, \eta) \in L^2(\Omega) \times H^{1/2}(\Gamma) \times H^{3/2}(\Gamma)$ and we set $\hat{w}_0 := \mathcal{B}_0^{-1}\widetilde{w}$. Then, for all $\hat{z}_0 = (z, \zeta_0) \in \widetilde{V}$, we have:

$$\mathcal{B}_0(\hat{w}_0, \hat{z}_0) = \mathcal{B}_0(\mathcal{B}_0^{-1}\widetilde{w}, \hat{z}_0) = (\mathcal{B}_0\mathcal{B}_0^{-1}\widetilde{w})(\hat{z}_0) = \langle \zeta_0, \eta \rangle_{H^{-3/2}(\Gamma)\times H^{3/2}(\Gamma)}. \tag{4.14}$$

From the continuity of $\mathcal{B}_0^{-1} : L^2(\Omega) \times H^{1/2}(\Gamma) \times H^{3/2}(\Gamma) \to H^2(\Omega) \times H^{1/2}(\Gamma)$ (see Remark 3.3), it follows that:

$$\left\|\hat{w}_0\right\|_{H^2(\Omega)\times H^{1/2}(\Gamma)} \lesssim \left\|\eta\right\|_{H^{3/2}(\Gamma)}. \tag{4.15}$$

Therefore, by choosing $\hat{z}_0 = \hat{v}_{0h} - \hat{v}_0$ in (4.14), we can write

$$\begin{aligned}\langle \mu_{0h} - \mu_0, \eta \rangle_{H^{-3/2}(\Gamma)\times H^{3/2}(\Gamma)} &= \mathcal{B}_0(\hat{w}_0, \hat{v}_{0h} - \hat{v}_0) \\ &= \mathcal{B}_0(\hat{w}_0 - \hat{w}_{0h}, \hat{v}_{0h} - \hat{v}_0) + \mathcal{B}_0(\hat{w}_{0h}, \hat{v}_{0h}) - \mathcal{B}_0(\hat{w}_{0h}, \hat{v}_0).\end{aligned}$$

Since $\hat{v}_{0h} \in \widetilde{V}_h^k$ is the solution of (4.11), we rewrite the previous identity as follows:

$$\begin{aligned}&\left|\langle \mu_{0h} - \mu_0, \eta \rangle_{H^{-3/2}(\Gamma)\times H^{3/2}(\Gamma)}\right| \\ &\quad = \left|\mathcal{B}_0(\hat{w}_0 - \hat{w}_{0h}, \hat{v}_{0h} - \hat{v}_0) + \mathcal{B}_0(\hat{w}_{0h}, \hat{v}_{0h}) - \mathcal{B}_{0,h}(\hat{w}_{0h}, \hat{v}_{0h})\right| \\ &\quad \leq \left|\mathcal{B}_0(\hat{w}_0 - \hat{w}_{0h}, \hat{v}_{0h} - \hat{v}_0)\right| + \left|\mathcal{B}_0(\hat{w}_{0h}, \hat{v}_{0h}) - \mathcal{B}_{0,h}(\hat{w}_{0h}, \hat{v}_{0h})\right| =: I + II.\end{aligned} \tag{4.16}$$

Choosing in (4.16) $\hat{w}_{0h} = \hat{w}_{0h}^I$, the interpolant of $\hat{w}_0 \in \widetilde{V}$ in $\widetilde{V}_h^k$, due to Lemma 4.3 with $s = 1$ and (4.12a), we can estimate II as follows:

$$II \lesssim h \left\|\hat{v}_0\right\|_V \left\|\hat{w}_0\right\|_{H^2(\Omega)\times H^{1/2}(\Gamma)}. \tag{4.17}$$

Combining the continuity of $\mathcal{B}_0$, (H1.a) and (H1.c) with $s = 1$, and (4.12a), we have:

$$\begin{aligned}I \lesssim \left\|\hat{v}_{0h} - \hat{v}_0\right\|_V \left\|\hat{w}_0 - \hat{w}_{0h}^I\right\|_V &\lesssim h \left\|\hat{v}_{0h} - \hat{v}_0\right\|_V \left\|\hat{w}_0\right\|_{H^2(\Omega)\times H^{1/2}(\Gamma)} \\ &\lesssim h \left\|\hat{v}_0\right\|_V \left\|\hat{w}_0\right\|_{H^2(\Omega)\times H^{1/2}(\Gamma)}.\end{aligned} \tag{4.18}$$

Finally, from (4.15) and (4.16) together with (4.17) and (4.18), inequality (4.13) directly follows. Inequality (4.12c) is proved similarly to (4.12b). Indeed, if we consider $\widetilde{w} := (v_h - v, 0, 0) \in L^2(\Omega) \times H^{1/2}(\Gamma) \times H^{3/2}(\Gamma)$ and $\hat{w}_0 = \mathcal{B}_0^{-1}\widetilde{w}$ in (4.14), we get

$$\mathcal{B}_0(\hat{w}_0, \hat{z}_0) = (v_h - v, z)_{L^2(\Omega)} \qquad \forall\, \hat{z}_0 = (z, \zeta_0) \in \widetilde{V}. \tag{4.19}$$

Then, choosing $\hat{z}_0 = \hat{v}_{0h} - \hat{v}_0$ in (4.19), we have

$$\mathcal{B}_0(\hat{w}_0, \hat{v}_{0h} - \hat{v}_0) = (v_h - v, v_h - v)_{L^2(\Omega)} = \left\|v_h - v\right\|^2_{L^2(\Omega)}.$$

Finally, by taking into account the continuity of $\mathcal{B}_0^{-1}$, we obtain

$$\left\|\hat{w}_0\right\|_{H^2(\Omega)\times H^{1/2}(\Gamma)} \lesssim \left\|v_h - v\right\|_{L^2(\Omega)}$$

and, proceeding as we did to estimate (4.16), we write

$$\|v_h - v\|^2_{L^2(\Omega)} \lesssim h\,\|\hat{w}_0\|_{H^2(\Omega)\times H^{1/2}(\Gamma)}\,\|\hat{v}_0\|_V \lesssim h\,\|v_h - v\|_{L^2(\Omega)}\,\|\hat{v}_0\|_V\,,$$

whence (4.12c) follows. □

Lemma 4.7. *Assume* (H1.a)*,* (H1.c)*,* (H2.a)*,* (H2.b)*,* (H3.b) *and* (H3.c)*. Let* $\hat{v} = (v,\mu) \in V$*. There exists* $\hat{v}_h = (v_h,\mu_h) \in V_h^k$ *such that*

$$\mathcal{B}_{0,h}(\hat{w}_h,\hat{v}_h) = \mathcal{B}_0(\hat{w}_h,\hat{v}) + \mathcal{B}_0((0,\bar{\eta}_h),\hat{v}_h - \hat{v}) \quad \forall \hat{w}_h \in (w_h,\eta_h) \in V_h^k$$

where $\bar{\eta}_h = {}^{\langle \eta_h,1\rangle_\Gamma}/_{\langle 1,1\rangle_\Gamma}$*. Moreover, it holds*

$$\|\hat{v}_h\|_V \lesssim \|\hat{v}\|_V\,, \tag{4.20a}$$
$$\|\mu_h - \mu\|_{H^{-3/2}(\Gamma)} \lesssim h\,\|\hat{v}\|_V\,, \tag{4.20b}$$
$$\|v_h - v\|_{L^2(\Omega)} \lesssim h\,\|\hat{v}\|_V\,. \tag{4.20c}$$

Proof. Let us consider $\hat{v}_0 = (v,\mu_0) := (v,\mu-\bar{\mu})$, with $\bar{\mu} = {}^{\langle \mu,1\rangle_\Gamma}/_{\langle 1,1\rangle_\Gamma}$, and $\hat{w}_{0h} := (w_h,\eta_h-\bar{\eta}_h) \in \widetilde{V}_h^k$. Then we have

$$\mathcal{B}_0(\hat{w}_h,\hat{v}) = \mathcal{B}_0(\hat{w}_{0h},\hat{v}) + \mathcal{B}_0((0,\bar{\eta}_h),\hat{v}) = \mathcal{B}_0(\hat{w}_{0h},\hat{v}_0) + \mathcal{B}_0((0,\bar{\eta}_h),\hat{v}) + \mathcal{B}_0(\hat{w}_{0h},(0,\bar{\mu})).$$

According to Lemma 4.6 applied to the first term at the right hand side of the above equality, there exists a unique $\hat{v}_{0h} = (v_h,\mu_{0h}) \in \widetilde{V}_h^k$ such that

$$\mathcal{B}_0(\hat{w}_h,\hat{v}) = \mathcal{B}_{0,h}(\hat{w}_{0h},\hat{v}_{0h}) + \mathcal{B}_0((0,\bar{\eta}_h),\hat{v}) + \mathcal{B}_0(\hat{w}_{0h},(0,\bar{\mu})).$$

