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Periodic and Non-Periodic Brainwaves Emerging via Stochastic Syncronization of Closed Loops of Firing Neurons

Piero Mazzetti and Anna Carbone *

Politecnico di Torino, 10129 Torino, Italy
* Correspondence: anna.carbone@polito.it

Abstract: Periodic and non-periodic components of electrophysiological signals are modelled in terms of syncronized sequences of closed loops of firing neurons correlated according to a Markov chain. Single closed loops of firing neurons reproduce fundamental and harmonic components, appearing as lines in the power spectra at frequencies ranging from 0.5 Hz to 100 Hz. Further interesting features of the brainwave signals emerge by considering multiple syncronized sequences of closed loops. In particular, we show that fluctuations in the number of syncronized loops lead to the onset of a broadband power spectral component. By the effects of these fluctuations and the emergence of a broadband component, a highly distorted waveform and nonstationarity of the signal are observed, consistent with empirical EEG and MEG signals. The amplitudes of the periodic and aperiodic components are evaluated by using typical firing neuron pulse amplitudes and durations.

Keywords: brainwaves power spectra; Markov-chain models; stochastic point processes

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1. Introduction

Periodic components of brain signals and their frequency bands (*delta* (1–3 Hz), *theta* (4–8 Hz), *alpha* (9–12 Hz), *beta* (12–30 Hz), *gamma* (>30 Hz)) are central to neuroscience basic research and clinical protocols [1–3]. Aperiodic components, initially disregarded in comparison to periodic ones as considered to be just background noise, represent a significant part of signals. They manifest with power spectral densities varying approximately as $1/f^{\beta}$ and have been related to brain critical states [4,5]. Recent studies have suggested that simultaneous changes in aperiodic and periodic brainwave components can underpin changes in functional and behavioural features, with broadband components modulated by task performance and correlated with neuronal spiking activity. Synchronization between different neuronal groups may also manifest within arrhythmic brain activity with no apparent periodicity [6–13]. To keep pace with these findings, algorithms are being developed with the purpose of breaking complex electrophysiological signals down and transferring scientific findings into clinical practices [14,15]. The issue of separating periodic and aperiodic components has also become increasingly relevant to the development of brain–machine interfaces [16]. Despite remarkable advances in the interpretation and quantification of neurological signals, several problems still remain unsolved, mainly related to the dynamics of brain at various scales [17].

In this work, a unified framework to quantify periodic and aperiodic power spectral components of electro- and magneto-encephalograms is developed based on a *Markov-chain* description. The power spectral density is estimated in terms of time sequences according to a statistical approach originally pioneered in the communication and information-theory context [18–24]. Line spectral components are generated by *Markov matrices* corresponding to closed-loop sequences of firing neurons characterized by a set of heterogeneous states. The oscillatory frequencies, observed in the EEG and MEG from 0.5 Hz to 100 Hz, can be reproduced by closed loops involving a few hundred neurons down to a few neurons with firing intervals of the order of a few milliseconds. The closed loop operates as an electric circuit where the neuron behaves as a rectifying diode, producing unidirectional currents; the synapses act as dissipative elements and the ionic currents as current generators. The formation of closed loops dissipates the excess energy (heat) accumulated in an active region of the brain by the local increase in circulating blood. A realistic description of the neurological signals requires bunches of synchronized sequences of closed loops of firing neurons spontaneously formed in regions of high density, where neurons are connected to thousands of other neurons nearby, that, if in a critical state, may be simultaneously fired.