Using (H3.c) with $\hat{q} = (0,\bar{\mu})$, we can write

$$\begin{aligned}\mathcal{B}_0(\hat{w}_h,\hat{v}) &= \mathcal{B}_{0,h}(\hat{w}_{0h},\hat{v}_{0h}) + \mathcal{B}_0((0,\bar{\eta}_h),\hat{v}) + \mathcal{B}_{0,h}(\hat{w}_{0h},(0,\bar{\mu}))\\ &= \mathcal{B}_{0,h}(\hat{w}_{0h},\hat{v}_h) + \mathcal{B}_0((0,\bar{\eta}_h),\hat{v}),\end{aligned}$$

where we have set $\hat{v}_h = (v_h,\mu_h) := (v_h,\mu_{0h}+\bar{\mu}) \in V_h^k$. Moreover, by adding and subtracting in this latter the term $\mathcal{B}_0((0,\bar{\eta}_h),\hat{v}_h)$ and using (H3.c), we get

$$\mathcal{B}_0(\hat{w}_h,\hat{v}) = \mathcal{B}_{0,h}(\hat{w}_h,\hat{v}_h) + \mathcal{B}_0((0,\bar{\eta}_h),\hat{v}-\hat{v}_h).$$

By using the Cauchy-Schwarz inequality and (4.12a), we prove (4.20a) as follows:

$$\|\hat{v}_h\|_V = \|\hat{v}_{0h} + (0,\bar{\mu})\|_V \lesssim \|\hat{v}_0\|_V + \|\bar{\mu}\|_{H^{-1/2}(\Gamma)} \lesssim \|\hat{v}\|_V + \|\mu\|_{H^{-1/2}(\Gamma)} \lesssim \|\hat{v}\|_V\,.$$

Finally, from (4.12b)-(4.12c) we easily prove (4.20b) and (4.20c):

$$\begin{aligned}&\|\mu_h - \mu\|_{H^{-3/2}(\Gamma)} = \|\mu_{0h} - \mu_0\|_{H^{-3/2}(\Gamma)} \lesssim h\,\|\hat{v}_0\|_V \lesssim h\,\|\hat{v}\|_V\,,\\ &\|v_h - v\|_{L^2(\Omega)} \lesssim h\,\|\hat{v}_0\|_V \lesssim h\,\|\hat{v}\|_V\,.\end{aligned}$$ □

4.2. Error estimate in the energy norm. In the present section we show the validity of the inf-sup condition for the discrete bilinear form $\mathcal{A}_{\kappa,h}$, with $\kappa > 0$, and we prove that the discrete scheme has the optimal approximation order, providing for the optimal error estimate in the V-norm.

THEOREM 4.8. *Assume* (H1.a), (H1.c), (H2.a), (H2.b) *and* (H3.a)-(H3.c). *Moreover, assume that κ^2 is not an eigenvalue of the Laplacian in $\Omega \cup \overline{\Omega}_0$ endowed with a Dirichlet boundary condition on Γ. Then, for h small enough,*

$$\sup_{\substack{\hat{v}_h \in V_h^k \\ \hat{v}_h \neq 0}} \frac{\mathcal{A}_{\kappa,h}(\hat{w}_h, \hat{v}_h)}{\|\hat{v}_h\|_V} \gtrsim \|\hat{w}_h\|_V \quad \forall\, \hat{w}_h \in V_h^k.$$

Proof. Given $\hat{w}_h \in V_h^k$, let $\hat{v} := \mathcal{A}_\kappa^{*-1} J\hat{w}_h \in V$ where $J : V \to V'$ denotes the canonical continuous map $(J\hat{w})(\hat{z}) := (\hat{w}, \hat{z})_V$. Therefore we can write:

$$\begin{aligned} \mathcal{A}_\kappa(\hat{z}, \hat{v}) &= \mathcal{A}_\kappa(\hat{z}, \mathcal{A}_\kappa^{*-1} J\hat{w}_h) = (\mathcal{A}_\kappa \hat{z})(\mathcal{A}_\kappa^{*-1} J\hat{w}_h) = (\mathcal{A}_\kappa^{*} \mathcal{A}_\kappa^{*-1} J\hat{w}_h)(\hat{z}) \\ &= (J\hat{w}_h)(\hat{z}) = (\hat{w}_h, \hat{z})_V, \qquad \forall\, \hat{z} \in V. \end{aligned} \tag{4.21}$$

Moreover, according to the continuity of $\mathcal{A}_\kappa^{*-1}$ (see Remark 3.2) and of J, we obtain

$$\|\hat{v}\|_V \lesssim \|\hat{w}_h\|_V. \tag{4.22}$$

By Lemma 4.7, writing $\hat{v} = (v, \mu) \in V$, there exists $\hat{v}_h = (v_h, \mu_h) \in V_h^k$ such that

$$\mathcal{B}_{0,h}(\hat{w}_h, \hat{v}_h) = \mathcal{B}_0(\hat{w}_h, \hat{v}) + \mathcal{B}_0((0, \bar{\eta}_h), \hat{v}_h - \hat{v}) \quad \forall\, \hat{w}_h = (w_h, \eta_h) \in V_h^k \tag{4.23}$$

where $\bar{\eta}_h = {}^{\langle \eta_h, 1\rangle_\Gamma}/_{\langle 1,1\rangle_\Gamma}$ and such that (4.20a)-(4.20c) hold. Proceeding as in Theorem 5.2 of [36], and recalling the definitions of $\mathcal{A}_{0,h}$ and $\mathcal{A}_{\kappa,h}$ and of $\mathcal{D}_{\kappa,h}$ and $\mathcal{D}_\kappa$ (see assumption (H3.a) and Remark 4.4), we rewrite $\mathcal{A}_{\kappa,h}$ as follows:

$$\begin{aligned} \mathcal{A}_{\kappa,h} &= \mathcal{A}_{0,h} + (\mathcal{A}_k - \mathcal{A}_0) + (\mathcal{A}_0 - \mathcal{A}_{0,h}) + (\mathcal{A}_{\kappa,h} - \mathcal{A}_\kappa) \\ &= \mathcal{A}_{0,h} + (\mathcal{A}_k - \mathcal{A}_0) + (\mathcal{B}_0 - \mathcal{B}_{0,h}) + (\mathcal{B}_{\kappa,h} - \mathcal{B}_\kappa) \\ &= \mathcal{A}_{0,h} + (\mathcal{A}_k - \mathcal{A}_0) + (\mathcal{D}_{\kappa,h} - \mathcal{D}_\kappa). \end{aligned} \tag{4.24}$$

Using Lemma 4.5 with $s = 0$, we have

$$|(\mathcal{D}_{\kappa,h} - \mathcal{D}_\kappa)(\hat{w}_h, \hat{v}_h)| \lesssim h\|\hat{w}_h\|_V \|\hat{v}_h\|_V. \tag{4.25}$$

Recalling (3.4a)–(3.4c) and (4.5), and using (4.23), we get:

$$\begin{aligned} \mathcal{A}_{0,h}(\hat{w}_h, \hat{v}_h) &= \mathcal{B}_{0,h}(\hat{w}_h, \hat{v}_h) + \mathcal{K}_0(\hat{w}_h, \hat{v}_h) = \mathcal{B}_0(\hat{w}_h, \hat{v}) + \mathcal{B}_0((0, \bar{\eta}_h), \hat{v}_h - \hat{v}) + \mathcal{K}_0(\hat{w}_h, \hat{v}_h) \\ &= \mathcal{B}_0(\hat{w}_h, \hat{v}) + \mathcal{K}_0(\hat{w}_h, \hat{v}) + \mathcal{B}_0((0, \bar{\eta}_h), \hat{v}_h - \hat{v}) + \mathcal{K}_0(\hat{w}_h, \hat{v}_h) - \mathcal{K}_0(\hat{w}_h, \hat{v}) \\ &= \mathcal{A}_0(\hat{w}_h, \hat{v}) + \mathcal{B}_0((0, \bar{\eta}_h), \hat{v}_h - \hat{v}) - 2\langle \mu_h - \mu, \mathrm{K}_0 w_h \rangle_\Gamma. \end{aligned} \tag{4.26}$$

By applying the Hölder inequality and (4.20b), we estimate the last term in (4.26) as

$$\begin{aligned} |\langle \mu_h - \mu, \mathrm{K}_0 w_h \rangle_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)}| &\lesssim \|\mu_h - \mu\|_{H^{-3/2}(\Gamma)} \|\mathrm{K}_0 w_h\|_{H^{3/2}(\Gamma)} \\ &\lesssim h\|\hat{v}\|_V \|\mathrm{K}_0 w_h\|_{H^{3/2}(\Gamma)}. \end{aligned} \tag{4.27}$$