The impulse generated by the bunch of syncronized sequence is represented by a Gaussian time function. The width of the Gaussian is related to the duration of the impulse emitted by the bunch and, in turn, to the characteristic cut-off filtering out the harmonic components of the power spectrum at high frequency. The amplitude of the Gaussian depends on the number of syncronized loops N_i. By taking into account the fluctuations in N_i in the Markov-chain model, a mixed power spectrum is obtained where lines and continuous spectral components co-exist. It is shown that the broadband component causes the line amplitude to change (while its frequency remains unchanged, being only dependent on the reciprocal duration of the loops) and the distortion of the signal is consistent with what is observed in the empirical EEG and MEG records. This distortion affects the lowest rather than the highest frequency components: thus, *delta waves* with harmonics at frequencies lower than the cut-off are more distorted than *gamma waves*. In general, the waves reported in electrophysiological graphs are at least distorted by second harmonic components, resulting in asymmetric triangular forms of the wave.

Tha manuscript is organized as follows. In Section 2, the general expression of the power spectral density of a sequence of Markov correlated events is recalled. The conditions required for the onset of open or closed loops in the framework of the Markov-chain description are also provided. Then, the approach is extended to an arbitrary number of synchronized closed loops, whose fluctuations cause the emergence of a broadband noise component. In Section 3, the proposed mathematical framework is used as the background to reproduce brainwaves signal features in relation to the mixed power spectrum. In Section 4, the model results are discussed. Conclusions and suggestions for future directions of the work are also provided.

2. Mathematical Framework

In this section, the mathematical background for the calculation of the power spectral density of a sequence of events correlated according to a Markov chain and the conditions to yield a closed loop are recalled. The amplitude of the periodic and aperiodic components are derived for a single loop in Section 2.1 and for an arbitrary number of synchronized closed loops in the Section 2.2.

Consider a neuron, in a state labelled α_1, making a firing to a neuron, in a state labelled α_2, in a sequence $\alpha_1, \alpha_2, \ldots, \alpha_N$ of N states. Let $n_1, n_2, \ldots, n_N$ indicate the numbers of neurons, respectively, in the states $\alpha_1, \alpha_2 \ldots \alpha_N$. The firings of n_1 synchronized neurons result in the subsequent firing of n_2 neurons, yielding a total of $n_1 \cdot n_2$ synchronized neurons, and so on. Hence, the number of synchronized neurons may reach a value of the order of several thousands in a relatively short time. For the sake of the example, if each of the $n_1, n_2, \ldots$ produces a pair of simultaneously firing neurons with a firing characteristic time of 5 ms, a total number N_i of about 10^7 synchronized neurons would be produced in only 0.1 s. The same process occurs on the neurons in the sequence $\alpha_2, \alpha_3, \ldots, \alpha_N$ of the N connected states. Thus, a bunch of N groups of N_i synchronized neurons for each state α of the sequence could be expected.

The neuron firing pulses can be described in terms of functions $F_{\alpha_i}(t)$, where t is the time and α_i is one of the N states which completely characterizes the firing. The superposition of individual firing $F_{\alpha_i}(t - t_i)$, where t_i is the time origin arbitrarily chosen for each $F_{\alpha_i}(t)$ defines the relevant neurological signal as $I(t) = \sum_{i=-\infty}^{\infty} F_{\alpha_i}(t - t_i)$. The time interval between subsequent events $F_{\alpha_i}(t)$ and $F_{\alpha_{i+1}}(t)$ is indicated by the variable u_i, which depends on the states α through the distribution function $q_{\alpha_i,\alpha_{i+1}}(u_i)$, accounting for the correlation between the time intervals u_i.

The α states are represented by an homogeneous Markov chain characterised by the $N \times N$ matrix m with entries defined as the conditional probability that if the event i is in a state α, the event $i+1$ will be in the state α', where α and α' are a pair of N states:

$$m_{\alpha_i,\alpha_{i+1}} = P(\alpha_{i+1} = \alpha' | \alpha_i = \alpha) \tag{1}$$

Then, the matrix $M(u)$ can be built with entries:

$$M_{\alpha,\alpha'}(u) = m_{\alpha,\alpha'} q_{\alpha,\alpha'}(u), \tag{2}$$

where α and α' are the states of a pair of successive events of the sequence and $q_{\alpha,\alpha'}(u)$ the distribution function of the time interval between the pair. The Fourier transform of $M_{\alpha,\alpha'}(u)$ is defined as:

$$M_{\alpha,\alpha'}(\omega) = m_{\alpha,\alpha'} Q_{\alpha,\alpha'}(\omega), \tag{3}$$

where:

$$Q_{\alpha,\alpha'}(\omega) = \int_0^\infty q_{\alpha,\alpha'}(u) \exp(i\omega u) du, \tag{4}$$

$M_{\alpha,\alpha'}(\omega)$ defines the entries of the correlation matrix $M(\omega)$ of the Markov process.

The power spectrum of random processes correlated according to a Markov chain is given by:

$$\Phi(\omega) = \nu \overline{|S_\alpha(\omega)|^2} + 2\nu \, \mathrm{Re} \sum_{\alpha\alpha'} S_\alpha^*(\omega) S_{\alpha'}(\omega) p_\alpha K(\omega)_{\alpha\alpha'} \quad , \tag{5}$$

where ν is the average number of events per unit time, $S_\alpha(\omega)$ is the Fourier transform of $F_\alpha(t)$, with the overline indicating the average over all the pulses in the sequence. $S^*_{\alpha'}(\omega)$ is the conjugate of the Fourier transform of $F_{\alpha'}(t)$ for the state α'. The quantity p_α indicates the fraction of states α in the sequence. Re means the real part. The matrix $K(\omega)$ is defined as:

$$K(\omega) = M(\omega) \cdot (I - M(\omega))^{-1} \tag{6}$$

with $M(\omega)$ the correlation matrix of the Markov process with entries defined by Equation (3) and I the identity matrix.

2.1. Line Power Spectral Density

In this subsection, we will show how to derive the power spectral density of a single closed loop of Markov-chain correlated pulses. When the Markov matrix m in Equation (1) describes random events organized in closed loop sequences, the correlation matrix M yields physically sound line spectra as those observed in brain waves. The lines in the power spectrum correspond to singularities of Equation (6) when $\det(I - M(\omega)) = 0$. The singularities occur for $q_{\alpha,\alpha'}(u) = \delta(u - u_{\alpha,\alpha'})$ where $u_{\alpha,\alpha'}$ is the time interval between consecutive events characterized by the states α, α'. Then:

$$Q_{\alpha,\alpha'}(\omega) = \exp\left(i\omega u_{\alpha,\alpha'}\right) \tag{7}$$

and Equation (3) takes the form:

$$M_{\alpha,\alpha'}(\omega) = m_{\alpha,\alpha'} \exp\left(i\omega u_{\alpha,\alpha'}\right) \quad . \tag{8}$$

A closed loop involving N states α can be expressed by a $N \times N$ matrix $M(\omega)$ where all the rows are made up of zeroes except one entry equal to 1 (with $m_{\alpha,\alpha'} = 0$ except if

$\alpha_{i+1} = \alpha'$ and $\alpha_i = \alpha \; m_{\alpha,\alpha'} = 1$). The condition $m_{N-1,N} = m_{N,1} = 1$ ensures the closeness of the loops. With the above choices, the relationship $\det(I - M(\omega)) = 0$ becomes:

$$1 - \exp[i\omega(u_{1,2} + u_{2,3} + \ldots + u_{N,1})] = 0 \tag{9}$$

with solutions:

$$\omega = \omega_n = n\omega_o = \frac{2\pi}{\sum_{i=1}^{N} u_{i,i+1}} \tag{10}$$