Then, using the continuity of $\mathrm{K}_0 : H^{1/2}(\Gamma) \to H^{3/2}(\Gamma)$ (see [40], formula (2.11)) and the trace theorem, we obtain

$$\|\mathrm{K}_0 w_h\|_{H^{3/2}(\Gamma)} \lesssim \|w_h\|_{H^{1/2}(\Gamma)} \lesssim \|w_h\|_{H^1(\Omega)} \leq \|\hat{w}_h\|_V,$$

and, hence, combining this latter with (4.27), it follows that

$$|\langle \mu_h - \mu, \mathrm{K}_0 w_h \rangle_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)}| \lesssim h\|\hat{v}\|_V \|\hat{w}_h\|_V. \tag{4.28}$$

Then, from (4.26) and (4.28), we obtain

$$\mathcal{A}_{0,h}(\hat{w}_h, \hat{v}_h) \gtrsim \mathcal{A}_0(\hat{w}_h, \hat{v}) + \mathcal{B}_0((0, \bar{\eta}_h), \hat{v}_h - \hat{v}) - h\|\hat{v}\|_V \|\hat{w}_h\|_V. \tag{4.29}$$

By explicitly writing

$$\begin{aligned}\mathcal{B}_0((0, \bar{\eta}_h), \hat{v}_h - \hat{v}) &= -\langle \bar{\eta}_h, v_h - v\rangle_\Gamma + 2\langle \mu_h - \mu, \mathrm{V}_0 \bar{\eta}_h\rangle_\Gamma \\ &= -\bar{\eta}_h \langle 1, v_h - v\rangle_\Gamma + 2\bar{\eta}_h \langle \mu_h - \mu, \mathrm{V}_0 1\rangle_\Gamma =: I + II,\end{aligned}$$

and using the Cauchy-Schwarz inequality to bound $|\bar{\eta}_h| \lesssim \|\eta_h\|_{H^{-1/2}}$, we can estimate II by using the Hölder inequality, the continuity of $\mathrm{V}_0 : H^{1/2}(\Gamma) \to H^{3/2}(\Gamma)$ (see Remark 2.2) and (4.20b):

$$|II| \lesssim \|\eta_h\|_{H^{-1/2}(\Gamma)} \|\mu_h - \mu\|_{H^{-3/2}(\Gamma)} \|\mathrm{V}_0 1\|_{H^{3/2}(\Gamma)} \lesssim h\|\hat{v}\|_V \|\hat{w}_h\|_V.$$

To estimate the term I, we use the Hölder inequality and the trace theorem (see e.g. Eq. (2.1) of [26]) and we obtain, for $0 < \varepsilon < 1/2$:

$$|I| \lesssim \|\eta_h\|_{H^{-1/2}(\Gamma)} |\langle 1, v_h - v\rangle_\Gamma| \lesssim \|\hat{w}_h\|_V \|v_h - v\|_{H^\varepsilon(\Gamma)} \lesssim \|\hat{w}_h\|_V \|v_h - v\|_{H^{1/2+\varepsilon}(\Omega)}.$$

Then, using the characterization of the fractional Sobolev space $H^{1/2+\varepsilon}(\Omega)$ as the real interpolation between $L^2(\Omega)$ and $H^1(\Omega)$, by a standard result concerning the norm of real interpolation spaces (see Prop. 2.3 of [42]), it holds

$$\|v_h - v\|_{H^{1/2+\varepsilon}(\Omega)} \leq \|v_h - v\|_{L^2(\Omega)}^{1/2-\varepsilon} \|v_h - v\|_{H^1(\Omega)}^{1/2+\varepsilon}.$$

Hence, by applying (4.20a) and (4.20c), we finally get:

$$\begin{aligned}|I| &\lesssim h^{1/2-\varepsilon} \|\hat{v}\|_V^{1/2-\varepsilon} \|v_h - v\|_{H^1(\Omega)}^{1/2+\varepsilon} \|\hat{w}_h\|_V \lesssim h^{1/2-\varepsilon} \|\hat{v}\|_V^{1/2-\varepsilon} (\|\hat{v}\|_V + \|\hat{v}_h\|_V)^{1/2+\varepsilon} \|\hat{w}_h\|_V \\ &\lesssim h^{1/2-\varepsilon} \|\hat{v}\|_V^{1/2-\varepsilon} \|\hat{v}\|_V^{1/2+\varepsilon} \|\hat{w}_h\|_V = h^{1/2-\varepsilon} \|\hat{v}\|_V \|\hat{w}_h\|_V.\end{aligned}$$

Combining (4.29) with the bounds for I and II, we can write

$$\begin{aligned}\mathcal{A}_{0,h}(\hat{w}_h, \hat{v}_h) &\gtrsim \mathcal{A}_0(\hat{w}_h, \hat{v}) - h^{1/2-\varepsilon} \|\hat{v}\|_V \|\hat{w}_h\|_V - h\|\hat{v}\|_V \|\hat{w}_h\|_V \\ &\gtrsim \mathcal{A}_0(\hat{w}_h, \hat{v}) - h^{1/2-\varepsilon} \|\hat{v}\|_V \|\hat{w}_h\|_V.\end{aligned} \tag{4.30}$$

Starting from (4.24), using (4.25) and (4.30), it follows

$$\begin{aligned}\mathcal{A}_{\kappa,h}(\hat{w}_h, \hat{v}_h) &\gtrsim \mathcal{A}_{0,h}(\hat{w}_h, \hat{v}_h) + (\mathcal{A}_k - \mathcal{A}_0)(\hat{w}_h, \hat{v}_h) - h\|\hat{w}_h\|_V \|\hat{v}_h\|_V \\ &\gtrsim \mathcal{A}_0(\hat{w}_h, \hat{v}) - h^{1/2-\varepsilon} \|\hat{v}\|_V \|\hat{w}_h\|_V + (\mathcal{A}_k - \mathcal{A}_0)(\hat{w}_h, \hat{v}_h) \\ &= \mathcal{A}_\kappa(\hat{w}_h, \hat{v}) - h^{1/2-\varepsilon} \|\hat{v}\|_V \|\hat{w}_h\|_V + (\mathcal{A}_\kappa - \mathcal{A}_0)(\hat{w}_h, \hat{v}_h) + (\mathcal{A}_0 - \mathcal{A}_\kappa)(\hat{w}_h, \hat{v}) \\ &= \|\hat{w}_h\|_V^2 - h^{1/2-\varepsilon} \|\hat{v}\|_V \|\hat{w}_h\|_V + (\mathcal{A}_\kappa - \mathcal{A}_0)(\hat{w}_h, \hat{v}_h - \hat{v})\end{aligned} \tag{4.31}$$

having used (4.21) in the last equality. Concerning the last term in (4.31),

$$(\mathcal{A}_\kappa - \mathcal{A}_0)(\hat{w}_h, \hat{v}_h - \hat{v}) = -\kappa^2 m(w_h, v_h - v) + 2\langle \mu_h - \mu, (\mathrm{V}_\kappa - \mathrm{V}_0)\eta_h - (\mathrm{K}_\kappa - \mathrm{K}_0) w_h\rangle_\Gamma$$

by using the continuity of m, the Hölder inequality and the continuity of $\mathrm{V}_\kappa - \mathrm{V}_0 : H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma)$ and of $\mathrm{K}_\kappa - \mathrm{K}_0 : H^{1/2}(\Gamma) \to H^{3/2}(\Gamma)$ (see Lemma 2.1), we get

$$\begin{aligned}|(\mathcal{A}_\kappa - \mathcal{A}_0)(\hat{w}_h, \hat{v}_h - \hat{v})| &\lesssim \|\hat{w}_h\|_V \|v_h - v\|_{L^2(\Omega)} + \|\hat{w}_h\|_V \|\mu_h - \mu\|_{H^{-3/2}(\Gamma)} \\ &\lesssim h\|\hat{w}_h\|_V \|\hat{v}_h\|_V,\end{aligned} \tag{4.32}$$

having used, in the last bound, (4.20b) and (4.20c). Finally, combining (4.31) with (4.32) and (4.22) we get

$$\mathcal{A}_{\kappa,h}(\hat{w}_h, \hat{v}_h) \gtrsim \|\hat{w}_h\|_V^2 - h^{1/2-\varepsilon}\|\hat{w}_h\|_V\|\hat{v}\|_V \gtrsim (1 - h^{1/2-\varepsilon})\|\hat{w}_h\|_V^2$$

whence, for h small enough, the claim follows. □

We conclude by proving the convergence error estimate for Problem (4.5).

THEOREM 4.9. *Assume* (H1.a)-(H1.c), (H2.a), (H2.b), (H3.a)-(H3.c) *hold and that* κ^2 *is not an eigenvalue of the Laplacian in* $\Omega \cup \overline{\Omega}_0$ *endowed with a Dirichlet boundary condition on* Γ. *Furthermore, assume that for all* $1 \leq s \leq k$, *there exists* $\sigma_s : L^2(\Omega) \to \mathbf{R}^+$ *such that*

(H4.a) $|\mathcal{L}_f(\hat{v}_h) - \mathcal{L}_{f,h}(\hat{v}_h)| \lesssim h^s \|\hat{v}_h\|_V \ \sigma_s(f) \qquad \forall\, \hat{v}_h \in V_h^k.$

Then, for h *small enough, Problem* (4.5) *admits a unique solution* $\hat{u}_h \in V_h^k$ *and if* $\hat{u}$, *solution of Problem* (3.3), *satisfies* $\hat{u} \in H^{s+1}(\Omega) \times H^{s-1/2}(\Gamma)$ *for* $1 \leq s \leq k$, *it holds*

$$\|\hat{u} - \hat{u}_h\|_V \lesssim h^s \left(\|u\|_{H^{s+1}(\Omega)} + \sigma_s(f)\right).$$

Proof. Existence and uniqueness of $\hat{u}_h$ follows from the discrete inf-sup condition of Theorem 4.8. Let $\hat{u}_h^I \in V_h^k$ be the interpolant of $\hat{u}$. By virtue of Theorem 4.8 there exists $\hat{v}_h^* = (v_h^*, \mu_h^*) \in V_h^k$ such that

$$\|\hat{u}_h - \hat{u}_h^I\|_V \lesssim \frac{\mathcal{A}_{\kappa,h}(\hat{u}_h - \hat{u}_h^I, \hat{v}_h^*)}{\|\hat{v}_h^*\|_V}.$$

Since $\hat{u}$ and $\hat{u}_h$ are solution of (3.3) and (4.5) respectively, we have

$$\begin{aligned}
\|\hat{u}_h - \hat{u}_h^I\|_V\|\hat{v}_h^*\|_V &\lesssim \mathcal{A}_{\kappa,h}(\hat{u}_h - \hat{u}_h^I, \hat{v}_h^*) = \mathcal{A}_{\kappa,h}(\hat{u}_h, \hat{v}_h^*) - \mathcal{A}_{\kappa,h}(\hat{u}_h^I, \hat{v}_h^*)\\
&= \mathcal{L}_{f,h}(\hat{v}_h^*) - \mathcal{A}_{\kappa,h}(\hat{u}_h^I, \hat{v}_h^*) + [\mathcal{A}_\kappa(\hat{u}, \hat{v}_h^*) - \mathcal{L}_f(\hat{v}_h^*)]\\
&= [\mathcal{L}_{f,h}(\hat{v}_h^*) - \mathcal{L}_f(\hat{v}_h^*)] + \mathcal{A}_\kappa(\hat{u} - \hat{u}_h^I, \hat{v}_h^*) + [\mathcal{A}_\kappa(\hat{u}_h^I, \hat{v}_h^*) - \mathcal{A}_{\kappa,h}(\hat{u}_h^I, \hat{v}_h^*)]\\
&= [\mathcal{L}_{f,h}(\hat{v}_h^*) - \mathcal{L}_f(\hat{v}_h^*)] + \mathcal{A}_\kappa(\hat{u} - \hat{u}_h^I, \hat{v}_h^*) + [(\mathcal{B}_\kappa - \mathcal{B}_{\kappa,h})(\hat{u}_h^I, \hat{v}_h^*)].
\end{aligned}$$

Then, by using Assumption (H4.a), the continuity of $\mathcal{A}_\kappa$ and Lemma 4.3, we obtain

$$\|\hat{u}_h - \hat{u}_h^I\|_V\|\hat{v}_h^*\|_V \lesssim h^s \|\hat{v}_h^*\|_V \ \sigma_s(f) + \|\hat{u} - \hat{u}_h^I\|_V\|\hat{v}_h^*\|_V + h^s\|u\|_{H^{s+1}(\Omega)}\|\hat{v}_h^*\|_V,$$

whence it easily follows

$$\|\hat{u} - \hat{u}_h\|_V \leq \|\hat{u} - \hat{u}_h^I\|_V + \|\hat{u}_h - \hat{u}_h^I\|_V \lesssim \|\hat{u} - \hat{u}_h^I\|_V + h^s\|u\|_{H^{s+1}(\Omega)} + h^s\sigma_s(f). \tag{4.33}$$

Finally, combining (H1.a), (H1.b) and (4.33), the assertion is proved. □

5. The discrete scheme. In this section we introduce the discrete CVEM-BEM scheme for the coupling procedure (3.2). We start by briefly describing the main tools of the VEM, referring the interested reader to [1, 6, 11] for a deeper presentation. In what follows, for each element $E \in \mathcal{T}_h$ we denote by $e_1, \ldots, e_{n_\mathrm{E}}$ the n_E edges of its boundary ∂E. For simplicity of presentation, we assume that at most one edge is curved while the remaining ones are straight. We identify the curved edge by $e_1 \subset \partial\Omega$, to which we associate a regular invertible parametrization $\gamma_E : I_E \to e_1$, where $I_E \subset \mathbf{R}$ is a closed interval.

In what follows we will show that all the assumptions, used to obtain the theoretical results in Section 4.1, are satisfied.

5.1. The discrete spaces Q_h^k, X_h^k and $\widetilde{X}_h^k$: validity of (H1.a)–(H1.c)**.** To describe the discrete space Q_h^k, introduced in Section 4.1 in a generic setting, we preliminarily consider for each $E \in \mathcal{T}_h$ the following local finite dimensional *augmented* virtual space $\widetilde{Q}_h^k(E)$ and the local *enhanced* virtual space $Q_h^k(E)$ defined as follows:

$$\widetilde{Q}_h^k(E) := \left\{ v_h \in H^1(E) \ : \ \Delta v_h \in P_k(E), \ v_h\,{|_{e_1}} \in \widetilde{P}_k(e_1), \ v_h\,{|_{e_i}} \in P_k(e_i)\,, \, i \geq 2 \right\},$$
$$Q_h^k(E) := \left\{ v_h \in \widetilde{Q}_h^k(E) \ : \ \left(\Pi_k^{\nabla,E} v_h - \Pi_k^{0,E} v_h\right) \in P_{k-2}(E) \right\},$$

where $\widetilde{P}_k(e_1) := \left\{ \widetilde{q} = q \circ \gamma_E^{-1} \ : \ q \in P_k(I_E) \right\}$. For details on such spaces, we refer the reader to [11] (see Remark 2.6) and to [1] (see Section 3). It is possible to prove (see Proposition 2 in [1] and Proposition 2.2 in [11]) that the dimension of $Q_h^k(E)$ is $n := \dim(Q_h^k(E)) = kn_{\rm E} + k(k-1)/2$ and that a generic element v_h of $Q_h^k(E)$ is uniquely determined by the following n conditions:

- its values at the $n_{\rm E}$ vertices of E;
- its values at the $k-1$ internal points of the $(k+1)-$point Gauss-Lobatto quadrature rule on every straight edge $e_2, \ldots, e_{n_E} \in \partial E$;
- its values at the $k-1$ internal points of e_1 that are the images, through γ_E, of the $(k+1)-$point Gauss-Lobatto quadrature rule on I_E;
- the internal $k(k-1)/2$ moments of v_h against a polynomial basis $\mathcal{M}_{k-2}(E)$ of $P_{k-2}(E)$ defined for $k \geq 2$, as:

$$\frac{1}{|E|} \int_E v_h(\mathbf{x}) p(\mathbf{x}) \, \mathrm{d}\mathbf{x} \qquad \forall\, p \in \mathcal{M}_{k-2}(E) \text{ with } \|p\|_{L^\infty(E)} \lesssim 1.$$

Choosing an arbitrary but fixed ordering of the degrees of freedom such that these are indexed by $i = 1, \cdots, n$, we introduce the operator $\mathrm{dof}_i : Q_h^k(E) \to \mathbf{R}$, defined as

$$\mathrm{dof}_i(v_h) := \text{the value of the } i\text{-th local degree of freedom of } v_h.$$

The basis functions $\{\Phi_j\}_{j=1}^n$ chosen for the space $Q_h^k(E)$ are the standard Lagrangian ones, such that $\mathrm{dof}_i(\Phi_j) = \delta_{ij}, \ i, j = 1, \ldots, n$, δ_{ij} being the Kronecker delta. On the basis of the definition of the local enhanced virtual space $Q_h^k(E)$, we are allowed to construct the global enhanced virtual space

$$Q_h^k := \left\{ v_h \in H^1_{0,\Gamma_0}(\Omega) \ : \ v_{h_{|E}} \in Q_h^k(E) \quad \forall E \in \mathcal{T}_h \right\}.$$

Remark 5.1. We remark that the global enhanced virtual space Q_h^k defined above is slightly different from that introduced in the pioneering paper on CVEM [11], the latter being defined for the solution of the Laplace problem. However, as highlighted in Remark 2.6 in [11], the theoretical analysis therein contained can be extended to our context by following the ideas of [1].