Equation (10) yields the fundamental frequency ω_o and the harmonics of the periodic component. n is an integer. ω_o decreases as N and $u_{i,i+1}$ increase. The amplitude of the spectral lines at $\omega = \omega_n$ is given by:

$$A_n = \frac{2\pi\nu^2}{N^2} \operatorname{Re} \sum_{\alpha\alpha'} S^*_\alpha(\omega_n) S_{\alpha'}(\omega_n) C_{\alpha\alpha'}(\omega_n) \tag{11}$$

where ν and $S^*_\alpha(\omega)S_{\alpha'}(\omega)$ have been defined after Equation (5) and $C_{\alpha\alpha'}$ are the entries of the adjoint matrix: $C(\omega_n) = \operatorname{adj}(I - M(\omega_n))$. The amplitude given by Equation (11) holds only at the singular frequency values $\omega = \omega_n$, when the determinant is zero. At frequencies ω different from the singularities ω_n, the amplitude is equal to zero as expected for strictly periodic functions. Thus, the power spectral density of the periodic components is:

$$\Phi_\ell(\omega) = \sum_{n=-\infty}^{\infty} A_n \, \delta(\omega - \omega_n) \tag{12}$$

The amplitude A_n has been estimated for different values of the parameters ν, N and Fourier transform of the single pulse S_α in [23].

The power spectral density of an arbitrary Markov-correlated pulse sequence Equation (5) was worked out by averaging over time from $-\infty$ to ∞, by assuming stationarity. In the case of a single closed-loop sequence, the power spectrum of a perfectly periodic signal is obtained, i.e., a line power spectrum taking discrete positive values at $n\,\omega_0$ and zero at any other frequency, and no broadband noise is generated. The expression within square brackets in Equation (5) corresponds to the real part of the sum of N elements of the principal diagonal of the matrix $K(\omega)$ multiplied by $S_\alpha(\omega)S_\alpha(\omega)^*$ and by p_α, while the off-diagonal elements do not contribute. The real parts of the diagonal elements are equal to $-1/2$, multiplied by $p_\alpha = 1/N$ and by 2, and give $-\overline{|S(\omega)|^2}$, which summed to the first term $\overline{|S(\omega)|^2}$ of the same equation, cancel each other. The term $-\overline{|S(\omega)|^2}$ is the average over all the N states α of the square modulus of the product of the complex conjugate transforms of a couple of identical impulses symmetric with respect to zero within the sequence extending from $-\infty$ to ∞, while similar products within square brackets in the same equation refer to couples of different impulses relative to the same state α along the sequence. When each state α remains identical along the sequence from $-\infty$ to ∞ and only their position along the sequence changes according to the conditional probability expressed by the Markov matrix m, averaging over the statistical ensemble does not change the above conclusions. The line spectrum, generated by a periodic function, remains unchanged by averaging along the sequence.

2.2. Broadband Power Spectral Density

In this subsection, the Markov matrix approach is extended to an arbitrary number N_i of synchronized closed loops of firing neurons. Under the assumption of fluctuations in N_i, the expression of the power spectrum containing a broadband component is derived. The syncronized firing of neurons belonging to the same state α will be described by the superposition of individual firing functions in terms of $F_\alpha(t) = \sum_{i=1}^{N_i} F_{i,\alpha}(t) = N_i \langle F_{i,\alpha}(t) \rangle$, where the brackets $\langle\ \rangle$ refer to the average over the ensemble of N_i loops. Owing to the large number of loops N_i and to the imperfect synchronization of the elementary impulses belonging to the same α state, the firing events are described as Gaussian functions; hence, $\langle F_{i,\alpha}(t)\rangle$ can be written as:

$$\langle F_{i,\alpha}(t)\rangle = \frac{\langle A_\alpha\rangle}{\sigma_\alpha\sqrt{2\pi}}\exp\left(-\frac{t^2}{2\sigma_\alpha^2}\right) \quad , \tag{13}$$

with variance σ_α, amplitude $\langle A_\alpha\rangle = \int_{-\infty}^{\infty}\langle F_{i,\alpha}(t)\rangle dt$ and the time origin of the impulses taken at the maximum of the Gaussian function relative to every state α. The Fourier transform of the function $F_\alpha(t)$ can be written as $S_\alpha(\omega) = N_i\langle S_{i,\alpha}(\omega)\rangle$, where:

$$\langle S_{i,\alpha}(\omega)\rangle = \langle A_\alpha\rangle \exp\left(-\frac{\omega^2\sigma_\alpha^2}{2}\right) \quad . \tag{14}$$