In the following lemma we prove that Assumption (H1.a) holds for the space Q_h^k.

LEMMA 5.2. *Let $v \in H^{s+1}(\Omega)$ with $1/2 < s \leq k$. Then*

$$\inf_{v_h \in Q_h^k} \|v - v_h\|_{H^1(\Omega)} \lesssim h^s \|v\|_{H^{s+1}(\Omega)}.$$

Proof. By virtue of Theorem 3.7 in [11] and Theorem 11 in [21] there exists v_h^I, interpolant of v in Q_h^k, such that

$$|v - v_h^I|_{H^1(E)} \lesssim h_E^s \|v\|_{H^{s+1}(E)}.$$

Moreover, by using the Poincaré-Friedrichs inequality (see (2.11) in [16]), we can write

$$\begin{aligned}\|v - v_h^I\|_{L^2(E)} &\lesssim h_E |v - v_h^I|_{H^1(E)} + \left|\int_{\partial E} \left[v(\mathbf{x}) - v_h^I(\mathbf{x})\right] ds\right| \\ &\lesssim h_E^{s+1}\|v\|_{H^{s+1}(E)} + \int_{\partial E} \left|v(\mathbf{x}) - v_h^I(\mathbf{x})\right| ds.\end{aligned}$$

Then, by applying the Hölder inequality, Lemma 3.2 and (3.20) in [11], we can estimate the second term in the right hand side of the above inequality as follows

$$\begin{aligned}\int_{\partial E} \left|v(\mathbf{x}) - v_h^I(\mathbf{x})\right| ds &= \sum_{e\subset\partial E}\int_e \left|v(\mathbf{x}) - v_h^I(\mathbf{x})\right| ds \le \sum_{e\subset\partial E} |e|^{1/2}\|v - v_h^I\|_{L^2(e)} \\ &\lesssim h_E^{1/2}\sum_{e\subset\partial E} h_E^{s+1/2}\|v\|_{H^{s+1/2}(e)} \lesssim h_E^{s+1}\|v\|_{H^{s+1}(E)}\end{aligned}$$

and consequently

$$\|v - v_h^I\|_{L^2(E)} \lesssim h_E^{s+1}\|v\|_{H^{s+1}(E)}.$$

Combining the local bounds for the L^2-norm and for the H^1-seminorm of $v - v_h^I$ on E, the assertion easily follows. ∎

Finally, we introduce the boundary element space X_h^k associated with Γ

$$X_h^k := \left\{\lambda \in L^2(\Gamma)\ : \lambda_{|_e} \in \widetilde{P}_k(e),\ \forall\, e \in \mathcal{T}_h^\Gamma\right\}.$$

By Theorem 4.3.20 in [48], X_h^k satisfies the interpolation property (H1.b). For what concerns the space $\widetilde{X}_h^k = X_h^k \cap H_0^{-1/2}(\Gamma)$ and the corresponding hypothesis (H1.c), we refer to (3.2b) in [40]. Moreover, a natural basis for the space X_h^k consists in the choice of the functions $\Phi_{j_{|_\Gamma}}$, which are the restriction of Φ_j on Γ.

5.2. The discrete bilinear form $\mathcal{B}_{\kappa,h}$: validity of (H2.a), (H2.b) and (H3.a)–(H3.c). To define a computable discrete local bilinear form $a_h^{\mathrm{E}} : Q_h^k(E)\times Q_h^k(E) \to \mathbf{C}$, following [6] and by using the definition of $\Pi_k^{\nabla,E}$, we split a^{E} in a part that can be computed exactly (up to the machine precision) and in one that will be approximated:

$$a^{\mathrm{E}}(u_h, v_h) = a^{\mathrm{E}}\left(\Pi_k^{\nabla,E}u_h, \Pi_k^{\nabla,E}v_h\right) + a^{\mathrm{E}}\left(\left(I - \Pi_k^{\nabla,E}\right)u_h, \left(I - \Pi_k^{\nabla,E}\right)v_h\right). \tag{5.1}$$

Following [7], the second term in (5.1) is approximated by the stabilization term:

$$s^{\mathrm{E}}(w_h, v_h) := \sum_{j=1}^{n} \mathrm{dof}_j(w_h)\mathrm{dof}_j(v_h).$$

Therefore, we define a_h^{E} the approximation of a^{E} as

$$a_h^{\mathrm{E}}(u_h, v_h) := a^{\mathrm{E}}\left(\Pi_k^{\nabla,E}u_h, \Pi_k^{\nabla,E}v_h\right) + s^{\mathrm{E}}\left(\left(I - \Pi_k^{\nabla,E}\right)u_h, \left(I - \Pi_k^{\nabla,E}\right)v_h\right).$$

As discussed in [11], this bilinear form satisfies the following properties:

- k-consistency: for all $v_h \in Q_h^k(E)$ and for all $q \in P_k(E)$:

$$a_h^{\mathrm{E}}(v_h, q) = a^{\mathrm{E}}(v_h, q) \tag{5.2}$$

- stability: for all $v_h \in Q_h^k(E)$:

$$a^{\mathrm{E}}(v_h, v_h) \lesssim a_h^{\mathrm{E}}(v_h, v_h) \lesssim a^{\mathrm{E}}(v_h, v_h). \tag{5.3}$$

The bilinear form m^{E}, which involves the non-constant term $\theta(\mathbf{x})$, is approximated, according to [8], as follows:

$$m_h^{\mathrm{E}}(u_h, v_h) := m^{\mathrm{E}}\left(\Pi_{k-1}^{0,E} u_h, \Pi_{k-1}^{0,E} v_h\right). \tag{5.4}$$

In particular, following Section 5 of [8], it is possibile to show that, for $u \in H^{s+1}(E)$, with $0 \le s \le k$, it holds

$$\left| m_h^{\mathrm{E}}(\Pi_k^{0,E} u, v_h) - m^{\mathrm{E}}(\Pi_k^{0,E} u, v_h) \right| \lesssim h_E^{s+1} \|u\|_{H^{s+1}(E)} \|v_h\|_{H^1(E)}. \tag{5.5}$$

It is worth to point out that the analysis of our method requires the ellipticity property only for the bilinear form $\mathcal{B}_{0,h}$ (see (H3.b)). Moreover, the bilinear form m appears in $\mathcal{B}_\kappa$ with negative sign. Hence, a stabilizing term is not needed for m_h^E.

The global approximate bilinear forms $a_h, m_h : Q_h^k \times Q_h^k \to \mathbf{C}$ are then defined by summing up the local contributions as follows:

$$a_h(u_h, v_h) := \sum_{E \in \mathcal{T}_h} a_h^{\mathrm{E}}(u_h, v_h) \qquad \text{and} \qquad m_h(u_h, v_h) := \sum_{E \in \mathcal{T}_h} m_h^{\mathrm{E}}(u_h, v_h).$$

From the smoothness of θ and the continuity of Π_{k-1}^0, it immediately follows that

$$m_h(v_h, v_h) \lesssim \|v_h\|_{L^2(\Omega)}^2 \qquad \forall\, v_h \in Q_h^k$$

while, combining (5.3) with the Poincaré-Friedrichs inequality, we get

$$\|v_h\|_{H^1(\Omega)}^2 \lesssim a_h(v_h, v_h) \lesssim \|v_h\|_{H^1(\Omega)}^2 \qquad \forall v_h \in Q_h^k. \tag{5.6}$$

The characterization of the virtual element space Q_h^k and the boundary element space X_h^k allows then us to formally define the bilinear form $\mathcal{B}_{\kappa,h} : V_h^k \times V_h^k \to \mathbf{C}$,

$$\mathcal{B}_{\kappa,h}(\hat{u}_h, \hat{v}_h) := a_h(u_h, v_h) - \kappa^2 m_h(u_h, v_h) - \langle \lambda_h, v_h \rangle_\Gamma + \langle \mu_h, u_h \rangle_\Gamma + 2\langle \mu_h, \mathrm{V}_\kappa \lambda_h \rangle_\Gamma$$

for $\hat{u}_h = (u_h, \lambda_h), \hat{v}_h = (v_h, \mu_h) \in V_h^k$.

From the k-consistency (5.2) of the discrete bilinear forms a_h^{E} and (5.5) of m_h^{E}, it immediately follows that $\mathcal{B}_{\kappa,h}$ satisfies (H2.a). Furthermore, the continuity of the bilinear forms a_h and m_h, as well as the continuity of V_κ, ensure (H2.b). Analogously, (H3.a), is a consequence of the continuity of m_h and of Lemma 2.1.

To verify the $\widetilde{V}_h^k$-ellipticity (H3.b), we focus on the term

$$\mathcal{B}_{0,h}(\hat{v}_{0h}, \hat{v}_{0h}) = a_h(v_h, v_h) + 2\langle \mu_{0h}, \mathrm{V}_0 \mu_{0h} \rangle_\Gamma$$

for $\hat{v}_{0h} = (v_h, \mu_{0h}) \in \widetilde{V}_h^k$. In order to bound from below the first and the second term in the above sum, we use (5.6) and Theorem 6.22 in [50] respectively, and we get

$$\mathcal{B}_{0,h}(\hat{v}_{0h}, \hat{v}_{0h}) \gtrsim \|v_h\|_{H^1(\Omega)}^2 + \|\mu_{0h}\|_{H^{-1/2}(\Gamma)}^2 = \|\hat{v}_{0h}\|_V^2 .$$

Finally, Assumption (H3.c) is a direct consequence of the k-consistency of a_h^{E}.

5.3. The discrete linear operator $\mathcal{L}_{f,h}$: validity of (H4.a). In the present section, we define the discrete linear operator $\mathcal{L}_{f,h} : V_h^k \to \mathbf{C}$ such that

$$\mathcal{L}_{f,h}(\hat{v}_h) := \begin{cases} \sum\limits_{E\in\mathcal{T}_h} (f, \Pi_1^{0,E} v_h)_{L^2(E)} & k = 1, 2\,, \\ \sum\limits_{E\in\mathcal{T}_h} (f, \Pi_{k-2}^{0,E} v_h)_{L^2(E)} & k \geq 3. \end{cases}$$

Assuming $f \in H^{s-1}(\Omega)$ with $1 \leq s \leq k$, in [16] (see Lemma 3.4) it has been proved

$$|\mathcal{L}_f(\hat{v}_h) - \mathcal{L}_{f,h}(\hat{v}_h)| \lesssim h^s \, |f|_{H^{s-1}(\Omega)} \, \|v_h\|_{H^1(\Omega)} \, .$$

Hence, Assumption (H4.a) is fulfilled with $\sigma_s(f) = |f|_{H^{s-1}(\Omega)}$.

6. Numerical results. In this section, we present some numerical test to validate the theoretical results and to show the effectiveness of the proposed method. We start by considering the coarse mesh associated with the level of refinement zero (lev. 0) and all the successive refinements are obtained by halving each side of its elements.

Recalling that the approximate solution u_h is not known inside the polygons, as suggested in [11] we compute the H^1-seminorm and L^2-norm errors, and the corresponding Estimated Order of Convergence (EOC), by means of the following formulas:

- H^1-seminorm error $\varepsilon_{\text{lev}}^{\nabla,k} := \sqrt{\dfrac{\sum_{E\in\mathcal{T}_h} \left|u - \Pi_k^{\nabla,E} u_h\right|^2_{H^1(E)}}{\sum_{E\in\mathcal{T}_h} |u|^2_{H^1(E)}}}$ and $\text{EOC} := \log_2\left(\dfrac{\varepsilon_{\text{lev}+1}^{\nabla,k}}{\varepsilon_{\text{lev}}^{\nabla,k}}\right)$;
- L^2-norm error $\varepsilon_{\text{lev}}^{0,k} := \sqrt{\dfrac{\sum_{E\in\mathcal{T}_h} \left\|u - \Pi_k^{0,E} u_h\right\|^2_{L^2(E)}}{\sum_{E\in\mathcal{T}_h} \|u\|^2_{L^2(E)}}}$ and $\text{EOC} := \log_2\left(\dfrac{\varepsilon_{\text{lev}+1}^{0,k}}{\varepsilon_{\text{lev}}^{0,k}}\right)$.

In the above formulas the superscript $k = 1, 2$ refers to the linear or quadratic order approximation of u, the subscript lev refers to the refinement level and, we recall, $\Pi_k^{\nabla,E}$ and $\Pi_k^{0,E}$ are the local H^1 and L^2-projector defined in (4.1) and (4.2), respectively.

6.1. Example 1. Constant coefficient θ. We consider Problem (2.1) with $f(\mathbf{x}) = 0$, $\theta(\mathbf{x}) = 1$ and Dirichlet condition

$$g(\mathbf{x}) = \frac{\imath}{4} H_0^{(1)} \left(\kappa |\mathbf{x} - \mathbf{x}_0|\right) \quad \text{with} \quad \mathbf{x}_0 = (0, 0), \;\mathbf{x} \in \Gamma_0, \tag{6.1}$$

$H_0^{(1)}$ being the 0-th order Hankel function of the first kind. The exact solution $u(\mathbf{x})$ is the field produced by the point source $\mathbf{x}_0$ and its expression is given by (6.1) for every $\mathbf{x} \in \mathbf{R}^2$. Let us consider the unbounded region Ω_e, external to the unitary disk. The artificial boundary Γ is the circumference of radius 2.

In Table 1, we report the total number of the degrees of freedom (d.o.f.) associated with the CVEM space, corresponding to each refinement level and the approximation orders $k = 1, 2$. We remark that due to the technical computer specifications, the maximum level of refinement we have considered is lev. 7 for $k = 1$, whose number of d.o.f. coincides with that of lev. 6 for $k = 2$. In what follows the symbol $\times$ means that the corresponding simulation has not been performed. In Tables 2 and 3, we report the errors $\varepsilon_{\text{lev}}^{\nabla,k}$ and $\varepsilon_{\text{lev}}^{0,k}$ and the corresponding EOC. As we can see, for both $\kappa = 1$ and $\kappa = 10$, the H^1-seminorm error confirms the convergence order k of the method. Although we did not provide the L^2-norm error estimate, the reported numerical results show the expected convergence order $k + 1$.

Remark 6.1. As pointed out in Remark 3.1, the assumption that κ^2 is not an eigenvalue of the Laplace problem in $\Omega \cup \overline{\Omega}_0$, endowed with a Dirichlet boundary

	lev. 0	lev. 1	lev. 2	lev. 3	lev. 4	lev. 5	lev. 6	lev. 7
$k=1$	104	368	$1,376$	$5,312$	$20,864$	$82,688$	$329,216$	$1,313,792$
$k=2$	368	$1,376$	$5,312$	$20,864$	$82,688$	$329,216$	$1,313,792$	$5,267,456$

Table 1: Example 1 (circular annulus). Total number of d.o.f. for $k=1,2$ and different lev.

		L^2-norm				H^1-seminorm			
lev.	h	$\varepsilon_{\text{lev}}^{0,1}$	EOC	$\varepsilon_{\text{lev}}^{0,2}$	EOC	$\varepsilon_{\text{lev}}^{\nabla,1}$	EOC	$\varepsilon_{\text{lev}}^{\nabla,2}$	EOC
0	$8.02e-01$	$1.64e-02$		$5.83e-04$		$5.22e-02$		$6.07e-03$	
			1.9		3.0		1.0		2.0
1	$4.28e-01$	$4.52e-03$		$7.23e-05$		$2.59e-02$		$1.54e-03$	
			1.9		3.0		1.0		2.0
2	$2.22e-01$	$1.18e-03$		$9.00e-06$		$1.29e-02$		$3.88e-04$	
			2.0		3.0		1.0		2.0
3	$1.13e-01$	$3.00e-04$		$1.12e-06$		$6.44e-03$		$9.72e-05$	
			2.0		3.0		1.0		2.0
4	$5.68e-02$	$7.56e-05$		$1.40e-07$		$3.22e-03$		$2.42e-05$	
			2.0		3.0		1.0		2.0
5	$2.85e-02$	$1.90e-05$		$1.75e-08$		$1.61e-03$		$6.07e-06$	
			2.0		3.0		1.0		2.0
6	$1.43e-02$	$4.75e-06$		$2.20e-09$		$8.04e-04$		$1.52e-06$	
			2.0		$\times$		1.0		$\times$
7	$7.14e-03$	$1.19e-06$		$\times$		$4.02e-04$		$\times$	

Table 2: Example 1 (circular annulus). Relative errors and EOC for $\kappa = 1$.

condition on Γ, turns out to be necessary for the success of the proposed method. Indeed, if κ^2 is close to one of the above mentioned eigenvalues, the accuracy of the proposed method deteriorates, the loss of precision depending on the closeness of κ^2 to the eigenvalue. To support this claim, taking into account that the spectrum of the Dirichlet-Laplace problem associated with the circular contour Γ of radius R is given by $\{\lambda_i = ({z_i}/{R})^2\}$, $\{z_i\}$ being the positive zeros of the first-kind Bessel functions, in our test we have chosen $\kappa^2 \in \{1.4457, 1.44579649, 1.44579649073669\}$, the latter value being very close to the smallest eigenvalue λ_1. For the refinement lev. 0 and the approximation order $k=1$, the corresponding L^2-norm relative errors are $2.11e-02$, $2.68e-02$ and $3.88e-01$, respectively. We note that the magnitude order of the first two errors is comparable with that reported in Table 2, while for the critical value of κ^2 very close to λ_1 we have obtained a larger error. To overcome this drawback, we exploit the property that, for a generic given Γ and a dilation factor $t>0$, if λ_i is the i-th eigenvalue associated with Γ, then ${\lambda_i}/{t^2}$ is the i-th associated with $\Gamma_t := \{t\,\widetilde{\mathbf{x}}, \widetilde{\mathbf{x}} \in \Gamma\}$. In fact, by choosing for example $t = 1.01$, we retrieve that the L^2-norm relative errors computed in the annulus bounded by Γ_0 and Γ_t, for the above choices of κ^2 are all equal to $2.19e-02$. Hence, a small variation on Γ allowed us to retrieve the expected error.