When the fluctuations in the number N_i of synchronized loops are taken into account in the general expression of the power spectal density, the quantity $\overline{S_\alpha(\omega)^* S_\alpha(\omega)}$ is written as $(\overline{N_i})^2\overline{(<|S_{i,\alpha}(\omega)|>)^2}$. The first term in the same equation is $\overline{(N_i^2)}\ \overline{(<|S_{i,\alpha}(\omega)|>)^2}$. Hence, the power spectrum is:

$$\Phi(\omega) = \nu\left(\overline{N_i^2} - \overline{N_i}^2\right)\overline{\langle|S_{i,\alpha}(\omega)|\rangle^2} \quad . \tag{15}$$

The quantity $\overline{N_i^2} - \overline{N_i}^2$, yielding the fluctuations in N_i around its average value $\overline{N_i}$, can be estimated by assuming that N_i is a stationary random variable described by a normalized Gaussian probability function $P(N_i)$:

$$P(N_i) = \frac{1}{\sigma_{N_i}\sqrt{2\pi}} \cdot \exp\frac{-(N_i - \overline{N_i})^2}{2\sigma_{N_i}^2} \quad , \tag{16}$$

with $(\overline{N_i})^2 = \left(\int_{-\infty}^{\infty}(N_i \cdot P(N_i))dN_i\right)^2$ and $\overline{(N_i^2)} = \int_{-\infty}^{\infty}(N_i^2 \cdot P(N_i))dN_i = (\overline{N_i})^2 + \sigma_{N_i}^2$ and the variance is σ_{N_i}. Hence, Equation (15) is:

$$\Phi(\omega) = \nu\sigma_{N_i}^2\overline{\langle|S_{i,\alpha}(\omega)|\rangle^2} \quad , \tag{17}$$

which yields the broadband component of the noise power spectrum, whose intensity has been estimated for normally distributed fluctuations in the number of syncronized loops. The amplitude of the broadband component depends on the fluctuations in N_i through the variance σ_{N_i}.

The width of the Gaussian σ_α for each state α in Equation (14) is related to the cut-off frequency of the power spectrum and, thus, to the cut-off of the highest harmonic frequencies of the periodic components. If the pulses are Gaussian with Fourier transform given by Equation (14), the cut-off frequency may be estimated as $\omega_c = 2\pi f_c = 1/\overline{\sigma}_\alpha$, corresponding to a reduction factor of $1/e \approx 0.36$. For instance, $\overline{\sigma}_\alpha = 2 \cdot 10^{-3}$ s yields a cut-off frequency $\omega_c = 500$ rad s^{-1} and $f_c = 79.5$ Hz, which, for an *alpha* wave of 8 Hz, would allow about 10 harmonics to stand in the power spectrum up to 80 Hz with only a slight reduction, while for a wave of 20 Hz a reduction in the amplitude would occur after the 4th harmonic. At frequencies lower than ω_c, the number and amplitude of the harmonics merely depend on the states characterizing the neurons forming the loops.

In summary, the main effect of the fluctuations in N_i is the onset of a mixed power spectrum, which includes spectral lines at the angular frequencies $n \cdot \omega_0$ and continuous broadband noise. Fluctuations in the parameters intrisic to a single loop give rise to changes in the synchronization with the other loops by changing the number N_i. As observed in empirical EEG records, amplitude and waveform vary almost at every period of the detected signal. According to the model proposed in this work, this change may be due to fluctuations in the number of synchronized loops N_i.