On the basis of what observed for circular artificial boundaries, since the spectrum associated to a generic contour Γ is not known, we provide a fully numerical strategy, which consists in detecting whether κ^2 is close to an eigenvalue of the corresponding interior problem and, in the case, in modifying the artificial boundary Γ in such a way that all the eigenvalues associated with the new one are sufficiently far away from κ^2. This procedure allows us to avoid a full remeshing of the computational domain and, hence, to keep as low as possible the computational cost of the global scheme.

		L^2-norm				H^1-seminorm			
lev.	h	$\varepsilon_{\text{lev}}^{0,1}$	EOC	$\varepsilon_{\text{lev}}^{0,2}$	EOC	$\varepsilon_{\text{lev}}^{\nabla,1}$	EOC	$\varepsilon_{\text{lev}}^{\nabla,2}$	EOC
0	$8.02e-01$	$6.03e-01$		$2.57e-01$		$5.77e-01$		$3.07e-01$	
			0.8		2.7		0.6		1.8
1	$4.28e-01$	$3.52e-01$		$4.00e-02$		$3.92e-01$		$8.59e-02$	
			1.5		3.2		1.1		2.0
2	$2.22e-01$	$1.33e-01$		$4.37e-03$		$1.84e-01$		$2.18e-02$	
			1.8		3.2		1.2		2.0
3	$1.13e-01$	$3.76e-02$		$4.71e-04$		$7.88e-02$		$5.49e-03$	
			1.9		3.1		1.1		2.0
4	$5.68e-02$	$9.74e-03$		$5.51e-05$		$3.65e-02$		$1.38e-03$	
			2.0		3.0		1.0		2.0
5	$2.85e-02$	$2.46e-03$		$6.75e-06$		$1.78e-02$		$3.44e-04$	
			2.0		3.0		1.0		2.0
6	$1.43e-02$	$6.16e-04$		$8.39e-07$		$8.86e-03$		$8.61e-05$	
			2.0		$\times$		1.0		$\times$
7	$7.14e-03$	$1.54e-04$		$\times$		$4.42e-03$		$\times$	

Table 3: Example 1 (circular annulus). Relative errors and EOC for $\kappa = 10$.

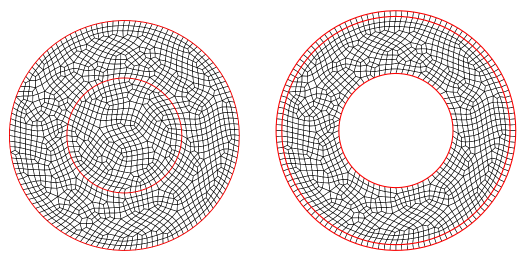

Fig. 1: Example 1. Representative meshes generated by the Algorithm 6.1.

Algorithm 6.1 relies on the property that, for any smooth boundary Γ, the eigenvalues of the Dirichlet-Laplace problem constitute a divergent sequence of positive real numbers. Moreover, it exploits some approximation properties of the continuous spectrum by means of the discrete one associated with the generalized eigenvalue problem (see for example [34] for a reference in the VEM case). In particular, assuming the set of eigenvalues in increasing order and denoting by λ_i and $\lambda_{i,h}$ the exact and the approximate i-th eigenvalue, the estimate $|\lambda_i - \lambda_{i,h}| \leq C(i)h^{2\min\{k,r\}}$ holds, k being the order of the method, r the regularity of the i-th eigenfunction and $C(i)$ an increasing function with respect to i. In the forthcoming algorithm, we will use the parameters M, tol_1 and tol_2. In particular, M denotes an initial guess number such that $\kappa^2 \leq \lambda_M$, tol_1 is the accuracy required in the computation of the discrete spectrum and, after the dilation of the artificial contour, the distance of κ^2 from the new set of eigenvalues is at least tol_2. The algorithm is structured in such a way that the output dilation parameter t satisfies $t > 1$, which guarantees both that Γ_t surrounds Ω_0 and that the computed mesh and the associated CVEM matrices, can be used in the subsequent global CVEM-BEM scheme for solving the original Helmholtz problem in the annulus Ω_t, bounded by Γ_0 and Γ_t.

We point out that, for the latter aim, it is sufficient to add a mesh discretization of the thin annulus bounded by Γ and Γ_t, with mesh size $\tilde{h}$ (see Fig. 1 (b)), and to complete the matrices $\mathbb{A}_{\tilde{h}}$ and $\mathbb{M}_{\tilde{h}}$ with the rows/columns entries corresponding

Algorithm 6.1 **input**: κ, Γ_0, Γ, M, tol_1, tol_2, **output:** t, $\mathcal{T}_{\tilde{h}}$, $\mathbb{A}_{\tilde{h}}$ and $\mathbb{M}_{\tilde{h}}$

1: **generate** two meshes, $\mathcal{T}_{0,h}$ and $\mathcal{T}_h$ of Ω_0 and Ω, both having mesh size $h \approx 1/\kappa$ and matching on the interface Γ_0 (see, e.g. Fig. 1 (a))
2: **compute** the CVEM mass $\mathbb{M}_h$ and stiffness $\mathbb{A}_h$ matrices associated to the Dirichlet-Laplace problem on the union of the two meshes
3: **refine** the current mesh and denote by $\tilde{h}$ the corresponding mesh size
4: **repeat** line 2 and retrieve $\mathbb{A}_{\tilde{h}}, \mathbb{M}_{\tilde{h}}$ with the new mesh
5: $i = 1$, $\lambda_{0,h} = \lambda_{0,\tilde{h}} = 0$
6: **while** true **do**
7: **compute** the smallest eigenvalues $\{\lambda_{j,h}\}_{j=1}^{iM}$ and $\{\lambda_{j,\tilde{h}}\}_{j=1}^{iM}$ of the generalized eigenvalue problems associated with matrices $\mathbb{A}_h, \mathbb{M}_h$ and $\mathbb{A}_{\tilde{h}}, \mathbb{M}_{\tilde{h}}$
8: **if** there exists $j \in \mathbf{N}$ such that $\lambda_{j,\tilde{h}} \leq \kappa^2 \leq \lambda_{j+1,\tilde{h}}$ **then**
9: **if** $|\lambda_{j+1,\tilde{h}} - \lambda_{j+1,h}| \leq tol_1$ **then**
10: **if** $\kappa^2 - \lambda_{j,\tilde{h}} < tol_1$ **then**
11: **find** the smallest $\ell \in \mathbf{N}$ such that $\lambda_{j+\ell+1,\tilde{h}} - \lambda_{j+\ell,\tilde{h}} > 2tol_2$
12: **if** $|\lambda_{j+\ell+1,\tilde{h}} - \lambda_{j+\ell+1,h}| \leq tol_1$ **then**
13: $t = \sqrt{\lambda_{j+\ell,\tilde{h}}/(\kappa^2 - tol_2)}$
14: **break**
15: **else**
16: $h \leftarrow \tilde{h}$; **go to** line 3
17: **end if**
18: **else if** $\lambda_{j+1,\tilde{h}} - \kappa^2 < tol_1$ **then**
19: **go to** line 11, with $\ell \in \mathbf{N} \setminus \{0\}$
20: **else**
21: $t = 1$
22: **end if**
23: **else**
24: $h \leftarrow \tilde{h}$; **go to** line 3
25: **end if**
26: **else if** $|\lambda_{iM,\tilde{h}} - \lambda_{iM,h}| > tol_1$ **then**
27: $h \leftarrow \tilde{h}$; **go to** line 3
28: **else**
29: $i = i + 1$
30: **end if**
31: **end while**

to the added degrees of freedom. Regarding line 7 of Algorithm 6.1, we remark that, the computation of a subset of the discrete spectrum instead of the whole one, is efficiently performed by using appropriate algorithms, e.g. the function *eigs* of Matlab. Finally, it is worth to point out that, if we choose the most widely used elliptic or rectangular artificial contours, for which the continuous spectrum is known, the algorithmic procedure is not necessary and the dilation factor t can be computed a priori following the criterium of the algorithm, without generating meshes and computing the CVEM matrices.