3. Discussion

In this section, the power spectrum generated by the synchronized closed-loop sequences of Markov-chain-correlated firing neurons will be confronted with typical features observed in EEG and MEG measurements. As discussed in Section 2, the sequences of electric and magnetic impulses received by the sensors on the scalp at every cycle correspond to N groups of N_i syncronized impulses generated by the firing of a single neuron in each loop. The average time interval u between subsequent neuron firings along the loop multiplied by N yields the loop duration, while its reciprocal yields the lowest frequency component of the mixed power spectrum. In principle N_i could be estimated by considering the intensity of either the elementary electric or the magnetic impulses, which are almost simultaneously produced by a single firing neuron in each sequence. More accurate estimates are obtained by using the magnetic component of the impulse, which is not attenuated by the cerebral matter and the scalp, contrarily to the electrical components of the signal [25–27]. The magnetic and electric field components $\vec{B}$ and $\vec{E}$ generated by the firing of a single neuron obey the Maxwell equation $\nabla \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \, \delta \vec{E}/\delta t$, with $\vec{J}$ and $\mu_0 \epsilon_0 \, \delta \vec{E}/\delta t$, respectively, the conduction and displacement current density. The conduction charges in the axon move at speeds ranging between 0.5 ms^{-1} and 5.0 ms^{-1}, i.e., much faster than the charges moving in the outer regions. Therefore, the electric field in the axon is partly screened by the conductive cerebral matter. The magnetic field generated by the conduction charge inside the axon can be estimated using the relationship valid for metallic conductors:

$$\vec{B}(t) = \frac{\mu_0}{4\pi} I(t)\, \Delta\ell \, \frac{\vec{u}_A \times \vec{u}_D}{r_s^2} \; ,$$

with $I(t)$ the current intensity; r_s the distance between the midpoint of the axon and the point where $\vec{B}(t)$ is measured; and $\Delta\ell$ the length of the axon, which, for neurons connected in the same area of the brain, ranges between 50 μm and 200 μm. The unit vector $\vec{u}_A$ and $\vec{u}_D$ indicate, respectively, the directions of $\Delta\ell$ and r_s. The amplitude of the magnetic field $\vec{B}(t)$ is:

$$B(t) = \frac{\mu_0}{4\pi} \, I(t)\, \Delta\ell \, \frac{\sin\theta}{r_s^2} \quad , \tag{18}$$

where θ is the angle between $\vec{u}_A$ and $\vec{u}_D$. $I(t)$ can be estimated by considering the charge transferred by the firing process to nearby neurons. When the neuron receives positive inputs from its dendrites over a short time interval, its resting membrane potential increases from about -75 mV to a critical value of about -45 mV. At this point, a positive charge Q enters the soma from the ionic channels making the membrane potential slightly positive. This process lasts about 1 ms. During a subsequent time interval of approximately the same duration (1 ms), the excess positive charge Q is ejected through the axon, restoring the membrane potential to the resting value (-75 mV) after a small over-shut. As a good approximation, the ejection of this charge corresponds to a variation in the membrane potential of about 100 mV. By approximating the soma as a sphere of radius R with uniform inner charge density ρ, the electric field can be written as $E(r) = \rho \frac{4}{3}\pi r^3 / 4\pi\epsilon_0 r^2 = \rho r / 3\epsilon_0$ at $r \leq R$ with r the distance from the center of the sphere. The membrane potential V_m is:

$$V_m = \int_0^R E(r) dr = \frac{\rho}{3\epsilon_0} \int_0^R r dr = \frac{\rho}{6\epsilon_0} R^2 = \frac{Q}{8\pi\epsilon_0 R} \tag{19}$$

where $Q = \rho \frac{4}{3}\pi R^3$ is the total charge within the sphere. A membrane potential of about $V_m = 100$ mV and a soma radius of about $R = 25$ μm result in an ejected charge $Q = 8\pi\epsilon_0 R V_m = 5.55 \cdot 10^{-16}$ C. If the ejected charge Q crosses the axon in about 1 ms, the current impulse $I(t)$ can be described by a Gaussian function of time with variance $\sigma = 1$ ms:

$$I(t) = \frac{Q}{\sigma\sqrt{2\pi}} \exp\left(\frac{t^2}{2\,\sigma^2} \right) \tag{20}$$

with the condition $\int_{-\infty}^{\infty} I(t)dt = Q$. The current $I(t)$ can be introduced in the expression of the magnetic field (Equation (18)), i.e.,

$$B(t) = \frac{\mu_0}{4\pi} \frac{Q}{\sigma\sqrt{2\pi}} \exp\left(\frac{t^2}{2\,\sigma^2}\right) \Delta\ell \frac{\sin(\theta)}{r_s^2} \quad . \tag{21}$$

The maximum value of the magnetic field amplitude is achieved at $t = 0$ with $\sin\theta = 1$. This situation, where $\sin(\theta) = 1$, occurs when the axon of the firing neuron is tangent to the surface of the skull in the position of the magnetic sensor. In the cortical area of the brain, this may happen when the firing neuron is in one of the numerous regions called gyri [28,29]. By assuming an average value of $\Delta\ell = 100$ µm and an average value of the distance of the magnetic sensor detecting the impulse of $r_s = 2 \cdot 10^{-2}$ m, an average peak intensity $B(0) = 0.5 \cdot 10^{-20}\, T$ is obtained. To generate magnetic impulses of the order of $10^{-13}\, T$, a value generally found in MEG and EEG, about $N_i = 10^7$ synchronized firing neurons would be needed (the value $10^{-13}\, T$ is close to the sensitivity limit of the SQUID sensors (about $10^{-15}\, T$)).

While it is generally accepted that the magnetic signal generated by firing neurons (MEG) is due to the ejected excess charge within the axon, the origin of the electric signal (EEG) is more controversial. A common assumption is that the signal is generated in correspondence of the chemical synapses connecting the firing neuron to the dendrites of numerous postsynaptic ones. The charge emitted from the axon of the firing neuron is split in hundreds or thousands of fractions which generate ionic currents external to the synapses, and are then detected by the sensors of the EEG setup. The synchronized firing of a large population of neurons and the role played by the excitatory and inhibitory synapses yield a complex situation of small current impulses, where spontaneous oscillations can be generated under suitable assumptions [30,31]. An alternative interpretation relates the electric signals to the transient potential impulses radially emitted externally to the soma of the firing neurons, due to the rapid variations in the external electric field generated by the charge variations inside the neuron. Within this description, the potential impulse peak value, detected by an electrode internal to the soma during the firing event, is assumed of the order of 100 mV, the electrode external to the soma peak value of about 0.1 mV, and the distance of the second electrode from the membrane of the neuron of the order of 200 µm. In order to evaluate the intensity of the electric potential impulse, we consider again the neuron with the spherical soma of radius $R = 25$ µm and the electric field $E(r)$ at $r < R$ used above for the calculation of the magnetic impulse. If the conductive medium within the cranial bone is neglected, the electric field at $r > R$ is given by $E(r) = Q/4\pi\epsilon_0 r^2$ where Q is the whole charge within the soma. The electric potential, at distance $r > R$ from the center of the sphere, is given by $V(r) = Q/4\pi\epsilon_0 r$, with $V = 0$ as $r = \infty$. The field $E(r)$ in the presence of conductive liquids, as it is the case for the brain, in stationary condition would be zero. A charge layer equal to the internal charge but with the opposite sign, would be attracted and surround the membrane of the soma. This charge with spherical symmetry cancels out the field for $r > R$. When a fast transient process occurs to the charge within the soma, as during the firing of the neuron, and the conductive liquid within the cranial bone contains positive and negative ions, the screening of the charge inside the soma is expected to occur only in part, which justifies the impulse [30]. The amplitude of this impulse, compared to the amplitude expected from Equation (19), allows to evaluate the effective charge Q_{eff} at the center of the sphere, and, thus, the amplitude of the impulse at an arbitrary distance $r > R$. By assuming a distance r from the center of the sphere of the order of 200 µm and $Q = 5.55 \cdot 10^{-16}\, C$, Equation (19) gives $V = 22.28$ mV, exceeding the typical measured value $0.1 mV$. This result could imply that the charge internal to the soma is screened by an external opposite charge which reduces the Q value to Q_{eff}. A value of Q_{eff} of the order of $10^{-18} C$, i.e., two orders of magnitude lower than Q, would yield a potential value of the order of 0.1 mV at a distance of about 200 µm and a potential value of the order of 10^{-6} V at a distance of about 10^{-2} m. It should be also considered that the electrometer input impedance is close to ∞. The impulse amplitude is expected to be reduced by the cranial bone and by the input impedance of the operational amplifiers connected to the electric sensors of the EEG set up, with a gain inversely proportional to the input resistance, which must be kept low but large enough to drive the signal recorder. By taking into account all these effects, a smaller value of the potential could be obtained.