To validate the proposed algorithm, we apply the quadratic CVEM to the benchmark example in which Γ is the circle of radius 2 and $\kappa^2 = 81.6408382321$, a value close to the eigenvalue λ_{74}. We fix the parameters $tol_1 = 0.05$, $tol_2 = 0.1$ and $M = 82 \approx \kappa^2$; these revealed to be reasonable and good choices for all the numerical tests we have performed, beyond the ones reported. The dilation factor computed by the algorithm is $t \approx 1.0006130018$. The relative L^2-norm and H^1-seminorm errors associated with the mesh size $\tilde{h} = 5.03e-02$, are $3.83e-01$ and $3.50e-01$ in the domain Ω, while they are $4.06e-05$ and $1.11e-03$ in the new domain Ω_t. These latter values are coherent with those obtained in Table 3.

We conclude by pointing out that alternative methods are available in literature for the "drum problem" (see, e.g. [5]) which could be in principle used as well. However, the strategy proposed here is properly tuned for our approach, since it is based on the main ingredients of the global CVEM-BEM scheme, which are successively exploited.

		L^2-norm				H^1-seminorm			
lev.	h	$\varepsilon_{\text{lev}}^{0,1}$	EOC	$\varepsilon_{\text{lev}}^{0,2}$	EOC	$\varepsilon_{\text{lev}}^{\nabla,1}$	EOC	$\varepsilon_{\text{lev}}^{\nabla,2}$	EOC
0	$7.60e-01$	$1.71e-02$		$8.34e-04$		$1.57e-01$		$1.66e-02$	
			2.0		3.0		1.1		2.0
1	$3.85e-01$	$4.37e-03$		$1.01e-04$		$7.57e-02$		$4.07e-03$	
			2.0		3.0		1.0		2.0
2	$1.94e-01$	$1.10e-03$		$1.26e-05$		$3.78e-02$		$1.02e-03$	
			2.0		3.0		1.0		2.0
3	$9.73e-02$	$2.74e-04$		$1.57e-06$		$1.89e-02$		$2.56e-04$	
			2.0		3.0		1.0		2.0
4	$4.87e-02$	$6.86e-05$		$1.96e-07$		$9.46e-03$		$6.40e-05$	
			2.0		3.0		1.0		2.0
5	$2.44e-02$	$1.71e-05$		$2.46e-08$		$4.73e-03$		$1.60e-05$	
			2.0		2.9		1.0		2.0
6	$1.22e-02$	$4.29e-06$		$3.35e-09$		$2.36e-03$		$4.03e-06$	
			2.0		$\times$		1.0		$\times$
7	$6.10e-03$	$1.07e-06$		$\times$		$1.18e-03$		$\times$	

Table 4: Example 1 (square annulus). Relative errors and EOC for $\kappa = 1$.

Remark 6.2. Even if we have not provided theoretical results for the CVEM-BEM coupling in the case of a non sufficiently smooth artificial boundary Γ, we show that the proposed method allows us to obtain the optimal convergence order also when a polygonal Γ is considered. To this aim, let $\Omega_0 := [-1,1]^2$ and Γ be the contour of the square $[-2,2]^2$. As we can see from Table 4, the expected convergence order of the VEM-BEM approach for both H^1-seminorm and L^2-norm errors are confirmed.

6.2. Example 2. Non-constant coefficient θ. We consider here the same problem solved in Section 5 of [41] and in Section 8 of [44]. In particular, we consider Problem (2.1) with $\theta(\mathbf{x}) = 1 + \psi(\|\mathbf{x}\|)$, where $\psi(t) = (1-t^4)^2$ for $0 \le t \le 1$ and null elsewhere. The total wave, solution of the associated Helmholtz problem in the whole $\mathbf{R}^2$ with null source and incident wave $w(\mathbf{x}) = e^{\imath\kappa\mathbf{x}\cdot(\cos 1, \sin 1)}$, admits the following polar coordinates representation

$$u^{\text{tot}}(r\cos\phi, r\sin\phi) = \sum_{n=-\infty}^{\infty} y_n(r) e^{\imath n(\phi-1)},$$

whose coefficients y_n satisfy the integral equation

$$y_n(r) = \imath^n J_n(\kappa r) - \frac{\imath\kappa^2\pi}{2}\int_0^1 \mathcal{G}_n(r,\rho) y_n(\rho)(1-\theta(\rho))\rho d\rho, \tag{6.2}$$

where $J_n(\cdot)$ denotes the n-th first-kind Bessel function and

$$\mathcal{G}_n(r,\rho) := \begin{cases} J_n(\kappa r) H_n^{(1)}(\kappa\rho) & \rho \ge r, \\ J_n(\kappa\rho) H_n^{(1)}(\kappa r) & \rho < r. \end{cases}$$

By choosing $f(\mathbf{x}) = (\theta(\mathbf{x})-1)\kappa^2 w(\mathbf{x})$ and $g(\mathbf{x}) = u^{\text{tot}}(\mathbf{x}) - w(\mathbf{x})$, the solution of (2.1) is $u_e(\mathbf{x}) = u^{\text{tot}}(\mathbf{x}) - w(\mathbf{x})$. To compute u_e we apply a Nyström method to (6.2), with n large enough to guarantee an accurate approximation of u^{tot}.

We apply our method by choosing Γ_0 and Γ circumferences of radius 0.5 and 1.1, respectively. In Figure 2 we report the behaviour of the L^∞-norm absolute error, obtained for $\kappa = 2$, $k = 1,2$ and refinement lev. 0, 1, 2, 3. As we can see, the order of convergence turns out to be $k+1$. Finally, in Figure 3 we show the approximated solution and the corresponding absolute error for $\kappa = 20$, $k = 2$ and lev. 3.

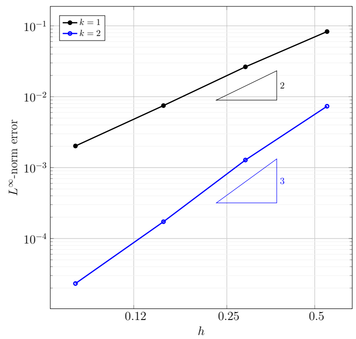

Fig. 2: Example 2. L^∞-norm absolute error for $\kappa = 2$.

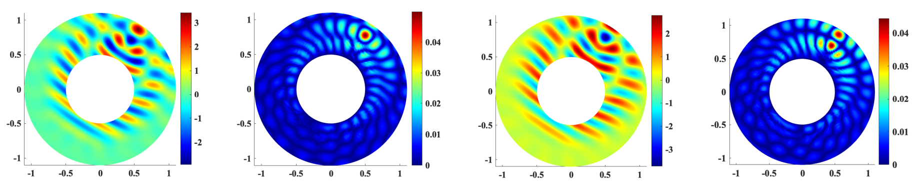

Fig. 3: Example 2. Real (first plot) and imaginary (third plot) part of the numerical solution u_e and corresponding absolute errors (second and fourth plots), for $\kappa = 20$, lev. 3 and $k = 2$.

7. Conclusions and perspectives. We have proposed a novel numerical approach for the solution of 2D Helmholtz problems defined in unbounded regions, external to bounded obstacles. This consists in reducing the unbounded domain to a finite computational one and in the coupling of the CVEM with the one equation BEM, by means of the Galerkin approach. While the VEM/CVEM has been extensively and successfully applied to interior problems, its application to exterior problems is still at an early stage and, to the best of the authors' knowledge, the CVEM has been applied in this paper for the first time to solve exterior frequency-domain wave propagation problems in the Galerkin context. We remark that the above mentioned coupling has been proposed here in a conforming approach context, so that the order of the CVEM and BEM approximation spaces have been chosen with the same polynomial degree of accuracy, and the grid used for the BEM discretization is the one inherited by the interior CVEM. It is worth noting that it is possible in principle to decouple the CVEM and the BEM discretization, both in terms of degree of accuracy and of non-matching grids. This would lead to alternative approaches such as the use of CVEM with intermediate boundary element nodes (that do not require a significantly modification of the interior mesh) or the coupling by mortar like techniques (see for instance [13]). The latter approach would offer the further advantage of coupling different types of approximation spaces and of using fast techniques for the discretization of the BEM (see for example the very recent papers [23, 14, 29, 30]). This will be the subject of a future investigation.

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