4. Concluding Remarks

A mathematical framework, leading to the unified description of the discrete and continuous power spectral components of the EEG and MEG signals, was proposed. The model is based on the assumption that the complex network of interacting neurons spontaneously form closed-loops chains of N firing neurons, giving rise to the repetitive emission of electric and magnetic impulses. By describing the electric and magnetic impulses generated by the firing of a neuron as Gaussians of a given amplitude and width, the line power spectrum of this periodic function is obtained. Fundamental and harmonic components are shown to depend on the distribution of the amplitude and variance in the emitted impulses and on the distribution of the time intervals between their emission, which are expressions of the N states α of the sequence. A general matrix equation gives the line power spectrum.

Furthermore, by considering the fluctuations in the number N_i of synchronized firing neurons belonging to different closed loops, a broadband component emerges of the power spectrum. The syncronized pulses are represented as Gaussian time functions whose amplitude and variance depend on the degree of syncronization of the firing of the neurons belonging to different single loops but in the same state, α.

As discussed in Section 3, the fluctuations in the number of synchronized loops N_i may change the amplitude of the signal and its waveform, which is determined by the harmonic components generated by the loops. It is worthy of note that, even if the fluctuations change the waveform of the signal, the frequency of the wave remains unchanged, as it depends only on the duration of the loop $\sum_i u_i$ which is unchanged by the fluctuations in N_i. An additional effect of the broadband component concerns the onset of a cut off in the line power spectrum, which is related to the width of the electric impulses represented by the variance in the Gaussian time functions. A larger variance implies a lower cut-off frequency and smaller number of harmonic components, thus filtering out a signal closer and becoming closer to a pure sinusoid.

The main features explained by the proposed model are summarized here, below. The broad range of discrete frequencies observed in the EEG and MEG signals is obtained by considering different durations of closed-loops sequences of firing neurons. By assuming an average firing interval between successive neurons of $5 \cdot 10^{-3}$ s, sequences ranging from a few hundred neurons to only a few neurons cover the range from 0.5 Hz (the lowest frequency of *delta* waves) to 50 Hz (the highest frequency of *gamma* waves). The presence of several harmonic components of the fundamental sinusoidal wave in the graph of EEG is also accounted for, particularly at low frequency, as the *delta* waves, where a pure sinusoid is never observed. High-frequency waves are less distorted since harmonics stand at frequencies multiple of the fundamental one, and the cut-off frequency of the detected power spectrum cuts the harmonics exceeding that frequency. In a few particular cases, all harmonics are cut off, or strongly reduced, giving a nearly pure sinusoidal wave. Part of the distortion can be due to the broadband noise created by the fluctuation in the number of N_i. One of the effects discussed above, and observed in almost all EEG graphs, is the continuous change in the waveform, practically at every period, of the received signal produced by the loops. This change, which is associated to a heavy distortion of the fundamental wave, is due to the presence of several harmonic waves within the periodic signal. A change in the amplitude or the phase of one or a few of these harmonics during the fluctuation in N_i is enough to change the waveform over the period. A signal constituted by a fundamental wave and several harmonics is expected from a sequence of heterogeneous electric impulses characterized by different states α, as considered in the present paper.

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